01 Jan 2023

Three Solutions For Discrete Anisotropic Kirchhoff-type Problems

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Recommended Citation

M. Bohner et al., "Three Solutions For Discrete Anisotropic Kirchhoff-type Problems," *Demonstratio Mathematica*, vol. 56, no. 1, article no. 20220209, De Gruyter, Jan 2023.
The definitive version is available at https://doi.org/10.1515/dema-2022-0209

Research Article

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Three solutions for discrete anisotropic Kirchhoff-type problems

https://doi.org/10.1515/dema-2022-0209
received July 17, 2022; accepted February 7, 2023

Abstract: In this article, using critical point theory and variational methods, we investigate the existence of at least three solutions for a class of double eigenvalue discrete anisotropic Kirchhoff-type problems. An example is presented to demonstrate the applicability of our main theoretical findings.

Keywords: discrete boundary value problem, Kirchhoff-type equation, critical point theory, variational methods

MSC 2020: 39A05, 35B38

1 Introduction

Let $N \in \mathbb{N} \setminus \{1\}$ and put $[1, N]_{\mathbb{Z}} = [1, N] \cap \mathbb{Z}$. Consider the anisotropic discrete Kirchhoff-type problem

$$\begin{cases} -M(\rho(u))\Delta(\phi_{p(n-1)}(\Delta u(n-1))) = \lambda f(n, u(n)) + \mu g(n, u(n)), & n \in [1, N]_{\mathbb{Z}}, \\ u(0) = u(N+1) = 0, \end{cases} \tag{1.1}$$

where $\phi_{p(n)}(t) = |t|^{p(n)-2}t$, $(n, t) \in [1, N]_{\mathbb{Z}} \times \mathbb{R}$, $\lambda, \mu > 0$ are two parameters, Δ is the usual forward difference operator (defined by $\Delta u(n) = u(n+1) - u(n)$), $f, g : [1, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, g is nonnegative throughout this article, $\rho : \mathbb{R} \to [0, \infty)$ is a functional defined by

$$\rho(u) := \sum_{n=1}^{N+1} \frac{|\Delta u(n-1)|^{p(n-1)}}{p(n-1)},$$

$p : [0, N]_{\mathbb{Z}} \to [2, \infty)$ is a function with

$$p^- := \min_{n \in [0,N]_{\mathbb{Z}}} p(n) \le \max_{n \in [0,N]_{\mathbb{Z}}} p(n) =: p^+,$$

and $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function that satisfies
(M_0) there exist $m_0, m_1 > 0$ with $m_0 \le M(t) \le m_1$ for all $t \ge 0$.

The difference equation in (1.1) can be considered a discrete analogue of Kirchhoff's equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \mathrm{d}x \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

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which Kirchhoff studied in 1883 (see [1]) and which extends d'Alembert's wave equation, by considering the effect during vibrations when the length of the string is varied. In (1.2), the parameter L denotes string length, h stands for the cross-sectional area, E is the material's Young modulus, ρ is the mass density, and ρ_0 is the initial tension. A special feature of the Kirchhoff equation is that (1.2) contains the nonlocal coefficient

$$\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x,$$

depending on the average $\frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x$ of kinetic energy $\frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2$ on $[0, L]$, and therefore (1.2) is not a pointwise identity. On the other hand, the stationary analogue of (1.2) is given as follows:

$$\begin{cases} \left(a + b\displaystyle\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\triangle u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which was studied extensively only after Lions [2] initiated an abstract setting for this problem. Some related, interesting, and important results can be seen, e.g., in [3–7].

Difference equations are generally understood as the first theory to appear with the systematic growth of mathematics, and they can be found in biological neural networks, economy, signal processing, computer engineering, genetics, medicine, ecology, and digital control. In the last decades, many researchers around the globe have used variational methods and critical point theory to study the existence and multiplicity of solutions for discrete boundary value problems, as referenced in [8–12]. We also refer the reader to [13–16], where discrete Kirchhoff-type equations were studied. However, as to the problem (1.1), it contains the Kirchhoff term $\sum_{n=1}^{N+1}\frac{|\Delta u(n-1)|^{p(n-1)}}{p(n-1)}$, which makes it much more complicated to work with, and there are some studies [17–23], that discuss the existence of solutions for some discrete boundary value problems of $p(k)$-Kirchhoff-type using variational methods and critical point theory.

Inspired by the above results, in this article, we investigate the existence of three solutions for (1.1). In this case, we apply suitable conditions and create intervals for the two parameters λ and μ. We also give Example 3.3 to show the use of our proven theorems.

2 Preliminaries and basic notation

In this article, X denotes a finite-dimensional real Banach space and $I_\lambda : X \to \mathbb{R}$ is a functional satisfying the following structure hypothesis:

> $I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$ for all $u \in X$, where $\Phi, \Psi : X \to \mathbb{R}$ are two functions of class C^1 on X such that Φ is coercive, i.e., $\lim_{\|u\|\to\infty}\Phi(u) = \infty$, and λ is a positive real parameter.

In this framework, a finite-dimensional variant of [24, Theorem 3.3] (see also [24, Corollary 3.1 and Remark 3.9]) is as follows.

For all r, r_1, and r_2 with $r_2 > r_1$ and $r_2 > \inf_X\Phi$, and all $r_3 > 0$, we define

$$\begin{aligned} \varphi(r) &:= \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\sup_{v\in\Phi^{-1}(-\infty,r)}\Psi(v) - \Psi(u)}{r - \Phi(u)}, \\ \beta(r_1, r_2) &:= \inf_{u\in\Phi^{-1}(-\infty,r_1)} \sup_{v\in\Phi^{-1}[r_1,r_2)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)}, \\ \gamma(r_2, r_3) &:= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\Psi(u)}{r_3}, \\ \alpha(r_1, r_2, r_3) &:= \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\}. \end{aligned}$$

Theorem 2.1. (See [24, Theorem 3.3]) *Assume that*

(a_1) Φ *is convex and* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$,

(a_2) *for every* $u_1, u_2 \in X$ *such that* $\Psi(u_1) \geq 0$ *and* $\Psi(u_2) \geq 0$, *one has*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are three positive constants r_1, r_2, *and* r_3 *with* $r_1 < r_2$ *such that*

(a_3) $\varphi(r_1) < \beta(r_1, r_2)$,

(a_4) $\varphi(r_2) < \beta(r_1, r_2)$,

(a_5) $\gamma(r_2, r_3) < \beta(r_1, r_2)$.

Then, for each $\lambda \in \left(\frac{1}{\beta(r_1,r_2)}, \frac{1}{\alpha(r_1,r_2,r_3)}\right)$, *the functional* $\Phi - \lambda\Psi$ *admits three distinct critical points* u_1, u_2, *and* u_3 *such that*

$$u_1 \in \Phi^{-1}(-\infty, r_1), \quad u_2 \in \Phi^{-1}[r_1, r_2), \quad u_3 \in \Phi^{-1}(-\infty, r_2 + r_3).$$

For an application of Theorem 2.1 to the discrete case, see [18]. Furthermore, we refer the reader to [25–28] for situations of successful employment of results such as Theorem 2.1 in order to prove the existence of solutions for various boundary value problems. We introduce the N-dimensional Hilbert space

$$X := \{u : [0, N+1] \to \mathbb{R} : u(0) = u(N+1) = 0\}$$

endowed with the norm

$$\|u\| = \left(\sum_{n=1}^{N+1} |\Delta u(n-1)|^2\right)^{\frac{1}{2}}.$$

Lemma 2.2. *If* $u \in X$, *then*

$$\sum_{n=1}^{N+1} |\Delta u(n-1)|^{p(n-1)} \geq (N+1)^{\frac{2-p^-}{2}} \|u\|^{p^-}.$$

Proof. This inequality is a consequence of [29, (A.6)], as in our setting $p : [1, N]_{\mathbb{Z}} \to [2, \infty)$. □

Definition 2.3. We say that $u \in X$ solves (1.1) provided

$$\begin{aligned} &M(\rho(u)) \sum_{n=1}^{N+1} |\Delta u(n-1)|^{p(n-1)-2} \Delta u(n-1)\Delta v(n-1) \\ &\quad = \lambda \sum_{n=1}^{N} f(n, u(n))v(n) + \mu \sum_{n=1}^{N} g(n, u(n))v(n) \quad \text{for all } v \in X. \end{aligned}$$

Set

$$\Phi(u) = \widehat{M}(\rho(u)), \quad \Psi(u) = \sum_{n=1}^{N} \left(F(n, u(n)) + \frac{\mu}{\lambda} G(n, u(n))\right),$$

where, for $t \in \mathbb{R}$ and $n \in [1, N]_{\mathbb{Z}}$,

$$\widehat{M}(t) = \int_0^t M(\xi)\mathrm{d}\xi, \quad F(n, t) = \int_0^t f(n, s)\mathrm{d}s, \quad G(n, t) = \int_0^t g(n, s)\mathrm{d}s.$$

3 Main results

In this section, we formulate our main results based on the existence of at least three solutions for the problem (1.1). Set

$$G^{\theta} \coloneqq \sum_{n=1}^{N} \max_{|t|\le\theta} G(n,t) \text{ for } \theta > 0$$

and

$$G_{\eta} \coloneqq \sum_{n=1}^{N} \inf_{t\in[0,\eta]} G(n,t) \text{ for } \eta > 0.$$

Theorem 3.1. *Assume there exist constants $\theta_1, \theta_2, \theta_3 > 0$, $\eta \ge 1$, and $n_0 \in [1,N]_{\mathbb{Z}}$ with*

$$\theta_1 < \eta 2^{1/p^-}\sqrt{N+1}, \tag{3.1}$$

$$\theta_2 > \left(\frac{2m_1p^+}{m_0p^-}\right)^{1/p^-}(N+1)^{1-1/p^-}\eta^{p^+/p^-}, \tag{3.2}$$

$$F(n_0,\eta) > \sum_{n=1}^{N} F(n,\theta_1) \tag{3.3}$$

and $\theta_2 < \theta_3$ such that
(A_1) $f(n,t) \ge 0$ *for each* $(n,t) \in [1,N]_{\mathbb{Z}} \times [-\theta_3,\theta_3]$,
(A_2) *the inequality*

$$\max\left\{\frac{\sum_{n=1}^{N}F(n,\theta_1)}{\theta_1^{p^-}}, \frac{\sum_{n=1}^{N}F(n,\theta_2)}{\theta_2^{p^-}}, \frac{\sum_{n=1}^{N}F(n,\theta_3)}{\theta_3^{p^-}-\theta_2^{p^-}}\right\} < \frac{m_0p^-(N+1)^{1-p^-}}{2m_1p^+\eta^{p+}}\left(F(n_0,\eta) - \sum_{n=1}^{N}F(n,\theta_1)\right)$$

holds.

Then, for every

$$\lambda \in \left(\frac{\frac{2m_1}{p^-}\eta^{p+}}{F(n_0,\eta)-\sum_{n=1}^{N}F(n,\theta_1)}, \frac{m_0(N+1)^{1-p^-}}{p^+}\min\left\{\frac{\theta_1^{p^-}}{\sum_{n=1}^{N}F(n,\theta_1)}, \frac{\theta_2^{p^-}}{\sum_{n=1}^{N}F(n,\theta_2)}, \frac{\theta_3^{p^-}-\theta_2^{p^-}}{\sum_{n=1}^{N}F(n,\theta_3)}\right\}\right)$$

and for all

$$\begin{aligned}\mu \in \Bigg[0, \min\Bigg\{&\frac{\frac{2m_1}{p^-}\eta^{p+} - \lambda\left(F(n_0,\eta)-\sum_{n=1}^{N}F(n,\theta_1)\right)}{G_\eta - G^{\theta_1}},\\ &\min\Bigg\{\frac{\frac{m_0}{p^+}(N+1)^{1-p^-}\theta_1^{p^-} - \lambda\sum_{n=1}^{N}F(n,\theta_1)}{G^{\theta_1}}, \frac{\frac{m_0}{p^+}(N+1)^{1-p^-}\theta_2^{p^-} - \lambda\sum_{n=1}^{N}F(n,\theta_2)}{G^{\theta_2}},\\ &\frac{\frac{m_0}{p^+}(N+1)^{1-p^-}(\theta_3^{p^-}-\theta_2^{p^-}) - \lambda\sum_{n=1}^{N}F(n,\theta_3)}{G^{\theta_3}}\Bigg\}\Bigg\}\Bigg),\end{aligned}$$

the problem (1.1) *has at least three nonnegative solutions u_1, u_2, and u_3 satisfying*

$$\max_{n\in[1,N]_{\mathbb{Z}}}|u_1(n)| < \theta_1, \quad \max_{n\in[1,N]_{\mathbb{Z}}}|u_2(n)| < \theta_2, \quad \max_{n\in[1,N]_{\mathbb{Z}}}|u_3(n)| < \theta_3.$$

Proof. Our aim is to apply Theorem 2.1 to the problem. We consider the auxiliary problem

$$\begin{cases}-M(\rho(u))\Delta(\phi_{p(n-1)}(\Delta u(n-1))) = \lambda\hat{f}(n,u(n)) + \mu g(n,u(n)), & n \in [1,N]_{\mathbb{Z}},\\ u(0) = u(N+1) = 0,\end{cases} \tag{3.4}$$

where $\hat{f} : [1, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function defined by

$$\hat{f}(n, \xi) = \begin{cases} f(n, -\theta_3) & \text{if } \xi < -\theta_3, \\ f(n, \xi) & \text{if } -\theta_3 \le \xi \le \theta_3, \\ f(n, \theta_3) & \text{if } \xi > \theta_3. \end{cases}$$

If (3.4) has a solution u satisfying $-\theta_3 \le u(n) \le \theta_3$ for all $n \in [1, N]_{\mathbb{Z}}$, then u is a solution of (1.1). This estimate is obtained at the end of this proof. Therefore, for our goal, it is enough to show that our conclusion holds for (3.4). Let the functions Φ and Ψ, for every $u \in X$, be defined as follows:

$$\Phi(u) = \widehat{M}(\rho(u)), \quad \Psi(u) = \sum_{n=1}^{N} \left[F(n, u(n)) + \frac{\mu}{\lambda} G(n, u(n)) \right].$$

Furthermore, let us denote by I_λ the energy functional associated with problem (1.1), i.e., $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for every $u \in X$. By standard arguments, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse. On the other hand, Ψ is well defined, continuously Gâteaux differentiable and with compact derivative. More precisely, one has

$$\Psi'(u)v = \sum_{n=1}^{N} \left[f(n, u(n))v(n) + \frac{\mu}{\lambda} g(n, u(n))v(n) \right]$$

and

$$\Phi'(u)v = M(\rho(u)) \sum_{n=1}^{N+1} |\Delta u(n-1)|^{p(n-1)-2} \Delta u(n-1) \Delta v(n-1)$$

for every $u, v \in X$. By the definition of Φ and thanks to Lemma 2.2, we have

$$\Phi(u) \ge \frac{m_0}{p^+}(N+1)^{\frac{2-p^-}{2}} \|u\|^{p^-}, \tag{3.5}$$

so Φ is coercive. Therefore, the assumptions on Φ and Ψ, as requested in Theorem 2.1, are verified. We know the critical points of the function $\Phi - \lambda\Psi$ are the solutions to the problem (1.1). Let $w(n_0) = \eta$ for a fixed integer $n_0 \in [1, N]_{\mathbb{Z}}$ and $w(n) = 0$ for $n \in [1, N]_{\mathbb{Z}} \setminus \{n_0\}$. Clearly $w \in X$. Since $\|w\| = \eta\sqrt{2} \ge 1$, (3.5) shows

$$\frac{2m_0}{p^+}(N+1)^{(2-p^-)/2} \eta^{p^-} \le \Phi(w) \le \frac{2m_1}{p^-} \eta^{p+}. \tag{3.6}$$

We set

$$r_1 = \frac{m_0}{p^+}(N+1)^{1-p^-} \theta_1^{p^-}, \quad r_2 = \frac{m_0}{p^+}(N+1)^{1-p^-} \theta_2^{p^-}, \quad r_3 = \frac{m_0}{p^+}(N+1)^{1-p^-} (\theta_3^{p^-} - \theta_2^{p^-}).$$

From (3.1), (3.2), and (3.6), we deduce

$$r_1 < \Phi(w) < r_2.$$

By first using the Cauchy-Schwarz inequality, then (3.5), and then the estimate $\Phi(u) < r_1$, we obtain

$$|u(n)|^{p^-} \le (N+1)^{\frac{p^-}{2}} \|u\|^{p^-} \le \frac{p^+}{m_0}(N+1)^{p^- - 1} \Phi(u) \le \frac{p^+}{m_0}(N+1)^{p^- - 1} r_1 = \theta_1^{p^-}.$$

From the definition of r_1, we obtain

$$\Phi^{-1}(-\infty, r_1) = \{u \in X : \Phi(u) < r_1\} \subseteq \{u \in X : |u| \le \theta_1\}.$$

By using the assumption (A_1), one has

$$\sup_{u \in \Phi^{-1}(-\infty, r_1)} \sum_{n=1}^{N} F(n, u(n)) \le \sum_{n=1}^{N} \max_{|t| \le \theta_1} F(n, t) \le \sum_{n=1}^{N} F(n, \theta_1).$$

In a similar way, we obtain

$$\sup_{u\in\Phi^{-1}(-\infty,r_2)}\sum_{n=1}^{N}F(n,u(n))\leq\sum_{n=1}^{N}F(n,\theta_2)$$

and

$$\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\sum_{n=1}^{N}F(n,u(n))\leq\sum_{n=1}^{N}F(n,\theta_3).$$

Therefore, since $0\in\Phi^{-1}(-\infty,r_1)$ and $\Phi(0)=\Psi(0)=0$, one has

$$\begin{aligned}\varphi(r_1)&=\inf_{u\in\Phi^{-1}(-\infty,r_1)}\frac{\sup_{v\in\Phi^{-1}(-\infty,r_1)}\Psi(v)-\Psi(u)}{r_1-\Phi(u)}\\&\leq\frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)}{r_1}\\&=\frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\sum_{n=1}^{N}\Big[F(n,u(n))+\frac{\mu}{\lambda}G(n,u(n))\Big]}{r_1}\\&\leq\frac{\sum_{n=1}^{N}\max_{|t|<\theta_1}\Big[F(n,t)+\frac{\mu}{\lambda}G(n,t)\Big]}{r_1}\\&=\frac{\sum_{n=1}^{N}\max_{|t|<\theta_1}F(n,t)+\frac{\mu}{\lambda}G^{\theta_1}}{\frac{m_0}{p^+}(N+1)^{1-p^-}\theta_1^{p^-}}.\end{aligned}$$

In addition,

$$\begin{aligned}\varphi(r_2)&\leq\frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)}{r_2}\\&=\frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\sum_{n=1}^{N}\Big[F(n,u(n))+\frac{\mu}{\lambda}G(n,u(n))\Big]}{r_2}\\&\leq\frac{\sum_{n=1}^{N}\max_{|t|<\theta_2}F(n,t)+\frac{\mu}{\lambda}G^{\theta_2}}{\frac{m_0}{p^+}(N+1)^{1-p^-}\theta_2^{p^-}}\end{aligned}$$

and

$$\begin{aligned}\gamma(r_2,r_3)&=\frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\Psi(u)}{r_3}\\&=\frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\sum_{n=1}^{N}\Big[F(n,u(n))+\frac{\mu}{\lambda}G(n,u(n))\Big]}{r_3}\\&\leq\frac{\sum_{n=1}^{N}\max_{|t|\leq\theta_3}\Big[F(n,t)+\frac{\mu}{\lambda}G(n,t)\Big]}{r_3}\\&=\frac{\sum_{n=1}^{N}\max_{|t|\leq\theta_3}F(n,t)+\frac{\mu}{\lambda}G^{\theta_3}}{\frac{m_0}{p^+}(N+1)^{1-p^-}(\theta_3^{p^-}-\theta_2^{p^-})}.\end{aligned}$$

Now, for $u\in\Phi^{-1}(-\infty,r_1)$, we have

$$\begin{aligned}\beta(r_1,r_2)&\geq\frac{F(n_0,\eta)-\sum_{n=1}^{N}F(n,\theta_1)+\frac{\mu}{\lambda}[G_\eta-G^{\theta_1}]}{\Phi(w)-\Phi(u)}\\&\geq\frac{F(n_0,\eta)-\sum_{n=1}^{N}F(n,\theta_1)+\frac{\mu}{\lambda}[G_\eta-G^{\theta_1}]}{\frac{2m_1}{p^-}\eta^{p+}}.\end{aligned}$$

Due to the condition (A_2), we obtain

$$\alpha(r_1, r_2, r_3) < \beta(r_1, r_2).$$

Therefore, the assumptions (a_3), (a_4), and (a_5) of Theorem 2.1 are verified. Since $\hat{f}$ and g are nonnegative, the solutions of the problem (1.1) are nonnegative. Indeed, let u_* be a nontrivial solution of the problem (1.1). Then, u_* is nonnegative. Arguing by contradiction, assume that the discrete interval $\mathcal{A} := \{n \in [1, N]_{\mathbb{Z}} : u_*(n) < 0\} \neq \varnothing$. Put $\bar{v}(n) = \min\{u_*(n), 0\}$ for $n \in [1, N]_{\mathbb{Z}}$. Clearly, $\bar{v} \in X$, and one has

$$M(\rho(u_*)) \sum_{n=1}^{N+1} |\Delta u_*(n-1)|^{p(n-1)-2} \Delta u_*(n-1) \Delta \bar{v}(n-1) = \lambda \sum_{n=1}^{N} \hat{f}(n, u_*(n)) \bar{v}(n) + \mu \sum_{n=1}^{N} g(n, u_*(n)) \bar{v}(n).$$

By choosing $\bar{v} = u_*$, we have

$$0 \leq m_0 (N+1)^{1-p^-/2} \|u_*\| \leq M(\rho(u_*)) \sum_{n=1}^{N+1} |\Delta u_*(n-1)|^{p(n-1)} \leq 0,$$

i.e.,

$$\|u_*\| \leq 0.$$

Thus, $u_* = 0$ in $\mathcal{A}$, which is absurd. Hence, u_* is nonnegative. Now, we show that the functional I_λ satisfies the assumption (a_2) of Theorem 2.1. Let u_1 and u_2 be two local minima for I_λ (see [24, proof of Theorem 3.3]). Then u_1 and u_2 are critical points for I_λ, and so, they are nonnegative solutions to the problem (1.1). Then, we have $u_1, u_2 \geq 0$. Thus, it follows that

$$su_1 + (1-s)u_2 \geq 0 \quad \text{for all } s \in [0,1].$$

Therefore, $\left(f + \frac{\mu}{\lambda} g\right)(n, su_1 + (1-s)u_2) \geq 0$ for all $s \in [0,1]$. Consequently,

$$\Psi(su_1 + (1-s)u_2) \geq 0 \quad \text{for all } s \in [0,1].$$

Hence, Theorem 2.1 implies that for every λ in the interval given in the statement and for every μ in the interval given in the statement, the functional I_λ has three critical points $u_1, u_2, u_3 \in X$ with $\Phi(u_1) < r_1$, $\Phi(u_2) < r_2$, and $\Phi(u_3) < r_2 + r_3$, i.e.,

$$\max_{n \in [1,N]_{\mathbb{Z}}} |u_1(n)| < \theta_1, \quad \max_{n \in [1,N]_{\mathbb{Z}}} |u_2(n)| < \theta_2, \quad \max_{n \in [1,N]_{\mathbb{Z}}} |u_3(n)| < \theta_3.$$

This completes the proof. □

Remark 3.2. Note that (3.3) is satisfied with large η if, for example, f is positive and then F is increasing with respect to the second variable.

We now present the following example to illustrate Theorem 3.1.

Example 3.3. Consider the problem

$$\begin{cases} -(2 + \sin(\rho(u)))\Delta(\phi_{p(n-1)}(\Delta u(n-1))) = \lambda f(n, u(n)) + \mu g(n, u(n)), & n \in [1,4]_{\mathbb{Z}}, \\ u(0) = u(5) = 0, \end{cases} \tag{3.7}$$

which is in the form (1.1) with

$$N = 4, \quad p(n) = \frac{n}{4} + 2, \quad M(\xi) = 2 + \sin\xi.$$

Here,

$$p^- = 2, \quad p^+ = 3, \quad m_0 = 1, \quad m_1 = 3.$$

Let

$$f(n,t)=\begin{cases} t^6 & \text{for } t\le 10^{10},\\ \dfrac{10^{70}}{t} & \text{for } t>10^{10}\end{cases}$$

for all $n\in[1,4]_{\mathbb{Z}}$. Thus, we have

$$F(n,t)=\begin{cases} \dfrac{t^7}{7} & \text{for } t\le 10^{10},\\ 10^{70}\left(\ln t+\dfrac{1}{7}-10\ln(10)\right) & \text{for } t>10^{10}\end{cases}$$

for all $n\in[1,4]_{\mathbb{Z}}$. Let now

$$\theta_1=10,\quad \theta_2=10^{100},\quad \theta_3=10^{200},\quad \eta=10^{10}.$$

Then

$$0<\theta_1^2<10^{21},\quad \theta_3^2>\theta_2^2>45\cdot 10^{30}$$

and $f(n,t)\ge 0$ for each $(n,t)\in[1,4]_{\mathbb{Z}}\times\mathbb{R}$. Taking into account

$$\max\left\{\frac{\sum_{n=1}^4 F(n,10)}{10^2},\frac{\sum_{n=1}^4 F(n,10^{100})}{10^{200}},\frac{\sum_{n=1}^4 F(n,10^{200})}{10^{400}-10^{200}}\right\}$$
$$=\max\left\{\frac{4\cdot\frac{10^7}{7}}{10^2},\frac{4\cdot 10^{70}\left(90\ln(10)+\frac{1}{7}\right)}{10^{200}},\frac{4\cdot 10^{70}\left(190\ln(10)+\frac{1}{7}\right)}{10^{400}-10^{200}}\right\}=\frac{4}{7}\cdot 10^5,$$

so (A_2) of Theorem 3.1 is verified. Note

$$\frac{4}{7}\cdot 10^5<\frac{1}{45\cdot 10^{30}}\left(\frac{10^{70}}{7}-\frac{4\cdot 10^7}{7}\right).$$

Then, for every

$$\lambda\in\left(\frac{21\cdot 10^{30}}{10^{70}-4\cdot 10^7},\frac{7}{60}\cdot 10^{-5}\right)$$

and for every nonnegative continuous function $g:[1,4]_{\mathbb{Z}}\times\mathbb{R}\to\mathbb{R}$, there exists δ such that, for each $\mu\in[0,\delta)$, the problem (3.7) has at least three nonnegative solutions u_1, u_2, and u_3 satisfying

$$\max_{n\in[1,4]_{\mathbb{Z}}}|u_1(n)|<10,\quad \max_{n\in[1,4]_{\mathbb{Z}}}|u_2(n)|<10^{100},\quad \max_{n\in[1,4]_{\mathbb{Z}}}|u_3(n)|<10^{200}.$$

Remark 3.4. If either $f(n,t)\ne 0$ for some $n\in[1,N]_{\mathbb{Z}}$ or $g(n,t)\ne 0$ for some $n\in[1,N]_{\mathbb{Z}}$, or both hold true, then the solutions from Theorem 3.1 are not trivial.

We now deduce the following consequence of Theorem 3.1.

Theorem 3.5. *Assume there exist constants $\theta_1,\theta_4>0$, $\eta\ge 1$, and $n_0\in[1,N]_{\mathbb{Z}}$ with*

$$\theta_1^{p^-}<\min\{\eta^{p+},2(N+1)^{p^-/2}\eta^{p^-}\}$$

and

$$\theta_4^{p^-}>\max\left\{\eta^{p^-},\frac{4m_1p^+}{m_0p^-}(N+1)^{p^--1}\eta^{p+}\right\}$$

such that

(A_5) *$f(n,t) \geq 0$ for each $(n,t) \in [1,N]_{\mathbb{Z}} \times [-\theta_4, \theta_4]$,*
(A_6) *the inequality*

$$\max\left\{\frac{\sum_{n=1}^{N} F(n,\theta_1)}{\theta_1^{p^-}}, \frac{2\sum_{n=1}^{N} F(n,\theta_4)}{\theta_4^{p^-}}\right\} < \frac{m_0 p^-(N+1)^{1-p^-}}{2m_1 p^+ + m_0 p^-(N+1)^{1-p^-}} \cdot \frac{F(n_0,\eta)}{\eta^{p+}}$$

holds.

Then, for every

$$\lambda \in \left(\eta^{p+} \frac{\frac{m_0 p^-(N+1)^{1-p^-}}{2} + m_1 p^+}{F(n_0,\eta)}, \frac{m_0 p^-(N+1)^{1-p^-}}{2} \min\left\{\frac{\theta_1^{p^-}}{\sum_{n=1}^{N} F(n,\theta_1)}, \frac{\theta_4^{p^-}}{2\sum_{n=1}^{N} F(n,\theta_4)}\right\}\right)$$

and for all

$$\mu \in \left[0, \min\left\{\frac{\frac{2m_1}{p^-}\eta^{p+} - \lambda\left(F(n_0,\eta) - \sum_{n=1}^{N} F(n,\theta_1)\right)}{G_\eta - G^{\theta_1}},\right.\right.$$
$$\left.\left.\min\left\{\frac{\frac{m_0}{p^+}(N+1)^{1-p^-}\theta_1^{p^-} - \lambda\sum_{n=1}^{N} F(n,\theta_1)}{G^{\theta_1}}, \frac{\frac{m_0}{p^+}(N+1)^{1-p^-}\theta_4^{p^-} - 2\lambda\sum_{n=1}^{N} F\left(n, \frac{\theta_4}{\sqrt[p^-]{2}}\right)}{2G^{\frac{\theta_4}{\sqrt[p^-]{2}}}},\right.\right.\right.$$
$$\left.\left.\left.\frac{\frac{m_0}{p^+}(N+1)^{1-p^-}\theta_4^{p^-} - 2\lambda\sum_{n=1}^{N} F(n,\theta_4)}{2G^{\theta_4}}\right\}\right\}\right),$$

the problem (1.1) *has at least three nonnegative solutions u_1, u_2, and u_3 satisfying*

$$\max_{n\in[1,N]_{\mathbb{Z}}} |u_1(n)| < \theta_1, \quad \max_{n\in[1,N]_{\mathbb{Z}}} |u_2(n)| < \frac{\theta_4}{\sqrt[p^-]{2}}, \quad \max_{n\in[1,N]_{\mathbb{Z}}} |u_3(n)| < \theta_4.$$

Proof. Choose $\theta_2 = \frac{\theta_4}{\sqrt[p^-]{2}}$ and $\theta_3 = \theta_4$. So from (A_6), we obtain

$$\begin{aligned}\frac{\sum_{n=1}^{N} F(n,\theta_2)}{\theta_2^{p^-}} &= \frac{2\sum_{n=1}^{N} F\left(n, \frac{\theta_4}{\sqrt[p^-]{2}}\right)}{\theta_4^{p^-}} \\ &\leq \frac{2\sum_{n=1}^{N} F(n,\theta_4)}{\theta_4^{p^-}} \\ &< \frac{m_0 p^-(N+1)^{1-p^-}}{2m_1 p^+ + m_0 p^-(N+1)^{1-p^-}} \cdot \frac{F(n_0,\eta)}{\eta^{p+}}\end{aligned} \tag{3.8}$$

and

$$\frac{\sum_{n=1}^{N} F(n,\theta_3)}{\theta_3^{p^-} - \theta_2^{p^-}} = \frac{2\sum_{n=1}^{N} F(n,\theta_4)}{\theta_4^{p^-}} < \frac{m_0 p^-(N+1)^{1-p^-}}{2m_1 p^+ + m_0 p^-(N+1)^{1-p^-}} \cdot \frac{F(n_0,\eta)}{\eta^{p+}}. \tag{3.9}$$

From (A_6) and taking into account $\theta_1 < \eta^{\frac{p^+}{p^-}}$, we have

$$\begin{aligned}&\frac{m_0 p^-(N+1)^{1-p^-}}{2m_1 p^+ \eta^{p+}}\left(F(n_0,\eta) - \sum_{n=1}^{N} F(n,\theta_1)\right) \\ &> \frac{m_0 p^-(N+1)^{1-p^-}}{2m_1 p^+ \eta^{p+}} F(n_0,\eta) - \frac{m_0 p^-(N+1)^{1-p^-}}{2m_1 p^+ \theta_1^{p^-}} \sum_{n=1}^{N} F(n,\theta_1) \\ &> \frac{m_0 p^-(N+1)^{1-p^-}}{2m_1 p^+ \eta^{p+}} F(n_0,\eta) - \frac{(m_0 p^-(N+1)^{1-p^-})^2}{2m_1 p^+(2m_1 p^+ + m_0 p^-(N+1)^{1-p^-})\eta^{p+}} F(n_0,\eta) \\ &= \frac{m_0 p^-(N+1)^{1-p^-}}{(2m_1 p^+ + m_0 p^-(N+1)^{1-p^-})\eta^{p+}} F(n_0,\eta).\end{aligned}$$

Hence, from (3.8), (3.9), and (A_6), the assumption (A_6) of Theorem 3.1 is satisfied, and it follows the conclusion. □

Here, we present a simple consequence of Theorem 3.5 in the case when f does not depend upon n.

Theorem 3.6. *Assume that there exist constants $\theta_1, \theta_4 > 0$, and $\eta \geq 1$ with*

$$\theta_1^{p^-} < \min\{\eta^{p+}, 2(N+1)^{p^-/2}\eta^{p^-}\}$$

and

$$\theta_4^{p^-} > \max\left\{\eta^{p^-}, \frac{4m_1 p^+}{m_0 p^-}(N+1)^{p^- -1}\eta^{p+}\right\}$$

such that

(A_7) *$f(t) \geq 0$ for each $t \in [-\theta_4, \theta_4]$,*

(A_8) *the inequality*

$$\max\left\{\frac{F(\theta_1)}{\theta_1^{p^-}}, \frac{2F(\theta_4)}{\theta_4^{p^-}}\right\} < \frac{m_0 p^-(N+1)^{1-p^-}}{2m_1 p^+ + m_0 p^-(N+1)^{1-p^-}} \cdot \frac{F(\eta)}{\eta^{p+}}$$

holds.

Then, for every

$$\lambda \in \left(\left(\frac{m_0 p^-(N+1)^{1-p^-}}{2} + m_1 p^+\right)\frac{\eta^{p+}}{F(\eta)}, \frac{m_0 p^-(N+1)^{1-p^-}}{2}\min\left\{\frac{\theta_4^{p^-}}{F(\theta_1)}, \frac{\theta_4^{p^-}}{2F(\theta_4)}\right\}\right)$$

and for all

$$\mu \in \left[0, \min\left\{\frac{\frac{2m_1}{p^-}\eta^{p+} - \lambda(F(\eta) - F(\theta_1))}{G_\eta - G^{\theta_1}},\right.\right.$$
$$\min\left\{\frac{\frac{m_0}{p^+}(N+1)^{1-p^-}\theta_1^{p^-} - \lambda F(\theta_1)}{G^{\theta_1}}, \frac{\frac{m_0}{p^+}(N+1)^{1-p^-}\theta_4^{p^-} - 2\lambda F\left(\frac{1}{\sqrt[p^-]{2}}\theta_4\right)}{2G^{\frac{1}{\sqrt[p^-]{2}}\theta_4}},\right.$$
$$\left.\left.\left.\frac{\frac{m_0}{p^+}(N+1)^{1-p^-}\theta_4^{p^-} - 2\lambda F(\theta_4)}{2G^{\theta_4}}\right\}\right\}\right),$$

the problem (1.1) *has at least three nonnegative solutions u_1, u_2, and u_3 satisfying*

$$\max_{n\in[1,N]_{\mathbb{Z}}}|u_1(n)| < \theta_1, \quad \max_{n\in[1,N]_{\mathbb{Z}}}|u_2(n)| < \frac{\theta_4}{\sqrt[p^-]{2}}, \quad \max_{n\in[1,N]_{\mathbb{Z}}}|u_3(n)| < \theta_4.$$

Finally, we provide the following simple consequence of Theorem 3.5 when $\mu = 0$.

Theorem 3.7. *Let $f : [1, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ be continuous with $tf(n,t) > 0$* for all $(n,t) \in [1,N]_{\mathbb{Z}} \times (\mathbb{R}\setminus\{0\})$. *Assume that*

(A_9) *we have*

$$\lim_{t\to 0}\frac{f(n,t)}{|t|^{p^- -1}} = \lim_{t\to\infty}\frac{f(n,t)}{|t|^{p^- -1}} = 0.$$

Then, for every $\lambda > \bar{\lambda}$, where

$$\bar{\lambda} = \left(\frac{m_0 p^-(N+1)^{1-p^-}}{2} + m_1 p^+\right)\max\left\{\inf_{\eta \geq 1}\frac{\eta^{p+}}{F(n_0,\eta)}, \inf_{0<\eta<1}\frac{\eta^{p^-}}{F(n_0,\eta)}, \inf_{-1<\eta<0}\frac{(-\eta)^{p^-}}{F(n_0,\eta)}, \inf_{\eta\leq -1}\frac{(-\eta)^{p+}}{F(n_0,\eta)}\right\},$$

the problem (1.1), *in the case $\mu = 0$, possesses at least four distinct nontrivial solutions.*

Proof. Put

$$f_1(n,t) = \begin{cases} f(n,t) & \text{if } (n,t) \in [1,N]_{\mathbb{Z}} \times [0,\infty), \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_2(n,t) = \begin{cases} -f(n,-t) & \text{if } (n,t) \in [1,N]_{\mathbb{Z}} \times [0,\infty), \\ 0 & \text{otherwise,} \end{cases}$$

and define

$$F_1(n,t) = \int_0^t f(n,\xi)\,d\xi \quad \text{for every } (n,t) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}.$$

Fix $\lambda > \bar{\lambda}$, and let $\eta > 1$ be such that

$$\lambda > \left(\frac{m_0 p^-(N+1)^{1-p^-}}{2} + m_1 p^+\right)\frac{\eta^{p+}}{F_1(n_0,\eta)}.$$

From

$$\lim_{t\to 0}\frac{f_1(n,t)}{|t|^{p^- -1}} = \lim_{t\to\infty}\frac{f_1(n,t)}{|t|^{p^- -1}} = 0,$$

there is $\theta_1 > 0$ such that

$$\theta_1^{p^-} < \min\{\eta^{p+}, 2(N+1)^{p^-/2}\eta^{p^-}\}$$

and

$$\frac{\sum_{n=1}^N F_1(n,\theta_1)}{\theta_1^{p^-}} < \frac{m_0 p^-(N+1)^{1-p^-}}{2\lambda},$$

and there is $\theta_4 > 0$ such that

$$\max\left\{\eta^{p^-}, \frac{4m_1 p^+}{m_0 p^-}(N+1)^{p^- -1}\eta^{p+}\right\} < \theta_4^{p^-}$$

and

$$\frac{\sum_{n=1}^N F_1(n,\theta_4)}{\theta_4^{p^-}} < \frac{m_0 p^-(N+1)^{1-p^-}}{2\lambda}.$$

Then the condition (A_6) in Theorem 3.5 is satisfied, and

$$\lambda \in \left(\left(\frac{m_0 p^-(N+1)^{1-p^-}}{2} + m_1 p^+\right)\frac{\eta^{p+}}{F_1(n_0,\eta)}, \frac{m_0 p^-(N+1)^{1-p^-}}{2}\min\left\{\frac{\theta_1^{p^-}}{\sum_{n=1}^N F_1(n,\theta_1)}, \frac{\theta_4^{p^-}}{\sum_{n=1}^N F_1(n,\theta_4)}\right\}\right).$$

Hence, the problem (1.1), in the case $\mu = 0$, admits two positive solutions u_1 and u_2, which are positive solutions. Next, arguing in the same way, from

$$\lim_{t\to 0}\frac{f_2(n,t)}{|t|^{p^- -1}} = \lim_{t\to\infty}\frac{f_2(n,t)}{|t|^{p^- -1}} = 0,$$

we ensure the existence of two positive solutions u_3 and u_4 for the problem (1.1), in the case $\mu = 0$. Clearly, $-u_3$ and $-u_4$ are negative solutions to the problem (1.1), in the case $\mu = 0$, and the conclusion is achieved. □

Remark 3.8. As is customary, difference schemes are used to approximate the solutions of differential equations. The idea is to take N sufficiently large. But here, because of the term $(N+1)^{1-p^-}$, the intervals that contain λ and μ then become smaller, obscuring the importance of our results. Example 3.3 for $N = 4$ is an illustration of this fact.

Acknowledgements: The authors would like to thank the three anonymous referees and the handling editor for many useful comments and suggestions, leading to a substantial improvement in the presentation of this article.

Funding information: None declared.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state that there is no conflict of interest.

Data availability statement: Not applicable.

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