3-6-2012

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Recommended Citation

Byrne DP, Donner MJ & Sibley TQ. 2013. Groups of graphs of groups. *Beiträge zur Algebra und Geometrie* 54, 323–332. https://doi.org/10.1007/s13366-012-0093-7

Groups of Graphs of Groups

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Abstract. We classify all groups of color preserving automorphisms (isometries) of edge colored complete graphs derived from finite groups.

For any group G we construct a complete edge colored graph $\Gamma(G)$ with vertices the elements of G. Denote the edge from a to b by $\overline{ab}$, as opposed to the product ab. Define a *color function* $\rceil\ \lceil : G \to \mathcal{P}(G)$ by $\rceil g \lceil = \{g, g^{-1}\}$. For $a, b \in G$, define the *color* of the edge $\overline{ab}$ to be $\rceil ab^{-1} \lceil$. These colored graphs are related to Cayley graphs, although a Cayley graph is not colored and need not be complete. (See [4] and [5].) If $g = g^{-1} \neq e$, the identity of G, then $\rceil g \lceil = \{g\}$ and the edges of color $\rceil g \lceil$ form a regular graph of degree 1. If $g \neq g^{-1}$, the edges of color $\rceil g \lceil$ form a regular graph of degree 2. Colored graphs generalize the notion of distance and $\rceil\ \lceil$ generalizes absolute value for the real numbers. These colored graphs are related to homogeneous symmetric coherent configurations, but generalize them in a different way. (See [2] for more on coherent configurations.) The original proofs of these results appear in the honors theses of the first two authors ([1] and [3].)

Definitions. An *isometry* of G is a bijection σ of G such that for all $a, b \in G$, $\rceil \sigma(a)\sigma(b)^{-1} \lceil = \rceil ab^{-1} \lceil$. The set of all isometries of G is $I(G)$ and $I_e(G)$ is the subset fixing e.

Example 1. The colored graph for the cyclic group $\mathbb{Z}_n$ can be represented as a regular n-gon, as Figure 1 illustrates. For $n \geq 3$, $I(\mathbb{Z}_n) \cong D_n$. That is, $I(\mathbb{Z}_n)$ is isomorphic to the dihedral group for a regular n-gon. Also, $I_e(\mathbb{Z}_n)$ has two isometries: the identity permutation ε and the bijection μ taking each element to its inverse. Similarly, $\mathbb{Z}$ has $I(\mathbb{Z}) \cong D_{\mathbb{Z}}$, the infinite dihedral group and $I_e(\mathbb{Z})$ has the two isometries ε and μ.

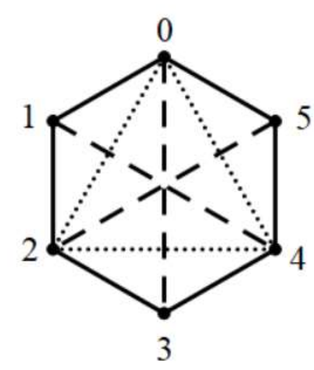

Figure 1. The cyclic group $\mathbb{Z}_6$.

For any group G, $I(G)$ is the group of color preserving automorphisms of the colored graph $\Gamma(G)$. For all $\alpha \in I_e(G)$ and $g \in G$, to preserve the color $\rceil g \lceil$, $\alpha(g)$ is either g or g^{-1}. Hence if $\alpha \in I_e(G)$, α^2 is the identity isometry ε. (See [6].) Thus $I_e(G)$ is an elementary abelian 2-group and so, if finite, is of the form $(\mathbb{Z}_2)^n$, for some $n \geq 0$. For any finite group G, the Classification Theorem, stated below, classifies the group $I(G)$, the group of the graph of the group. Elementary abelian 2-groups, which we will abbreviate as Boolean, play an important role in the classification theorem as well as the isometry groups $I_e(G)$.

Classification Theorem. Let G be any finite group. Then $I(G) \cong G$ unless one of the following occurs.

a) G is abelian and not a Boolean group. In this case, $I(G) \cong G \rtimes \mathbb{Z}_2$.

b) For some $n \geq 0$, $G \cong Q_8 \times (\mathbb{Z}_2)^n$, where Q_8 is the quaternion group. In this case, $I(G)$ has eight times as many elements as G and is isomorphic to $I(Q_8) \times (\mathbb{Z}_2)^n$, where $I(Q_8) \cong \mathbb{Z}_2 \wr (\mathbb{Z}_2)^2$.

c) G is not in previous cases and is isomorphic to Y or to $C \rtimes Y$, where C is an abelian, non-Boolean group and Y is either $\mathbb{Z}_4$ or a dicyclic group Q_{2^k}, for some power 2^k. In this case, $I(G) \cong G \rtimes \mathbb{Z}_2$.

While this classification only applies to finite groups, many of the lemmas and theorems leading up to it apply to all groups. We will indicate which properties apply only for finite groups. The first lemma uses Cayley's Theorem to find a subgroup of $I(G)$ isomorphic to G.

Lemma 1. For a group G and $g \in G$, the mapping $\psi_g : G \to G$ given by $\psi_g(x) = xg$ is an isometry of G. The set $\overline{G} = \{\psi_g : g \in G\}$ forms a subgroup of $I(G)$ isomorphic to G. If G is finite, then $|I(G)| = |G| \cdot |I_e(G)|$.

Proof. For $x, y, g \in G$, $\rceil \psi_g(x)\psi_g(y)^{-1} \lceil = \rceil (xg)(yg)^{-1} \lceil = \rceil xgg^{-1}y^{-1} \lceil = \rceil xy^{-1} \lceil$. Hence, ψ_g is an isometry. The set $\overline{G} = \{\psi_g : g \in G\}$ gives the right regular representation of G in Cayley's Theorem. Thus $\overline{G}$ is a subgroup of $I(G)$ and is isomorphic to G. Further, the orbit of e under $I(G)$ is the entire group. So if G is finite, the Orbit Stabilizer Theorem gives $|I(G)| = |G| \cdot |I_e(G)|$. □

Remark. Even if G is infinite, every element of $I(G)$ can be written in a unique way as the composition $\psi_g \circ \beta$, where $\psi_g \in \overline{G}$ and $\beta \in I_e(G)$.

An element g with $g = g^{-1}$ can't be moved by an isometry in $I_e(G)$ since there is only one edge $\overline{xe}$ with color $\rceil g \lceil$, namely $\overline{ge}$. An induction argument shows that any group G generated by elements of order 2 has $I(G) = \overline{G}$ and so is isomorphic to G. For example, Boolean groups, dihedral groups and symmetric groups have isometry groups isomorphic to the original group. We look for conditions on the group G so that $I_e(G)$ is nontrivial, implying it has more isometries than ε and so $I(G)$ has isometries besides those in $\overline{G}$. In Example 1, the group $\mathbb{Z}_n$ has $2n$ isometries and $I_e(\mathbb{Z}_n)$ has a nontrivial isometry. We generalize this situation to all abelian groups and a little more broadly.

Definitions. Define $\mu : G \to G$ by $\mu(x) = x^{-1}$. A group G is *nearly commutative* iff for all $x, y \in G$, $\rceil xy \lceil = \rceil yx \lceil$.

Lemma 2. A group G is nearly commutative iff μ is an isometry.

Proof. Let G be nearly commutative and $a,\ b \in G$. Then $\rceil\mu(a)\mu(b)^{-1}\lceil\ =\ \rceil a^{-1}b\lceil$ $=\ \rceil ba^{-1}\lceil\ =\ \rceil(ab^{-1})^{-1}\lceil\ =\ \rceil ab^{-1}\lceil$ since an element and its inverse have the same color. Conversely, suppose μ is an isometry for G. Then $\rceil ab\lceil\ =\ \rceil\mu(a)\mu(b)\lceil\ =\ \rceil a^{-1}b^{-1}\lceil\ =\ \rceil(a^{-1}b^{-1})^{-1}\lceil\ =\ \rceil ba\lceil$. □

We will determine the nearly commutative groups later in this paper. They will include parts a) and b) in the Classification Theorem, as well as Boolean groups. By Lemma 2, an abelian group G has the isometries μ and ε in $I_e(G)$. If G is Boolean, every element is its own inverse and so $\mu = \varepsilon$ in this case. Indeed, in this Boolean case, $I_e(G) = \{\varepsilon\}$. All other abelian groups G have $\mu \neq \varepsilon$ and so have at least two isometries in $I_e(G)$. Theorem 9 implies that abelian groups have no other isometries in $I_e(G)$. To do so, Theorem 9 determines which groups have more than two isometries in $I_e(G)$. Surprisingly, we show that all groups with more than two isometries in $I_e(G)$ have eight isometries and form a family of direct product groups of the quaternion group Q_8 with Boolean groups. Further, Theorem 10 shows that groups in this family are also the only non-abelian nearly commutative groups. Theorem 3.3 in [2] identifies several properties that match with the class of groups we call nearly commutative.

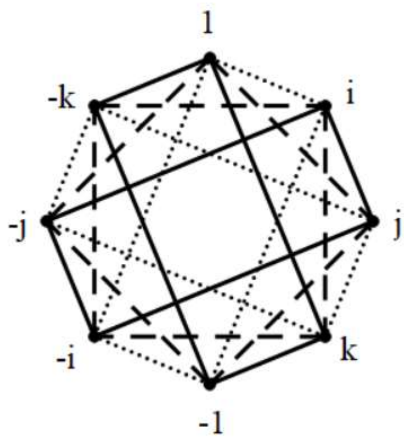

Figure 2. The Quaternions, Q_8.

Example 2. Figure 2 gives (most of) a colored graph for the eight element quaternion group Q_8. (For clarity the figure omits the edges $\overline{-x\,x}$, which form a fourth color.) Recall that Q_8 can be presented abstractly as $\langle i,j\ :\ i^2 = j^2,\ ji = i(j^{-1})\rangle$. We build the isometries of $I_1(Q_8)$ from three generators, which move $\pm i$, $\pm j$ and $\pm k$ independently. An inspection of Figure 2 confirms that the bijection α fixing all of Q_8 except for switching i and $-i$ is an isometry. Similarly β and γ are isometries, where β fixes all elements except for switching j and $-j$ and γ fixes all elements except for switching k and $-k$. Any isometry of $I_1(Q_8)$ fixes ± 1. So the isometries of $I_1(Q_8)$ are determined by where they move $\pm i$, $\pm j$ and $\pm k$. Hence α, β and γ generate $I_1(Q_8)$, which is isomorphic to $(\mathbb{Z}_2)^3$. Thus $I(Q_8)$ has 64 elements. We describe $I(Q_8)$ using three additional generators. For δ the isometry switching ± 1 and fixing all other elements, $\langle \alpha, \beta, \gamma, \delta \rangle \cong (\mathbb{Z}_2)^4$. Let $\theta = \psi_i \circ \alpha \circ \beta$ and $\lambda = \psi_j \circ \beta \circ \gamma$. Then θ and λ act on $\{1, i, j, k\}$ as $(1, i)(j, k)$ and $(1, j)(i, k)$, respectively, without affecting the $\pm$ signs. Hence $\langle \theta, \lambda \rangle \cong (\mathbb{Z}_2)^2$ and this group acts on $\{\alpha, \beta, \gamma, \delta\}$ by conjugation. Hence $I(Q_8) \cong \mathbb{Z}_2 \wr (\mathbb{Z}_2)^2$.

We next extend Example 2 using direct products.

Lemma 3. Let G and H be groups with $a, b \in G$, $h \in H$ and $h = h^{-1}$. If $\rceil a \lceil \; = \; \rceil b \lceil$, then $\rceil (a,h) \lceil \; = \; \rceil (b,h) \lceil$.

Proof. From $\rceil a \lceil \; = \; \rceil b \lceil$, we have two cases: $a = b$ or $a = b^{-1}$. If $a = b$, then $(a,h) = (b,h)$ and $\rceil (a,h) \lceil \; = \; \rceil (b,h) \lceil$. If $a = b^{-1}$, then $(a,h)^{-1} = (a^{-1}, h^{-1}) = (b,h)$ and the result follows from the definition of $\rceil \;\; \lceil$. $\square$

Theorem 4. For any group G, $I_e(G \times \mathbb{Z}_2) \cong I_e(G)$ and $I(G \times \mathbb{Z}_2) \cong I(G) \times \mathbb{Z}_2$.

Proof. We find a subgroup T of $I(G \times \mathbb{Z}_2)$ isomorphic to $I(G)$. We use T to determine all of $I(G \times \mathbb{Z}_2)$. For a group G and $\sigma \in I(G)$, we first show that $\tau \in I(G \times \mathbb{Z}_2)$, where $\tau(x,y) = (\sigma(x), y)$. Now $\rceil \tau(a,b)(\tau(c,d))^{-1} \lceil \; = \; \rceil (\sigma(a), b)(\sigma(c), d)^{-1} \lceil \; = \; \rceil (\sigma(a)\sigma(c)^{-1}, b-d) \lceil$ for $(a,b), (c,d) \in G \times \mathbb{Z}_2$. Since σ is an isometry, $\rceil (\sigma(a)\sigma(c)^{-1} \lceil \; = \; \rceil ac^{-1} \lceil$. By Lemma 3 τ is an isometry. Then the subgroup $T = \{\tau \in I(G \times \mathbb{Z}_2) \, : \, \exists \sigma \in I(G)$ with $\tau(x,y) = (\sigma(x), y)\}$ is isomorphic to $I(G)$ and $T \cap I_e(G \times \mathbb{Z}_2) \cong I_e(G)$.

Next we show that $I_e(G \times \mathbb{Z}_2) \subseteq T$. Let $\beta \in I_e(G \times \mathbb{Z}_2)$. Since $\beta(x) = x$ or $\beta(x) = x^{-1}$ for $x \in G$, β maps each subgroup to itself. Thus β leaves $G \times \{0\} = \{(g, 0\} \, : \, g \in G\}$ stable and so is an isometry of $G \times \{0\}$, which is isomorphic to G. Thus there is $\gamma \in I_e(G)$ such that for all $g \in G$, $\beta(g, 0) = (\gamma(g), 0)$. Note that $(e, 1)$ has order two and for all $g \in G$, $\rceil (e,1) \lceil \; = \; \rceil (g,0)(g,1)^{-1} \lceil \; = \; \rceil \beta(g,0)\beta(g,1)^{-1} \lceil \; = \; \rceil (\gamma(g), 0)\beta(g,1)^{-1} \lceil$. Further, $\rceil (\gamma(g), 0)(\gamma(g), 1)^{-1} \lceil \; = \; \rceil (e,1) \lceil$. Since $(e, 1)$ is of order two, $\beta(g,1)^{-1} = (\gamma(g), 1)^{-1}$. Hence for all $x \in \mathbb{Z}_2$, $\beta(g,x) = (\gamma(g), x)$ and $I_e(G \times \mathbb{Z}_2) \subseteq T$. Thus $I_e(G \times \mathbb{Z}_2) = T \cap I_e(G \times \mathbb{Z}_2)$, which we saw is isomorphic to $I_e(G)$.

The previous equality and the remark after Lemma 1 show that T is the subgroup of all isometries in $I(G \times \mathbb{Z}_2)$ that take $G \times \{0\}$ to itself. Now any isometry of $G \times \mathbb{Z}_2$ takes $G \times \{0\}$ to itself or its one other coset $G \times \{1\}$. (See [6].) Hence T has index two in $I(G \times \mathbb{Z}_2)$. Then ρ defined on $G \times \mathbb{Z}_2$ by $\rho(x,y) = (x, y+1)$ is an isometry. Further, for $\tau \in T$, $\rho \circ \tau = \tau \circ \rho$. Thus $I(G)$ is the internal direct product $T \times \langle \rho \rangle$. Hence $I(G \times \mathbb{Z}_2) \cong I(G) \times \mathbb{Z}_2$. $\square$

Corollary 5. For any natural number n, the group $Q_8 \times (\mathbb{Z}_2)^n$ has eight isometries fixing the identity and $I(Q_8 \times (\mathbb{Z}_2)^n)$ is isomorphic to $I(Q_8) \times (\mathbb{Z}_2)^n \cong (\mathbb{Z}_2 \wr (\mathbb{Z}_2)^2) \times (\mathbb{Z}_2)^n$.

Theorem 9 will show that Q_8 and the direct products in Corollary 5 are the only ones with more than two isometries fixing the identity.

Lemma 6. In a group G, suppose that $\alpha \in I_e(G)$ and $g \in G$ such that $\alpha(g) = g^{-1} \neq g$ but $\alpha \neq \mu$. Then g has order four and for all $h \in G$, if $\alpha(h) = h$, then

$$gh^{-1} = hg \qquad \text{(Equation 1).}$$

Proof. Suppose that $\alpha(g) = g^{-1} \neq g$ but $\alpha(h) = h$. Then $\rceil hg^{-1} \lceil \; = \; \rceil \alpha(h)\alpha(g)^{-1} \lceil \; = \; \rceil hg \lceil$. By definition of $\rceil \;\; \lceil$, either $hg^{-1} = hg$ or $(hg^{-1})^{-1} = hg$. Since $g^{-1} \neq g$, the first case can't happen. Thus $gh^{-1} = hg$.

The isometry α acts on the cyclic subgroup $\langle g \rangle$ as the isometries in Example 1. Since $\alpha(e) = e$ and $\alpha(g) = g^{-1} \neq g$, from Example 1 we have $\alpha(g^i) = g^{-i}$, for any i. For a contradiction, suppose g doesn't have order four (or order one or two). Then $g^2 \neq g^{-2}$. As in the previous paragraph, $g^2h^{-1} = hg^2$. We substitute Equation 1 into both sides of this last equality to get $ghg = gh^{-1}g$. By cancellation, $h = h^{-1}$. Since we assumed that $\alpha \neq \mu$, there is some $h \in G$ such that $\alpha(h) = h$, but $h \neq h^{-1}$, giving a contradiction. So g has order four. □

Lemma 7. In a group G let $H = \{h \in G : \forall \beta \in I_e(G),\, \beta(h) = h\}$. Then H is a subgroup of G.

Proof. By definition of $I_e(G)$, $e \in H$. Further, if $h \in H$, then $h^{-1} \in H$ because β either switches an element and its inverse or it leaves both fixed. To show closure, let $h,j \in H$. Suppose for a contradiction that $\gamma \in I_e(G)$ and $\gamma(hj) = (hj)^{-1} = j^{-1}h^{-1} \neq hj$. Then $\omega = \psi_j \circ \psi_h \circ \gamma \circ \psi_j \circ \psi_h$ fixes e and so $\omega \in I_e(G)$. Then $\omega(h^{-1}) = h^{-1}$. But $\gamma(j) = j$ and so $\psi_j \circ \psi_h \circ \gamma \circ \psi_j \circ \psi_h(h^{-1}) = jhj$. This gives $h^{-1} = jhj$ or $j^{-1}h^{-1} = hj$, a contradiction. Thus $hj \in H$ and H is a subgroup. □

Lemma 8. Suppose G is a group with $a,b \in G$, $\langle a,b \rangle \cong Q_8$ and there is an isometry $\beta \in I_e(G)$ such that $\beta(a) = a$ and $\beta(b) = b^{-1}$. Then for all $x \in G$, either $\langle x,a \rangle \cong Q_8$ or $x \in C(a)$, the centralizer of a.

Proof. Let $x \in G$ with all the hypotheses holding. From the presentation of Q_8 we know that $ab = ba^{-1}$. Either $\beta(x) = x$ or $\beta(x) = x^{-1}$. We will further split each of these cases into two subcases.

Suppose $\beta(x) = x$. By Equation 1, $xb = bx^{-1}$. Suppose in addition, $\beta(xa) = xa$. Then $(xa)b = b(a^{-1}x^{-1})$. Also $xab = xba^{-1} = bx^{-1}a^{-1}$. Cancellation of $ba^{-1}x^{-1} = bx^{-1}a^{-1}$ gives $a^{-1}x^{-1} = x^{-1}a^{-1}$, showing $x \in C(a)$ in this subcase. For the other subcase, suppose $\beta(xa) = a^{-1}x^{-1} \neq xa$. Equation 1 gives $x(xa) = (xa)x^{-1}$ and so $xa = ax^{-1}$, one of the relations to show $\langle a,x \rangle \cong Q_8$. Now $\beta(a) = a$, so similarly $a(xa) = (xa)a^{-1} = x$ or $xa = a^{-1}x$. Then $ax^{-1} = a^{-1}x$, giving $a^2 = x^2$, the other relation to show that $\langle a,x \rangle \cong Q_8$.

Next we assume that $\beta(x) = x^{-1} \neq x$. Then $ax = xa^{-1}$. For the next subcase, suppose that $\beta(bx) = bx$. Then $b = (bx)x^{-1} = x^{-1}(x^{-1}b^{-1})$ and $b^2 = x^{-2}$. Further, x has order four, so $b^2 = x^2$. Since $a^2 = b^2 = x^2$ and $ax = xa^{-1}$, $\langle a,x \rangle \cong Q_8$. For the final subcase, suppose that $\beta(bx) = x^{-1}b^{-1} \neq bx$. Then $a(bx) = (bx)a^{-1}$. Also $ab = ba^{-1}$. Then $ba^{-1}x = bxa^{-1}$. Cancelling and rearranging gives $ax = xa$. But $ax = xa^{-1}$, contradicting $a \neq a^{-1}$, eliminating this case. Thus in each subcase either $\langle a,x \rangle \cong Q_8$ or $x \in C(a)$. □

Theorem 9. Let G be a finite group with more than two isometries in $I_e(G)$. Then there is $n \geq 0$ such that $G \cong Q_8 \times (\mathbb{Z}_2)^n$.

Proof. Suppose G is a group with more than two isometries in $I_e(G)$, say, ε, α and β are in $I_e(G)$. Further, $I_e(G)$ is a Boolean group, so $\alpha \circ \beta$ is a fourth element of $I_e(G)$. Without loss of generality, there are $a,\ b \in G$ such that $\alpha(a) = a^{-1} \neq a$, $\alpha(b) = b \neq b^{-1}$, $\beta(b) = b^{-1} \neq b$ and so $(\alpha \circ \beta)(b) = b^{-1}$. Also, $\beta(a) \neq (\alpha \circ \beta)(a)$, so one is a and one is a^{-1}. Without loss of generality, $\beta(a) = a$ and $(\alpha \circ \beta)(a) = a^{-1}$. Using α and Equation 1 we have $ab^{-1} = ba$, one relation needed to show that $\langle a, b \rangle \cong Q_8$.

Next, for $a^2 = b^2$, Lemma 6 shows that a has order four. Replacing α with β, we see that $ba^{-1} = ab$ and b is also of order four. Also, a and b don't commute: If $ab = ba$ and $ab^{-1} = ba$, we have $b = b^{-1}$, contradicting the order of b. For $\alpha \circ \beta$ we have $\rceil ab \lceil = \rceil (\alpha \circ \beta)(a)(\alpha \circ \beta)(b) \lceil = \rceil a^{-1}b^{-1} \lceil$. Then either $ab = a^{-1}b^{-1}$ or $ab = (a^{-1}b^{-1})^{-1} = ba$. But the second equality is impossible and the first one gives $a^2 = b^2$. So $\langle a, b \rangle \cong Q_8$.

If $\langle a, b \rangle = G$, we are done. Suppose $\langle a, b \rangle \subsetneq G$. Note for $i,\ j \in \{1, \ldots,\ n\}$ that $\left\langle a, b, t_i\ :\ a^2 = b^2,\ ab^{-1} = ba,\ t_i^2 = e,\ at_i = t_i a,\ bt_i = t_i b,\ t_i t_j = t_j t_i \right\rangle$ gives a presentation of $Q_8 \times (\mathbb{Z}_2)^n$. To start verifying this presentation for G, we show that if $g, h \in G$ and $h = h^{-1}$, then $gh = hg$. Thus any elements of order two can be used as t_i in the presentation. Let $g,\ h \in G$ and $h = h^{-1}$. By Equation 1, if g is moved by any isometry of $I_e(G)$, then $gh^{-1} = hg$ and so h commutes with g. Now suppose all isometries of $I_e(G)$ fix g. By Lemma 7, gh is also fixed. Since a is not, by Equation 1 $(gh)a = a(gh)^{-1}$ as well as $ag^{-1} = ga$ and $ah = h^{-1}a = ha$. Then $(gh)a = a(gh)^{-1} = ah^{-1}g^{-1} = ahg^{-1} = hag^{-1} = hga$. By cancellation, $gh = hg$ and h commutes with all of G.

Suppose for a contradiction that G is not of the required form and H is a maximal subgroup of G generated by a, b and some (possibly empty) set of t_i satisfying the abstract presentation of Q_8 or $Q_8 \times (\mathbb{Z}_2)^n$. So there is some $x \in G$ with $x \notin H$. We consider three cases for x and find t_{n+1} satisfying the relations of the t_i so that x is generated by H and t_{n+1}, giving a contradiction.

First suppose that $x \in C(a) \cap C(b)$. Either $\alpha(x) = x^{-1}$ or $\alpha(x) = x$. For the first option, Equation 1 gives $xb^{-1} = bx$. But $x \in C(b)$, so $xb^{-1} = xb$, giving $b^{-1} = b$, which is impossible. So $\alpha(x) = x$. From Equation 1, $ax^{-1} = xa$. But $x \in C(a)$, so $ax^{-1} = ax$ and $x^{-1} = x$. Then x commutes with all of G and can be t_{n+1}.

Next, without loss of generality, suppose that $x \in C(a)$, but $x \notin C(b)$. By Lemma 8, $\langle x, b \rangle \cong Q_8$ and so $x^2 = b^2$. Then xa has order two: $xaxa = x^2a^2 = x^2b^2 = b^4 = e$. Thus xa commutes with all of G and can be t_{n+1}.

Finally suppose that $x \notin C(a)$ and $x \notin C(b)$. By Lemma 8, $\langle x, a \rangle \cong Q_8$, $\langle x, b \rangle \cong Q_8$ and $\langle a, b \rangle \cong Q_8$. Then xab has order two: $xabxab = xax^{-1}bab = x^2abab = x^2ba^{-1}ab = b^4 = e$. Pick $xab = t_{n+1}$. In each case, H is not maximal and so G must have the required form. □

Theorem 10. Let G be a finite non-abelian, nearly commutative group. Then $G \cong Q_8 \times (\mathbb{Z}_2)^n$ for some $n \geq 0$.

Proof. Suppose G is a finite non-abelian nearly commutative group. We first show that G contains a subgroup isomorphic to Q_8. There are $a,\ b \in G$ such that $ab \neq ba$. Since G is nearly commutative, we know $\rceil ab \lceil = \rceil ba \lceil$. Hence either $ab = ba$, which is false, or $ab = (ba)^{-1} = a^{-1}b^{-1}$. This last gives $a^2 = b^2$, one relation needed to show that $\langle a, b \rangle \cong Q_8$. Also from nearly commutative we have $\rceil (ab)a^{-1} \lceil = \rceil a^{-1}(ab) \lceil = \rceil b \lceil$. There are two options, but $aba^{-1} = b$ gives $ab = ba$, a contradiction. The other option $aba^{-1} = b^{-1}$ gives $ab = b^{-1}a$, showing that $\langle a,b\rangle \cong Q_8$.

If $\langle a,b\rangle$ is all of G, we are done. Otherwise we follow the structure of the proof of Theorem 9. The only difference is the case when $x \in C(a)$ and $x \in C(b)$. Note that $a \notin C(b)$, so $xa \notin C(b)$. Then we repeat the argument of the previous paragraph to show that $\langle ax,b\rangle \cong Q_8$. Then $b^2 = (ax)^2 = axax = a^2x^2 = b^2x^2$. Thus $e = x^2$, which brings us back to the argument of this case in Theorem 9. Hence $G \cong Q_8 \times (\mathbb{Z}_2)^n$, for some $n \in \mathbb{N}$. □

As a consequence of Theorem 9, $I_e(G)$ has order one, two or eight. Those with order eight are specified in Theorem 9. Those with μ and ε as isometries are nearly commutative. By Theorem 10 non-abelian nearly commutative groups exactly coincide with the groups of Theorem 9. Hence the abelian groups must have only μ and ε as isometries. Thus the previous theorems classify all isometry groups $I(G)$ except the case when $I_e(G)$ has two isometries and $\mu \notin I_e(G)$. The dicyclic groups Q_{4n} with $n > 2$ satisfy these conditions, as Example 3 below illustrates. Example 4 follows and gives a more general form and Theorem 12 shows these examples cover all groups with these conditions.

Example 3. The presentation of the dicyclic group Q_{4n} with $4n$ elements is $\langle a,b|\ a^n = b^2,\ ba^{-1} = ab\rangle$. These equations force $a^{2n} = e = b^4$. Also, every element can be written in the form a^i or a^ib, where $0 \le i < 2n$. Further, $(a^ib)(a^ib) = (ba^{-i})(a^ib) = b^2 = a^n$, so each of the $2n$ elements a^ib is of order four. (See Figure 3 for a partial illustration of Q_{12}. The elements a^i form the cyclic subgroup on the outside and the elements a^ib are all part of four cycles.) Define the bijection α on Q_{4n} by $\alpha(a^i) = a^i$ and $\alpha(a^ib) = a^{n+i}b$, which is the inverse of a^ib.

We use cases to show that α is an isometry. The color of the edge $\overline{a^ia^k}$ clearly doesn't change. For the edge $\overline{(a^ib)(a^k)}$, note that $\rceil\alpha(a^ib)\alpha(a^k)^{-1}\lceil = \rceil a^{n+i}ba^{-k}\lceil = \rceil a^{n+i}a^kb\lceil = \rceil a^{n+i+k}b\lceil$ and $\rceil a^ib(a^k)^{-1}\lceil = \rceil a^{i+k}b\lceil$. Since $a^{n+i+k}b$ and $a^{i+k}b$ are inverses, color is preserved. For the edge $\overline{(a^ib)(a^kb)}$, similarly $\rceil\alpha(a^ib)\alpha(a^kb)^{-1}\lceil = \rceil a^{n+i}ba^kb\lceil = \rceil a^{n+i}a^{-k}b^2\lceil = \rceil a^{n+i-k}a^n\lceil = \rceil a^{i-k}\lceil$ and $\rceil a^ib(a^kb)^{-1}\lceil = \rceil a^iba^{n+k}b\lceil = \rceil a^ia^{-n-k}a^n\lceil = \rceil a^{i-k}\lceil$.

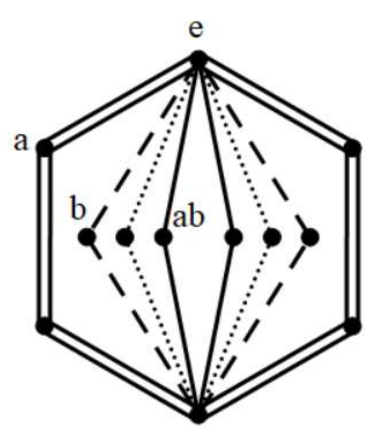

Figure 3. The dicyclic group Q_{12}.

Remark: In Q_{4n}, if $n = 2$, then Q_8 has other isometries. However, by Theorem 9, if $n > 2$, then there are at most two isometries in $I_e(G)$. Also note that $\mathbb{Z}_4$ acts like it is a dicyclic group "Q_4," where we take $a = 2$ and $b = 1$.

Example 4. Let C be any abelian non-Boolean group and Y be $\mathbb{Z}_4$ or Y be a dicyclic group Q_{2^n} with 2^n elements. Note that μ is an isometry of C, and since C is not Boolean, $\mu \neq \varepsilon$. As in Example 3, we take a and b to be the generators of Y. The mapping ϕ taking Y into $\{\varepsilon, \mu\}$ given by $\phi(a^i) = \varepsilon$ and $\phi(a^i b) = \mu$ is readily shown to be a homomorphism. Define the semidirect product $C \rtimes Y$ by $(c, y)(d, z) = (c\phi_y(d), yz)$, where $\phi_y = \phi(y)$. Then the subgroup $C \rtimes \{a^i\} = \{(c, a^i) : c \in C \text{ and } 0 \leq i < 2^{n-1}\}$ is abelian and $(c, a^i b)(c, a^i b) = (cc^{-1}, (a^i b)^2) = (e, b^2)$ and so $(c, a^i b)$ is of order four. Define the bijection α on $C \rtimes Y$ by $\alpha(c, a^i) = (c, a^i)$ and $\alpha(c, a^i b) = (c, a^i b)^{-1} = (c, a^{2^{n-2}+i} b)$. We verify that α is an isometry as in Example 3 . Further, since C is not a Boolean group, there is some c such that $c^2 \neq e$. Then $(c, ab)(c, e) = (cc^{-1}, ab) = (e, ab)$, while $(c, e)(c, ab) = (c^2, ab)$. Thus $\rceil(c, ab)(c, e)\lceil$ does not equal $\rceil(c, e)(c, ab)\lceil$, showing that $C \rtimes Y$ is not nearly commutative. Hence $I_e(C \rtimes Y)$ has only the two isometries ε and α.

Lemma 11. Let G be any non-nearly commutative group and suppose that $I_e(G)$ has exactly two isometries. Then $H = \{h \in G : \forall \beta \in I_e(G),\ \beta(h) = h\}$ is an abelian subgroup of G with index two. Further, for all $x, y \notin H$, $x^2 = y^2$.

Proof. By Lemma 7 and the hypotheses, H is a proper subgroup of G. Let $h \in H$ and $x \in G \backslash H$. From Equation 1, $hx = xh^{-1}$ or $x^{-1}hx = h^{-1}$. Thus conjugation by x inverts all of H. Inversion is thus an automorphism of H, showing H is abelian. Further, for $x, y \in G \backslash H$, we have $(xy)^{-1}h(xy) = y^{-1}h^{-1}y = h$. Then $xy \in H$, showing H has index 2. Finally we show $x^2 = y^2$. For β the non-identity isometry in $I_e(G)$, $\rceil x(xy)^{-1}\lceil = \rceil\beta(x)\beta(xy)^{-1}\lceil = \rceil x^{-1}(xy)^{-1}\lceil$. Since $x \neq x^{-1}$, then $x(xy)^{-1} \neq x^{-1}(xy)^{-1}$. By the definition of $\rceil\ \lceil$, $(x(xy)^{-1})^{-1} = x^{-1}(xy)^{-1}$ or $xyx^{-1} = x^{-1}y^{-1}x^{-1}$. So $xy = x^{-1}y^{-1}$ and $x^2 = y^{-2} = y^2$ since y has order four by Lemma 6. □

Theorem 12. Suppose G is a finite non-nearly commutative group, $I_e(G)$ has two elements and $\mu \notin I_e(G)$. Then G is isomorphic to Y or to $C \rtimes Y$, where C is an abelian group and Y is either a dicyclic group Q_{2^n} with 2^n elements or Y is $\mathbb{Z}_4$.

Proof. Given such a group G, let $\beta \in I_e(G)$ with $\beta \neq \varepsilon$. By Lemma 11 the elements of G fixed by β form an abelian subgroup H of index two. Also, for a fixed $y \in G \backslash H$, $y^2 \in H$ and y^2 has order two. We can write H as the product of cyclic groups of prime power order: $H \cong C_1 \times C_2 \times \ldots \times C_k$ and have $y^2 \in C_1$. Since y^2 is of order two, C_1 has order 2^k for some k. Let d be a generator of C_1. Let C be $C_2 \times \ldots \times C_k$ and let Y be generated by y and d. By Lemma 6, $dy = yd^{-1}$ or $y^{-1}dy = d^{-1}$. Further C_1 has one element of order two, so $d^{2^{k-1}} = y^2$. Thus $Y \cong Q_{2^{k+1}}$, unless $k = 1$, in which case $Y \cong \mathbb{Z}_4$. If $C = \{e\}$, we're done.

Suppose that $C \neq \{e\}$. Note that $C \cap Y = \{e\}$. Also every element of G is in H or the coset yH. Thus every element of G is of the form cz, where $c \in C$ and $z \in Y$. Next we show that C is normal in G. Let $b \in C$ and $cz \in G$. Then $(cz)^{-1}b(cz) = z^{-1}bz$ since $b, c \in C$, an abelian group. By Lemma 6, $z^{-1}bz$ is b or b^{-1}, and both are in C.

Finally, we show $G \cong C \rtimes Y$. Since C is abelian, the inversion μ is an automorphism. Define $\phi : Y \to Aut(C)$ by $\phi(x) = \varepsilon$ if $x \in C_1$ and $\phi(x) = \mu$ if $x \notin C_1$. As in Example 4, ϕ is the homomorphism used to define $C \rtimes Y$. We prove the function $\xi : C \rtimes Y \to G$ given by $\xi((a,x)) = ax$ is an isomorphism. By the decomposition of H every element of G can be written in one of two forms, aw or awy, where $w \in C_1$. Clearly, $\xi((a,w)) = aw$ and $\xi((a,wy)) = awy$, so ξ is a bijection. We show operation preservation by cases. Let $c, d \in C$ and $w \in C_1$ and $z \in Y$. Then using Equation 1 in the middle we have $\xi((c,wy)(d,z)) = \xi((c\mu(d), wyz)) = cd^{-1}wyz = cw(d^{-1}y)z = cw(yd)z = \xi((c,wy))\xi((d,z))$. Similarly, $\xi((c,w)(d,z)) = \xi((cd,wz)) = cdwz = cwdz = \xi((c,w))\xi((d,z))$. □

Classification Theorem. Let G be any finite group. Then $I(G) \cong G$ unless one of the following occurs.

a) G is abelian and not a Boolean group. In this case, $I(G) \cong G \rtimes \mathbb{Z}_2$.

b) For some $n \geq 0$, $G \cong Q_8 \times (\mathbb{Z}_2)^n$, where Q_8 is the quaternion group. In this case, $I(G)$ has eight times as many elements as G and is isomorphic to $I(Q_8) \times (\mathbb{Z}_2)^n$, where $I(Q_8) \cong \mathbb{Z}_2 \wr (\mathbb{Z}_2)^2$.

c) G is not in previous cases and is isomorphic to Y or to $C \rtimes Y$, where C is an abelian, non-Boolean group and Y is either $\mathbb{Z}_4$ or a dicyclic group Q_{2^k}, for some power 2^k. In this case, $I(G) \cong G \rtimes \mathbb{Z}_2$.

Proof. Let G be a finite group. By Theorem 9, $I_e(G)$ has one, two or eight elements. The case with eight elements is determined by Example 2, Corollary 5 and Theorem 9, giving part b. For the situation when there are two isometries in $I_e(G)$, we consider first the case when $\mu \in I_e(G)$. By Lemma 2, G is nearly commutative. By Theorem 10 and Corollary 5, G is therefore abelian, giving part a. Theorem 12 and Example 3 and 4 give part c, the case with two isometries in $I_e(G)$ and $\mu \notin I_e(G)$. Otherwise, $I_e(G)$ has only one isometry and so by Lemma 1, $I(G) = \overline{G}$, which is isomorphic to G. □

While we have classified isometry groups of finite groups, several related questions remain open. First of all, can this classification be extended to infinite groups? Also, we can define a "similarity" of a group G to be a permutation of the elements of G that consistently permutes the colors of the edges of the graph of G. That is, σ is a *similarity* of G iff for all $a, b, c, d \in G$, if $\rceil ab^{-1} \lceil = \rceil cd^{-1} \lceil$, then $\rceil \sigma(a)\sigma(b^{-1}) \lceil = \rceil \sigma(c)\sigma(d^{-1}) \lceil$. It is known that all group automorphisms of a group are similarities. It is conjectured that all similarities are compositions of group automorphisms and isometries. (Remark. Cameron in [2] uses automorphism for what we call an isometry and weak automorphism for what we call similarity. Given the connection between group automorphisms and similarities, we feel it is advantageous to have different terms.) Next, one can modify the definition of $\rceil \;\; \lceil$. In particular, combining colors by allowing $\rceil x \lceil = \rceil y \lceil$ even if $y \notin \{x, x^{-1}\}$, gives a generalization of the notion of a norm in a finite vector space. One can then ask what the isometry and similarity groups are for these colored graphs. Finally, we can replace the group G with a "homogeneous loop," as defined in [6]. In particular, a Cayley loop C_n with 2^n elements generalizes the quaternions $Q_8 = C_3$ and, for $n > 3$, has many isometries: $I_e(C_n)$ has $2^{(2^{n-1}-1)}$ elements. Very little is known about the isometry groups and similarity groups of homogeneous loops.

Acknowledgements. We wish to thank the referees and Dr. Bret Benesh for their help in simplifying and clarifying some of the proofs.

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