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A Generalized Approach To Partitioning Weighted Points In A Plane

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A GENERALIZED APPROACH TO PARTITIONING WEIGHTED POINTS IN A PLANE

by

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I dedicate this work to St. Jacinta Marto,
and most especially to the Immaculate Heart of Mary.

ABSTRACT

A theorem commonly known as the Discrete Pancake Theorem states that for two finite disjoint sets S and T of points in a plane where $S \cup T$ contains no three collinear points, there exists a line that simultaneously bisects $|S|$ and $|T|$ within an error of at most one. This thesis considers a more general situation in which each point is assigned two non-negative weights and, instead of simply bisecting the plane to obtain a balance in the number of points, we prove there exists a line that simultaneously balances weight one and weight two accumulations within a prescribed tolerance. The Discrete Pancake Theorem is shown to be a special case of this Dual-Balanced Theorem, and a computational implementation of this generalization is applied to various examples.

1 Introduction

The process of congressional and state legislative redistricting is a political undertaking that has the potential to create imbalances in the political landscape. A United States Supreme Court ruling in the 1964 case *Wesberry v. Sanders* interpreted Article I Section II of the United States Constitution to mean that "as nearly as is practicable, one person's vote in a congressional election is to be worth as much as another's [27]." This ruling, which has been further supported by subsequent Supreme Court decisions [6, 7, 10, 19], solidified the understanding that considering population balance is paramount when redistricting congressional districts.

Many states reference a similar population balance requirement for state legislative districts. For example, Article IV Section II of the North Dakota State Constitution articulates that "the legislative assembly shall guarantee, as nearly as is practicable, that every elector is equal to every other elector in the state in the power to cast ballots for legislative candidates [16]." See the table in [11] for links to constitutions of other states in the United States and their reference to different criteria such as equal population, race/ethnicity, contiguity, compactness, etc. The United States Supreme Court has also decided that race cannot be the ultimate motivating factor in the determination of district lines [1, 4, 5, 15, 21].

1.1 Statement of the Problem

Given a region where each data point is assigned two non-negative numerical weights, we want to construct a directed line so that the region is balanced with respect to both weights.

Take, for example, this hypothetical scenario: After the 2020 census, a region underwent the process of redistricting, and the districts were created to satisfy the population balance requirement. However, because of unforeseen or perhaps unconsidered large population shifts over the course of 7-8 years, the population was unreasonably imbalanced among districts in 2028. Could there be a way to redistrict a population so as to also consider future population growth? If given a reasonable prediction of future population growth, could we establish districts that would both necessarily balance population now and balance a predicted population after a set period of time?

1.2 Definition of Balance and Dual-Balance

We now define the meaning of *balance* and *dual-balance*, both being essential to the purpose of this paper.

Definition 1.1: Balance

A directed line *balances* a region with respect to weight x values if the absolute difference of the sum of the weight x values on one side of the directed line and the sum of the weight x values on the other side of the directed line is less than or equal to the maximum weight x value.

Definition 1.2: Dual-balance

A directed line *dual-balances* a region with respect to weight y and weight z values if the directed line balances the region both with respect to weight y values and with respect to weight z values.

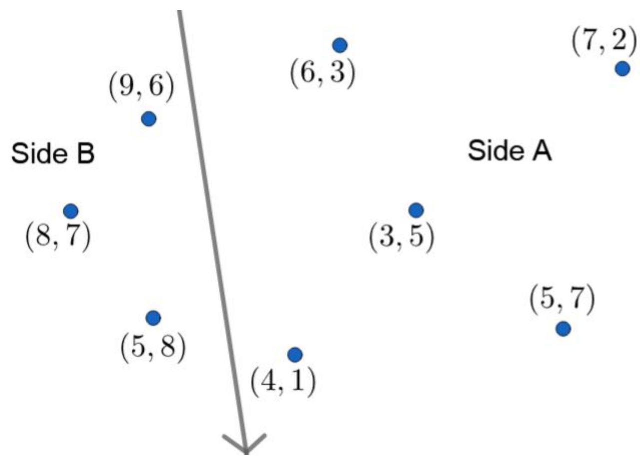


Figure 1: Dual-balancing example

As an example, consider the region above in Figure 1. For every ordered pair (a, b), the first coordinate a represents the weight one value of the corresponding point, and the second coordinate b represents the same point's weight two value. The directed line in Figure 1 separates the region into Side A and Side B, where each side accumulates weight one and weight two values. Note that the maximum weight one value is nine, and the maximum weight two value is eight. With the information given in Table 1, we see by definition that the directed line in Figure 1 *dual-balances* the region.

Table 1: Summary of weight accumulations by side in Figure 1

Side	Sum of weight one values	Sum of weight two values
A	25	18
B	22	21
|Difference|	3	3

1.3 Use of Census Blocks

In order to satisfy equal population requirements for congressional and state legislative districts as determined by state legislatures and judicial law [3, 10, 11], data must be gathered in order to ascertain population concentrations. The United States census, performed every ten years, gathers district, regional, and statewide information about population and other demographics. Census blocks, the "smallest level of geography you can get basic demographic data for, such as total population by age, sex, and race [20]," are determined by an automated process with boundaries that can change over time [20].

Census blocks, many of which do not have any population, are combined into block groups, one of which is a "combination of census blocks that is a subdivision of a census tract or block numbering area [23]." Relatively permanent census tracts can be utilized in the formation of districts based on demographic information such as population [23, 24]. As an example, in the 2010 census, North Dakota had 133,769 census blocks, 572 block groups, and 205 census tracts [25].

1.4 Region with the Usual Conditions

Consider a finite collection of points α_i in a plane, where $i \in \overline{n} = \{1, 2, 3, \ldots, n\}$. If it is not already the case, adjust the α_i locations only as necessary so that for every distinct $j, k, l \in \overline{n}$, the points α_j, α_k, and α_l are not collinear. Additionally, if it is not already the case, adjust the α_i locations only as necessary so that for every distinct $r, s \in \overline{n}$, the points α_r and α_s are not located on the same vertical line. Finally, every α_i in the region stores the same finite number of non-negative weights. In this paper, we will refer to a region constructed in this fashion as a *region with* *the usual conditions.*

Though the Dual-Balanced Theorem proves a more generalized result, note that this region definition allows for the consideration of census blocks. In particular, we may consider a finite collection of n census blocks in a finite region. For every census block c_i, where $i \in \overline{n}$, we could let α_i be a representative point of c_i located at the census block's centroid. We would certainly be able to adjust the representative points as necessary to satisfy the non-collinear requirements. Furthermore, the finite number of non-negative weights could be any census-provided demographic information from the particular census blocks.

1.5 Representative Points for Census Blocks

To achieve the outcome of a district division given a group of census blocks, we assign every census block c_i a representative point α_i that stores information from a finite number of non-negative weights. For the main result of this paper, we consider α_i being assigned two weights (w_{1i} and w_{2i}).

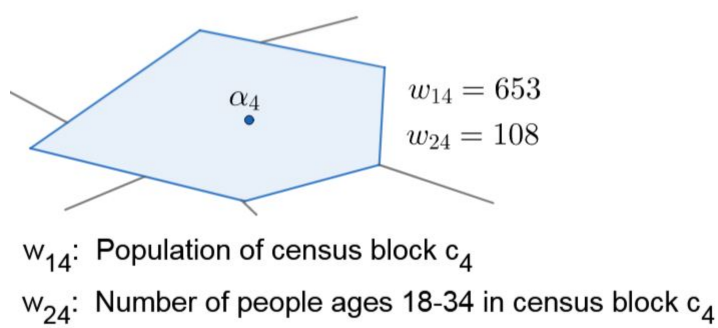


Figure 2: Census block c_4 with demographic information

In Figure 2, representative point α_4 of the census block c_4 stores two weight values. In particular, α_4 contains the information that the weight one value of total population in c_4 is 653, and the weight two value of number of people ages 18-34 in c_4 is 108. By considering representative points instead of census block boundaries, we are able to relate a group of census blocks to a region satisfying the usual conditions.

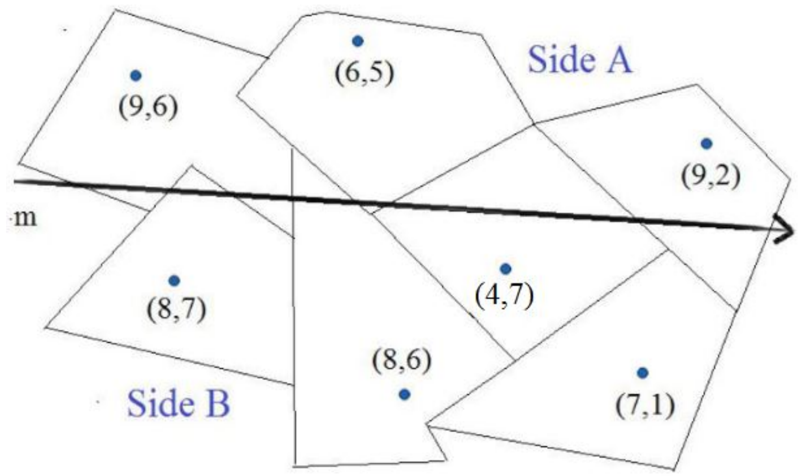


Figure 3: Directed line in a region with census blocks

Directed lines, such as the one seen in Figure 3, will be used to attempt to separate a region into balanced partitions. In the special case of census blocks and redistricting, note that the representative points and directed line are merely tools to achieve the ultimate outcome of a district division. Such an example is shown in Figure 4.

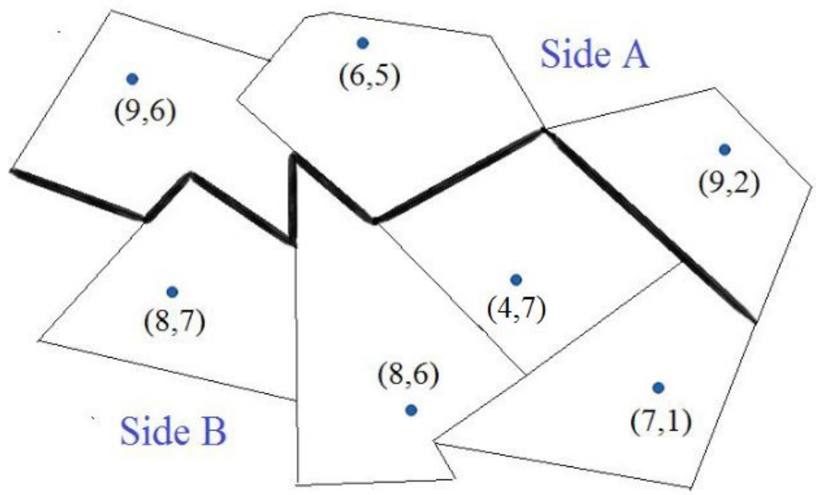


Figure 4: Resulting polygonal district division

Table 2: Summary of weight accumulations by side in Figure 4

Side	Sum of weight one values	Sum of weight two values
A	24	13
B	27	21
|Difference|	3	8

By examining the coordinates in Figure 4, we note that the weight one maximum value is nine, whereas the maximum weight two value is seven. We can see in Table 2 that although the directed line balances weight one values in the region, weight two values are imbalanced. Thus, the directed line seen in Figure 3 does not dual-balance the region.

1.6 Summary of Notation Used

For the remainder of this paper, the following notation will be used for a specified region with the usual conditions.

ε_1 : Maximum weight one value

ε_2 : Maximum weight two value

ε_3 : Maximum weight three value

S_a : Sub-region strictly to the left of the directed line

S_b : Sub-region strictly to the right of the directed line

S_{1a} : Sum of weight one values assigned to S_a

S_{2a} : Sum of weight two values assigned to S_a

S_{3a} : Sum of weight three values assigned to S_a

S_{1b} : Sum of weight one values assigned to S_b

S_{2b} : Sum of weight two values assigned to S_b

S_{3b} : Sum of weight three values assigned to S_b

c_i : A census block in the region

α_i : A point in the region

w_{ji} : The j^{th} weight value for point α_i

S_1: Sum of all weight one values in the region

S_2: Sum of all weight two values in the region

S_3: Sum of all weight three values in the region

x_1: $S_{1a} - S_{1b}$; difference in weight one accumulations between S_a and S_b

x_2: $S_{2a} - S_{2b}$; difference in weight two accumulations between S_a and S_b

x_3: $S_{3a} - S_{3b}$; difference in weight three accumulations between S_a and S_b

1.7 Tolerance Level for Balance

The Dual-Balanced Theorem will show that any region with the usual conditions can be *dual-balanced* by at least one directed line. With the notation given above, this means there exists a line so that

1. $|x_1| = |S_{1a} - S_{1b}| \leq \varepsilon_1$, and

2. $|x_2| = |S_{2a} - S_{2b}| \leq \varepsilon_2$.

Given the example below of a non-trivial region satisfying the usual conditions, we see there can be issues for a balance tolerance less than the maximum weight value. In particular, with each weight one value being ten in Figure 5, there does not exist a line so that $|x_1| = |S_{1a} - S_{1b}| < 10$. This shows that our defined tolerance level for balance is reasonable.

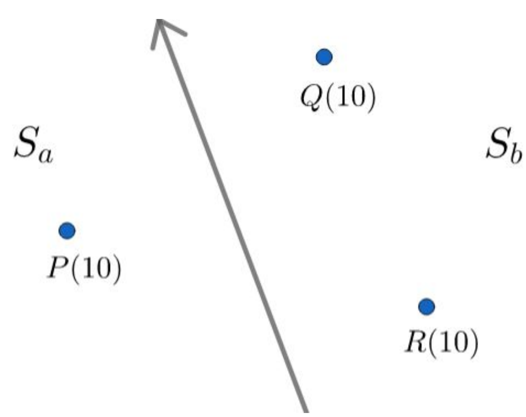


Figure 5: Non-trivial example showing reasonableness of defined tolerance level for balance

1.8 Initial Weight-One Balancing

Consider any arbitrary region R with the usual conditions, along with the weight one values w_{1i} for each point α_i. Note that in the special case of considering census blocks, we will disregard the boundaries of the census blocks and focus on the representative points until the desired directed line is constructed.

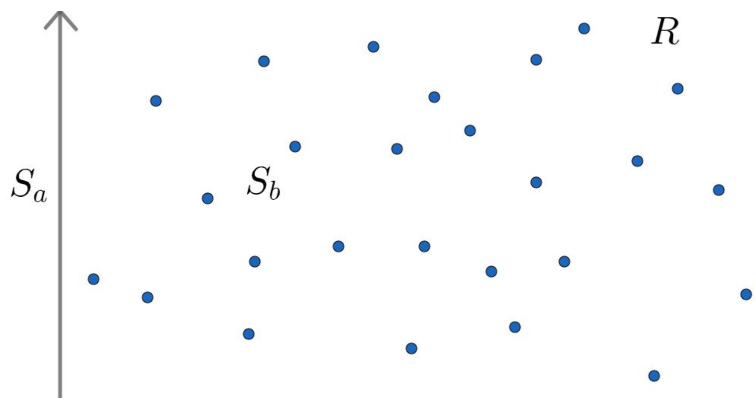


Figure 6: Initial weight-one balancing primary construction

To begin the initial weight-one balancing process, we first construct a vertical directed line l directed up so that all points in the region are to the right. It is certainly the case that the entire population S_1 is strictly to the right of the directed line, or in S_b (see Figure 6). As the vertical directed line l horizontally translates to the right, as shown in Figure 7, more and more weight one values will be accumulating on the left side of the directed line, or in S_a.

Since every vertical line in region R contains at most one α_i by assumption, there must be a point $\alpha_p \in R$ for $p \in \overline{n}$ such that if line l crosses α_p, it is the case for the first time that the sum of the weight one values of points strictly to the left of line l is greater than the sum of the weight one values of points strictly to the right. Note

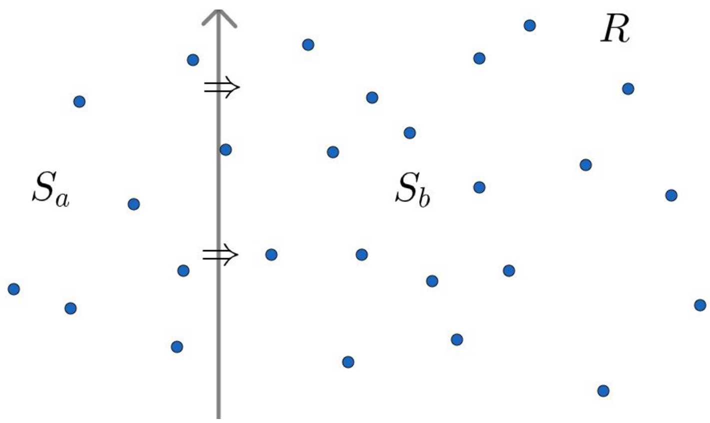


Figure 7: Initial weight-one balancing horizontal translation

that since each α_i can have a different weight one value, the point α_p does not have to be centrally located.

Let the directed vertical line l contain α_p. Also let S'_{1a} denote the sum of the weight one values of points strictly to the left of line l, and let S'_{1b} denote the sum of the weight one values of points strictly to the right of line l. The only weight one value not assigned to either S'_{1a} or S'_{1b} is w_{1p}, the weight one value of α_p. Figure 8 gives an illustration of this step.

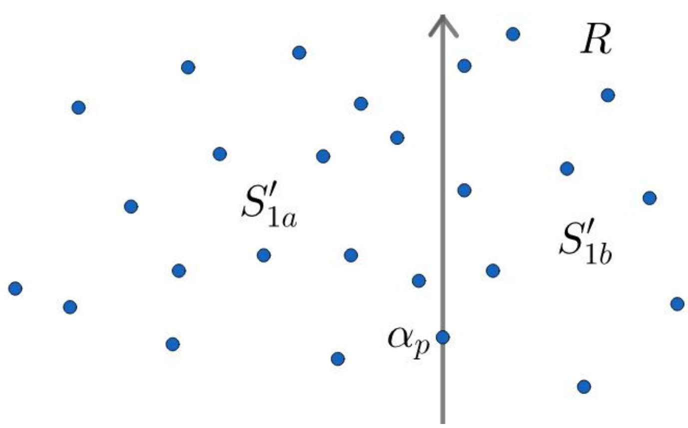


Figure 8: Directed line in initial weight-one balancing containing α_p before w_{1p} assignment

To conclude the initial weight-one balancing process, we will assign w_{1p} to S_{1a} if $S'_{1a} \leq S'_{1b}$, and we will assign w_{1p} to S_{1b} if $S'_{1a} > S'_{1b}$.

Note that by definitions 1.3, 1.4, and 1.5 below, the initial weight-one balancing process generates the inequalities

$$S'_{1a} + w_{1p} > S'_{1b} \qquad \text{and} \qquad S'_{1b} + w_{1p} \geq S'_{1a}.$$

Because an initial weight-one balancing process will be regularly assumed, these consequential inequalities will prove to be useful in upcoming proofs.

Definition 1.3: α_p

Given a vertical directed line directed up in a region with the usual conditions, initially constructed so all points are to the right and followed by a rightward horizontal translation, α_p is the point in the region such that if the directed line shifts past it, for the first time it is the case that the sum of the weight one values of points strictly to the left of the directed vertical line is greater than the sum of the weight one values of points strictly to the right.

Definition 1.4: S'_{1a}

The sum of the weight one values of points strictly to the left of the directed vertical line directed up that contains α_p

Definition 1.5: S'_{1b}

The sum of the weight one values of points strictly to the right of the directed vertical line directed up that contains α_p

Definition 1.6: Initial weight-one balancing

In a region with the usual conditions, the construction of a vertical directed line directed up that contains α_p, where w_{1p} is assigned to S_{1a} if $S'_{1a} \leq S'_{1b}$, and w_{1p} is assigned to S_{1b} if $S'_{1a} > S'_{1b}$

1.9 Assigning Weight Values to S_a and S_b

The directives below will be adhered to when considering whether weight values will be assigned to S_a or S_b.

1. If a point α_i is completely contained in a side (either S_a or S_b), then the weight values for α_i will be assigned to that side.

2. If a point α_i is contained on the directed line between S_a and S_b, then the weight values for α_i can be assigned to either side of the directed line, though all weights for α_i must be assigned to the same side.

3. A point α_i can only switch sides by decision during a clockwise rotation when there are two points on the directed line.

4. If α_p is assigned to S_a after an initial weight-one balancing process, then α_p is assigned to S_b when the directed line contains α_p and is vertically directed down. If α_p is assigned to S_b after an initial weight-one balancing process, then α_p is assigned to S_a when the directed line contains α_p and is vertically directed down.

Consider the example in Figure 9, where the directed line is rotating clockwise about point L. By directive one above, the weight value for point J is assigned to

S_a, and the weight values for points K and M are assigned to S_b. In Table 3, we see the weight value for point L is assigned to S_a. By directive three, the weight value for point L cannot switch sides by decision until the directed line intersects another point.

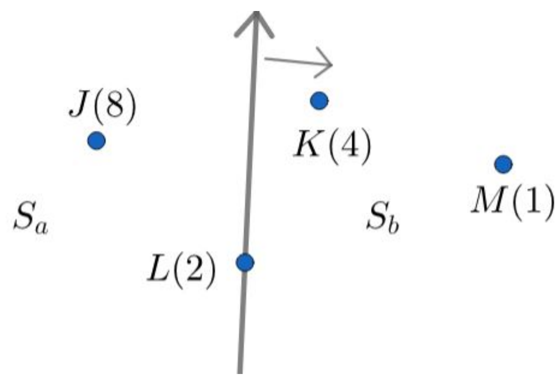


Figure 9: Clockwise rotation about point L

Sub-region	Point assignment	Weight total
S_a	J, L	10
S_b	K, M	5
|Difference|		5

Table 3: Summary of weight accumulations by side in Figure 9

As the directed line rotates clockwise about L, it eventually intersects point K. This moment is shown in Figure 10, where by directive three above we now can switch the designations of K and L by decision. In Table 4, we see both points K and L on the directed line were assigned to S_b. Since the maximum weight one value is eight, we have maintained weight one balance thus far in the clockwise rotation.

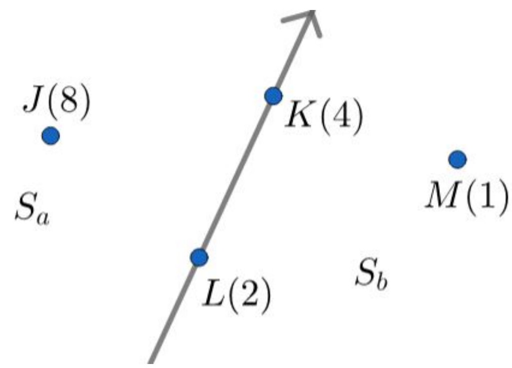


Figure 10: Directed line rotating clockwise contains two points

Sub-region	Point assignment	Weight total
S_a	J	8
S_b	K, L, M	7
|Difference|		1

Table 4: Summary of weight accumulations by side in Figure 10

At the moment shown in Figure 10, continuing the clockwise rotation either implies rotating about point K or about point L. Though either are legitimate possibilities, we can see in Figure 11 that the clockwise rotation continues by rotating about point K. Note in Table 5 that a clockwise rotation about K results in a balanced region, whereas a clockwise rotation about point L would not.

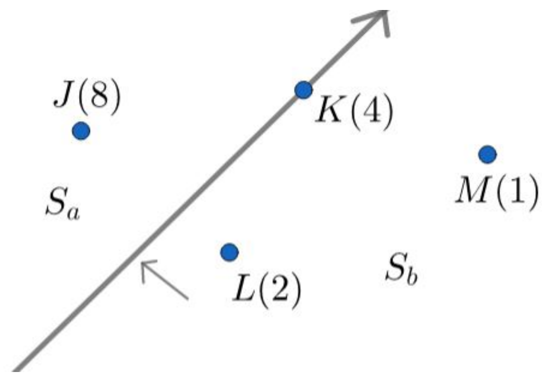


Figure 11: Clockwise rotation continues about point K

Sub-region	Point assignment	Weight total
S_a	J	8
S_b	K, L, M	7
|Difference|		1

Table 5: Summary of weight accumulations by side in Figure 11

1.10 Summary

The main result of this paper, the Dual-Balanced Theorem, proves that any region with the usual conditions can be dual-balanced by at least one line. Processes such as an initial weight-one balancing and the assignment of weight values have been established, along with the definitions of balance, dual-balance, and a region with the usual conditions. Acceptable clockwise rotations of a directed line are defined to be rotations about a point, and the point about which to rotate can be chosen when two points lie on the directed line. It was shown through an example that a tolerance level for balance less than the one described can cause problems in a region with the usual conditions, proving the given tolerance level for balance is reasonable. Finally, a summary of the notation to be used was listed.

In the United States, population balance is necessary in the process of redistricting, both for congressional and state legislative districts. It was shown that redistricting and the consideration of census blocks provide an application where the Dual-Balanced Theorem can be utilized.

2 Review of Literature

Political redistricting, when non-biased, can be considered a form of balanced partitioning. As a result, there is a wide variety of literature pertaining to this topic, ranging from the pure mathematical interest of creating balanced partitions of points in a plane to the prevention of redistricting in a way so as to create an unfair advantage for one political party. Since our main result pertains to dual-balancing a region with the usual conditions, we naturally first consider a proof of the Pancake Theorem.

2.1 Borsuk-Ulam Theorem in One Dimension

Consider two disjoint subsets $P_1, P_2 \subseteq \mathbb{R}^2$. The Pancake Theorem asserts the existence of a line that simultaneously bisects P_1 and P_2 into regions of equal area. To prove this, we will first show a proof for the Borsuk-Ulam Theorem in one dimension.

Lemma 2.1

If ϕ is adjusted as necessary so that $0 \leq \phi < 2\pi$, then the function $h : S^1 \to [0, 2\pi)$ given by $h(\cos\phi, \sin\phi) = \phi$ is a homeomorphism.

Proof. Consider the function $h : S^1 \to [0, 2\pi)$ given by $h(\cos\phi, \sin\phi) = \phi$, where ϕ was adjusted as necessary so that $0 \leq \phi < 2\pi$. Note that $[0, 2\pi)$ is Hausdorff, being a subspace of Hausdorff $\mathbb{R}$. We also know S^1, being closed and bounded in $\mathbb{R}^2$, is compact in $\mathbb{R}^2$.

Let $t \in [0, 2\pi)$. Then $h(\cos t, \sin t) = t$, implying h is surjective.

Consider any $h(\cos\phi_1, \sin\phi_1) = h(\cos\phi_2, \sin\phi_2)$, where ϕ_1 and ϕ_2 have been adjusted as necessary. Since $h(\cos\phi_1, \sin\phi_1) = \phi_1$, and $h(\cos\phi_2, \sin\phi_2) = \phi_2$, it follows that $\phi_1 = \phi_2$, showing h is bijective.

Let $\mathbf{x}(\lambda) = (\cos\lambda, \sin\lambda) \in S^1$, where λ was adjusted as necessary. Consider any neighborhood V of $h(\cos\lambda, \sin\lambda) = \lambda$. Then there exists some basis element (a, b) in the subspace topology on $[0, 2\pi)$ so that $\lambda \in (a, b) \subset V$. But then $\mathbf{x}(a, b)$ is a neighborhood of $\mathbf{x}(\lambda)$ so that

$$h(\mathbf{x}(a, b)) = (a, b) \subset V,$$

proving h is continuous.

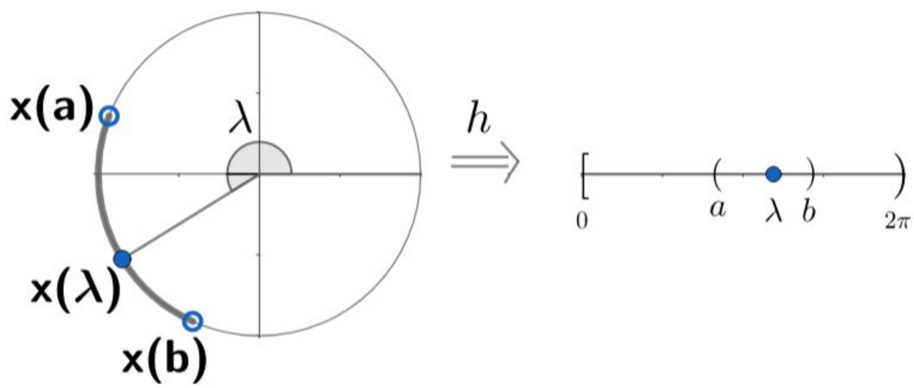


Figure 12: Continuous function h

Let C be any closed set in S^1. It follows that C is compact in S^1 as closed sets in compact spaces are compact. Since h is continuous, $h(C)$ is compact in $[0, 2\pi)$. With $h(C)$ being compact in Hausdorff space $[0, 2\pi)$, $h(C)$ is closed in $[0, 2\pi)$, showing h^{-1} is also continuous.

Proving h to be bijective and continuous so that h^{-1} is also continuous, we have shown that h is a homeomorphism. $\square$

Theorem 2.2: Borsuk-Ulam Theorem in One Dimension

If $f : S^1 \to \mathbb{R}$ is continuous, then there exists an $\mathbf{x} \in S^1$ so that $f(-\mathbf{x}) = f(\mathbf{x})$.

Proof. Let $f : S^1 \to \mathbb{R}$ be continuous, and define $p(\theta) : [0, 2\pi) \to S^1$ so that $p(\theta) = (\cos\theta, \sin\theta)$. By Lemma 2.1, p is continuous.

Note that $f \circ p : [0, 2\pi) \to \mathbb{R}$ is a continuous function as a composition of continuous functions is continuous. If we define $g(\theta) : [0, 2\pi) \to \mathbb{R}$ so that $g(\theta) = (f \circ p)(\theta + \pi) - (f \circ p)(\theta)$, then g is also continuous. Furthermore, $g(0) = -g(\pi)$, as

$$\begin{aligned} -g(\pi) &= -[(f \circ p)(0) - (f \circ p)(\pi)] \\ &= (f \circ p)(\pi) - (f \circ p)(0) \\ &= g(0). \end{aligned}$$

Since $f \circ p$ is continuous, $[0, 2\pi)$ is a connected space, and since $\mathbb{R}$ is an ordered set in the order topology, we can utilize the Intermediate Value Theorem. In particular, $g(\pi) = g(\pi)$, $g(0) = -g(\pi)$, and, as a result, there exists some $0 < r < \pi$ so that $g(r) = 0$. Since $g(r) = (f \circ p)(r + \pi) - (f \circ p)(r)$, it follows that

$$\begin{aligned} 0 &= (f \circ p)(r + \pi) - (f \circ p)(r) \\ (f \circ p)(r) &= (f \circ p)(r + \pi) \\ f[p(r)] &= f[p(r + \pi)]. \end{aligned}$$

Note $f[p(r)] = f(\cos r, \sin r)$, and

$$\begin{aligned} f[p(r+\pi)] &= f[(\cos(r+\pi), \sin(r+\pi)] \\ &= f(-\cos r, -\sin r) \\ &= f[-(\cos r, \sin r)]. \end{aligned}$$

By transitivity, we have

$$\begin{aligned} f[p(r)] &= f[p(r+\pi)] \\ f(\cos r, \sin r) &= f[-(\cos r, \sin r)]. \end{aligned}$$

Consequentially, as $(\cos r, \sin r) = \mathbf{x}$ for some $\mathbf{x} \in S^1$, we have found a value $\mathbf{x} \in S^1$ so that $f(-\mathbf{x}) = f(\mathbf{x})$. □

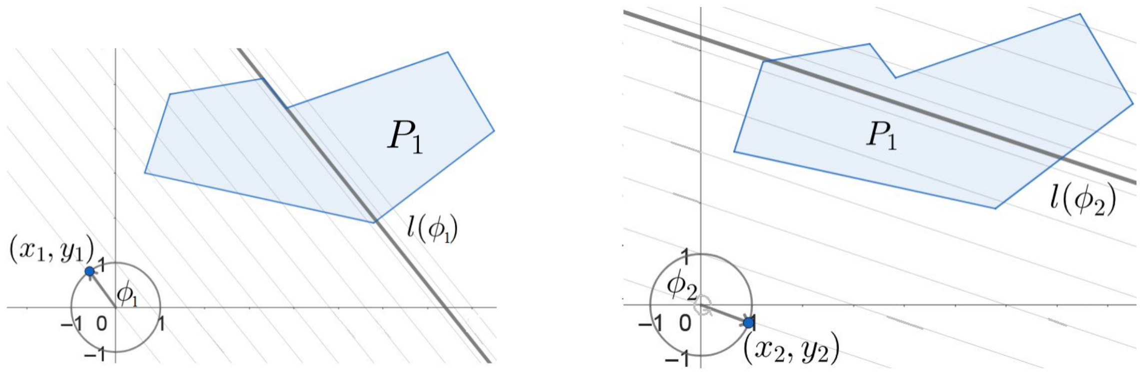
2.2 Proof of the Pancake Theorem

We now develop an appropriate continuous function $f : S^1 \to \mathbb{R}$ to prove the Pancake Theorem.

Theorem 2.3: Pancake Theorem

If P_1, P_2 are disjoint subsets in $\mathbb{R}^2$, then there exists a line l that simultaneously bisects both P_1 and P_2 into regions of equal area.

Proof. Let P_1 and P_2 be disjoint subsets in $\mathbb{R}^2$. Consider the unit circle S^1, and let $\mathbf{x} = (\cos\phi, \sin\phi) \in S^1$ with ϕ adjusted as necessary so that $0 \leq \phi < 2\pi$. The $x_1 = \cos\phi$ and $y_1 = \sin\phi$ coordinates of $\mathbf{x}$ on the unit circle can be utilized as coordinates of a unit vector $\langle x_1, y_1 \rangle$ with slope $\tan\phi$. Furthermore, for any $0 \leq \phi < 2\pi$, there exists a directed line $l(\phi)$ with slope $\tan\phi$ so that there is an equal amount of area of P_1 on either side of the line. Figures 13a and 13b show examples of two such lines $l(\phi_1)$ and $l(\phi_2)$ for a given P_1 and for directions ϕ_1 and ϕ_2.



(a) Line $l(\phi_1)$ for direction ϕ_1

(b) Line $l(\phi_2)$ for direction ϕ_2

Figure 13: Two examples of lines with an equal amount of Area(P_1) on either side

Define $h_1 : S^1 \to [0, 2\pi)$ so that $h_1(\cos\phi, \sin\phi) = \phi$, where ϕ was adjusted as necessary to satisfy $0 \leq \phi < 2\pi$. By Lemma 2.1, h_1 is continuous. Next, define $g_1 : [0, 2\pi) \to \mathbb{R}$ so that $g_1(\phi) = \text{Area}(P_2 \cap R[l(\phi)])$, where $R[l(\phi)]$ denotes the right side of the line $l(\phi)$.

Let $\sigma \in [0, 2\pi)$, and consider any neighborhood V_σ of $\text{Area}(P_2 \cap R[l(\sigma)]) \in \mathbb{R}$. Then there exists some basis element (c, d) in the standard topology on $\mathbb{R}$ so that $\text{Area}(P_2 \cap R[l(\sigma)]) \in (c, d) \subset V_\sigma$. Certainly there are perturbations small enough about σ so that the image $g_1(\sigma_1, \sigma_2)$ is contained in (c, d), proving g_1 is continuous.

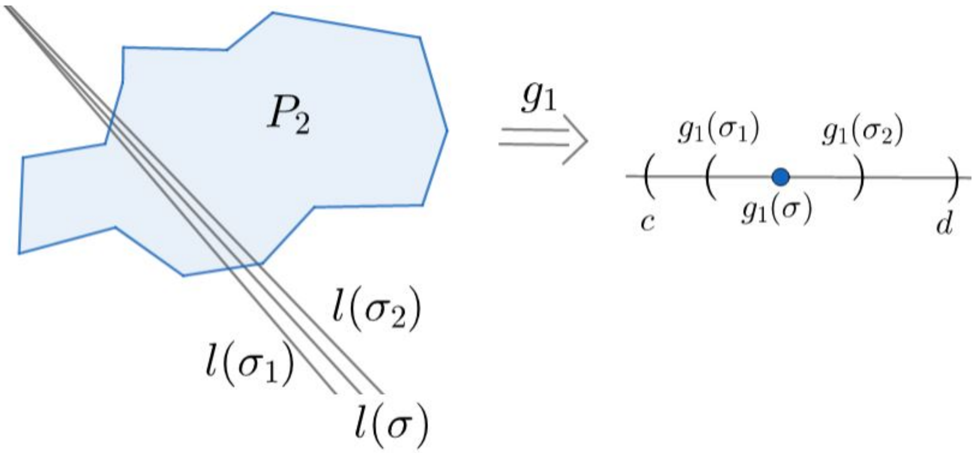


Figure 14: Continuous function g_1

We have shown $h_1 : S^1 \to [0, 2\pi)$ and $g_1 : [0, 2\pi) \to \mathbb{R}$ are both continuous. Thus, $f_1 \coloneqq g_1 \circ h_1$ is also continuous being the composition of continuous functions. Note that $f_1 : S^1 \to \mathbb{R}$, and by the Borsuk-Ulam Theorem above, there must exist a $\mathbf{y} = (\cos\mu, \sin\mu) \in S^1$ so that $f_1(-\mathbf{y}) = f_1(\mathbf{y})$. Adjust μ as necessary so that $0 \leq \mu < 2\pi$. By definition of the function f_1, this implies there exists a $\mathbf{y}$ so that

$$
\begin{aligned}
f_1(-\mathbf{y}) &= f_1(\mathbf{y}) \\
g_1 \circ h_1(-\mathbf{y}) &= g_1 \circ h_1(\mathbf{y}) \\
g_1 \circ h_1(-(\cos\mu, \sin\mu)) &= g_1 \circ h_1(\cos\mu, \sin\mu) \\
g_1 \circ h_1(-\cos\mu, -\sin\mu)) &= g_1 \circ h_1(\cos\mu, \sin\mu) \\
g_1 \circ h_1(\cos(\pi+\mu), \sin(\pi+\mu)) &= g_1 \circ h_1(\cos\mu, \sin\mu).
\end{aligned}
$$

If we adjust $\pi + \mu$ in the usual way as necessary to be μ', then

$$\begin{aligned} g_1 \circ h_1(\cos(\pi + \mu), \sin(\pi + \mu)) &= g_1 \circ h_1(\cos \mu, \sin \mu) \\ g_1 \circ h_1(\cos(\mu'), \sin(\mu')) &= g_1 \circ h_1(\cos \mu, \sin \mu) \\ g_1(\mu') &= g_1(\mu) \\ \text{Area}(P_2 \cap R[l(\mu')]) &= \text{Area}(P_2 \cap R[l(\mu)]). \end{aligned}$$

Furthermore, this means there exists a line $l(\mu)$ so that

$$\text{Area}(P_2 \cap L[l(\mu)]) = \text{Area}(P_2 \cap R[l(\mu)]).$$

Since line $l(\mu)$ by construction also bisects P_1, we have determined a line $l(\mu)$ that simultaneously bisects both P_1 and P_2 into regions of equal area.

$\square$

In Figure 15, we see an example of two disjoint subsets in $\mathbb{R}^2$ with 12 lines that bisect P_1 into regions of equal area. Specifically, we will consider $l(0), l(\pi/6), l(\pi/3)$, $l(\pi/2), \ldots, l(5\pi/3)$, and $l(11\pi/6)$. Though each of these lines bisects P_1, their effect on P_2 differs.

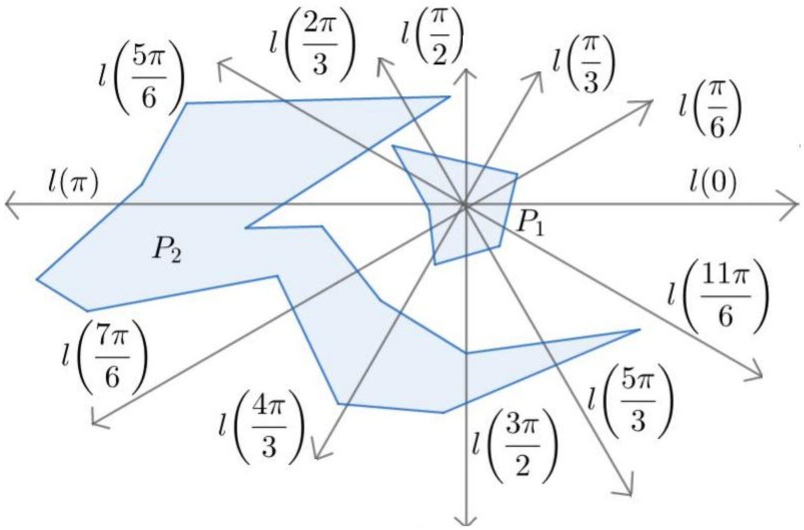


Figure 15: Example of disjoint $P_1, P_2 \in \mathbb{R}^2$ with 12 lines that bisect P_1

Note that the image of g_1 (Area$(P_2 \cap R[l(\phi)])$) has an upper bound $b \leq$ Area(P_2) and a lower bound $a \geq 0$. In Figure 16, we see an approximation of the relationship between the pre-image and image of g_1 based on the example in Figure 15. Every such graph will have the same bounded and continuous features.

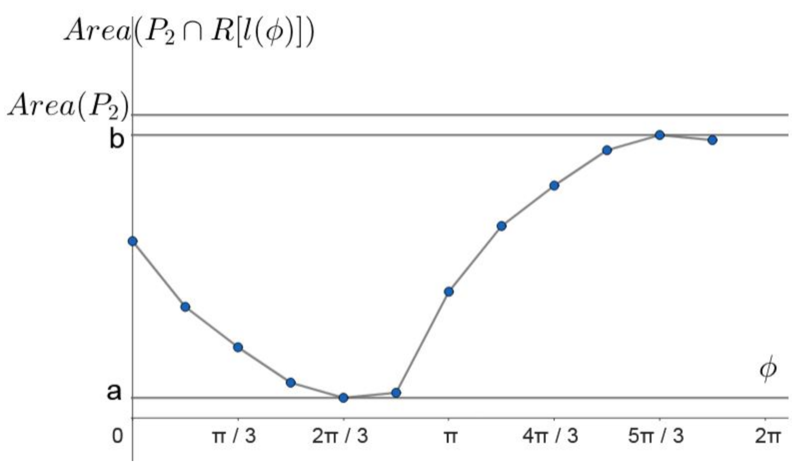
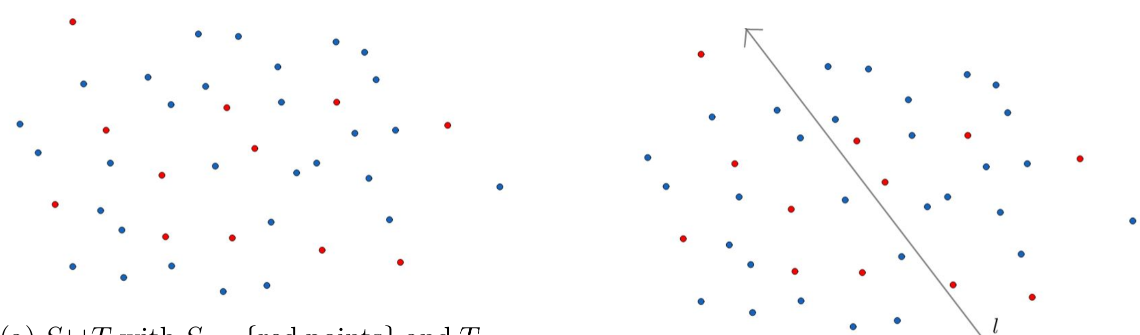


Figure 16: Approximate relationship between pre-image and image of g_1 based on Figure 15

2.3 Generalizations of the Discrete Pancake Theorem

In Theorem 2.3, the continuous version of the Pancake Theorem is proven. Instead of determining a line that simultaneously balances areas of two disjoint sub-regions P_1 and P_2, the discrete version of the Pancake Theorem considers two disjoint sets of points S and T in a plane with no three collinear points, and the determination of a line that simultaneously balances $|S|$ and $|T|$ to within an error of at most one. Though the continuous version of the Pancake Theorem is more well-known, utilizing the Pancake Theorem in a discrete manner is not uncommon, and certain generalizations of the Discrete Pancake Theorem have been established.

Both generalizations of the Discrete Pancake Theorem that follow involve separating a plane containing two disjoint sets of points into partitions which contain a specified number of points from each set. In [9], two disjoint sets of points S and T in a plane are considered, where $S \cup T$ contains no three collinear points. The number of points in S and T are constrained to be of the form $|S| = 2q$ and $|T| = mq$, where $m \geq 2$ and $q \geq 1$ are integers. It is proven that the collection of points $S \cup T$ can be partitioned into q disjoint convex hulls so that each convex hull contains two points from S and m points from T.



(a) $S \cup T$ with $S = \{$red points$\}$ and $T = \{$blue points$\}$

(b) Bisecting line l for $S \cup T$

Figure 17: Example for generalization in [9]

In Figure 17a, a plane has been generated with S representing the set of all red points and T representing the set of all blue points. By construction, S and T are disjoint sets. Furthermore, $|S| = 12$ with $q = 6$ and $|T| = 30$ with $m = 5$.

When considering q being even in the inductive step of the proof in [9], the Pancake Theorem was used to show there exists at least one line simultaneously bisecting $|S|$ and $|T|$. An example of this is shown in Figure 17b. Note that the number of red points to the right and to the left of the directed line l is $S \cap R(l) = S \cap L(l) = 6$, and that the number of blue points to the right and to the left of the directed line l is $T \cap R(l) = T \cap L(l) = 15$. Kaneko and Kano [9] then utilized a strong inductive hypothesis to prove the desired disjoint convex hulls exist in the case where q is even.

Figure 18 demonstrates the desired partitioning of the example plane into six disjoint convex hulls, where each convex hull contains two red points and five blue points.

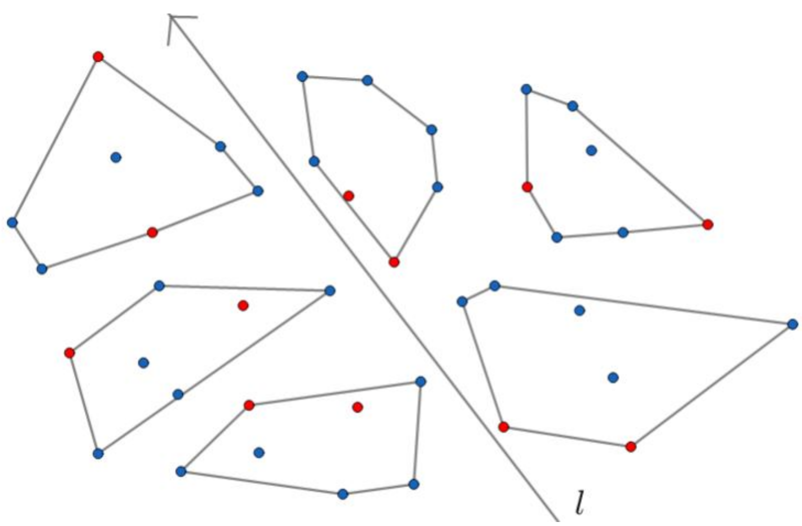


Figure 18: Desired partitions in example for generalization in [9]

For the inductive step where q is odd, Kaneko and Kano [9] demonstrated the desired convex hulls exist through the exploration of cases and the use of other proven lemmas. Thus, with certain given constraints, [9] is able to show that the Discrete Pancake Theorem can be generalized.

An even more generalized version of the Discrete Pancake Theorem is given in [2]. The same noncollinear assumptions are made about the points in the plane, though in this generalization $|S| = gn$ and $|T| = gm$, where $m \geq 2, n \geq 2$, and g is a positive integer. Note these constraints are similar to those in [9], though in [9] the value of n was pre-determined to be two. A similar goal is to be achieved, namely separating the points in the plane into g disjoint convex polygons so that each disjoint convex polygon contains n points from S and m points from T.

Bespamyatnikh, Kirkpatrick, and Snoeyink [2] make use of what they call an *equitable 2-cutting* and an *equitable 3-cutting*, with the latter having greater influence in the remainder of the proof. An equitable 3-cutting separates the plane into three partitions by constructing three rays from a common endpoint. Furthermore, each ith partition contains a proportional number of points from S and points from T, namely $g_i n$ points from S and $g_i m$ points from T. It naturally follows that for these $0 \leq g_1, g_2, g_3 < g$, we have $g_1 + g_2 + g_3 = g$. A critical theorem in [2] - Theorem 2 - proves there exists an equitable 3-cutting of $S \cup T$ for any gn points from S and gm points from T where $g \geq 2$. To prove this critical theorem, Bespamyatnikh et al. [2] utilizes the Pancake Theorem when $g = 2$, and a combination of lemmas for all other cases.

To prove the desired result, Bespamyatnikh et al. [2] explain that Theorem 2 can be used recursively in each subsequent partition to attain the outcome where each ith partition contains n points from S and m points from T. In the example below, we consider where $g = 10$, $n = 3$, and $m = 4$. As a result, $|S| = 30$, $|T| = 40$, and our goal is to divide the set of points into $g = 10$ partitions such that each partition contains three points from the set S and four points from the set T. In Figure 19, we see $S \cup T$, where as before S is the set of all red points and T is the set of all blue points.

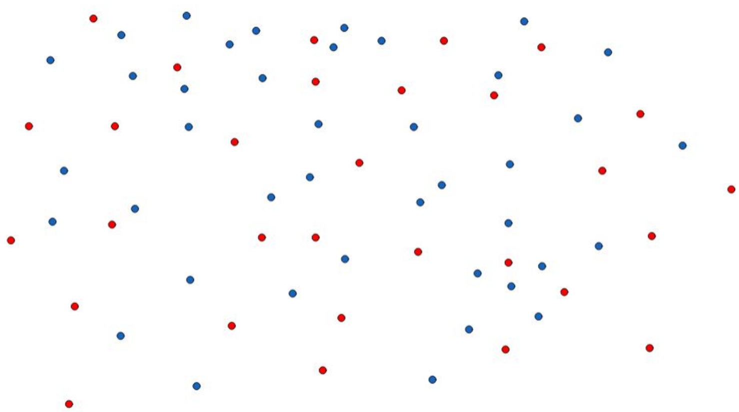


Figure 19: Example for generalization in [2]

By Theorem 2 in [2], we know there exists an equitable 3-cutting. In Figure 20, the three rays with a common vertex p generate the three wedges W, Y, and Z.

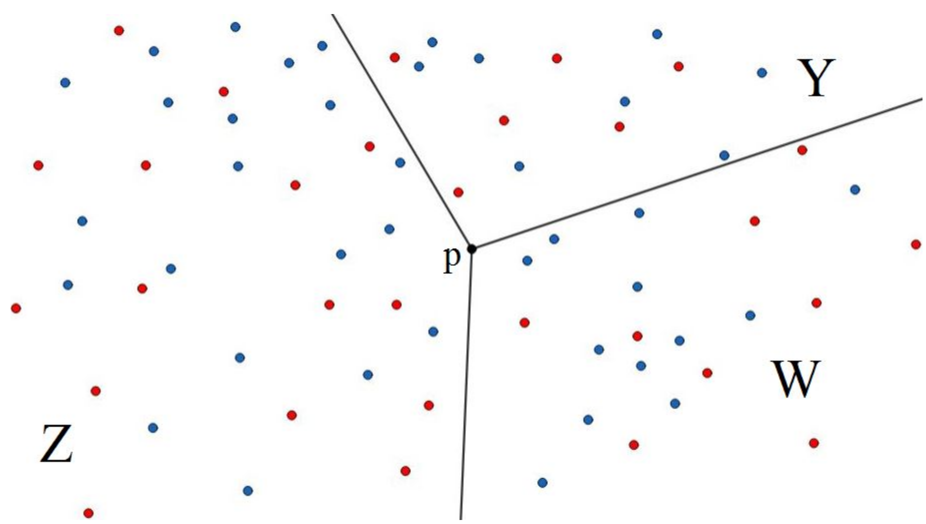


Figure 20: Equitable 3-cutting in example for generalization in [2]

Note that wedge W has a g_1 value of $g_1 = 3$, as $n = 3$ and the number of red points in W is

$$|S_W| = g_1 \cdot n = 3 \cdot 3 = 9.$$

In a similar fashion, $m = 4$ and the number of blue points in W is

$$|T_W| = g_1 \cdot m = 3 \cdot 4 = 12.$$

Note also that Y has a g_2 value of $g_2 = 2$, and the number of red points in this wedge is

$$|S_Y| = g_2 \cdot n = 2 \cdot 3 = 6.$$

The number of blue points in Y is $|T_Y| = 8$, consistent with the value $m = 4$.

Finally, consider wedge Z and its g_3 value of $g_3 = 5$. The number of red points in Z is

$$|S_Z| = g_3 \cdot n = 5 \cdot 3 = 15,$$

and the number of blue points in Z is

$$|T_Z| = g_3 \cdot m = 5 \cdot 4 = 20.$$

Thus, each wedge consists of a proportional number of red and blue points, particularly in the red to blue ratio of $3 : 4$. For $g_1 = 3$, $g_2 = 2$, $g_3 = 5$, and $g = 10$, we have both $0 \leq g_1, g_2, g_3 < g$ and $g_1 + g_2 + g_3 = g$. By definition, wedges W, Y, and Z seen in Figure 20 are equitable.

We can now perform a recursive process on the remaining wedges W, Y, and Z. In particular, since $g_2 = 2$ for Y, the Pancake Theorem can be implemented to determine an equitable 2-cutting. Wedge Y, with sub-wedges Y_1 and Y_2, now satisfies the goal of containing $n = 3$ points of S and $m = 4$ points of T in each partition. This can be seen in Figures 21 and 22.

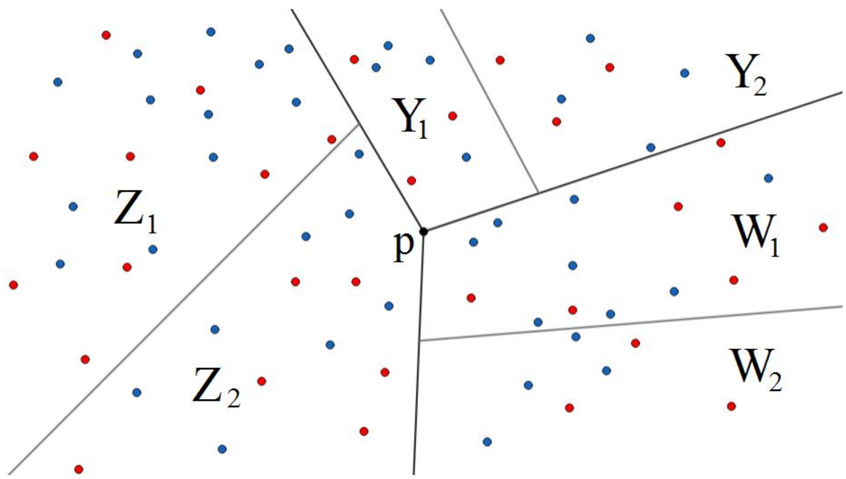


Figure 21: Recursive equitable 3-cutting in example for generalization in [2]

Bespamyatnikh et al. noted in [2] that any equitable 2-cutting can also be considered as an equitable 3-cutting. As can be seen in Figure 21, an equitable 3-cutting in wedge W and an equitable 3-cutting in wedge Z can separate wedge W into two equitable sub-wedges, W_1 and W_2, and can separate wedge Z into two equitable sub-wedges, Z_1 and Z_2. Sub-wedge W_2 now has the desired proportion and number of red and blue points.

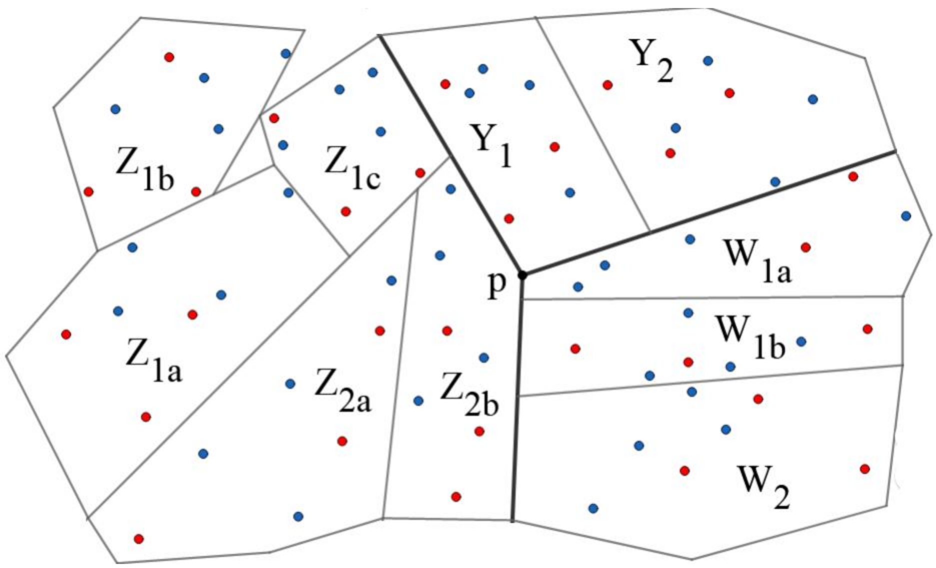


Figure 22: Desired partitioning in example for generalization in [2]

One final recursive step can be taken in our example to establish the desired partitioning of $S \cup T$ into ten disjoint convex polygons, where each convex polygon contains three red points and four blue points. In particular, an equitable 3-cutting and an equitable 2-cutting in wedge Z determine appropriately balanced polygons $Z_{1a}, Z_{1b}, Z_{1c}, Z_{2a}$, and Z_{2b}. An equitable 2-cutting in wedge W_1 does the same by establishing balanced polygons W_{1a} and W_{1b}. Figure 22 demonstrates this desired partitioning of the work by Bespamyatnikh et al. [2]. Though there still are certain constraints for the number of points in the disjoint sets S and T, Bespamyatnikh et al. succeeded in establishing a generalization for the Discrete Pancake Theorem.

2.4 Packing and Cracking Gerrymander Examples

A recent survey of this topic summarizes the result from [2] in a discussion about the possibilities of drawing biased political maps, often called *gerrymandering*. *Packing* and *cracking* are described in [22] as the two forms of gerrymandering which can be implemented. In particular, packing involves "concentrating a group in a single district where they win by a large margin, thereby minimizing the impact of their votes [22]." Contrast this with cracking, which "refers to dispersing a group across many districts, thereby diluting the impact of their votes [22]." The results from [9] and [2] would exemplify the cracking approach to gerrymandering, as the majority group would be able to hold the proportional majority in each and every district, leaving the minority group with a dilution of their vote.

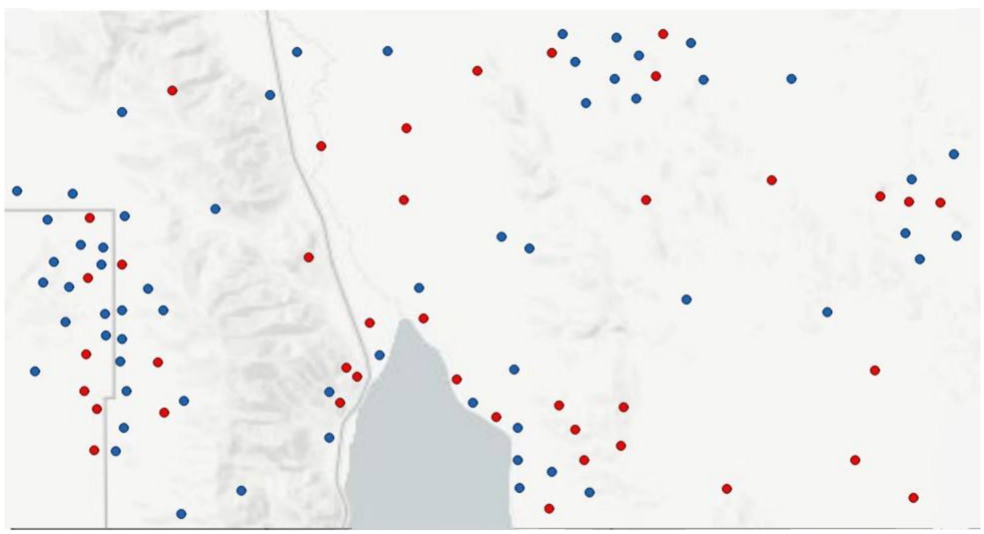


Figure 23: Hypothetical region with two distinct political choices

Consider the hypothetical region shown in Figure 23, where red and blue points represent individuals from two different political parties. Suppose we wanted to divide this region into six districts, where each district will determine a representative for the citizens within its boundary. There are many options to do such a task, though we will adhere to the usual constraint that the population of each district must be either equal or reasonably close. In this example, we will consider "equal or reasonably close" to be the state legislative general range constraint of 10% based on the Supreme Court ruling in *Brown v. Thomson* [3]. Let n be the total number of points in a given region. When separating a region into d districts, the number of points within each district must consequentially be between

$$\frac{n}{d} - \frac{0.1n}{2} \quad \text{and} \quad \frac{n}{d} + \frac{0.1n}{2},$$

as the maximum range between the number of points would then be

$$\begin{aligned}\left(\frac{n}{d} + \frac{0.1n}{2}\right) - \left(\frac{n}{d} - \frac{0.1n}{2}\right) &= \frac{n}{d} + \frac{0.1n}{2} - \frac{n}{d} + \frac{0.1n}{2} \\ &= 0.1n.\end{aligned}$$

Note that the number of points in the region in Figure 23 is $n = 100$, where the number of red points is $|S| = 40$ and the number of blue points is $|T| = 60$. Therefore, for each of the $d = 6$ districts that will be created, we will expect the number of points to be between

$$\begin{array}{rcl} \dfrac{n}{6} - \dfrac{0.1n}{2} & \text{and} & \dfrac{n}{6} + \dfrac{0.1n}{2} \\ \dfrac{100}{6} - \dfrac{0.1(100)}{2} & \text{and} & \dfrac{100}{6} + \dfrac{0.1(100)}{2} \\ 16.\overline{6} - 5 & \text{and} & 16.\overline{6} + 5 \\ 11.\overline{6} & \text{and} & 21.\overline{6} \end{array}$$

In the three examples below, certainly districts 1-6, A-F, and i-vi contain an appropriate number of points. Furthermore, an attempt was made to make the district populations as equal as possible.

The terms *packing* and *cracking* were previously discussed as two forms of gerrymandering. We will next show an example of each with the hypothetical region in Figure 23, where we want to split the region into six districts. First, consider the example of gerrymandering by packing shown in Figure 24.

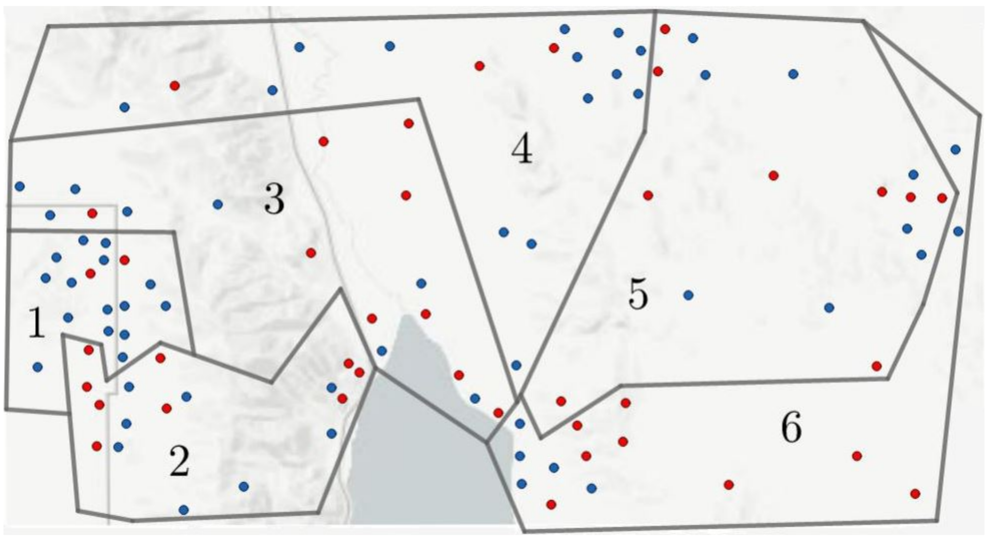


Figure 24: Gerrymandering by packing example

Table 6: Summary of packed gerrymander case from Figure 24

District	No. of red points	No. of blue points	Majority Choice
1	2	15	Blue
2	9	8	Red
3	9	8	Red
4	3	14	Blue
5	9	8	Red
6	8	7	Red

Districts 1 and 4 are particularly overloaded with blue dots, allowing the red choice to maintain a majority in the other four districts. As a result, although there are

60 blue points and 40 red points, gerrymandering by packing results in four districts being represented by the red choice and only two districts being represented by the blue choice. A summary of these results is found in Table 6. It is clear to see that the districts drawn in Figure 24 are partisan in favor of red.

Next we will consider an alternative redistricting plan in Figure 25, where this plan represents gerrymandering by cracking.

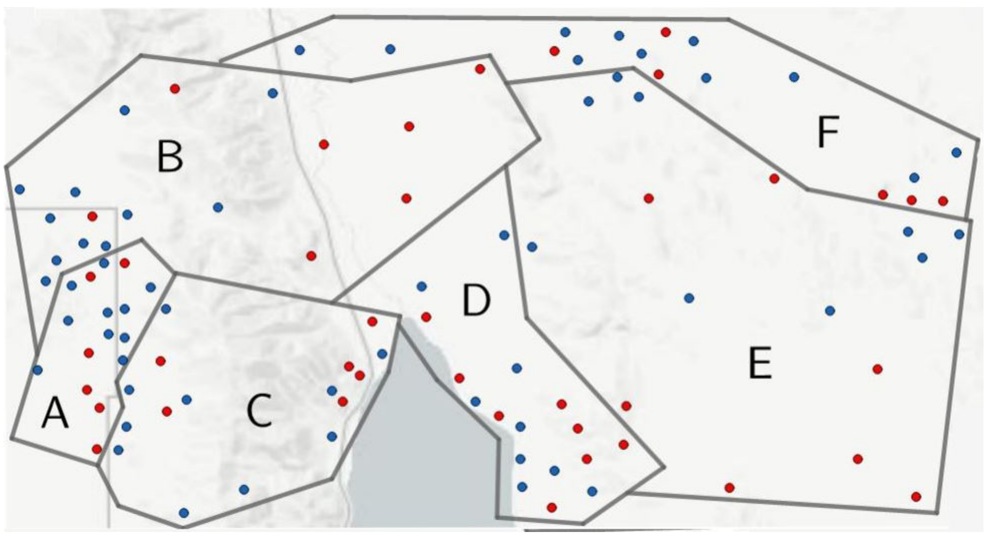


Figure 25: Gerrymandering by cracking example

Table 7: Summary of cracked gerrymander case from Figure 25

District	No. of red points	No. of blue points	Majority Choice
A	6	10	Blue
B	7	11	Blue
C	6	10	Blue
D	8	9	Blue
E	7	9	Blue
F	6	11	Blue

As mentioned previously, gerrymandering by cracking relates to the work in [9] and [2], where the minority choice numbers are spread out proportionally to the extent where they receive no representation in the region. Table 7 shows this result, where we can see that every district almost proportionally chose blue over red. Clearly, the districts drawn in Figure 25 are partisan in favor of blue.

Note that in both examples above, many of the districts are irregular and concave. Some of the districts, such as District 6, resemble animals or objects. In fact, the term *gerrymander* came from a discussion in Massachusetts in 1812 regarding district maps and their strange shapes. In [8], a description of a dinner party is detailed, where guests were mocking the shapes of the local districts. Some were likening the shapes to flying creatures, whereas "when a name for the figure was called for, some one proposed the term salamander [8]." At the time, the governor of Massachusetts was Elbridge Gerry, a member of the Continental Congress and a signer of the Declaration of Independence. As governor, Gerry approved the apportionment law of 1812 for the state of Massachusetts, and it appeared the districts were created to support the party to which Gerry belonged. As a result, the term *gerrymander* was developed and became well-known [8].

2.5 Geographic Compactness Measures

In order to draw six districts for the hypothetical region in Figure 23 without partisan bias, we would presume that our outcome would be generally proportional to the ratio of blue points to red points. In particular, since there are 60 blue points and 40 red points, we would expect there to be either three or four of the six districts with a blue majority. Furthermore, we will want to consider measures of geographic compactness such as the Reock score and the Polsby-Popper score. The Reock score is a measure of geographic compactness that is the ratio formed by dividing the area of the district by the area of its circumscribing circle. As a result, the Reock score of a district is between zero and one, where a higher Reock score is a result of the district being more circular [17, 26].

The Polsby-Popper score is also a measure of geographic compactness. Popper and Polsby contend in [17] that the Reock score is not able to take into account rough edges of a district. Its definition, they argue, would "register the silhouette of a circular saw blade as almost perfectly compact. Those serrated edges, which could be quite useful to a gerrymanderer, would essentially be ignored [17]." The Polysby-Popper score, as a result, considers the ratio of the area of the district to the area of a differently defined circle. In particular, this score is "determined by dividing the area of the shape by the area of a circle with a perimeter of equal length [17]." In other words, it is the ratio of the area of the district to the area of a circle with a circumference equal to the perimeter of the district. This score is then better able to judge the smoothness of the district boundary lines. As before, the resulting values are between zero and one, with values closer to one representing greater compactness. A site providing both descriptions and helpful visualizations of the Reock and Polsby-Popper scores can be found at [12].

Using this information about compactness, we can attempt to draw six districts for the hypothetical region that are not gerrymandered. Furthermore, by the metrics in [26], we will also attempt to keep communities together and counties in the same district. Note in Figure 26 that half of the districts are convex polygons, in contrast to the other two gerrymander examples where every district was a concave polygon. By the definitions of the Reock and Polsby-Popper scores, concave polygonal districts would necessarily have lower compactness scores relative to the compactness scores of similar convex polygonal districts. Further note in Figure 26 that towns are kept together in the same district as much as possible, in contrast to the gerrymander cases.

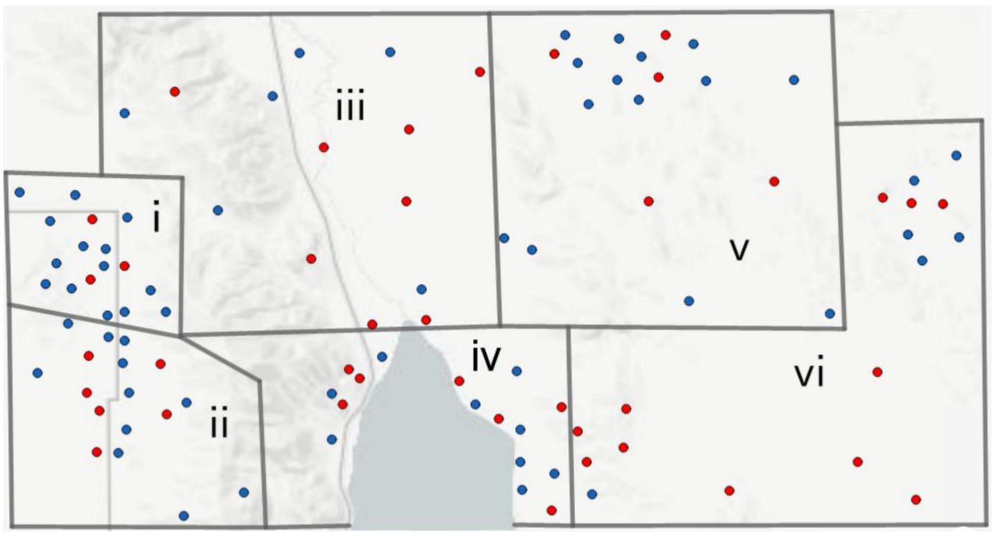


Figure 26: Non-partisan and compact districting example

Table 8: Summary of non-partisan and compact case from Figure 26

District	No. of red points	No. of blue points	Majority Choice
i	3	14	Blue
ii	6	11	Blue
iii	8	6	Red
iv	7	9	Blue
v	5	14	Blue
vi	11	6	Red

It also appears that the division lines could be interpreted as county lines, where in the gerrymander figures there appears to be many cases of split counties. Finally, it appears in Figure 26 that the representatives from each district would understand who they are representing. For example, District iv generally represents those who live by the lake, District i and ii generally represent those who live in the city limits of the city on the left side of the figure, and District iii generally represents those who live in the rural areas between the two larger cities.

The summary of Figure 26 located in Table 8 indicates that the region satisfies equal population requirements among districts. Furthermore, the majority choice being blue in four districts agrees with the ratio of 60 blue points to 40 red points.

2.6 Further Research into Gerrymandering

Research into the topic of gerrymandering consists of both mathematical scholarship such as the work being performed at the MGGG Redistricing Lab, as well as groups performing metric analyses on district maps that are currently functioning. The MGGG Redistricting Lab [14], which is affiliated with Tisch College of Tufts University, provides information articles regarding the mathematics of gerrymandering, research articles, reports, publicly-available election data, code, and interactives meant to "put the tools of redistricting in the hands of the public [13]."

The Gerrymandering Project at Princeton University [26] allows participants to explore metrics of the current state house, state senate, and congressional maps from each state of the United States of America. Metrics provided include: partisan fairness, number of competitive districts, number of county splits, partisan composition, minority composition, metrics related to Reock scores and Polsby-Popper scores, and a split pair metric. Methodologies for the metrics are explained, and the data presented can be downloaded. Proposed redistricting plans are graded, and the Princeton Gerrymandering Project identifies whether a plan is fair, scores well on compactness metrics, or if a plan allows for competitive districts. In Wisconsin, as an example, there were 23 redistricting plans proposed by various entities within the government of Wisconsin between October 2021 and March 2022. The Gerrymandering Project at Princeton University was able to grade each of these plans in such a timely fashion as to be potentially useful in the Wisconsin redistricting debate [26].

Quantifying Gerrymandering, affiliated with Duke University, is another research group focused on the discussion of gerrymandering. The articles generated by members of this research group analyze and evaluate selected state redistricting plans, comment on gerrymandering court cases, and discuss various mathematical and data science considerations as they relate to redistricting and gerrymandering [18].

3 The Dual-Balanced Theorem

Before arriving at the main result of this paper, multiple lemmas will be proven with regard to an initial weight-one balancing process and the rotation of a directed line in a region with the usual conditions. The consequences of these proofs will assist in accomplishing the main result of this paper, namely proving that any region with the usual conditions can be dual-balanced by at least one directed line. Dual-balance will not be considered, however, until weight one balance can be proven to be preserved throughout a 180° rotation.

3.1 Initial Weight-One Balancing Considerations

The first of three lemmas regarding an initial weight-one balancing is given below.

Lemma 3.1

Any vertical directed line that has undergone an initial weight-one balancing process in a region with the usual conditions balances weight one values.

Proof. Let R be a region with the usual conditions, and let the initial weight-one balancing process be performed, where l is the vertical directed line directed up that contains α_p. Recall the consequential inequalities of an initial weight-one balancing below.

$$S'_{1a} + w_{1p} > S'_{1b} \qquad \text{and} \qquad S'_{1b} + w_{1p} \geq S'_{1a} \tag{1}$$

We will consider the two cases of $S'_{1a} \leq S'_{1b}$ and $S'_{1a} > S'_{1b}$.

Case I: $S'_{1a} \leq S'_{1b}$

If $S'_{1a} \leq S'_{1b}$, we know by Definition 1.6 that w_{1p} is assigned to S_{1a}. As a result, we have now assigned every point in region R, with

$$S_{1a} = S'_{1a} + w_{1p} \quad \text{and} \quad S_{1b} = S'_{1b}. \tag{2}$$

Since $S'_{1a} \leq S'_{1b}$ in this case, and as $w_{1p} \leq \varepsilon_1$, it follows by use of (1) that

$$S'_{1b} < S'_{1a} + w_{1p} \leq S'_{1b} + \varepsilon_1.$$

Substitution from (2) yields

$$\begin{aligned} S_{1b} < S_{1a} \leq S_{1b} + \varepsilon_1 \\ 0 < S_{1a} - S_{1b} \leq \varepsilon_1, \end{aligned}$$

implying $|x_1| = |S_{1a} - S_{1b}| \leq \varepsilon_1$, and we have shown balance with respect to weight one values by Definition 1.1.

Case II: $S'_{1a} > S'_{1b}$

If $S'_{1a} > S'_{1b}$, Definition 1.6 dictates that w_{1p} is assigned to S_{1b}. In this case, our values for S_{1a} and S_{1b} are given as

$$S_{1a} = S'_{1a} \quad \text{and} \quad S_{1b} = S'_{1b} + w_{1p}. \tag{3}$$

Since $S'_{1a} > S'_{1b}$, and as $w_{1p} \leq \varepsilon_1$, it follows by use of (1) that

$$S'_{1a} \leq S'_{1b} + w_{1p} < S'_{1a} + \varepsilon_1.$$

By substitution from (3),

$$
\begin{aligned}
S_{1a} \leq S_{1b} &< S_{1a} + \varepsilon_1 \\
0 \leq S_{1b} - S_{1a} &< \varepsilon_1 \\
-\varepsilon_1 < S_{1a} - S_{1b} &\leq 0,
\end{aligned}
$$

implying $|x_1| = |S_{1a} - S_{1b}| \leq \varepsilon_1$, and balance with respect to weight one values has been achieved. Thus, we have shown that any directed line l that has undergone an initial weight-one balancing process in a region with the usual conditions balances weight one values. $\square$

We will now show, more particularly, that the vertical directed line generated by the initial weight-one balancing process minimizes $|x_1| = |S_{1a} - S_{1b}|$ for vertical directed lines. Note that, without loss of generality, we only need to consider vertical directed lines directed up, as each corresponding vertical directed line directed down has the same value of $|x_1|$.

Lemma 3.2

The vertical directed line generated by the initial weight-one balancing process in a region with the usual conditions minimizes the weight one absolute difference $|x_1| = |S_{1a} - S_{1b}|$ *for all vertical directed lines in the region.*

Proof. Let T be a region with the usual conditions, and let line n be the directed line generated by an initial weight-one balancing process. By Definition 1.6, line n contains α_p, and α_p is either assigned to the left side of the directed line (S_a) or to the right side of the directed line (S_b).

Suppose, for the sake of argument, that line n does not minimize $|x_1|$ for vertical directed lines in region T. Then there would exist another vertical directed line m with an absolute difference closer to zero. To differentiate, let $|x_n|$ denote the weight one absolute difference generated by the directed line n, and let $|x_m|$ denote the weight one absolute difference generated by the directed line m. Without loss of generality, we will choose directed line m so that it does not intersect any points. We will also horizontally translate line n either slightly to the left or slightly to the right of α_p so that line n does not cross any other points and $|x_n|$ is not altered.

Define U_{nR} to be the set of all points strictly to the right of directed line n, and define U_{nL} to be the set of all points strictly to the left of line n. Furthermore, define V_{mR} to be the set of all points strictly to the right of directed line m, and similarly define V_{mL} to be the set of all points strictly to the left of line m.

Without loss of generality, suppose α_p is assigned to S_a, implying $S'_{1a} \leq S'_{1b}$ by Definition 1.6. Recall that an initial weight-one balancing process generates the inequality $S'_{1a} + w_{1p} > S'_{1b}$. This implies

$$S'_{1a} + w_{1p} - S'_{1b} > 0, \tag{4}$$

and

$$\begin{aligned} S'_{1a} &\leq S'_{1b} \\ S'_{1a} - S'_{1b} &\leq 0 \\ S'_{1a} + w_{1p} - S'_{1b} &\leq w_{1p}. \end{aligned} \tag{5}$$

Since α_p is assigned to S_a, the weight one difference generated by the directed line n is $x_n = S'_{1a} + w_{1p} - S'_{1b}$, which we have shown in lines (4) and (5) to have a value in the interval below.

$$0 < x_n \leq w_{1p} \tag{6}$$

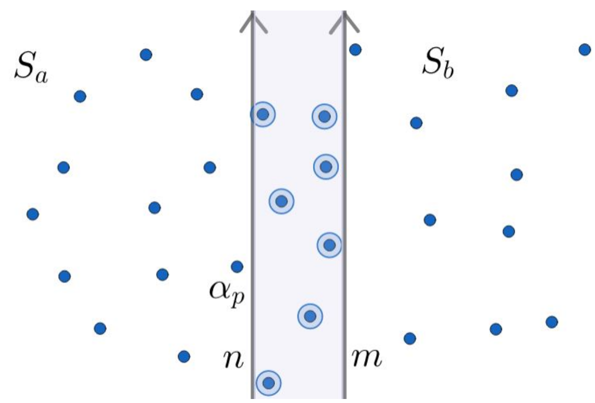


Figure 27: Shaded region $U_{nR} \cap V_{mL} \neq \emptyset$, where m is to the right of n

Suppose the directed line m is to the right of n. Since $|x_m| < |x_n|$ by assumption, it follows that $U_{nR} \neq V_{mR}$. In particular, $U_{nR} \cap V_{mL} \neq \emptyset$, and $U_{nR} \cap V_{mL}$ contains some point α_R with a positive weight one value w_{1R}. The shaded region in Figure 27 is an illustration of nonempty $U_{nR} \cap V_{mL}$ with $\alpha_p \in S_a$. It follows that the value of $|x_m|$ would have the relationship below.

$$\begin{aligned} |x_m| &\geq |S'_{1a} + w_{1p} + w_{1R} - (S'_{1b} - w_{1R})| \\ &= |S'_{1a} + w_{1p} - S'_{1b} + 2w_{1R}| \\ &= |x_n + 2w_{1R}| \end{aligned}$$

Since $2w_{1R} > 0$ and $x_n > 0$ in this case,

$$|x_m| \geq |x_n| + 2w_{1R},$$

implying $|x_m| > |x_n|$, and contradicting our assumption $\rightarrow\leftarrow$.

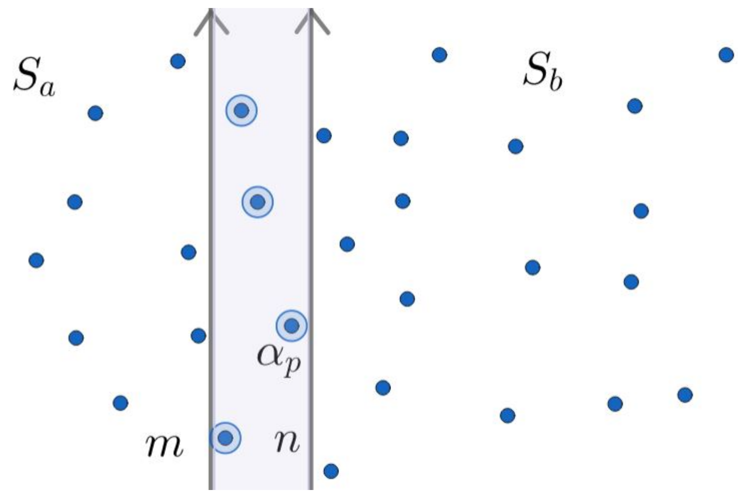


Figure 28: Shaded region $U_{nL} \cap V_{mR} \neq \emptyset$, where m is to the left of n

Suppose the directed line m is to the left of line n. As before, we can use our assumption $|x_m| < |x_n|$ to conclude $U_{nL} \cap V_{mR} \neq \emptyset$. Thus, $U_{nL} \cap V_{mR}$ contains some point α_L with a positive weight one value w_{1L}. Note that $\alpha_p \in U_{nL} \cap V_{mR}$ in this case, as demonstrated in Figure 28. We also still have the relationships $S'_{1a} - S'_{1b} \leq 0$ by assumption and $0 < x_n \leq w_{1p}$ from (4) and (5).

The line m will certainly assign w_{1p} to S_{1b}, and since $S'_{1a} - S'_{1b} \leq 0$, the weight one difference

$$\begin{aligned} x_m &\leq S'_{1a} - (S'_{1b} + w_{1p}) \\ &= S'_{1a} - S'_{1b} - w_{1p} \\ &\leq -w_{1p}. \end{aligned}$$

Thus, the weight one absolute difference generated by the line m will be

$$|x_m| \geq w_{1p},$$

where $|x_n| \leq w_{1p}$. But this contradicts the claim that $|x_m| < |x_n| \rightarrow\leftarrow$. A similar, mirrored argument would show similar contradictions in the case where α_p was assigned to S_b. As a result, we have proven that the directed line generated by the initial weight-one balancing process in a region with the usual conditions minimizes $|x_1|$ for vertical directed lines. $\square$

As a consequence of Lemma 3.2, any other vertical directed line in a region with the usual conditions with the same weight one absolute difference as the vertical directed line generated by the initial weight-one balancing process also minimizes $|x_1|$ for vertical directed lines.

Lemma 3.3

At least two vertical directed lines directed up in a region with the usual conditions contain a point and minimize the weight one absolute difference for vertical directed lines.

Proof. Let region T_1 have the usual conditions, and let n_1 be the vertical directed line generated by an initial weight-one balancing process. By Lemma 3.2, we know n_1 minimizes the weight one absolute difference for vertical directed lines. Without loss of generality, suppose α_p was assigned to S_a by the initial weight-one balancing process. Horizontally translate n_1 to the right to contain the next point, and define this vertical directed line as n_1'. Maintain weight assignments for α_p and for the point $\alpha_{n_1'}$ contained on vertical directed line n_1'.

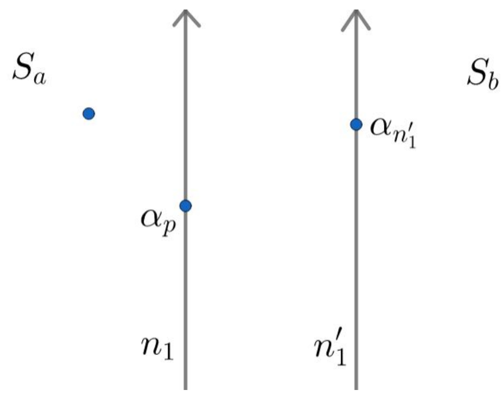


Figure 29: Two vertical directed lines minimizing $|x_1|$ for vertical directed lines in T_1

As a result, the weight one absolute difference of T_1 generated by n_1 would be equivalent to the weight one absolute difference of T_1 generated by n_1', and thus we have shown there exists at least two vertical directed lines directed up in a region with the usual conditions that both contain a point and minimize the weight one absolute difference for vertical directed lines.

□

3.2 Maintaining Balance with a Rotating Directed Line

Transitioning from the horizontal shifts of the initial weight-one balancing process, we will next rotate our directed line clockwise about the point α_p. In general, rotating clockwise implies rotating about a point. When two points α_y and α_z lie on the directed line, the directed line may exclusively either continue to rotate clockwise about α_y or about α_z.

While the proof below is more general, keep in mind that we will later assume an initial weight-one balancing process has occurred. As a result, we will begin the clockwise rotation process about α_p, and hence begin the clockwise rotation process when weight one values are balanced.

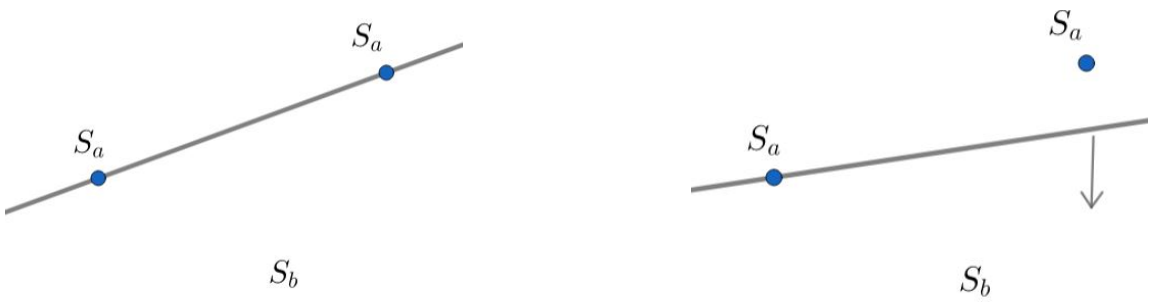
Theorem 3.4

Consider any directed line p that balances a region with the usual conditions being rotated clockwise about a point β. If p intersects another point γ in its clockwise rotation, there exists a clockwise rotational path about either β or γ so that balance is maintained.

Proof. Let T_2 be a region with the usual conditions, and let n_2 be a directed line that balances T_2. Also assume n_2 is being rotating clockwise about a point β in T_2, and that n_2 intersects another point γ in its clockwise rotation. Four cases are to be considered regarding the assignments of β and γ. Since n_2 balances T_2, we only need to show that in each case, either a clockwise rotation can be performed so that no change in weight one absolute difference occurs, or a clockwise rotation can be performed so that the change in weight one absolute difference does not cause imbalance.

Case I: β and γ are both assigned to S_a

If the directed line n_2 intersects two points assigned to S_a, then maintain each assignment and continue to rotate clockwise about the point on the left. As we rotate clockwise, the point on the right will become fully contained in S_a, preserving S_a assignment. Thus, no change in the weight one absolute difference $|x_1|$ occurs in this case.



(a) Directed line n_2 intersects two points assigned to S_a

(b) Clockwise rotation action for Case I

Figure 30: β and γ are both assigned to S_a

<u>Case II: β and γ are both assigned to S_b</u>

If the directed line n_2 intersects two points assigned to S_b, then maintain each assignment and continue to rotate clockwise about the point on the right. The point on the left will become fully contained in S_b, preserving S_b assignment. No change in the weight one absolute difference $|x_1|$ occurs in this case as well.

(a) Directed line n_2 intersects two points assigned to S_b

(b) Clockwise rotation action for Case II

Figure 31: β and γ are both assigned to S_b

<u>Case III: Directed line n_2 intersects a point assigned to S_b on the left and a point assigned to S_a on the right</u>

If the directed line n_2 intersects a point assigned to S_b on the left and a point assigned to S_a on the right as in Figure 32, maintain each assignment and rotate clockwise about either point.

Figure 32: n_2 intersects a point from S_b on the left and a point from S_a on the right

If we continue the clockwise rotation about the point on the left, then the point on the right will become fully contained in S_a, preserving assignment and resulting in no change in the weight one absolute difference $|x_1|$. Figure 33a demonstrates this action.

(a) Clockwise rotation of n_2 about the point on the left

(b) Clockwise rotation of n_2 about the point on the right

Figure 33: Clockwise rotation actions for Case III

Similarly, if we continue the clockwise rotation about the point on the right, as is shown in Figure 33b, then the point on the left will become fully contained in S_b and preserve assignment. This also results in no change in the weight one absolute difference $|x_1|$.

Case IV: Directed line n_2 intersects a point assigned to S_a on the left and a point assigned to S_b on the right

If the directed line n_2 intersects a point assigned to S_a on the left and a point assigned to S_b on the right as in Figure 34, an issue arises. Recall that in the previous three cases, a clockwise rotation about a point could be determined so that the weight one absolute difference did not change. In this case, however, a clockwise rotation about either point is likely to alter the weight one absolute difference. As a result, we must assess three subcases. For these three subcases, let α_m be the point on the left and let α_n be the point of the right with respective weight one values w_{1m} and w_{1n}. By assumption, we know the weight one values are balanced prior to this case, implying

$$|x_1| = |S_{1a} - S_{1b}| \leq \varepsilon_2.$$

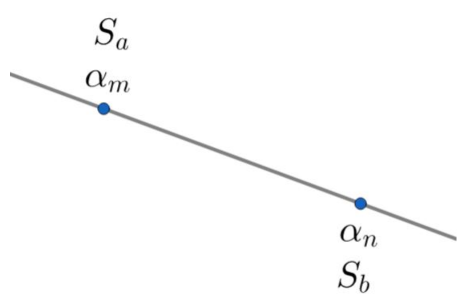


Figure 34: n_2 intersects a point from S_a on the left and a point from S_b on the right

Subcase IV.a: Maintaining assignment for α_m and rotating clockwise about α_m preserves balance

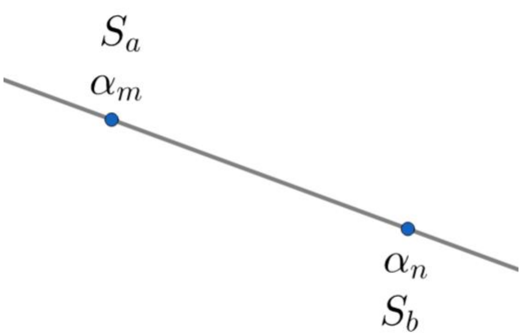
Maintaining assignment for α_m and rotating n_2 clockwise about α_m can possibly change $|x_1|$, as this action causes α_n to switch designation from S_b to S_a. Thus, the weight one value w_{1n} would be subtracted from S_{1b} and added on to S_{1a}. The new weight one difference, then, is

$$\begin{aligned} S_{1a} + w_{1n} - (S_{1b} - w_{1n}) &= S_{1a} + w_{1n} - S_{1b} + w_{1n} \\ &= S_{1a} - S_{1b} + 2 \cdot w_{1n} \\ &= x_1 + 2 \cdot w_{1n}. \end{aligned}$$

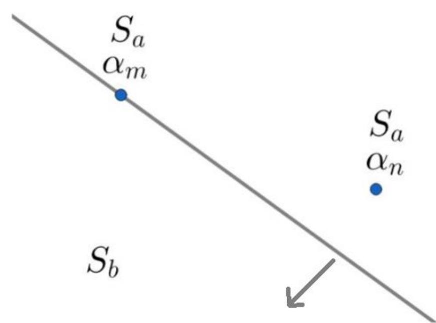
Assuming a preservation of balance with respect to weight one values, it must be the case that

$$x_1 + 2 \cdot w_{1n} \leq \varepsilon_1$$

and $|x_1 + 2 \cdot w_{1n}| \leq \varepsilon_1$. Note the visualization of this subcase in Figure 35b.



(a) Original picture for Case IV



(b) Ability to rotate n_1 clockwise about the left point and maintain balance

Figure 35: Clockwise rotation action for Subcase IV.a

<u>Subcase IV.b: Maintaining assignment for α_n and rotating clockwise about α_n preserves balance</u>

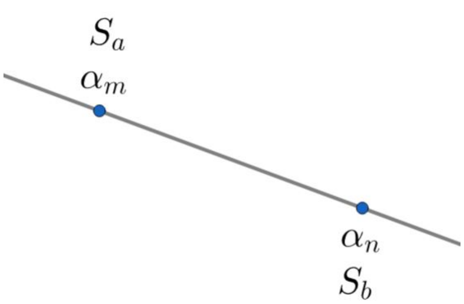
Maintaining assignment for α_n and rotating n_1 clockwise about α_n also can possibly change $|x_1|$, as α_m is forced to switch designation from S_a to S_b. The weight one value w_{1m} would be subtracted from S_{1a} and added on to S_{1b}, generating the new weight one difference below.

$$\begin{aligned} S_{1a} - w_{1m} - (S_{1b} + w_{1m}) &= S_{1a} - w_{1m} - S_{1b} - w_{1m} \\ &= S_{1a} - S_{1b} - 2 \cdot w_{1m} \\ &= x_1 - 2 \cdot w_{1m}. \end{aligned}$$

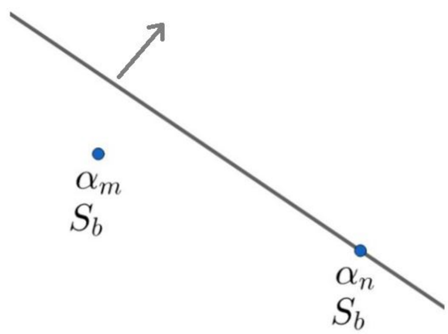
Weight one balance is assumed to be preserved, and so

$$x_1 - 2 \cdot w_{1m} \geq -\varepsilon_1$$

and $|x_1 - 2 \cdot w_{1m}| \leq \varepsilon_1$. This case is demonstrated in Figure 36b.



(a) Original picture for Case IV



(b) Ability to rotate n_1 clockwise about the right point and maintain balance

Figure 36: Clockwise rotation action for Subcase IV.b

<u>Subcase IV.c: Both Subcase IV.a and Subcase IV.b actions cause imbalance</u>

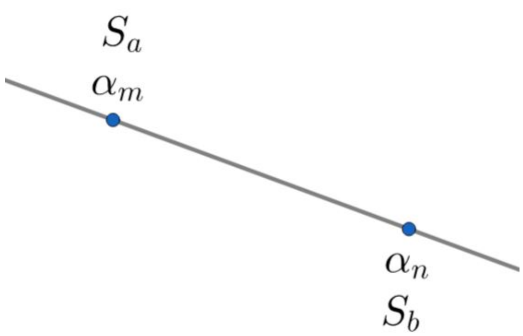
In the case where the Subcase IV.a and Subcase IV.b actions cause weight one imbalance, we know both

$$x_1 + 2 \cdot w_{1n} > \varepsilon_1 \tag{7}$$

and

$$x_1 - 2 \cdot w_{1m} < -\varepsilon_1. \tag{8}$$

In an attempt to preserve balance, we will switch the designations of the two points on the directed line n_1. We intend to show that if the Subcase IV.a and Subcase IV.b actions cause weight one imbalance, then switching the designations of the two points on the directed line preserves weight one balance.



(a) Original picture for Case IV

(b) Switching designations of the left and right points on the directed line

Figure 37: Switching designations action for Subcase IV.c

The change in weight one difference when the designations of the two points on the directed line are switched is given below. See Figures 37a and 37b for reference.

$$\begin{aligned}S_{1a}+w_{1n}-w_{1m}-(S_{1b}+w_{1m}-w_{1n}) &= S_{1a}+w_{1n}-w_{1m}-S_{1b}-w_{1m}+w_{1n}\\ &= S_{1a}-S_{1b}+2\cdot w_{1n}-2\cdot w_{1m}\\ &= x_1+2\cdot w_{1n}-2\cdot w_{1m}\end{aligned}$$

Recall that in this subcase, the inequalities (7) and (8) must be true. Consider (8) below, where we then add $2\cdot\varepsilon_1$ to both sides of the inequality.

$$\begin{aligned}x_1-2\cdot w_{1m} &< -\varepsilon_1\\ x_1+2\cdot\varepsilon_1-2\cdot w_{1m} &< \varepsilon_1\end{aligned}$$

Since $0\leq w_{1n}\leq\varepsilon_1$, it follows that $2\cdot w_{1n}\leq 2\cdot\varepsilon_1$, and

$$\begin{aligned}&x_1+2\cdot w_{1n}-2\cdot w_{1m}\leq x_1+2\cdot\varepsilon_1-2\cdot w_{1m}<\varepsilon_1\\ &x_1+2\cdot w_{1n}-2\cdot w_{1m}<\varepsilon_1. \qquad (9)\end{aligned}$$

In particular, we have shown that the new weight one difference is less than ε_1.

Now let us consider (7), where this time we will be subtracting $2\cdot\varepsilon_1$ from both sides of the inequality.

$$\begin{aligned}x_1+2\cdot w_{1n} &> \varepsilon_1\\ x_1+2\cdot w_{1n}-2\cdot\varepsilon_1 &> -\varepsilon_1\end{aligned}$$

Since $2 \cdot w_{1m} \leq 2 \cdot \varepsilon_1$,

$$-\varepsilon_1 < x_1 + 2 \cdot w_{1n} - 2 \cdot \varepsilon_1 \leq x_1 + 2 \cdot w_{1n} - 2 \cdot w_{1m}$$

$$-\varepsilon_1 < x_1 + 2 \cdot w_{1n} - 2 \cdot w_{1m}, \tag{10}$$

and we have also shown the new weight one difference is greater than $-\varepsilon_1$. Combining the inequalities from (9) and (10) results in

$$-\varepsilon_1 < x_1 + 2 \cdot w_{1n} - 2 \cdot w_{1m} < \varepsilon_1,$$

$|x_1 + 2 \cdot w_{1n} - 2 \cdot w_{1m}| \leq \varepsilon_1$, and the designation switch results in a preservation of balance for weight one values. Note that the new scenario in Figure 37b is now Case III, where we can rotate clockwise about either α_m or α_n without affecting the weight one absolute difference.

Thus exhausting all cases, we have shown that for any directed line p that balances a region with the usual conditions, where the directed line is being rotated clockwise about a point β until it intersects another point γ, there exists a clockwise rotation of the directed line about either β or γ so that balance is maintained.

$\square$

Corollary 3.5

The directed line generated by the initial weight-one balancing process in a region with the usual conditions can indefinitely rotate clockwise about points so as to always maintain balance. In particular, there exists a 180° *clockwise rotational path the directed line can undergo to always maintain balance.*

Proof. Let n_3 be the directed line generated by an initial weight-one balancing process in a region T_3 with the usual conditions. By Lemma 3.1, n_3 balances T_3. We can repeatedly utilize Theorem 3.4 indefinitely to show there exists a clockwise rotational path for n_3 that maintains balance. Certainly, it follows that there exists a 180° clockwise rotational path where n_3 maintains balance in T_3. $\square$

3.3 Example of Weight-One Balancing with a Directed Line Rotating 180° (Example A)

Consider region R_1 in Figure 38, where no three points are collinear and no two points lie on the same vertical line. Additionally, note the weight one values written near each of the corresponding points. We intend to illustrate both the initial weight-one balancing process as well as the procedures to guarantee continual weight-one balancing with a directed line rotating 180° clockwise. We will be considering the sum of the weight one values $S_1 = 22$, and especially the maximum weight one value $\varepsilon_1 = 6$ when evaluating whether or not region R_1 is balanced.

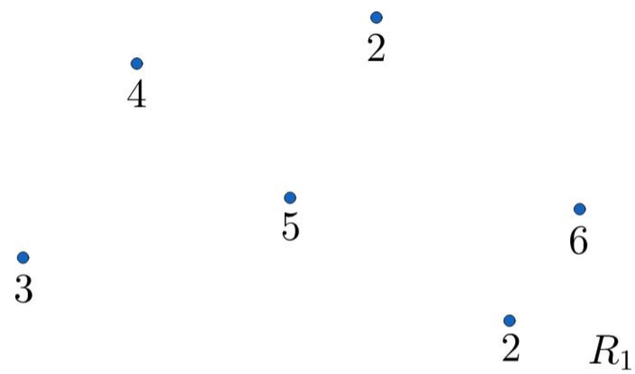
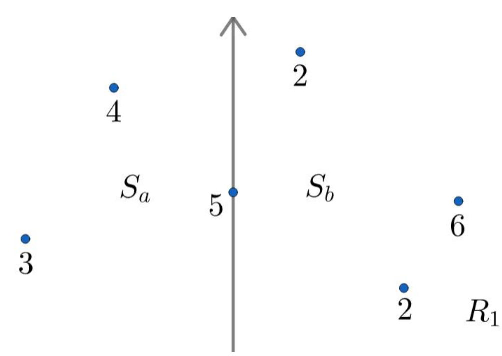


Figure 38: Region R_1 for Example A

By Definition 1.3, the point with a weight one value of five is α_p. The initial weight-one balancing process constructs directed line l_1 to contain α_p, where $S'_{1a} = 4 + 3 = 7$ and $S'_{1b} = 2 + 6 + 2 = 10$, implying α_p is assigned to S_a by Definition 1.6. This is demonstrated in Figure 39, where weight one balance is reflected in the table.

Figure 39: Initial weight-one balancing of region R_1



S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
12	10	2	✓

3.3.1 Scenarios and Actions During Clockwise Rotations

We will now begin rotating the directed line l_1 clockwise about α_p, where our greatest concern will be with the decisions regarding which point to rotate about when two points lie on the directed line. To avoid confusing the cases in the proof of Theorem 3.4 with the cases below, but yet to preserve consistency, we will define the subsequent scenarios and actions as follows:

Definition 3.6: Case i

Scenario: Both points on the directed line are assigned to S_a

Action: Rotate clockwise about the point on the left

Definition 3.7: Case ii

Scenario: Both points are assigned to S_b

Action: Rotate clockwise about the point on the right

Definition 3.8: Case iii

Scenario: The point on the left is assigned to S_b and the point on the right is assigned to S_a

Subcase iii.L action: Rotate clockwise about the point on the left

Subcase iii.R action: Rotate clockwise about the point on the right

Definition 3.9: Case iv

Scenario: The point on the left is assigned to S_a and the point on the right is assigned to S_b

Subcase iv.a action: Equivalent to Case i action

Subcase iv.b action: Equivalent to Case ii action

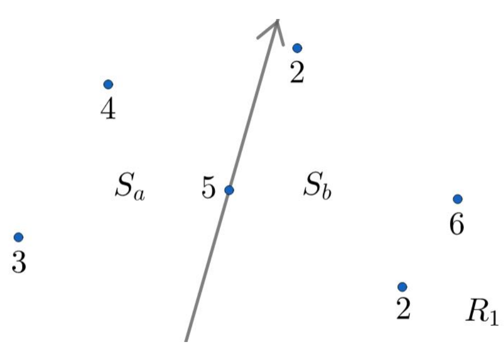
Subcase iv.c action: Switch designations of the points; rotation action equivalent to Case iii actions

Given any directed line containing two points that balances a region with the usual conditions, note that there may be multiple actions above that maintain weight one balance. In Example A, we will choose a clockwise rotational path that minimizes the balance at each step.

3.3.2 Example A Continued

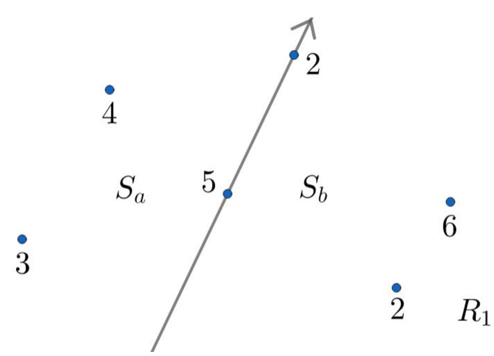
As the process of rotating the directed line l_1 clockwise about α_p begins, notice in Figure 40b that the next point encountered by l_1 is a point in S_b with a weight value of two. To maintain balance in this Case iv scenario, we can choose to either accept the Subcase iv.a or Subcase iv.c action. The Subcase iv.c action is accepted, as it generates a weight one absolute difference closer to zero. As a result, we will switch the designations of the two points on l_1 and rotate clockwise about the point on the right (equivalently the Case iii.R action). This transition is illustrated in the second transitionary clockwise rotation in Figure 41a.

Figure 40: Example A clockwise rotation actions



(a) 1st transitionary clockwise rotation

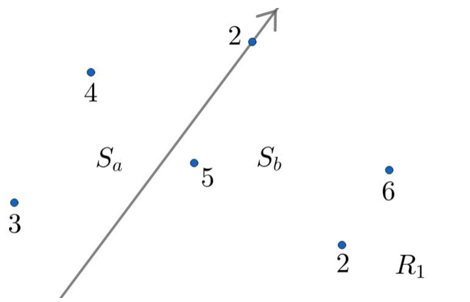
S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
12	10	2	✓



(b) Subcase iv.c & Case iii.R actions

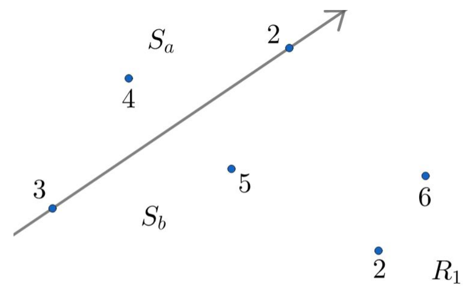
S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
12	10	2	✓

Figure 41: Example A clockwise rotation actions cont.



(a) 2nd transitionary clockwise rotation

S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
9	13	4	✓

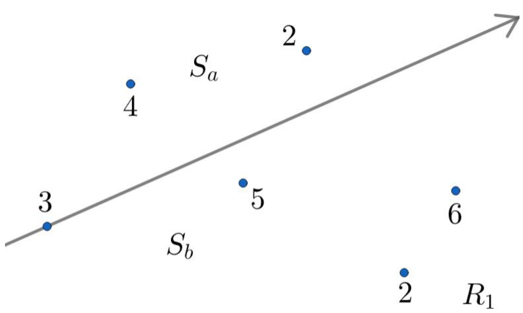


(b) Case i action

S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
9	13	4	✓

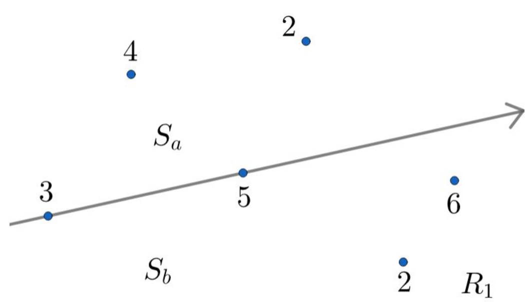
In Figure 41b, we see that the next point to intersect l_1 is the point with a weight value of three. In this case, the only balanced path preserves assignment in S_a, resulting in a Case i scenario. The Case i action dictates l_1 to rotate clockwise about the point on the left. Figure 42a illustrates this step.

Figure 42: Example A clockwise rotation actions cont.



(a) 3rd transitionary clockwise rotation

S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
9	13	4	✓

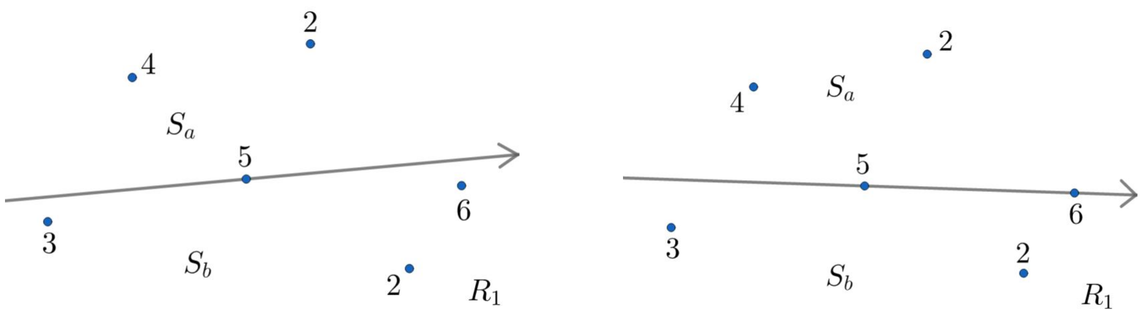


(b) Subcase iv.c & Case iii.R actions

S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
9	13	4	✓

Another Case iv scenario occurs in Figure 42b, and both Subcase iv.a and Subcase iv.c actions maintain weight one balance. Once again, the Subcase iv.c action, in combination with the Case iii.R action, results in a weight one absolute difference closer to zero; Figure 43a demonstrates both actions.

Figure 43: Example A clockwise rotation actions cont.



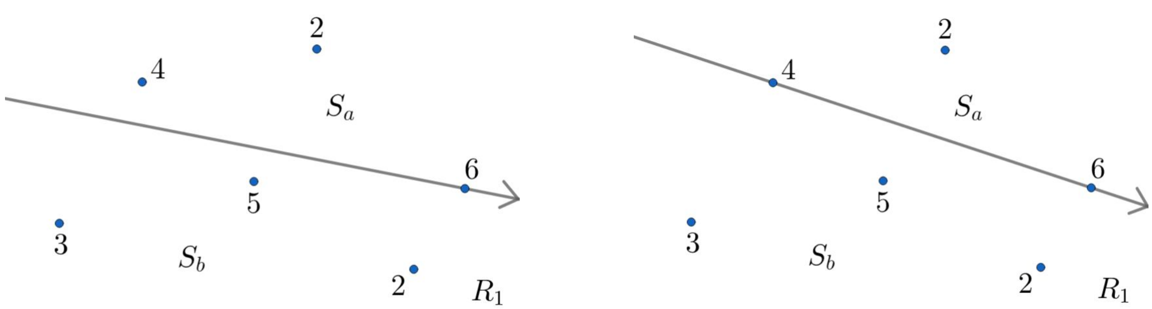
(a) 4th transitionary clockwise rotation

(b) Subcase iv.c & Case iii.R actions

S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
11	11	0	✓

S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
11	11	0	✓

Figure 44: Example A clockwise rotation actions cont.



(a) 5th transitionary clockwise rotation

(b) Case i action

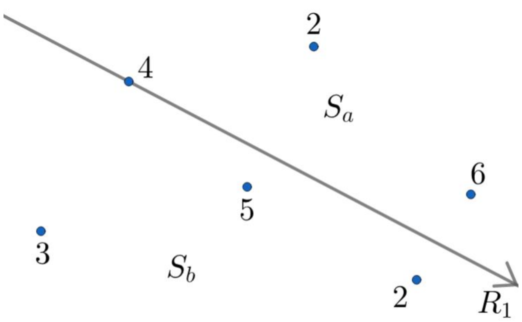
S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
12	10	2	✓

S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
12	10	2	✓

Yet another Subcase iv.c action presents itself as the best Case iv option in Figure 43b, and after switching the designations of the points, we will continue rotating clockwise about the point on the right.

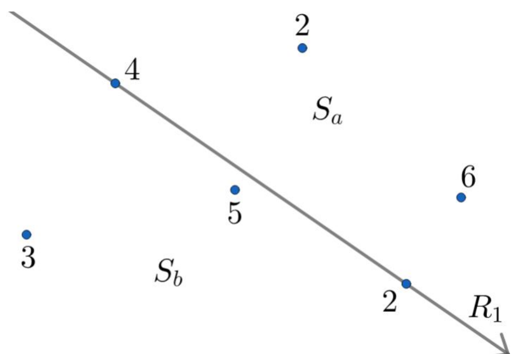
The clockwise rotation process will continue with another Case i scenario in Figure 44b, followed by a Subcase iv.c and Case iii.R action in Figure 45b. Each of these actions minimizes the balance at their respective steps.

Figure 45: Example A clockwise rotation actions cont.



(a) 6th transitionary clockwise rotation

S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
12	10	2	✓

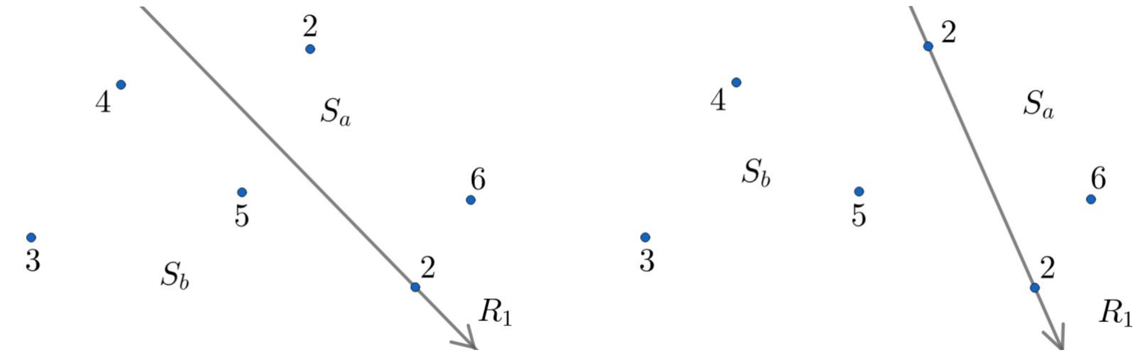


(b) Subcase iv.c & Case iii.R action

S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
12	10	2	✓

The final steps in the process involve one more Case i action before the directed line has completed its 180° rotation in Figure 47b, now being centered at the point with a weight one value of two. Note that, as expected, the directed line balanced region R_1 at every step of the clockwise rotation.

Figure 46: Example A clockwise rotation actions cont.



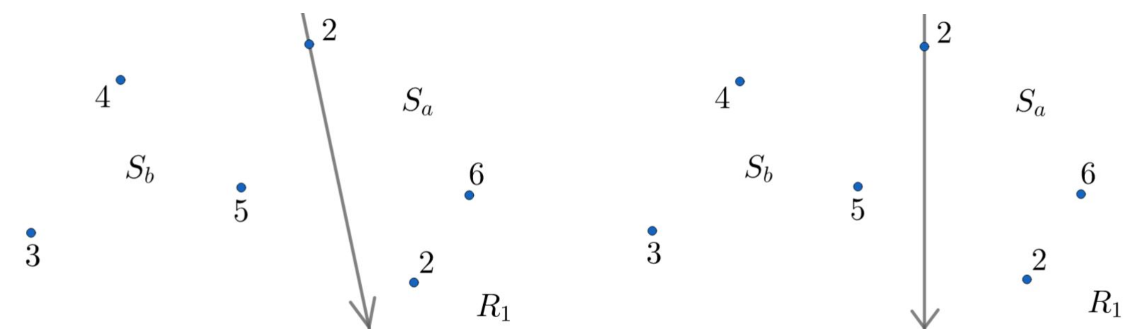
(a) 7th transitionary clockwise rotation

(b) Case i action

S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
10	12	2	✓

S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
10	12	2	✓

Figure 47: Example A conclusion



(a) 8th transitionary clockwise rotation

(b) Completed 180° clockwise rotation

S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
10	12	2	✓

S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
10	12	2	✓

3.4 Balanced Path and Minimized Balanced Path

We define a *balanced path* in a region with the usual conditions to have the following qualifications.

Definition 3.10: Balanced path

1. The initial vertical directed line directed up balances the region and contains a point
2. A clockwise rotational path about points commences
3. When two points are located on the directed line, any clockwise rotation action about a point that generates balance in the region is accepted.
4. The directed line rotates clockwise 180°.

Though a balanced path will be the only requirement in the upcoming lemma, the definition of a minimized balanced path will help determine code to demonstrate the existence of a dual-balanced line in a region with the usual conditions.

Definition 3.11: Minimized balanced path

1. The initial vertical directed line directed up balances the region, minimizes the weight one absolute difference for vertical directed lines, and contains a point

2. A clockwise rotational path about points commences

3. When two points are located on the directed line, the accepted clockwise rotation action generates balance in the region and minimizes $|x_1|$.

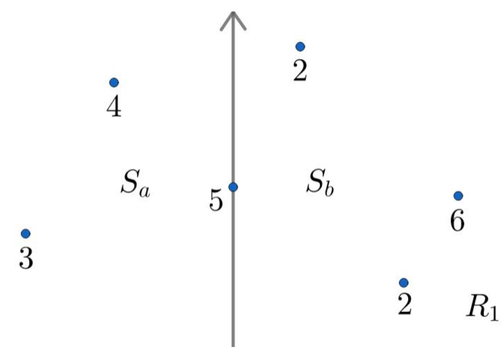
4. The directed line rotates clockwise 180°.

Note there could be many minimized balanced paths in a region with the usual conditions, as at any step there could be multiple actions that minimize the balance. Moreover, we have proven in Lemma 3.3 that there are at least two vertical directed lines that satisfy qualification number one for a minimized balanced path. Example A is an example of a minimized balanced path by definition, as we began the process with an initial weight-one balancing, the directed line rotated clockwise 180°, and we chose a clockwise rotation action about a point at every step that generated the minimized balance.

3.5 Alternative Minimized Balanced Path for Region R_1 Containing α_p at Its Conclusion (Example B)

Recall region R_1 and its initial weight-one balancing below.

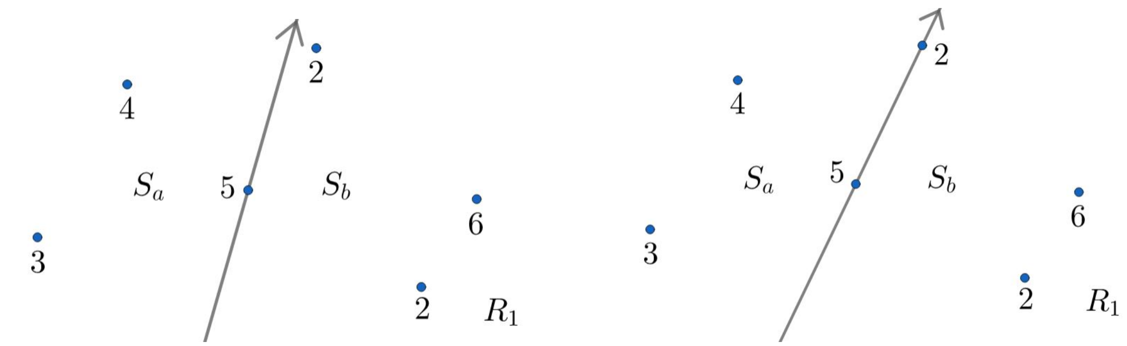
Figure 48: Initial weight-one balancing of region R_1



S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
12	10	2	✓

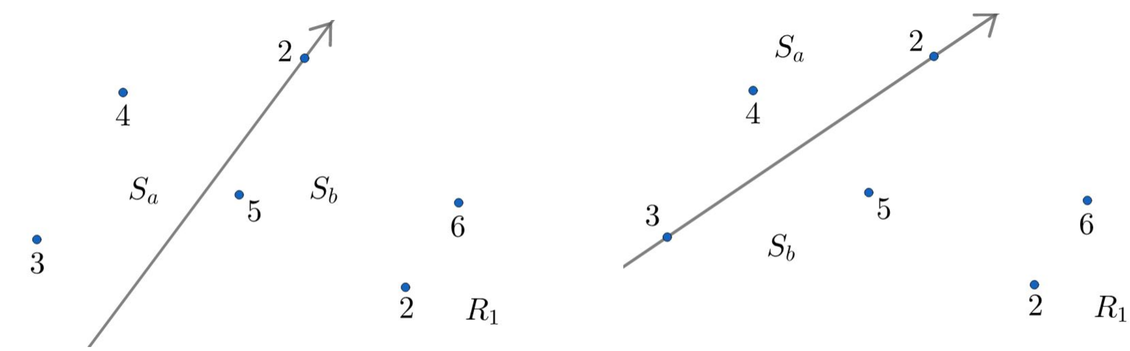
What follows is a list of figures showing consecutive clockwise rotation actions that generate a minimized balanced path different from the minimized balanced path in Example A. Note that Figures 48 - 53, in addition to Figure 54a, are the first twelve steps of Example A. The alternative path of Example B deviates from the path of Example A beginning in Figure 54b.

Figure 49: Example B clockwise rotation actions



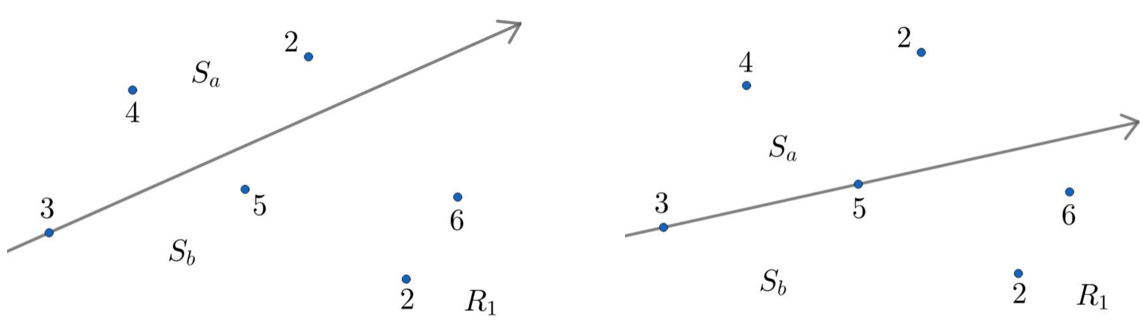
(a) 1st transitionary clockwise rotation (b) Subcase iv.c & Case iii.R actions

Figure 50: Example B clockwise rotation actions cont.



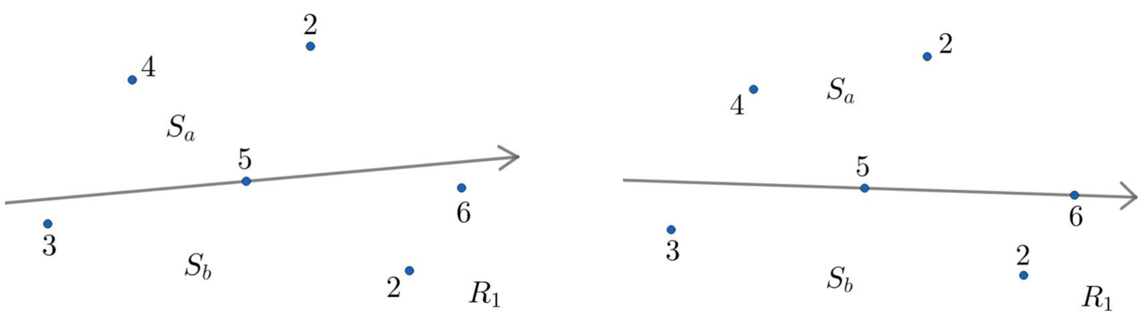
(a) 2nd transitionary clockwise rotation (b) Case i action

Figure 51: Example B clockwise rotation actions cont.



(a) 3rd transitionary clockwise rotation (b) Subcase iv.c & Case iii.R actions

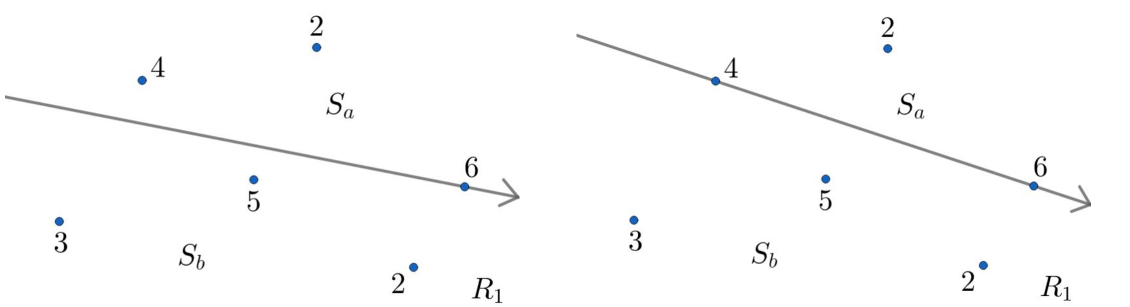
Figure 52: Example B clockwise rotation actions cont.



(a) 4th transitionary clockwise rotation

(b) Subcase iv.c & Case iii.R actions

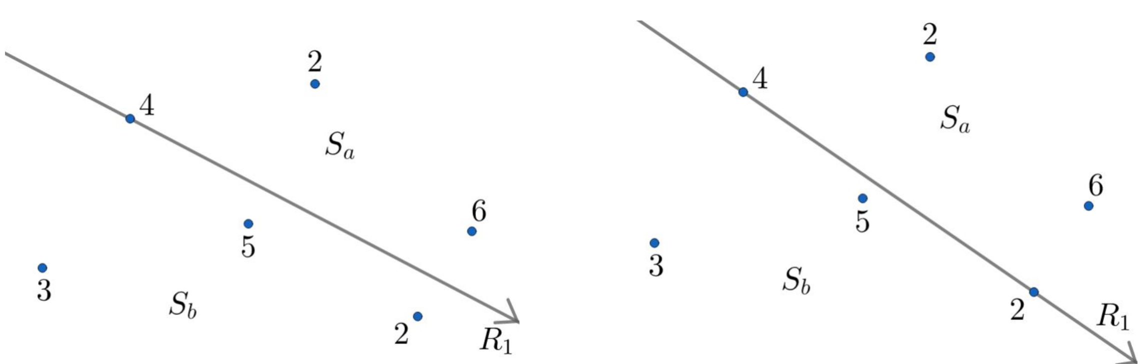
Figure 53: Example B clockwise rotation actions cont.



(a) 5th transitionary clockwise rotation

(b) Case i action

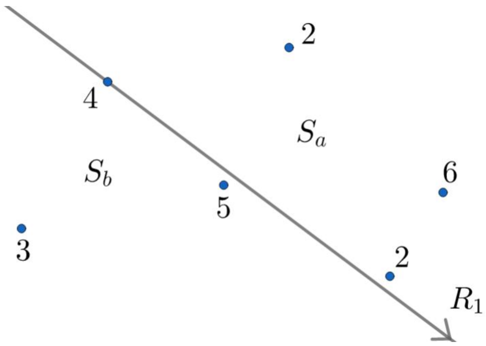
Figure 54: Example B clockwise rotation actions cont.



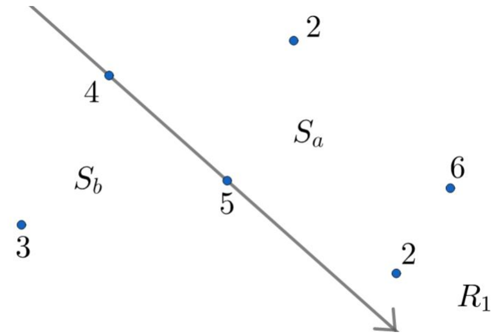
(a) 6th transitionary clockwise rotation

(b) **Subcase iv.c & Case iii.L actions**

Figure 55: Example B clockwise rotation actions cont.



(a) **7th transitionary clockwise rotation**

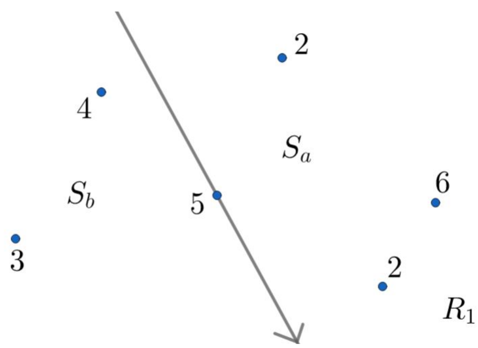


(b) **Case ii action**

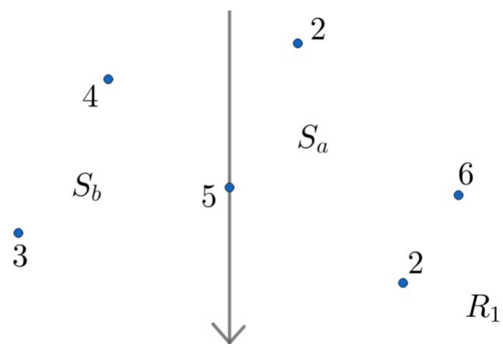
S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
10	12	2	✓

S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
10	12	2	✓

Figure 56: Example B conclusion



(a) **8th transitionary clockwise rotation**



(b) **Completed** $180°$ **clockwise rotation**

S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
10	12	2	✓

S_{1a}	S_{1b}	$\lvert x_1 \rvert$	Balanced?
10	12	2	✓

In Example A and Example B, we see that it is possible both for a minimized balanced path to contain α_p at its conclusion and for a minimized balanced path to not contain α_p at its conclusion. The next section will consider a horizontal translation of the directed line that does not contain α_p at its conclusion, and whether or not balance can be maintained as it translates toward α_p.

3.6 Maintaining Balance with a Horizontal Translation of a Balanced Path Not Containing α_p at Its Conclusion

We now consider cases such as Example A, where the concluding vertical directed line of a balanced path does not contain α_p.

Lemma 3.12

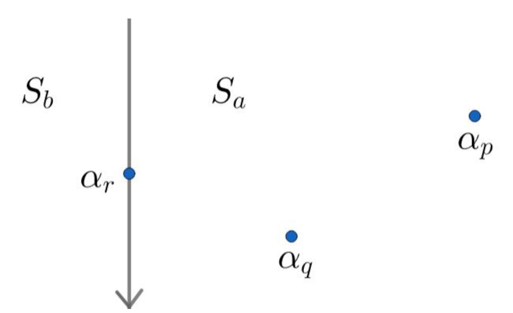
Any directed line that has undergone a balanced path in a region with the usual conditions can be horizontally translated to contain α_p while maintaining balance the entirety of the horizontal translation.

Proof. Let l_2 be the directed vertical line generated by an initial weight-one balancing process in a region R_2 with the usual conditions. Suppose l_2 has undergone a balanced path. If l_2 contains α_p at its conclusion, then we are done.

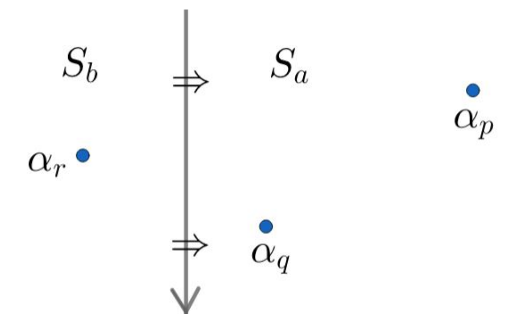
If l_2 does not contain α_p at its conclusion, but some other point α_r, then we will consider horizontally translating l_2 toward α_p. There are two cases: where α_r is to the left of α_p and where α_r is to the right of α_p.

Case I: α_r is to the left of α_p

If α_r is to the left of α_p as in Figure 57a, then we can begin horizontally translating the directed line to the right toward α_p. This action is shown in Figure 57b.



(a) Concluding directed line contains a point to the left of α_p



(b) Right horizontal shift of the directed line

Figure 57: Horizontal translation to the right from α_r to α_p

Let B be the set of all points with positive weight one values that will switch designations as we horizontally translate the directed line to the right toward α_p. Note it is possible for $B = \emptyset$, in which case we are done by the assumption l_2 had undergone a balanced path.

Suppose the set $B \neq \emptyset$. Then a horizontal shift of the directed line to the right toward α_p would eventually have an effect on the weight one absolute difference. Let $\alpha_q \in B$ be the first point that will switch its designation as the directed line translates to the right. As the directed line crosses over α_q, the weight one value w_{1q} is subtracted from S_a and added to S_b. Hence, the new weight one difference would be

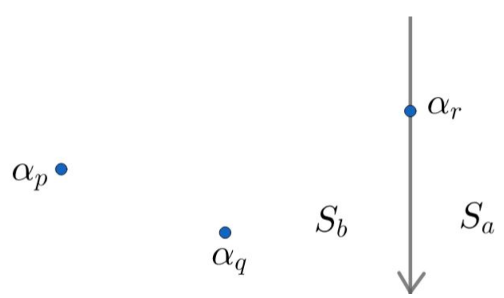
$$\begin{aligned} S_{1a} - w_{1q} - (S_{1b} + w_{1q}) &= S_{1a} - w_{1q} - S_{1b} - w_{1q} \\ &= S_{1a} - S_{1b} - 2 \cdot w_{1q} \\ &= x_1 - 2 \cdot w_{1q} \end{aligned}$$

For the sake of argument, suppose that somewhere along the horizontal translation from α_r to α_p, the value of x_1 is less than $-\varepsilon_1$, implying the region is imbalanced.

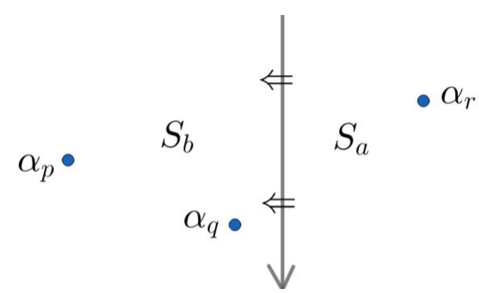
Then as the value of x_1 decreases monotonically as the directed line translates to the right, the directed line containing α_p also generates an imbalance with $x_1 < -\varepsilon_1$. But by Lemma 3.1, the directed line containing α_p balances the region $\rightarrow\leftarrow$. Thus, the value of x_1 remains in the interval $[-\varepsilon_1, \varepsilon_1]$ as it translates horizontally to the right from α_r to α_p. This implies weight one balance is maintained during the entire horizontal shift of the directed line to the right from α_r to α_p.

Case II: α_r is to the right of α_p

Consider the case where a left horizontal shift of the directed line is necessary for the concluding directed line to contain α_p. See Figures 58a and 58b for a visualization.



(a) Concluding directed line contains a point to the right of α_p



(b) Left horizontal shift of the directed line

Figure 58: Horizontal translation to the left from α_r to α_p

Similar to the previous case, there is no change in the balance unless a point with a positive weight one value switches designation. Let α_q again be the first such point that will switch its designation as the directed line translates to the left. As the directed line crosses over α_q, the weight one value of α_q is subtracted from S_b and added to S_a. There would then be a $+2 \cdot w_{1q}$ net change in the value of x_1.

For the sake of argument, suppose that somewhere along the horizontal translation from α_r to α_p, the value of x_1 is greater than ε_1, implying the region is imbalanced. Since the value of x_1 increases monotonically as the directed line translates to the left, the directed line containing α_p also generates an imbalance with $x_1 > \varepsilon_1$. This creates a similar contradiction as in Case I, as the directed line containing α_p balances the region by Lemma 3.1 $\rightarrow\leftarrow$. It is also true in this case that the value of x_1 remains in the interval $[-\varepsilon_1, \varepsilon_1]$ as the directed line translates horizontally to the left from α_r to α_p. Thus, whether the concluding directed line after a balanced path contains α_p or a point other than α_p, we are able to shift the directed line horizontally to contain α_p while maintaining weight one balance for the duration of the translation. $\square$

3.7 Proof of the Existence of a Specified Balanced Path

Finally, we are able to coalesce the previous results to prove the valuable theorem below.

Theorem 3.13

For any region with the usual conditions, there exists a directed line that maintains balance for a $180°$ clockwise rotation about points in the plane, initially containing a point when directed up, and then containing the same point at its conclusion when directed down.

Proof. Let R_3 be a region with the usual conditions. Perform an initial weight-one balancing process, which will identify $\alpha_p \in R_3$ and generate the vertical directed line l_3 containing α_p. By Lemma 3.1, we know the vertical directed line l_3 balances region R_3 with respect to weight one values, satisfying the first qualification of a balanced path. We can utilize Corollary 3.5 next to state there exists a $180°$ clockwise rotational path l_3 can undergo about points to always maintain balance. This satisfies the second, third, and fourth qualifications to be a balanced path. Finally, Lemma

3.12 assures us that, if necessary, we can horizontally translate l_3 after its balanced $180°$ clockwise rotational path to contain α_p while maintaining balance the entirety of the horizontal translation. □

Corollary 3.14

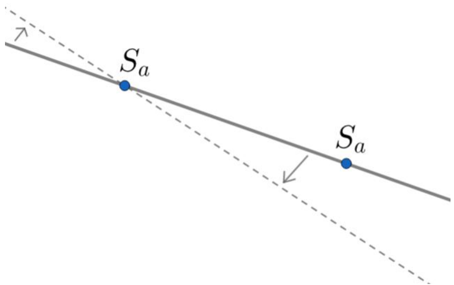
For any region with the usual conditions, the vertical directed line generated by the initial weight-one balancing process, initially containing α_p when directed up, can perform a $180°$ clockwise rotation about points in the plane with a balanced horizontal translation as necessary to contain α_p again at its conclusion when directed down.

Proof. See proof of Theorem 3.13 above. □

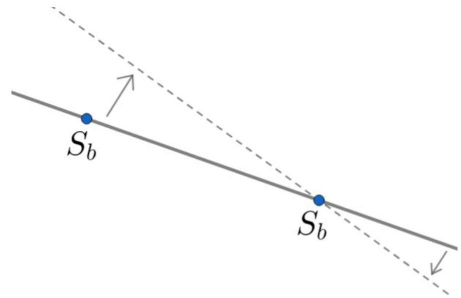
3.8 Unique Minimized Balanced Path

We have already defined a minimized balanced path. In this final section solely regarding the balancing of one weight value, we consider the definition of a *unique minimized balanced path.* In addition to satisfying the qualifications of a minimized balanced path, which can be found in Definition 3.11, the unique minimized balanced path will specify which minimized path to take whenever multiple actions minimize the balance.

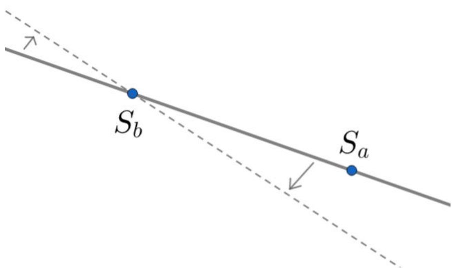
In essence, we only need to consider the Case i, Case ii, and Case iii actions, all of which are shown in Figure 59.



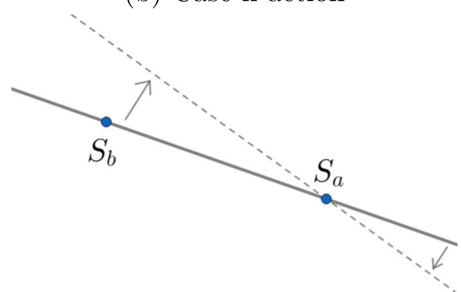
(a) Case i action



(b) Case ii action



(c) Case iii.L action



(d) Case iii.R action

Figure 59: Four clockwise rotation actions to consider

Definition 3.15: Unique minimized balanced path

A minimized balanced path is considered the *unique minimized balanced path* if the initial vertical directed line is the vertical directed line generated by the initial weight-one balancing process, and if the following priority list is adhered to at every step whenever two or more actions result in a minimized balance.

1. Case iii.R action
2. Case ii action
3. Case i action
4. Case iii.L action

This means that if the Case iii.R action and Case i action generate the same minimized balance, then the unique minimized balanced path would accept the Case iii.R action. One can check that Example A is the unique minimized balance path for region R_1, as the qualifications for a minimized balanced path are satisfied and the priority list above is respected.

On the other hand, Example B is a minimized balanced path but not the *unique* minimized balanced path for region R_1. In Figure 54b, we see that the Case iii.L action is prioritized over the Case iii.R action, contradicting the priority list of the unique minimized balanced path above.

3.9 The Dual-Balanced Theorem

For the main result of this paper, we now consider where points in a region with the usual conditions are assigned two non-negative weight values. Refer to Definition 1.2 for a definition of dual-balance as needed.

Theorem 3.16: The Dual-Balanced Theorem

Any region with the usual conditions can be dual-balanced by at least one directed line.

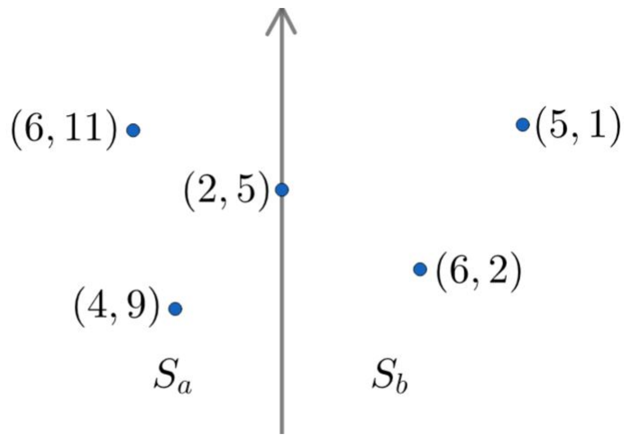
Proof. Let R_4 be a region with the usual conditions, where each point is assigned two non-negative weight values. Perform an initial weight-one balancing process, where l_4 is the resulting vertical directed line that contains α_p. By Lemma 3.1, directed line l_4 balances region R_4 with respect to weight one values. If l_4 balances region R_4 with respect to weight two values, then we are done.

Suppose the initial weight-one balancing process results in an imbalance for weight two values. This means

$$|x_2| = |S_{2a} - S_{2b}| > \varepsilon_2$$

after the initial weight-one balancing. Without loss of generality, assume the imbalance is caused by an over-accumulation of weight two values in S_a, the left side of the vertical directed line l_4. In Figure 60, we see an example of this situation. Recall that for the coordinates (a, b), a represents the point's weight one value, and b represents the point's weight two value.

Figure 60: Initial weight-one balancing resulting in a weight two imbalance

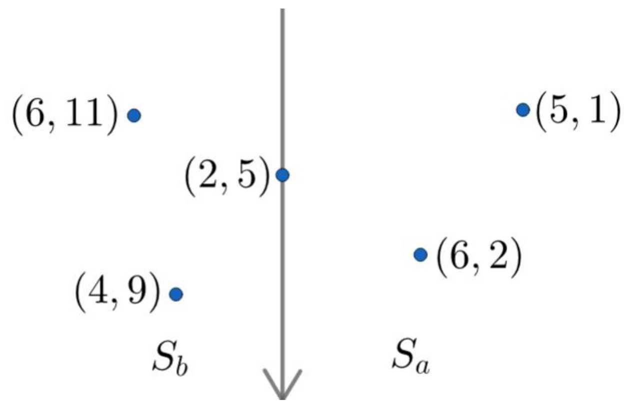


S_{1a}	S_{1b}	x_1	Weight 1 balanced?	S_{2a}	S_{2b}	x_2	Weight 2 balanced?
12	11	1	✓	25	3	22	✗

Next, perform a balanced 180° clockwise rotational path with respect to weight one values about points with a balanced horizontal translation as necessary so that l_4 contains α_p again at its conclusion. We know such a balanced clockwise rotational path exists by Corollary 3.14. By Rule 4 concerning the assignment of weight values, α_p will be assigned so that both the weight one difference $x_1 = S_{1a} - S_{1b}$ and the weight two difference $x_2 = S_{2a} - S_{2b}$ will have the opposite value relative to when l_4 contained α_p and was directed up.

Figures 60 and 61 exemplify this, as the value of x_1 changes from 1 to -1, and the value of x_2 changes from 22 to -22.

Figure 61: Example of concluding directed line showing opposite values of x_1 and x_2 as in Figure 60

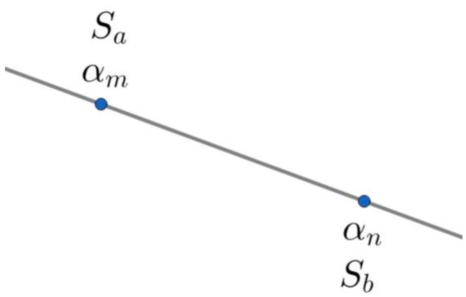


S_{1a}	S_{1b}	x_1	Weight 1 balanced?	S_{2a}	S_{2b}	x_2	Weight 2 balanced?
11	12	-1	✓	3	25	-22	✗

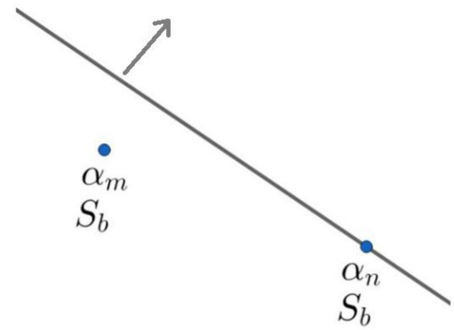
Because the directed line was being rotated in a continuous fashion, and since no three points are collinear, it must be the case that at some stage of the clockwise rotation, the value of x_2 changed from being positive or zero to being negative. Moreover, this event must have occurred when one or two points switched designations so that the weight two difference decreased.

Case I: The value of x_2 first becomes negative when one point switches designation

During a balanced path, it is possible for x_2 to decrease when only one point switches designation. Consider again the Case iv scenario shown below in Figure 62a, where the directed line intersects a point from S_a on the left and a point from S_b on the right. In what was considered a Subcase iv.b action, or, equivalently, a Case ii action, we were able to rotate clockwise about the point on the right and preserve balance. The point on the left would then switch designation from S_a to S_b, either decreasing or keeping the value of x_2 constant.



(a) Original picture for Case IV



(b) Ability to rotate clockwise about the right point

Figure 62: Case I scenario related to a Subcase iv.b action

Additionally, since no two points are on the same vertical line, one point at a time switches designation from S_a to S_b when the vertical directed line directed down is horizontally translated toward α_p from the right, as shown in Figure 63. The following proof will satisfy both of these situations.

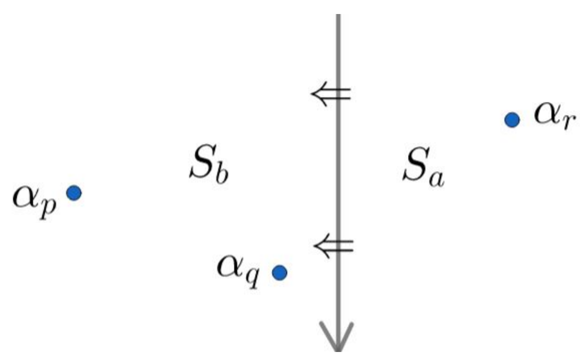


Figure 63: Case I scenario related to a horizontal translation of the directed line

Suppose that the value of x_2 first becomes negative when one point switches designation from S_a to S_b. We will show that in this case, we cannot avoid balance for weight two values.

Let α_m be the point switching designation from S_a to S_b. Note that since we are undergoing a balanced path, weight one balance will be preserved. To differentiate, let x_2 be the weight two difference before the action, and let x'_2 be the weight two difference after. Suppose, for the sake of argument, that the weight two values are imbalanced before and after this action. Since our case is considering when the value of x_2 first becomes negative, it must follow that $x_2 > \varepsilon_2$ and $x'_2 < -\varepsilon_2$.

When the action that switches the designation of α_m occurs, the weight two value w_{2m} would be subtracted from S_{2a} and added to S_{2b}, with the resultant weight two difference below.

$$\begin{aligned} x'_2 &= S_{2a} - w_{2m} - (S_{2b} + w_{2m}) \\ &= S_{2a} - w_{2m} - S_{2b} - w_{2m} \\ &= S_{2a} - S_{2b} - 2 \cdot w_{2m} \\ &= x_2 - 2 \cdot w_{2m} \end{aligned}$$

Consequentially, $x_2 - 2 \cdot w_{2m} < -\varepsilon_2$. But since $w_{2m} \leq \varepsilon_2$,

$$\begin{aligned} x_2 - 2 \cdot \varepsilon_2 \leq x_2 - 2 \cdot w_{2m} &< -\varepsilon_2 \\ x_2 - 2 \cdot \varepsilon_2 &< -\varepsilon_2 \\ x_2 &< \varepsilon_2, \end{aligned}$$

contradicting the assumption that both $x_2 > \varepsilon_2$ and $x'_2 < -\varepsilon_2 \rightarrow\leftarrow$. Hence, either $0 \leq x_2 \leq \varepsilon_2$ or $-\varepsilon_2 \leq x'_2 < 0$. By definition, then, either x_2 or x'_2 signifies weight two balance. Without loss of generality, suppose x'_2 signifies balance with respect to weight two values. Since weight one balance would have been preserved through this action by definition of balanced path, the directed line with weight two absolute difference $|x_2'|$ dual-balances R_4.

Case II: The value of x_2 first becomes negative when two points switch designation

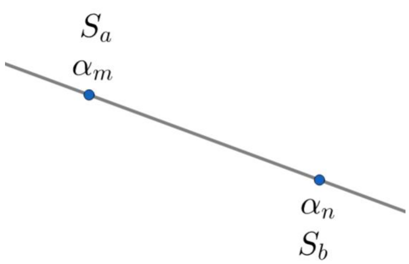
Two possibilities exist where two points switch designation and x_2 is subsequently decreased: two points from S_a could switch designation to S_b, or one point could switch designation from S_a to S_b while the other point switches designation from S_b to S_a.

Subcase II.a: The value of x_2 first becomes negative when two points switch designation from S_a to S_b

During a balanced path, when two points on the directed line are assigned to S_a, we may preserve assignment and rotate about the point on the left to maintain balance. In this way, we avoid this subcase altogether.

Subcase II.b: The value of x_2 first becomes negative when one point switches designation from S_a to S_b and the other point switches designation from S_b to S_a

Switching the designations of the two points on the directed line is a Subcase iv.c action, which would then be followed by a Case iii action. Since a Case iii action does not entail a change in weight two balance, we will only consider the designation switch, demonstrated again in Figure 64b. Recall that this designation switch would maintain balance with respect to weight one values by definition of balanced path.



(a) Original picture for Case IV

(b) Switching designations of the left and right points

Figure 64: Subcase II.b scenario related to a Subcase iv.c action

Let α_m and α_n be the two points on the directed line, with α_m on the left assigned to S_a and α_n on the right assigned to S_b. We will again let x_2 be the weight two difference before the designation switch, and x_2' the weight two difference after. Suppose, for the sake of argument, that the weight two values are imbalanced before and after the designation switch. Then it must be the case that $x_2 > \varepsilon_2$ and $x_2' < -\varepsilon_2$. When we consider α_m instead being assigned to S_b and α_n instead being assigned to S_a, the weight two difference x_2 would be affected as follows.

$$\begin{aligned}
x_2' &= S_{2a} - w_{2m} + w_{2n} - (S_{2b} + w_{2m} - w_{2n}) \\
&= S_{2a} - w_{2m} + w_{2n} - S_{2b} - w_{2m} + w_{2n} \\
&= S_{2a} - S_{2b} + 2 \cdot w_{2n} - 2 \cdot w_{2m} \\
&= x_2 + 2 \cdot w_{2n} - 2 \cdot w_{2m}
\end{aligned}$$

By substitution, $x_2 + 2 \cdot w_{2n} - 2 \cdot w_{2m} < -\varepsilon_2$, and

$$x_2 < -\varepsilon_2 + 2 \cdot w_{2m} - 2 \cdot w_{2n}.$$

But since

$$
\begin{aligned}
2 \cdot w_{2m} - 2 \cdot w_{2n} &= 2 \cdot (w_{2m} - w_{2n}) \\
&< 2 \cdot \varepsilon_2,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
x_2 &< -\varepsilon_2 + 2 \cdot w_{2m} - 2 \cdot w_{2n} \\
&< -\varepsilon_2 + 2 \cdot \varepsilon_2 = \varepsilon_2,
\end{aligned}
$$

showing $x_2 < \varepsilon_2$ and contradicting the assumption that both $x_2 > \varepsilon_2$ and $x_2' < -\varepsilon_2$ $\rightarrow\leftarrow$. Thus, either $0 \leq x_2 \leq \varepsilon_2$ or $-\varepsilon_2 \leq x_2' < 0$. By definition, either x_2 or x_2' signifies weight two balance. Without loss of generality, suppose x_2' signifies weight two balance. Since weight one balance would have been preserved by definition of balanced path, the directed line with weight two absolute difference $|x_2'|$ dual-balances R_4.

Thus, we used the fact that we are able to determine a balanced path with respect to weight one values in any region with the usual conditions to prove there must exist a directed line that dual-balances the region by definition. $\square$

3.10 Discrete Pancake Theorem Generalization

A theorem commonly known as the Discrete Pancake Theorem states that for two finite disjoint sets S and T of points in a plane where $S \cup T$ contains no three collinear points, there exists a line that simultaneously bisects $|S|$ and $|T|$ within an error of at most one. As is often the case, the disjoint sets are represented so that the points from one set are colored blue, and the points from the other set are colored red.

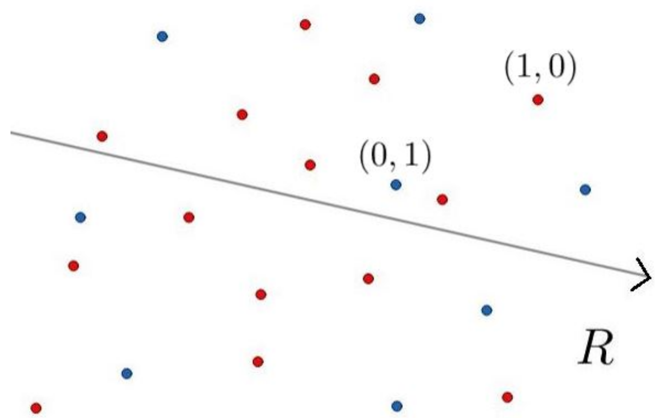


Figure 65: Blue and red points instead being assigned two non-negative weights

We can consider the Discrete Pancake Theorem to be but a special case of the Dual-Balanced Theorem, where each point is instead assigned two non-negative weights. In particular, the points colored red could be assigned a weight one value of one and a weight two value of zero, and the points colored blue could then be assigned a weight one value of zero and a weight two value of one. The definition of balance with respect to each weight would be the same as in Definition 1.1, namely

$$|S_{1a} - S_{1b}| \leq \varepsilon_1 \quad \text{and} \quad |S_{2a} - S_{2b}| \leq \varepsilon_2.$$

S_{1a} and S_{1b} would respectively be the number of red points on side S_a and side S_b, and S_{2a} and S_{2b} would respectively be the number of blue points on side S_a and side S_b. Furthermore, $\varepsilon_1 = \varepsilon_2 = 1$, and dual-balance would be satisfied if the absolute difference of the number of red points on either side of the line, and if the absolute difference of the number of blue points on either side of the line, both have values of at most one. This weighted scenario precisely describes the Discrete Pancake Theorem. Thus, we have shown that the Discrete Pancake Theorem is but a special case of the Dual-Balanced Theorem, and that the Dual-Balanced Theorem generalizes the Discrete Pancake Theorem.

3.11 Further Examples

Given a region with the usual conditions, we have shown through the Dual-Balanced Theorem that we can always determine a directed line so that the region is dual-balanced. This raises a natural question: can we always determine a directed line so that a region with the usual conditions is *triple-balanced*?

We define triple-balanced as one would expect, where the points in a region are now each being assigned three non-negative weights.

Definition 3.17: Triple-balanced

A directed line *triple-balances* a region with respect to weight x, y, and z values if the directed line balances the region with respect to weight x values, with respect to weight y values, and with respect to weight z values.

Certainly, in some regions, a directed line can be generated to triple-balance the region. However, in this section, we will provide a counterexample showing that it is not always possible to determine a directed line that triple-balances a region with the usual conditions.

Example 3.18

Given a region with the usual conditions, there does not always exist a directed line that triple-balances the region.

Proof. Consider region T_4 in Figure 66, where we note the locations of points α_i for $i \in \overline{5}$ imply region T_4 satisfies the usual conditions. Each coordinate (w_{1i}, w_{2i}, w_{3i}) for every α_i communicates the three weight values w_{1i}, w_{2i}, and w_{3i}. For example, $w_{25} = 4$, as the second weight value of α_5 is four.

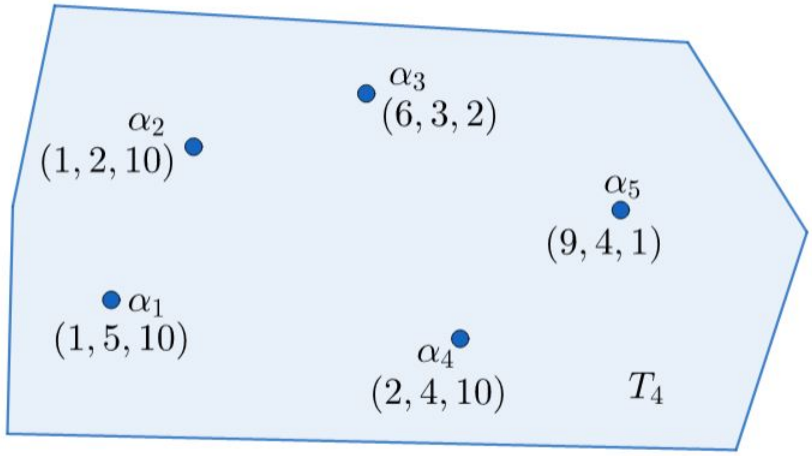


Figure 66: Region T_4

We intend to show region T_4 is an example of a region with the usual conditions where it is impossible to determine a directed line that simultaneously balances the weight one, weight two, and weight three values. We will consider the critical values in Table 9 which influence whether or not the region is balanced by definition.

	Weight One	Weight Two	Weight Three
Total	$S_1 = 19$	$S_2 = 18$	$S_3 = 33$
Maximum	$\varepsilon_1 = 9$	$\varepsilon_2 = 5$	$\varepsilon_3 = 10$

Table 9: Weight totals and maximum values in region T_4

The weight three values particularly invite imbalance. We will focus our attention specifically where $|S_{3a} - S_{3b}| \leq \varepsilon_3$. In Table 10, all possible S_{3a} and S_{3b} values are given, along with an evaluation of whether or not each difference signifies weight three balance. Note there are only four values of S_{3a} that determine weight three balance.

S_{3a}	S_{3b}	x_3	$\lvert x_3\rvert \leq \varepsilon_3$
33	0	+33	✗
32	1	+31	✗
31	2	+29	✗
30	3	+27	✗
23	10	+13	✗
22	11	+11	✗
21	12	+9	✓
20	13	+7	✓
13	20	-7	✓
12	21	-9	✓
11	22	-11	✗
10	23	-13	✗
3	30	-27	✗
2	31	-29	✗
1	32	-31	✗
0	33	-33	✗

Table 10: Possible S_{3a} and S_{3b} values for region T_4

We will now only consider directed lines that balance weight three values. In Figure 67, we separate T_4 into wedges, where each wedge is defined to be a gap from A to E. We will use these gaps to determine all of the possible ways to balance weight three values, where we can then demonstrate that any directed line balancing weight three values does not simultaneously balance weight one and weight two values.

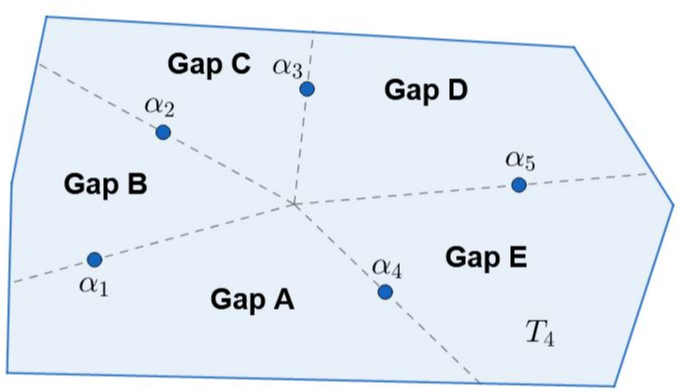
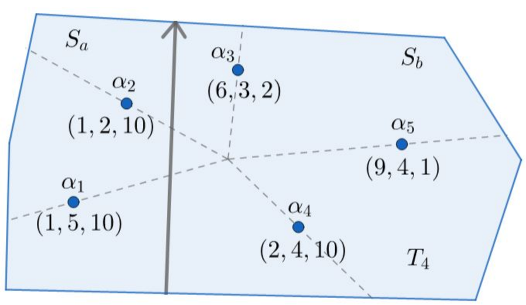


Figure 67: Region T_4 separated into gap wedges

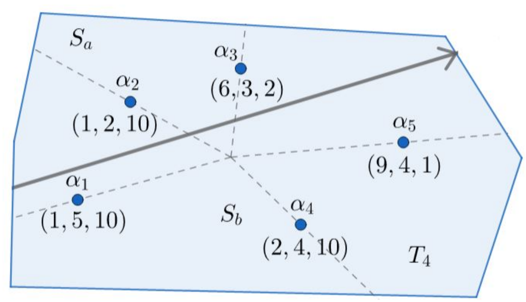
The only path to balance weight three values from Gap A directs toward Gap C. Note there are infinitely many directed lines being directed from Gap A to Gap C, though every directed line directed from Gap A which balances weight three values will maintain the designations seen in Figure 68a. Thus, while there exists directed lines that dual-balance region T_4, no directed line directed from Gap A triple-balances region T_4.

Figure 68: Directed line gap considerations



(a) From Gap A to Gap C

Absolute Differences	Balanced?
$\lvert x_1\rvert = 15$	✗
$\lvert x_2\rvert = 4$	✓
$\lvert x_3\rvert = 7$	✓



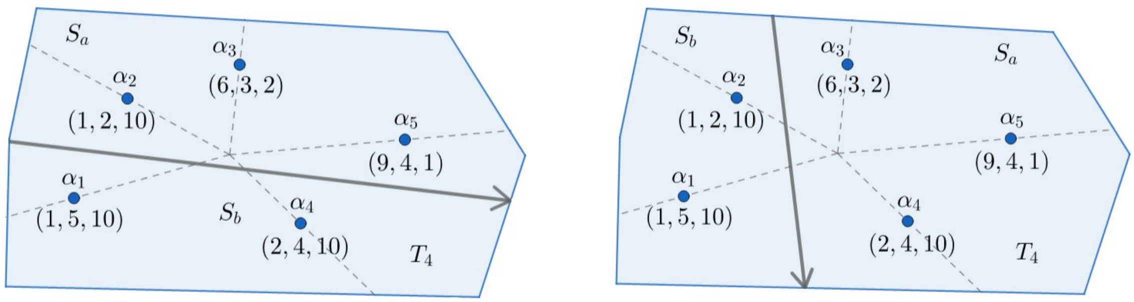
(b) From Gap B to Gap D

Absolute Differences	Balanced?
$\lvert x_1\rvert = 5$	✓
$\lvert x_2\rvert = 8$	✗
$\lvert x_3\rvert = 9$	✓

In Figures 68b and 69a, we see that every directed line being directed from Gap B which balances weight three values either has an imbalance for weight two values or an imbalance for weight one values, implying no directed line directed from Gap B triple-balances the region.

Figures 69b, 70a, and 70b show the only paths to balance weight three values being directed respectively from Gap C, Gap D, and Gap E. Weight one values are imbalanced whenever weight three values are balanced in the Gap C and Gap E cases, and weight two values are imbalanced whenever weight three values are balanced in the Gap D case. No directed line directed from Gap C, Gap D, or Gap E triple-balance T_4.

Figure 69: Directed line gap considerations cont.



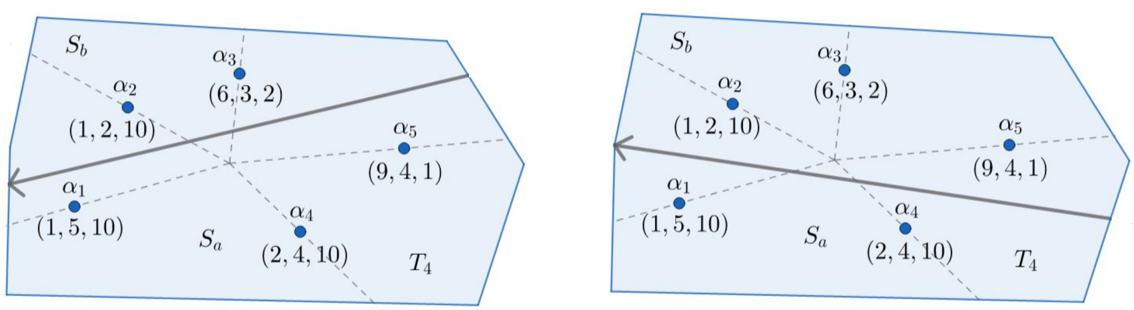
(a) From Gap B to Gap E

(b) From Gap C to Gap A

Absolute Differences	Balanced?
$\lvert x_1 \rvert = 13$	✗
$\lvert x_2 \rvert = 0$	✓
$\lvert x_3 \rvert = 7$	✓

Absolute Differences	Balanced?
$\lvert x_1 \rvert = 15$	✗
$\lvert x_2 \rvert = 4$	✓
$\lvert x_3 \rvert = 7$	✓

Figure 70: Directed line gap considerations cont.



(a) From Gap D to Gap B

(b) From Gap E to Gap B

Absolute Differences	Balanced?
$\lvert x_1 \rvert = 5$	✓
$\lvert x_2 \rvert = 8$	✗
$\lvert x_3 \rvert = 9$	✓

Absolute Differences	Balanced?
$\lvert x_1 \rvert = 13$	✗
$\lvert x_2 \rvert = 0$	✓
$\lvert x_3 \rvert = 7$	✓

Thus, we have shown that wherever weight three values are balanced, either weight one or weight two values are imbalanced. We have proven, then, that there does not exist a directed line in region T_4 that triple-balances the region. As a result, given a region with the usual conditions, there does not always exist a directed line that triple-balances the region. $\square$

3.12 Code-Generated Unique Minimized Balanced Path

Coding was implemented in Python to determine the unique minimized balanced path for any specified region with the usual conditions. In the proof for the Dual-Balanced Theorem, we provided an example to demonstrate some of the expressed statements. The region of that example is displayed again in Figure 71.

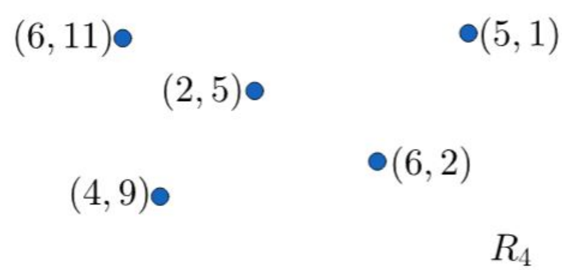
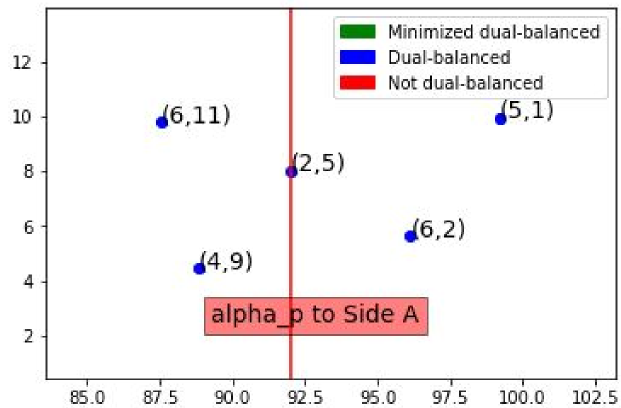


Figure 71: Region R_4

We now will use the generated code to identify α_p for region R_4, where the initial weight-one balancing can be seen in Figure 72. Note that the weight one and weight two values are written by each point, and the assignment decision for the point(s) on the directed line is written in a text box.

Figure 72: Code-generated initial weight-one balancing for region R_4




```
S1a = 12        S1b = 11        x1 = 1         Max Weight 1 Value = 6
Balanced weight one!
S2a = 25        S2b = 3        x2 = 22         Max Weight 2 Value = 11
Not balanced weight two!
Not dual-balanced!
```

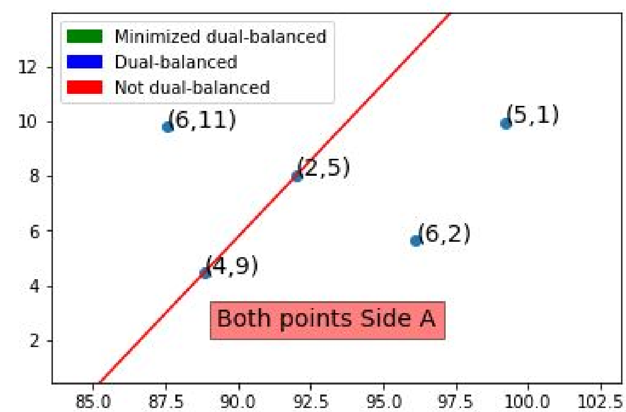
As one can check, the coding is consistent with the initial weight-one balancing in Figure 60, where weight two imbalance was caused by an over-accumulation of weight two values on side S_a. The directed line generated by the initial weight-one balancing process in Figure 72 does not dual-balance region R_4 as signified by the red line. We also see a new term titled *minimized dual-balanced* in the legend of Figure 72. Defined below, a directed line that minimizes dual-balance for a region is colored green, while a directed line that dual-balances the region but does not minimize the dual-balance is colored blue.

Definition 3.19: Minimized dual-balanced

A directed line *minimizes dual-balance* for a region with the usual conditions if the qualifications below are met:

1. The directed line was generated by a unique minimized balanced path
2. The directed line dual-balances the region
3. $|x_1| + |x_2|$ is minimized

After the first clockwise rotation, we see in Figure **??** that there is still imbalance with respect to the weight two values. Note that the code is correctly selecting a Case i action, as the weight one balance is minimized by preserving S_a assignments.

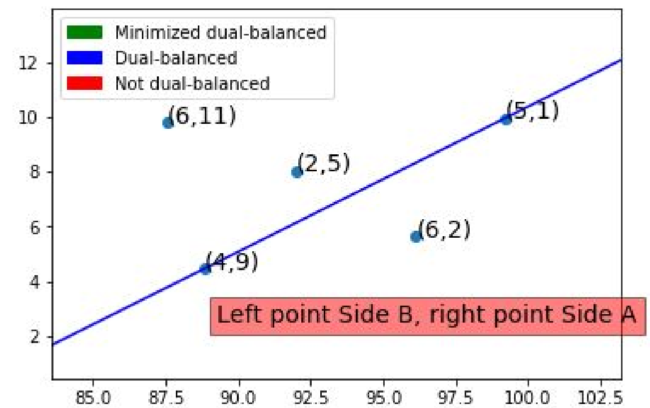



```
S1a = 12        S1b = 11        x1 = 1        Max Weight 1 Value = 6
Balanced weight one!
S2a = 25        S2b = 3        x2 = 22          Max Weight 2 Value = 11
Not balanced weight two!
Not dual-balanced!
```

Figure 73: First code-generated clockwise rotation

The first directed line during this unique minimized balanced path that dual-balances region R_4 is seen in Figure 74, where a Case iv scenario results in a designation switch and a clockwise rotation about the point on the right.

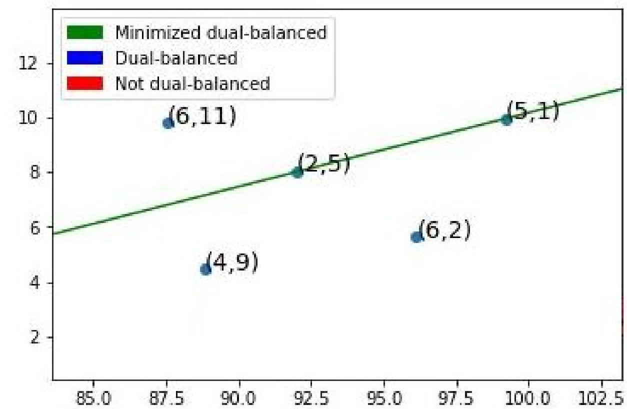
Figure 74: Second code-generated clockwise rotation




```
S1a = 13        S1b = 10        x1 = 3        Max Weight 1 Value = 6
Balanced weight one!
S2a = 17        S2b = 11        x2 = 6          Max Weight 2 Value = 11
Balanced weight two!
Dual-balanced!
```

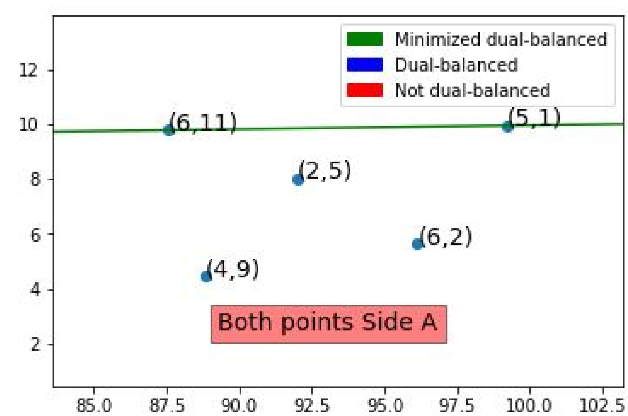
A Subcase iv.b action is accepted in Figure 75, as rotating past the point on the left will generate the minimized balance. We also notice a green line for the first time, implying the minimum value of $|x_1| + |x_2|$ is five for dual-balanced lines along this path. It is also the case for the first time that x_2 is negative. We demonstrated in the proof for Theorem 3.16 that this occurrence must be accompanied by a dual-balanced line, which is verified in Figure 75 for region R_4.

Figure 75: Third code-generated clockwise rotation




```
S1a = 11        S1b = 12        x1 = -1         Max Weight 1 Value = 6
Balanced weight one!
S2a = 12        S2b = 16        x2 = -4           Max Weight 2 Value = 11
Balanced weight two!
Dual-balanced!
```

Both points preserve S_a assignment in the fourth clockwise rotation in Figure 76, resulting in the same minimized dual-balance, but this time with a clockwise rotation about the point on the left.

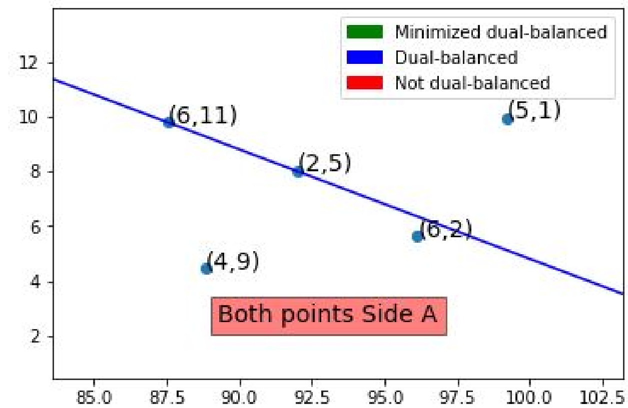



```
S1a = 11        S1b = 12        x1 = -1         Max Weight 1 Value = 6
Balanced weight one!
S2a = 12        S2b = 16        x2 = -4           Max Weight 2 Value = 11
Balanced weight two!
Dual-balanced!
```

Figure 76: Fourth code-generated clockwise rotation

Yet another dual-balanced situation presents itself in Figure 77, where a Case i action is accepted as the minimized balanced path.

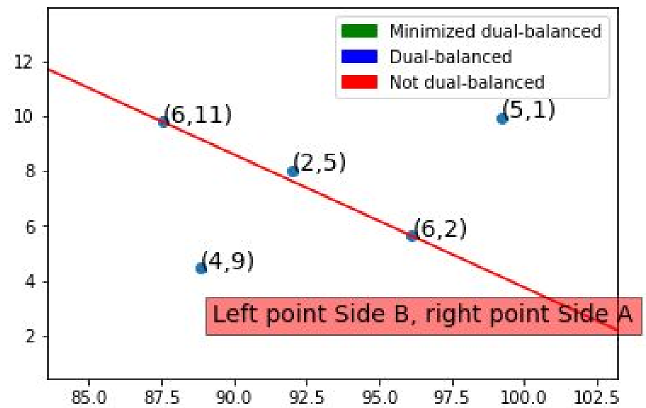
Figure 77: Fifth code-generated clockwise rotation




```
S1a = 13        S1b = 10        x1 = 3        Max Weight 1 Value = 6
Balanced weight one!
S2a = 17        S2b = 11        x2 = 6          Max Weight 2 Value = 11
Balanced weight two!
Dual-balanced!
```

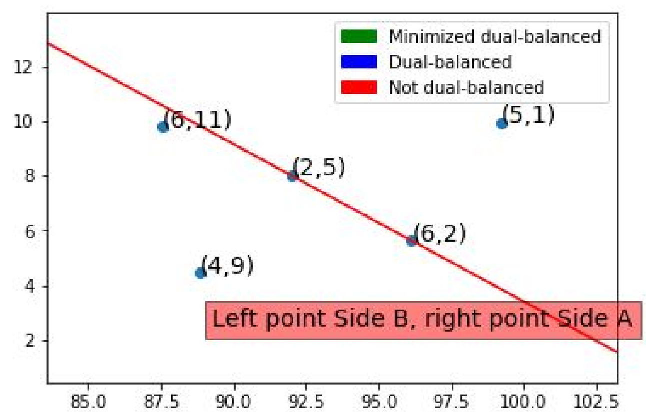
A designation switch in Figure 78 results in a directed line that does not dual-balance the region, and it appears an imbalance for weight two values will continue.

Figure 78: Sixth code-generated clockwise rotation




```
S1a = 13        S1b = 10        x1 = 3         Max Weight 1 Value = 6
Balanced weight one!
S2a = 8        S2b = 20        x2 = -12         Max Weight 2 Value = 11
Not balanced weight two!
Not dual-balanced!
```

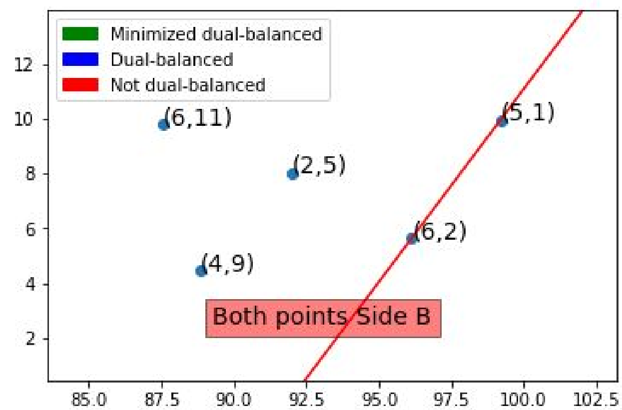
A final designation switch and clockwise rotation about the point on the right will complete the 180° unique minimized balanced path. Note that the values of x_1 and x_2 in Figure 79 are opposite that of their respective initial values in Figure 72.




```
S1a = 11        S1b = 12        x1 = -1        Max Weight 1 Value = 6
Balanced weight one!
S2a = 3        S2b = 25        x2 = -22         Max Weight 2 Value = 11
Not balanced weight two!
Not dual-balanced!
```

Figure 79: Seventh code-generated clockwise rotation

The concluding vertical directed line will contain the point with a weight two value of two, entirely consistent with the definition of a unique minimized balanced path.




```
S1a = 12        S1b = 11        x1 = 1         Max Weight 1 Value = 6
Balanced weight one!
S2a = 25        S2b = 3        x2 = 22          Max Weight 2 Value = 11
Not balanced weight two!
Not dual-balanced!
```

Figure 80: Eighth code-generated clockwise rotation rotates past 180°

4 Application with Grand Forks Census Data

United States census block population data is free to access, and this information will be considered in the following application regarding population disbursement in the city of Grand Forks, North Dakota. Each census block will have a representative point located at its centroid, and the first weight assigned to each representative point is the population of the census block recorded by the 2010 census. The second weight is the population of the census block recorded by the 2020 census.

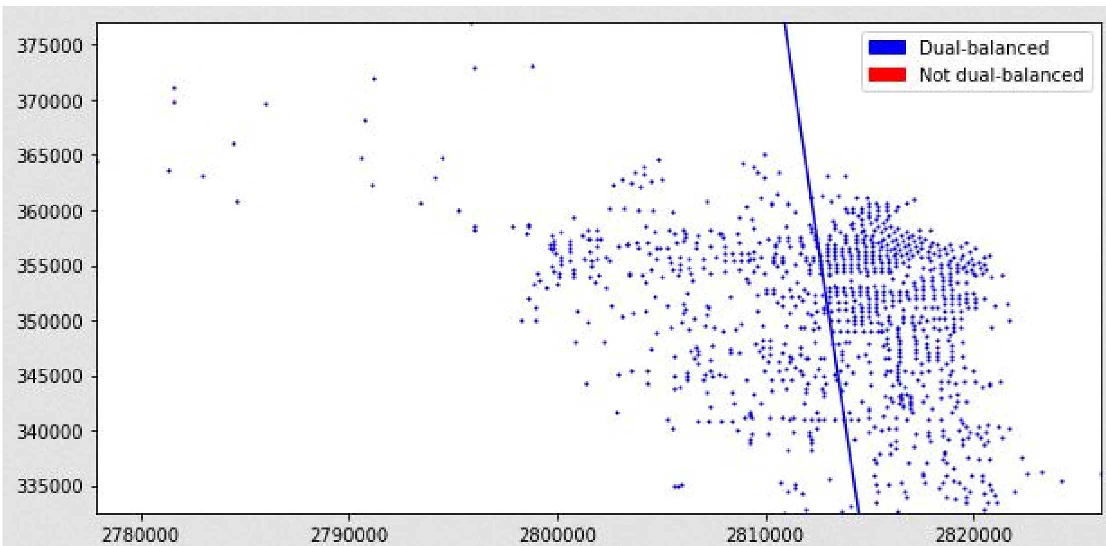
As a result of the Dual-Balanced Theorem, there exists some line so that the 2010 and 2020 populations are balanced. In this case, S_{1a} and S_{1b} respectively describe the population recorded in Grand Forks according to the 2010 census on the left and right of the directed line. S_{2a} and S_{2b} are defined similarly, though they refer to the 2020 census. Finally, ε_1 is the maximum census block population in Grand Forks in 2010, and ε_2 is the maximum census block population in Grand Forks in 2020. We wish to find a line where $|x_1| = |S_{1a} - S_{2b}| \leq \varepsilon_1$ and $|x_2| = |S_{2a} - S_{2b}| \leq \varepsilon_2$.

4.1 Directed Line that Dual-Balances the Region

By the Dual-Balanced Theorem, any region with the usual conditions can be dual-balanced by some line. In this census data context, that means there always exists a line where the 2010 and 2020 populations balance. This directed line would separate the region into sub-regions that perhaps had a similar growth rate over the span of ten years. Note that the path generated by the code in the following examples is a unique minimized balanced path, and, as a result, the weight one values (2010 census population) will always be balanced.

In Figure 81, we see an example of a directed line dual-balancing the region of

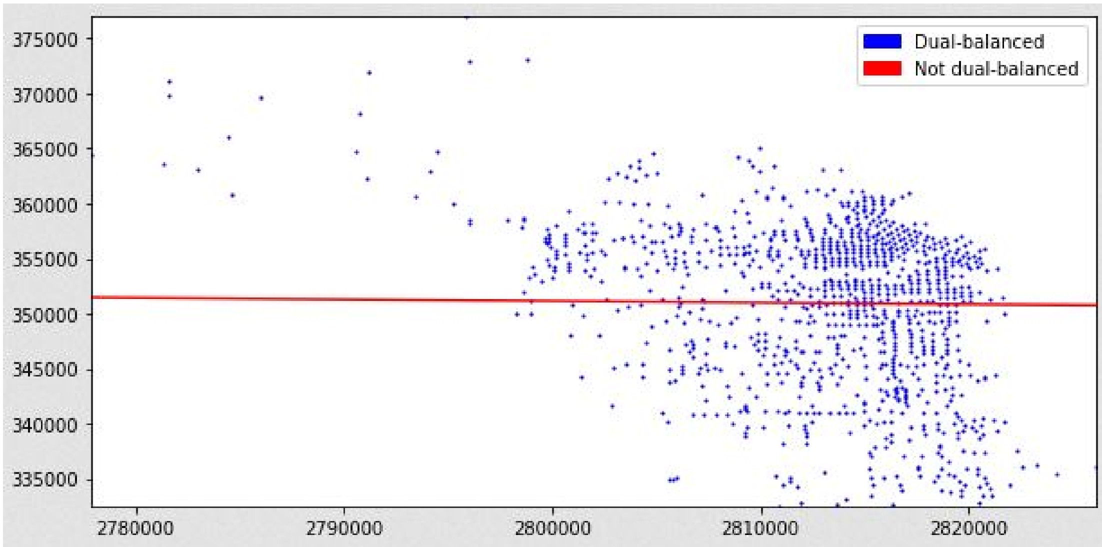
Grand Forks, though the value of $|x_2| = 966$ is rather high, and close to the maximum census block population value from the 2020 census being $\varepsilon_2 = 1379$. With the values below the graph, note that the greatest change was an increase in population on the western side of the directed line from 2010 to 2020. Otherwise, it appears the eastern and western portions of Grand Forks had similar growth trends from 2010 to 2020.




```
S1a = 26683        S1b = 26673        x1 = 10        Max Weight 1 Value = 1219
Balanced weight one!
S2a = 29100        S2b = 30066        x2 = -966        Max Weight 2 Value = 1379
Balanced weight two!
Dual-balanced!
```

Figure 81: Line dual-balancing 2010 and 2020 census block data

4.2 Directed Line that Does Not Dual-Balance the Region



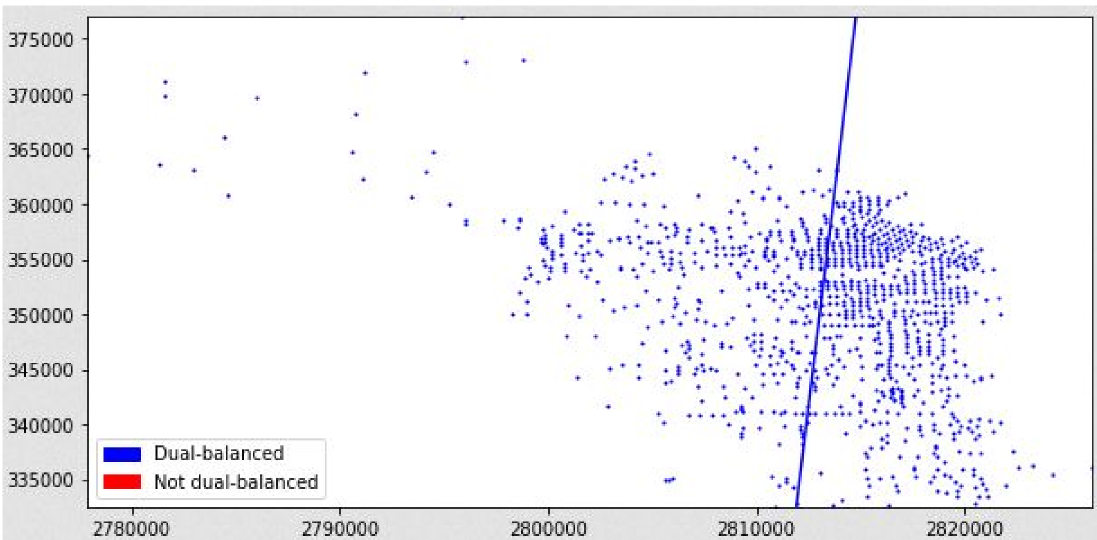

```
S1a = 26682        S1b = 26674        x1 = 8        Max Weight 1 Value = 1219
Balanced weight one!
S2a = 26180        S2b = 32986        x2 = -6806        Max Weight 2 Value = 1379
Not balanced weight two!
Not dual-balanced!
```

Figure 82: Line balancing 2010 but not 2020 census block data

The imbalanced region in Figure 82 demonstrates the population growth of the south end of Grand Forks between the years 2010 and 2020. The directed line certainly balances population disbursement in 2010, but there is a drastic imbalance in 2020. If district boundaries were created in 2010 without a consideration for such a population boom on the south end of town, then possible imbalances may have existed in 2020 before new district lines were drawn. Anticipating and predicting such future population shifts could avoid district imbalances in the future, where we can utilize population prediction models and the Dual-Balanced Theorem to balance population disbursement both now and at a set time in the future.

4.3 Particularly Good Dual-Balancing

Finally, we consider a separation of the Grand Forks region where $|x_1| + |x_2|$ is particularly close to zero. Note in Figure 83 that $x_1 = 28$, $x_2 = -54$, and that both absolute values are relatively low.




```
S1a = 26692        S1b = 26664        x1 = 28        Max Weight 1 Value = 1219
Balanced weight one!
S2a = 29556        S2b = 29610        x2 = -54        Max Weight 2 Value = 1379
Balanced weight two!
Dual-balanced!
```

Figure 83: Line dual-balancing 2010 and 2020 census block population particularly well

Because of how well the above separation dual-balances the region of Grand Forks, we constructed the corresponding district division of Grand Forks, seen in Figure 84.

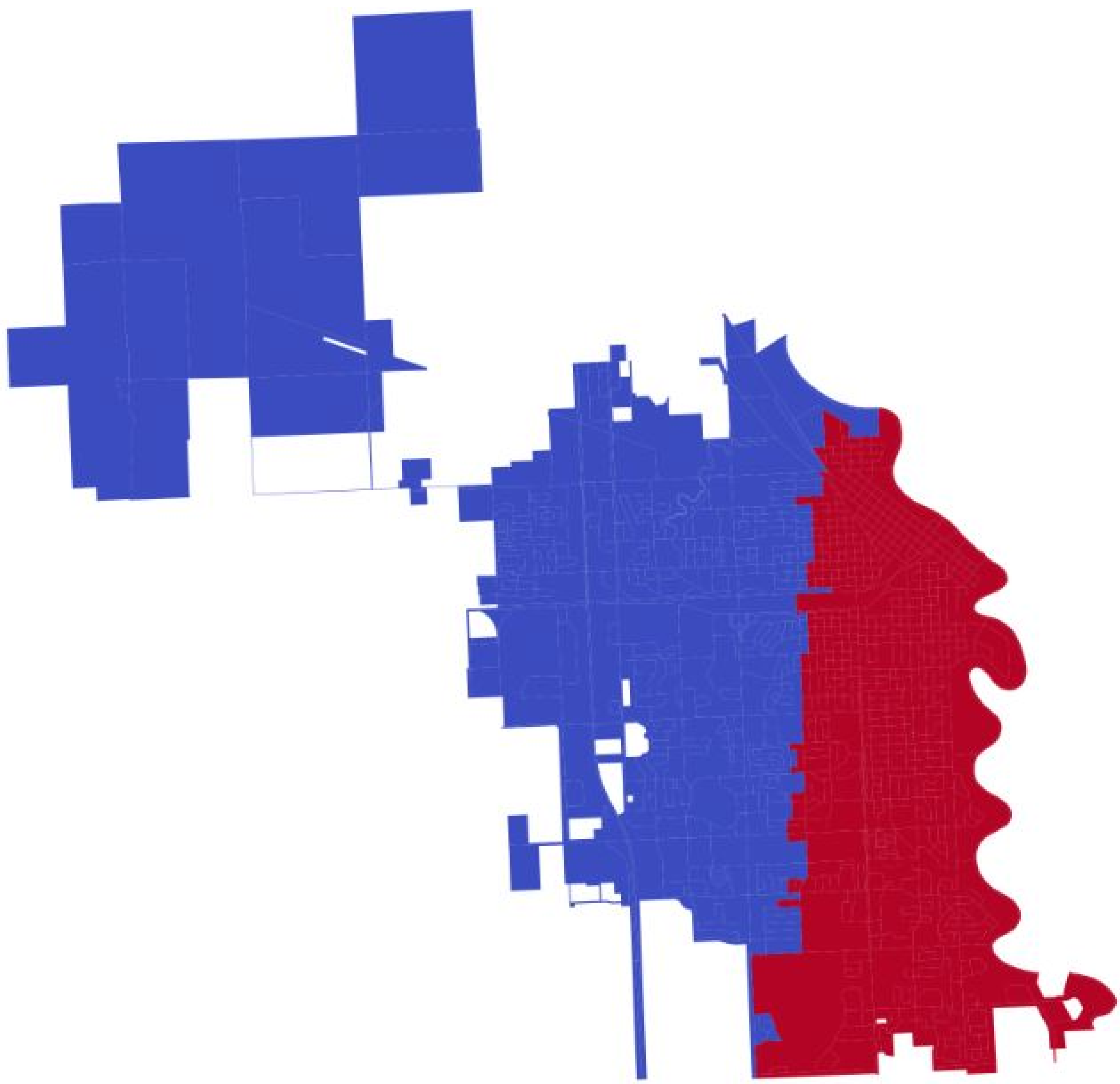


Figure 84: Resulting district division of Grand Forks from directed line in Figure 83

4.4 Application Discussion

The Dual-Balanced Theorem helped to determine a district division that dual-balances census block population data from 2010 and 2020. This same technique could be used if one wanted to use 2020 census data for weight one assignment and implement a population prediction model for weight two assignment. Instead of dual-balancing 2010 and 2020 populations, we would then be able to dual-balance a relatively current population and an estimated future population.

From analyzing the unique minimized balanced path, I noticed that all of the lines that dual-balanced the Grand Forks region more or less separated the region into east and west sub-regions. If one wanted to use the Dual-Balanced Theorem again in a recursive process, it might be difficult to generate lines to dual-balance the sub-regions that do not cause the Grand Forks region to be cut into vertical strips. Continuing the process of dual-balancing may result in an undesirable division of the region that could score low on geographic compactness scores. Though there may be negative effects of this process, a recursive use of the Dual-Balanced Theorem is a natural extension of this work, and this idea is mentioned below in suggestions for future research.

5 Conclusion

By the Dual-Balanced Theorem, if a region has the usual conditions and each point in the region is assigned two non-negative weights, then there exists a line that dual-balances the region. In the determination of this theorem, we defined what we call a *unique minimized balanced path* to demonstrate the feasibility of the Dual-Balanced Theorem through various examples with the aid of Python code.

5.1 Suggestions for Future Research

In the research expressed in [9] and [2], disjoint sub-regions were constructed so that each sub-region contained a proportional number of red points and blue points. A natural extension of this work would be to consider a recursive use of the Dual-Balanced Theorem and how dual-balance would be affected by multiple divisions. Perhaps some of the techniques from [9] and [2], such as an equitable 3-cutting, may prove useful. It would be particularly beneficial to be able to create any number of disjoint, dual-balanced sub-regions given a set of weighted points, as in the census block scenario we would then be able to create a desired number of dual-balanced districts. If a direct generalization in constructing dual-balanced sub-regions is not possible in the weighted scenario, it may be worth considering constraints where progress could be made. In both [9] and [2], constraints were placed on $|S|$ and $|T|$. Perhaps one could consider maximum weight value constraints, a weight variability constraint, or a constraint on the number of sub-regions that could be constructed.

The Dual-Balanced Theorem proves there exists a line that dual-balances a region in a two-dimensional plane. Could this idea be extended to three dimensions, where we could determine a plane that would dual-balance a region in three-dimensional space?

In section four, the population data from 2010 and 2020 were respectively used as weight one and weight two assignments. With more research into current population trends, I think it would have been interesting to use 2020 population data as the weight one assignment, and to have the results of a population prediction model be the weight two assignment. This consideration opens another window of research, in combining the idea of dual-balance with population prediction modeling.

Finally, this paper stated that no two points could be on the same vertical line. Is this stipulation necessary? One could consider writing a proof for the Dual-Balanced Theorem without this condition.

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