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An effective algorithm for computing the asymptotes of an implicit curve

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ABSTRACT

In this paper, we first summarize the algorithm presented in Blasco and Pérez-Díaz (2014) for computing the *generalized asymptotes* of algebraic curves implicitly defined. This algorithm is based on the computation of Puiseux series. The main and very important contribution of this paper is a new and efficient method that allows to easily compute all the *generalized asymptotes* of an algebraic plane curve implicitly defined by just solving a triangular system of equations. The method can be easily generalized to the case of algebraic curves implicitly defined in the n-dimensional space.

1. Introduction

An algebraic curve may have more general curves than lines describing the status of a branch at the points with sufficiently large coordinates. Intuitively speaking, we say that a curve $\widetilde{\mathcal{C}}$ is a *generalized asymptote* (or *g-asymptote*) of another curve $\mathcal{C}$ if the distance between $\widetilde{\mathcal{C}}$ and $\mathcal{C}$ tends to zero as they tend to infinity, and $\mathcal{C}$ cannot be approached by a new curve of lower degree. This notion, introduced and studied by S. Pérez-Díaz in some previous papers [1–3], generalizes the classical concept of an asymptote of a curve $\mathcal{C}$ defined as a line such that the distance between $\mathcal{C}$ and the line approaches zero as they tend to infinity (see e.g. [4–6]).

Generalized asymptotes contain much of the information about the behavior of the curves in the large and additionally they are an important tool for instance, for sketching its graph. This motivates our interest in efficiently computing these entities and some important and efficient algorithms for the case of curves parametrically defined are presented in [7,8]. Additionally, some important properties concerning generalized asymptotes for implicit algebraic curves are obtained in [9] and a initial generalization for surfaces of these new concepts are presented in [10].

However, although the implicit case for algebraic curves is studied in [2,3] no efficient methods of computation are provided. More precisely, the algorithm presented in [2] is based on the computation of all the infinity branches by means of Puiseux series which turns to be very expensive and inefficient. For this purpose, the great contribution of this paper is a new and efficient method that allows to easily compute all the *generalized asymptotes* of an algebraic curve implicitly defined by just solving a triangular system of equations constructed from the polynomials defining the input curve.

In Section 2, we recall the theory of infinity branches and introduce the notions of convergent branches (branches that get closer as they tend to infinity) and approaching curves (see [1]). Section 3 provides the fundamental concepts of *perfect*

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curve (a curve of degree d that cannot be approached by any curve of degree less than d) and *g-asymptote* (a perfect curve that approaches another curve at an infinity branch), and we present an algorithm which is developed in [1,3], which is based on the computation of Puiseux series, that returns all the asymptotes of a given curve implicitly defined (see Section 3.1).

In Section 4, we develop a new and very efficient method that allows to easily compute all the generalized asymptotes of an implicitly defined algebraic curve by only determining the solutions of a triangular system of equations constructed from the implicit polynomial. The results presented are concerned with algebraic plane curves but, as we remark in the paper, they can trivially be adapted for dealing with algebraic curves in n-dimensional space (see Example 4). The method proposed implies the computation of roots of some univariate polynomials, which sometimes may require the use of algebraic numbers. For this reason, we have included an example that shows how to overcome this problem by using conjugate points and polynomial remainders (see Example 5). In Section 4.1, we show the advantage of the new algorithm and we report running times for the algorithm presented and the previous algorithm presented in Section 3.1. For this purpose, we consider ten input curves defined implicitly. Finally, a section of conclusions and future work is presented (see Section 5).

2. Notation and previous results

In this section, we introduce the notion of *infinity branch*, *convergent branches* and *approaching curves*, and we present some properties which allow us to compare the behavior of two implicit algebraic plane curves at infinity. For more details on these concepts and results, we refer to [2] (Sections 3 and 4).

Let $\mathcal{C}$ be an irreducible algebraic affine plane curve over $\mathbb{C}$ defined by the irreducible polynomial $f(x, y) \in \mathbb{R}[x, y]$. $\mathcal{C}^*$ denotes its corresponding projective curve defined by the homogeneous polynomial

$$F(x, y, z) = f_d(x, y) + zf_{d-1}(x, y) + z^2 f_{d-2}(x, y) + \cdots + z^d f_0 \in \mathbb{R}[x, y, z],$$

where $d := \deg(\mathcal{C})$ and $f_j(x, y)$ are the homogeneous forms of degree j, for $j = 0, \ldots, d$. Throughout the paper, we assume that (0:1:0) is not an infinity point of $\mathcal{C}^*$ (otherwise, we may consider a linear change of coordinates).

In order to get the infinity branches of $\mathbb{C}$, we consider the curve defined by the polynomial $g(y, z) = F(1 : y : z)$ and we compute the series expansion for the solutions of $g(y, z) = 0$ around $z = 0$. There exist exactly $\deg_y(g)$ solutions given by different Puiseux series that can be grouped into conjugacy classes. More precisely, if $\varphi(z) = m + a_1 z^{N_1/N} + a_2 z^{N_2/N} + a_3 z^{N_3/N} + \cdots \in \mathbb{C}\langle\langle z\rangle\rangle$, $a_i \neq 0$, $\forall i \in \mathbb{N}$, where $N \in \mathbb{N}$, $N_i \in \mathbb{N}$, $i \in \mathbb{N}$, and $0 < N_1 < N_2 < \cdots$, is a Puiseux series such that $g(\varphi(z), z) = 0$, and $\nu(\varphi) = N$ (N is the called *ramification index* of φ), the series $\varphi_j(z) = m + a_1 c_j^{N_1} z^{N_1/N} + a_2 c_j^{N_2} z^{N_2/N} + a_3 c_j^{N_3} z^{N_3/N} + \cdots$, where $c_j^N = 1$, $j \in \{1, \ldots, N\}$, are called the *conjugates* of φ. The set of all the conjugates of φ is called the *conjugacy class* of φ and it contains $\nu(\varphi)$ different series.

Since $g(\varphi(z), z) = 0$ in some neighborhood of $z = 0$ where $\varphi(z)$ converges, there exists $M \in \mathbb{R}^+$ such that $F(1 : \varphi(t) : t) = g(\varphi(t), t) = 0$ for $t \in \mathbb{C}$ and $|t| < M$, which implies that $F(t^{-1} : t^{-1}\varphi(t) : 1) = f(t^{-1}, t^{-1}\varphi(t)) = 0$, for $t \in \mathbb{C}$ and $0 < |t| < M$. We set $t^{-1} = z$, and we obtain that $f(z, r(z)) = 0$ for $z \in \mathbb{C}$ and $|z| > M^{-1}$ where

$$r(z) = z\varphi(z^{-1}) = mz + a_1 z^{1-N_1/N} + a_2 z^{1-N_2/N} + a_3 z^{1-N_3/N} + \cdots, \quad a_i \neq 0,\ \forall i \in \mathbb{N}$$

$N, N_i \in \mathbb{N}$, $i \in \mathbb{N}$, and $0 < N_1 < N_2 < \cdots$.

One may reason similarly with the N different series in the conjugacy class, $\varphi_1, \ldots, \varphi_N$. Since in [2], we prove that all the results hold independently on the chosen series in the conjugacy class in the following, we consider any representant in the conjugacy class and we introduce the following definition.

Definition 1. An *infinity branch of a plane curve* $\mathcal{C}$ associated to the infinity point $P = (1 : m : 0)$, $m \in \mathbb{C}$, is a set $B = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\}$, $M \in \mathbb{R}^+$,

$$r(z) = z\varphi(z^{-1}) = mz + a_1 z^{1-N_1/N} + a_2 z^{1-N_2/N} + a_3 z^{1-N_3/N} + \cdots, \tag{2.1}$$

where $N, N_i \in \mathbb{N}$, $i \in \mathbb{N}$, and $0 < N_1 < N_2 < \cdots$.

Now, we introduce the notions of convergent branches and approaching curves. Intuitively speaking, two infinity branches converge if they get closer as they tend to infinity. This concept will allow us to analyze whether two curves approach each other.

Definition 2. Two infinity branches, $B = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\} \subset B$ and $\overline{B} = \{(z, \overline{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > \overline{M}\} \subset \overline{B}$, are convergent if $\lim_{z\to\infty}(\overline{r}(z) - r(z)) = 0$.

Theorem 1 provides a characterization for the convergence of two infinity branches.

Theorem 1. *Two branches $B = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\}$ and $\overline{B} = \{(z, \overline{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > \overline{M}\}$ are convergent if and only if the terms with non negative exponent in the series $r(z)$ and $\overline{r}(z)$ are the same. Hence, two convergent infinity branches are associated to the same infinity point.*

This paper is concerned with the study of the asymptotes of an implicit algebraic curve. The classical concept of asymptote has to be with a line that *approaches* a given curve at the infinity. In the following we generalize this idea in the sense that two curves *approach each other* if they have two infinity branches that converge (see Definition 3 and Theorem 2 below).

Definition 3. Let $\mathcal{C}$ be an algebraic plane curve with an infinity branch B. We say that a curve $\overline{\mathcal{C}}$ *approaches* $\mathcal{C}$ at B if $\lim_{z\to\infty} d((z, r(z)), \overline{\mathcal{C}}) = 0$.

Theorem 2. *Let $\mathcal{C}$ be a plane algebraic curve with an infinity branch B. A plane algebraic curve $\overline{\mathcal{C}}$ approaches $\mathcal{C}$ at B if and only if $\overline{\mathcal{C}}$ has an infinity branch, $\overline{B}$, such that B and $\overline{B}$ are convergent.*

It is clear that $\mathcal{C}_1$ approaches $\mathcal{C}_2$ if and only if $\mathcal{C}_2$ approaches $\mathcal{C}_1$. When it happens, we say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *approaching curves* or that they *approach each other*. In the next section we use this concept to generalize the classical notion of asymptote of a curve.

3. Asymptotes of an algebraic curve

Let $\mathcal{C}$ be an algebraic plane curve and B an infinity branch of $\mathcal{C}$. In Section 2, we have described how $\mathcal{C}$ can be approached at B by a second curve $\overline{\mathcal{C}}$. Let us assume that $\deg(\overline{\mathcal{C}}) < \deg(\mathcal{C})$. Then, one may say that $\mathcal{C}$ *degenerates* since it behaves at infinity as a curve of smaller degree. For instance, a hyperbola is a curve of degree two that has two real asymptotes, which implies that the hyperbola degenerates, at infinity, to two lines. Similarly, an ellipse has two asymptotes that, in this case, are complex lines. However, the asymptotic behavior of a parabola is different since it cannot be approached at infinity by any line. This motivates the definition of *perfect curve*.

Definition 4. An algebraic curve of degree d is a *perfect curve* if it cannot be approached by any curve of degree less than d.

A curve that is not perfect can be approached by other curves of smaller degree. If these curves are perfect, we call them *g-asymptotes*.

Definition 5. Let $\mathcal{C}$ be a curve with an infinity branch B. A *g-asymptote* (generalized asymptote) of $\mathcal{C}$ at B is a perfect curve that approaches $\mathcal{C}$ at B.

The notion of *g-asymptote* is similar (in fact it is a generalization) to the classical concept of asymptote. The difference is that a g-asymptote is not necessarily a line, but a perfect curve (see Definition 4). Throughout the paper we refer sometimes to *g-asymptote* simply as *asymptote*.

We remark that the degree of an g-asymptote is less than or equal to the degree of the curve it approaches. In fact, a g-asymptote of a curve $\mathcal{C}$ at a branch B has minimal degree among all the curves that approach $\mathcal{C}$ at B.

In Section 3.1, we show that every infinity branch of a given algebraic plane curve implicitly defined has, at least, one asymptote and we show how to compute it. For this purpose, we rewrite Eq. (2.1) defining a branch B (see Definition 1) as

$$r(z) = mz + a_1 z^{1-n_1/n} + \cdots + a_k z^{1-n_k/n} + a_{k+1} z^{1-N_{k+1}/N} + \cdots \tag{3.1}$$

where $0 < N_1 < \cdots < N_k \leq N < N_{k+1} < \cdots$ and $\gcd(N, N_1, \ldots, N_k) = b$, $N = n \cdot b$, $N_j = n_j \cdot b$, $j \in \{1, \ldots, k\}$. That is, we simplify the non negative exponents such that $\gcd(n, n_1, \ldots, n_k) = 1$. Note that $0 < n_1 < n_2 < \cdots$, and $n_k \leq n$, and $N < N_{k+1}$, i.e. the terms $a_j z^{1-N_j/N}$ with $j \geq k+1$ are those which have negative exponent. We denote these terms as $A(z) := \sum_{\ell=k+1}^{\infty} a_\ell z^{-q_\ell}$, where $q_\ell = 1 - N_\ell/N \in \mathbb{Q}^+$, $\ell \geq k+1$.

Under these conditions, we say that n is *the degree of* B, and we denote it by $\deg(B)$.

3.1. Construction of a g-asymptote

Taking into account Theorems 1 and 2, we have that any curve $\overline{\mathcal{C}}$ approaching $\mathcal{C}$ at B should have an infinity branch $\overline{B} = \{(z, \overline{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > \overline{M}\}$ such that the terms with non negative exponent in $r(z)$ and $\overline{r}(z)$ are the same. In the simplest case, if $A = 0$ (there are no terms with negative exponent, see Eq. (3.1)), we get

$$\tilde{r}(z) = mz + a_1 z^{1-n_1/n} + a_2 z^{1-n_2/n} + \cdots + a_k z^{1-n_k/n}, \tag{3.2}$$

where $a_1, a_2, \ldots \in \mathbb{C} \setminus \{0\}$, $m \in \mathbb{C}$, $n, n_1, n_2 \ldots \in \mathbb{N}$, $\gcd(n, n_1, \ldots, n_k) = 1$, and $0 < n_1 < n_2 < \cdots$. Note that $\tilde{r}$ has the same terms with non negative exponent as r, and $\tilde{r}$ does not have terms with negative exponent.

Let $\widetilde{\mathcal{C}}$ be the plane curve containing the branch $\widetilde{B} = \{(z, \tilde{r}(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > \widetilde{M}\}$ (note that $\widetilde{\mathcal{C}}$ is unique since two different irreducible algebraic curves have finitely many common points). Observe that

$$\widetilde{\mathcal{Q}}(t) = (t^n, mt^n + a_1 t^{n-n_1} + \cdots + a_k t^{n-n_k}) \in \mathbb{C}[t]^2,$$

where $n, n_1, \dots, n_k \in \mathbb{N}$, $\gcd(n, n_1, \dots, n_k) = 1$, and $0 < n_1 < \cdots < n_k$, is a polynomial parametrization of $\widetilde{\mathcal{C}}$, and it is proper (see Lemma 3 in [1]). In Theorem 2 in [1], we prove that $\widetilde{\mathcal{C}}$ is a g-asymptote of $\mathcal{C}$ at B.

From this construction, we obtain the following algorithm that computes an asymptote for each infinity branch of a given plane curve. We illustrate it with an example. We assume that we have prepared the input curve $\mathcal{C}$, by means of a suitable linear change of coordinates, such that (0:1:0) is not an infinity point of $\mathcal{C}$.

Algorithm *Asymptotes Construction-Implicit Case.*

Given a plane curve $\mathcal{C}$ implicitly defined by an irreducible polynomial $f(x, y) \in \mathbb{R}[x, y]$, the algorithm computes one asymptote for each of its infinity branches.

1. Compute the infinity points of $\mathcal{C}$. Let $P_1, \dots, P_n$ be these points.
2. For each $P_i := (1 : m_i : 0)$ do:

2.1. Compute the infinity branches of $\mathcal{C}$ associated to P_i. Let $B_{ij} = \{(z, r_{ij}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_{ij}\}$, $j \in \{1, \dots, s_i\}$, be these branches, where r_{ij} is written as in Eq. (3.1). That is,

$$r_{ij}(z) = m_i z + a_{1,i,j} z^{1-n_{1,i,j}/n_{ij}} + \cdots + a_{k_{ij},i,j} z^{1-n_{k_{ij},i,j}/n_{ij}} + A_{ij}(z),$$

$$A_{ij}(z) = \sum_{\ell=k_{ij}+1}^{\infty} a_{\ell,i,j} z^{-q_{\ell,i,j}}, \quad q_{\ell,i,j} = 1 - N_{\ell,i,j}/N_{ij} \in \mathbb{Q}^+, \quad \ell \ge k_{ij} + 1,$$

$a_{1,i,j}, a_{2,i,j}, \dots \in \mathbb{C} \setminus \{0\}$, $n_{ij}, n_{1,i,j}, \dots \in \mathbb{N}$, $0 < n_{1,i,j} < n_{2,i,j} < \cdots$, $n_{k_{ij}} \le n_{ij}$, $N_{ij} < n_{k_{ij}+1}$, and $\gcd(n_{ij}, n_{1,i,j}, \dots, n_{k_{ij},i,j}) = 1$.

2.2. For each branch B_{ij}, $j \in \{1, \dots, s_i\}$ do:

2.2.1. Consider $\tilde{r}_{ij}$ as in Eq. (3.2). That is,

$$\tilde{r}_{ij}(z) = m_i z + a_{1,i,j} z^{1-n_{1,i,j}/n_{ij}} + \cdots + a_{k_{ij},i,j} z^{1-n_{k_{ij},i,j}/n_{ij}}$$

Note that $\tilde{r}_{ij}$ has the same terms with non negative exponent as r_{ij}, and $\tilde{r}_{ij}$ does not have terms with negative exponent.

2.2.2. Return the asymptote $\widetilde{\mathcal{C}}_{ij}$ defined by the proper parametrization $\widetilde{\mathcal{Q}}_{ij}(t) = (t^{n_{ij}}, \tilde{r}_{ij}(t^{n_{ij}})) \in \mathbb{C}[t]^2$.

Example 1. Let $\mathcal{C}$ be the curve of degree $d = 6$ defined by the irreducible polynomial $f(x, y) = -9xy^5 + 2y^6 - 144x^3y^2 - 400x^2y^3 + 159xy^4 - 24y^5 - 360x^3y + 2872x^2y^2 - 929xy^3 + 53y^4 - 225x^3 + 9303x^2y + 2855xy^2 + 114y^3 + 6360x^2 + 4966xy + 508y^2 - 508x \in \mathbb{R}[x, y]$.

We apply algorithm Asymptotes Construction-Implicit Case to compute the asymptotes of $\mathcal{C}$.

Step 1: The infinity points are $P_1 = (1 : 0 : 0)$ and $P_2 = (2 : 9 : 0)$.

We first consider P_1:

Step 2.1: There are three branches associated to P_1, $B_{1j} = \{(z, r_{1j}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_1\}$, $j = 1, 2, 3$, where

$$r_{11}(z) = -5/4 + 207/128z^{-1} - 19757/8192z^{-2} + 4386031/1048576z^{-3} \cdots,$$
$$r_{12}(z) = -5/4 + 121/1152z^{-1} - 6413/663552z^{-2} + 958441/764411904z^{-3} + \cdots,$$
$$r_{13}(z) = 19/6 + z^{1/3}16^{2/3} - z^{2/3}16^{1/3} + 5/96z^{-1/3}16^{4/3} + 437/36864z^{-2/3}16^{5/3} + 1/2z^{-1} + \cdots$$

(we compute r_{1j}, $j = 1, 2, 3$, using the algcurves package included in the computer algebra system Maple; in particular, we use the command puiseux).

Step 2.2.1: We compute $\tilde{r}_{1j}(z)$, $j = 1, 2, 3$, and we have that

$$\tilde{r}_{11}(z) = -5/4, \;\; \tilde{r}_{12}(z) = -5/4, \;\; \tilde{r}_{13}(z) = 19/6 + z^{1/3}16^{2/3} - z^{2/3}16^{1/3}.$$

Step 2.2.2: The parametrizations of the asymptotes $\widetilde{\mathcal{C}}_j$, $j = 1, 2, 3$, are given by

$$\widetilde{\mathcal{Q}}_1(t) = (t, \, -5/4), \quad \widetilde{\mathcal{Q}}_2(t) = (t, \, -5/4), \quad \widetilde{\mathcal{Q}}_3(t) = (t^3, 19/6 + 16^{2/3}t - 16^{2/3}t^2).$$

Note that the curve $\widetilde{\mathcal{C}}_1$ is the same than the curve $\widetilde{\mathcal{C}}_2$. One may compute the polynomial defining implicitly $\widetilde{\mathcal{C}}_3$ (apply e.g. Chapter 4 in [11]), and we have that

$$\tilde{f}_1(x, y) = -216y^3 - 3456x^2 - 10368xy + 2052y^2 + 88128x - 6498y + 6859 \in \mathbb{R}[x, y].$$

Now, we analyze the point P_2:

Step 2.1: We have only one infinity branch associated to P_2 is $B_2 = \{(z, r_2(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M_2\}$, where $r_2(z) = 5 + 9/2z - 29/9z^{-1} + 16/81z^{-2} - 5230/729z^{-3} + \cdots$.

Step 2.2.1: We obtain that $\tilde{r}_2(z) = 5 + 9/2z$.

Step 2.2.2: The parametrization of the asymptote $\widetilde{\mathcal{C}}_4$ is given by $\widetilde{\mathcal{Q}}_4(t) = (t, \, 5 + 9/2t) \in \mathbb{R}[t]^2$. One may compute the polynomial defining implicitly $\widetilde{\mathcal{C}}_4$, and we have $\tilde{f}_4(x, y) = 9x + 10 - 2y \in \mathbb{R}[x, y]$.

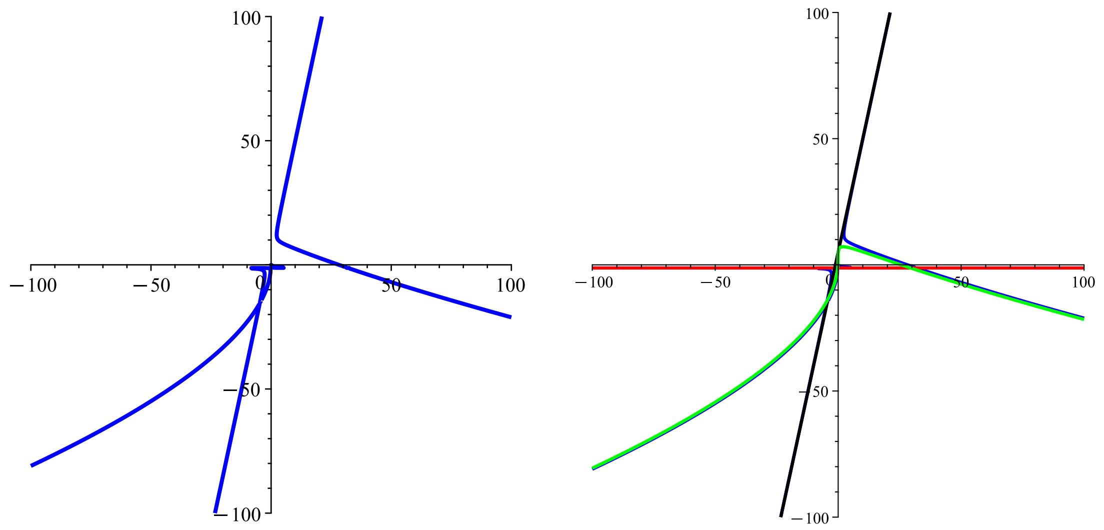

Fig. 1. Curve $\mathcal{C}$ (left), and curve and asymptotes (right).

In Fig. 1, we plot the curve $\mathcal{C}$, and the asymptotes $\widetilde{\mathcal{C}}_1$, $\widetilde{\mathcal{C}}_3$ and $\widetilde{\mathcal{C}}_4$.

4. New efficient method for the implicit algebraic curves

In this section, we present an improvement of the method described above, which avoids the computation of infinity branches and Puiseux series. Instead, we simply have to determine the solutions of a triangular system of equations constructed from the implicit polynomial.

The results presented are concerned with algebraic plane curves but, as we remark, they can trivially be adapted for dealing with algebraic curves in n-dimensional space (see Example 4).

We recall that we are assuming that $(0{:}1{:}0)$ is not an infinity point; otherwise, we consider a change of coordinates.

We start with the main theorem that provides a constructive method for determining all the asymptotes associated to the infinity points.

Theorem 3. *Let $\mathcal{C}$ be a plane algebraic curve defined by the irreducible polynomial $f(x,y) \in \mathbb{R}[x,y]$ of degree d. The g-asymptotes of $\mathcal{C}$ are defined by the parametrizations*

$$\widetilde{\mathcal{Q}}(t) = (t^k,\ b_0t^k + b_1t^{k-1} + \cdots + b_k), \quad 1 \le k \le d$$

with $b_i \in \mathbb{C},\ i = 0,\ldots,k$ satisfying that

$$\Lambda_i(b_0,\ldots,b_i) = 0,\ i = 0,\ldots,k, \qquad \Lambda_i(b_0,\ldots,b_k) = 0,\ i = k+1,\ldots,u,\ u \ge k+1$$

where $f(\widetilde{\mathcal{Q}}(t)) =$

$$\Lambda_0(b_0)t^{d\cdot k} + \Lambda_1(b_0,b_1)t^{d\cdot k-s_1} + \cdots + \Lambda_u(b_0,b_1,\ldots,b_k)t^{d\cdot k-s_u} + \sum_{j=u+1}^{v}\Lambda_j(b_0,b_1,\ldots,b_k)t^{d\cdot k-s_j}$$

$0 < s_1 < s_2 < \cdots < s_v \le d\cdot k$, $s_j \in \mathbb{N}$, $j = 1,\ldots,v$. Furthermore, $b_0 = m$ is a root of $\Lambda_0(b_0) = 0$ if and only if $(1:m:0)$ is an infinity point.

Proof. The idea of the proof is that given a branch, the substitution in the implicit function $f(x,y)$ must converge at infinity and this implies that the terms of positive exponents of this substitutions must be 0 (see equality (4.1)). We do not use the complete branch (because in reality this branch cannot be calculated in its entirety) but the truncated branch, and from it and its substitution in $f(x,y)$, we obtain certain coefficients that must be zero (see equality (4.3)).

For this purpose, we first recall that $B = \{(z, r(z)) \in \mathbb{C}^2 : z \in \mathbb{C},\ |z| > M\}$, $M \in \mathbb{R}^+$, and

$$r(z) = z\varphi(z^{-1}) = b_0z + b_1z^{1-N_1/N} + b_2z^{1-N_2/N} + b_3z^{1-N_3/N} + \cdots$$

where $N, N_i \in \mathbb{N},\ i \in \mathbb{N}, 0 < N_1 < \cdots < N_k = N < N_{k+1} < \cdots,$

$$\varphi(z) = b_0 + b_1 z^{N_1/N} + b_2 z^{N_2/N} + b_3 z^{N_3/N} + \cdots .$$

Let

$$\gamma(z) = \varphi(z^N) = b_0 + b_1 z^{N_1} + b_2 z^{N_2} + b_3 z^{N_3} + \cdots$$

and $\Gamma(z, w)$ be the homogenization of $\gamma(z)$. It holds that $F(z^N, z^N\gamma(z^{-1}), 1) =$

$$F(z^N,\ b_0 z^N + b_1 z^{N-N_1} + b_2 z^{N-N_2} + b_3 z^{N-N_3} + \cdots + b_k + b_{k+1} z^{N-N_{k+1}} + \cdots,\ 1) = 0$$

which is equivalent to

$$F(1, \Gamma(1, w), w^N) = F(1,\ b_0 + b_1 w^{N_1} + b_2 w^{N_2} + \cdots + b_k w^N + b_{k+1} w^{N_{k+1}} + \cdots,\ w^N) = 0 \tag{4.1}$$

where $N - N_j \geq 0,\ j = 1, \ldots, k$ and $N - N_j < 0,\ j \geq k+1$. Let us denote

$$\mathcal{E}_1(w) := F(1, \Gamma(1, w), w^N).$$

Now, we consider the truncated branch. More precisely, let

$$r^\star(z) = b_0 z^N + b_1 z^{N-N_1} + b_2 z^{N-N_2} + \cdots + b_k$$

and $R^\star(z, w)$ its homogenization (note that $r^\star(z)$ provides the same asymptote than considering the polynomial $\tilde{r}(z)$, see Eq. (3.2), except that possibly the asymptote is not proper since $\gcd(N, N_1, \ldots, N_k) = b$ and we have not simplified the non negative exponents; see Eq. (3.1)). Then, we consider the polynomial $F(z^N, R^\star(z, w), w^N)$ (which is not identically zero since we have truncated the branch), and let

$$\mathcal{E}_2(w) := F(1, R^\star(1, w), w^N) = F(1, b_0 + b_1 w^{N_1} + b_2 w^{N_2} + \cdots + b_k w^N, w^N). \tag{4.2}$$

Finally, we consider the previous equality in the affine chart,

$$\mathcal{E}_3(z) := F(z^N, r^\star(z), 1) = f(z^N, b_0 z^N + b_1 z^{N-N_1} + b_2 z^{N-N_2} + \cdots + b_k) =$$

$$\Lambda_0(b_0) z^{N\cdot d} + \Lambda_1(b_0, b_1) z^{N\cdot d - s_1} + \cdots + \Lambda_u(b_0, b_1, \ldots, b_k) z^{N\cdot d - s_u} + \sum_{j=u+1}^{v} \Lambda_j(b_0, b_1, \ldots, b_k) z^{N\, d - s_j} \tag{4.3}$$

$$0 < s_1 < s_2 < \cdots < s_v \leq N\cdot d,\quad s_j \in \mathbb{N},\ j = 1, \ldots, v.$$

Now, from equalities (4.1) and (4.2), we get the coefficients Λ_i in equality (4.3) that are the numbers we are looking for constructing the asymptote $\widetilde{\mathcal{Q}}$ introduced in the statement of the theorem. More precisely, we observe that equalities (4.1) and (4.2) can be written as

$$\mathcal{E}_2(w) = \Lambda_0(b_0) + \Lambda_1(b_0, b_1) w^{s_1} + \cdots + \Lambda_u(b_0, b_1, \ldots, b_k) w^{s_u} + \cdots + \Lambda_v(b_0, b_1, \ldots, b_k) w^{s_v}$$

$$0 = \mathcal{E}_1(w) = \mathcal{E}_2(w) + \Lambda_{v+1}(b_0, b_1, \ldots, b_k) w^{s_{v+1}} + \cdots,$$

where $0 < s_1 < s_2 < \cdots,\ \ s_j \in \mathbb{N}$. Hence,

$$\Lambda_0(b_0) = \Lambda_1(b_0, b_1) = \cdots = \Lambda_j(b_0, b_1, \ldots, b_k) = 0, \quad j = 1, \ldots, v.$$

Note that we show that all $\Lambda_j(b_0, b_1, \ldots, b_k) = 0, \quad j = 1, \ldots, v$. However, some equations may become redundant and that a subset of equations up to index $u \leq v$ may be enough.

Finally, we observe that

$$0 = F(1, \varphi(0), 0) = F(1, \Gamma(1, 0), 0) = F(1, R^\star(1, 0), 0) = F(1, b_0, 0)$$

which is equivalent to $0 = \mathcal{E}_1(0) = \mathcal{E}_2(0) = \Lambda_0(b_0)$. From this equation, we obtain the values of b_0 and we deduce that $b_0 = m$ is a root of $\Lambda_0(b_0) = 0$ if and only if $(1 : m : 0)$ is an infinity point. □

Remark 1. Since each root of $\Lambda_0(b_0) = 0$ provides an infinity point then, for each root of $\Lambda_0(b_0) = 0$, we get the infinity branches associated to this infinity point and then, the corresponding asymptotes.

We note that redundant equations are obtained (some Λ_{i_0} could be identically zero) and that is why additional equations ($\Lambda_i(b_0, \ldots, b_k) = 0,\ i = k+1, \ldots, u,\ \ u \geq k+1$) must be considered until all the indeterminate coefficients can be computed. The authors have not been able to determine the exact value of u that would fix the number of equations to consider. Although it is known that u is intimately related to the possible singular character of the infinity point, it is left as a future work to be able to determine the exact number u and therefore, the exact equations to be considered.

Remark 2.

1. Note that we do not know the degree of the asymptote but a bound ($1 \leq k \leq d$). Therefore, in order to compute the parametrization of the asymptote $\widetilde{\mathcal{Q}}(t)$, we can start with $k = d$ and the constructive method presented in Theorem 3 provides the proper parametrizations of the asymptotes of degree d (if they exist) and all the non-proper parametrizations of the asymptotes of degree $k^\star \in \mathbb{N}$, where $k^\star$ divides d (if they exist). In this case, a reparametrization of the original parametrization provides a proper parametrization of these asymptotes of degree $k^\star \in \mathbb{N}$. Afterward, one repeat the process with $2 \leq k \leq d-1$, and $k \notin \{k^\star, d\}$. Also we can reason as we suggest in the following statement.
   In order to reparametrize properly a rational parametrization of an algebraic curve one may apply the method presented in [12].
2. Let $f_d(x,y) = (y-mx)^\ell \prod_{i=1}^{\kappa}(y-m_ix)^{\ell_i}$. We consider the factor $(y-mx)^\ell$ that provides the infinity point $(1:m:0)$ (we reason similarly for the factors $(y-m_ix)^{\ell_i}$, $i = 1,\dots,\kappa$). It holds that $\ell = n_1 + \cdots + n_r$, where r are the number of branches associated to $(1:m:0)$, and n_i is the degree of each asymptote obtained from each branch (see [2,7]). We can know in advance the number of branches r (use, for instance, the algcurves package included in the computer algebra system Maple) and then we have a bound for n_i. More precisely, if $n_1 \leq n_2 \leq \cdots \leq n_r$, we have that
   $$n_1 \leq \lfloor \ell/r \rfloor \leq n_r.$$
   Hence, in order to apply the constructive method presented in Theorem 3, one consider first asymptotes of degree $\lfloor \ell/r \rfloor$ and we determine, at least, one asymptote of degree n_1. Once we have n_1, we compute n_2 reasoning as before and considering $\ell - n_1 = n_2 + \cdots + n_r$. Observe that in this case, we obtain proper parametrizations defining the asymptotes (see [1]).
   Finally, we note that if $\ell = 1$ then $r = 1$ and $n_i = 1$.

In the following, we introduce the Algorithm Improvement Asymptotes Construction-Implicit Case I, which uses the above results for computing the g-asymptotes of a plane curve. In fact, we apply Remark 1 and statement 2 in Remark 2, and we compute the number of branches for each infinity point.

Recall that we have assumed that (0:1:0) is not an infinity point; otherwise, we consider a change of coordinates and afterward we undo it in the obtained parametrization of the asymptote.

Algorithm *Improvement Asymptotes Construction-Implicit Case I.*

Given an irreducible algebraic plane curve $\mathcal{C}$ defined by a polynomial $f(x,y) \in \mathbb{R}[x,y]$ of degree d, the algorithm outputs all the asymptotes of $\mathcal{C}$.

1. Compute the homogeneous form of maximum degree of $F(x,y,z)$. Let $f_d(x,y) = \prod_{i=1}^{\kappa}(y-m_ix)^{\ell_i}$. The infinity points are $P_i = (1:m_i:0)$, $i = 1,\dots,\kappa$.
2. For each $P_i := (1:m_i:0)$ do: determine the number of branches, r_i, associated to P_i and let $\ell := \ell_i$ and $r := r_i$.

2.1. Consider $k := \lfloor \ell/r \rfloor$ and

$$\widetilde{\mathcal{Q}}(t) = (t^k,\ m_it^k + b_1t^{k-1} + \cdots + b_k),$$

where $b_i \in \mathbb{C}$, $i = 1,\dots,k$ are undetermined coefficients. Compute $f(\widetilde{\mathcal{Q}}(t)) =$

$$\Lambda_0(b_0)t^{d\cdot k} + \Lambda_1(b_0,b_1)t^{d\cdot k-s_1} + \cdots + \Lambda_u(b_0,b_1,\dots,b_k)t^{d\cdot k-s_u} + \sum_{j=u+1}^{v}\Lambda_u(b_0,b_1,\dots,b_k)t^{d\cdot k-s_j}.$$

(Note that $\Lambda_0(m_i) = 0$).

2.2. Solve the triangular system of equation

$$\Lambda_i(b_0,\dots,b_i) = 0,\ i = 0,\dots,k, \qquad \Lambda_i(b_0,\dots,b_k) = 0,\ i = k+1,\dots,u$$

and substitute the solutions in $\widetilde{\mathcal{Q}}(t)$. Let $\widetilde{\mathcal{Q}}_{ij}(t)$, $j = 1,\dots,h$ be these reparametrizations and $\deg(\widetilde{\mathcal{Q}}_{ij}) = n_j$.

2.3. If $h < r_i$ let $\ell := \ell - n_1 - \cdots - n_h$ and $r := r - h$ and go to Step 2.1. Otherwise, go to Step 3.

3. Return the asymptotes defined by all the proper parametrizations $\widetilde{\mathcal{Q}}_{ij}(t)$, $j = 1,\dots,r_i$, $i = 1,\dots,\kappa$.

By applying Algorithm Improvement Asymptotes Construction-Implicit Case I, we can easily obtain all the g-asymptotes of any plane curve, as the following example shows.

Example 2. Consider the plane curve $\mathcal{C}$ introduced in Example 1 and defined by the irreducible polynomial

$f(x,y) = -9xy^5 + 2y^6 - 144x^3y^2 - 400x^2y^3 + 159xy^4 - 24y^5 - 360x^3y + 2872x^2y^2 - 929xy^3 + 53y^4 - 225x^3 + 9303x^2y + 2855xy^2 + 114y^3 + 6360x^2 + 4966xy + 508y^2 - 508x \in \mathbb{R}[x,y]$.

We apply Algorithm Improvement Asymptotes Construction-Implicit Case I.

Step 1: The homogeneous form of maximum degree of $F(x,y,z)$ is $-y^5(9x-2y)$. Hence, the infinity points are $P_1 = (1:0:0)$ and $P_2 = (1:9/2:0)$.

Step 2: We start with the infinity point P_1 (which is a singular point) that has three branches, i.e. $r_1 = 3$ and $\ell_1 = 5$.

2.1. We have that $k = 1$, and thus we consider $\widetilde{\mathcal{Q}}(t) = (t,\ b_1)$, where b_1 is an undetermined coefficient. Compute $f(\widetilde{\mathcal{Q}}(t))$.
2.2. Solve the equation obtained from the first non-zero coefficient of maximum degree of the above polynomial. We get a double root $b_1 = -5/4$ and thus, we have one (double) asymptote of degree 1 defined by the proper parametrization

$$\widetilde{\mathcal{Q}}_{11}(t) = (t,\ -5/4).$$

2.3 We go to Step 2.1 with $k = 3$. Reasoning similarly, we get an asymptote of degree 3 defined by the proper parametrization

$$\widetilde{\mathcal{Q}}_{12}(t) = (t^3, -2 \cdot 2^{1/3}t^2 + 4 \cdot 2^{2/3}t + 19/6).$$

We may compute the implicit polynomial defining this asymptote and we have that $6859/216 + 408x - 16x^2 - 48xy - y^3 + 19/2y^2 - 361/12y$ (see Chapter 4 in [11]).

Step 2: We reason with the infinity point P_2 which is a simple point. Thus, we have one only branch of degree one.

2.1. Consider $\widetilde{\mathcal{Q}}(t) = (t,\ 9/2t + b_1)$, where b_1 is an undetermined coefficient. Compute $f(\widetilde{\mathcal{Q}}(t))$.
2.2. Solve the equation obtained from the first non-zero coefficient of maximum degree of the above polynomial. We get

$$59049/16b_1 - 295245/16 = 0 \quad \Rightarrow \quad b_1 = 5.$$

We substitute the solution in $\widetilde{\mathcal{Q}}(t)$ and we get the proper parametrization

$$\widetilde{\mathcal{Q}}_{21}(t) = (t, 9/2t + 5).$$

Step 3: The asymptotes of the input curve $\mathcal{C}$ are defined by the proper parametrizations

$$\widetilde{\mathcal{Q}}_{11}(t) = (t,\ -5/4), \quad \widetilde{\mathcal{Q}}_{12}(t) = (t^3, -2 \cdot 2^{1/3}t^2 + 4 \cdot 2^{2/3}t + 19/6), \quad \text{and} \quad \widetilde{\mathcal{Q}}_{21}(t) = (t, 9/2t + 5).$$

The input curve, $\mathcal{C}$, and its three asymptotes have been plotted in Fig. 1.

Additionally to the previous algorithm, we present Algorithm Improvement Asymptotes Construction-Implicit Case II, where we do not need to compute the number of branches. That is, we apply statement 1 in Remark 2.

Algorithm *Improvement Asymptotes Construction-Implicit Case II.*
Given an irreducible algebraic plane curve $\mathcal{C}$ defined by a polynomial $f(x, y) \in \mathbb{R}[x, y]$ of degree d and $\mathcal{L} = \{\emptyset\}$, the algorithm outputs all the asymptotes of $\mathcal{C}$.
1. Set $k = d$ and do:

1.1. Consider

$$\widetilde{\mathcal{Q}}(t) = (t^k,\ b_0t^k + b_1t^{k-1} + \cdots + b_k),$$

where $b_i \in \mathbb{C},\ i = 0, \dots, k$ are undetermined coefficients and $f(\widetilde{\mathcal{Q}}(t)) =$

$$\Lambda_0(b_0)t^{d\cdot k} + \Lambda_1(b_0, b_1)t^{d\cdot k-s_1} + \cdots + \Lambda_u(b_0, b_1, \dots, b_k)t^{d\cdot k-s_u} + \sum_{j=u+1}^{v} \Lambda_j(b_0, b_1, \dots, b_k)t^{d\cdot k-s_j}.$$

1.2. Solve the triangular system of equation

$$\Lambda_i(b_0, \dots, b_i) = 0,\ i = 0, \dots, k, \qquad \Lambda_i(b_0, \dots, b_k) = 0,\ i = k + 1, \dots, u$$

and substitute the solutions in $\widetilde{\mathcal{Q}}(t)$.
1.3. Reparametrize properly $\widetilde{\mathcal{Q}}(t)$ (apply [12]). Let $\widetilde{\mathcal{Q}}_{ij}(t)$ be the proper reparametrizations obtained of degree k_{ij} and associated to the infinity points $(1 : m_i : 0)$ (recall that each m_i is a different root of $\Lambda_0(b_0)$).
1.4. Let $\mathcal{L} = \{\mathcal{L} \cup \widetilde{\mathcal{Q}}_{ij}(t)\}$.

2. Go to Step 1. and consider $2 \le k \le d - 1$, such that $k \notin \{k_{ij}, d\}$. Repeat this step till such a k does not exists and then, go to Step 3.
3. Return the asymptotes defined by all the proper parametrizations given in the set $\mathcal{L}$.

In the following example, we apply Algorithm Improvement Asymptotes Construction-Implicit Case II to compute all the g-asymptotes of the plane algebraic curve introduced in Example 1.

Example 3. We consider the curve $\mathcal{C}$ of degree $d = 6$ of Example 1 defined by the polynomial
$f(x, y) = -9xy^5 + 2y^6 - 144x^3y^2 - 400x^2y^3 + 159xy^4 - 24y^5 - 360x^3y + 2872x^2y^2 - 929xy^3 + 53y^4 - 225x^3 + 9303x^2y + 2855xy^2 + 114y^3 + 6360x^2 + 4966xy + 508y^2 - 508x \in \mathbb{R}[x, y]$.

We apply algorithm Improvement Asymptotes Construction-Implicit Case II. Let $\mathcal{L} = \{\emptyset\}$.

Step 1: Set $k = 6$.

1.1. Consider $\widetilde{\mathcal{Q}}(t) = (t^6,\ b_0t^6 + b_1t^5 + \cdots + b_6)$, and

$$f(\widetilde{\mathcal{Q}}(t)) = \sum_{j=0}^{6} \Lambda_j(b_0, \ldots, b_j)t^{d\cdot k-j} + \sum_{j=7}^{36} \Lambda_j(b_0, \ldots, b_6)t^{d\cdot k-j}.$$

1.2. We solve the triangular system of equations

$$\Lambda_i(b_0, \ldots, b_6) = 0,\ i = 0, \ldots, \ell$$

for some $\ell \in \mathbb{N}$ such that we get all the solutions for the parameters b_j, $j = 1, \ldots, 6$. More precisely we have that

$$\Lambda_0(b_0) = b_0^5(2b_0 - 9) = 0 \quad \Rightarrow \quad b_0 = 0,\ b_0 = 9/2.$$

Note that every value of b_0 provide the infinity points $P_1 = (1 : 0 : 0)$ and $P_2 = (1 : 9/2 : 0)$.

1.2.1 We first compute the asymptotes for $P_1 = (1 : 0 : 0)$ (i.e. $b_0 = 0$). We have that $\Lambda_i(0, b_1, \ldots, b_4) = 0$, $i = 1, \ldots, 4$, and

$$\Lambda_5(0, b_1, \ldots, b_5) = -9b_1^5 \quad \Rightarrow \quad b_1 = 0.$$

Now, we have that $\Lambda_i(0, 0, b_2, \ldots, b_6) = 0$, $i = 6, \ldots, 9$, and $\Lambda_{10}(0, 0, b_2, \ldots, b_6) = -9b_2^2(b_2^3 + 16) \quad \Rightarrow$

$$b_2 = 0, \quad b_2 = -2\cdot 2^{1/3}, \quad b_2 = 2^{1/3} - \sqrt{3}\cdot 2^{1/3}I, \quad b_2 = 2^{1/3} + \sqrt{3}\cdot 2^{1/3}I.$$

1.2.1.1 Now, we compute the solutions for $b_0 = b_1 = b_2 = 0$. We have that

$$\Lambda_{12}(0, 0, 0, b_3, \ldots, b_6) = -144b_3^2 \quad \Rightarrow \quad b_3 = 0.$$

Furthermore

$$\Lambda_{14}(0, 0, 0, 0, b_4, b_5, b_6) = -144b_4^2 \quad \Rightarrow \quad b_4 = 0,$$

$$\Lambda_{16}(0, 0, 0, 0, 0, b_5, b_6) = -144b_5^2 \quad \Rightarrow \quad b_5 = 0,$$

and

$$\Lambda_{18}(0, 0, 0, 0, 0, 0, b_6) = -9(4b_6 + 5)^2 \quad \Rightarrow \quad b_6 = -5/4.$$

We substitute the solutions in $\widetilde{\mathcal{Q}}(t)$ and we get the first asymptote defined by the parametrization $(t^6, -5/4)$. We reparametrize this parametrization properly and we get the proper parametrization given as

$$\widetilde{\mathcal{Q}}_{11}(t) = (t,\ -5/4).$$

1.2.1.2 Now, we compute the solutions for $b_0 = b_1 = 0$ and $b_2 = -2\cdot 2^{1/3}$. We get that

$$\Lambda_{11}(0, 0, -2\cdot 2^{1/3}, b_3, \ldots, b_6) = -864\cdot 2^{1/3}b_3 \quad \Rightarrow \quad b_3 = 0,$$

$$\Lambda_{12}(0, 0, -2\cdot 2^{1/3}, 0, b_4, b_5, b_6) = 864\cdot 2^{1/3}(4\cdot 2^{2/3} - b_4) \quad \Rightarrow \quad b_4 = 4\cdot 2^{2/3},$$

$$\Lambda_{13}(0, 0, -2\cdot 2^{1/3}, 0, 4\cdot 2^{2/3}, b_5, b_6) = -864\cdot 2^{1/3}b_5 \quad \Rightarrow \quad b_5 = 0,$$

$$\Lambda_{14}(0, 0, -2\cdot 2^{1/3}, 0, 4\cdot 2^{2/3}, 0, b_6) = -144\cdot 2^{1/3}(6b_6 - 19) \quad \Rightarrow \quad b_6 = 19/6.$$

We substitute the solutions in $\widetilde{\mathcal{Q}}(t)$, and we get the second asymptote defined by $(t^6, 19/6 - 2\cdot 2^{1/3}t^4 + 4\cdot 2^{2/3}t^2)$. We reparametrize properly this parametrization and we get the proper parametrization of the second asymptote corresponding to the infinity point P_1

$$\widetilde{\mathcal{Q}}_{12}(t) = (t^3, 19/6 - 2\cdot 2^{1/3}t^2 + 4\cdot 2^{2/3}t).$$

1.2.1.3 Now, we compute the solutions for $b_0 = b_1 = 0$ and $b_2 = 2^{1/3} - \sqrt{3}\cdot 2^{1/3}I$. Reasoning as before we get again the parametrization $\widetilde{\mathcal{Q}}_{12}$.

1.2.1.4 Now, we compute the solutions for $b_0 = b_1 = 0$ and $b_2 = 2^{1/3} - \sqrt{3}\cdot 2^{1/3}I$. Reasoning as before we get again $\widetilde{\mathcal{Q}}_{12}$.

1.2.2 Now, we compute the asymptotes for $P_2 = (1 : 9/2 : 0)$ (i.e. $b_0 = 9/2$). We get the equations

$$\Lambda_1(9/2, b_1) = 0 \quad \Rightarrow \quad b_1 = 0,$$

$$\Lambda_2(9/2, 0, b_2) = 0 \quad \Rightarrow \quad b_2 = 0,$$

$$\Lambda_3(9/2, 0, 0, b_3) = 0 \quad \Rightarrow \quad b_3 = 0,$$

$$\Lambda_4(9/2, 0, 0, 0, b_4) = 0 \quad \Rightarrow \quad b_4 = 0,$$

$$\Lambda_5(9/2, 0, 0, 0, 0, b_5) = 0 \quad \Rightarrow \quad b_5 = 0,$$

$$\Lambda_6(9/2, 0, 0, 0, 0, 0, b_6) = 0 \quad \Rightarrow \quad b_6 = 5.$$

We substitute the solutions in $\widetilde{\mathcal{Q}}(t)$ and we get the third asymptote defined by $(t^6, 9/2+5t^6)$. We reparametrize this parametrization properly and we get the proper parametrization given as

$$\widetilde{\mathcal{Q}}_{21}(t) = (t, 9/2 + 5t).$$

1.4. Let $\mathcal{L} = \{\widetilde{\mathcal{Q}}_{11}(t), \widetilde{\mathcal{Q}}_{12}(t), \widetilde{\mathcal{Q}}_{21}(t)\}$.

Step 2: This step does not provide new asymptotes. Observe that we have to check if there exists asymptotes of degree 5, and 4 (note that these degrees do not divide 6 and thus it would not appear in the previous process). When we consider $k = 5$ (similarly for $k = 4$), we obtain again a non-proper parametrization for the asymptotes of degree 1 obtained in Step 1.

Step 3: The algorithm returns the asymptotes defined by the proper parametrizations

$$\widetilde{\mathcal{Q}}_{11}(t) = (t,\ -5/4), \quad \widetilde{\mathcal{Q}}_{12}(t) = (t^3, -2 \cdot 2^{1/3}t^2 + 4 \cdot 2^{2/3}t + 19/6), \quad \text{and} \quad \widetilde{\mathcal{Q}}_{21}(t) = (t, 9/2t + 5).$$

The input curve, $\mathcal{C}$, and its three asymptotes have been plotted in Fig. 1.

The method above described may be trivially adapted for dealing with algebraic curves $\mathcal{C}$ in the n-dimensional space defined by irreducible polynomials $f_{j-1}(x_1, x_j)$, $j = 2, \ldots, n$. For instance, if $n = 3$, and we have a curve $\mathcal{C}$ defined by the irreducible polynomials $f_1(x_1, x_2)$ and $f_2(x_1, x_3)$, we can compute the asymptotes defined by the parametrizations

$$\widetilde{\mathcal{Q}} = (t^k,\ a_0t^k + a_1t^{k-1} + \cdots + a_k,\ b_0t^k + b_1t^{k-1} + \cdots + b_k)$$

by successively applying the algorithm to the first and second component, and then to the first and third component. Note that as in the planar case, we also must assume that $(0 : 1 : 0 : 0)$ and $(0 : 0 : 1 : 0)$ are not infinity points (otherwise, we consider a change of coordinates and afterward we undo it in the obtained parametrization of the asymptote).

Observe that a curve $\mathcal{C}$ in the n-dimensional space defined by $n - 1$ polynomials can be assumed to be defined by irreducible polynomials of the form $f_{j-1}(x_1, x_j)$, $j = 2, \ldots, n$ (see [3]). Example 4 illustrates this idea (see Fig. 2).

Example 4. Let $\mathcal{C}$ be the space curve defined by the polynomials

$$f_1(x_1, x_2) = -9x_1^2x_2 + 12x_1x_2^2 - 4x_2^3 + 9x_1^2 + 21x_1x_2 + 9x_2^2 - 6x_1 - 6x_2 + 1,$$

$$f_2(x_1, x_3) = -2x_1x_3^2 - 2x_3^3 + 4x_1^2 + 2x_1x_3.$$

We apply Algorithm Improvement Asymptotes Construction-Implicit Case I for each plane curve, $\mathcal{C}_1$ and $\mathcal{C}_2$, defined by $f_1(x_1, x_2)$ and $f_2(x_1, x_3)$, respectively. We start with $\mathcal{C}_1$.

Step 1: The homogeneous form of maximum degree of $F_1(x_1, x_2, x_4)$ is $x_2(3x_1 - 2x_2)^2$. Hence, the infinity points are $P_{11} = (1 : 3/2 : 0)$ and $P_{12} = (1 : 0 : 0)$.

Step 2: We start with the simple point P_{11}. Thus, we have one branch of degree two.

2.1. Consider $\widetilde{\mathcal{Q}}(t) = (t^2,\ 3/2t^2 + b_1t^1 + b_2)$, where b_1, b_2 are undetermined coefficients. Compute $f_1(\widetilde{\mathcal{Q}}(t))$.

2.2. Solve the triangular system obtained from the first two non-zero coefficients of maximum degree of the above polynomial. We get that

$$243/4 - 6b_1^2 = 0 \quad \Rightarrow \quad b_1 = -9/4\sqrt{2},\ b_1 = 9/4\sqrt{2},$$

$$-4b_1(b_1^2 + 3b_2 - 12) = 0 \quad \Rightarrow \quad b_2 = 5/8.$$

We substitute the solutions in $\widetilde{\mathcal{Q}}(t)$, and we get the proper parametrization

$$\widetilde{\mathcal{Q}}_{11}(t) = (t^2, 3/2t^2 + 9/4\sqrt{2}t + 5/8).$$

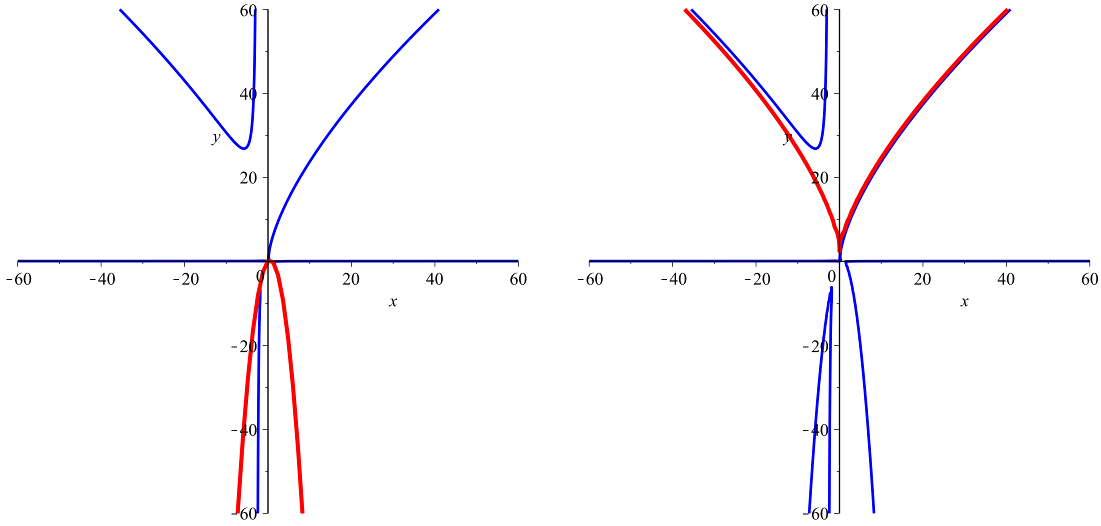

Fig. 2. Curve $\mathcal{C}$ and asymptote $\widetilde{\mathcal{C}}_1$ (left), curve $\mathcal{C}$ and asymptote $\widetilde{\mathcal{C}}_2$ (right).

Step 2: We reason now with the simple point P_{12}. Thus, we have one branch of degree one.

2.1. Consider $\widetilde{\mathcal{Q}}(t) = (t,\ b_1)$, where b_1 is an undetermined coefficient. Compute $f_1(\widetilde{\mathcal{Q}}(t))$.
2.2. Solve the equation obtained from the first non-zero coefficient of maximum degree of the above polynomial. We get that

$$-9b_1 + 9 = 0 \quad \Rightarrow \quad b_1 = 1.$$

We substitute the solutions in $\widetilde{\mathcal{Q}}(t)$, and we get the proper parametrization

$$\widetilde{\mathcal{Q}}_{12}(t) = (t, 1).$$

Now, we reason with the plane curve $\mathcal{C}_2$.

Step 1: The homogeneous form of maximum degree of $F_2(x_1, x_3, x_4)$ is $-2x_3^2(x_1 + x_3)$. Hence, the infinity points are $P_{21} = (1 : 0 : 0)$ and $P_{22} = (1 : -1 : 0)$.

Step 2: We start with the simple point P_{21} and reasoning similarly as above, we get the proper parametrization

$$\widetilde{\mathcal{Q}}_{21}(t) = (t^2, \sqrt{2}t - 1/2).$$

Step 2: For the simple point P_{22}, we get the proper parametrization

$$\widetilde{\mathcal{Q}}_{22}(t) = (t, -t + 1).$$

Step 3: The asymptotes of the input space curve, $\mathcal{C}$, are defined by the proper rational parametrizations

$$(t^2, 3/2t^2 + 9/4\sqrt{2}t + 5/8, \sqrt{2}t - 1/2), \quad \text{and} \quad (t, 1, -t + 1).$$

We observe that the infinity points of $\mathcal{C}$ are (1:0:−1:0) and (1:3/2:0:0). Therefore, the point P_{11} is associated with the point P_{21} (both points provide the infinity point of the input curve (1:0:−1:0)), and the point P_{12} is associated with the point P_{22} (both points provide the infinity point of the input curve (1:3/2:0:0)). We use this remark to combine appropriately the asymptotes of $\mathcal{C}_1$ and $\mathcal{C}_2$ to get the asymptotes of $\mathcal{C}$.

Algorithms Improvement Asymptotes Construction-Implicit Case I and II allow us to easily obtain all the generalized asymptotes of an algebraic curve implicitly defined. However, one has to determine the roots of some given equations, which may entail certain difficulties if algebraic numbers are involved. For this purpose, we use the notion of *conjugate points* (see [11]), which will help us to overcome this problem.

The idea is to collect points whose coordinates depend algebraically on all conjugate roots of the same irreducible polynomial, say $m(t) \in \mathbb{R}[t]$. This will imply that the computations on such families can be carried out by using the defining polynomial $m(t)$ of these algebraic numbers. That is, one applies the formulae presented in Theorem 3, but modulo $m(t)$, i.e. we use the polynomial $m(t)$ to carry out the arithmetic by computing polynomial remainders.

The following example shows this method based on the conjugate points to obtain the asymptotes of a plane curve.

Example 5. We consider the algebraic plane curve $\mathcal{C}$ defined by the irreducible polynomial $f(x, y) = 6xy^5 + 2y^6 + 6x^3y^2 - 70x^2y^3 - 96xy^4 - 24y^5 - 30x^3y + 992x^2y^2 + 421xy^3 + 53y^4 + 75x^3 - 4037x^2y - 1335xy^2 + 114y^3 + 8010x^2 + 4966xy + 508y^2 - 508x \in \mathbb{R}[x, y]$.

We apply algorithm Improvement Asymptotes Construction-Implicit Case I.

Step 1: The homogeneous form of maximum degree is $2y^5(3x + y)$ and hence, the infinity points are $P_1 = (1 : 0 : 0)$ and $P_2 = (1 : -3 : 0)$.

Step 2: We start with the singular point P_1 that has three branches, i.e. $r_1 = 3$ and $\ell_1 = 5$.

2.1. We have that $k = 1$, and thus, we consider $\widetilde{\mathcal{Q}}(t) = (t,\ b_1)$, where b_1 is an undetermined coefficient. Compute $f(\widetilde{\mathcal{Q}}(t))$.
2.2. Solve the equation obtained from the first non-zero coefficient of maximum degree of the above polynomial. We get that $b_1 = \alpha$, where $m_1(\alpha) = 0$ and $m_1(t) = 6t^2 - 30t + 75$. Thus, we have two asymptotes, each of degree one, given by the proper parametrization $\widetilde{\mathcal{Q}}_{11}(t) = (t,\ \alpha),\quad m_1(\alpha) = 0$.
2.3 We go to Step 2.1 with $k = 3$. Reasoning similarly as above, we get the asymptote of degree 3 defined by the proper parametrization $\widetilde{\mathcal{Q}}_{12}(t) = (t^3, \beta t^2 + 4t/\beta + 7/3)$, where $m_2(\beta) = 0$ and $m_2(t) = t^2 - t + 1$. One may check that the implicit polynomial defining this asymptote is

$$27y^3 + 27x^2 - 324xy - 189y^2 + 2484x + 441y - 343.$$

Step 2: Now, we reason with the simple point P_2. Thus, we only have one branch of degree one.

2.1. Consider $\widetilde{\mathcal{Q}}(t) = (t,\ -3t + b_1)$, where b_1 is an undetermined coefficient. Compute $f(\widetilde{\mathcal{Q}}(t))$.
2.2. Solve the equation obtained from the first non-zero coefficient of maximum degree of the above polynomial. We have

$$-486b_1 = 0 \quad \Rightarrow \quad b_1 = 0.$$

We substitute the solution in $\widetilde{\mathcal{Q}}(t)$ and we get the proper parametrization

$$\widetilde{\mathcal{Q}}_{21}(t) = (t, -3t).$$

Step 3: The proper rational parametrizations defining the asymptotes are

$$\widetilde{\mathcal{Q}}_{11}(t) = (t,\ \alpha),\quad m_1(\alpha) = 0$$

where $m_1(t) = 6t^2 - 30t + 75$,

$$\widetilde{\mathcal{Q}}_{12}(t) = (t^3, \beta t^2 + 4t/\beta + 7/3),\quad m_2(\beta) = 0$$

where $m_2(t) = t^2 - t + 1$, and

$$\widetilde{\mathcal{Q}}_{21}(t) = (t, -3t).$$

4.1. *Experimental times*

We finish this section by comparing the performance of Algorithm Asymptotes Construction-Implicit Case, presented in [3] (method 1) and Algorithm Improvement Asymptotes Construction-Implicit Case I (method 2). The performance of Algorithms Improvement Asymptotes Construction-Implicit Case I and Improvement Asymptotes Construction-Implicit Case II provides similar results.

We have implemented the algorithms using Maplesoft 2022 on a Lenovo ThinkPad Intel(R) Core(TM) i7-10510UU CPU @ 2.30 GHz and 16 GB of RAM, OS-Windows 11 Pro. We have run these algorithms on a set of ten arbitrary implicit curves with different degrees and different numbers of infinity branches (these properties are displayed in the next table). For each of these curves, we show the degree, the number of monomials, the number of infinity branches, and the running time (given in seconds of CPU) spent by each of the two methods. All these data are shown in the following table:

Curve	Degree	Nops	Infinity branches	Method 1	Method 2
$\mathcal{C}_1$	8	28	4	0.078	0.062
$\mathcal{C}_2$	14	39	6	0.172	0.047
$\mathcal{C}_3$	36	29	6	7.093	3.500
$\mathcal{C}_4$	24	33	7	3.469	0.031
$\mathcal{C}_5$	3	4	3	3.703	4.750
$\mathcal{C}_6$	18	34	6	1.625	0.109
$\mathcal{C}_7$	32	26	8	18.281	1.125
$\mathcal{C}_8$	20	6	2	18.281	0.062
$\mathcal{C}_9$	40	5	8	6.954	0.125
$\mathcal{C}_{10}$	30	35	3	7.875	0.047

In order to compare the methods, we have marked in red the longer running time for each curve. We observe that, in general, the new method (method 2) is better than method 1 which is based on the computation of Puiseux series. In fact, we have noted that we get a significant improvement when we deal with high degree curves having a large number of monomials and when there exist families of conjugate points.

Note also that these running times are related to the degree of the curve and the number of infinity branches as well as the number of conjugate points that provide the branches. It makes sense, since the number of coefficients we need to compute for getting the asymptotes depends on these parameters.

5. Conclusion

The main result of this paper, Theorem 3, provides a method to determine the generalized asymptotes of a curve by only computing the solutions of a triangular system of equations constructed from the implicit polynomial defining the input algebraic curve. From this theorem, we develop an efficient algorithm which determines all the g-asymptotes avoiding the laborious computation of Puiseux series and infinity branches. In fact, the comparison with the other existing method shows that this one reduces significantly the computation time, specially when we deal with high degree curves and when families of conjugate points exist.

Thus, the present paper yields a remarkable improvement of the methodology developed in [3] (see Section 5). Furthermore, this procedure can be trivially applied for dealing with algebraic curves in the *n*-dimensional space. All these techniques are proved to work on several illustrative examples.

As a future work, we aim to extend the notion of g-asymptote to the study of the asymptotic behavior of algebraic surfaces. We look for surfaces which approach a given one of higher degree, when "moving to infinity", that is, when some of the coordinates take infinitely large values. The ideas introduced in this paper might provide the foundations for efficient methods that allow us to compute those "asymptotic surfaces".

Data availability

No data was used for the research described in the article.

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Appendix

In this appendix, we present the implicit curves used in the study of the algorithms presented in Section 4.1.

$$\begin{aligned} f_1(x_1,x_2) = {} & -729x_1^3x_2^5 + 486x_1^2x_2^6 - 108x_1x_2^7 + 8x_2^8 - 11664x_1^5x_2^2 - 21384x_1^4x_2^3 + 33102x_1^3x_2^4 - 13615x_1^2x_2^5 + 758x_1x_2^6 \\ & + 88x_2^7 - 16200x_1^5x_2 + 463374x_1^4x_2^2 - 356142x_1^3x_2^3 + 81460x_1^2x_2^4 + 10019x_1x_2^5 + 2717x_2^6 - 2025x_1^5 \\ & + 720963x_1^4x_2 + 990531x_1^3x_2^2 + 360506x_1^2x_2^3 + 26086x_1x_2^4 - 538x_2^5 + 91170x_1^4 + 151414x_1^3x_2 \\ & + 68580x_1^2x_2^2 + 10160x_1x_2^3 + 508x_2^4 - 508x_1^3. \end{aligned}$$

$$\begin{aligned} f_2(x_1,x_2) = {} & 8683257856x_1^2x_2^{12} + 323348480x_1x_2^{13} + 3010225x_2^{14} + 64703758336x_1^2x_2^{11} + 862548160x_1x_2^{12} \\ & + 16170050x_2^{13} + 232442176512x_1^2x_2^{10} + 48053996x_1x_2^{11} + 43580990x_2^{12} + 528605828352x_1^2x_2^9 \\ & - 4439796536x_1x_2^{10} + 73326664x_2^{11} + 841835426768x_1^2x_2^8 - 11331043129x_1x_2^9 + 84195424x_2^{10} \\ & + 983078241348x_1^2x_2^7 - 15767456944x_1x_2^8 + 66445920x_2^9 + 8479744x_1^3x_2^5 + 858325540303x_1^2x_2^6 \\ & - 13731543555x_1x_2^7 + 35276319x_2^8 + 18668608x_1^3x_2^4 + 561575753843x_1^2x_2^5 - 7598293305x_1x_2^6 \\ & + 11576043x_2^7 + 15965033x_1^3x_2^3 + 271978409572x_1^2x_2^4 - 2421613296x_1x_2^5 + 1807272x_2^6 \\ & + 7462357x_1^3x_2^2 + 94728130531x_1^2x_2^3 - 279328716x_1x_2^4 + 1805013x_1^3x_2 + 22440615102x_1^2x_2^2 \\ & + 38555136x_1x_2^3 + 170667x_1^3 + 3209154048x_1^2x_2 + 205627392x_1^2. \end{aligned}$$

$$
\begin{aligned}
f_3(x_1,x_2) = {} & x_2^{36} - 12x_1x_2^{34} + 84x_1^2x_2^{32} - 346x_1^3x_2^{30} + 1116x_1^4x_2^{28} - 2703x_1^5x_2^{26} + 7167x_1^6x_2^{24} - 12432x_1^7x_2^{22} + 26268x_1^8x_2^{20} \\
& - 19780x_1^9x_2^{18} + 92451x_1^{10}x_2^{16} - 36x_1^9x_2^{17} + 46746x_1^{11}x_2^{14} - 1728x_1^{10}x_2^{15} + 183499x_1^{12}x_2^{12} - 20658x_1^{11}x_2^{13} \\
& + 101730x_1^{13}x_2^{10} - 93984x_1^{12}x_2^{11} + 95988x_1^{14}x_2^{8} - 189576x_1^{13}x_2^{9} + 32705x_1^{15}x_2^{6} - 177504x_1^{14}x_2^{7} \\
& + 15903x_1^{16}x_2^{4} - 75036x_1^{15}x_2^{5} + 2673x_1^{17}x_2^{2} - 12670x_1^{16}x_2^{3} + 729x_1^{18} - 594x_1^{17}x_2 - x_1^{17}.
\end{aligned}
$$

$$
\begin{aligned}
f_4(x_1,x_2) = {} & x_2^{24} - 824x_1x_2^2 1 - 6x_1x_2^{20} - 135056x_1^2x_2^{18} - 1090032x_1^2x_2^{17} - 305878948x_1^3x_2^{15} + 15x_1^2x_2^{16} - 332513805x_1^3x_2^{14} \\
& + 29561428776x_1^4x_2^{12} - 21896514x_1^3x_2^{13} + 84368663334x_1^4x_2^{11} - 20x_1^3x_2^{12} - 7336125742241x_1^5x_2^{9} \\
& + 105311475102x_1^4x_2^{10} - 13905265469868x_1^5x_2^{8} - 47228926x_1^4x_2^{9} + 899624912729920x_1^6x_2^{6} \\
& - 20445470980563x_1^5x_2^{7} + 15x_1^4x_2^{8} + 879421250107854x_1^6x_2^{5} - 432927678855x_1^5x_2^{6} \\
& - 4954629145755 1243x_1^7x_2^{3} + 110672614997661 0x_1^6x_2^{4} - 15188802x_1^5x_2^{5} + 11658438494227425x_1^7x_2^{2} \\
& - 302031801287456x_1^6x_2^{3} - 6x_1^5x_2^{4} - 2982955494854578697x_1^{8} + 34873482038112303x_1^7x_2 \\
& + 30252411978x_1^6x_2^{2} - 38859097616298524x_1^{7} - 361023x_1^6x_2 + x_1^{6}.
\end{aligned}
$$

$$
f_5(x_1,x_2) = 1591619472413x_1^3 - 673417059x_1^2x_2 + 695664x_1x_2^2 - 508x_2^3
$$

$$
\begin{aligned}
f_6(x_1,x_2) = {} & 16613x_2^{18} - 2740x_2^{17} - 12419798x_1x_2^{15} + 6437x_2^{16} + 3582232x_1x_2^{14} + 2746x_2^{15} + 1952563681x_1^2x_2^{12} \\
& - 1467646x_1x_2^{13} + 508x_2^{14} - 885502736x_1^2x_2^{11} - 977636x_1x_2^{12} + 108700209375x_1^3x_2^{9} - 58319519x_1^2x_2^{10} \\
& - 228600x_1x_2^{11} - 46834562944x_1^3x_2^{8} + 88713464x_1^2x_2^{9} + 13105756970066x_1^4x_2^{6} - 30599270332x_1^3x_2^{7} \\
& + 30109256x_1^2x_2^{8} + 623044982020x_1^4x_2^{5} - 10387555930x_1^3x_2^{6} - 1016x_1^2x_2^{7} + 275372079901108x_1^5x_2^{3} \\
& - 605624249977x_1^4x_2^{4} - 1150332004x_1^3x_2^{5} - 18962869580694x_1^5x_2^{2} - 129176285888x_1^4x_2^{3} - 504952x_1^3x_2^{4} \\
& + 3482432688234375x_1^{6} - 8645033171511x_1^5x_2 + 7759907085x_1^4x_2^{2} - 2585419862778x_1^{5} \\
& - 4626150x_1^4x_2 + 508x_1^{4}.
\end{aligned}
$$

$$
\begin{aligned}
f_7(x_1,x_2) = {} & x_2^{32} - 16x_1x_2^{30} + 120x_1^2x_2^{28} + 61940x_1^3x_2^{26} + 11001820x_1^4x_2^{24} + 161289382x_1^5x_2^{22} - 983098242x_1^6x_2^{20} \\
& + 268005051060x_1^7x_2^{18} + 51939536575370x_1^8x_2^{16} + 1353401803363560x_1^9x_2^{14} - 4882812500000x_1^8x_2^{15} \\
& + 1146045003018613 3x_1^{10}x_2^{12} - 170898437500000x_1^9x_2^{13} - 2885063914609481 18x_1^{11}x_2^{10} \\
& + 574340820312500x_1^{10}x_2^{11} + 4632123972939345 4945x_1^{12}x_2^{8} + 1490966796875000 00x_1^{11}x_2^{9} \\
& + 1300907094338145405690x_1^{13}x_2^{6} - 9101544189453125000x_1^{12}x_2^{7} + 1134304694512821093870x_1^{14}x_2^{4} \\
& - 91102025756835937500x_1^{13}x_2^{5} - 5113796344091952849875 16x_1^{15}x_2^{2} + 3738908596801757812500x_1^{14}x_2^{3} \\
& + 2620324752242133844655000 1x_1^{16} + 1208195419262695312500 00x_1^{15}x_2 \\
& - 232830643653869628906 25x_1^{15}.
\end{aligned}
$$

$$
f_8(x_1,x_2) = 669124x_2^{20} - x_2^{19} - 134152x_1x_2^{15} + 5088x_1^2x_2^{10} + 164x_1^3x_2^5 + x_1^4.
$$

$$
f_9(x_1,x_2) = 5x_2^{40} - x_2^{39} + x_1x_2^{35} + x_1^4x_2^{20} + x_1^8.
$$

$$
\begin{aligned}
f_{10}(x_1,x_2) = {} & -5x_2^{30} + x_2^{29} - 742024584461x_1x_2^{20} + 7683405853352x_1x_2^{19} - 37988459635463x_1x_2^{18} + 116162290454294x_1x_2^{17} \\
& - 244084938363007x_1x_2^{16} + 365869703752040x_1x_2^{15} - 397726639601195x_1x_2^{14} + 311055201529068x_1x_2^{13} \\
& - 173145728975362x_1x_2^{12} - 4105193062724807793 33x_1^2x_2^{10} + 68741967824024x_1x_2^{11} \\
& + 2166842986886772629632x_1^2x_2^{9} - 19684811300898x_1x_2^{10} - 4739724078829277416759x_1^2x_2^{8} \\
& + 4119511886657x_1x_2^{9} + 3272371230339620137063x_1^2x_2^{7} - 636885234774x_1x_2^{8} \\
& + 5858951312742883334666x_1^2x_2^{6} + 73211082962x_1x_2^{7} - 1671317407524840617937 9x_1^2x_2^{5} \\
& - 6262657946x_1x_2^{6} + 1138353543631435690669 8x_1^2x_2^{4} + 396213107x_1x_2^{5} + 1073292744115136447588x_1^2x_2^{3} \\
& - 18252147x_1x_2^{4} - 1108815472467847502158 1x_1^2x_2^{2} + 594220x_1x_2^{3} - 5706888270894414462096282000 1x_1^{3} \\
& - 1555468653865985432600x_1^2x_2 - 12941x_1x_2^{2} + 7327623438850494819 05x_1^{2} + 169x_1x_2 - x_1.
\end{aligned}
$$

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