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UNIQUENESS FOR AN INVERSE QUANTUM-DIRAC PROBLEM WITH GIVEN WEYL FUNCTION

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ABSTRACT. In this work, we consider a boundary value problem for a q-Dirac equation. We prove orthogonality of the eigenfunctions, realness of the eigenvalues, and we study asymptotic formulas of the eigenfunctions. We show that the eigenfunctions form a complete system, we obtain the expansion formula with respect to the eigenfunctions, and we derive Parseval's equality. We construct the Weyl solution and the Weyl function. We prove a uniqueness theorem for the solution of the inverse problem with respect to the Weyl function.

1. Introduction

Quantum calculus is equivalent to traditional infinitesimal calculus without the notion of limits. The q-derivative notion was introduced by Jackson [27] in 1910. The book by Kac and Cheung [28] covers many of the aspects of quantum calculus and also q-special functions. Quantum calculus has a lot of applications in calculus of variations, orthogonal polynomials, theory of relativity, quantum theory, and statistical physics (see [3,32,39] and the references therein). Inspired by the formal work of Exton [20], Annaby and Mansour [15] provided a detailed study of q-calculus. Their results are extended to different versions of boundary value problems for both q-Sturm–Liouville [2, 6, 10–12, 14, 17, 19, 29, 30, 34, 37] and q-Dirac operators [4, 5, 7–9, 16, 36]. In this paper, we study the boundary

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value problem

$$\begin{cases} D_q y_2(t) + p(t)y_1(t) + r(t)y_2(t) = \lambda y_1(t), \\ -D_q y_1^\sigma(t) + r(t)y_1(t) - p(t)y_2(t) = \lambda y_2(t), \end{cases} \quad t \in [0, a], \tag{1}$$

$$U_1(y) := y_1(0) = 0, \tag{2}$$

$$U_2(y) := y_1(a) = 0, \tag{3}$$

where $p, r \in \mathcal{L}_q^1(0, a)$ are real-valued functions defined on $[0, a]$ and continuous at zero, $q \in (0, 1)$ is fixed, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $y^\sigma(t) := y(q^{-1}t)$, and λ is a spectral parameter.

The structure of the paper is as follows. In Section 2, we introduce notations, definitions, and preliminary facts which are used throughout the paper. Section 3 and Section 4 are devoted to the so-called direct problem of spectral analysis. In Section 3, we establish spectral properties of the boundary value problem (1)–(3). In particular, we prove orthogonality of the eigenfunctions, realness of the eigenvalues, and we study asymptotic formulas of the eigenfunctions. In Section 4, we show that the eigenfunctions form a complete system, we obtain the expansion formula with respect to the eigenfunctions, and we derive Parseval's equality. Section 5 is devoted to constructing the inverse problem for the boundary value problem (1)–(3). We construct the Weyl solution and the Weyl function. We prove a uniqueness theorem for the solution of the inverse problem with respect to the Weyl function.

Although great success in inverse spectral theory has been achieved for boundary value problems involving the classical Sturm–Liouville and Dirac differential equations, to the best of our knowledge, no work has been reported on inverse spectral theory of boundary value problems generated by q-Sturm–Liouville and q-Dirac differential equations. The motivation of this paper is to fill this gap and to initiate further research in inverse spectral theory of q-Sturm–Liouville and q-Dirac operators.

2. Preliminaries

In this section, we introduce some of the q-notations that will be used throughout the paper. We use the standard notations found in [13–15].

The set of nonnegative integers is denoted by $\mathbb{N}_0$, and the set of positive integers is denoted by $\mathbb{N}$. For $t > 0$,

$$A_{q,t} := \{tq^n : n \in \mathbb{N}_0\}, \qquad A_{q,t}^* := A_{q,t} \cup \{0\}.$$

When $t = 1$, we simply use A_q and A_q^* to denote $A_{q,1}$ and $A_{q,1}^*$, respectively.

A set $S \subseteq \mathbb{R}$ is called a q-geometric set if, for every $t \in S$, we have $qt \in S$. Let y be a real or complex valued function defined on a q-geometric set S. The q-difference operator is defined by

$$D_q y(t) := \frac{y(t) - y(qt)}{t(1-q)}, \qquad t \neq 0.$$

If $0 \in S$, then the q-derivative of a function y at zero is defined as

$$D_q y(0) := \lim_{n\to\infty} \frac{y(tq^n) - y(0)}{tq^n}, \quad t \in S,$$

if the limit exists and does not depend on t.

Let us define the function $y^\rho(t) := y(qt)$. Note that $(y^\sigma)^\rho = (y^\rho)^\sigma = y$. The following lemma can be proven similarly to [28, (1.12)].

Lemma 2.1. *The q-product rule holds:*

$$D_q(yz)(t) = D_q y(t) z(t) + y(t) D_q z^\sigma(t). \tag{4}$$

Lemma 2.2. *The equation is always valid:*

$$D_q\left(y_2 z_1 - z_2 y_1\right) = D_q y_2 z_1 + y_2 D_q z_1^\sigma - D_q z_2 y_1 - D_q y_1^\sigma z_2. \tag{5}$$

As a right inverse of the q-difference operator, q-integration is defined by Jackson [27] as

$$\int_0^t y(s)\,\mathrm{d}_q s := t(1-q)\sum_{n=0}^{\infty} q^n y(tq^n), \qquad t \in S,$$

if the series converges. In general, we have

$$\int_a^b y(s)\,\mathrm{d}_q s := \int_0^b y(s)\,\mathrm{d}_q s - \int_0^a y(s)\,\mathrm{d}_q s, \quad a, b \in S.$$

There is no unique canonical choice for q-integration over $[0,\infty)$. Hahn [23] defined the q-integration for a function y over $[0,\infty)$ by

$$\int_0^\infty y(s)\,\mathrm{d}_q s = (1-q)\sum_{n=-\infty}^{\infty} q^n y(q^n),$$

while Matsuo [35, (2.2)] defined q-integration on the interval $[0,\infty)$ by

$$\int_0^{b\infty} y(s)\,\mathrm{d}_q s := b(1-q)\sum_{n=-\infty}^{\infty} q^n y(bq^n), \quad b > 0,$$

provided that the series converges. Consequently, q-integration of a function y defined on $\mathbb{R}$ can be defined as

$$\int_{-\infty/b}^{\infty/b} y(s)\,\mathrm{d}_q s = \frac{1-q}{b}\sum_{n=-\infty}^{\infty} q^n \big(y(q^n/b)+y(-q^n/b)\big), \quad b>0,$$

provided that the series converges absolutely.

Definition 2.3. Let y be a function defined on a q-geometric set S. We say that y is q-integrable on S if and only if $\int_0^t y(s)\,\mathrm{d}_q s$ exists for all $t \in S$.

Let S^* be a q-geometric set containing zero. A function y defined on S^* is called q-regular at zero if

$$\lim_{n\to\infty} y(tq^n) = y(0)$$

holds for all $t \in S^*$.

Let $\mathrm{C}(S^*)$ denote the space of all functions that are q-regular at zero and defined on S^* with values in $\mathbb{R}$. $\mathrm{C}(S^*)$, associated with the norm

$$\|y\| = \sup\left\{|y(tq^n)| : \ t\in S^*,\ n\in\mathbb{N}_0\right\},$$

is a normed space. The q-integration by parts rule [13] reads

$$\int_a^b z(t)D_q y(t)\,\mathrm{d}_q t = \left.(yz)\right|_a^b + \int_a^b y(qt)D_q z(t)\,\mathrm{d}_q t, \quad a,b\in S^*$$

for $y,z \in \mathrm{C}(S^*)$. Let y be the function that is q-regular at zero and defined on the q-geometric set S^*. Define

$$Y(z) := \int_c^z y(t)\,\mathrm{d}_q t, \quad z\in S^*,$$

where c is a fixed point in S^*. Then Y is q-regular at zero. Furthermore, $D_qY(z)$ exists for every $z \in S^*$, and $D_qY(z) = y(z)$ for every $z \in S^*$. Conversely, if a and b are two points in S^*, then

$$\int_a^b D_q y(t)\,\mathrm{d}_q t = y(b)-y(a).$$

For $p > 0$ and X equal to $A_{q,t}$ or $A^*_{q,t}$, the space $L^q_p(X)$ is the normed space of all functions defined on X such that

$$\|y\|_p := \left(\int_0^x |y(t)|^p\,\mathrm{d}_q t\right)^{1/p} < \infty.$$

If $p = 2$, then $L_q^2(X)$ associated with the inner product

$$\langle y, z\rangle := \int_0^x y(t)\overline{z(t)}\,\mathrm{d}_q t$$

is a Hilbert space. The space of all q-absolutely functions on $A_{q,t}^*$ is denoted by $\mathrm{AC}_q(A_{q,t}^*)$ and defined as the space of all functions y that are q-regular at zero and satisfy

$$\sum_{j=0}^{\infty} \left|y(tq^j) - y(tq^{j+1})\right| \le K$$

for all $t \in A_{q,t}^*$, and K is a constant depending on the function y (c.f., [15, Definiton 4.3.1, p.118]), i.e., $\mathrm{AC}_q(A_{q,t}^*) \subseteq \mathrm{C}_q(A_{q,t}^*)$. The space $\mathrm{AC}_q^{(n)}(A_{q,t}^*)$, $n \in \mathbb{N}$, is the space of all functions y defined on S^* such that $y, D_q y, \ldots, D_q^{n-1} y$ are q-regular at zero and $D_q^{n-1} y \in \mathrm{AC}_q(A_{q,t}^*)$ (c.f., [15, Definition 4.3.2, p. 119]). For $n \in \mathbb{N}_0$ and $\alpha, a_1, \ldots, a_n \in \mathbb{C}$, the q-shifted factorial, the multiple q-shifted factorial, and the q-binomial coefficients are defined to be

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{k=0}^{n-1}(1 - aq^k), \qquad (a;q)_\infty := \prod_{k=0}^{\infty}(1 - aq^k),$$

$$(a_1, a_2, \ldots, a_k; q)_n := \prod_{j=1}^{k}(a_j; q)_n,$$

and

$$\begin{bmatrix}\alpha\\0\end{bmatrix}_q := 1, \quad \begin{bmatrix}\alpha\\n\end{bmatrix}_q := \frac{(1-q^\alpha)(1-q^{\alpha-1})\cdots(1-q^{\alpha-n+1})}{(q;q)_n},$$

respectively [4, 14, 38]. The generalized q-shifted factorial is defined by

$$(a;q)_\nu = \frac{(a;q)_\infty}{(aq^\nu;q)_\infty}, \quad \nu \in \mathbb{R}$$

(see [4, 14, 38]). The q-Gamma function is defined by

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \quad z \in \mathbb{C}, \quad |q| < 1$$

(see [4, 22, 26]). The third type of the q-Bessel functions of Jackson of order $\nu > -1$ is defined to be

$$J_v(z;q) = z^\nu \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2}}{(q;q)_n (q^{\nu+1};q)_n} z^{2n}, \quad z \in \mathbb{C}$$

(see [24, 25]). This function is entire in z of order zero with a countable set of real and simple zeros only, cf. [31]. The basic trigonometric functions $\cos(z;q)$ and

$\sin(z;q)$ are defined on $\mathbb{C}$ by

$$\begin{aligned}\cos(z;q) &:= \sum_{n=0}^{\infty}(-1)^n \frac{q^{n^2}(z(1-q))^{2n}}{(q;q)_{2n}} \\ &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\big(zq^{-1/2}(1-q)\big)^{1/2} J_{-1/2}\big(z(1-q)/\sqrt{q};q^2\big), \\ \sin(z;q) &:= \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)}(z(1-q))^{2n+1}}{(q;q)_{2n+1}} \\ &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\big(z(1-q)\big)^{1/2} J_{1/2}\big(z(1-q);q^2\big)\end{aligned}$$

(see [1,4]).

3. Spectral properties of the problem

In the Hilbert space $\mathcal{L}_{2,q}(0,a,\mathbb{C}^2)$, let an inner product be defined by

$$\langle y,z\rangle := \int_0^a \big(y_1(t)\overline{z_1(t)} + y_2(t)\overline{z_2(t)}\big)\,\mathrm{d}_q t,$$

where

$$y = (y_1,y_2)^T \in \mathcal{L}_{2,q}(0,a,\mathbb{C}^2), \qquad z = (z_1,z_2)^T \in \mathcal{L}_{2,q}(0,a,\mathbb{C}^2).$$

Assume that the boundary value problem under consideration has a nontrivial solution

$$y(t,\lambda_0) = \begin{pmatrix} y_1(t,\lambda_0) \\ y_2(t,\lambda_0) \end{pmatrix}$$

for certain λ_0, called an eigenvalue, and the corresponding solution $y(t,\lambda_0)$ is called a vector-valued eigenfunction.

Lemma 3.1. *The vector-valued eigenfunctions $y(t,\lambda_1)$ and $z(t,\lambda_2)$ corresponding to different eigenvalues $\lambda_1 \neq \lambda_2$ are orthogonal.*

Proof. Since $y(t,\lambda_1)$ and $z(t,\lambda_2)$ are solutions of the system (1), we have

$$\begin{aligned} D_q y_2(t,\lambda_1) + p(t)y_1(t,\lambda_1) + r(t)y_2(t,\lambda_1) &= \lambda_1 y_1(t,\lambda_1), \\ -D_q y_1^\sigma(t,\lambda_1) + r(t)y_1(t,\lambda_1) - p(t)y_2(t,\lambda_1) &= \lambda_1 y_2(t,\lambda_1), \\ D_q z_2(t,\lambda_2) + p(t)z_1(t,\lambda_2) + r(t)z_2(t,\lambda_2) &= \lambda_2 z_1(t,\lambda_2), \\ -D_q z_1^\sigma(t,\lambda_2) + r(t)z_1(t,\lambda_2) - p(t)z_2(t,\lambda_2) &= \lambda_2 z_2(t,\lambda_2). \end{aligned}$$

Multiplying these equalities by $z_1(t,\lambda_2)$, $z_2(t,\lambda_2)$, $-y_1(t,\lambda_1)$, $-y_2(t,\lambda_1)$, respectively, and adding the resulting equations together, we obtain

$$\begin{aligned}
&D_q y_2(t,\lambda_1) z_1(t,\lambda_2) + y_2(t,\lambda_1) D_q z_1^\sigma(t,\lambda_2) \\
&\quad - D_q z_2(t,\lambda_2) y_1(t,\lambda_1) - D_q y_1^\sigma(t,\lambda_1) z_2(t,\lambda_2) \\
&= (\lambda_1-\lambda_2)\big(y_1(t,\lambda_1)z_1(t,\lambda_2) + y_2(t,\lambda_1)z_2(t,\lambda_2)\big) \\
&\overset{(5)}{=} D_q\big(y_2(t,\lambda_1)z_1(t,\lambda_2) - z_2(t,\lambda_2)y_1(t,\lambda_1)\big) \\
&= (\lambda_1-\lambda_2)\big(y_1(t,\lambda_1)z_1(t,\lambda_2) + y_2(t,\lambda_1)z_2(t,\lambda_2)\big).
\end{aligned}$$

Now, integrating from 0 to a, we find

$$(\lambda_1-\lambda_2)\langle y,z\rangle = \big(y_2(t,\lambda_1)z_1(t,\lambda_2) - z_2(t,\lambda_2)y_1(t,\lambda_1)\big)\Big|_0^a,$$

where the right-hand side vanishes because of the boundary conditions (2), (3). □

Corollary 3.1. The eigenvalues of the boundary value problem (1)–(3) are real.

Let $\varphi(\cdot,\lambda) = \begin{pmatrix} \varphi_1(\cdot,\lambda) \\ \varphi_2(\cdot,\lambda) \end{pmatrix}$ and $\psi(\cdot,\lambda) = \begin{pmatrix} \psi_1(\cdot,\lambda) \\ \psi_2(\cdot,\lambda) \end{pmatrix}$ be the solutions of (1) satisfying the initial conditions

$$\varphi_1(0,\lambda)=0, \qquad \varphi_2(0,\lambda)=1, \qquad \psi_1(a,\lambda)=0, \qquad \psi_2(a,\lambda)=1. \tag{6}$$

Clearly,

$$U_1(\varphi) = U_2(\psi) = 0. \tag{7}$$

Denote

$$\Delta(\lambda) = \varphi_2(\cdot,\lambda)\psi_1(\cdot,\lambda) - \varphi_1(\cdot,\lambda)\psi_2(\cdot,\lambda). \tag{8}$$

The function $\Delta(\lambda)$ is called the characteristic function of the boundary value problem (1)–(3). It can be easily seen that the characteristic function does not depend on t. Indeed, taking ψ for z and φ for y in (5) yields

$$\begin{aligned}
D_q\Delta(\lambda) &= D_q\varphi_2(\cdot,\lambda)\psi_1(\cdot,\lambda) + \varphi_2(\cdot,\lambda)D_q\psi_1^\sigma(\cdot,\lambda) \\
&\quad - \varphi_1(\cdot,\lambda)D_q\psi_2(\cdot,\lambda) - D_q\varphi_1^\sigma(\cdot,\lambda)\psi_2(\cdot,\lambda) \\
&= \left[\lambda\varphi_1(\cdot,\lambda) - p(\cdot)\varphi_1(\cdot,\lambda) - r(\cdot)\varphi_2(\cdot,\lambda)\right]\psi_1(\cdot,\lambda) \\
&\quad + \varphi_2(\cdot,\lambda)\left[-\lambda\psi_2(\cdot,\lambda) + r(\cdot)\psi_1(\cdot,\lambda) - p(\cdot)\psi_2(\cdot,\lambda)\right] \\
&\quad - \varphi_1(\cdot,\lambda)\left[\lambda\psi_1(\cdot,\lambda) - p(\cdot)\psi_1(\cdot,\lambda) - r(\cdot)\psi_2(\cdot,\lambda)\right] \\
&\quad - \psi_2(\cdot,\lambda)\left[-\lambda\varphi_2(\cdot,\lambda) + r(\cdot)\varphi_1(\cdot,\lambda) - p(\cdot)\varphi_2(\cdot,\lambda)\right] = 0.
\end{aligned}$$

Substituting $t=0$ and $t=a$ into (8), we get

$$\Delta(\lambda) = -U_1(\psi) = U_2(\varphi). \tag{9}$$

The function $\Delta(\lambda)$ is entire in λ, and it has an at most countable set of zeros $\{\lambda_n\}$.

Theorem 3.2 (See [16, Theorem 7]). *The eigenvalues of the boundary value problem* (1)–(3) *coincide with the simple zeros of* $\Delta(\lambda)$. *The functions* $\varphi(\cdot,\lambda_n)$ *and* $\psi(\cdot,\lambda_n)$ *are eigenfunctions, and there exists a sequence* $\{k_n\}$ *such that*

$$\psi(t,\lambda_n)=k_n\varphi(t,\lambda_n),\quad k_n\neq 0. \tag{10}$$

In what follows, we study asymptotic formulas for the eigenfunctions of the boundary value problem (1)–(3). For this task, we first construct the integral representations of the solutions of the above-mentioned boundary value problem. After providing some necessary results for the next section, we finally give the asymptotic formula for the characteristic function. The functions

$$\theta_1(t,\lambda)=\begin{pmatrix}\cos(\lambda t;q)\\ \frac{\sin(\lambda t;q)}{\lambda}\end{pmatrix}\qquad\text{and}\qquad\theta_2(t,\lambda)=\begin{pmatrix}-\frac{\sin(\lambda t;q)}{\lambda}\\ \cos(\lambda t;q)\end{pmatrix}$$

form a fundamental set of (1) when $p(t)=r(t)\equiv 0$, with the q-Wronskian

$$W_q\big(\theta_1(\cdot,\lambda),\theta_2(\cdot,\lambda)\big)\equiv 1$$

(see [15, Proof of Theorem 3.2, p. 76]). Applying the method of variation of parameters, which was introduced in [1, Example 4.1], the function $\varphi(\cdot,\lambda)$ is given by

$$\begin{aligned}\varphi(t,\lambda)=&c_1\theta_1(t,\lambda)+c_2\theta_2(t,\lambda)\\ &+q\int_0^t\left[\theta_2(t,\lambda)\theta_1^T(qs,\lambda)-\theta_1(t,\lambda)\theta_2^T(qs,\lambda)\right]V(qt)\varphi(qt,\lambda)\,\mathrm{d}_qt,\end{aligned}$$

where $V(t)=\begin{pmatrix}p(t)&r(t)\\ r(t)&-p(t)\end{pmatrix}$, $t\in[0,a]$, $\lambda\in\mathbb{C}$, and T denotes the matrix transpose. From here, we have

$$\begin{aligned}\varphi_1(t,\lambda)&=c_1\theta_{11}(t,\lambda)+c_2\theta_{21}(t,\lambda)\\ &\quad+q\int_0^t\big(a_{11}(t,s,\lambda)\varphi_1(qt,\lambda)+a_{12}(t,s,\lambda)\varphi_2(qt,\lambda)\big)\,\mathrm{d}_qt,\end{aligned}\tag{11}$$

$$\begin{aligned}\varphi_2(t,\lambda)&=c_1\theta_{12}(t,\lambda)+c_2\theta_{22}(t,\lambda)\\ &\quad+q\int_0^t\big(a_{21}(t,s,\lambda)\varphi_1(qt,\lambda)+a_{22}(t,s,\lambda)\varphi_2(qt,\lambda)\big)\,\mathrm{d}_qt,\end{aligned}\tag{12}$$

where

$$\begin{aligned}a_{11}(t,s,\lambda)=&\left(-\frac{\sin(\lambda t;q)}{\lambda}\cos(\lambda qs;q)+\cos(\lambda t;q)\frac{\sin(\lambda qs;q)}{\lambda}\right)p(qt)\\ &-\left(\frac{\sin(\lambda t;q)}{\lambda}\frac{\sin(\lambda qs;q)}{\lambda}+\cos(\lambda t;q)\cos(\lambda qs;q)\right)r(qt),\end{aligned}$$

$$
\begin{aligned}
a_{12}(t,s,\lambda) &= \left(-\frac{\sin(\lambda t;q)}{\lambda}\cos(\lambda qs;q) + \cos(\lambda t;q)\frac{\sin(\lambda qs;q)}{\lambda}\right) r(qt) \\
&\quad + \left(\frac{\sin(\lambda t;q)}{\lambda}\frac{\sin(\lambda qs;q)}{\lambda} + \cos(\lambda t;q)\cos(\lambda qs;q)\right) p(qt), \\
a_{21}(t,s,\lambda) &= \left(\cos(\lambda t;q)\cos(\lambda qs;q) - \frac{\sin(\lambda t;q)}{\lambda}\frac{\sin(\lambda qs;q)}{\lambda}\right) p(qt) \\
&\quad + \left(\cos(\lambda t;q)\frac{\sin(\lambda qs;q)}{\lambda} - \frac{\sin(\lambda t;q)}{\lambda}\cos(\lambda qs;q)\right) r(qt), \\
a_{22}(t,s,\lambda) &= \left(\cos(\lambda t;q)\cos(\lambda qs;q) - \frac{\sin(\lambda t;q)}{\lambda}\frac{\sin(\lambda qs;q)}{\lambda}\right) r(qt) \\
&\quad - \left(\cos(\lambda t;q)\frac{\sin(\lambda qs;q)}{\lambda} - \frac{\sin(\lambda t;q)}{\lambda}\cos(\lambda qs;q)\right) p(qt).
\end{aligned}
$$

Substituting (11) and (12) in (6), we get

$$
\begin{aligned}
\varphi_1(t,\lambda) &= -\frac{\sin(\lambda t;q)}{\lambda} \\
&\quad + q\int_0^t \big(a_{11}(t,s,\lambda)\varphi_1(qt,\lambda) + a_{12}(t,s,\lambda)\varphi_2(qt,\lambda)\big)\,\mathrm{d}_q t, \qquad (13) \\
\varphi_2(t,\lambda) &= \cos(\lambda t;q) \\
&\quad + q\int_0^t \big(a_{21}(t,s,\lambda)\varphi_1(qt,\lambda) + a_{22}(t,s,\lambda)\varphi_2(qt,\lambda)\big)\,\mathrm{d}_q t. \qquad (14)
\end{aligned}
$$

We are now ready to proceed giving asymptotic formulas for the eigenfunctions.

The following theorems can be proven in an exactly similar way to the proofs of [14, Theorem 5.3, Theorem 5.4], respectively.

Theorem 3.3 (See [14, Theorem 5.3])**.** *Let $\lambda \in \mathbb{C}$. Then for each $t \in [0,a]$, we have the asymptotic formulas*

$$
\varphi_1(t,\lambda) = \cos(\lambda t;q) + O\left(\frac{E(|\lambda|t;q)}{|\lambda|}\right), \qquad (15)
$$

$$
\varphi_2(t,\lambda) = \frac{\sin(\lambda t;q)}{\lambda} + O\left(\frac{E(|\lambda|t;q)}{|\lambda|^2}\right) \qquad (16)
$$

as $|\lambda| \to \infty$, where for each $t \in (0,a]$, the O-terms are uniform on $\{tq^n : n \in \mathbb{N}\}$. Moreover, if $p(\cdot)$ and $r(\cdot)$ are bounded on $[0,a]$, then the O-terms (15), (16) *are uniform throughout $[0,a]$.*

Theorem 3.4 (See [14, Theorem 5.4]). *As* $|\lambda| \to \infty$, *we have*

$$\varphi_1(t,\lambda) = \cos(\lambda t; q) + O\left(|\lambda|^{-1} \exp\left(\frac{-(\log|\lambda| t(1-q))^2}{\log q}\right)\right), \tag{17}$$

$$\varphi_2(t,\lambda) = \frac{\sin(\lambda t; q)}{\lambda} + O\left(|\lambda|^{-2} \exp\left(\frac{-(\log|\lambda| t(1-q))^2}{\log q}\right)\right), \tag{18}$$

where for each $t \in (0, a]$, *the* O*-terms are uniform on* $\{tq^n : n \in \mathbb{N}\}$. *Moreover, if* $p(\cdot)$ *and* $r(\cdot)$ *are bounded on* $[0, a]$, *then the* O*-terms* (17), (18) *are uniform throughout* $[0, a]$.

The solution $\varphi(t,\lambda)$ given by (13), (14) is a nontrivial solution of (1) satisfying the boundary condition (2) for any λ. We therefore find the eigenvalues by substituting $\varphi(t,\lambda)$ in the second condition (3). Thus, we have the equation

$$\Delta(\lambda) = \varphi_1(a,\lambda). \tag{19}$$

We use the following notation for the weight numbers $\{\alpha_n\}$ of the boundary value problem (1)–(3)

$$\alpha_n^2 := \int_0^a \left[\varphi_1^2(t,\lambda_n) + \varphi_2^2(t,\lambda_n)\right] \mathrm{d}_q t. \tag{20}$$

Lemma 3.5. *The following relation holds:*

$$k_n \alpha_n = \dot{\Delta}(\lambda_n), \tag{21}$$

where the numbers k_n *are defined by* (10), *and* $\dot{\Delta}(\lambda) = \frac{d}{d\lambda}\Delta(\lambda)$.

Proof. Since $\varphi(t,\lambda_n)$ and $\varphi(t,\lambda)$ are solutions of the boundary value problem (1)–(3), we have:

$$D_q\varphi_2(t,\lambda_n) + p(t)\varphi_1(t,\lambda_n) + r(t)\varphi_2(t,\lambda_n) = \lambda_n\varphi_1(t,\lambda_n),$$

$$-D_q\varphi_1^\sigma(t,\lambda_n) + r(t)\varphi_1(t,\lambda_n) - p(t)\varphi_2(t,\lambda_n) = \lambda_n\varphi_2(t,\lambda_n),$$

$$D_q\psi_2(t,\lambda) + p(t)\psi_1(t,\lambda) + r(t)\psi_2(t,\lambda) = \lambda\psi_1(t,\lambda),$$

$$-D_q\psi_1^\sigma(t,\lambda) + r(t)\psi_1(t,\lambda) - p(t)\psi_2(t,\lambda) = \lambda\psi_2(t,\lambda).$$

Multiplying these equations by $\psi_1(t,\lambda)$, $\psi_2(t,\lambda)$, $-\varphi_1(t,\lambda_n)$, $-\varphi_2(t,\lambda_n)$, respectively, adding the resulting equations together, and taking (5) into account, we obtain

$$D_q\big(\varphi_2(t,\lambda_n)\psi_1(t,\lambda) - \psi_2(t,\lambda)\varphi_1(t,\lambda_n)\big) = (\lambda_n - \lambda)\left[\varphi_1(t,\lambda_n)\psi_1(t,\lambda) + \varphi_2(t,\lambda_n)\psi_2(t,\lambda)\right]\Big|_0^a.$$

Integrating the last equation from 0 to a, using boundary conditions (2), (3), making necessary calculations, and using (9), we get

$$\begin{aligned}
&(\lambda_n - \lambda) \int_0^a \big(\varphi_1(t,\lambda_n)\psi_1(t,\lambda) + \varphi_2(t,\lambda_n)\psi_2(t,\lambda)\big)\,\mathrm{d}_q t \\
&= [\varphi_2(t,\lambda_n)\psi_1(t,\lambda) - \psi_2(t,\lambda)\varphi_1(t,\lambda_n)]|_0^a \\
&= -\varphi_1(a,\lambda_n) - \psi_1(0,\lambda) \overset{(9)}{=} \Delta(\lambda_n) - \Delta(\lambda).
\end{aligned}$$

For $\lambda \to \lambda_n$, this yields

$$\int_0^a [\varphi_1(t,\lambda_n)\psi_1(t,\lambda_n) + \varphi_2(t,\lambda_n)\psi_2(t,\lambda_n)]\,\mathrm{d}_q t = \dot{\Delta}(\lambda_n).$$

Using (10) and (20), we arrive at (21). □

Corollary 3.2. The eigenvalues of boundary value problem (1)–(3) are simple, i.e., $\dot{\Delta}(\lambda) \neq 0$.

Theorem 3.6. *As $|\lambda| \to \infty$, the characteristic function $\Delta(\lambda)$ satisfies the asymptotic relation*

$$\Delta(\lambda) = \cos(\lambda a; q) + O\left(|\lambda|^{-1} \exp\left(\frac{-(\log|\lambda|a(1-q))^2}{\log q}\right)\right).$$

4. Expansion theorem, Parseval equation

In this section, we prove that the eigenfunctions of the boundary value problem (1)–(3) form a complete system in $\mathcal{L}_{2,q}(0,a,\mathbb{C}^2)$. We establish the expansion theorem with respect to the eigenfunctions, and we obtain the Parseval equation. Here, it should be mentioned that the proof of Theorem 4.1 differs very little from the proof of [21, Theorem 1.2.1, p. 15], and the proofs of Theorem 4.2 and Theorem 4.3 differ very little from the proofs of [33, Theorem 7.3.2, p. 199] and [33, Theorem 7.3.3, p. 200]. Denote

$$G(t,s;\lambda) = \frac{1}{\Delta(\lambda)} \begin{cases} \psi(t,\lambda)\varphi^T(s,\lambda), & s < t, \\ \varphi(t,\lambda)\psi^T(s,\lambda), & s > t, \end{cases}$$

and consider the function

$$y(t,\lambda)=\int_0^a G(t,s;\lambda)f(t)\,\mathrm{d}_q t=\frac{1}{\Delta(\lambda)}\Bigg(\psi(t,\lambda)\int_0^t\big(\varphi_1(s,\lambda)f_1(s)+\varphi_2(s,\lambda)f_2(s)\big)\,\mathrm{d}_q s$$
$$+\varphi(t,\lambda)\int_t^a\big(\psi_1(s,\lambda)f_1(s)+\psi_2(s,\lambda)f_2(s)\big)\,\mathrm{d}_q s\Bigg).$$

The function $G(t,s;\lambda)$ is called Green's function for the boundary value problem (1)–(3). $G(t,s;\lambda)$ is the kernel of the q-Dirac operator, i.e., $y(t,\lambda)$ is the solution of the boundary value problem

$$D_q y_2(t)+(p(t)-\lambda)y_1(t)+r(t)y_2(t)+f(t)=0, \tag{22}$$

$$-D_q y_1^{\sigma}(t)+r(t)y_1(t)-\big(p(t)+\lambda\big)y_2(t)+f(t)=0, \tag{23}$$

$$U_1(y)=0, \qquad U_2(y)=0. \tag{24}$$

Taking (10) into account and using Corollary 3.2, we calculate

$$\begin{aligned}\mathrm{Res}_{\lambda=\lambda_n}y(t,\lambda)&=\frac{1}{\dot{\Delta}(\lambda)}\psi(t,\lambda_n)\int_0^t\big(\varphi_1(s,\lambda_n)f_1(s)+\varphi_2(s,\lambda_n)f_2(s)\big)\,\mathrm{d}_q s\\&\quad+\frac{1}{\dot{\Delta}(\lambda)}\varphi(t,\lambda_n)\int_0^t\big(\psi_1(s,\lambda_n)f_1(s)+\psi_2(s,\lambda_n)f_2(s)\big)\,\mathrm{d}_q s\\&=\frac{k_n}{\dot{\Delta}(\lambda)}\varphi(t,\lambda_n)\int_0^a\big(\varphi_1(s,\lambda_n)f_1(s)+\varphi_2(s,\lambda_n)f_2(s)\big)\,\mathrm{d}_q s.\end{aligned}$$

By virtue of (21),

$$\mathrm{Res}_{\lambda=\lambda_n}y(t,\lambda)=\frac{1}{\alpha_n}\varphi(t,\lambda_n)\int_0^a\big(\varphi_1(s,\lambda_n)f_1(s)+\varphi_2(s,\lambda_n)f_2(s)\big)\,\mathrm{d}_q s. \tag{25}$$

Let $f(\cdot)\in\mathcal{L}_{2,q}(0,a,\mathbb{C}^2)$ and assume that

$$\langle\varphi(s,\lambda_n),f(s)\rangle=0,$$

i.e.,

$$\int_0^a\big(\varphi_1(s,\lambda_n)f_1(s)+\varphi_2(s,\lambda_n)f_2(s)\big)\,\mathrm{d}_q s=0, \quad n\geq 0.$$

Then, in view of (25),

$$\mathrm{Res}_{\lambda=\lambda_n}y(t,\lambda)=0,$$

and consequently, for each fixed $t\in[0,a]$, the function $y(t,\lambda)$ is entire in λ.

Furthermore, for $\lambda \in G_\delta = \{\lambda : |\lambda - \lambda_n| \geq \delta\}$ and $|\lambda| \geq \lambda^*$, where $\delta > 0$ is fixed and $\lambda^* > 0$ is sufficiently large, we have [30]

$$|\Delta(\lambda)| \geq C_\delta e^{|\tau|a}, \qquad \lambda \in G_\delta, \qquad \lambda = \sigma + i\tau,$$

and consequently,

$$|y(t,\lambda)| \leq C_\delta |\lambda|^{-1}, \quad \lambda \in G_\delta.$$

Using the maximum principle [18, p. 128] and Liouville's theorem [18, p. 77], we conclude that $y(t,\lambda) \equiv 0$. From this and (22)–(24), it follows that $f(s) = 0$ a.e. on $(0,a)$. Thus, we have proved the following theorem.

Theorem 4.1. *The system of eigenfunctions $\{\varphi(t,\lambda_n)\}_{n\geq 0}$ of the boundary value problem (1)–(3) is complete in $\mathcal{L}_{2,q}(0,a,\mathbb{C}^2)$.*

Denote by $\lambda_0, \lambda_{\mp 1}, \lambda_{\mp 2}, \ldots, \lambda_{\mp n}, \ldots$ the eigenvalues of the boundary value problem and by $y_0, y_{\mp 1}, y_{\mp 2}, \ldots, y_{\mp n}, \ldots$ the corresponding normalized vector--valued eigenfunctions. Consider the matrix kernel

$$G(t,\xi) = \sum_{n=-\infty}^{\infty} \frac{y_n(t) y_n^T(\xi)}{\lambda_n}. \tag{26}$$

Put

$$D_q f_2(t) + p(t) f_1(t) + r(t) f_2(t) =: h_1(t),$$

$$-D_q f_1^\sigma(t) + r(t) f_1(t) - p(t) f_2(t) =: h_2(t).$$

We then have $f(t) = \int_0^a G(t,\xi) h(\xi)\, \mathrm{d}_q\xi$, $h(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}$, and using the expression from (26),

$$f(t) = \sum_{n=-\infty}^{\infty} y_n(t) \cdot \frac{1}{\lambda_n} \int_0^a h^T(\xi) y_n^T(\xi)\, \mathrm{d}_q\xi = \sum_{n=-\infty}^{\infty} a_n y_n(t).$$

Furthermore, it follows that $f^T(t) = \sum_{-\infty}^{\infty} a_n y_n^T(t)$, which multiplied on the right by $y_n(t)$ and integrated from 0 to a yields

$$a_n = \int_0^a f^T(t) y_n(t)\, \mathrm{d}_q t, \tag{27}$$

since $y_n(t)$ are orthonormalized, and the following theorem is thus proved.

Theorem 4.2. *If $D_q f(t)$ is q-regular at zero and fulfills the boundary conditions (2), (3), then $f(t)$ can be expanded into an absolutely and uniformly convergent Fourier series of the vector-valued eigenfunctions of the boundary value problem (1)–(3), viz.*

$$f(t) = \sum_{n=-\infty}^{\infty} a_n(t) y_n(t), \qquad a_n = \int_0^a f^T(t) y_n(t)\, \mathrm{d}_q t. \tag{28}$$

Theorem 4.3. *For any square integrable vector-valued function f in an interval $[0,a]$, the Parseval equation*

$$\int_0^a f^2(t)\,\mathrm{d}_q t = \sum_{n=-\infty}^{\infty} a_n^2, \qquad f^2(t) = f_1^2(t) + f_2^2 \tag{29}$$

holds.

Proof. If $f(\cdot)$ fulfills the conditions of Theorem 4.2, then the equality (29) immediately follows from the uniform convergence of the series (28). Indeed, multiplying the expansion (28) on the left by $f^T(t)$ and integrating from 0 to a, we obtain, also relying on (27) that

$$\int_0^a f^T(t)f(t)\,\mathrm{d}_q t = \int_0^a f^2(t)\,\mathrm{d}_q t = \sum_{n=-\infty}^{\infty} a_n \int_0^a f^T(t)y_n(t)\,\mathrm{d}_q t = \sum_{n=-\infty}^{\infty} a_n^2,$$

i.e., the equality (29). □

5. Weyl solution, Weyl function

Let the function $\Phi(t,\lambda)$ be the solution of (1) satisfying the boundary conditions

$$U_1(\Phi) = 1, \qquad U_2(\Phi) = 0.$$

We set $M(\lambda) := \Phi(0,\lambda)$. The functions $\Phi(t,\lambda)$ and $M(\lambda)$ are called the Weyl solution and the Weyl function for the boundary value problem (1)–(3). Clearly,

$$\Phi(t,\lambda) = \frac{\psi(t,\lambda)}{\Delta(\lambda)} = C(t,\lambda) + M(\lambda)\varphi(t,\lambda). \tag{30}$$

The Weyl solution and Weyl function are meromorphic functions having simple poles at the points λ_n, the eigenvalues of problem (1)–(3).

In this section, we consider the following inverse problem.

Inverse Problem 1. Given the Weyl function, construct the boundary value problem (1)–(3).

Theorem 5.1. *The boundary value problem* (1)–(3) *is uniquely determined by the Weyl function $M(\lambda)$.*

Proof. Describe the matrix $P(t,\lambda) = [P_{i,j}(t,\lambda)]_{i,j=1,2}$ by the formula

$$P(t,\lambda)\begin{bmatrix}\tilde\varphi_1(t,\lambda) & \tilde\Phi_1(t,\lambda)\\ \tilde\varphi_2(t,\lambda) & \tilde\Phi_2(t,\lambda)\end{bmatrix} = \begin{bmatrix}\varphi_1(t,\lambda) & \Phi_1(t,\lambda)\\ \varphi_2(t,\lambda) & \Phi_2(t,\lambda)\end{bmatrix}, \tag{31}$$

and define the q-Wronskian of the solutions $\tilde\varphi(t,\lambda)$ and $\tilde\Phi(t,\lambda)$ by

$$W_q[\tilde\varphi(t,\lambda),\tilde\Phi(t,\lambda)] = \tilde\varphi_1(t,\lambda)\tilde\Phi_2(t,\lambda) - \tilde\varphi_2(t,\lambda)\tilde\Phi_1(t,\lambda) = -1. \tag{32}$$

Using (31) and (32), we find

$$\begin{cases} P_{j1}(t,\lambda) = \varphi_j(t,\lambda)\tilde{\Phi}_2(t,\lambda) - \Phi_j(t,\lambda)\tilde{\varphi}_2(t,\lambda), \\ P_{j2}(t,\lambda) = \Phi_j(t,\lambda)\tilde{\varphi}_1(t,\lambda) - \varphi_j(t,\lambda)\tilde{\Phi}_1(t,\lambda), \end{cases} \tag{33}$$

$$\begin{cases} \varphi_1(t,\lambda) = P_{11}(t,\lambda)\tilde{\varphi}_1(t,\lambda) + P_{12}\tilde{\varphi}_2(t,\lambda), \\ \varphi_2(t,\lambda) = P_{21}(t,\lambda)\tilde{\varphi}_1(t,\lambda) + P_{22}\tilde{\varphi}_2(t,\lambda), \\ \Phi_1(t,\lambda) = P_{11}(t,\lambda)\tilde{\Phi}_1(t,\lambda) + P_{12}\tilde{\Phi}_2(t,\lambda), \\ \Phi_2(t,\lambda) = P_{21}(t,\lambda)\tilde{\Phi}_1(t,\lambda) + P_{22}\tilde{\Phi}_2(t,\lambda). \end{cases} \tag{34}$$

It follows from (33), (30), and (32) that

$$P_{11}(t,\lambda) = 1 + \frac{\tilde{\psi}_2(t,\lambda)}{\tilde{\Delta}(\lambda)}\left[\varphi_1(t,\lambda) - \tilde{\varphi}_1(t,\lambda)\right] + \tilde{\varphi}_2(t,\lambda)\left[\frac{\tilde{\psi}_1(t,\lambda)}{\tilde{\Delta}(\lambda)} - \frac{\psi(t,\lambda)}{\Delta(\lambda)}\right],$$

$$P_{12}(t,\lambda) = \frac{\psi_1(t,\lambda)}{\Delta(\lambda)}\left[\tilde{\varphi}_1(t,\lambda) - \varphi_1(t,\lambda)\right] + \varphi_1(t,\lambda)\left[\frac{\psi_1(t,\lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_1(t,\lambda)}{\tilde{\Delta}(\lambda)}\right],$$

$$P_{21}(t,\lambda) = \frac{\psi_2(t,\lambda)}{\Delta(\lambda)}\left[\varphi_2(t,\lambda) - \tilde{\varphi}_2(t,\lambda)\right] + \varphi_2(t,\lambda)\left[\frac{\tilde{\psi}_2(t,\lambda)}{\tilde{\Delta}(\lambda)} - \frac{\psi_2(t,\lambda)}{\tilde{\Delta}(\lambda)}\right],$$

$$P_{22}(t,\lambda) = 1 + \frac{\tilde{\psi}_1(t,\lambda)}{\tilde{\Delta}(\lambda)}\left[\tilde{\varphi}_2(t,\lambda) - \varphi_2(t,\lambda)\right] + \tilde{\varphi}_1(t,\lambda)\left[\frac{\psi_2(t,\lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_2(t,\lambda)}{\tilde{\Delta}(\lambda)}\right].$$

Using $|\Delta(\lambda)| \geq C_\delta e^{|\tau|a}$ yields

$$|P_{11}(t,\lambda) - 1| \leq C_\delta, \quad |P_{12}(t,\lambda)| \leq C_\delta, \quad |P_{21}(t,\lambda)| \leq C_\delta, \quad |P_{22}(t,\lambda) - 1| \leq C_\delta. \tag{35}$$

Substituting the right-hand side of (30) in (33), we have

$$\begin{aligned} P_{11}(t,\lambda) &= \varphi_1(t,\lambda)\tilde{C}_2(t,\lambda) - C_1(t,\lambda)\tilde{\varphi}_2(t,\lambda) \\ &\qquad + \varphi_1(t,\lambda)\tilde{\varphi}_2(t,\lambda)\left[\tilde{M}(\lambda) - M(\lambda)\right], \\ P_{12}(t,\lambda) &= C_1(t,\lambda)\tilde{\varphi}_1(t,\lambda) - \varphi_1(t,\lambda)\tilde{C}_1(t,\lambda) \\ &\qquad + \varphi_1(t,\lambda)\tilde{\varphi}_1(t,\lambda)\left[M(\lambda) - \tilde{M}(\lambda)\right], \\ P_{21}(t,\lambda) &= \varphi_2(t,\lambda)\tilde{C}_2(t,\lambda) - C_2(t,\lambda)\tilde{\varphi}_2(t,\lambda) \\ &\qquad + \varphi_2(t,\lambda)\tilde{\varphi}_2(t,\lambda)\left[\tilde{M}(\lambda) - M(\lambda)\right], \\ P_{22}(t,\lambda) &= C_2(t,\lambda)\tilde{\varphi}_1(t,\lambda) - \varphi_2(t,\lambda)\tilde{C}_1(t,\lambda) \\ &\qquad + \varphi_2(t,\lambda)\tilde{\varphi}_1(t,\lambda)\left[M(\lambda) - \tilde{M}(\lambda)\right]. \end{aligned}$$

Thus, if $M(\lambda) \equiv \tilde{M}(\lambda)$, then for each fixed t, the functions $P_{ij}(t,\lambda)$, $i,j = 1,2$, are entire in λ. Then from (35), we find

$$P_{11}(t,\lambda) \equiv 1, \qquad P_{12}(t,\lambda) \equiv 0, \qquad P_{21}(t,\lambda) \equiv 0, \qquad P_{22}(t,\lambda) \equiv 1.$$

Substituting these into (34), we get

$$\varphi_1(t,\lambda) \equiv \tilde{\varphi}_1(t,\lambda), \qquad \varphi_2(t,\lambda) \equiv \tilde{\varphi}_2(t,\lambda),$$

$$\Phi_1(t,\lambda) \equiv \tilde{\Phi}_1(t,\lambda), \qquad \Phi_2(t,\lambda) \equiv \tilde{\Phi}_2(t,\lambda)$$

for every t and λ. Hence, we arrive at

$$p(t) \equiv \tilde{p}(t), \qquad r(t) \equiv \tilde{r}(t).$$

□

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