On the stability and convergence of a semi-discrete discontinuous Galerkin scheme to the kinetic Cucker–Smale model

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Abstract
We study analytical properties of a semi-discrete discontinuous Galerkin (DG) scheme for the kinetic Cucker–Smale (CS) equation. The kinetic CS equation appears in the mean-field limit of the particle CS model and it corresponds to the dissipative Vlasov type equation approximating the large particle CS system. For this proposed DG scheme, we show that it exhibits analytical properties such as the conservation of mass, L^2-stability and convergence to the sufficiently regular solution, as the mesh-size tends to zero. In particular, we verify that the convergence rate of the DG numerical solution to the sufficiently regular kinetic solution is dependent on the Sobolev regularity of the kinetic soluiton. We also present several numerical simulations for low-dimensional cases.

Keywords Flocking · Clusters · Discontinuous Galerkin method · Kinetic model

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1 Introduction

Flocking of self-propelled particles (agents) denotes a collective motion in which particles are organized into an ordered state from a disordered state only using the limited environmental information and simple rules. It appears in natural and man-made systems [1, 4, 5, 47, 49], e.g., flocking of birds, drones and robots, flocking of birds, swarming of fish and herding of sheep, etc. Despite of its ubiquitous presence, modeling and analysis for flocking were begun only several decades ago. After Reynolds' boid model [45], Vicsek et al. [48] proposed a simple discrete planar model with a unit speed constraint. As far as the authors know, there is no rigorous convergence proof for the Vicsek model (see [37] for a convergence proof under a priori connected assumption for each instant). To circumvent this a priori connectedness assumption, Cucker and Smale introduced a second-order Newton type model [22, 23] for position and velocity which we call it as the Cucker–Smale (in short CS) model. In fact, the CS model uses a weighted sum of relative velocities as an force. For a brief introduction on CS flocking, we refer to a survey article [13]. To set up the stage, we begin with a brief discussion for the (particle) CS model.

Let x_i and ξ_i be the position and velocity of the ith CS particle on a spatial domain $\Omega^x (\subset \mathbb{R}^d)$, respectively. Then, the CS model reads as follows.

$$\begin{cases} \dot{x}_i = \xi_i, \quad t > 0, \quad i \in [N] := \{1, \ldots, N\}, \\ \dot{\xi}_i = \frac{1}{N} \sum_{j=1}^{N} \psi(x_j, x_i)(\xi_j - \xi_i), \end{cases} \tag{1}$$

where N and ψ denote the total number of particles and Lipschitz continuous communication weight function satisfying symmetry and boundedness, respectively: there exists a positive constant ψ_M such that

$$\psi(x, y) = \psi(y, x), \quad 0 < \psi(x, y) \le \psi_M, \quad \forall x, y \in \Omega^x. \tag{2}$$

The global well-posedness for (1)–(2) is guaranteed by the standard Cauchy–Lipschitz theory. Hence, most literature for (1) are concerned with the emergence of flocking under various contexts, e.g., collision avoidance [18, 19], stochastic environment [2, 21, 25, 26, 33, 44], time-delay [27], network topologies [20, 24, 38, 39], relativistic and thermodynamic effects [32, 36], etc.

On the other hand, when the number of particles is sufficiently large (namely a mesoscopic regime), direct integration of (1) will be too expensive to describe the motion of CS ensemble with $N \gg 1$. Hence, as an effective mean-field approximation of a large particle system (1), we can use the corresponding mean-field kinetic equation (see [34] for the mean-field limit). More precisely, let $f = f(t, x, \xi)$ be the one-particle distribution function at position x, velocity ξ at time t. Then, the temporal-

phase space evolution of f is governed by the following initial boundary value problem to the kinetic CS equation on a spatial domain $\Omega^x (\subset \mathbb{R}^d)$:

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f + \nabla_\xi \cdot (F_a(f)f) = 0, & t > 0, \quad (x, \xi) \in \Omega^x \times \mathbb{R}^d, \\ f(t, x, \xi)\Big|_{t=0+} = f_0(x, \xi), \end{cases} \tag{3}$$

subject to suitable boundary conditions on $\partial(\Omega^x \times \mathbb{R}^d)$. Here $F_a(f)$ is a velocity alignment force whose explicit form is given as follows:

$$F_a(f)(t, x, \xi) = -\int_{\Omega^x \times \mathbb{R}^d} \psi(x, x_*)(\xi - \xi_*) f(t, x_*, \xi_*) dx_* d\xi_*. \tag{4}$$

The global well-posedness and flocking dynamics of the Cauchy problem (3)–(4) have been extensively studied in [11, 29, 34, 35]. However due to the non-local nature of (4), numerical implementation of (3) is less investigated. Recently, structure and positivity preserving schemes are proposed for the related collective models such as the continuum Kuramoto model [8] and kinetic flocking model [46]. In particular, the latter work [46] introduces a fully discrete DG scheme for (1) and flocking model [40] in one-dimensional setting and the author also showed that his DG scheme exhibits a positivity preserving property which results in the stability in L^1 under the suitable CFL type condition on time-step and mesh size. The DG method, proposed in [46], can be reduced to a classical high order finite volume method, which is proved to be positive preserving, for (3), where the transport term is neglected and the velocity alignment (4) is independent on the space variable. As long as there is no confusion, we will use the jargons "kinetic model" and "kinetic equation" interchangeably.

The main results of this paper are two-fold. First, we introduce a local discontinuous Galerkin method for the computation of the approximate solution with high order approximations in time, space and velocity (see Sect. 2.2. Indeed, the preservation of high order accuracy allows us to investigate complicate structures in space, as it has already been observed for macroscopic models. Discontinuous Galerkin methods [6, 15, 17, 30, 31] are particularly suited for transport type equations with several attractive properties, such as their easiness for adaptivity and parallel computation, and their nice stability properties. We refer to the survey paper [16] and the references therein for a discontinuous Galerkin methods. For discontinuous Galerkin methods solving kinetic type equations we refer to [3, 12]. Discontinuous Galerkin methods are particularly suitable for transport type equations with several attractive properties, such as their easiness for adaptivity and parallel computation, and their nice stability properties. Second, we study L^2-stability, consistency and convergence of numerical solutions given by the discontinuous Galerkin method for (3) and (4) (see Theorem 2.5).

The paper after this introduction is organized as follows. In Sect. 2, we study basic properties and well-posedness of the kinetic CS model, and discuss a semi discrete discontinuous Galerkin numerical approximation and main results. In Sect. 3, we study the mass conservation and L^2-stability of the numerical solutions. In Sect. 4, we provide an L^2-convergence of numerical solutions to the regular solutions to the kinetic CS model. In Sect. 5, we present several numerical implementations in one and two

dimensional settings. Finally, Sect. 6 is devoted to a brief summary of our main results and some remaining issues for a future work.

Notation In what follows, we often suppress domain depence $\Omega^x \times \mathbb{R}^d$ for Lebesgue space $L^p(U \times \mathbb{R}^d)$:

$$\| \cdot \|_{L^p} := \| \cdot \|_{L^p(\Omega^x \times \mathbb{R}^d)}, \quad 1 \le p \le \infty.$$

As long as there is no confusion, we use the subscript and superscript to denote the particle number and component, respectively, i.e., x_i^j, ξ_i^j are the jth spatial and velocity component of the ith particle:

$$x_i = (x_i^1, \cdots, x_i^d) \in \Omega^x, \quad \xi_i = (\xi_i^1, \cdots, \xi_i^d) \in \mathbb{R}^d.$$

2 Preliminaries and main results

In this section, we first present basic properties of the kinetic Cucker–Smale model and review the global well-posedness of the Cauchy problem to (3)–(4), and then we delineate a semi-discrete discontinuous Galerkin method and provide our main results on the L^2-stability and convergence of numerical solutions.

2.1 The kinetic CS equation on $\Omega^x = \mathbb{R}^d$

Consider the Cauchy problem to the kinetic CS equation on the whole space $\mathbb{R}^{2d}$:

$$\begin{cases} \partial_t f + \nabla_x \cdot (\xi f) + \nabla_\xi \cdot (F_a(f)f) = 0, & t > 0, \ (x, \xi) \in \mathbb{R}^{2d}, \\ f(t, x, \xi)\Big|_{t=0+} = f_0(x, \xi), \end{cases} \tag{5}$$

In what follows, we discuss the propagation of the first three velocity moments: for $t \ge 0$,

$$m_0(t) := \int_{\mathbb{R}^{2d}} f(t, z)dz, \quad m_1(t) := \int_{\mathbb{R}^{2d}} \xi f(t, z)dz, \quad m_2(t) := \int_{\mathbb{R}^{2d}} |\xi|^2 f(t, z)dz,$$

where $z = (x, \xi)$ and $dz = d\xi dx$.

Lemma 2.1 *Let $f = f(t, z)$ be a global classical solution to (5) which decays to zero sufficiently fast at infinity in the phase space. Then, velocity moments m_i, $i = 0, 1, 2$ satisfy the following relations: for $t \ge 0$,*

(i) $m_0(t) = m_0(0), \quad m_1(t) = m_1(0).$

(ii) $m_2(t) = m_2(0) - \int_0^t \int_{\mathbb{R}^{4d}} \psi(x, x_*)|\xi - \xi_*|^2 f(s, z_*) f(s, z)dz_* dz ds.$

Proof (i) The first relation directly follows from the integration of $(5)_1$ over $\mathbb{R}^{2d}$ using the divergence theorem and fast decay of f at infinity. For the second relation, we multiply ξ to $(5)_1$ to find

$$\partial_t(\xi f) + \nabla_x \cdot (\xi \otimes \xi f) + \nabla_\xi \cdot (\xi \otimes F_a(f)f) = d F_a(f)f.$$

Again we integrate the above equation over $\mathbb{R}^{2d}$ using the symmetry property of ψ, and use the divergence theorem to find the desired relation.

(ii) We use $(5)_1$ to find

$$
\begin{aligned}
\partial_t(|\xi|^2 f) = |\xi|^2 \partial_t f &= -|\xi|^2 \Big(\nabla_x \cdot (\xi f) + \nabla_\xi \cdot (F_a(f)f) \Big) \\
&= -\nabla_x \cdot (|\xi|^2 \xi f) - \nabla_\xi \cdot (|\xi|^2 F_a(f)f) + 2(\xi \cdot F_a(f))f.
\end{aligned}
\tag{6}
$$

Now, we integrate (6) over $\mathbb{R}^{2d}$ and use index exchange transformation $(x, \xi) \leftrightarrow (x_*, \xi_*)$ to get

$$
\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^{2d}} |\xi|^2 f dz &= 2 \int_{\mathbb{R}^{2d}} (\xi \cdot F_a(f)) f dz \\
&= -2 \int_{\mathbb{R}^{4d}} \psi(x, x_*) \xi \cdot (\xi - \xi_*) f(t, z_*) f(t, z) dz_* dz \\
&= 2 \int_{\mathbb{R}^{4d}} \psi(x, x_|) \xi_* \cdot (\xi - \xi_*) f(t, z_*) f(t, z) dz_* dz \\
&= - \int_{\mathbb{R}^{4d}} \psi(x, x_*) |\xi - \xi_*|^2 f(t, z_*) f(t, z) dz_* dz.
\end{aligned}
$$

Again, we integrate the above relation in time to get the desired estimate. $\qquad\square$

Remark 1 Note that if we assume that initial total mass is unity, then total mass is conserved in time:

$$\int_{\mathbb{R}^{2d}} f(t, z) dz = 1, \quad t > 0.$$

Next, we discuss the dynamics of particle trajectories (or bi-characteristics) corresponding to (5). For this, we rewrite $(5)_1$ into a quasi-linear form:

$$\partial_t f + \xi \cdot \nabla_x f + F_a(f) \cdot \nabla_\xi f = -(\nabla_\xi \cdot F_a(f))f, \quad t > 0. \tag{7}$$

Note that the coefficient $-(\nabla_\xi \cdot F_a(f))$ in the R. H. S. of (7) can be rewritten as

$$-(\nabla_\xi \cdot F_a(f)) = d \int_{\mathbb{R}^{2d}} \psi(x, x_*) f(t, x_*, \xi_*) dx_* d\xi_*.$$

This yields

$$\|\nabla_\xi \cdot F_a(f)\|_{L^\infty} \le d\psi_M m_0(t) = d\psi_M m_0(0).$$

For $(x, \xi) \in \text{supp}_{(x,\xi)} f_0$, we define bi-charteristics (forward particle trajectory):

$$(x(t), \xi(t)) = (x(t; 0, x, \xi), (\xi(t : 0, x, \xi)))$$

as a solution to the following ODE system:

$$\begin{cases} \dot{x}(t) = \xi(t), \quad t > 0, \\ \dot{\xi}(t) = -\displaystyle\int_{\mathbb{R}^{2d}} \psi(x(t), x_*)(\xi(t) - \xi_*) f(t, x_*, \xi_*) dx_* d\xi_*, \\ (x(0), \xi(0)) = (x, \xi). \end{cases} \tag{8}$$

Lemma 2.2 *Suppose that the communication weight function ψ satisfies an extra condition together with (2): there exists positive constants ψ_m and ψ_M such that*

$$0 < \psi_m \le \psi(r) \le \psi_M, \quad \forall r \ge 0, \tag{9}$$

and let $(x(t), \xi(t))$ be the particle trajectory of (5) issued from $(x, \xi) \in \text{supp} f_0$ at time 0. Then, the ith velocity component $\xi_i(t)$ satisfies

$$\underline{\xi}^i(t) \le \xi^i(t) \le \overline{\xi}^i(t), \quad t \ge 0.$$

Here uniform lower and upper bounds $\underline{\xi}^i$ and $\overline{\xi}^i$ are defined as follows.

$$\underline{\xi}^i(t) := -\frac{\psi_m}{\psi_M} \sqrt{\frac{m_2(0)}{m_0(0)}} + \left(\xi^i(0) + \frac{\psi_m}{\psi_M} \sqrt{\frac{m_2(0)}{m_0(0)}}\right) e^{-\psi_M m_0 t}, \quad l \in [d],$$

$$\overline{\xi}^i(t) := \frac{\psi_M}{\psi_m} \sqrt{\frac{m_2(0)}{m_0(0)}} + \left(\xi^i(0) - \frac{\psi_M}{\psi_m} \sqrt{\frac{m_2(0)}{m_0(0)}}\right) e^{-\psi_m m_0 t}. \tag{10}$$

Proof We use the same argument in [35]. For $(x, \xi) \in \text{supp}_{(x,\xi)} f_0$ at time 0, we set

$$x(t) = x(t; 0, x, \xi) \quad \text{and} \quad \xi(t) = \xi(t; 0, x, \xi), \quad t \ge 0.$$

Then, it follows from $(8)_2$ that for $l \in [d]$,

$$\begin{aligned} \frac{d\xi^l(t)}{dt} &= -\int_{\mathbb{R}^{2d}} \psi(x^l(t), x_*)(\xi^l(t) - \xi_*) f(t, x_*, \xi_*) dx_* d\xi_* \\ &= -\left[\int_{\mathbb{R}^{2d}} \psi(x^l(t), x_*) f(t, x_*, \xi_*) dx_* d\xi_*\right] \xi^l(t) \\ &\quad + \int_{\mathbb{R}^{2d}} \psi(x^l(t), x_*) \xi_*^l f(t, x_*, \xi_*) dx_* d\xi_* \\ &=: -\alpha_1(t)\xi^l(t) + \alpha_2(t). \end{aligned} \tag{11}$$

Below, we estimate α_i, $i = 1, 2$ one by one.

• (Estimate of α_1): we apply the extra condition (9) to find

$$\psi_m m_0 \leq \alpha_1(t) \leq \psi_M m_0, \quad t \geq 0. \tag{12}$$

• (Estimate of α_2): again, we use (9), the Cauchy-Schwarz inequality and Lemma 2.1 to obtain

$$
\begin{aligned}
|\alpha_2(t)| &\leq \psi_M \int_{\mathbb{R}^{2d}} |\xi_*^l| f(t, x_*, \xi_*) dx_* d\xi_* \\
&\leq \psi_M \Big(\int_{\mathbb{R}^{2d}} |\xi_*^l|^2 f(t, x_*, \xi_*) dx_* d\xi_* \Big)^{\frac{1}{2}} \cdot \Big(\int_{\mathbb{R}^{2d}} f(t, x_*, \xi_*) dx_* d\xi_* \Big)^{\frac{1}{2}} \\
&= \psi_M \sqrt{m_0(t) m_2(t)} \leq \kappa \psi_M \sqrt{m_0(0) m_2(0)}.
\end{aligned}
\tag{13}
$$

Now, in (11) we combine all the estimates (12) and (13) to get differential inequalities:

$$-\psi_M m_0 \xi^l(t) - \psi_M \sqrt{m_0(0) m_2(0)} \leq \frac{d\xi^l(t)}{dt} \leq -\psi_m m_0 \xi^l(t) + \psi_M \sqrt{m_0(0) m_2(0)}.$$

Then, we use the Gronwall type arguments to derive the desired estimates. □

Remark 2 Below, we briefly comment on the result of Lemma 2.2.

1. Note that the explicit relations (10) imply

$$
\min \Big\{ \min_{1 \leq i \leq d} \xi^i(0), \ -\frac{\psi_m}{\psi_M} \sqrt{\frac{m_2(0)}{m_0(0)}} \Big\} \leq \inf_{0 \leq t < \infty} \xi^i(t), \ \lim_{t \to \infty} \underline{\xi}^i(t) = -\frac{\psi_m}{\psi_M} \sqrt{\frac{m_2(0)}{m_0(0)}},
$$

$$
\sup_{0 \leq t < \infty} \overline{\xi}^i(t) \leq \max \Big\{ \max_{1 \leq i \leq d} \xi^i(0), \ \frac{\psi_M}{\psi_m} \sqrt{\frac{m_2(0)}{m_0(0)}} \Big\}, \ \lim_{t \to \infty} \overline{\xi}^i(t) = \frac{\psi_M}{\psi_m} \sqrt{\frac{m_2(0)}{m_0(0)}}.
$$

2. Suppose that communication weight and initial datum satisfy

$$\psi \equiv 1, \quad m_0(0) < \infty, \quad |m_1(0)| < \infty.$$

Then, it follows from (11) that particle trajectory $\xi^i(t)$ satisfies

$$\frac{d\xi(t)}{dt} = -m_0(0)\xi(t) + m_1(0), \quad t > 0.$$

By direct calculation, one has

$$\xi(t) = \frac{m_1(0)}{m_0(0)} + \Big(\xi - \frac{m_1(0)}{m_0(0)} \Big) e^{-m_0(0)t},$$

$$x(t) = x + \frac{m_1(0)}{m_0(0)} t + \frac{1}{m_0(0)} \Big(\xi - \frac{m_1(0)}{m_0(0)} \Big) (1 - e^{-m_0(0)t}).$$

Therefore we have

$$\lim_{t \to \infty} \left| \xi(t) - \frac{m_1(0)}{m_0(0)} \right| = 0, \quad \lim_{t \to \infty} \left| x(t) - x - \frac{m_1(0)}{m_0(0)}t - \frac{1}{m_0(0)} \left(\xi - \frac{m_1(0)}{m_0(0)} \right) \right| = 0.$$

Next, we return to the global well-posedness of (5). Then, the global well-posedness of (6) follows from a priori $W^{k,\infty}$-estimates along the particle trajectory.

Theorem 2.3 [35] *Suppose that the initial datum f_0 is compactly supported in the phase space and sufficiently regular such that*

$$f_0 \in W^{k,\infty}(\mathbb{R}^{2d}), \quad \text{for some integer } k \geq 2.$$

Then for any $T \in (0, \infty)$, there exists a unique classical solution $f(t) \in W^{k,\infty}(\mathbb{R}^{2d})$ for $t \in (0, T)$ and a positive constant $C(T)$ such that

$$\sup_{0 \leq t \leq T} \| f(t) \|_{W^{k,\infty}} \leq C(T) \| f_0 \|_{W^{k,\infty}}.$$

Proof The proof is basically based on a priori estimate on the control of $W^{k,\infty}$-norm for f along bi-characteristics: for $T \in (0, \infty)$, let $(x(t), \xi(t))$ be a particle trajectory defined by (8). Then, we have

$$\frac{d}{dt} f(t, x(t), \xi(t)) \leq d\psi_M m_0 f(t, x(t), \xi(t)), \quad t > 0.$$

This yields

$$\| f(t) \|_{L^\infty} \leq e^{d\psi_M m_0 t} \| f_0 \|_{L^\infty}, \quad t \in (0, T).$$

For the higher-order $W^{k,\infty}$-estimate, we use the same arguments to get

$$\sup_{0 \leq t \leq T} \| f(t) \|_{W^{k,\infty}} \leq C(T) \| f_0 \|_{W^{k,\infty}}. \tag{14}$$

Then we combine the above a priori estimate (14) and standard local existence result to derive a global existence of classical solution for $k \geq 2$. We refer to [35] for details. $\square$

Next, we discuss a global well-posedness of a measure-valued solution to the Cauchy problem (5). For the concept of measure-valued solutions, we refer to [34]. Now, we briefly recall some jargons. For the particle solution $(x_i(t), \xi_i(t))$ to (1), we introduce the associated empirical measure $\mu^N(t)$:

$$\mu^N(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_i(t), \xi_i(t))}.$$

Then we can show that μ^N satisfies the Eq. (5) in the sense of distributions, i.e., μ^N is a measure-valued solution to (5).

Let $\mathcal{M}(\mathbb{R}^{2d})$ be the set of positive Radon measures and we fix $T > 0$. Then, we define the set $\mathcal{S}$ of test functions and a bounded Lipschitz distance $d_{BL}(\mu_1, \mu_2)$ on $\mathcal{S}$ as follows:

$$\mathcal{S} := \left\{ h : \mathbb{R}^{2d} \to \mathbb{R} \;:\; \|h\|_{L^\infty} \leq 1 \quad \text{and} \quad \text{Lip(h)} := \sup_{x \neq y} \frac{|h(x) - h(y)|}{\|x - y\|} \leq 1 \right\},$$

$$d_{BL}(\mu_1, \mu_2) := \sup_{h \in \mathcal{S}} \left\{ \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} h \, d\mu_1 - \int_{\mathbb{R}^d \times \mathbb{R}^d} h \, d\mu_2 \right| \;:\; \text{for } \mu_1, \mu_2 \in \mathcal{M}(\mathbb{R}^{2d}) \right\}.$$

Theorem 2.4 *[34] Suppose that the initial measure $f_0 dx d\xi \in \mathcal{M}(\mathbb{R}^{2d})$ is compactly supported, and we take a sequence of μ_0^N of measures of the form $\mu_0^N = \frac{1}{N} \sum_{i=1}^N \delta_{(x_i(0), \xi_i(0))}$ such that*

$$\lim_{N \to \infty} d_{BL}(\mu_0^N, f_0 d\xi dx) = 0.$$

Define the empirical measure $\mu_N(t)$ made of particle solution (x_i, ξ_i) with initial data $(x_i(0), \xi_i(0))$. Then there exists a unique measure-valued solution f to (5) with the initial datum f_0 such that

$$\lim_{N \to \infty} d_{BL}(\mu^N(t), f(t) dx d\xi) = 0 \quad \text{for} \quad t \geq 0.$$

2.2 A semi-discrete DG scheme

In this subsection, we describe a semi-discrete DG scheme for (5) on the spatial periodic domain Ω^x.

Suppose that ψ and initial datum f_0 are spatially periodic with the same period, and compactly supported in the velocity variable. Then, it is easy to see that the solution f is periodic in spatial variable and compactly supported in velocity variable as well. Before we describe the DG scheme, we first discuss finite-dimensional function spaces. Let $\Omega = \Omega^x \times \Omega^\xi \subset \mathbb{R}^d \times \mathbb{R}^d$ be an open bounded set such that

$$\text{supp}_\xi f(t, x, \cdot) \subset \Omega^\xi, \quad t > 0, \quad x \in \Omega^x,$$

and we set $\partial \Omega^x$ and $\partial \Omega^\xi$ to be the boundaries of Ω^x and Ω^ξ, respectively. Let $\mathcal{P}_h(\Omega^x) = \{\Omega_h^x\}$ and $\mathcal{P}_h(\Omega^\xi) = \{\Omega_h^\xi\}$ be the partitions of Ω^x and Ω^ξ with maximal amplitude h, respectively and we define a partition of Ω as follows:

$$\mathcal{P}_h(\Omega^x \times \Omega^\xi) := \left\{ \Omega_h^x \times \Omega_h^\xi \subseteq \Omega^x \times \Omega^\xi \;:\; \Omega_h^x \in \mathcal{P}_h(\Omega_x), \; \Omega_h^\xi \in \mathcal{P}_h(\Omega_\xi) \right\}.$$

On the other hand, for a nonnegative integer k, let $P^k(\Omega_h^x \times \Omega_h^\xi)$ be the set of polynomials of total degree *at most* k on $\Omega_h^x \times \Omega_h^\xi$. We define the discrete P-type space:

$$\mathcal{G}_h^k = \left\{ g \in L^2(\Omega^x \times \Omega^\xi) : \left. g \right|_{\Omega_h^x \times \Omega_h^\xi} \in P^k(\Omega_h^x \times \Omega_h^\xi), \ \Omega_h^x \times \Omega_h^\xi \in \mathcal{P}_h(\Omega^x \times \Omega^\xi) \right\}$$

to be used for the approximation of the kinetic function f. We also recall that one can replace the space $\mathcal{G}_h^k$ by the space $P^k(\Omega_h^x) \times P^k(\Omega_h^\xi)$. Moreover, we use the Q-type space, $Q^k(\Omega_h^x \times \Omega_h^\xi)$ which is the set of polynomials of degree at most k in each variable in $\Omega_h^x \times \Omega_h^\xi$, then

$$\mathcal{H}_h^k = \left\{ g \in L^2\left(\Omega^x \times \Omega^\xi\right) : \left. g \right|_{\Omega_h^x \times \Omega_h^\xi} \in Q^k(\Omega_h^x \times \Omega_h^\xi), \ \Omega_h^x \times \Omega_h^\xi \in \mathcal{P}_h(\Omega^x \times \Omega^\xi) \right\},$$

or the space $Q^k(\Omega_h^x) \times Q^k(\Omega_h^\xi)$. The presented results hold for each space defined above in the same manner. For the simplicity of notation, we formulate our results in the space $\mathcal{G}_h^k$ only, but the same arguments can be done for Q-type space. We will make this more precise in Sect. 5.

Next, we are ready to delineate a semi-discrete DG scheme for (5). For fixed k, h and $\Omega_h^x \times \Omega_h^\xi \in \mathcal{P}_h(\Omega^x \times \Omega^\xi)$, we look for $f_h(t, \cdot, \cdot) \in \mathcal{G}_h^k$ such that

$$\begin{cases} \displaystyle\int_{\Omega_h^x \times \Omega_h^\xi} \frac{\partial f_h}{\partial t} g \, dx d\xi - \int_{\Omega_h^x \times \Omega_h^\xi} f_h \xi \cdot \nabla_x g \, dx d\xi - \int_{\Omega_h^x \times \Omega^\xi} F_a(f_h) f_h \cdot \nabla_\xi g \, dx d\xi \\[4pt] \quad + \displaystyle\int_{\partial\Omega_h^x \times \Omega_h^\xi} \widehat{f_h \xi} \cdot n_x g^- \, ds_x d\xi + \int_{\Omega_h^x \times \partial\Omega_h^\xi} \widehat{f_h F_a(f_h)} \cdot n_\xi g^- \, dx ds_\xi = 0, \quad \text{for all } g \in \mathcal{G}_h^k, \\[4pt] F_a(f_h)(t, x, \xi) = -\displaystyle\int_{\Omega_h^x \times \Omega_h^\xi} \psi(|x - x_*|)(\xi - \xi_*) f_h(t, x_*, \xi_*) \, dx_* d\xi_*, \end{cases}$$
$$(15)$$

where n_x and n_ξ are the outward unit normal vectors of $\partial\Omega_h^x$ and $\partial\Omega_h^\xi$, resepctively and all hat functions are numerical fluxes determined by upwind condition (see upwind standard formulation [6, 12]):

$$\begin{aligned} \widehat{f_h \xi} \cdot n_x &= \xi \cdot n_x \{f_h\}_x - \frac{|\xi \cdot n_x|}{2} [f_h]_x , \\[4pt] \widehat{f_h F_a(f_h)} \cdot n_\xi &= F_a(f_h) \cdot n_\xi \{f_h\}_\xi - \frac{|F_a(f_h) \cdot n_\xi|}{2} [f_h]_\xi , \end{aligned} \qquad (16)$$

where

$$\{f_h\}_x := \frac{f_h(t, x^+, \xi) + f_h(t, x^-, \xi)}{2} \quad \text{and} \quad [f_h]_x := f_h(t, x^+, \xi) - f_h(t, x^-, \xi)$$

are the average and the jump across the edge $\Omega_h^{x^+} \cap \Omega_h^{x^-}$ for a piecewise functions f_h in x, with $x^+ \in \Omega_h^{x^+}$ and $x^- \in \Omega_h^{x^-}$, respectively. The expressions $\{f_h\}_\xi$ and $[f_h]_\xi$ can be defined similarly.

2.3 Main results

Define the local mass density ρ_h as

$$\rho_h(t, x) := \int_{\Omega^\xi} f_h(t, x, \xi)d\xi.$$

Now, we are ready to state our main results in the following theorem.

Theorem 2.5 *Suppose that initial datum* $f_h(0, \cdot, \cdot)$ *is compactly supported and lies in the space* $W^{2,\infty}(\Omega^x \times \Omega^\xi)$, *and let* f *be a classical solution to* (5) *such that*

$$f \in H^{k+2}((0, T) \times \Omega^x \times \Omega^\xi), \quad \text{for } k \geq 0,$$

and let $f_h \in \mathcal{G}_h^k$ *be a numerical solution given by the semi-discrete DG scheme supplemented with periodic boundary conditions in* $\Omega^x \times \Omega^\xi$. *Then, for any* $h_0, T \in (0, \infty)$, *there exists a positive constant* $C_T = C(f, T, h_0)$ *such that for* $h < h_0$ *and* $t \in [0, T]$, *the following assertions hold:*

1. *(Mass conservation and* L^2*-stability):*

$$\frac{d}{dt} \int_{\Omega^x} \rho_h(t, x)dx = 0 \quad and \quad \|f_h(t)\|_{L^2(\Omega)} \leq C_T. \tag{17}$$

2. *(*L^2*-convergence):*
$$\|f(t) - f_h(t)\|_{L^2(\Omega)} \leq C_T h^{k+1/2}. \tag{18}$$

Proof Since the proof is very lengthy, we leave its proof in Sects. 3 and 5. □

3 Mass conservation and L^2-stability

In this section, we provide the proof of the first assertion (17) on the mass conservation and the L^2-stability of the numerical solution f_h to the semi-discrete DG scheme described in Sect. 2.2.

3.1 Mass conservation

For fixed h and k, let $f_h \in \mathcal{G}_h^k$ be the numerical solution to the semi-discrete nonlinear Galerkin method for (5) supplemented with periodic boundary conditions. We choose the test function

$$g \equiv 1.$$

Then, it is easy to see that

$$g \in \mathcal{G}_h^k, \quad \text{for each } k \geq 0, \quad \nabla_x g = 0 \quad \text{and} \quad \nabla_\xi g = 0.$$

This and $(15)_1$ yield

$$\frac{d}{dt} \int_{\Omega_h^x \times \Omega_h^\xi} f_h dx d\xi + \int_{\partial\Omega_h^x \times \Omega_h^\xi} \widehat{f_h \xi} \cdot n_x ds_x d\xi + \int_{\Omega_h^x \times \partial\Omega_h^\xi} \widehat{f_h F_a(f_h)} \cdot n_\xi dx ds_\xi = 0.$$
(19)

We sum (19) over all the partitions using the periodic boundary condition to find the mass conservation:

$$\frac{d}{dt} \int_{\Omega^x \times \Omega^\xi} f_h(t, x, \xi) dx d\xi = \frac{d}{dt} \int_{\Omega^x} \rho_h(t, x) dx = 0.$$

3.2 L^2-stability

Let h, k and $T > 0$ be fixed. Suppose that the initial datum $f_h(0, \cdot, \cdot) \in (C^1 \cap W^{2,\infty})(\Omega^x \times \Omega^\xi)$ and let $f_h \in \mathcal{G}_h^k$ be a numerical solution to (15) with periodic boundary conditions. Then, $(17)_2$ can be divided into two steps.

• Step A (Differential inequality for $\|f_h\|_{L^2}^2$): we claim that

$$\frac{d}{dt} \sum \int_{\Omega_h^x \times \Omega_h^\xi} f_h^2 dx d\xi + \sum \int_{\Omega_h^x \times \Omega_h^\xi} \nabla_\xi \cdot F_a(f_h) f_h^2 dx d\xi \leq 0,$$
(20)

for each $t > 0$.

Proof of claim (20) In $(15)_1$, we choose $g = f_h$ as a test function to find

$$
\begin{aligned}
0 = & \int_{\Omega_h^x \times \Omega_h^\xi} \frac{\partial f_h}{\partial t} f_h dx d\xi - \int_{\Omega_h^x \times \Omega_h^\xi} f_h \xi \cdot \nabla_x f_h dx d\xi - \int_{\Omega_h^x \times \Omega_h^\xi} f_h F_a(f_h) \cdot \nabla_\xi f_h dx d\xi \\
& + \int_{\partial\Omega_h^x \times \Omega_h^\xi} \widehat{f_h \xi} \cdot n_x f_h^- ds_x d\xi + \int_{\Omega_h^x \times \partial\Omega_h^\xi} \widehat{f_h F_a(f_h)} \cdot n_\xi f_h^- dx ds_\xi \\
=: & \sum_{i=1}^5 \mathcal{I}_{1i}.
\end{aligned}
$$
(21)

In what follows, we estimate each term $\mathcal{I}_{1i}$ one by one. □

◇ Case A.1 (Estimates of $\mathcal{I}_{11}$): by direct calculation, we have

$$\sum \int_{\Omega_h^x \times \Omega_h^\xi} \frac{\partial f_h}{\partial t} f_h dx d\xi = \frac{1}{2} \frac{d}{dt} \int_\Omega f_h^2 dx d\xi.$$
(22)

◇ Case A.2 (Estimate of $\mathcal{I}_{12}$ and $\mathcal{I}_{13}$): by integration by parts, we have

$$\sum \int_{\Omega_h^x \times \Omega_h^\xi} f_h \xi \cdot \nabla_x f_h \, dx d\xi = \sum \frac{1}{2} \int_{\Omega_h^x \times \Omega_h^\xi} \xi \cdot \nabla_x f_h^2 \, dx d\xi$$

$$= \sum \int_{\Omega_h^x \times \Omega_h^\xi} \nabla_x \cdot \left(\xi \frac{f_h^2}{2} \right) dx d\xi = -\sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} n_x \cdot \xi \left(\frac{f_h^2}{2} \right)_x^- ds_x d\xi$$

$$= -\sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} n_x \cdot \xi \{f_h\}_x [f_h]_x \, ds_x d\xi.$$

(23)

Similarly, we have

$$\sum \int_{\Omega_h^x \times \Omega_h^\xi} f_h F_a(f_h) \cdot \nabla_\xi f_h \, dx d\xi$$

$$= \sum \int_{\Omega_h^x \times \Omega_h^\xi} \frac{f_h^2}{2} \nabla_\xi \cdot F_a(f_h) dx d\xi + \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} n_\xi \cdot F_a(f_h) \{f_h\}_\xi [f_h]_\xi \, dx ds_\xi.$$

(24)

◇ Case A.3 (Estimate of $\mathcal{I}_{14}$ and $\mathcal{I}_{15}$): by direct calculations, we have

$$\sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} \widehat{f_h \xi} \cdot n_x f_h^- \, ds_x d\xi = -\sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} \widehat{f_h \xi} \cdot n_x [f_h]_x \, ds_x d\xi,$$

$$\sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \widehat{f_h F_a(f_h)} \cdot n_\xi f_h^- \, dx ds_\xi = -\sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \widehat{f_h F_a(f_h)} \cdot n_\xi [f_h]_\xi \, dx ds_\xi.$$

(25)

In (21), we combine (22), (23), (24), (25) and use (16) to get the desired estimate:

$$\frac{1}{2} \frac{d}{dt} \int_\Omega f_h^2 dx d\xi + \frac{1}{2} \int_\Omega f_h^2 \nabla_\xi \cdot F_a(f_h) \, d\xi dx$$

$$= \sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} \left(\widehat{\xi f_h} \cdot n_x - \xi \cdot n_x \{f_h\}_x \right) [f_h]_x \, ds_x d\xi$$

$$+ \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \left(\widehat{f_h F_a(f_h)} \cdot n_\xi - F_a(f_h) \cdot n_\xi \{f_h\}_\xi \right) [f_h]_\xi \, dx ds_\xi$$

$$= -\frac{1}{2} \sum \int_{\partial \Omega_h^x \times \Omega_\xi} |\xi \cdot n_x| [f_h]_x^2 \, ds_x dx - \frac{1}{2} \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} |F_a(f_h)| [f_h]_\xi^2 \, dx ds_\xi$$

$$\leq 0,$$

where we used (16) in the last equality.

• Step B (A bound for $\|f_h\|_{L^2}^2$): it follows from (20) that

$$\frac{d}{dt} \|f_h\|_{L^2}^2 \leq \left\| \nabla_\xi \cdot F_a(f_h) \right\|_{L^\infty} \|f_h\|_{L^2}^2 \leq d\psi_M \|f_h(0)\|_{L^1} \|f_h\|_{L^2}^2, \qquad (26)$$

where we used (2), (15)$_2$ and conservation of mass to find

$$\left\| \nabla_\xi \cdot F_a\left(f_h\right) \right\|_{L^\infty(\Omega^x \times \Omega^\xi \times [0,T])} \leq d\psi_M \|f_h(0)\|_{L^1}.$$

Then, we apply Gronwall's lemma for (26) to obtain, for $t \in [0, T]$,

$$\|f_h(\cdot, \cdot, t)\|_{L^2}^2 \leq \|f_h(\cdot, \cdot, 0)\|_{L^2}^2 \, e^{d\psi_M \|f_h(0)\|_{L^1} t} \leq \|f_h(\cdot, \cdot, 0)\|_{L^2}^2 \, e^{d\psi_M \|f_h(0)\|_{L^1} T}.$$

Now we set

$$C_T := e^{\frac{d\psi_M \|f_h(0)\|_{L^1} T}{2}}$$

to get the desired L^2-stability estimate:

$$\sup_{0 \leq t < T} \|f_h(t)\|_{L^2(\Omega^x \times \Omega^\xi)} \leq C_T.$$

4 Preparatory lemmas for convergence analysis

In this section, we study several preparatory estimates to be used in the L^2-convergence of the numerical solution f_h for (15) to the classical solution f for (5).

4.1 Error functional

Let f an g be piecewise C^1 functions in each partition box Ω_h^x and Ω_h^ξ, and we also assume that f has integrable first time-derivative. Then, we set

$$
\begin{aligned}
\mathcal{E}(f, g) &:= \sum \int_{\Omega_h^x \times \Omega_h^\xi} \frac{\partial f}{\partial t} g \, dx d\xi - \sum \int_{\Omega_h^x \times \Omega_h^\xi} f\xi \cdot \nabla_x g \, dx d\xi \\
&\quad - \sum \int_{\Omega_h^x \times \Omega_h^\xi} f F_a(f) \cdot \nabla_\xi g \, dx d\xi \\
&\quad + \sum \int_{\partial\Omega_h^x \times \Omega_h^\xi} \widehat{f\xi} \cdot n_x g^- \, ds_x d\xi + \sum \int_{\Omega_h^x \times \partial\Omega_h^\xi} \widehat{f F_a(f)} \cdot \xi g^- \, dx ds_\xi.
\end{aligned}
$$

For a classical solution f and semi-disctete DG solution f_h, one has

$$\mathcal{E}(f, g) = 0; \quad \mathcal{E}(f_h, g) = 0, \quad \forall \, g \in \mathcal{G}_h^k.$$

Note that

$$
\begin{aligned}
\mathcal{E} &:= \mathcal{E}(f, g) - \mathcal{E}(f_h, g) \\
&= \underbrace{\sum \int_{\Omega_h^x \times \Omega_h^\xi} \frac{\partial}{\partial t}(f - f_h) g \, dx d\xi - \sum \int_{\Omega_h^x \times \Omega_h^\xi} (f - f_h)\xi \cdot \nabla_x g \, dx d\xi}
\end{aligned}
$$

$$+ \sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} \underbrace{(\widehat{f - f_h})\xi \cdot n_x g^- \, ds_x d\xi}$$

$$=: \mathcal{L}(f - f_h, g)$$

$$- \sum \int_{\Omega_h^x \times \Omega_h^\xi} \underbrace{f F_a(f) \cdot \nabla_\xi g \, dx d\xi + \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \widehat{f F_a(f)} \cdot \xi g^- \, dx ds_\xi}$$

$$=: \mathcal{N}(f, g)$$

$$- \left[\underbrace{- \sum \int_{\Omega_h^x \times \Omega_h^\xi} f_h F_a(f_h) \cdot \nabla_\xi g \, dx d\xi + \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \widehat{f_h F_a(f_h)} \cdot \xi g^- \, dx ds_\xi} \right]$$

$$=: \mathcal{N}(f_h, g)$$

$$=: \mathcal{L}(f - f_h, g) + \mathcal{N}(f, g) - \mathcal{N}(f_h, g), \tag{27}$$

where $\mathcal{L}$ is the linear part of $\mathcal{E}$, while the terms $\mathcal{N}$ and $\mathcal{N}_h$ are nonlinear.

4.2 Estimates for $\mathcal{L}$ and $\mathcal{N}$

In this subsection, we study reduced expressions for the linear and nonlinear functionals introduced in (27). We denote the L^2-projection onto $\mathcal{G}_h^k$ by $\mathbb{P}_h$, and we set

$$\sigma_h := f - \mathbb{P}_h f, \quad \delta_h := \mathbb{P}_h (f - f_h), \quad \varepsilon_h := \sigma_h + \delta_h = f - f_h : \text{ total error.} \tag{28}$$

Lemma 4.1 *The linear functional $\mathcal{L}$ defined in* (27) *satisfies*

$$\mathcal{L}(f - f_h, \delta_h) = \frac{1}{2} \frac{d}{dt} \sum \int_{\Omega_h^x \times \Omega_h^\xi} |\delta_h|^2 dx d\xi + \frac{1}{2} \sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} |\xi \cdot n_x| \, [\delta_h]_x^2 \, ds_x d\xi + \mathcal{K}, \tag{29}$$

where $\mathcal{K}$ is given by

$$\mathcal{K} = \sum \int_{\Omega_h^x \times \Omega_h^\xi} \frac{\partial \sigma_h}{\partial t} \delta_h dx d\xi - \sum \int_{\Omega_h^x \times \Omega_h^\xi} \sigma_h \xi \cdot \nabla_x \delta_h dx d\xi$$

$$+ \sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} \widehat{\sigma_h \xi} \cdot n_x \, [\delta_h]_x \, ds_x d\xi . \tag{30}$$

Proof It follows from (28) that

$$f - f_h = \sigma_h + \delta_h. \tag{31}$$

Then, we use (31) to get

$$
\begin{aligned}
\mathcal{L}\left(f - f_h, \delta_h\right) &= \mathcal{L}\left(\sigma_h, \delta_h\right) + \mathcal{L}\left(\delta_h, \delta_h\right) \\
&= \sum \int_{\Omega_h^x \times \Omega_h^\xi} \frac{\partial \sigma_h}{\partial t} \delta_h \, dx d\xi - \sum \int_{\Omega_h^x \times \Omega_h^\xi} \sigma_h \xi \cdot \nabla_x \delta_h dx d\xi \\
&\quad + \sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} \widehat{\sigma_h \xi} \cdot n_x \delta_h^- ds_x d\xi \\
&\quad + \sum \int_{\Omega_h^x \times \Omega_h^\xi} \frac{\partial \delta_h}{\partial t} \delta_h dx d\xi - \sum \int_{\Omega_h^x \times \Omega_h^\xi} \delta_h \xi \cdot \nabla_x \delta_h dx d\xi \\
&\quad + \sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} \widehat{\delta_h \xi} \cdot n_x \delta_h^- ds_x d\xi \\
&= \sum \int_{\Omega_h^x \times \Omega_h^\xi} \left(\frac{\partial \sigma_h}{\partial t} + \frac{\partial \delta_h}{\partial t} \right) \delta_h dx d\xi \\
&\quad + \frac{1}{2} \sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} |\xi \cdot n_x| \, [\delta_h]_x^2 \, ds_x d\xi \\
&\quad - \sum \int_{\Omega_h^x \times \Omega_h^\xi} \sigma_h \xi \cdot \nabla_x \delta_h dx d\xi + \sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} \widehat{\sigma_h \xi} \cdot n_x \delta_h^- ds_x d\xi \\
&= \sum \int_{\Omega_h^x \times \Omega_h^\xi} \left(\frac{\partial \sigma_h}{\partial t} + \frac{\partial \delta_h}{\partial t} \right) \delta_h dx d\xi \\
&\quad + \frac{1}{2} \sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} |\xi \cdot n_x| \, [\delta_h]_x^2 \, ds_x d\xi \\
&\quad - \sum \int_{\Omega_h^x \times \Omega_h^\xi} \sigma_h \xi \cdot \nabla_x \delta_h dx d\xi + \sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} \widehat{\sigma_h \xi} \cdot n_x \, [\delta_h]_x \, ds_x d\xi \\
&=: \sum \int_{\Omega_h^x \times \Omega_h^\xi} \frac{\partial \delta_h}{\partial t} \delta_h dx d\xi + \frac{1}{2} \sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} |\xi \cdot n_x| \, [\delta_h]_x^2 \, ds_x d\xi + \mathcal{K},
\end{aligned}
$$

where $\mathcal{K}$ is given by (30). $\qquad\square$

Lemma 4.2 *The functional $\mathcal{N}$ defined in* (27) *satisfies*

$$
\begin{aligned}
\mathcal{N}(f, \delta_h) &- \mathcal{N}_h(f_h, \delta_h) \\
&= \sum \int_{\Omega_h^x \times \Omega_h^\xi} \nabla_\xi \cdot \left(F_a(f - f_h) \, f \right) \delta_h dx d\xi \\
&\quad + \frac{1}{2} \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} |F_a(f_h)| \, [\delta_h]^2 \, dx ds_\xi + \mathcal{H},
\end{aligned}
\tag{32}
$$

where $\mathcal{H}$ is given by

$$
\begin{aligned}
\mathcal{H} = &- \sum \int_{\Omega_h^x \times \Omega_h^\xi} F_a(f_h) \, \sigma_h \cdot \nabla_\xi \delta_h \, dx d\xi \\
&+ \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \left(\widehat{\sigma_h F_a(f_h)} \right) \cdot n_\xi \, [\delta_h] \, dx ds_\xi.
\end{aligned}
\tag{33}
$$

Proof Recall that

$$\mathcal{N}(f, \delta_h) = -\sum \int_{\Omega_h^x \times \Omega_h^\xi} f \, F_a(f) \cdot \nabla_\xi \delta_h dx d\xi$$

$$+ \sum \int_{\Omega_h^x \times \partial\Omega_h^\xi} \widehat{f \, F_a(f)} \cdot n_\xi \delta_h^- dx ds_\xi,$$

$$\mathcal{N}_h(f_h, \delta_h) = -\sum \int_{\Omega_h^x \times \Omega_h^\xi} f_h F_a(f_h) \cdot \nabla_\xi \delta_h dx d\xi$$

$$+ \sum \int_{\Omega_h^x \times \partial\Omega_h^\xi} \widehat{f_h F_a(f_h)} \cdot n_\xi \delta_h^- dx ds_\xi.$$

Then, these yield

$$\mathcal{N}(f, \delta_h) - \mathcal{N}_h(f_h, \delta_h)$$

$$= -\sum \int_{\Omega_h^x \times \Omega_h^\xi} [f \, F_a(f) - f_h F_a(f_h)] \cdot \nabla_\xi \delta_h dx d\xi$$

$$+ \sum \int_{\Omega_h^x \times \partial\Omega_h^\xi} \left(\widehat{f \, F_a(f)} - \widehat{f_h F_a(f_h)} \right) \cdot n_\xi \delta_h^- dx ds_\xi$$

$$= \mathcal{I}_{21} + \mathcal{I}_{22}.$$

Next, we estimate the term $\mathcal{I}_{2i}$ one by one.

- (Estimate of $\mathcal{I}_{21}$): note that

$$-\mathcal{I}_{21} = \sum \int_{\Omega_h^x \times \Omega_h^\xi} [f \, F_a(f) - f_h F_a(f_h)] \cdot \nabla_\xi \delta_h dx d\xi$$

$$= \sum \int_{\Omega_h^x \times \Omega_h^\xi} (F_a(f) - F_a(f_h)) \, f \cdot \nabla_\xi \delta_h dx d\xi$$

$$+ \sum \int_{\Omega_h^x \times \Omega_h^\xi} F_a(f_h) \, (f - f_h) \cdot \nabla_\xi \delta_h dx d\xi$$

$$= \sum \int_{\Omega_h^x \times \Omega_h^\xi} (F_a(f) - F_a(f_h)) \, f \cdot \nabla_\xi \delta_h dx d\xi$$

$$+ \sum \int_{\Omega_h^x \times \Omega_h^\xi} F_a(f_h) \, (\sigma_h + \delta_h) \cdot \nabla_\xi \delta_h dx d\xi$$

$$= \sum \int_{\Omega_h^x \times \Omega_h^\xi} (F_a(f - f_h)) \, f \cdot \nabla_\xi \delta_h dx d\xi$$

$$+ \sum \int_{\Omega_h^x \times \Omega_h^\xi} F_a(f_h) \sigma_h \cdot \nabla_\xi \delta_h dx d\xi$$

$$+ \sum \int_{\Omega_h^x \times \Omega_h^\xi} F_a(f_h) \delta_h \cdot \nabla_\xi \delta_h dx d\xi$$

$$= -(\mathcal{I}_{211} + \mathcal{I}_{212} + \mathcal{I}_{213}). \tag{34}$$

Below, we estimate the term $\mathcal{I}_{21i}$, $i = 1, 2, 3$ one by one.

◇ (Estimate of $-\mathcal{I}_{211}$ and $-\mathcal{I}_{213}$): note that

$$
\begin{aligned}
-\mathcal{I}_{211} = &-\sum \int_{\Omega_h^x \times \Omega_h^\xi} \nabla_\xi \cdot (F_a(f - f_h) f) \, \delta_h dx d\xi \\
&+ \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} (F_a(\widehat{f - f_h}) f) \cdot n_\xi \delta_h^- dx ds_\xi \\
&- \sum \int_{\Omega_h^x \times \Omega_h^\xi} \nabla_\xi \cdot (F_a(f - f_h) f) \, \delta_h \, dx d\xi \\
&+ \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} (F_a(\widehat{f - f_h}) f) \cdot n_\xi \, [\delta_h] dx ds_\xi,
\end{aligned}
\tag{35}
$$

and

$$
\begin{aligned}
-\mathcal{I}_{213} = &- \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} F_a(f_h) \cdot n_\xi \left(\frac{\delta_h^2}{2} \right)^- dx ds_\xi \\
= &-\frac{1}{2} \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \widehat{F_a(f_h)} \cdot n_\xi \left[\delta_h^2 \right] dx ds_\xi.
\end{aligned}
\tag{36}
$$

● (Estimate of $\mathcal{I}_{22}$): by direct estimate, we have

$$
\begin{aligned}
\mathcal{I}_{22} = &\sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \left(\widehat{f F_a(f)} - \widehat{f F_a(f_h)} \right) \cdot n_\xi \delta_h^- dx ds_\xi \\
&+ \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \left((f - \widehat{f_h}) F_a(f_h) \right) \cdot n_\xi \delta_h^- dx ds_\xi \\
= &\sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \left(\widehat{f F_a(f)} - \widehat{f F_a(f_h)} \right) \cdot n_\xi \, [\delta_h] dx ds_\xi \\
&+ \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \left((f - \widehat{f_h}) F_a(f_h) \right) \cdot n_\xi \, [\delta_h] dx ds_\xi \\
= &\sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \left(\widehat{f F_a(f)} - \widehat{f F_a(f_h)} \right) \cdot n_\xi \, [\delta_h] dx ds_\xi \\
&+ \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \left(\sigma_h \widehat{F_a(f_h)} \right) \cdot n_\xi \, [\delta_h] dx ds_\xi \\
&- \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \left(\delta_h \widehat{F_a(f_h)} \right) \cdot n_\xi \, [\delta_h] dx ds_\xi.
\end{aligned}
\tag{37}
$$

Finally, we combine all the estimates (34), (35), (36) and (37) to obtain

$$
\begin{aligned}
&\mathcal{N}(f, \delta_h) - \mathcal{N}_h(f_h, \delta_h) \\
&= \sum \int_{\Omega_h^x \times \Omega_h^\xi} \nabla_\xi \cdot (F_a(f - f_h) \, f) \, \delta_h dx d\xi \\
&\quad - \sum \int_{\Omega_h^x \times \partial\Omega_h^\xi} (F_a(\widehat{f - f_h}) \, f) \cdot n_\xi \, [\delta_h] \, dx ds_\xi \\
&\quad - \sum \int_{\Omega_h^x \times \Omega_h^\xi} F_a(f_h) \, \sigma_h \cdot \nabla_\xi \delta_h dx d\xi - \frac{1}{2} \sum \int_{\Omega_h^x \times \partial\Omega_h^\xi} \widehat{F_a(f_h)} \cdot n_\xi \, [\delta_h^2] \, dx ds_\xi \\
&\quad + \sum \int_{\Omega_h^x \times \partial\Omega_h^\xi} \left(\widehat{f F_a(f)} - \widehat{f F_a(f_h)} \right) \cdot n_\xi \, [\delta_h] \, dx ds_\xi \\
&\quad + \sum \int_{\Omega_h^x \times \partial\Omega_h^\xi} \left(\widehat{\sigma_h F_a(f_h)} \right) \cdot n_\xi \, [\delta_h] \, dx ds_\xi \\
&\quad - \sum \int_{\Omega_h^x \times \partial\Omega_h^\xi} \left(\widehat{\delta_h F_a(f_h)} \right) \cdot n_\xi \, [\delta_h] \, dx ds_\xi \\
&= \sum \int_{\Omega_h^x \times \Omega_h^\xi} \nabla_\xi \cdot (F_a(f - f_h) \, f) \, \delta_h dx d\xi - \sum \int_{\Omega_h^x \times \Omega_h^\xi} F_a(f_h) \, \sigma_h \cdot \nabla_\xi \delta_h dx d\xi \\
&\quad - \frac{1}{2} \sum \int_{\Omega_h^x \times \partial\Omega_h^\xi} \widehat{F_a(f_h)} \cdot n_\xi \, [\delta_h^2] \, dx ds_\xi + \sum \int_{\Omega_h^x \times \partial\Omega_h^\xi} \left(\widehat{\sigma_h F_a(f_h)} \right) \cdot n_\xi \, [\delta_h] \, dx ds_\xi \\
&\quad - \sum \int_{\Omega_h^x \times \partial\Omega_h^\xi} \left(\widehat{\delta_h F_a(f_h)} \right) \cdot n_\xi \, [\delta_h] \, dx ds_\xi,
\end{aligned}
\tag{38}
$$

where we used

$$
(F_a(\widehat{f - f_h}) \, f) = \left(\widehat{f F_a(f)} - \widehat{f F_a(f_h)} \right).
$$

Moreover, we have

$$
\begin{aligned}
&-\frac{1}{2} \sum \int_{\Omega_h^x \times \partial\Omega_h^\xi} \widehat{F_a(f_h)} \cdot n_\xi \, [\delta_h^2] \, dx ds_\xi + \sum \int_{\Omega_h^x \times \partial\Omega_h^\xi} \left(\widehat{\delta_h F_a(f_h)} \right) \cdot n_\xi \, [\delta_h] \, dx ds_\xi \\
&= \frac{1}{2} \sum \int_{\Omega_h^x \times \partial\Omega_h^\xi} |F_a(f_h)| \, [\delta_h]^2 \, dx ds_\xi.
\end{aligned}
\tag{39}
$$

We combine (38) and (39) to obtain

$$
\begin{aligned}
&\mathcal{N}(f, \delta_h) - \mathcal{N}_h(f_h, \delta_h) \\
&= \sum \int_{\Omega_h^x \times \Omega_h^\xi} \nabla_\xi \cdot (F_a(f - f_h) \, f) \, \delta_h dx d\xi - \sum \int_{\Omega_h^x \times \Omega_h^\xi} F_a(f_h) \, \sigma_h \cdot \nabla_\xi \delta_h dx d\xi \\
&\quad + \sum \int_{\Omega_h^x \times \partial\Omega_h^\xi} \left(\widehat{\sigma_h F_a(f_h)} \right) \cdot n_\xi \, [\delta_h] \, dx ds_\xi + \frac{1}{2} \sum \int_{\Omega_h^x \times \partial\Omega_h^\xi} |F_a(f_h)| \, [\delta_h]^2 \, dx ds_\xi.
\end{aligned}
\tag{40}
$$

$\square$

4.3 L^∞-estimates for $\mathcal{K}$ and $\mathcal{H}$

In this subsection, we study uniform bound estimate for functionals defined in (30) and (33). For this, we first recall a lemma regarding approximation and Poincare type inequality. We set

$$\Omega := \Omega^x \times \Omega^\xi, \qquad \Omega_h := \Omega_h^x \times \Omega_h^\xi \in \mathcal{P}_h(\Omega).$$

Lemma 4.3 [15] *Let k and h be a nonnegative integer and positive real number, respectively, and let $\Omega = \Omega^x \times \Omega^\xi$ be a periodic domain in x and ξ-variables. We denote the L^2-projection onto $\mathcal{G}_h^k$ by $\mathbb{P}_h$. Then, there exists a positive constant $C = C(k, \Omega) > 0$ independent of h such that the following assertions hold.*

1. *(Approximation properties): for any $g \in H^{k+1}(\Omega)$ and $\Omega_h \in \mathcal{P}_h(\Omega)$,*

$$\|g - \mathbb{P}_h g\|_{L^2(\Omega_h)} + h^{1/2}\|g - \mathbb{P}_h g\|_{L^2(\partial\Omega_h)} \le Ch^{k+1}\|g\|_{H^{k+1}(\Omega)}, \qquad (41)$$

2. *(Poincare type inequality): for any $g \in P^k(\Omega_h)$ with $\Omega_h \in \mathcal{P}_h(\Omega)$,*

$$\|\nabla_x g\|_{L^2(\Omega_h)} \le \frac{C}{h}\|g\|_{L^2(\Omega_h)}, \quad \|\nabla_\xi g\|_{L^2(\Omega_h)} \le \frac{C}{h}\|g\|_{L^2(\Omega_h)}. \qquad (42)$$

Now, we are ready to provide L^∞-estimates for $\mathcal{K}$ and $\mathcal{H}$ in the following two lemmas.

Lemma 4.4 *For a positive constant h_0, if $h \le h_0$, there exits a positive constant $C = C(k, \Omega)$ independent of the mesh sizes h such that*

$$\|\mathcal{K}\|_{L^\infty} \le C\left(h^{2k+2}\left\|\frac{\partial f}{\partial t}\right\|_{H^{k+1}} + h^{2k+2}\|f\|_{H^{k+1}}^2 + \|\delta_h\|_{L^2}^2\right)$$
$$+ \frac{1}{2}\sum \int_{\partial\Omega_h^x \times \Omega_h^\xi} |\xi \cdot n_x|\,[\delta_h]_x^2\,ds_x d\xi. \qquad (43)$$

Proof Note that

$$\mathcal{K} = \sum \int_{\Omega_h^x \times \Omega_h^\xi} \frac{\partial\sigma_h}{\partial t}\delta_h\,dxd\xi - \sum \int_{\Omega_h^x \times \Omega_h^\xi} \sigma_h\xi \cdot \nabla_x\delta_h dxd\xi$$
$$+ \sum \int_{\partial\Omega_h^x \times \Omega_h^\xi} \widehat{\sigma_h\xi} \cdot n_x\,[\delta_h]_x\,ds_x d\xi \qquad (44)$$
$$=: \mathcal{I}_{31} + \mathcal{I}_{32} + \mathcal{I}_{33}.$$

Below, we estimate the term $\mathcal{I}_{3i}$, $i = 1, 2, 3$ one by one.
• (Estimate of $\mathcal{I}_{31}$): now, we use (41) and Young's inequality to get

$$\left|\sum \int_{\Omega_h^x \times \Omega_h^\xi} \frac{\partial\sigma_h}{\partial t}\delta_h dxd\xi\right| \le \frac{1}{2}\left\|\frac{\partial\sigma_h}{\partial t}\right\|_{L^2}^2 + \frac{1}{2}\|\delta_h\|_{L^2}^2 \le \frac{C}{2}h^{2k+2}\left\|\frac{\partial f}{\partial t}\right\|_{H^{k+1}} + \frac{1}{2}\|\delta_h\|_{L^2}^2. \qquad (45)$$

• (Estimate of $\mathcal{I}_{32}$): let ξ_0 be the L^2 projection of ξ onto the piecewise constant space with respect to $\mathcal{P}_h(\Omega)$. Then, we have

$$\sum \int_{\Omega_h^x \times \Omega_h^\xi} \sigma_h \xi \cdot \nabla_x \delta_h dx d\xi = \sum \int_{\Omega_h^x \times \Omega_h^\xi} \sigma_h (\xi - \xi_0) \cdot \nabla_x \delta_h dx d\xi$$
$$+ \sum \int_{\Omega_h^x \times \Omega_h^\xi} \sigma_h \xi_0 \cdot \nabla_x \delta_h dx d\xi. \tag{46}$$

By definition of σ_h in (28), it follows from [28] that

$$\sum \int_{\Omega_h^x \times \Omega_h^\xi} \sigma_h \xi_0 \cdot \nabla_x \delta_h dx d\xi = 0. \tag{47}$$

Then, we use (41), (42), (46) and (47) to find

$$\left| \sum \int_{\Omega_h^x \times \Omega_h^\xi} \sigma_h \xi \cdot \nabla_x \delta_h dx d\xi \right| = \left| \sum_{\Omega_h^x \times \Omega_h^\xi} \sigma_h (\xi - \xi_0) \cdot \nabla x \delta_h dx d\xi \right|$$
$$\leq \|\xi - \xi_0\|_{L^\infty} \|\sigma_h\|_{L^2} \|\nabla_x \delta_h\|_{L^2} \leq C h^{k+1} \|f\|_{H^{k+1}} \|\delta_h\|_{L^2}$$
$$\leq \frac{1}{2} C \left(h^{2k+2} \|f\|_{H^{k+1}}^2 + \|\delta_h\|_{L^2}^2 \right), \tag{48}$$

where we used $\|\xi - \xi_0\|_{L^\infty} \leq Ch$ in [28].

• (Estimate of $\mathcal{I}_{33}$): we use (40) to find

$$\left| \sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} \widehat{\sigma_h \xi} \cdot n_x [\delta_h]_x \, ds_x d\xi \right| \leq \sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} |\sigma_h| |\xi \cdot n_x| |[\delta_h]_x| \, ds_x d\xi$$
$$\leq \frac{1}{2} \sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} |\xi \cdot n_x| \sigma_h^2 \, ds_x d\xi + \frac{1}{2} \sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} |\xi \cdot n_x| [\delta_h]_x^2 \, ds_x d\xi$$
$$\leq C \|f\|_{H^{k+1}}^2 h^{2h+1} + \frac{1}{2} \sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} |\xi \cdot n_x| [\delta_h]_x^2 \, ds_x d\xi. \tag{49}$$

In (44), we combine all the estimates (45), (48) and (49) to find the desired estimate.
□

Recall that the quantity $\|F_a(f_h)\|_\infty$ is bounded for each time $t \in [0, T]$ as $f_h \in \mathcal{P}_h^k(\Omega)$.

Lemma 4.5 *For $h_0 > 0$, if $h \leq h_0$, there exits a positive constant $C = C(k, \Omega)$ independent of the mesh sizes h such that*

$$\|\mathcal{H}\|_{L^\infty} \leq C\left(h^k \|F_a(f_h)\|_\infty \|f\|_{H^{k+1}} \|\delta_h\|_{L^2} + \|f\|_{H^{2k+1}}^2 h^{2k+1}\right)$$
$$+ \frac{1}{2} \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \left|F_a(f_h) \cdot n_\xi\right| [\delta_h]^2 \, dxds_\xi. \tag{50}$$

Proof It follows from (33) that

$$|\mathcal{H}| \leq \left|\sum \int_{\Omega_h^x \times \Omega_h^\xi} F_a(f_h) \, \sigma_h \, \cdot \nabla_\xi \delta_h dx d\xi\right|$$
$$+ \left|\sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \left(\widehat{\sigma_h F_a(f_h)}\right) \cdot n_\xi \, [\delta_h] \, dxds_\xi\right| \tag{51}$$
$$=: \mathcal{I}_{41} + \mathcal{I}_{42}.$$

Below, we estimate the term $\mathcal{I}_{4i}$ one by one.

• (Estimate of $\mathcal{I}_{42}$): we use (40) and (41) to obtain

$$\mathcal{I}_{42} \leq \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \left|F_a(f_h) \cdot n_\xi\right| |\sigma_h| |[\delta_h]| \, dxds_\xi$$
$$\leq \frac{1}{2} \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \left|F_a(f_h) \cdot n_\xi\right| |\sigma_h|^2 \, dxds_\xi$$
$$+ \frac{1}{2} \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \left|F_a(f_h) \cdot n_\xi\right| [\delta_h]^2 \, dxds_\xi \tag{52}$$
$$\leq C \|f\|_{H^{2k+1}}^2 h^{2k+1} + \frac{1}{2} \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} \left|F_a(f_h) \cdot n_\xi\right| [\delta_h]^2 \, dxds_\xi.$$

• (Estimate of $\mathcal{I}_{41}$): similarly, one has

$$\mathcal{I}_{41} \leq \|F_a(f_h)\|_\infty \|\sigma_h\|_{L^2} \left\|\nabla_\xi \delta_h\right\|_{L^2} \leq Ch^k \|F_a(f_h)\|_\infty \|f\|_{H^{k+1}} \|\delta_h\|_{L^2}. \tag{53}$$

In (51), we combine (52) and (53) to get the desired estimate. $\qquad \square$

In the following lemma, we give an L^2-estimate of the velocity divergence of the interaction term:

Lemma 4.6 *For $h_0 > 0$, if $h \leq h_0$, there exits a positive constant $C = C(k, \Omega)$ independent of the mesh sizes h such that*

$$\|\nabla_\xi \cdot (F_a(f - f_h))\|_{L^2}^2 \leq C\left(h^{2k+2} \|f\|_{H^{k+1}}^2 + \|\delta_h\|_{L^2}^2\right). \tag{54}$$

Proof We use (41) and (42) to find the desired estimate:

$$\|\nabla_\xi \cdot (F_a(f - f_h))\|_{L^2}^2$$

$$\leq \kappa d \int_{\Omega^x \times \Omega^\xi} \int_{\Omega^x \times \Omega^\xi} \psi(|x - x_*|)^2 |(f - f_h)(t, x_*, \xi_*|^2 \, d\xi_* dx_* d\xi dx$$

$$\leq C \left(\|\sigma_h\|_{L^2}^2 + \|\delta_h\|_{L^2}^2 \right) \leq C \left(h^{2k+2} \|f\|_{H^{k+1}}^2 + \|\delta_h\|_{L^2}^2 \right).$$

$\square$

5 L^2-convergence

In this section, we provide the proof of the second part in Theorem 2.3 on L^2-convergence. Suppose that the initial data $f_h(0)$ lies in $(C^1 \cap W^{k,\infty})(\Omega^x \times \Omega^\xi)$, $k \geq 2$ and f is the corresponding classical solution to (3)–(4) satisfying the regularity assumption:

$$f \in H^{k+2} \left([0, T) \times \Omega^x \times \Omega^\xi \right), \quad \text{for } k \geq 0.$$

Let $f_h \in \mathcal{G}_h^k$ be the numerical solution given by the semi-discrete DG scheme supplemented with periodic boundary conditions in $\Omega^x \times \Omega^\xi$. Then, we claim that for any $h_0 > 0$, there exists a positive constant $C_T = C(f, T, h_0) > 0$ such that for $h < h_0$

$$\|f(t) - f_h(t)\|_{L^2} \leq C_T h^{k+1/2}, \quad t \in [0, T]. \tag{55}$$

In what follows, we provide a derivation of (55). For this, we use (29), (32) and

$$\mathcal{E} = \mathcal{L}(f - f_h, g) + \mathcal{N}(f, g) - \mathcal{N}(f_h, g)$$

to find

$$\frac{1}{2} \frac{d}{dt} \sum \int_{\Omega_h^x \times \Omega_h^\xi} \delta_h^2 dx d\xi + \frac{1}{2} \sum \int_{\partial \Omega_h^x \times \Omega_h^\xi} |\xi \cdot n_x| \, [\delta_h]_x^2 \, ds_x d\xi + \mathcal{K}$$

$$+ \sum \int_{\Omega_h^x \times \Omega_h^\xi} \nabla_\xi \cdot (F_a(f - f_h) \, f) \, \delta_h \, dx d\xi \tag{56}$$

$$+ \frac{1}{2} \sum \int_{\Omega_h^x \times \partial \Omega_h^\xi} |F_a(f_h)| \, [\delta_h]^2 \, dx ds_\xi + \mathcal{H} = 0,$$

where $\mathcal{K}$ and $\mathcal{H}$ are given in (30) and (33), respectively. We use (43), (50) and (54) to rewrite (56) as

$$\frac{1}{2}\frac{d}{dt}\sum\int_{\Omega_h^x\times\Omega_h^\xi}\delta_h^2 dxd\xi$$

$$+\frac{1}{2}\sum\int_{\partial\Omega_h^x\times\Omega_h^\xi}|\xi\cdot n_x|\,[\delta_h]_x^2\,ds_xd\xi+\frac{1}{2}\sum\int_{\Omega_h^x\times\partial\Omega_h^\xi}|F_a(f_h)|\,[\delta_h]^2\,dxds_\xi$$

$$\leq\|f\|_{L^\infty}\sum\int_{\Omega_h^x\times\Omega_h^\xi}|\nabla_\xi\cdot(F_a(f-f_h))|\,|\delta_h|\,dxd\xi+|\mathcal{K}|+|\mathcal{H}|$$

$$\leq\frac{1}{2}\|f\|_{L^\infty}\left(\|\nabla_\xi\cdot(F_a(f-f_h))\|_{L^2}^2+\|\delta_h\|_{L^2}^2\right)+Ch^{2k+2}\left\|\frac{\partial f}{\partial t}\right\|_{H^{k+1}}$$

$$+Ch^{2k+2}\|f\|_{H^{k+1}}^2+C\,\|\delta_h\|_{L^2}^2+\frac{1}{2}\sum\int_{\partial\Omega_h^x\times\Omega_h^\xi}|\xi\cdot n_x|\,[\delta_h]_x^2\,ds_xd\xi$$

$$+h^k\,\|F_a(f_h)\|_{L^\infty}\,\|f\|_{H^{k+1}}\,\|\delta_h\|_{L^2}+C\,\|f\|_{H^{2k+1}}^2\,h^{2k+1}$$

$$+\frac{1}{2}\sum\int_{\Omega_h^x\times\partial\Omega_h^\xi}|F_a(f_h)\cdot n_\xi|\,[\delta_h]^2\,dxds_\xi$$

$$\leq C\frac{1}{2}\|f\|_{L^\infty}h^{2k+2}\|f\|_{H^{k+1}}^2+C\|f\|_{L^\infty}\|\delta_h\|_{L^2}^2+Ch^{2k+2}\left\|\frac{\partial f}{\partial t}\right\|_{H^{k+1}}$$

$$+Ch^{2k+2}\|f\|_{H^{k+1}}^2+C\,\|\delta_h\|_{L^2}^2+\frac{1}{2}\sum\int_{\partial\Omega_h^x\times\Omega_h^\xi}|\xi\cdot n_x|\,[\delta_h]_x^2\,ds_xd\xi$$

$$+h^k\,\|F_a(f_h)\|_{L^\infty}\,\|f\|_{H^{k+1}}\,\|\delta_h\|_{L^2}$$

$$+C\,\|f\|_{H^{2k+1}}^2\,h^{2k+1}+\frac{1}{2}\sum\int_{\Omega_h^x\times\partial\Omega_h^\xi}|F_a(f_h)\cdot n_\xi|\,[\delta_h]^2\,dxds_\xi.$$

This yields

$$\frac{1}{2}\frac{d}{dt}\|\delta_h\|_{L^2}^2\leq\frac{1}{2}\|f\|_{L^\infty}Ch^{2k+2}\|f\|_{H^{k+1}}^2+\|f\|_{L^\infty}C\|\delta_h\|_{L^2}^2+Ch^{2k+2}\left\|\frac{\partial f}{\partial t}\right\|_{H^{k+1}}$$

$$+Ch^{2k+2}\|f\|_{H^{k+1}}^2+C\,\|\delta_h\|_{L^2}^2+h^k\,\|F_a(f_h)\|_{L^\infty}\,\|f\|_{H^{k+1}}\,\|\delta_h\|_{L^2}$$

$$+C\,\|f\|_{H^{2k+1}}^2\,h^{2k+1}.$$

$$(57)$$

Now, we estimate $\|F_a(f_h)\|_\infty$ as follows.

$$|F_a(f_h)|\leq\int_{\Omega^x\times\Omega^\xi}|\psi(|x-x_*|)|\,|\xi-\xi_*|\,|f_h(t,x_*,\xi_*)|\,dx_*d\xi_*$$

$$\leq h\,\|f_h\|_{L^1}\leq\kappa h\left(\|\delta_h\|_{L^1}+\|\sigma_h\|_{L^1}+\|f\|_{L^1}\right)$$

$$\leq Ch\left(\|\delta_h\|_{L^2}+\|\sigma_h\|_{L^2}+\|f\|_{L^1}\right)$$

$$\leq C\left(h\,\|\delta_h\|_{L^2}+h^{k+2}\,\|f\|_{H^{k+1}}+h\,\|f\|_{L^1}\right).$$

Since $h < h_0$, we can chose a positive constant C such that one can rewrite (57) as

$$\frac{1}{2}\frac{d}{dt}\|\delta_h\|_{L^2}^2$$

$$\leq \frac{1}{2}\|f\|_{L^\infty}Ch^{2k+2}\|f\|_{H^{k+1}}^2 + \|f\|_{L^\infty}C\|\delta_h\|_{L^2}^2 + Ch^{2k+2}\left\|\frac{\partial f}{\partial t}\right\|_{H^{k+1}}$$

$$+ Ch^{2k+2}\|f\|_{H^{k+1}}^2 + C\|\delta_h\|_{L^2}^2 + Ch^{k+1}\|f\|_{H^{k+1}}\|\delta_h\|_{L^2}^2 + Ch^{2k+2}\|f\|_{H^{k+1}}\|\delta_h\|_{L^2}$$

$$+ Ch^{h+1}\|f\|_{L^1}\|f\|_{H^{k+1}}\|\delta_h\|_{L^2} + C\|f\|_{H^{2k+1}}^2 h^{2k+1}$$

$$\leq \left(C\|f\|_{L^\infty} + C + Ch^{k+1}\|f\|_{H^{k+1}} + \frac{1}{2}Ch^{2k+2} + \frac{1}{2}\right)\|\delta_h\|_{L^2}^2$$

$$+ \frac{1}{2}\|f\|_{L^\infty}Ch^{2k+2}\|f\|_{H^{k+1}}^2 + Ch^{2k+2}\left\|\frac{\partial f}{\partial t}\right\|_{H^{k+1}} + Ch^{2k+2}\|f\|_{H^{k+1}}^2$$

$$+ \frac{C}{2}h^{2k+2}\|f\|_{H^{k+1}}^4 + \frac{C}{2}h^{2h+2}\|f\|_{L^1}^2\|f\|_{H^{k+1}}^2 + C\|f\|_{H^{2k+1}}^2 h^{2k+1}.$$

Again, this yields

$$\frac{1}{2}\frac{d}{dt}\|\delta_h\|_{L^2}^2 \leq C\|\delta_h\|_{L^2}^2 + Ch^{2k+1}. \tag{58}$$

Finally, we apply Gronwall's lemma to (58) and use the projection inequality (41) to find

$$\|f - f_h\|_{L^2}^2 \leq \|\delta_h\|_{L^2}^2 + \|\delta_h\|_{L^2}^2 + \|\sigma_h\|_{L^2}^2 \leq Ch^{2k+1}.$$

This completes the proof. $\qquad\square$

In the sequel, we provide a corollary which improves the L^2-stability estimate and the L^∞ bound of the interaction function in term of h, and the L^2-norm of the classical solution.

Corollary 5.1 *Suppose that the same assumptions in Theorem 2.5 hold, and let f and $f_h \in \mathcal{G}_h^k$ be a classical solution to (5) and a numerical solution given by the semi-discrete DG scheme supplemented with periodic boundary conditions in $\Omega^x \times \Omega^\xi$. Then, we have the following estimates:*

$$(i)\ \|f_h\|_{L^2} \leq Ch^{k+1/2} + \|f\|_{L^2}.$$
$$(ii)\ \|F_a(f_h)\|_{L^\infty} \leq Ch\left(h^{k+1/2} + \|f\|_{L^2}\right).$$
$$(iii)\ \|\nabla_\xi \cdot F_a(f_h)\|_{L^\infty} \leq C\left(h^{k+1/2} + \|f\|_{L^2}\right).$$

Proof (i) Note that

$$\|f_h\|_{L^2} \leq \|f - f_h\|_{L^2} + \|f\|_{L^2}.$$

Then, we use the above relation and (18) to get the desired estimate.

(ii) We use the result (i) and the relation

$$|F_a(f_h)| \leq Ch\|f_h\|_{L^1} \leq Ch\|f_h\|_{L^2} \leq Ch\left(Ch^{k+1/2} + \|f\|_{L^2}\right).$$

The third estimate can be treated analogously. □

6 Numerical simulations

In this section, we present several numerical results. Let ϕ_j, for $j = 1, \ldots, N_k$ be the N_k-polynomials of k almost degree in the box $\Omega_h^x \times \Omega_h^\xi$ (for example Lagrange interpolant polynomials or Legendre polynomial expansion). Then, we set

$$f_h(t, x, \xi) = \sum_{j=1}^{N_k} f_h^j(t)\phi_j(x, \xi), \quad \mathbf{f}_h = (f_h^1, \ldots, f_h^{N_k}).$$

On each box of the partition, we have the following system of ODEs:

$$\mathcal{M}\frac{d\mathbf{f}_h}{dt} = \mathcal{S}(\mathbf{f}_h),$$

where the square matrices $\mathcal{M} = (\mathcal{M}_{ij})$, $\mathcal{S}^1 = (\mathcal{S}_{ij}^1)$ and a vector $\mathcal{S}^2(\mathbf{f}_h)$ are defined as follows.

$$\begin{cases} \mathcal{M}_{ij} = \int_{\Omega_h^x \times \Omega_h^\xi} \phi_i(x, \xi)\phi_j(x, \xi)dxd\xi, \quad \mathcal{S}(\mathbf{f}_h)_i = \mathcal{S}^1\mathbf{f}_h + \mathcal{S}^2(\mathbf{f}_h), \\ \mathcal{S}_{ij}^1 = \int_{\Omega_h^x \times \Omega_h^\xi} \phi_i(x, \xi)\xi \cdot \nabla_x\phi_j(x, \xi)dxd\xi + \int_{\partial\Omega_h^x \times \Omega_h^\xi} \widehat{\phi_j\xi} \cdot n_x\phi_i^- ds_x d\xi, \\ \mathcal{S}^2(\mathbf{f}_h)_i = \sum_{j=1}^{N_k}\left(\int_{\Omega_h^x \times \Omega_h^\xi} \phi_j F_a(f_h) \cdot \nabla_\xi\phi_i dxd\xi\right)f_h^j \\ \quad + \sum_{j=1}^{N_k}\left(\int_{\Omega_h^x \times \partial\Omega_h^\xi} \phi_j\widehat{F_a(f_h)} \cdot n_\xi\phi_i^- dxds_\xi\right)f_h^j, \end{cases}$$

and the integral in the interaction term $F_a(f_h)$ is computed by the Cavalieri-Simpson rule on a grid which ensures the h^{2k} convergence. To solve the method of lines ODE resulting from the semi-discrete DG scheme ($\mathcal{M}$ is invertible and $\mathcal{R} = \mathcal{M}^{-1}\mathcal{S}$):

$$\frac{d\mathbf{f}_h}{dt} = \mathcal{R}(\mathbf{f}_h),$$

we use the total variation diminishing third order Runge–Kutta method (see [30, 31]):

$$\begin{cases} \mathbf{f}_h^{(1)} = \mathbf{f}_h^n + \Delta t \mathcal{R}(\mathbf{f}_h^n), \\ \mathbf{f}_h^{(2)} = \dfrac{1}{4}\left(3\mathbf{f}_h^n + \mathbf{f}_h^{(1)} + \Delta t \mathcal{R}(\mathbf{f}_h^{(1)})\right), \\ \mathbf{f}_h^{n+1} = \dfrac{1}{3}\left(\mathbf{f}_h^n + 2\mathbf{f}_h^{(2)} + 2\Delta t \mathcal{R}(\mathbf{f}_h^{(2)})\right), \end{cases}$$

where $\mathbf{f}_h^n$ represents a numerical approximation of the solution at discrete time t^n. Such time stepping methods are the convex combinations of the Euler forward time discretization. We now present a numerical experiment.

Consider the domain

$$\Omega^x \times \Omega^\xi = [-\pi, \pi] \times [-2, 2]$$

with boundary condition in the x domain and zero at the boundary of the velocity domain. The region is divided into N rectangles, and we choose Lagrange polynomial of degree less then 2 ($N_2 = 6$), on each rectangle. Consider the following initial datum:

$$\begin{aligned} f_0(x, \xi) &= 0.1\, e^{-20\left[(x+1.5)^2+(\xi+0.2)^2\right]} + 0.05\, e^{-15\left[(x+0.1)^2+(\xi+0.5)^2\right]} \\ &\quad + 0.3\, e^{-50\left[(x+0.7)^2+(\xi+0.4)^2\right]}. \end{aligned}$$

As the initial datum is rapidly decaying, it is almost zero at the boundaries of the computational domain considered. So that we can assume that it is periodic in x and with compact support in ξ. As a communication weight function, we consider

$$\psi(|x - y|) = \frac{1}{(1 + (x - y)^2)^{\frac{\beta}{2}}},$$

with $\beta = 2$ (see [13]). The numerical computation is performed up to a time such that the numerical solution is compactly supported in the ξ variable. In our case this time is $t \approx 2.5$. The order of accuracy of the method is summarized in the table below, where the errors and the orders are computed at time $t = 1$:
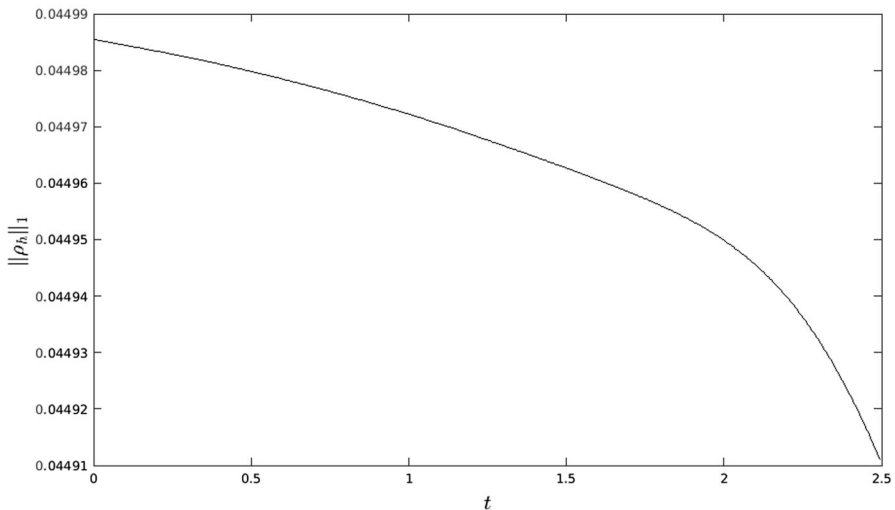
	N	L^2 error	Order	L^∞ error	Order
$k = 2$	16	0.113	1.98	0.0855	2.73
	64	0.0381	2.22	0.0253	2.94
	256	0.0142	2.48	0.0105	3.01

In Fig. 1, we show the temporal evolution of the mass density function

$$\rho_h(t, x) = \int_{[-2,2]} f_h(t, x, \xi)\, d\xi$$
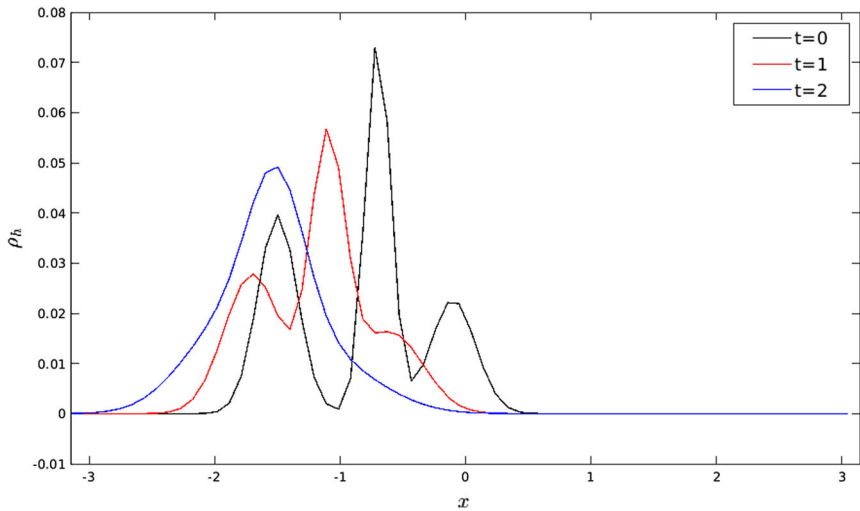
(a) Temporal evolution of local mass density

(b) Temporal evolution of total mass

Fig. 1 Temporal evolutions of ρ_h and $\|\rho_h\|_{L^1}$

and the evolution in time of the total mass $\int_{[-\pi,\pi]} \rho_h(t,x)dx$, which is shown to be constant up to fifth digit order.

7 Conclusion

In this paper, we have presented a semi-discrete DG scheme tor the kinetic Cucker–Smale equation. The kinetic Cucker–Smale equation is a dissipative Vlasov type equation whose total energy is non-increasing along the solution. This is a contrasted

difference with the classical Vlasov equation. From the viewpoint of numerics, due to the non-local nature of velocity alignment forcing, its numerical studies are vary few, compared to extensive analytical studies. We showed that our proposed semi-discrete DG scheme exhibits three crucial properties such as the total mass conservation, L^2-stability estimate and L^2-convergence of the numerical solution to the corresponding classical solution to the kinetic CS equation. Moreover, we showed that the convergence is at most $k + \frac{1}{2}$, as long as the target classical solution lies in H^{k+2}. In this paper, we assume that the communication weight function is bounded. However, there are several analytical studies [7, 9, 10, 14, 41–43] for the particle and kinetic CS models with singular communication weights. Thus, it would be interesting to extend current DG method to the setting with a singular communication weight. We leave this interesting problem for a future work.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. Acebrón, J.A., Bonilla, L.L., Pérez Vicente, C.J.P., Ritort, F., Spigler, R.: The Kuramoto model: a simple paradigm for synchronization phenomena. Rev. Mod. Phys. **77**, 137–185 (2005)
2. Ahn, S., Ha, S.-Y.: Stochastic flocking dynamics of the Cucker–Smale model with multiplicative white noises. J. Math. Phys. **51**, 103301 (2010)
3. Ayuso De Dios, B., Carrillo, J.A., Shu, C.-W.: Discontinuous Galerkin methods for the multi-dimensional Vlasov–Possion problem. Math. Models Methods Appl. Sci. **22**, 1250042 (2012)
4. Ballerini, M., Cabibbo, N., Candelier, R., Cavagna, A., Cisbani, E., Giardina, I., Lecomte, V., Orlandi, A., Parisi, G., Procaccini, A., Viale, M., Zdravkovic, V.: Interaction ruling animal collective behavior depends on topological rather than metric distance: evidence from a field study. Proc. Natl. Acad. Sci. USA **105**, 1232–1237 (2008)
5. Bellomo, N., Ha, S.-Y.: A quest toward a mathematical theory of the dynamics of swarms. Math. Models Methods Appl. Sci. **27**, 745–770 (2017)
6. Brezzi, F., Cockburn, B., Marini, L.D., Suli, E.: Stabilization mechanisms in discontinuous Galerkin finite element methods. Comput. Methods Appl. Mech. Eng. **195**, 3293–3310 (2006)

7. Byeon, J., Ha, S.-Y., Kim, J.: Asymptotic flocking dynamics of a relativistic Cucker–Smale flock under singular communications. J. Math. Phys. **63**, 012702 (2022)
8. Carrillo, J.A., Choi, Y.-P., Pareschi, L.: Structure preserving schemes for the continuum Kuramoto model: phase transitions. J. Comput. Phys. **376**, 365–389 (2019)
9. Carrillo, J.A., Choi, Y.-P., Hauray, M.: Local well-posedness of the generalized Cucker–Smale model with singular kernels. In: Mathematical Modeling of Complex Systems, ESAIM: Proceedings and Surveys, vol. 47, pp. 17–35. EDP Science, Les Ulis (2014)
10. Carrillo, J.A., Choi, Y.-P., Mucha, P.B., Peszek, J.: Sharp conditions to avoid collisions in singular Cucker–Smale interactions. Nonlinear Anal. Real World Appl. **37**, 317–328 (2017)
11. Carrillo, J.A., Fornasier, M., Rosado, J., Toscani, G.: Asymptotic flocking dynamics for the kinetic Cucker–Smale model. SIAM J. Math. Anal. **42**, 218–236 (2010)
12. Cheng, Y., Gamba, I.M., Majorana, A., Shu, C.-W.: A discontinuous Galerkin solver for Boltzmann–Poisson systems in nano devices. Comput. Methods Appl. Mech. Eng. **198**, 3130–3150 (2009)
13. Choi, Y.-P., Ha, S.-Y., Li, Z.: Emergent dynamics of the Cucker–Smale flocking model and its variants. In: Bellomo, N., Degond, P., Tadmor, E. (eds.) Active Particles vol 1-Theory, Models, Applications. Modeling and Simulation in Sciences and Technology. Springer, Birkhauser (2017)
14. Choi, Y.P., Zhang, X.: One dimensional singular Cucker–Smale model: uniform-in-time mean-field limit and contractivity. J. Differ. Equ. **287**, 428–459 (2021)
15. Ciarlet, P.-G.: The Finite Element Methods for Elliptic Problems. North-Holland, Amsterdam (1975)
16. Cockburn, B., Shu, C.-W.: Runge–Kutta discontinuous Galerkin methods for convection-dominated problems. J. Sci. Comput. **16**, 173–261 (2001)
17. Cockburn, B., Shu, C.-W.: TVB Runge–Kutta local projection discontinuous Galerkin finite element method for conservation law II: general framework. Math. Comput. **52**, 411–435 (1989)
18. Cucker, F., Dong, J.-G.: Avoiding collisions in flocks. IEEE Trans. Automat. Control **55**, 1238–1243 (2010)
19. Cucker, F., Dong, J.-G.: On flocks influenced by closest neighbors. Math. Models Methods Appl. Sci. **26**, 2685–2708 (2016)
20. Cucker, F., Dong, J.-G.: On flocks under switching directed interaction topologies. SIAM J. Appl. Math. **79**, 95–110 (2019)
21. Cucker, F., Mordecki, E.: Flocking in noisy environments. J. Math. Pures Appl. **89**, 278–296 (2008)
22. Cucker, F., Smale, S.: On the mathematics of emergence. Jpn. J. Math. **2**, 197–227 (2007)
23. Cucker, F., Smale, S.: Emergent behavior in flocks. IEEE Trans. Automat. Control **52**, 852–862 (2007)
24. Dong, J.-G., Qiu, L.: Flocking of the Cucker–Smale model on general digraphs. IEEE Trans. Automat. Control **62**, 5234–5239 (2017)
25. Dalmao, F., Mordecki, E.: Cucker–Smale flocking under hierarchical leadership and random interactions. SIAM J. Appl. Math. **71**, 1307–1316 (2011)
26. Dalmao, F., Mordecki, E.: Hierarchical Cucker–Smale model subject to random failure. IEEE Trans. Automat. Control **57**, 1789–1793 (2012)
27. Erban, R., Haskovec, J., Sun, Y.: On Cucker–Smale model with noise and delay. SIAM J. Appl. Math. **76**, 1535–1557 (2016)
28. Filbet, F., Shu, C.-W.: Discontinuous Galerkin methods for a kinetic model of self-organized dynamics. Math. Models Methods Appl. Sci. **28**(6), 1171–1197 (2018)
29. Fornasier, M., Haskovec, J., Toscani, G.: Fluid dynamic description of flocking via Povzner–Boltzmann equation. Physica D **240**, 21–31 (2011)
30. Gottlieb, S., Shu, C.-W.: Total variation diminishing Runge–Kutta schemes. Math. Comput. **67**, 73–85 (1998)
31. Gottlieb, S., Shu, C.-W., Tadmor, E.: Strong stability preserving high-order time discretization methods. SIAM Rev. **43**, 89–112 (2001)
32. Ha, S.-Y., Kim, J., Ruggeri, T.: From the relativistic mixture of gases to the relativistic Cucker–Smale flocking. Arch. Ration. Mech. Anal. **235**, 1661–1706 (2020)
33. Ha, S.-Y., Lee, K., Levy, D.: Emergence of time-asymptotic flocking in a stochastic Cucker–Smale system. Commun. Math. Sci. **7**, 453–469 (2009)
34. Ha, S.-Y., Liu, J.-G.: A simple proof of Cucker–Smale flocking dynamics and mean field limit. Commun. Math. Sci. **7**, 297–325 (2009)
35. Ha, S.-Y., Tadmor, E.: From particle to kinetic and hydrodynamic description of flocking. Kinet. Relat. Models **1**, 415–435 (2008)

36. Ha, S.-Y., Ruggeri, T.: Emergent dynamics of a thermodynamically consistent particle model. Arch. Ration. Mech. Anal. **223**, 1397–1425 (2017)
37. Jadbabaie, A., Lin, J., Morse, A.S.: Coordination of groups of mobile autonomous agents using nearest neighbor rules. IEEE Trans. Autom. Control **48**, 988–1001 (2003)
38. Li, Z., Ha, S.-Y.: On the Cucker–Smale flocking with alternating leaders. Q. Appl. Math. **73**, 693–709 (2015)
39. Li, Z., Xue, X.: Cucker–Smale flocking under rooted leadership with fixed and switching topologies. SIAM J. Appl. Math. **70**, 3156–3174 (2010)
40. Motsch, S., Tadmor, E.: A new model for self-organized dynamics and its flocking behavior. J. Stat. Phys. **144**, 923–947 (2011)
41. Mucha, P.B., Peszek, J.: The Cucker–Smale equation: singular communication weight, measure-valued solutions and weak-atomic uniqueness. Arch. Ration. Mech. Anal. **227**, 273–308 (2018)
42. Peszek, J.: Existence of piecewise weak solutions of a discrete Cucker–Smale's flocking model with a singular communication weight. J. Differ. Equ. **257**, 2900–2925 (2014)
43. Poyato, D., Soler, J.: Euler-type equations and commutators in singular and hyperbolic limits of kinetic Cucker–Smale models. Math. Models Methods Appl. Sci. **27**, 1089–1152 (2017)
44. Ru, L., Li, Z., Xue, X.: Cucker–Smale flocking with randomly failed interactions. J. Frankl. Inst. **352**, 1099–1118 (2015)
45. Reynolds, C.W.: Flocks, herds, and schools. Comput. Graph. **21**, 25–34 (1987)
46. Tan, C.: A discontinuous Galerkin method on kinetic flocking models. Math. Models Methods Appl. Sci. **27**, 1199–1221 (2017)
47. Toner, J., Tu, Y.: Flocks, herds and Schools: a quantitative theory of flocking. Phys. Rev. E **58**, 4828–4858 (1988)
48. Vicsek, T., Czirók, A., Ben-Jacob, E., Cohen, I., Shochet, O.: Novel type of phase transition in a system of self-driven particles. Phys. Rev. Lett. **75**, 1226–1229 (1995)
49. Vicsek, T., Zefeiris, A.: Collective motion. Phys. Rep. **517**, 71–140 (2012)