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Beauty Bare

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The two words of my title may conjure up images of Botticelli's "The Birth of Venus," but they actually come from a sonnet extolling the austere beauty of mathematics.

Euclid Alone Has Looked On Beauty Bare
by Edna St. Vincent Millay

Euclid alone has looked on Beauty bare.
Let all who prate of Beauty hold their peace,
And lay them prone upon the earth and cease
to ponder on themselves, the while they stare
At nothing, intricately drawn nowhere
In shapes of shifting lineage; let geese
Gabble and hiss, but heroes seek release
From dusty bondage into luminous air.
O blinding hour, O holy, terrible day,
When first the shaft into his vision shone
Of light anatomized! Euclid alone
Has looked on Beauty bare. Fortunate they
Who, though once only and then but far away,
Have heard her massive sandal set on stone.

The image of the ultimate in beauty as "nothing, intricately drawn nowhere" may seem odd at first, but it catches the essential abstractness of mathematics. Mathematics emphasizes form (or structure) above all. Nevertheless, pure form, even for mathematicians, needs connections with our intuitions. When structure unites with intuition in new and revealing ways, the thrilling insight is, I feel, the "Beauty bare" which Millay evokes. I hope to share with you a sense of the beauty and insight I find in mathematics, especially in my research.

It seems particularly appropriate to start with Euclid, in honor of both Edna St. Vincent Millay's tribute and his seminal role in the history of mathematics. Indeed, some have claimed that Euclid's *Elements* is the second most important book in Western civilization, after the Bible. It can legitimately be said that all Westerners since Euclid have seen geometry in the clothing of Euclid's exposition. This perhaps justifies Millay's saying that "Euclid alone has looked on Beauty bare." By starting with Euclid, I also hope to ground my discussion of structure and intuition in more familiar material before turning to my research.

Figure 1 shows Euclid's diagram for the Pythagorean Theorem, undoubtedly one of the greatest theorems in all of mathematics. If you remember this theorem at all, it may be only as a memorized formula "$a^2 + b^2 = c^2$." However, its meaning is geometrical, not algebraic. The areas of the two smaller squares in the figure always equal the area of the larger square, provided the angle at C is a right angle. Euclid provides a clever, structural proof of this theorem, which I will omit here. Instead, let me give a quick summary of what I mean by structure in mathematics.

Modern structural mathematics focuses on a united body of theorems, justified by formal reasoning, rather than diagrams or examples, and these theorems support a variety of interpretations.

Euclid's great work, *The Elements*, was the premier example of structural mathematics from his day until the nineteenth century. His text consists of hundreds of theorems with their proofs, building a tightly united system without any word of explanation or intuition. However, from a modern point of view, Euclid depended rather strongly on unstated assumptions which seemed obvious because of his diagrams. I will consider the notion of alternative interpretations later; for now let me weave the second main theme of this talk into the discussion with an intuitive argument for the Pythagorean Theorem.

Consider the large square in Figure 2 with its four shaded right triangles, all with sides of length a and b and hypotenuse c. The unshaded part is a square with side c, the "c^2" part of the Pythagorean Theorem. By sliding three of the triangles, as in Figure 3, we transform the unshaded part into two squares with sides a and b. Thus these two squares have area equal to the area of the square with side c: $a^2 + b^2 = c^2$. Beautiful!

What qualifies as intuitive? While intuition, unlike structure, has a personal component to it, there are some common aspects of intuition. Above

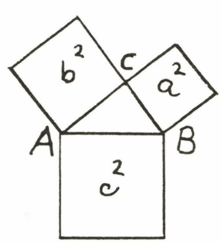

Figure 1. Euclid's diagram for the Pythagorean Theorem.

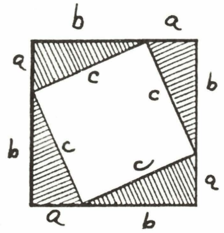

Figure 2. The four triangles leave an area of c^2 in this arrangement.

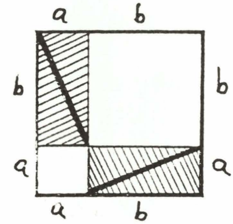

Figure 3. The four triangles leave two areas totaling $a^2 + b^2$ this way.

all, intuition needs to be clear and immediate. It should guide our thinking. A good intuition can create the feeling that the details are unnecessary in order to understand the heart of the idea - the insight is already internalized. Intuition often depends on an insightful example, a well-drawn diagram or a novel way of looking at the question— in short the particular, instead of the general or the abstract. Many sources can inspire our intuition in mathematics—geometry, physics, numerical patterns and even symbolic formulations. Moreover, intuition deepens with increased understanding. Since I am at heart a geometer, I will concentrate on geometrical intuition.

The Pythagorean Theorem is important not because it was proven long ago or because it has intuitive or structural proofs. Rather, it is itself a key structural component in mathematics. We use this theorem frequently to build more mathematics. Practical people centuries ago used the relationship $a^2 + b^2 = c^2$ to ensure that they had built a right angle. Students for 4000 years have solved word problems that rely on this relationship. A particularly revealing use of the Pythagorean Theorem from over 140 years ago emphasizes Edna St. Vincent Millay's image of mathematics as "nothing, intricately drawn nowhere." By 1850, a few daring mathematicians were using the Pythagorean Theorem as the basis for exploring geometry in four and more dimensions. They used their intuition about geometry to extend the two-dimensional formula $a^2 + b^2 = c^2$ and the three-dimensional formula $a^2 + b^2 + c^2 = d^2$ (which is also correct) to $a^2 + b^2 + c^2 + d^2 = e^2$ in four dimensions, where they simply defined this to be correct. Neither they nor any one else can see in more than three dimensions, yet their intuition led them to a profound structure which encompasses many real-world applications. Nowadays, the bold insight which led to higher dimensional geometry is used as a building block in linear algebra, which in turn is used in fields as varied as biology, business, economics and physics.

The structural role of the Pythagorean Theorem illustrates a remark of the mathematician George F. Simmons. "If [structural mathematics] is to justify itself, it must possess aesthetic qualities akin to those of a good piece of architecture. It should have a solid foundation...each part should bear a meaningful relation to every other part, and its towers and pinnacles should exalt the mind." Certainly the investigation of four and more dimensions qualifies as exalting the mind.

Intuition and structure are powerful together—and both are essential. By itself, intuition can lead us astray all too easily. Rather than recite important historical examples, I wish to give a more personal one. During my dissertation research, I came up with a conjecture that excited me. My description to my adviser of my intuition so convinced him that he exclaimed, "I'm morally certain that is true." Two days later, I had shown it was false. When I reminded him of his earlier statement, he laughed and said, "This just shows that ethics has no place in mathematics." Nevertheless, intuition does have a place. Structure alone too easily can become empty formulas and so can lose its meaning. Even more, without intuition, mathematicians could create very few conjectures, let alone find proofs of them.

My research on equidistance relations has drawn inspiration from several mathematical areas. The original source was a famous paper written in 1869 by the English mathematician Arthur Cayley. After I read his paper, I realized I could turn one of his intuitive geometrical ideas into a formal proof using mathematics developed since 1950. On this geometrical base, I found that important parts of algebra, graph theory and statistical design theory all fit together fruitfully.

Consider a simple object—a regular hexagon, shown in Figure 4. I want to look at the relationship of the distances between the six corners. These distances fall into three categories:

AB BC CD DE EF FA   AC CE EA BD DF FB   AD BE CF

So far, so unexciting. Now look at the distances between the six corners of a triangular prism, as shown in Figure 5. When we consider the actual distances, this shape is quite different from a regular hexagon. However, the categories for the distances mimic those for the hexagon:

ab bc cd de ef fa   ac ce ea bd df fb   ad be cf

Thus we see there is a common underlying structure of these two geometrical objects which is easily overlooked when we know too much. By ignoring the actual distances, we can focus on a deeper structure, equidistance. This structural connection between these particular shapes is not just coincidental. It is perhaps not obvious, but the notion of symmetry, central to many areas of mathematics and its applications, depends only on equidistance, not on the actual distances. Each motion of the prism

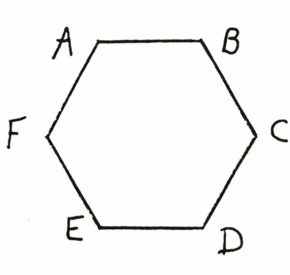

Figure 4. A hexagon.

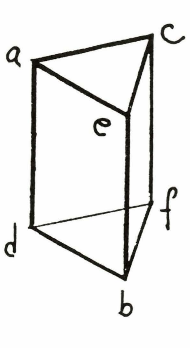

Figure 5. A triangular prism.

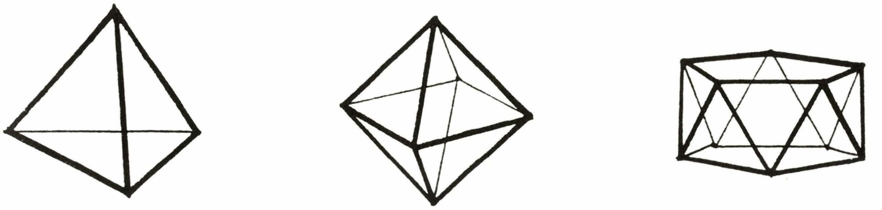

Figure 6. One-point homogeneous figures.

which brings it back onto itself corresponds to a motion of the hexagon. In technical language, their groups of symmetries are isomorphic; in more straight-forward terms, they have matching symmetries. The perfect fit between the equidistance structure and the structure of the symmetries of these two figures is mathematically beautiful.

Geometric shapes that people find pleasing usually possess many symmetries—indeed enough symmetries so that each point looks like each other point. Such shapes are called one-point homogeneous. (See Figure 6 above for several examples.)

Definition. A set is *one-point homogeneous* provided there are symmetries which move any given point of the set to every other point of the set.

My thesis adviser often talked about the importance of finding the correct definition. At first I thought he meant just that we needed to make our intuitive notions precise. For example, compare the preceding definition of one-point homogeneous with a common description of our student bodies here at Saint Benedict's and Saint John's as homogeneous. Later I came to understand that a good definition can unlock a string of interconnected results—of beautiful structure—that otherwise may have remained hidden. This happened to me when I developed the definition of a closed substructure.

Consider the eight corners of a cube. We have been taught to describe the cube as built from a bunch of "square substructures glued together." After all, that is how we make a model of a cube. However, when looking only at the distances of the corners from one another, it is far from obvious that these squares are any more fundamental than, say, the rather curious hexagon shown in Figure 7. (In fact, this hexagon is even one-point homogeneous.) The problem is that the edge distance connects the corners together in many ways, making it impossible to know how to separate one

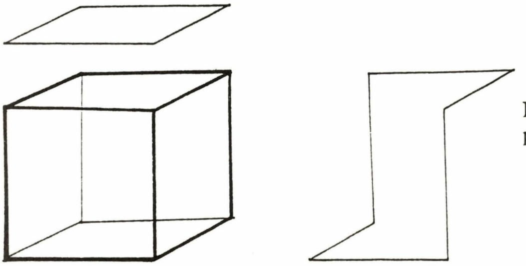

Figure 7. A cube and two possible "substructures."

potential substructure from another. However, there is a substructure of the cube that does have a natural separation from the rest of the cube. In Figure 8 four of the corners are connected to form a triangular pyramid, all of whose sides are the same length. The distance from one corner of this pyramid to another corner of the same pyramid can never lead us out of the pyramid. The definition of a closed substructure below generalizes this situation.

Definition. A subset Y is a *closed substructure* of X provided that no distances which connect two points within Y also connect a point in Y with a point outside Y.

My inspiration for this notion came when I noted that the curious porperty I just described about the triangular pyramid within the cube matched the intuition of closure in algebra. For example, among all of the numbers, the integers (..., –2, –1, 0, 1, 2, 3, ...) are closed under addition. That is, the sum of two integers is always an integer. This closure does not hold for every collection of numbers. For example the set of odd numbers, such as 1 and –7, is not closed under addition since 1 + (–7) = –6, an even integer.

I have found a sequence of results that focus on these notions of closure and homogeneity, especially one-point and two-point homogeneity. Albert Chiu, a student working with me last year, extended some of my results for higher levels of homogeneity. The most important of these results generalize some key theorems of group theory (a part of algebra) to geometric structures. I will describe the least complicated of these results, LaGrange's Theorem, and I will even indulge your curiosity and prove it. After all, what is a mathematics essay without proofs? Incidentally, Joseph-Louis LaGrange (1736–1813) was a noted mathematician who contributed to many areas of mathematics. I will not digress to discuss the theorem he found.

LaGrange's Theorem for Geometry. If a finite, one-point homogeneous equidistance structure X has a closed substructure Y, then the number of points in X is a multiple of the number of points in Y.

The triangular pyramid inside the cube of Figure 8 illustrates this theorem since the total number of corners (8) is a multiple of the number of corners (4) of the pyramid. So does the triangle atop the triangular prism

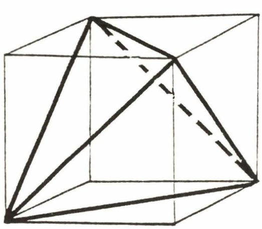

Figure 8. A triangular pyramid as a substructure of a cube.

in Figure 5. (I need to make a technical proviso: the length of the upright on the prism does not equal the side of the triangle.)

Proof. Suppose Y is a closed substructure of X, which is one-point homogeneous. By homogeneity, there must be copies of Y throughout X since every point in Y looks like every other point in X. However, Y is special: it is closed. That implies that two of the multiple copies of Y in X cannot include any of the same points. To see this, recall that closure guarantees that distances within Y cannot also be distances from Y to the rest of X. This means that every point of X must be in just one copy of Y. Of course, all the copies of Y must have the same number of points. Hence, the total number of points in X must be the product of the number of points in Y times the number of copies of Y in X. That is, the number of points in X is a multiple of the number of points in Y. Q.E.D.

This proof is very structural, depending on only the formal properties of the key terms. However, my realization that this was a theorem arose first intuitively, based on revealing examples. Furthermore, this theorem fits into a wider structure of mathematics. Rather than discussing other results I have shown, I want to try to relate some of these ideas to other areas. The amazing ability of mathematics to have fruitful applications is a manifestation of the interplay of structure and intuition. The structure provides the mathematical power, but a new intuition is needed to realize that this structure, developed in one context, fits with another context.

One-point homogeneous structures are not only pretty, they are important in chemistry, physics and other areas. Crystals, like diamonds and salt, can be modeled on the atomic level by lattices of points which are one-point homogeneous. Hermann Helmholtz, a noted 19th century scientist realized the importance of homogeneity for our understanding of physical space. He discussed the "facts" that were essential for us to function in space. His key fact is that we must be able to move and turn rigid bodies freely without distortion. This is equivalent to asserting the homogeneity of space. Originally, Helmholtz thought that among continuous spaces only traditional Euclidean geometry was homogeneous. However, mathematicians soon pointed out that hyperbolic and spherical geometries are just as

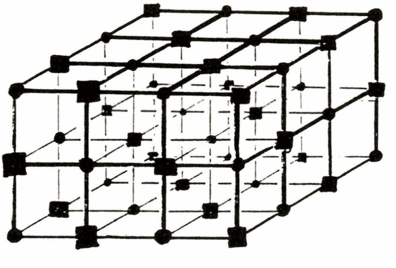

Figure 9. The structure of a salt crystal. The squares represent sodium ions and circles represent chlorine ions.

homogeneous. Somewhat more recently, Einstein's development of the special theory of relativity uses a homogeneous four-dimensional geometry.

Substructures, like homogeneity, have shown their worth in other fields. In particular, Henry Jacobowski use this idea in "The Biochemistry and Economics of Thrombosis" (p. 11 ff.). He writes about the goal of analyzing and even, he hoped, predicting the three dimensional folding of proteins using a much simplified model of the protein. In this model, each amino acid is represented by one of three signs (+, – or 0) depending on its electrical charge. In effect, we think of the amino acids as closed substructures of one of three kinds. We do not need to know the chemical information within these substructures, but only how they interrelate geometrically. It is an interesting and difficult mathematics problem and an important chemistry problem to determine what three dimensional shapes are compatible with the linear string of +, – and 0 signs.

I cannot claim any direct applications of my own research, but a closely related area, statistical design theory, does have important applications. Statistical design theory does not require as logically strong of a property as homogeneity. Basically, we need to insure that the design itself does not introduce any bias into an experiment. While this unbiasedness is weaker in principle than homogeneity, in practise almost all known designs are homogeneous. Homogeneous designs are much easier to create and, moreover, the structure of homogeneity allows a clear proof that bias is avoided. Consider an example developed by one of the founders of twentieth century statistics, Sir R. A. Fisher.

Imagine we want to test the interactions of five types of seeds (A, B, C, D, E) and five types of fertilizers (a, b, c, d, e). Clearly, we need to test each type of seed with each type of fertilizer, requiring twenty-five pairs: Aa, Ab, Ac, ..., Ba, ..., Ee. However, we need to be careful how we arrange these pairs in the rows of the field, so as to avoid unintentionally favoring one type of seed or fertilizer. One part of the field might have better drainage or better soil, etc. Hence to avoid bias, we require that each row and column in our experiment have each type of seed and each type of fertilizer just once.

Fisher solved this problem and the more general problem using homogeneous geometrical structures developed at the turn of the century from pursuing a mathematical intuition unrelated to any application. Figure 10 gives one possible solution. To see the pattern in Figure 10, think of the types of seeds lining up on "parallel lines" (which manage to wrap around the field) and the types of fertilizers lining up on another set of "parallel lines." This geometry even has its own algebra which fits it as perfectly as ordinary algebra fits Euclidean geometry.

In this example from statistics, the geometric intuition of lines has been expanded to achieve new insights. This illustrates my favorite quotation

Aa	Bb	Cc	Dd	Ee
Dc	Ed	Ae	Ba	Cb
Be	Ca	Db	Ec	Ad
Eb	Ac	Bd	Ce	Da
Cd	De	Ea	Ab	Bc

Figure 10. One solution where each type of seed and each type of fertilizer appear once in each row and column. Note that the top three As form a line sloping down from the left and that the other two As continue this pattern. The other capital letters continue this pattern, while the small letters form a different pattern of parallel lines that wrap around the square.

from Galileo: "... while logic is a most excellent guide in governing our reason, it does not, as regards stimulation to discovery, compare with the power of sharp distinction which belongs to geometry."

From my own personal point of view, the above example has another wonderful aspect—it contains in hidden form my equidistance relations. I was rather surprized when I realized that I could reinterpret equidistance relations in terms of lines. However, as a mathematician, I also realized that that is very much the nature of mathematical structure. One abstract structure can have a variety of interpretations. I was able to prove how to characterize lines in terms of equidistance relations. This means that all the properties of equidistance relations apply to lines. In fact, lines turn out in this interpretation to be closed substructures. Furthermore, since the entire geometrical structure is one-point homogeneous, LaGrange's Theorem holds. Thus the total number of points (the 25 pairs of seed and fertilizer types) must be a multiple of the number of points on each line (the five times each type of seed or type of fertilizer appears) .

Beautiful. The structure is flexible enough to fit in a totally unexpected situation. From an important vantage point, our intuitions of distances and lines have a common structure. One goal of mathematics is to make such common structure apparent. In the uniting of our intuitions under this common structure, we become like Euclid—the beholders of "Beauty bare."

***Tom Sibley** is associate professor of mathematics at Saint John's University. An earlier version of "Beauty Bare" was delivered as a Faculty Colloquium lecture on October 14, 1992.*

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