

Comparison results for a nonlocal singular elliptic problem

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Abstract

We provide symmetrization results in the form of mass concentration comparisons for fractional singular elliptic equations in bounded domains, coupled with homogeneous external Dirichlet conditions. Two types of comparison results are presented, depending on the summability of the right-hand side of the equation. The maximum principle arguments employed in the core of the proofs of the main results offer a nonstandard, flexible alternative to the ones described in [18, Theorem 31]. Some interesting consequences are L^p regularity results and nonlocal energy estimates for solutions.

1 Introduction

In this paper we consider the following singular nonlocal problem

$$(1.1) \quad \begin{cases} (-\Delta)^s u = \frac{f(x)}{u^\gamma} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Here $s \in (0, 1)$, $(-\Delta)^s$ stands for the fractional Laplacian operator, Ω is a bounded, open set in \mathbb{R}^N ($N \geq 2$) with Lipschitz boundary, $\gamma > 0$ and f is a nonnegative summable function.

Our aim is to use symmetrization techniques in order to get a comparison result between the weak solution to problem (1.1) and the weak solution v to a symmetrized problem, defined in the ball Ω^* centered at the origin having the same measure as Ω , which stays in the same class as the original one (namely singular and nonlocal).

After the seminal paper by Talenti [27], it is well-known that, if $u \in H_0^1(\Omega)$, $v \in H_0^1(\Omega^*)$ solve

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta v = f^* & \text{in } \Omega^* \\ v = 0 & \text{on } \partial\Omega^*, \end{cases}$$

respectively, then

$$(1.2) \quad u^*(x) \leq v(x), \quad x \in \Omega^*$$

(here $*$ stands for the Schwarz rearrangement of a function, see Section 2 for definition and properties). From (1.2) we immediately derive, for instance, that any Lebesgue norm of u is bounded from above by the same Lebesgue norm of v . Hence, the issue of estimating the solution u of a Dirichlet problem in Ω is solved once we can estimate the solution v of a symmetrized problem, which actually is a one-dimensional problem and clearly much easier to handle with.

For local operators, the approach described above has been extended through the years to uniformly elliptic equations with lower order terms, linear and nonlinear parabolic equations, non uniformly elliptic equations, and also to problems with boundary conditions other than Dirichlet. For a survey on the power of symmetrization techniques in both Calculus of Variations and PDEs

theory we refer the interested reader to [29]. In particular, symmetrization techniques have been applied to local, singular problems like (1.1) when, on the left-hand side, the Laplacian operator replaces the fractional one (see [9]).

In the framework of nonlocal problems, the effect of symmetrization on fractional elliptic problems has been investigated for the first time in [14] in a somewhat indirect way. Indeed, there it is used in an essential way the fact that a nonlocal problem involving the fractional Laplacian $(-\Delta)^s$, $s \in (0, 1)$, can be linked to a suitable, local extension problem, whose solution $\psi(x, y)$, an s -harmonic extension of the solution u to the nonlocal problem, is defined on the infinite cylinder $\mathcal{C}_\Omega = \Omega \times (0, +\infty)$, to which classical symmetrization techniques (with respect to the variable $x \in \Omega$) can be applied. For other results concerning the Neumann problems and the nonlocal Gaussian symmetrization see [32], [17], while symmetrization results for fractional parabolic equations of porous medium type have been achieved in [25, 30, 31, 32]. Moreover, we wish to mention here [19], where the case of a fractional nonlinear problem is considered, and [20], where a comparison type result in terms of the L^p norms of solutions (thus weaker than the mass concentration comparison) is established for solutions to equations involving a nonlocal operator with integral kernels, hence not covering the fractional Laplacian.

In this note we adopt the direct approach introduced in [18], where the authors deal with problem (1.1) in the case $\gamma = 0$. This direct approach does not employ the local interpretation of the fractional Laplacian described above, while it makes a clever use of a nonlocal version of the classical Pólya-Szegő inequality, plus a sophisticated representation of the fractional Laplacian of a spherical mean function in $(N + 2)$ dimensions, which allows to conclude by a maximum principle argument.

The real novelty of this paper is that the above mentioned interpretation is *avoided* in the proofs of our new results, thus in this sense they offer an alternative to the crucial part of the proof of [18, Theorem 31]. Furthermore, such a new technique seems to be rather flexible to be used in a broad variety of related contexts. Our main results are the following theorems.

Theorem 1.1. *Let $s \in (0, 1)$, $N \geq 2$, $\gamma > 0$ and assume that $f \in L^\infty(\Omega)$, $f \geq 0$. If u is the weak solution to problem (1.1) and v is the weak solution to the symmetrized problem*

$$(1.3) \quad \begin{cases} (-\Delta)^s v = \frac{\|f\|_{L^\infty(\Omega^*)}}{v^\gamma} & \text{in } \Omega^* \\ v > 0 & \text{in } \Omega^* \\ v = 0 & \text{on } \mathbb{R}^N \setminus \Omega^*, \end{cases}$$

then

$$(1.4) \quad \int_{B_r(0)} u^*(x) \, dx \leq \int_{B_r(0)} v^*(x) \, dx, \quad r > 0.$$

In order to obtain some regularity results depending on the value of γ and on the summability of f , we will also prove the following comparison result.

Theorem 1.2. *Let $s \in (0, 1)$, $N \geq 2$, $\gamma > 0$ and assume that $f \in L^1(\Omega)$, $f \geq 0$. If u is the weak solution to problem (1.1) and v is the weak solution to the following problem*

$$(1.5) \quad \begin{cases} (-\Delta)^s v = (\gamma + 1)f^* & \text{in } \Omega^* \\ v = 0 & \text{on } \mathbb{R}^N \setminus \Omega^*, \end{cases}$$

then

$$(1.6) \quad \int_{B_r(0)} u^*(x)^{\gamma+1} \, dx \leq \int_{B_r(0)} v^*(x) \, dx, \quad r > 0.$$

We stress that analogous estimates in the local case are proved in [9]. For example, in the same reference, instead of (1.4), a comparison result between mass concentrations of $u^*(x)^\gamma$ and $v^*(x)^\gamma$ is proved. It turns out that our result slightly improves the quoted ones when $\gamma > 1$ (see Remark 3.1 for details).

Moreover, some rather simple modifications to our arguments allow us to get comparison results for a singular fractional elliptic equation with a zero order term posed in Ω , of the form

$$(-\Delta)^s u + cu = \frac{f}{u^\gamma},$$

for some bounded coefficient $c \geq 0$, complemented with exterior Dirichlet boundary coefficients.

The paper is organized as follows. In Section 2 we provide the functional setting of the problem and we recall some basic notion about rearrangements. Section 3 is devoted to the proof of Theorem 1.1. In Section 4 we prove Theorem 1.2, which is the key ingredient to prove the regularity results contained in Section 5.

2 Notation and preliminaries

2.1 Functional setting

Let $s \in (0, 1)$. For any open set Ω and any measurable function u on Ω , we introduce the fractional Gagliardo seminorm

$$[u]_{H^s(\Omega)} = \left(\iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}.$$

Then we define the fractional Sobolev space $H^s(\Omega)$ as the space

$$H^s(\Omega) = \{u \in L^2(\Omega) : [u]_{H^s(\Omega)} < \infty\},$$

endowed with the norm

$$\|u\|_{H^s(\Omega)} = \|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)}.$$

We denote by $H_0^s(\Omega)$ the closure of $C_c^\infty(\Omega)$ in the $H^s(\Omega)$ topology. Moreover, we will define the space

$$H_{\text{loc}}^s(\Omega) = \{u : \Omega \rightarrow \mathbb{R} : u|_K \in H^s(K), \text{ for all } K \subset\subset \Omega\}.$$

There is a strict connection between the space $H^s(\mathbb{R}^N)$ and the fractional Laplacian operator $(-\Delta)^s$. For any $s \in (0, 1)$ and $u \in \mathcal{S}$ (the classical Schwartz class), the fractional Laplacian operator is defined as

$$(-\Delta)^s u = \gamma(N, s) \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

where

$$(2.1) \quad \gamma(N, s) = \frac{s 2^{2s} \Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-s)}.$$

The following result can be found in [15, Proposition 3.6].

Proposition 2.1. *Let $s \in (0, 1)$ and $u \in H^s(\mathbb{R}^N)$. Then*

$$[u]_{H^s(\mathbb{R}^N)}^2 = \frac{2}{\gamma(N, s)} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)}^2.$$

The analytic theory of the fractional Laplacian in the whole \mathbb{R}^N is nowadays considered classical and we refer the interested reader for example to [26].

We are interested in Dirichlet problems defined in bounded domains. To this aim, we consider the space $X_0^s(\Omega)$, defined as

$$X_0^s(\Omega) = \left\{ u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.$$

When Ω is an open bounded set with Lipschitz boundary, it can be proved that (see [12, Proposition B.1]) $X_0^s(\Omega)$ coincides with the completion of $C_c^\infty(\Omega)$ with respect to the seminorm $[\cdot]_{H^s(\mathbb{R}^N)}$. Moreover, when $2s \neq 1$, it can also be proved that $X_0^s(\Omega)$ coincides with $H_0^s(\Omega)$ (see [10, Proposition B.1]), while in general for $2s = 1$, we have a strict inclusion

$$X_0^s(\Omega) \subset H_0^s(\Omega)$$

(see [11, Remark 2.1]). Indeed, we have that $X_0^s(\Omega)$ coincides with the interpolation space $H_{00}^{1/2}(\Omega)$ (see [8, Appendix]).

A consequence of fractional Poincaré inequalities (see [10, Lemma 2.4]) is that we can equip the space $X_0^s(\Omega)$ with the Gagliardo seminorm

$$\|u\|_{X_0^s(\Omega)} = [u]_{H^s(\mathbb{R}^N)} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}.$$

From the definition of $X_0^s(\Omega)$ it easily follows that for each $u \in X_0^s(\Omega)$

$$\|u\|_{X_0^s(\Omega)} = \left(\iint_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}$$

where $Q = \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega = \mathbb{R}^N \setminus \Omega$.

Then we consider the *restricted* fractional Laplacian $(-\Delta|_\Omega)_{rest}$ on Ω , defined by duality on the space $X_0^s(\Omega)$. Since there will be no matter of confusion, we shall keep the classical notation $(-\Delta)^s$ for such operator. Moreover, denoted by $X^{-s}(\Omega)$ its dual, the operator

$$(-\Delta)^s : X_0^s(\Omega) \rightarrow X^{-s}(\Omega)$$

is continuous.

Finally, we recall that the following fractional Sobolev embedding holds true, see for instance [15, Theorem 6.5].

Theorem 2.1. *Let $s \in (0, 1)$ and $N > 2s$. There exists a positive constant $\mathfrak{S}(N, s)$ such that, for any measurable and compactly supported function $u : \mathbb{R}^N \rightarrow \mathbb{R}$, it holds*

$$\|u\|_{L^{2_s^*}(\mathbb{R}^N)}^2 \leq \mathfrak{S}(N, s) \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy,$$

where

$$2_s^* = \frac{2N}{N - 2s}$$

is the critical Sobolev exponent. In particular, if $u \in X_0^s(\Omega)$, we have

$$\|u\|_{L^{2_s^*}(\Omega)}^2 \leq \mathfrak{S}(N, s) \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy.$$

We end this subsection with an inequality that will turn out to be very useful in the sequel. We recall that, when we deal with fractional derivatives, the chain rule does not hold true. It can be replaced by an inequality where a convex or concave function is involved (see [23, Proposition 4]) and [11, Lemma 3.3].

Proposition 2.2. *Assume that $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz continuous, convex function, such that $\Phi(0) = 0$. Then, if $u \in X_0^s(\Omega)$, we have*

$$(-\Delta)^s \Phi(u) \leq \Phi'(u)(-\Delta)^s u \quad \text{weakly in } \Omega,$$

in the sense that for all nonnegative $\varphi \in X_0^s(\Omega)$ we have

$$(2.2) \quad \iint_{\mathbb{R}^{2N}} \frac{[\Phi(u(x)) - \Phi(u(y))][\varphi(x) - \varphi(y)]}{|x - y|^{N+2s}} dx dy \\ \leq \iint_{\mathbb{R}^{2N}} \frac{[u(x) - u(y)][\Phi'(u(x))\varphi(x) - \Phi'(u(y))\varphi(y)]}{|x - y|^{N+2s}} dx dy.$$

Analogously, if $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz continuous, concave function, such that $\Psi(0) = 0$, and $u \in X_0^s(\Omega)$, then

$$(-\Delta)^s \Psi(u) \geq \Psi'(u)(-\Delta)^s u \quad \text{weakly in } \Omega.$$

2.2 Schwarz symmetrization

We now recall some notions about Schwarz symmetrization and some related fundamental properties. For more details we refer the interested reader, for instance, to [4, 6, 21, 28].

Let u be a real measurable function on Ω . If u is such that its *distribution function* μ_u satisfies

$$\mu_u(t) := |\{x \in \Omega : |u(x)| > t\}| < +\infty, \quad \text{for every } t > 0,$$

we define the decreasing rearrangement of u as the generalized inverse of μ_u , that is

$$u^*(\sigma) = \sup\{t \geq 0 : \mu_u(t) > \sigma\}, \quad \sigma > 0.$$

The radially symmetric, decreasing rearrangement of u , also known as the Schwarz decreasing rearrangement of u , is hence defined as

$$u^*(x) = u^*(\omega_N |x|^N) \quad x \in \Omega^*,$$

where ω_N is the measure of the unitary ball in \mathbb{R}^N , and Ω^* is the ball (centered at the origin) having the same measure as Ω . From the definitions given above we can easily deduce that u , u^* and u^* are equi-distributed, that is

$$\mu_u = \mu_{u^*} = \mu_{u^*}.$$

Moreover, the following properties hold true.

Proposition 2.3. *Let $u, v : \Omega \rightarrow \mathbb{R}$ be two measurable functions satisfying*

$$(2.3) \quad \mu_u(t) < \infty, \quad \mu_v(t) < \infty, \quad \text{for every } t > 0.$$

Then

- (i) if $|v| \leq |u|$ a.e., then $v^* \leq u^*$;
- (ii) $(cu)^* = |c|u^*$, for every $c \in \mathbb{R}$;

(iii) if $H : [0, \infty] \rightarrow [0, \infty]$ is an increasing, continuous function, then $H(|u|)^* = H(u^*)$;

(iv) if $u \in L^p(\Omega)$, $1 \leq p \leq \infty$, then $u^* \in L^p(0, |\Omega|)$, $u^* \in L^p(\Omega^*)$ and

$$\|u\|_{L^p(\Omega)} = \|u^*\|_{L^p(0, |\Omega|)} = \|u^*\|_{L^p(\Omega^*)}.$$

Furthermore, the celebrated Hardy-Littlewood inequality holds true

$$(2.4) \quad \int_{\Omega} |u(x)v(x)| \, dx \leq \int_0^{|\Omega|} u^*(r)v^*(r) \, dr = \int_{\Omega^*} u^*(x)v^*(x) \, dx.$$

We can also define the maximal function of the rearrangement of u

$$u^{**}(\sigma) = \frac{1}{\sigma} \int_0^{\sigma} u^*(t) \, dt, \quad \sigma > 0.$$

It is easy to prove (see, for example, [6]) that the Lebesgue norms of u^* and u^{**} are equivalent, that is there exists $C > 0$ such that

$$\|u^*\|_{L^p(0, |\Omega|)} \leq \|u^{**}\|_{L^p(0, |\Omega|)} \leq C \|u^*\|_{L^p(0, |\Omega|)}.$$

Since we will prove comparison results between integrals of solutions to nonlocal problems, the following definition will play a fundamental role.

Definition 2.1. Let $u, v \in L^1_{\text{loc}}(\mathbb{R}^N)$. We say that u is less concentrated than v , and we write $u \prec v$, if for every $\sigma > 0$ we have

$$\int_0^{\sigma} u^*(t) \, dt \leq \int_0^{\sigma} v^*(t) \, dt,$$

or, equivalently, for every $r > 0$,

$$\int_{B_r(0)} u^*(x) \, dx \leq \int_{B_r(0)} v^*(x) \, dx.$$

Clearly, this definition can be adapted to functions defined in an open subset Ω of \mathbb{R}^N , by extending the functions to zero outside Ω . The partial order relationship \prec is called comparison of mass concentrations and it satisfies some nice properties (see [2]).

Proposition 2.4. Let $u, v \in L^1(\Omega)$ be two nonnegative functions. Then, the following statements are equivalent:

(a) $u \prec v$;

(b) for all nonnegative $\varphi \in L^\infty(\Omega)$

$$\int_{\Omega} u(x)\varphi(x) \, dx \leq \int_0^{|\Omega|} v^*(r)\varphi^*(r) \, dr = \int_{\Omega^*} v^*(x)\varphi^*(x) \, dx;$$

(c) for all convex, nonnegative, Lipschitz function Φ , such that $\Phi(0) = 0$,

$$\int_{\Omega} \Phi(u(x)) \, dx \leq \int_{\Omega} \Phi(v(x)) \, dx.$$

From Proposition 2.4 we immediately deduce that, if $u \prec v$, then

$$\|u\|_{L^p(\Omega)} \leq \|v\|_{L^p(\Omega)}, \quad 1 \leq p \leq +\infty.$$

Moreover, if $u, v \in L^p(\Omega)$ with $p > 1$, the inequality in point (b) above holds for all nonnegative $\varphi \in L^{p'}(\Omega)$.

We end this subsection by recalling the following generalization of the Riesz rearrangement inequality (see, for example, [1, Theorem 2.2]).

Proposition 2.5. *Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function such that $F(0, 0) = 0$ and*

$$F(u_1, v_1) + F(u_2, v_2) \geq F(u_1, v_2) + F(u_2, v_1)$$

whenever $u_2 \geq u_1 > 0$ and $v_2 \geq v_1 > 0$. Assume that u, v are two nonnegative, measurable functions on \mathbb{R}^N satisfying (2.3). Then

$$(2.5) \quad \iint_{\mathbb{R}^{2N}} F(u(x), v(y)) W(ax + by) \, dx \, dy \leq \iint_{\mathbb{R}^{2N}} F(u^*(x), v^*(y)) W(ax + by) \, dx \, dy$$

and

$$\int_{\mathbb{R}^N} F(u(x), v(x)) \, dx \leq \int_{\mathbb{R}^N} F(u^*(x), v^*(x)) \, dx,$$

for any nonnegative $W \in L^1(\mathbb{R}^N)$ and any choice of nonzero numbers a, b .

2.3 Two fundamental lemmata

We start by proving the following result, which will be fundamental in the crucial maximum principle arguments established in the proofs of Theorem 1.1 and Theorem 1.2. It is based on a technique introduced in [3, Theorem 1] and subsequently used in [30, Theorem 3.2].

Lemma 2.1. *Let u, v be two nonnegative, continuous functions on $[0, R]$. Let us define*

$$H_u(r) = \int_0^r u(\rho) \rho^{N-1} \, d\rho \quad H_v(r) = \int_0^r v(\rho) \rho^{N-1} \, d\rho$$

and

$$K_u(r) = \int_r^R u(\rho) \rho^{N-1} \, d\rho \quad K_v(r) = \int_r^R v(\rho) \rho^{N-1} \, d\rho.$$

Assume that $H_u(r) - H_v(r)$ admits a positive maximum point at $\bar{r} > 0$, that is,

$$0 < H_u(\bar{r}) - H_v(\bar{r}) = \max_{r \in [0, R]} (H_u(r) - H_v(r)).$$

Then, if $h \not\equiv 0$ is a positive, increasing bounded function on $(0, R)$, we have

$$(2.6) \quad \int_0^{\bar{r}} u(\rho) h(\rho) \rho^{N-1} \, d\rho - \int_0^{\bar{r}} v(\rho) h(\rho) \rho^{N-1} \, d\rho > 0.$$

Analogously, assume that $K_u(r) - K_v(r)$ admits a negative minimum point at $\bar{r} < R$, that is,

$$0 > K_u(\bar{r}) - K_v(\bar{r}) = \min_{r \in [0, R]} (K_u(r) - K_v(r)).$$

Then, if $h \not\equiv 0$ is a positive, decreasing bounded function on $(0, R)$, we have

$$(2.7) \quad \int_{\bar{r}}^R u(\rho) h(\rho) \rho^{N-1} \, d\rho - \int_{\bar{r}}^R v(\rho) h(\rho) \rho^{N-1} \, d\rho < 0.$$

Proof. It is enough to observe that (2.6) is an immediate consequence of the following integration by parts

$$\begin{aligned} & \int_0^{\bar{r}} u(\rho)h(\rho)\rho^{N-1} \, d\rho - \int_0^{\bar{r}} v(\rho)h(\rho)\rho^{N-1} \, d\rho \\ &= h(0)(H_u(\bar{r}) - H_v(\bar{r})) + \int_0^{\bar{r}} \left[(H_u(\bar{r}) - H_v(\bar{r})) - (H_u(\rho) - H_v(\rho)) \right] dh(\rho) > 0. \end{aligned}$$

Analogously, concerning (2.7), we have

$$\begin{aligned} & \int_{\bar{r}}^R u(\rho)h(\rho)\rho^{N-1} \, d\rho - \int_{\bar{r}}^R v(\rho)h(\rho)\rho^{N-1} \, d\rho \\ &= h(R)(K_u(\bar{r}) - K_v(\bar{r})) - \int_{\bar{r}}^R \left[(K_u(\bar{r}) - K_v(\bar{r})) - (K_u(\rho) - K_v(\rho)) \right] dh(\rho) < 0. \end{aligned}$$

□

Remark 2.1. We explicitly observe that, reasoning as in the proof of Lemma 2.1, we can prove that, if $\max_{r \in [0, R]} (H_u(r) - H_v(r)) = 0$, then

$$\int_0^{\bar{r}} u(\rho)h(\rho)\rho^{N-1} \, d\rho - \int_0^{\bar{r}} v(\rho)h(\rho)\rho^{N-1} \, d\rho \geq 0.$$

Analogously, if $\min_{r \in [0, R]} (K_u(r) - K_v(r)) = 0$, then

$$\int_{\bar{r}}^R u(\rho)h(\rho)\rho^{N-1} \, d\rho - \int_{\bar{r}}^R v(\rho)h(\rho)\rho^{N-1} \, d\rho \leq 0.$$

Remark 2.2. If $\bar{r} \in (0, R)$ is a maximum point for $H_u(r) - H_v(r)$, then \bar{r} is a non-positive minimum point for $K_u(r) - K_v(r)$. Indeed, it is easy to see that

$$\begin{aligned} K_u(r) - K_v(r) &= (H_u(R) - H_v(R)) - (H_u(r) - H_v(r)) \\ &\geq (H_u(R) - H_v(R)) - (H_u(\bar{r}) - H_v(\bar{r})) = K_u(\bar{r}) - K_v(\bar{r}). \end{aligned}$$

Being \bar{r} a maximum point for $H_u(r) - H_v(r)$ we have $K_u(\bar{r}) - K_v(\bar{r}) \leq 0$ and, by the above inequality,

$$\min_{r \in [0, R]} (K_u(r) - K_v(r)) = K_u(\bar{r}) - K_v(\bar{r}) \leq 0.$$

Lemma 2.1. Let $\gamma > 0$. Then, for every $a, b > 0$, we have

$$(2.8) \quad \frac{1}{a^\gamma} - \frac{1}{b^\gamma} \leq \frac{\gamma}{a^{\gamma+1}}(b - a).$$

Proof. It is immediate to check that, setting $g(t) = t^\gamma + \frac{\gamma}{t}$,

$$\min_{t > 0} g(t) = g(1) = \gamma + 1.$$

Choosing $t = \frac{a}{b}$ we get the claim. □

2.4 A key function

We end this section by discussing some properties of the function

$$\Theta_{N,s}(r, \rho) = \frac{1}{N\omega_N} \int_{|x'|=1} \left(\int_{|y'|=1} \frac{1}{|r x' - \rho y'|^{N+2s}} d\mathcal{H}^{N-1}(y') \right) d\mathcal{H}^{N-1}(x')$$

defined for $r, \rho > 0$. We observe that the internal integral does not depend on x' . So we can compute it by choosing any fixed x' and we obtain

$$\begin{aligned} \Theta_{N,s}(r, \rho) &= \int_{|y'|=1} \frac{1}{|r x' - \rho y'|^{N+2s}} d\mathcal{H}^{N-1}(y') \\ (2.9) \quad &= \frac{2\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} \int_0^\pi \frac{\sin^{N-2} \theta}{(r^2 - 2r\rho \cos \theta + \rho^2)^{\frac{N+2s}{2}}} d\theta. \end{aligned}$$

Identity (2.9) immediately infers that $\Theta_{N,s}(r, \rho)$ is symmetric, that is

$$\Theta_{N,s}(r, \rho) = \Theta_{N,s}(\rho, r), \quad r, \rho > 0.$$

Moreover,

$$(2.10) \quad \Theta_{N,s}(r, \rho) = \begin{cases} \frac{2\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} \rho^{-N-2s} {}_2F_1\left(\frac{N+2s}{2}, s+1; \frac{N}{2}; \frac{r^2}{\rho^2}\right) & \text{if } 0 \leq r < \rho < +\infty \\ \frac{2\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} r^{-N-2s} {}_2F_1\left(\frac{N+2s}{2}, s+1; \frac{N}{2}; \frac{\rho^2}{r^2}\right) & \text{if } 0 \leq \rho < r < +\infty, \end{cases}$$

where ${}_2F_1(a, b; c; x)$ is the hypergeometric function (see, for example, [22, Ch. 9]) defined by

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \tau^{b-1} (1-\tau)^{c-b-1} (1-x\tau)^{-a} d\tau, \quad c > b > 0, 0 < \tau < 1.$$

It is well-known that

$${}_2F_1'(a, b; c; x) = \frac{ab}{c} {}_2F_1(a+1, b+1; c+1; x).$$

Hence we immediately get that, if $\bar{r} > 0$, $\Theta_{N,s}(r, \rho)$ is increasing with respect to $r \in [0, \bar{r}]$ for any fixed $\rho > \bar{r}$, while it is decreasing with respect to $\rho > \bar{r}$ for any fixed $r \in [0, \bar{r}]$.

Finally, using (2.10), we have the following asymptotic behaviors:

$$(2.11) \quad \Theta_{N,s}(r, \rho) \sim \frac{1}{|r - \rho|^{1+2s}} \quad \text{as } |r - \rho| \rightarrow 0.$$

and

$$(2.12) \quad \Theta_{N,s}(r, \rho) \sim \frac{1}{r^{N+2s}} \quad \text{as } r \rightarrow +\infty, \quad \Theta_{N,s}(r, \rho) \sim \frac{1}{\rho^{N+2s}} \quad \text{as } \rho \rightarrow +\infty.$$

3 Proof of Theorem 1.1

Before proving our main result we need to specify the notion of solution to problem (1.1). Note that, due to the lack of regularity of solutions near the boundary, the notion of solution has to be understood in the weak distributional meaning, for test functions compactly supported in the domain. Furthermore, the nonlocal nature of the operator has to be taken into account.

We will adopt the following notion of solution contained in [13].

Definition 3.1. We say that a positive function $u \in H_{\text{loc}}^s(\Omega) \cap L^1(\Omega)$ is a weak solution to problem (1.1) if

$$u^{\max\{\frac{\gamma+1}{2}, 1\}} \in X_0^s(\Omega), \quad \frac{f}{u^\gamma} \in L_{\text{loc}}^1(\Omega),$$

and, for every nonnegative $\varphi \in C_c^\infty(\Omega)$, we have

$$\frac{\gamma(N, s)}{2} \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy = \int_{\Omega} \frac{f(x)}{u(x)^\gamma} \varphi(x) dx,$$

with $\gamma(N, s)$ defined in (2.1).

In [13, Theorem 1.2] (see also [5]), the authors prove the existence of a weak solution to problem (1.1) with $\text{essinf}_K u > 0$ for every compact set $K \subset\subset \Omega$, distinguishing two cases according to the value of γ : 1) (mildly singular) when $0 < \gamma \leq 1$ and $f \in L^p(\Omega)$, then there exists a solution $u \in X_0^s(\Omega)$; 2) (strongly singular) when $\gamma > 1$ and $f \in L^1(\Omega)$, then there exists a solution $u \in H_{\text{loc}}^s(\Omega) \cap L^1(\Omega)$ such that $u^{\frac{\gamma+1}{2}} \in X_0^s(\Omega)$. In the same paper the authors also discuss the uniqueness of such solutions. Since the way of understanding the boundary condition is not unambiguous, they start with the following:

Definition 3.2. Let u be such that $u = 0$ in $\mathbb{R}^N \setminus \Omega$. We say that $u \leq 0$ on $\partial\Omega$ if, for every $\varepsilon > 0$, it follows that

$$(u - \varepsilon)_+ \in X_0^s(\Omega).$$

We will say that $u = 0$ on $\partial\Omega$ if u is nonnegative and $u \leq 0$ on $\partial\Omega$.

Adopting such a definition, in [13, Theorem 1.4] the authors also show if $\gamma > 0$ and $f \in L^1(\Omega)$, there exists at most one weak solution to problem (1.1).

We can finally prove Theorem 1.1.

PROOF OF THEOREM 1.1.

We split the proof into different steps.

Step 1. Approximating problems

For every $k \in \mathbb{N}$ let us define

$$f_k := \min \{f(x), k\}$$

and let us consider the following sequence of nonsingular approximating problems

$$(3.1) \quad \begin{cases} (-\Delta)^s u_k = \frac{f_k}{(u_k + \frac{1}{k})^\gamma} & \text{in } \Omega \\ u_k > 0 & \text{in } \Omega \\ u_k = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases}$$

For every $k \in \mathbb{N}$ problem (3.1) has a nonnegative solution belonging to $X_0^s(\Omega) \cap L^\infty(\Omega)$ (see [5, Lemma 3.1]), which means that

$$(3.2) \quad \frac{\gamma(N, s)}{2} \iint_Q \frac{(u_k(x) - u_k(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy = \int_{\Omega} \frac{f_k(x)}{(u_k(x) + \frac{1}{k})^\gamma} \varphi(x) dx$$

for every $\varphi \in X_0^s(\Omega)$. Moreover, the sequence u_k is increasing, $u_k > 0$ in Ω , and, for every subset $\omega \subset\subset \Omega$, there exists a positive constant c_ω , independent of k , such that $u_k(x) \geq c_\omega > 0$ for every $x \in \omega$ and $k \in \mathbb{N}$ (see [5, Lemma 3.2]).

Step 2. Reduction to the radial case

Let $0 \leq t < \|u_k\|_{L^\infty(\Omega)}$ and $h > 0$. We consider the following test function

$$\varphi(x) = \mathcal{G}_{t,h}(u_k(x)),$$

where $\mathcal{G}_{t,h}(\theta)$ is defined as follows:

$$\mathcal{G}_{t,h}(\theta) = \begin{cases} h & \text{if } \theta > t + h \\ \theta - t & \text{if } t < \theta \leq t + h \\ 0 & \text{if } \theta \leq t. \end{cases}$$

We explicitly observe that $\mathcal{G}_{t,h}(\theta) \in X_0^s(\Omega)$, so we can use it in the weak formulation of solution (3.2), obtaining

$$(3.3) \quad \frac{\gamma(N, s)}{2} \iint_{\mathbb{R}^{2N}} \frac{[u_k(x) - u_k(y)] [\mathcal{G}_{t,h}(u_k(x)) - \mathcal{G}_{t,h}(u_k(y))]}{|x - y|^{N+2s}} dx dy = \int_{\Omega} \frac{f_k(x)}{(u_k(x) + \frac{1}{k})^\gamma} \mathcal{G}_{t,h}(u_k(x)) dx.$$

We first deal with the left-hand side in (3.3). All the arguments are contained in the proof of Theorem 3.1 in [18], but we summarize them here for the reader's convenience.

We start by writing

$$\iint_{\mathbb{R}^{2N}} \frac{[u_k(x) - u_k(y)] [\mathcal{G}_{t,h}(u_k(x)) - \mathcal{G}_{t,h}(u_k(y))]}{|x - y|^{N+2s}} dx dy = \frac{1}{\Gamma(\frac{N+2s}{2})} \int_0^\infty \mathcal{J}_\alpha[u_k, t, h] \alpha^{\frac{N+2s}{2}-1} d\alpha,$$

where

$$\mathcal{J}_\alpha[u_k, t, h] = \iint_{\mathbb{R}^{2N}} [u_k(x) - u_k(y)] [\mathcal{G}_{t,h}(u_k(x)) - \mathcal{G}_{t,h}(u_k(y))] e^{-\alpha|x-y|^2} dx dy.$$

Riesz inequality (2.5), with the choices

$$F(u_k, v_k) = u_k^2 + v_k^2 - (u_k - v_k) (\mathcal{G}_{t,h}(u_k) - \mathcal{G}_{t,h}(v_k)), \quad W_\alpha(x) = e^{-\alpha|x|^2}, \quad a = 1, \quad b = -1,$$

implies

$$\iint_{\mathbb{R}^{2N}} F(u_k(x), u_k(y)) W_\alpha(x - y) dx dy \leq \iint_{\mathbb{R}^{2N}} F(u_k^*(x), u_k^*(y)) W_\alpha(x - y) dx dy,$$

which immediately gives

$$\mathcal{J}_\alpha[u_k, t, h] \geq \mathcal{J}_\alpha[u_k^*, t, h].$$

Hence

$$(3.4) \quad \iint_{\mathbb{R}^{2N}} \frac{[u_k(x) - u_k(y)] [\mathcal{G}_{t,h}(u_k(x)) - \mathcal{G}_{t,h}(u_k(y))]}{|x - y|^{N+2s}} dx dy \geq \iint_{\mathbb{R}^{2N}} \frac{[u_k^*(x) - u_k^*(y)] [\mathcal{G}_{t,h}(u_k^*(x)) - \mathcal{G}_{t,h}(u_k^*(y))]}{|x - y|^{N+2s}} dx dy.$$

In order to simplify the notation, from now on $\mu_k(x) = \mu_k(|x|)$ will stand for $u_k^*(x)$. We change the variables in the right-hand side of (3.4) and we obtain

$$(3.5) \quad \iint_{\mathbb{R}^{2N}} \frac{[u_k^*(x) - u_k^*(y)] [\mathcal{G}_{t,h}(u_k^*(x)) - \mathcal{G}_{t,h}(u_k^*(y))]}{|x - y|^{N+2s}} dx dy = N\omega_N \int_0^{+\infty} \left(\int_0^{+\infty} [\mu_k(r) - \mu_k(\rho)] [\mathcal{G}_{t,h}(\mu_k(r)) - \mathcal{G}_{t,h}(\mu_k(\rho))] \Theta_{N,s}(r, \rho) \rho^{N-1} d\rho \right) r^{N-1} dr,$$

where $\Theta_{N,s}(r, \rho)$ is the function defined in (2.9).

We split the integral in the right-hand side of (3.5) into the sum

$$\mathcal{J}^1 + 2\mathcal{J}^2 + 2\mathcal{J}^3 + 2h\mathcal{J}^4,$$

where

$$\begin{aligned} \mathcal{J}^1 &= \int_{r(t+h)}^{r(t)} \left(\int_{r(t+h)}^{r(t)} (\mu_k(r) - \mu_k(\rho))^2 \Theta_{N,s}(r, \rho) \rho^{N-1} d\rho \right) r^{N-1} dr, \\ \mathcal{J}^2 &= \int_0^{r(t+h)} \left(\int_{r(t+h)}^{r(t)} (\mu_k(r) - \mu_k(\rho))(h - \mu_k(\rho) + t) \Theta_{N,s}(r, \rho) \rho^{N-1} d\rho \right) r^{N-1} dr, \\ \mathcal{J}^3 &= \int_{r(t)}^{+\infty} \left(\int_{r(t+h)}^{r(t)} (\mu_k(r) - \mu_k(\rho))(-\mu_k(\rho) + t) \Theta_{N,s}(r, \rho) \rho^{N-1} d\rho \right) r^{N-1} dr, \\ \mathcal{J}^4 &= \int_0^{r(t+h)} \left(\int_{r(t)}^{+\infty} (\mu_k(r) - \mu_k(\rho)) \Theta_{N,s}(r, \rho) \rho^{N-1} d\rho \right) r^{N-1} dr, \end{aligned}$$

with $\mu_k(r(t)) = t$ and $\mu_k(r(t+h)) = t+h$.

Concerning the integral \mathcal{J}^1 we observe that, since μ is decreasing along the radii, we get

$$|\mu_k(r) - \mu_k(\rho)| \leq \mu_k(r(t+h)) - \mu_k(r(t)) = h.$$

Recalling the asymptotic behavior of $\Theta_{N,s}(r, \rho)$ as $|\rho - r| \rightarrow 0$ given in (2.11), and that μ_k is $C^s(\mathbb{R}^N)$ (see [24, Proposition 1.1]) the integral \mathcal{J}^1 can be estimated in the following way

$$\mathcal{J}^1 \leq Ch \int_{r(t+h)}^{r(t)} \left(\int_{r(t+h)}^{r(t)} |r - \rho|^{-1-s} \rho^{N-1} d\rho \right) r^{N-1} dr,$$

being C a positive constant. It follows that

$$(3.6) \quad \frac{\mathcal{J}^1}{h} \rightarrow 0, \quad \text{as } h \rightarrow 0^+.$$

Similarly, we have

$$\mathcal{J}^2 \leq Ch \int_0^{r(t+h)} \left(\int_{r(t+h)}^{r(t)} |\rho - r|^{-1-s} \rho^{N-1} d\rho \right) r^{N-1} dr,$$

which implies that

$$(3.7) \quad \frac{\mathcal{J}^2}{h} \rightarrow 0, \quad \text{as } h \rightarrow 0^+.$$

We now consider the integral \mathcal{J}^3 and we observe that

$$\begin{aligned} \mathcal{J}^3 &\leq \int_{r(t)}^R \left(\int_{r(t+h)}^{r(t)} |t - \mu_k(\rho)| |r - \rho|^{-1-s} \rho^{N-1} d\rho \right) r^{N-1} dr \\ &\quad + \int_R^{+\infty} \left(\int_{r(t+h)}^{r(t)} \mu_k(\rho) |t - \mu_k(\rho)| \Theta_{N,s}(\rho, r) \rho^{N-1} d\rho \right) r^{N-1} dr = \mathcal{J}_1^3 + \mathcal{J}_2^3. \end{aligned}$$

For \mathcal{J}_1^3 we get

$$\mathcal{J}_1^3 \leq Ch \int_{r(t)}^R \left(\int_{r(t+h)}^{r(t)} |r - \rho|^{-1-s} d\rho \right) dr = Ch \left((R - r(t))^{1-s} - (R - r(t+h))^{1-s} \right),$$

so that

$$(3.8) \quad \frac{\mathcal{J}_1^3}{h} \rightarrow 0 \quad \text{as } h \rightarrow 0^+.$$

Recalling that $\Theta_{N,s}(r, \rho)$ is a symmetric function and it has the asymptotic behavior described in (2.12) as $r \rightarrow +\infty$, for \mathcal{J}_2^3 we have

$$\begin{aligned} \mathcal{J}_2^3 &\leq Ch \int_R^{+\infty} \left(\int_{r(t+h)}^{r(t)} \Theta_{N,s}(\rho, r) \rho^{N-1} d\rho \right) r^{N-1} dr \\ &= Ch \int_{r(t+h)}^{r(t)} \left(\int_R^{+\infty} \frac{1}{r^{N+2s}} r^{N-1} dr \right) \rho^{N-1} d\rho, \end{aligned}$$

so that \mathcal{J}_2^3 also satisfies

$$(3.9) \quad \frac{\mathcal{J}_2^3}{h} \rightarrow 0 \quad \text{as } h \rightarrow 0^+.$$

Gathering (3.6), (3.7), (3.8) and (3.9), from (3.5) we deduce

$$(3.10) \quad \begin{aligned} \lim_{h \rightarrow 0^+} \frac{1}{h} \iint_{\mathbb{R}^{2N}} \frac{[u_k^*(x) - u_k^*(y)] [\mathcal{G}_{t,h}(u_k^*(x)) - \mathcal{G}_{t,h}(u_k^*(y))]}{|x - y|^{N+2s}} dx dy \\ = 2N\omega_N \int_0^{r(t)} \left(\int_{r(t)}^{+\infty} (\mu_k(r) - \mu_k(\rho)) \Theta_{N,s}(r, \rho) \rho^{N-1} d\rho \right) r^{N-1} dr. \end{aligned}$$

We now focus on the right-hand side of (3.3). Since

$$\begin{aligned} \int_{\Omega} \frac{f_k(x)}{(u_k(x) + \frac{1}{k})^\gamma} \mathcal{G}_{t,h}(u_k(x)) dx \\ \leq h \|f\|_{L^\infty(\Omega)} \int_{u_k(x) > t+h} \frac{1}{(u_k(x) + \frac{1}{k})^\gamma} dx + \|f\|_{L^\infty(\Omega)} \int_{t < u_k(x) \leq t+h} \frac{u_k(x) - t}{(u_k(x) + \frac{1}{k})^\gamma} dx, \end{aligned}$$

we immediately get

$$(3.11) \quad \begin{aligned} \lim_{h \rightarrow 0^+} \frac{1}{h} \int_{\Omega} \frac{f_k(x)}{(u_k(x) + \frac{1}{k})^\gamma} \mathcal{G}_{t,h}(u_k(x)) dx &\leq \|f\|_{L^\infty(\Omega)} \int_{u_k(x) > t} \frac{1}{(u_k(x) + \frac{1}{k})^\gamma} dx \\ &= N\omega_N \|f\|_{L^\infty(\Omega)} \int_0^{r(t)} \frac{1}{(\mu_k(r) + \frac{1}{k})^\gamma} r^{N-1} dr. \end{aligned}$$

Assembling (3.3), (3.4), (3.5), (3.10) and (3.11), we finally obtain that, for $0 \leq t < \|u_k\|_{L^\infty(\Omega)}$, the following inequality holds true

$$\begin{aligned} \gamma(N, s) \int_0^{r(t)} \left(\int_{r(t)}^{+\infty} (\mu_k(r) - \mu_k(\rho)) \Theta_{N,s}(r, \rho) \rho^{N-1} d\rho \right) r^{N-1} dr \\ \leq \|f\|_{L^\infty(\Omega)} \int_0^{r(t)} \frac{1}{(\mu_k(r) + \frac{1}{k})^\gamma} r^{N-1} dr. \end{aligned}$$

Reasoning as in [18] we can actually prove that, for every $r \geq 0$,

$$(3.12) \quad \begin{aligned} \gamma(N, s) \int_0^r \left(\int_r^{+\infty} (\mu_k(\tau) - \mu_k(\rho)) \Theta_{N,s}(\tau, \rho) \rho^{N-1} d\rho \right) \tau^{N-1} d\tau \\ \leq \|f\|_{L^\infty(\Omega)} \int_0^r \frac{1}{(\mu_k(\tau) + \frac{1}{k})^\gamma} \tau^{N-1} d\tau. \end{aligned}$$

Step 3. Symmetrized approximating problems

Let v be the solution to the symmetrized problem (1.3). We denote by v_k the solution to the following problem

$$\begin{cases} (-\Delta)^s v_k = \frac{\|f\|_{L^\infty(\Omega)}}{(v_k + \frac{1}{k})^\gamma} & \text{in } \Omega^\star \\ v_k > 0 & \text{in } \Omega^\star \\ v_k = 0 & \text{on } \mathbb{R}^N \setminus \Omega^\star. \end{cases}$$

Due to the radial symmetry and the radial monotonicity (see [16]), the function v_k is such that $v_k(x) = v_k^\star(x)$ and, using the notation $\mathfrak{v}_k(x) = \mathfrak{v}_k(|x|) = v_k^\star(x)$, we have

$$(3.13) \quad \begin{aligned} \gamma(N, s) \int_0^r \left(\int_r^{+\infty} (\mathfrak{v}_k(\tau) - \mathfrak{v}_k(\rho)) \Theta_{N,s}(\tau, \rho) \rho^{N-1} d\rho \right) \tau^{N-1} d\tau \\ = \|f\|_{L^\infty(\Omega)} \int_0^r \frac{1}{(\mathfrak{v}_k(\tau) + \frac{1}{k})^\gamma} \tau^{N-1} d\tau. \end{aligned}$$

Step 4. Comparison result

Taking the difference between (3.12) and (3.13) we get

$$(3.14) \quad \begin{aligned} \int_0^r \left(\int_r^{+\infty} [(\mu_k(\tau) - \mathfrak{v}_k(\tau)) - (\mu_k(\rho) - \mathfrak{v}_k(\rho))] \Theta_{N,s}(\tau, \rho) \rho^{N-1} d\rho \right) \tau^{N-1} d\tau \\ \leq \|f\|_{L^\infty(\Omega)} \int_0^r \left(\frac{1}{(\mu_k(\rho) + \frac{1}{k})^\gamma} - \frac{1}{(\mathfrak{v}_k(\rho) + \frac{1}{k})^\gamma} \right) \tau^{N-1} d\tau. \end{aligned}$$

We want to prove that

$$(3.15) \quad \int_0^r \mu_k(\tau) \tau^{N-1} d\tau \leq \int_0^r \mathfrak{v}_k(\tau) \tau^{N-1} d\tau, \quad r \geq 0.$$

At this point, our approach greatly differs from the one used in the proof of [18, Theorem 31], which consists in the interpretation of the LHS of (3.15) as the difference of $N + 2$ dimensional fractional Laplacian of the spherical mean functions of μ_k, \mathfrak{v}_k . Indeed, we use now a qualitative contradiction argument based on Lemma 2.1. Suppose by contradiction that the function $\int_0^r (\mu_k(\tau) - \mathfrak{v}_k(\tau)) \tau^{N-1} d\tau$ has a positive maximum point at $\bar{r} \in (0, R]$, i.e.,

$$0 < \int_0^{\bar{r}} (\mu_k(\tau) - \mathfrak{v}_k(\tau)) \tau^{N-1} d\tau = \max_{r \in [0, R]} \int_0^r (\mu_k(\tau) - \mathfrak{v}_k(\tau)) \tau^{N-1} d\tau.$$

We recall that the function $\Theta_{N,s}(\tau, \rho)$ is increasing with respect to τ for any fixed $\rho > \bar{r}$. Hence, Lemma 2.1 provides that, for every $\rho > \bar{r}$,

$$(3.16) \quad \int_0^{\bar{r}} (\mu_k(\tau) - \mathfrak{v}_k(\tau)) \Theta_{N,s}(\tau, \rho) \tau^{N-1} d\tau > 0.$$

According to what we notice in Remark 2.2, if $\int_0^r (\mu_k(\tau) - \mathfrak{v}_k(\tau)) \tau^{N-1} d\tau$ has a point of positive maximum at \bar{r} , then \bar{r} is a point of non-positive minimum for $\int_r^R (\mu_k(\tau) - \mathfrak{v}_k(\tau)) \tau^{N-1} d\tau$. Hence, using what noticed in Remark 2.1 and the fact that $\Theta_{N,s}(\tau, \rho)$ is decreasing with respect to ρ for any fixed $\tau < \bar{r}$, we get that, for every $\tau < \bar{r}$,

$$(3.17) \quad \int_{\bar{r}}^R (\mu_k(\rho) - \mathfrak{v}_k(\rho)) \Theta_{N,s}(\tau, \rho) \rho^{N-1} d\rho \leq 0.$$

From (3.16) and (3.17) we immediately deduce that

$$(3.18) \quad \int_0^{\bar{r}} \left(\int_{\bar{r}}^{+\infty} (\mu_k(\tau) - \mu_k(\rho)) \Theta_{N,s}(\tau, \rho) \rho^{N-1} d\rho \right) \tau^{N-1} d\tau - \\ \int_0^{\bar{r}} \left(\int_{\bar{r}}^{+\infty} (\nu_k(\tau) - \nu_k(\rho)) \Theta_{N,s}(\tau, \rho) \rho^{N-1} d\rho \right) \tau^{N-1} d\tau > 0.$$

On the other hand, using Lemma 2.8 with the choice $a = \mu_k(\tau) + \frac{1}{k}$ and $b = \nu_k(\tau) + \frac{1}{k}$, we get that

$$\int_0^{\bar{r}} \left(\frac{1}{(\mu_k(\tau) + \frac{1}{k})^\gamma} - \frac{1}{(\nu_k(\tau) + \frac{1}{k})^\gamma} \right) \tau^{N-1} d\tau \leq \gamma \int_0^{\bar{r}} \frac{1}{(\mu_k(\tau) + \frac{1}{k})^{\gamma+1}} (\nu_k(\tau) - \mu_k(\tau)) \tau^{N-1} d\tau,$$

the last integral being negative via Lemma 2.1 since $\frac{1}{(\mu_k(\tau) + \frac{1}{k})^{\gamma+1}}$ is a positive, increasing function. This implies

$$(3.19) \quad \int_0^{\bar{r}} \left(\frac{1}{(\mu_k(\tau) + \frac{1}{k})^\gamma} - \frac{1}{(\nu_k(\tau) + \frac{1}{k})^\gamma} \right) \tau^{N-1} d\tau < 0.$$

Finally (3.18) and (3.19) contradict (3.14) at $r = \bar{r}$.

Step 5. Passing to the limit as $k \rightarrow +\infty$

In [5] the authors prove that the sequences u_k, v_k are bounded in $X_0^s(\Omega)$, resp. $X_0^s(\Omega^*)$. Hence, up to subsequences, u_k, v_k converge to functions $u \in X_0^s(\Omega)$, resp. $v \in X_0^s(\Omega^*)$, weakly in X_0^s , strongly in L^p for any $p \in [1, 2_s^*)$ and a.e. in Ω , resp. Ω^* . Moreover, u , resp. v , are solutions to problems (1.1), resp. (1.3). Hence we can pass to the limit in (3.15) getting

$$\int_0^r \mu(\tau) \tau^{N-1} d\tau \leq \int_0^r \nu(\tau) \tau^{N-1} d\tau, \quad r \geq 0,$$

where $\mu(x) = \mu(|x|) = u^*(x)$ and $\nu(x) = \nu(|x|) = v^*(x)$, which is equivalent to (1.4). \square

Remark 3.1. *In the local case, an analogous comparison result is proved in [9]. There the authors prove that*

$$(3.20) \quad \int_{B_r(0)} \frac{1}{u^*(x)^\gamma} dx \geq \int_{B_r(0)} \frac{1}{v^*(x)^\gamma} dx$$

and, consequently, by multiplying both the integrands by $u^*(x)^\gamma v^*(x)^\gamma$ and using property (b) in Proposition 2.4, they have

$$(3.21) \quad \int_{B_r(0)} u^*(x)^\gamma dx \leq \int_{B_r(0)} v^*(x)^\gamma dx.$$

Actually, by Lemma 2.1 applied with the choice $a = u^*(x)$ and $b = v^*(x)$, we get

$$u^*(x)^{\gamma+1} \left(\frac{1}{u^*(x)^\gamma} - \frac{1}{v^*(x)^\gamma} \right) \leq \gamma (v^*(x) - u^*(x))$$

so that (3.20) and property (b) in Proposition 2.4 imply

$$\int_{B_r(0)} u^*(x) dx \leq \int_{B_r(0)} v^*(x) dx,$$

which provides a more precise comparison result with respect to (3.21) when $\gamma > 1$.

Remark 3.2. When $\gamma = 0$, problem (1.1) coincides with the one discussed in [18]. If in (3.12) we replace the right-hand side with $\int_0^r f^*(\tau)\tau^{N-1} d\tau$, the subsequent arguments apply in order to gain an alternative proof of the mass concentration estimate (1.4). We stress that in this case there is no need of an approximation procedure.

4 An explicit comparison result: proof of Theorem 1.2

Theorem 1.1 allows us to compare the solution to problem (1.1) with the solution to a symmetrized problem having the same structure. As in the local case (see [9]), it is possible to compare u with the solution to a symmetrized problem whose solution can be explicitly computed. Such a comparison result is a key ingredient to prove further regularity results.

PROOF OF THEOREM 1.2.

We consider the same sequence of approximating problems (3.1) that we examined in the previous section and, for $0 \leq t < \|u_k\|_{L^\infty(\Omega)}$ and $h > 0$, we first prove that $\phi = u_k^\gamma \mathcal{G}_{t,h}(u_k^{\gamma+1})$ can be chosen as test function in (3.2). Indeed, we have that by the mean value theorem, the boundedness of u_k and the fact that $\mathcal{G}_{t,h}(\theta) \leq \theta$ yield

$$\begin{aligned} |\phi(x) - \phi(y)| &\leq u_k(x) |\mathcal{G}_{t,h}(u_k(x)^{\gamma+1}) - \mathcal{G}_{t,h}(u_k(y)^{\gamma+1})| + \mathcal{G}_{t,h}(u_k(y)^{\gamma+1}) |u_k(x)^\gamma - u_k(y)^\gamma| \\ &\leq C |u_k(x)^{\gamma+1} - u_k(y)^{\gamma+1}| + u_k(y)^{\gamma+1} |u_k(x)^\gamma - u_k(y)^\gamma| \\ &\leq C_\gamma |u_k(x) - u_k(y)| + \gamma [\max\{u_k(x), u_k(y)\}]^{2\gamma} |u_k(x) - u_k(y)| \end{aligned}$$

and the claim follows from the fact that $u_k \in X_0^s(\Omega)$.

Now putting ϕ in the weak formulation (3.2) we have

$$\begin{aligned} (4.1) \quad &\frac{\gamma(N, s)}{2} \iint_{\mathbb{R}^{2N}} \frac{[u_k(x) - u_k(y)] [u_k(x)^\gamma \mathcal{G}_{t,h}(u_k(x)^{\gamma+1}) - u_k(y)^\gamma \mathcal{G}_{t,h}(u_k(y)^{\gamma+1})]}{|x - y|^{N+2s}} dx dy \\ &= \int_{\Omega} \frac{f_k(x)}{(u_k(x) + \frac{1}{k})^\gamma} u_k(x)^\gamma \mathcal{G}_{t,h}(u_k(x)^{\gamma+1}) dx. \end{aligned}$$

By using (2.2) with the choices

$$\Phi(\theta) = \theta^{\gamma+1}, \quad \varphi(x) = \mathcal{G}_{t,h}(u_k(x)^{\gamma+1})$$

we can estimate the left hand side of (4.1) as follows

$$\begin{aligned} (4.2) \quad &\iint_{\mathbb{R}^{2N}} \frac{[u_k(x) - u_k(y)] [u_k(x)^\gamma \mathcal{G}_{t,h}(u_k(x)^{\gamma+1}) - u_k(y)^\gamma \mathcal{G}_{t,h}(u_k(y)^{\gamma+1})]}{|x - y|^{N+2s}} dx dy \\ &\geq \frac{1}{\gamma + 1} \iint_{\mathbb{R}^{2N}} \frac{[u_k(x)^{\gamma+1} - u_k(y)^{\gamma+1}] [\mathcal{G}_{t,h}(u_k(x)^{\gamma+1}) - \mathcal{G}_{t,h}(u_k(y)^{\gamma+1})]}{|x - y|^{N+2s}} dx dy. \end{aligned}$$

Reasoning as in the previous section, we can show that

$$\begin{aligned} (4.3) \quad &\iint_{\mathbb{R}^{2N}} \frac{[u_k(x)^{\gamma+1} - u_k(y)^{\gamma+1}] [\mathcal{G}_{t,h}(u_k(x)^{\gamma+1}) - \mathcal{G}_{t,h}(u_k(y)^{\gamma+1})]}{|x - y|^{N+2s}} dx dy \\ &\geq \iint_{\mathbb{R}^{2N}} \frac{[u_k^*(x)^{\gamma+1} - u_k^*(y)^{\gamma+1}] [\mathcal{G}_{t,h}(u_k^*(x)^{\gamma+1}) - \mathcal{G}_{t,h}(u_k^*(y)^{\gamma+1})]}{|x - y|^{N+2s}} dx dy \end{aligned}$$

and

$$(4.4) \quad \lim_{h \rightarrow 0^+} \frac{1}{h} \iint_{\mathbb{R}^{2N}} \frac{[u_k^*(x)^{\gamma+1} - u_k^*(y)^{\gamma+1}] [\mathfrak{G}_{t,h}(u_k^*(x)^{\gamma+1}) - \mathfrak{G}_{t,h}(u_k^*(y)^{\gamma+1})]}{|x-y|^{N+2s}} dx dy \\ = \int_0^{r(t)} \left(\int_{r(t)}^{+\infty} (\mu_k(r)^{\gamma+1} - \mu_k(\rho)^{\gamma+1}) \Theta_{N,s}(r, \rho) \rho^{N-1} d\rho \right) r^{N-1} dr,$$

where, as in the previous section, $\mu_k(x) = \mu_k(|x|)$ stands for $u_k^*(x)$. Regarding the right-hand side of (4.1), we observe that

$$\int_{\Omega} \frac{f_k(x)}{(u_k(x) + \frac{1}{k})^\gamma} u_k(x)^\gamma \mathfrak{G}_{t,h}(u_k(x)^{\gamma+1}) dx \\ = h \int_{u_k^{\gamma+1} > t+h} \frac{f_k(x)}{(u_k(x) + \frac{1}{k})^\gamma} u_k(x)^\gamma dx + \int_{t < u_k^{\gamma+1} \leq t+h} \frac{f_k(x)}{(u_k(x) + \frac{1}{k})^\gamma} u_k(x)^\gamma (u_k(x)^{\gamma+1} - t) dx$$

and

$$\int_{t < u_k^{\gamma+1} \leq t+h} \frac{f_k(x)}{(u_k(x) + \frac{1}{k})^\gamma} u_k(x)^\gamma (u_k(x)^{\gamma+1} - t) dx \leq h \|f\|_\infty \int_{t < u_k^{\gamma+1} \leq t+h} dx.$$

It follows that

$$(4.5) \quad \lim_{h \rightarrow 0^+} \frac{1}{h} \int_{\Omega} \frac{f_k(x)}{(u_k(x) + \frac{1}{k})^\gamma} u_k(x)^\gamma \mathfrak{G}_{t,h}(u_k(x)^{\gamma+1}) dx = \int_{u_k^{\gamma+1} > t} \frac{f_k(x)}{(u_k(x) + \frac{1}{k})^\gamma} u_k(x)^\gamma dx.$$

It is easy to observe that

$$(4.6) \quad \int_{u_k^{\gamma+1} > t} \frac{f_k(x)}{(u_k(x) + \frac{1}{k})^\gamma} u_k(x)^\gamma dx \leq \int_{u_k^{\gamma+1} > t} f(x) dx \leq \int_0^{r(t)} f^*(\rho) \rho^{N-1} d\rho,$$

being $\mu_k(r(t))^{\gamma+1} = t$. From (4.1)-(4.6) we deduce

$$\frac{\gamma(N, s)}{2(\gamma+1)} \int_0^{r(t)} \left(\int_{r(t)}^{+\infty} (\mu_k(r)^{\gamma+1} - \mu_k(\rho)^{\gamma+1}) \Theta_{N,s}(r, \rho) \rho^{N-1} d\rho \right) r^{N-1} dr \leq \int_0^{r(t)} f^*(\rho) \rho^{N-1} d\rho.$$

Reasoning as in [18] we can show that actually the following inequality holds true for every $r \geq 0$:

$$(4.7) \quad \frac{\gamma(N, s)}{2(\gamma+1)} \int_0^r \left(\int_r^{+\infty} (\mu_k(\tau)^{\gamma+1} - \mu_k(\rho)^{\gamma+1}) \Theta_{N,s}(\tau, \rho) \rho^{N-1} d\rho \right) \tau^{N-1} d\tau \leq \int_0^r f^*(\rho) \rho^{N-1} d\rho.$$

On the other hand, the solution to problem (1.5) satisfies

$$(4.8) \quad \frac{\gamma(N, s)}{2(\gamma+1)} \int_0^r \left(\int_r^{+\infty} (\mathfrak{v}(\tau) - \mathfrak{v}(\rho)) \Theta_{N,s}(\tau, \rho) \rho^{N-1} d\rho \right) \tau^{N-1} d\tau = \int_0^r f^*(\rho) \rho^{N-1} d\rho.$$

Subtracting (4.7) and (4.8), we can conclude as in the proof of Theorem 1.1.

5 Some regularity results

As an immediate consequence of Theorem 1.2 we can prove the following regularity results, depending on the value of γ and on the summability of f .

Theorem 5.1. *Let $s \in (0, 1)$, $N \geq 2$, $\gamma > 0$, and assume that $f \in L^p(\Omega)$, with $p \geq 2_s^*$, $f \geq 0$. If $u \in X_0^s(\Omega)$ is the weak solution to problem (1.1), the following estimates hold true.*

1. *If $p < \frac{N}{2s}$, then $u \in L^q(\Omega)$, with $q = \frac{Np(\gamma+1)}{N-2sp}$, and there exists a positive constant C such that*

$$\|u\|_{L^q(\Omega)} \leq C \|f\|_{L^p(\Omega)}^{1/(\gamma+1)}.$$

2. *If $p = \frac{N}{2s}$, then $u \in L_\Phi(\Omega)$, where $L_\Phi(\Omega)$ is the Orlicz space generated by the N -function*

$$\Phi(t) = \exp(|t|^{(\gamma+1)p'}) - 1.$$

Moreover, there exists a positive constant C such that

$$\|u\|_{L_\Phi(\Omega)} \leq C \|f\|_{L^p(\Omega)}^{1/(\gamma+1)}.$$

3. *If $p > \frac{N}{2s}$, then $u \in L^\infty(\Omega)$ and there exists a positive constant C such that*

$$\|u\|_{L^\infty(\Omega)} \leq C \|f\|_{L^p(\Omega)}^{1/(\gamma+1)}.$$

Proof. We simply observe that for $q > \gamma + 1$ we have

$$\|u\|_{L^q(\Omega)} = \|u^{\gamma+1}\|_{\frac{q}{\gamma+1}}^{1/(\gamma+1)},$$

therefore by Theorem 1.2

$$\|u\|_{L^q(\Omega)} \leq \|v\|_{L^q(\Omega^*)}^{1/(\gamma+1)}.$$

Then we can apply the regularity result [18, Theorem 3.2] to the solution v to the symmetrized problem (1.5). Moreover, in the limit case $p = N/2s$ we notice that Theorem 1.2 implies

$$(u^{\gamma+1})^{**} \leq v^{**}$$

and arguing as in the proof of [18, Theorem 3.2] the claim follows. □

Remark 5.1. *We stress that when $\gamma = 0$ we recover the estimates contained in [18], while when $s = 1$ we have the same estimates contained in [7, 9].*

We end the paper with the following energy estimate.

Proposition 5.1. *Let $s \in (0, 1)$, $N \geq 2$, $\gamma > 0$ and assume that $f \in L^{(2_s^*)'}(\Omega)$, $f \geq 0$. If $u \in X_0^s(\Omega)$ and $v \in X_0^s(\Omega^*)$ are the weak solutions to problems (1.1) and (1.5), respectively, then*

$$\|u^{\gamma+1}\|_{X_0^s(\Omega)} \leq \|v\|_{X_0^s(\Omega^*)}.$$

Proof. Let $k \in \mathbb{N}$ and let u_k be a solution to (3.1). Let $T > 1$. We consider the following function $\Phi_T : [0, +\infty) \rightarrow [0, +\infty)$ defined as

$$\Phi_T(\theta) = \begin{cases} \theta^{\gamma+1} & \text{if } 0 \leq \theta < T \\ (\gamma+1)T^\gamma\theta - \gamma T^{\gamma+1} & \text{if } \theta \geq T. \end{cases}$$

Since $\Phi_T(\theta)$ and $\Phi_T(\theta)\Phi'_T(\theta)$ are Lipschitz continuous functions, $\Phi_T(u_k)$ and $\Phi_T(u_k)\Phi'_T(u_k)$ belong to $X_0^s(\Omega)$. Inequality (2.2) with the choice $\varphi = \Phi_T(u_k)$ implies

$$\begin{aligned}
(5.1) \quad \|\Phi_T(u_k)\|_{X_0^s(\Omega)}^2 &= \iint_{\mathbb{R}^{2N}} \frac{|\Phi_T(u_k(x)) - \Phi_T(u_k(y))|^2}{|x - y|^{N+2s}} \, dx \, dy \\
&\leq \iint_{\mathbb{R}^{2N}} \frac{[u_k(x) - u_k(y)] [\Phi'_T(u_k(x))\Phi_T(u_k(x)) - \Phi'_T(u_k(y))\Phi_T(u_k(y))]}{|x - y|^{N+2s}} \, dx \, dy \\
&= \frac{2}{\gamma(N, s)} \int_{\Omega} \frac{f_k(x)}{(u_k(x) + \frac{1}{k})^\gamma} \Phi'_T(u_k(x))\Phi_T(u_k(x)) \, dx \\
&\leq \frac{2(\gamma + 1)}{\gamma(N, s)} \int_{\Omega} \frac{f_k(x)}{(u_k(x) + \frac{1}{k})^\gamma} u_k(x)^{\gamma+1} u_k(x)^\gamma \, dx \\
&\leq \frac{2(\gamma + 1)}{\gamma(N, s)} \int_{\Omega} f(x) u_k(x)^{\gamma+1} \, dx.
\end{aligned}$$

Recall now that the proof of Theorem 1.2 gives (1.6) being u replaced by u_k . Then using the Hardy-Littlewood inequality (2.4) and Proposition 2.4, we can estimate the right hand side of (5.1) as follows

$$\int_{\Omega} f(x) u_k(x)^{\gamma+1} \, dx \leq \int_{\Omega^*} f^*(x) v(x) \, dx,$$

then from (5.1) we conclude

$$(5.2) \quad \|\Phi_T(u_k)\|_{X_0^s(\Omega)}^2 \leq \frac{2(\gamma + 1)}{\gamma(N, s)} \int_{\Omega^*} f^*(x) v(x) \, dx = \|v\|_{X_0^s(\Omega^*)}^2.$$

Estimate (5.2) implies that the family $\Phi_T(u_k)$ is uniformly bounded with respect to $T > 1$ and $k \in \mathbb{N}$. Consequently, by the Sobolev embedding theorem we can extract a subsequence $T_\ell \rightarrow +\infty$ such that

$$\Phi_{T_\ell}(u_k) \rightharpoonup u_k^{\gamma+1} \quad \text{weakly in } X_0^s(\Omega), \quad \Phi_{T_\ell}(u_k) \rightarrow u_k^{\gamma+1} \quad \text{strongly in } L^q(\Omega), \, q < 2_s^*,$$

as $\ell \rightarrow +\infty$. Then we can pass to the limit in (5.2) and obtain

$$\|u_k^{\gamma+1}\|_{X_0^s(\Omega)} \leq \|v\|_{X_0^s(\Omega^*)}.$$

Thanks to the lower semicontinuity of the norm, we can pass to the limit in the previous inequality as k goes to $+\infty$ and we get the claim. \square

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