01 Oct 1988

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Recommended Citation

W. T. Ingram, "Concerning periodic points in mappings of continua," *Proceedings of the American Mathematical Society*, vol. 104, no. 2, pp. 643 - 649, American Mathematical Society, Oct 1988.
The definitive version is available at https://doi.org/10.1090/S0002-9939-1988-0962842-8

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 2, October 1988

CONCERNING PERIODIC POINTS IN MAPPINGS OF CONTINUA

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(Communicated by Dennis Burke)

ABSTRACT. In this paper we present some conditions which are sufficient for a mapping to have periodic points.

THEOREM. *If f is a mapping of the space X into X and there exist subcontinua H and K of X such that* (1) *every subcontinuum of K has the fixed point property,* (2) *$f[K]$ and every subcontinuum of $f[H]$ are in class W,* (3) *$f[K]$ contains H,* (4) *$f[H]$ contains $H \cup K$, and* (5) *if n is a positive integer such that $(f|H)^{-n}(K)$ intersects K, then $n = 2$, then K contains periodic points of f of every period greater than* 1.

Also included is a fixed point lemma:

LEMMA. *Suppose f is a mapping of the space X into X and K is a subcontinuum of X such that $f[K]$ contains K. If* (1) *every subcontinuum of K has the fixed point property, and* (2) *every subcontinuum of $f[K]$ is in class W, then there is a point x of K such that $f(x) = x$.*

Further we show that: If f is a mapping of $[0,1]$ into $[0,1]$ and f has a periodic point which is not a power of 2, then $\lim\{[0,1], f\}$ contains an indecomposable continuum. Moreover, for each positive integer i, there is a mapping of $[0,1]$ into $[0,1]$ with a periodic point of period 2^i and having a hereditarily decomposable inverse limit.

1. Introduction. In his book, *An Introduction to Chaotic Dynamical Systems* [**3**, Theorem 10.2, p. 62], Robert L. Devaney includes a proof of Sarkovskii s' Theorem. Consider the following order on the natural numbers: $3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2 \cdot 3 \rhd 2 \cdot 5 \rhd \cdots \rhd 2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd \cdots \rhd 2^3 \cdot 3 \rhd 2^3 \cdot 5 \rhd \cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1$. Suppose $f\colon R \to R$ is continuous. If $k \rhd m$ and f has a periodic point of prime period k, then f has a periodic point of period m. In working through a proof of this theorem for $k = 3$, the author discovered the main result of this paper—Theorem 2. For an alternate proof of Sarkovskii's Theorem for $k = 3$, see also [**7**]. For a further look at this theorem for ordered spaces see [**13**].

By a *continuum* we mean a compact connected metric space and by a *mapping* we mean a continuous function. By a *periodic point* of period n for a mapping f of a continuum M into M is meant a point x such that $f^n(x) = x$. The statement that x has prime period n means that n is the least integer k such that $f^k(x) = x$. A continuum M is said to have the *fixed point property* provided if f is a mapping of M into M there is a point x such that $f(x) = x$. A mapping f of a continuum X onto a continuum M is said to be *weakly confluent* provided for each subcontinuum K of

Received by the editors May 22, 1987 and, in revised form, September 14, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54F20, 54H20; Secondary 54F62, 54H25, 54F55.

Key words and phrases. Periodic point, fixed point property, class W, indecomposable continuum, inverse limit.

M some component of $f^{-1}(K)$ is thrown by f onto K. A continuum is said to be in *Class W* provided every mapping of a continuum onto it is weakly confluent. The continuum T is a *triod* provided there is a subcontinuum K of T such that $T-K$ has at least three components. A continuum is *atriodic* provided it does not contain a triod. A continuum M is *unicoherent* provided if M is the union of two subcontinua H and K, then the common part of H and K is connected. A continuum is *hereditarily unicoherent* provided each of its subcontinua is unicoherent. If f is a mapping of a space X into X, the *inverse limit* of the inverse limit sequence $\{X_i, f_i\}$ where, for each i, X_i is X and f_i is f will be denoted $\lim\{X, f\}$. For the inverse sequence $\{X_i, f_i\}$, the inverse limit is the subset of the product of the sequence of spaces $X_1, X_2, \ldots$ to which the point $(x_1, x_2, \ldots)$ belongs if and only if $f_i(x_{i+1}) = x_i$.

There has been considerable interest in periodic homeomorphisms of continua where a homeomorphism h is called periodic provided there is an integer n such that h^n is the identity. Wayne Lewis has shown [**8**] that for each n there is a chainable continuum with a periodic homeomorphism of period n. A theorem of Michel Smith and Sam Young [**14**] should be compared with Theorem 3 of this paper. Smith and Young show that if a chainable continuum M has a periodic homeomorphism of period greater than 2, then M contains an indecomposable continuum. In this paper we consider the question of the existence of periodic points in mappings of continua.

2. A fixed point theorem. The problem of finding a periodic point of period n for a mapping f is, of course, the same as the problem of finding a fixed point for f^n. Not surprisingly, we need a fixed point theorem as a lemma to the main theorem of this paper. The following theorem, which the author finds interesting in its own right, should be compared with an example of Sam Nadler [**11**] of a mapping with no fixed point of a disk to a containing disk. A corollary to Theorem 1 is the well-known corresponding result for mappings of intervals.

THEOREM 1. *Suppose X is a space, f is a mapping of X into X, and K is a subcontinuum of X such that $f[K]$ contains K. If (1) every subcontinuum of K has the fixed point property, and (2) every subcontinuum of $f[K]$ is in Class W, then there is a point x of K such that $f(x) = x$.*

PROOF. Since $f[K]$ is in Class W and K is a subset of $f[K]$, there is a subcontinuum K_1 of K such that $f[K_1] = K$. Then $f|K_1: K_1 \to K$ is weakly confluent since every subcontinuum of $f[K]$ is in Class W; thus there is a subcontinuum K_2 of K_1 such that $f[K_2] = K_1$. Since K_1 is in Class W, $f|K_2: K_2 \to K_1$ is weakly confluent; therefore there is a subcontinuum K_3 of K_2 such that $f[K_3] = K_2$. Continuing this process there exists a monotonic decreasing sequence $K_1, K_2, K_3, \ldots$ of subcontinua of K such that $f[K_{i+1}] = K_i$ for $i = 1, 2, 3, \ldots$. Let H denote the common part of all the terms of this sequence and note that $f[H] = H$, since $f[H] = f[\bigcap_{i>0} K_i] = \bigcap_{i>0} f[K_i] = \bigcap_{i>0} K_i = H$. Since $f|H$ throws H onto H and H has the fixed point property, there exists a point x of H (and therefore of K) such that $f(x) = x$.

REMARK. Note that (1) and (2) of the hypothesis of Theorem 1 are met if $f[K]$ is chainable ([**12**, Theorem 4, p. 236 and **4**], respectively), while (2) is met if $f[K]$ is atriodic and acyclic [**1**] and (1) is met by planar, tree-like continua such that each two points of a subcontinuum L lie in a weakly chainable subcontinuum of L [**10**].

3. Periodic points. In this section we prove the main result of the paper.

THEOREM 2. *If f is a mapping of the space X into X and there exist subcontinua H and K of X such that* (1) *every subcontinuum of K has the fixed point property,* (2) *$f[K]$ and every subcontinuum of $f[H]$ are in class W,* (3) *$f[K]$ contains H,* (4) *$f[H]$ contains $H \cup K$, and* (5) *if n is a positive integer such that $(f|H)^{-n}(K)$ intersects K, then $n = 2$, then K contains periodic points of f of every period greater than* 1.

PROOF. Suppose $n \geq 2$. There is a sequence $H_1, H_2, \ldots, H_{n-1}$ of subcontinua of H such that $f[H_1] = K$ (note that $f|H$ is weakly confluent) and $f[H_{i+1}] = H_i$ for $i = 1, 2, \ldots, n-2$ (in case $n > 2$). There is a subcontinuum K_n of K so that $f[K_n] = H_{n-1}$. Thus, $f^n[K_n] = K$ and so $f^n[K_n]$ contains K_n, so, by Theorem 1, there is a point x of K_n such that $f^n(x) = x$. We must show that if $j < n$ then $f^j(x)$ is not x. If $j < n$ and $f^j(x) = x$, then $j = n - 2$ and x is in H_2. Since $f^n(x) = x$ and $f^{n-2}(x) = x$, $f^2(x) = x$. Since x is in $(f|H)^{-2}(K)$, x is in $(f|H)^{-4}(K)$ and in K contrary to (5) of the hypothesis. Therefore, x is periodic of prime period n.

REMARK. If f is a mapping of the continuum M into itself and f has a periodic point of period k, then the mapping of $\lim\{M, f\}$ induced by f has periodic points of period k, e.g. $(x, f^{k-1}(x), \ldots, f(x), x, \ldots)$. Thus, although Theorem 2 does not directly apply to homeomorphisms, it may be used to conclude the existence of homeomorphisms with periodic points.

COROLLARY. *If M is a chainable continuum, f is a mapping of M into M, and there are subcontinua H and K of M such that $f[K] = H$, $f[H]$ contains $H \cup K$, and if $(f|H)^{-n}(K)$ intersects K then $n = 2$ then f has periodic points of every period.*

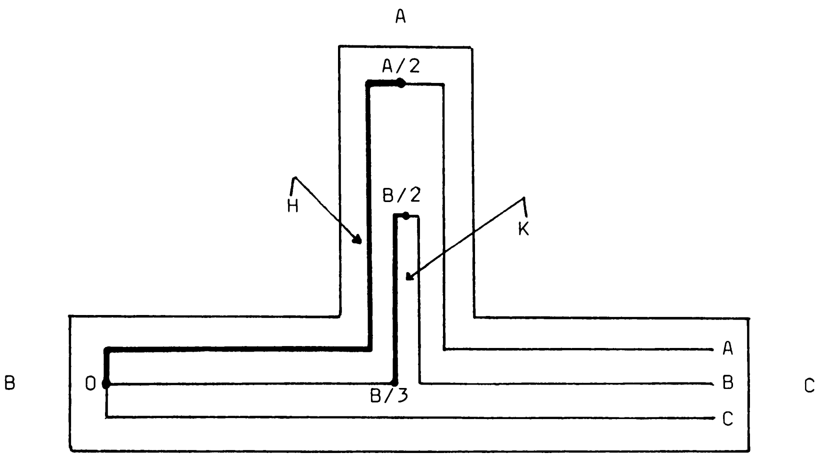

FIGURE 1

EXAMPLE. Let f be the mapping of the simple triod T to itself given in [**5**]. The mapping f is represented in Figure 1 above. Letting $H = [0, A/2]$ and $K = [B/3, B/2]$ it follows from Theorem 2 that f has periodic points of every period.

EXAMPLE. Let f be the mapping of the simple triod T to itself given in [**2**]. The mapping f is represented in Figure 2 below. Letting $H = [0, 3B/8]$ and $K = [C/32, C/8]$, it follows from Theorem 2 that f has periodic points of every period.

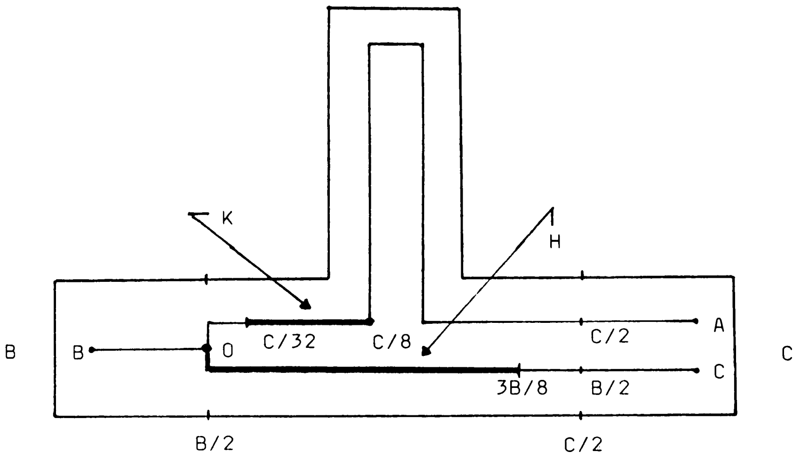

FIGURE 2

EXAMPLE. Let f be the mapping of the unit circle S^1 to itself given by $f(z) = z^2$. Letting $H = \{e^{i\theta} | 0 \leq \theta \leq 3\pi/4\}$ and $K = \{e^{i\theta} | \pi \leq \theta \leq 3\pi/2\}$, it follows from Theorem 2 that f has periodic points of every period. Similarly, if f is a mapping of S^1 onto itself which is homotopic to z^n for some $n > 1$, then f has periodic points of every period.

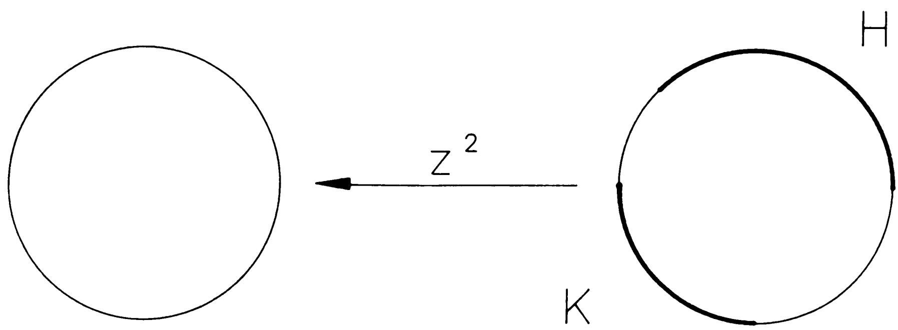

FIGURE 3

COROLLARY. *If f is a mapping of an interval to itself with a periodic point of period* 3, *then f has periodic points of every period.*

PROOF. To see this it is a matter of noting that the hypothesis of Theorem 2 is met. We indicate the proof for one of two cases and leave the second similar case to the reader. Suppose a, b and c are points of the interval with $a < b < c$ and $f(a) = b$, $f(b) = c$ and $f(c) = a$ [the other case is $f(a) = c$, $f(b) = a$ and $f(c) = b$].

If $f^{-1}(c)$ is nondegenerate, then there exist mutually exclusive intervals H and K lying in $[b, c]$ and $[a, b]$, respectively, so that $f[H]$ is $[a, c]$ and $f[K]$ is $[b, c]$ and Theorem 2 applies.

Suppose $f^{-1}(c) = \{b\}$. Choose K lying in $[a, b]$ and H lying in $[b, c]$ so that $f[K] = [b, c]$ and $f[H] = [a, c]$. For each i, denote by H_i the set $(f|H)^{-1}(K)$. Note that a is not in H_i for $i = 1, 2, 3, \ldots$ so c is not in H_i for $i = 2, 3, 4, \ldots$ and thus b is not in H_i for $i = 3, 4, \ldots$. Further, b is not in H_1 since c is not in K. Thus, if H_i intersects K, then $i = 2$. Consequently, the hypothesis of Theorem 2 is met.

REMARK. Condition (5) of Theorem 2 seems a bit artificial. A more natural condition the author experimented with in its place is a requirement that H and K be mutually exclusive. In fact, in each of the examples, the H and K given are mutually exclusive. However, replacing condition (5) with this proved to be undesirable in that the Sarkovskii Theorem for $k = 3$ is not a corollary to Theorem 2 if the alternate condition is used. That condition (5) may not be replaced by the assumption that H and K are mutually exclusive can be seen by the following. For the function $f\colon [0, 1] \to [0, 1]$, which is piecewise linear and contains the points $(0, \frac{1}{2})$, $(\frac{1}{2}, 1)$ and $(1, 0)$, there do not exist mutually exclusive intervals H and K such that $f[H]$ contains $H \cup K$ and $f[K]$ contains H. To see this suppose H and K are such mutually exclusive intervals. By Theorem 2, K contains a periodic point of f of period 3. Note that f^3 has only four fixed points: 0, $\frac{1}{2}$, $\frac{2}{3}$, and 1. Since $\frac{2}{3}$ is a fixed point for f, K must contain one of 0, $\frac{1}{2}$, and 1. We complete the proof by showing that each of these possibilities leads to a contradiction.

(1) Suppose 0 is in K. Then 1 is in H since $f^{-1}(0) = \{1\}$ and $f[H]$ contains K. But since $f^{-1}(1) = \{\frac{1}{2}\}$, $\frac{1}{2}$ is in both H and K.

(2) Suppose 1 is in K. Since $f^{-1}(1) = \{\frac{1}{2}\}$, $\frac{1}{2}$ is in H. Since $f^{-1}(\frac{1}{2}) = \{0, \frac{3}{4}\}$ and H and K do not intersect 0 is in H and $\frac{3}{4}$ is in K. But, $f^{-1}(0) = \{1\}$ so 1 is in H.

(3) Suppose $\frac{1}{2}$ is in K. As before, one of 0 and $\frac{3}{4}$ is in H. Since $f^{-1}(0) = \{1\}$, if 0 is in H then 1 is in both H and K. Thus $\frac{3}{4}$ is in H. Then $f^{-1}(\frac{3}{4})$ contains two points, $\frac{5}{8}$ and one less than $\frac{1}{2}$, so $P_1 = \frac{5}{8}$ is in H. Since $f^{-1}(P_1)$ contains two points, $\frac{1}{8}$ and one between $\frac{5}{8}$ and $\frac{3}{4}$, $\frac{1}{8}$ is in K. Thus, $f^{-1}(\frac{1}{8}) = \frac{15}{16}$ is in H. Since $f^{-1}(\frac{15}{16})$ contains two points, $\frac{17}{32}$ and one less than $\frac{1}{2}$, $P_2 = \frac{17}{32}$ is in H. Continuing this process, we get a sequence $P_1, P_2, \ldots$ of points of H which converges to $\frac{1}{2}$. Thus $\frac{1}{2}$ is in H.

4. Periodic points and indecomposability.

In this section we show that under certain conditions the existence of a periodic point of period three in a mapping of a continuum M to itself implies that $\lim\{M, f\}$ contains an indecomposable continuum. Of course the result is not true in general since a rotation of S^1 by 120 degrees yields a homeomorphism of S^1 and a copy of S^1 for the inverse limit.

THEOREM 3. *Suppose f is a mapping of the continuum M into itself and x is a point of M which is a periodic point of f of period three. If M is atriodic and hereditarily unicoherent, then* $\lim\{M, f\}$ *contains an indecomposable continuum.*

Moreover, the inverse limit is indecomposable if $\mathrm{cl}(\bigcup_{i>0} f^i[M_1]) = M$, *where* M_1 *is the subcontinuum of* M *irreducible from* x *to* $f(x)$.

PROOF. Suppose x is a periodic point of f of period three. Denote by M_1, M_2 and M_3 subcontinua of M irreducible from x to $f(x)$, $f(x)$ to $f^2(x)$ and $f^2(x)$ to x, respectively. Note that since M is hereditarily unicoherent, $M_1 \cap (M_2 \cup M_3) = (M_1 \cap M_2) \cup (M_1 \cap M_3)$ is a continuum, so there is a point p common to all three continua.

The three continua $M_1 \cap M_2$, $M_2 \cap M_3$ and $M_1 \cap M_3$ all contain the point p so, since M is atriodic, one of them is a subset of the union of the other two [**15**]. Suppose $M_1 \cap M_2$ is a subset of $(M_2 \cap M_3) \cup (M_1 \cap M_3) = M_3 \cap (M_1 \cup M_2) = M_3$. (The last equality follows since $M_3 \cap (M_1 \cup M_2)$ is a subcontinuum of M_3 containing x and $f^2(x)$ and M_3 is irreducible between x and $f^2(x)$). Then, $M_1 \cup M_2$ is a subset of M_3 for if not there is a point t of $M_1 \cup M_2$ such that t is not in M_3. Since $M_1 \cap M_2$ is a subset of M_3, t is in M_1 or in M_2 but not in $M_1 \cap M_2$. Suppose t is in $M_1 - (M_1 \cap M_2)$. Since t is not in M_3, t is in $M_1 - (M_1 \cap M_3)$ and thus t is in

$$M_1 - [(M_1 \cap M_2) \cup (M_1 \cap M_3)] = M_1 - [M_1 \cap (M_2 \cup M_3)].$$

But, $M_1 \cap (M_2 \cup M_3)$ is a subcontinuum of M_1 containing x and $f(x)$, so it contains M_1 since M_1 is irreducible between x and $f(x)$. Thus, $M_1 = M_1 \cap (M_2 \cup M_3)$ and so $M_1 \cup M_2$ is a subset of M_3.

Note that $f[M_1]$ is a continuum containing $f(x)$ and $f^2(x)$, so $f[M_1] \cap M_2$ is a subcontinuum of M_2 containing these two points. Since M_2 is irreducible from $f(x)$ to $f^2(x)$, $f[M_1] \cap M_2 = M_2$. Therefore, M_2 is a subset of $f[M_1]$. Similarly, $f[M_2]$ contains M_3 and $f[M_3]$ contains M_1. However, since M_3 contains $M_1 \cup M_2$, M_3 contains x, $f(x)$ and $f^2(x)$, so $f[M_3]$ contains $M_1 \cup M_2 \cup M_3$. Thus, $f^{n+2}[M_1]$ contains $f^{n+1}[M_2]$ which contains $f^n[M_3]$ which contains $M_1 \cup M_2 \cup M_3$ for $n = 1, 2, 3, \ldots$ and so $\mathrm{cl}(\bigcup_{i>0} f^i[M_1]) = \mathrm{cl}(\bigcup_{i>0} f^i[M_2]) = \mathrm{cl}(\bigcup_{i>0} f^i[M_3])$. Then, $H = \mathrm{cl}(\bigcup_{n>0} f^n[M_1])$ is a continuum such that $f|H: H \to H$. Denote by K the inverse limit, $\varprojlim\{H, f|H\}$. We show that K is indecomposable by showing the conditions of [**6**, Theorem 2, p. 267] are satisfied. Suppose n is a positive integer and e is a positive number. There is a positive integer k such that if t is in H then $d(t, f^k[M_3]) < e$. Suppose C is a subcontinuum of H containing two of the three points, x, $f(x)$ and $f^2(x)$. Then C contains one of M_1, M_2 and M_3. In any case $f^2[C]$ contains M_3, and thus, if $m = k+2$, $d(t, f^m[C]) < e$ for each t in H. By Kuykendall's Theorem, K is indecomposable.

THEOREM 4. *If* f *is a mapping of* $[0,1]$ *to* $[0,1]$ *and* f *has a periodic point whose period is not a power of* 2, *then* $\varprojlim\{([0,1], f)\}$ *contains an indecomposable continuum. Moreover, for each positive integer* i, *there exists a mapping which has a periodic point of period* 2^i *and hereditarily decomposable inverse limit.*[1]

PROOF. Suppose f has a periodic point which has period n and n is not a power of 2. Then, $n = 2^j\,(2k+1)$ for some $j, k \geq 0$, and f^{2^j} has a periodic point of period $2k+1$. By the Sarkovskii Theorem, f^{2^j} has a periodic point of period 6, so

[1] *Added in proof*: Theorem 4 first appeared, with a slightly different proof, as Theorem 1 of *Chaos, periodicity, and snakelike continua* by Marcy Barge and Joe Martin in a publication (MSRI 014-84) of the Mathematical Sciences Research Institute, Berkeley, California in January, 1984.

$g = (f^{2^j})^2$ has a periodic point of period 3. Since $\lim\{[0,1], f\}$ is homeomorphic to $\lim\{[0,1], g\}$, by Theorem 3 $\lim\{[0,1], f\}$ contains an indecomposable continuum.

In the family of maps $f_\mu(x) = \mu x(1-x)$, for $2 < \mu < \mu_c \sim 3.5699456\ldots$ all the inverse limits for μ in this range are hereditarily decomposable and for each power of 2, there is a map in this collection with a periodic point of period that power of 2. In fact for $2 < \mu < 3$ the inverse limit is an arc, for $3 < \mu < \mu_c$ the inverse limit becomes, as μ increases, first a sinusoid, then a sinusoid to a double sinusoid, etc. For more details on this, see [**9**].

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