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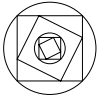
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POST-QUANTUM HERMITE–JENSEN–MERCER INEQUALITIES

MARTIN BOHNER, HÜSEYİN BUDAK AND HASAN KARA

The Jensen–Mercer inequality, which is well known in the literature, has an important place in mathematics and related disciplines. In this work, we obtain the Hermite–Jensen–Mercer inequality for post-quantum integrals by utilizing Jensen–Mercer inequalities. Then we investigate the connections between our results and those in earlier works. Moreover, we give some examples to illustrate our main results. This is the first paper about Hermite–Jensen–Mercer inequalities for post-quantum integrals.

1. Introduction

Inequalities and convex functions, a wide field of study in the literature, have been a focus of research mathematicians. Among them, Jensen’s inequality has pioneered many researches and contributed to the establishment of many inequalities. Let $0 < x_1 \leq x_2 \leq \dots \leq x_n$ and let $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ be nonnegative weights such that $\sum_{i=1}^n \mu_i = 1$. The Jensen inequality [8] states that if f is a convex function on the interval $[a, b]$, then

$$f\left(\sum_{i=1}^n \mu_i x_i\right) \leq \sum_{i=1}^n \mu_i f(x_i)$$

for all $x_i \in [a, b]$ and $\mu_i \in [0, 1]$, $i \in \{1, 2, \dots, n\}$.

Theorem 1.1 [18]. *If f is a convex function on $I = [a, b]$, then*

$$(1-1) \quad f\left(a + b - \sum_{i=1}^n \mu_i x_i\right) \leq f(a) + f(b) - \sum_{i=1}^n \mu_i f(x_i)$$

for each $x_i \in [a, b]$ and $\mu_i \in [0, 1]$, $i \in \{1, 2, \dots, n\}$ with $\sum_{i=1}^n \mu_i = 1$.

Inequality (1-1) is known as the Jensen–Mercer inequality. By using (1-1), the Hermite–Jensen–Mercer inequality was proved by Kian and Moslehian in [14]. Many mathematicians have done a lot of work using the Jensen and Jensen–Mercer inequalities. For example, Matković et al. [17] proved a variant of Jensen’s operator inequality which is a generalization of Mercer’s result. Also, Niezgoda generalized a result of the Mercer–Jensen inequality in [19]. Moreover, in [16], Ali and Khan established an integral version and then a refinement of Niezgoda’s extension of the variant of Jensen’s inequality given by Mercer. In addition, there is a lot of research on fractional integral versions of this inequality. To give examples of what has been done in recent years, Öğülmüş and Sarıkaya proved Hermite–Hadamard–Mercer inequalities for Riemann–Liouville fractional integrals in [21]. Studies of the Hermite–Jensen–Mercer inequality for k -Riemann–Liouville fractional integrals are also obtained in [5; 6].

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Quantum analysis is a very important subject in mathematics and related disciplines. Used in physics, philosophy, cryptology, computer science and mechanics, q -calculus became the focus of mathematicians in the theory of inequalities, numerical theory, basic hypergeometric functions, orthogonal polynomials.

Tariboon and Ntouyas defined the a -quantum difference operator and the a -quantum integral in 2013 [22]. Alp gave the Hermite–Hadamard inequality for a -quantum integrals in [2]. In 2020, the b -quantum difference operator and b -quantum integral were introduced by Bermudo et al. in [3]. The authors also gave the Hermite–Hadamard inequality for b -quantum integrals. In all these works, classical mathematical results are obtained when q approaches 1 from the left.

Sadjang generalized quantum calculus and introduced the notations of post-quantum calculus in [20]. Tunç and Göv gave the a -post-quantum difference operator and its integral from these notations [23]. In 2021, Chu et al. presented the b -post-quantum difference operator, Vivas-Cortes et al. presented the b -post-quantum integral version in [7] and [25], respectively. Kunt et al. [15] generalized the a -quantum Hermite–Hadamard inequality in [2] for post-quantum integrals. In [25], the generalized Hermite–Hadamard inequality was obtained for b -post-quantum integrals. For some other papers on inequalities for post post-quantum integrals, one can refer to [1; 11–13; 24]. In all these studies, when $p = 1$ in the post-quantum calculus, the results for quantum calculus are obtained.

This article consists of five sections. In Section 2, preliminary information is given for quantum and post-quantum integrals under two subsections. In Section 3, Hermite–Jensen–Mercer inequalities are obtained for the first time for post-quantum integrals. In Section 4, examples of the obtained inequalities are given. In Section 5, suggestions are made about the work that can be done in the future.

In [4], Budak and Kara obtained the Hermite–Jensen–Mercer inequality for quantum integrals by utilizing the Jensen–Mercer inequality. In this study, we extend the Hermite–Jensen–Mercer to post-quantum integrals. We also present some examples to illustrate our results.

2. Preliminaries

In this section, preliminary information about quantum calculus and post quantum calculus is given and their inequalities are presented.

q -calculus and some inequalities. In this part, we give some of the necessary explanations and related inequalities regarding q -calculus. Also, here and further we use the notation (see [10])

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad q \in (0, 1).$$

In [9], Jackson defined the q -Jackson integral from 0 to b for $0 < q < 1$ by

$$\int_0^b f(x) d_q x = (1 - q) b \sum_{n=0}^{\infty} q^n f(bq^n)$$

provided the sum converges absolutely. Moreover, in [9], the q -Jackson integral in a generic interval $[a, b]$ was introduced as

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

Definition 2.1 [22]. If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then the q_a -definite integral on $[a, b]$ is defined as

$$(2-1) \quad \int_a^b f(x) {}_a d_q x = (1-q)(b-a) \sum_{n=0}^{\infty} q^n f(q^n b + (1-q^n)a) = (b-a) \int_0^1 f((1-t)a + tb) d_q t.$$

In [2], Alp et al. proved the following q_a -Hermite–Hadamard inequality for convex functions in the setting of quantum calculus.

Theorem 2.2. *If $0 < q < 1$ and $f : [a, b] \rightarrow \mathbb{R}$ is convex and differentiable, then*

$$f\left(\frac{qa+qb}{1+q}\right) \leq \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \leq \frac{qf(a) + f(b)}{1+q}.$$

On the other hand, Bermudo et al. gave the following new definition and related Hermite–Hadamard-type inequality.

Definition 2.3 [3]. If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then the q^b -definite integral on $[a, b]$ is defined as

$$(2-2) \quad \int_a^b f(x) {}^b d_q x = (1-q)(b-a) \sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)b) = (b-a) \int_0^1 f(ta + (1-t)b) d_q t.$$

Theorem 2.4 [3]. *If $0 < q < 1$ and $f : [a, b] \rightarrow \mathbb{R}$ is convex, then*

$$f\left(\frac{a+qb}{1+q}\right) \leq \frac{1}{b-a} \int_a^b f(x) {}^b d_q x \leq \frac{f(a) + qf(b)}{1+q}.$$

(p, q)-calculus and some inequalities. In this part, we review some fundamental notions and notations of (p, q)-calculus. For $0 < q < p \leq 1$, the so-called (p, q)-integer $[n]_{p,q}$ is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

Definition 2.5 [25, Formula (10)]. The definite $(p, q)_a$ -integral of $f : [a, b] \rightarrow \mathbb{R}$ is given by

$$(2-3) \quad \int_a^x f(t) {}_a d_{p,q} t = (p-q)(x-a) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}x + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right)$$

for $x \in [a, pb + (1-p)a]$ and $0 < q < p \leq 1$.

Definition 2.6 [25, Formula (11)]. The definite $(p, q)^b$ -integral of $f : [a, b] \rightarrow \mathbb{R}$ is given by

$$(2-4) \quad \int_x^b f(t) {}^b d_{p,q} t = (p-q)(b-x) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}x + \left(1 - \frac{q^n}{p^{n+1}}\right)b\right)$$

for $x \in [pa + (1-p)b, b]$ and $0 < q < p \leq 1$.

Remark 2.7. It is evident that if we pick $p = 1$ in (2-3) and (2-4), then (2-3) and (2-4) change into (2-1) and (2-2), respectively.

In [15], Kunt et al. proved the following Hermite–Hadamard-type inequalities for convex functions via $(p, q)_a$ integral.

Theorem 2.8. *If $f : [a, b] \rightarrow \mathbb{R}$ is convex and differentiable, then*

$$f\left(\frac{qa + pb}{[2]_{p,q}}\right) \leq \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x) {}_a d_{p,q}x \leq \frac{qf(a) + pf(b)}{[2]_{p,q}},$$

where $0 < q < p \leq 1$.

In [25], Vivas-Cortez et al. proved the following Hermite–Hadamard-type inequality for convex functions via $(p, q)^b$ -integral.

Theorem 2.9. *If $f : [a, b] \rightarrow \mathbb{R}$ is convex and differentiable, then*

$$f\left(\frac{pa + qb}{[2]_{p,q}}\right) \leq \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b f(x) {}^b d_{p,q}x \leq \frac{pf(a) + qf(b)}{[2]_{p,q}},$$

where $0 < q < p \leq 1$.

3. Main results

In this section, we prove Hermite–Jensen–Mercer-type inequalities for post-quantum integrals.

Theorem 3.1. *If $0 < q < p \leq 1$ and $f : [a, b] \rightarrow \mathbb{R}$ is convex, then*

$$\begin{aligned} (3-1) \quad f\left(a+b-\frac{x+y}{2}\right) &\leq f(a) + f(b) - \frac{1}{2p(y-x)} \left[\int_{px+(1-p)y}^y f(t) {}^y d_{p,q}t + \int_x^{py+(1-p)x} f(t) {}_x d_{p,q}t \right] \\ &\leq f(a) + f(b) - \frac{1}{2} \left[f\left(\frac{px+qy}{[2]_{p,q}}\right) + f\left(\frac{qx+py}{[2]_{p,q}}\right) \right] \\ &\leq f(a) + f(b) - f\left(\frac{x+y}{2}\right) \end{aligned}$$

for all $x, y \in [a, b]$ with $x < y$.

Proof. By the Jensen–Mercer inequality (1-1),

$$f\left(a+b-\frac{u+v}{2}\right) \leq f(a) + f(b) - \frac{1}{2}[f(u) + f(v)]$$

for $u, v \in [a, b]$ with $u = tx + (1-t)y$ and $v = ty + (1-t)x$ for $t \in [0, 1]$, we obtain

$$\begin{aligned} (3-2) \quad f\left(a+b-\frac{x+y}{2}\right) &= f\left(a+b-\frac{u+v}{2}\right) \\ &\leq f(a) + f(b) - \frac{1}{2}[f(tx + (1-t)y) + f(ty + (1-t)x)]. \end{aligned}$$

By taking $(p, q)_a$ -integral on $[0, p]$ of both sides of (3-2), we have

$$\begin{aligned} (3-3) \quad pf\left(a+b-\frac{x+y}{2}\right) &\leq pf(a) + pf(b) - \frac{1}{2} \left[\int_0^p f(tx + (1-t)y) {}_0 d_{p,q}t + \int_0^p f(ty + (1-t)x) {}_0 d_{p,q}t \right]. \end{aligned}$$

Let us consider $g(t) = f(ty + (1 - t)x)$. From [Definition 2.5](#) and [Theorem 2.8](#), we get

$$\begin{aligned}
 (3-4) \quad \int_0^p f(ty + (1 - t)x) {}_0d_{p,q}t &= \int_0^p g(t) {}_0d_{p,q}t \\
 &= (p - q)p \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} g\left(\frac{q^n}{p^n}\right) \\
 &= (p - q)p \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} g\left(\frac{q^n}{p^n}y + \left(1 - \frac{q^n}{p^n}\right)x\right) \\
 &= \frac{1}{y - x} \int_x^{py+(1-p)x} f(t) {}_x d_{p,q}t \\
 &\geq pf\left(\frac{qx + py}{[2]_{p,q}}\right).
 \end{aligned}$$

Considering $h(t) = f(tx + (1 - t)y)$ and by using [Definition 2.6](#) and [Theorem 2.9](#), we have

$$\begin{aligned}
 (3-5) \quad \int_0^p f(tx + (1 - t)y) {}_0d_{p,q}t &= \int_0^p h(t) {}_0d_{p,q}t \\
 &= (p - q)p \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} h\left(\frac{q^n}{p^n}\right) \\
 &= (p - q)p \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} h\left(\left(\frac{q^n}{p^n}\right)x + \left(1 - \frac{q^n}{p^n}\right)y\right) \\
 &= \frac{1}{y - x} \int_{px+(1-p)y}^y f(t) {}^y d_{p,q}t \\
 &\geq pf\left(\frac{px + qy}{[2]_{p,q}}\right).
 \end{aligned}$$

By substituting the inequalities (3-4) and (3-5) into (3-3), we get

$$\begin{aligned}
 &pf\left(a + b - \frac{x + y}{2}\right) \\
 &\leq pf(a) + pf(b) - \frac{1}{2} \left[\frac{1}{y - x} \int_{px+(1-p)y}^y f(t) {}^y d_{p,q}t + \frac{1}{y - x} \int_x^{py+(1-p)x} f(t) {}_x d_{p,q}t \right] \\
 &\leq pf(a) + pf(b) - \frac{p}{2} \left[f\left(\frac{px + qy}{[2]_{p,q}}\right) + f\left(\frac{qx + py}{[2]_{p,q}}\right) \right],
 \end{aligned}$$

which, after dividing by p , gives the proof the first and second inequalities in (3-1). Moreover, since f is convex, we have

$$f\left(\frac{x + y}{2}\right) = f\left(\frac{1}{2} \frac{px + qy}{[2]_{p,q}} + \frac{1}{2} \frac{qx + py}{[2]_{p,q}}\right) \leq \frac{1}{2} \left[f\left(\frac{px + qy}{[2]_{p,q}}\right) + f\left(\frac{qx + py}{[2]_{p,q}}\right) \right].$$

This proves the last inequality in (3-1). □

Remark 3.2. If we take $p = 1$ in [Theorem 3.1](#), then we have the quantum Hermite–Jensen–Mercer inequalities

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) &\leq f(a) + f(b) - \frac{1}{2}\left[\frac{1}{y-x}\int_x^y f(t)_y \mathbf{d}_{p,q}t + \frac{1}{y-x}\int_x^y f(t)_x \mathbf{d}_{p,q}t\right] \\ &\leq f(a) + f(b) - \frac{1}{2}\left[f\left(\frac{x+qy}{[2]_q}\right) + f\left(\frac{qx+y}{[2]_q}\right)\right] \\ &\leq f(a) + f(b) - f\left(\frac{x+y}{2}\right), \end{aligned}$$

which are given by Budak and Kara in [\[4, Theorem 4\]](#).

Corollary 3.3. If we choose $x = a$ and $y = b$ in [Theorem 3.1](#), then we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq f(a) + f(b) - \frac{1}{2(b-a)}\left[\int_{pa+(1-p)b}^b f(t)_b \mathbf{d}_{p,q}t + \int_a^{pb+(1-p)a} f(t)_x \mathbf{d}_{p,q}t\right] \\ &\leq f(a) + f(b) - \frac{1}{2}\left[f\left(\frac{pa+qb}{[2]_{p,q}}\right) + f\left(\frac{qa+pb}{[2]_{p,q}}\right)\right] \\ &\leq f(a) + f(b) - f\left(\frac{a+b}{2}\right). \end{aligned}$$

Theorem 3.4. If $0 < q < p \leq 1$ and $f : [a, b] \rightarrow \mathbb{R}$ is convex, then

$$\begin{aligned} (3-6) \quad f\left(a+b-\frac{px+qy}{[2]_{p,q}}\right) &\leq \frac{1}{p(y-x)}\int_{a+b-y}^{a+b-(px+(1-p)y)} f(t)_{a+b-y} \mathbf{d}_{p,q}t \\ &\leq f(a) + f(b) - \frac{pf(x) + qf(y)}{[2]_{p,q}} \end{aligned}$$

for all $x, y \in [a, b]$ with $x < y$.

Proof. From [Definition 2.5](#) and [Theorem 2.8](#), we get

$$\begin{aligned} \int_0^p f(a+b-(tx+(1-t)y))_0 \mathbf{d}_{p,q}t &= \int_0^p f(t(a+b-x)+(1-t)(a+b-y))_0 \mathbf{d}_{p,q}t \\ &= \frac{1}{p(y-x)}\int_{a+b-y}^{a+b-(px+(1-p)y)} f(t)_{a+b-y} \mathbf{d}_{p,q}t \\ &\geq f\left(\frac{q(a+b-y)+p(a+b-x)}{[2]_{p,q}}\right) \\ &= f\left(a+b-\frac{px+qy}{[2]_{p,q}}\right), \end{aligned}$$

which gives proof of the first inequality in (3-6). By the Jensen–Mercer inequality (1-1), we obtain

$$\begin{aligned} \frac{1}{p(y-x)} \int_{a+b-y}^{a+b-(px+(1-p)y)} f(t) {}_{a+b-y}d_{p,q}t &= \int_0^p f(a+b-(tx+(1-t)y)) {}_0d_{p,q}t \\ &\leq \int_0^p f(a) + f(b) - [tf(x) + (1-t)f(y)] {}_0d_{p,q}t \\ &= f(a) + f(b) - \left[\frac{pf(x) + qf(y)}{[2]_{p,q}} \right]. \end{aligned}$$

This completes the proof. □

Remark 3.5. If we take $p = 1$ in Theorem 3.4, then we have the quantum Hermite–Jensen–Mercer inequalities

$$f\left(a+b - \frac{x+qy}{[2]_q}\right) \leq \frac{1}{y-x} \int_{a+b-y}^{a+b-x} f(t) {}_{a+b-y}d_qt \leq f(a) + f(b) - \frac{f(x) + qf(y)}{[2]_q},$$

which are given by Budak and Kara in [4, Theorem 5].

Remark 3.6. If we choose $x = a$ and $y = b$ in Theorem 3.4, then Theorem 3.4 reduces to Theorem 2.8.

Theorem 3.7. If $0 < q < p \leq 1$ and $f : [a, b] \rightarrow \mathbb{R}$ is convex, then

$$\begin{aligned} (3-7) \quad f\left(a+b - \frac{qx+py}{[2]_{p,q}}\right) &\leq \frac{1}{p(y-x)} \int_{a+b-(py+(1-p)x)}^{a+b-x} f(t) {}^{a+b-x}d_{p,q}t \\ &\leq f(a) + f(b) - \frac{qf(x) + pf(y)}{[2]_{p,q}} \end{aligned}$$

for all $x, y \in [a, b]$ with $x < y$.

Proof. From Definition 2.6 and Theorem 2.9, we get

$$\begin{aligned} \int_0^p f(a+b-(ty+(1-t)x)) {}_0d_{p,q}t &= \int_0^p f(t(a+b-y) + (1-t)(a+b-x)) {}_0d_{p,q}t \\ &= \frac{1}{p(y-x)} \int_{a+b-(py+(1-p)x)}^{a+b-x} f(t) {}^{a+b-x}d_{p,q}t \\ &\geq f\left(\frac{q(a+b-x) + p(a+b-y)}{[2]_{p,q}}\right) \\ &= f\left(a+b - \frac{py+qx}{[2]_{p,q}}\right), \end{aligned}$$

which proves the first inequality in (3-7). By the Jensen–Mercer inequality, we have

$$\begin{aligned} \int_0^p f(a+b-(ty+(1-t)x)) {}_0d_{p,q}t &\leq \int_0^p f(a) + f(b) - [tf(y) + (1-t)f(x)] {}_0d_{p,q}t \\ &= f(a) + f(b) - \left[\frac{pf(y) + qf(x)}{[2]_{p,q}} \right], \end{aligned}$$

which completes the proof. □

Remark 3.8. If we take $p = 1$ in [Theorem 3.4](#), then we have the quantum Hermite–Jensen–Mercer inequalities

$$f\left(a + b - \frac{qx + y}{[2]_q}\right) \leq \frac{1}{y - x} \int_{a+b-y}^{a+b-x} f(t)^{a+b-x} d_q t \leq f(a) + f(b) - \frac{qf(x) + f(y)}{[2]_q},$$

which are given by Budak and Kara in [\[4, Theorem 7\]](#).

Remark 3.9. If we choose $x = a$ and $y = b$ in [Theorem 3.7](#), then [Theorem 3.7](#) reduces to [Theorem 2.9](#).

4. Examples

In this section, we provide some examples of our main theorems.

Example 4.1. We define a convex function $f : [a, b] = [-2, 2] \rightarrow \mathbb{R}$ by $f(t) = t^2$. For $x = 0$, $y = 1$, $q = \frac{1}{2}$ and $p = \frac{3}{4}$, we have

$$f\left(a + b - \frac{x + y}{2}\right) = f\left(-\frac{1}{2}\right) = \frac{1}{4},$$

$$f(a) + f(b) = 8,$$

$$\int_{px+(1-p)y}^y f(t)^y d_{p,q} t = \int_{\frac{1}{4}}^1 t^{2 \cdot 1} d_{\frac{3}{4}, \frac{1}{2}} t = \frac{78}{380},$$

and

$$\int_x^{py+(1-p)x} f(t)_x d_{p,q} t = \int_0^{\frac{3}{4}} t^2 {}_0 d_{p,q} t = \frac{27}{76}.$$

Thus,

$$f(a) + f(b) - \frac{1}{2p(y-x)} \left[\int_{px+(1-p)y}^y f(t)^y d_q t + \int_x^{py+(1-p)x} f(t)_x d_q t \right] = 8 - \frac{2}{3} \left[\frac{78}{380} + \frac{27}{76} \right] = \frac{1449}{190}.$$

Now, we observe

$$f\left(\frac{px + qy}{[2]_{p,q}}\right) = f\left(\frac{2}{5}\right) = \frac{4}{25}, \quad f\left(\frac{qx + py}{[2]_{p,q}}\right) = f\left(\frac{3}{5}\right) = \frac{9}{25}$$

so that

$$f(a) + f(b) - \frac{1}{2} \left[f\left(\frac{px + qy}{[2]_{p,q}}\right) + f\left(\frac{qx + py}{[2]_{p,q}}\right) \right] = \frac{387}{50}.$$

Again, we observe

$$f(a) + f(b) - f\left(\frac{x + y}{2}\right) = \frac{31}{4}.$$

Finally, we conclude from the above that [\(3-1\)](#) is implied by

$$\frac{1}{4} < \frac{1449}{190} < \frac{387}{50} < \frac{31}{4}.$$

Example 4.2. For the same data as in [Example 4.1](#), we get

$$f\left(a + b - \frac{px + qy}{[2]_{p,q}}\right) = f\left(-\frac{2}{5}\right) = \frac{4}{25},$$

$$\frac{1}{p(y-x)} \int_{a+b-y}^{a+b-(px+(1-p)y)} f(t) {}_{a+b-y}d_{p,q}t = \frac{4}{3} \int_{-1}^{-\frac{1}{4}} t^2 {}_{-1}d_{p,q}t = \frac{4}{3} \cdot \frac{39}{190} = \frac{26}{95},$$

and

$$f(a) + f(b) - \frac{pf(x) + qf(y)}{[2]_{p,q}} = \frac{38}{5}.$$

From the above, (3-6) is implied by

$$\frac{4}{25} < \frac{26}{95} < \frac{38}{5}.$$

Example 4.3. For the same data as in [Example 4.1](#), we have

$$f\left(a + b - \frac{qx + py}{[2]_q}\right) = f\left(-\frac{3}{5}\right) = \frac{9}{25},$$

$$\frac{1}{p(y-x)} \int_{a+b-(py+(1-p)x)}^{a+b-x} f(t) {}^{a+b-x}d_{p,q}t = \frac{4}{3} \int_{-\frac{3}{4}}^0 t^2 {}_0d_{\frac{3}{4}, \frac{1}{2}}t = \frac{4}{3} \cdot \frac{27}{19} = \frac{9}{19},$$

and

$$f(a) + f(b) - \frac{qf(x) + pf(y)}{[2]_{p,q}} = \frac{37}{5}.$$

From the above, (3-7) is implied by

$$\frac{9}{25} < \frac{9}{19} < \frac{37}{5}.$$

5. Conclusion

In this paper, we generalize the Hermite–Jensen–Mercer inequality to the case of post-quantum integrals. This is the first study obtaining Hermite–Jensen–Mercer inequalities for post-quantum integrals. In future studies, one can try to generalize our results by utilizing different kinds of convex function classes.

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