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Digital simulation of multi-variate stochastic processes

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Abstract. Stochastic dynamic analysis of linear or nonlinear multi-degree-of-freedom systems excited by multi-variated processes is usually conducted by using digital Monte Carlo (MC) simulation. Since in structural systems few modal shapes contribute to the response in the nodal space, the computational burden of MC simulation is mainly related to the digital simulation of the input process. Usually, the generation of multi-variated samples of Gaussian input process is performed with the aid of the Shinozuka formula. However, since in this procedure the stochastic process is given as a summation of waves with random amplitude amplified by the square root of the power spectral density, the randomness is due to a random phase angle of each wave, therefore a very large number of waves is required to reach the Gaussianity, i.e. the process is only asymptotically stable. Moreover, the computational burden increases in case of multi-variated processes. The paper aims to drastically reduce the generation time of the input process through the use of a two-step procedure. In the first step, by using the Priestley formula, each wave is normally distributed. This first aspect allows to drastically reduce the computational effort for the mono-variate process since few waves are sufficient to reach the Gaussianity. In the second step, the multi-variate process is reduced as a summation of independent fully coherent vectors if the quadrature spectrum (q-spectrum) can be neglected. An application of digital simulation of the wind velocity field is discussed to prove the efficiency of the proposed approach.

Introduction

Monte Carlo (MC) simulation is commonly used in the stochastic dynamic analysis of linear or nonlinear multi-degree-of-freedom systems excited by multi-variated processes [1,2]. It consists, initially, in the generation of samples of an input process characterized in probabilistic setting; then the output process is calculated by performing a deterministic analysis [1, 2]. Finally, a probabilistic analysis of the results is performed [1, 2].

As far as the input process generation are concerning it can be used for different purposes [3-7] and, if the input process is Gaussian, then it is fully characterized in probabilistic setting by its Power Spectral Density (PSD). The generation of samples of a Gaussian multi-variate input process can be performed in different ways; such an example Auto Regressive (AR) or Auto Regressive Moving Average (ARMA) techniques can be used for this purposes [1,8]. Although AR or ARMA techniques lead to good results, these kind of techniques requires a very high computational burden [1]. Nowadays, the generation of samples of the input process is usually performed through the use of the Shinozuka's formula [9] that consists in a summation of harmonic waves each of which has a different frequency and a random phase angle. The latter is the realization of a random variable uniformly distributed in the range $[0 - 2\pi]$. However, since the phases are uniformly distributed random variables, a high number of waves is required to reach the Gaussianity [1,8]. For this reason, the Priestley's formula is preferable [8]. It is similar to the Shinozuka's formula but, in this case, each harmonic wave is multiplied by the realization of a zero-mean complex random variable having normal distribution. This difference, drastically

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reduces the number of waves required to reach the Gaussianity since the random variables in the Priestley's formula have normal distribution unlike the random phases present in the Shinozuka's formula that are uniformly distributed.

A multi-variate stochastic process can be decomposed as a summation of fully coherent vectors [1,8]. This decomposition can be performed by using different method to decompose the PSDs matrix. One of these ways has been consists of decomposing it into the basis of the eigenvectors of the PSDs matrix [1,8]. This way is preferable to the decomposition of the PSDs matrix into the product between two triangular matrices not only because it is faster than the second one, but also because, in the wind field simulation, it has a strong physical significance. In fact, the eigenvalues are related to the power of the process that will be generated, while the eigenvectors, usually called *blowing mode shapes*, represent the modal shapes associated with the wind field velocity [1,8].

In this paper an innovative method for the simulation of multi-variate stationary processes is proposed. It exploits the advantages deriving from the use of the Priestley's formula that allows to reach the Gaussianity with a little number of harmonics. Moreover, the decomposition of the PSDs matrix, performed by using its eigenvalues and its eigenvectors, is used to select only the first eigenvalue and the first eigenvector that give the major contribution to the total input process. However, since the contribution of the other eigenvalues and eigenvectors can be not negligible, a simple technique to preserve the variance of the input process is adopted. The proposed method allows to generate a multi-variate stochastic process with good precision and it is faster than the methods commonly used.

In order to prove the reliability of the proposed method a numerical simulation concerning the generation of multi-variate wind field velocity has been performed and the results obtained are discussed in detail.

Proposed Method

Let $\mathrm{V}(x,y,z;t)$ be the velocity field. It can be expressed as [1,8]

$$\mathrm{V}(x,y,z;t)=\bar{V}(z)+V(x,y,z;t) \tag{1}$$

in which $\bar{V}(z)$ represents the mean value and $V(x,y,z;t)$ is a zero-mean stationary process that represents the fluctuating component of the wind speed. The mean value $\bar{V}(z)$ can be expressed by the logarithmic law $\bar{V}(z)=u_*\ln(z/z_0)/k$ in which k is the Von Karman's constant, z_0 is the roughness length and u_* is the shear velocity [1,8]. In this paper, the mean value $\bar{V}(z)$ is expressed, without loss of generality, as the logarithmic law

$$\bar{V}(z)=\bar{V}^{(10)}\frac{\ln(z+1)}{\ln(11)} \tag{2}$$

being $\bar{V}^{(10)}$ the wind velocity at $z=10\mathrm{m}$.

The one-side PSD of $V_i=V(x_i,y_i,z_i;t)$ and $V_j=V(x_j,y_j,z_j;t)$ can be calculated, considering that its imaginary part can be neglected [1,10], as [1,8]

$$G_{V_iV_j}(\omega)=\sqrt{G_{V_iV_i}(\omega)G_{V_jV_j}(\omega)}\exp\left(-f_{ij}(\omega)\right) \tag{3}$$

in which $G_{V_iV_i}(\omega)$ is the auto-PSD of the process V_i and $f_{ij}(\omega)$ is expressed as [1,8]

$$f_{ij}(\omega)=|\omega|\frac{\sqrt{C_y^2\left(y_i-y_j\right)^2+C_z^2\left(z_i-z_j\right)^2}}{2\pi\left(\bar{V}(z_i)+\bar{V}(z_j)\right)}. \tag{4}$$

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In Eq.(4) C_y and C_z are decay coefficients that must be determined experimentally [1,8]. The auto-PSD $G_{V_jV_j}(\omega)$ can be calculated by using the Solari's formula [11], i.e.

$$G_{V_jV_j}(\omega)=\frac{6.868\sigma_V^2\gamma_j(\omega)L_V/z_j}{\frac{\omega}{2\pi}\left(1+10.302\gamma_j(\omega)L_V/z_j\right)^{\frac{5}{3}}} \tag{5}$$

in which L_V is the integral length scale of the turbulence, σ_V^2 is the variance of the longitudinal component of the velocity fluctuations and $\gamma_j(\omega)$ is the Monin coordinate expressed as [1,8]

$$\gamma_j(\omega)=\frac{\omega z_j}{2\pi\bar{V}(z_j)}. \tag{6}$$

Defining the multi-variate vector process $\mathbf{V}(t)=\left[V_1 \;\; V_2 \;\; \dots \;\; V_n\right]^T$, its PSDs matrix, denoted as $\mathbf{G}_\mathbf{V}(\omega)$, can be calculated through the use of Eq.(3) and then $\mathbf{G}_\mathbf{V}(\omega)$ can be decomposed as [1,8]

$$\mathbf{G}_\mathbf{V}(\omega)=\mathbf{\Psi}(\omega)\mathbf{\Lambda}(\omega)\mathbf{\Psi}^T(\omega) \tag{7}$$

in which $\mathbf{\Psi}(\omega)$ is a matrix whose columns are the eigenvectors of $\mathbf{G}_\mathbf{V}(\omega)$ while $\mathbf{\Lambda}(\omega)$ is a diagonal matrix that contains the eigenvalues of $\mathbf{G}_\mathbf{V}(\omega)$.

The vector process $\mathbf{V}(t)$ can be decomposed into a summation of multi-variate fully coherent normal vectors $\mathbf{Q}_j(t)$ in the form [1,8]

$$\mathbf{V}(t)=\sum_{j=1}^{n}\mathbf{Q}_j(t) \tag{8}$$

and each vector $\mathbf{Q}_j(t)$ can be generated through the use of the Priestley's formula, taking into account the decomposition of the PSDs matrix in Eq.(7), as [1,8]

$$\mathbf{Q}_j(t)=\sum_{r=1}^{N}\mathbf{\Psi}_j(\omega_r)\sqrt{2\Lambda_j(\omega_r)\Delta\omega}P_r^{(j)} \tag{9}$$

in which $\mathbf{\Psi}_j(\omega_r)$ is the j-th eigenvector of $\mathbf{G}_\mathbf{V}(\omega_r)$, $\Lambda_j(\omega_r)$ is the j-th eigenvalue of $\mathbf{G}_\mathbf{V}(\omega_r)$, N is the number of harmonics waves considered in the summation, $N\Delta\omega$ is the cut-off frequency and $P_r^{(j)}$ is the r-th realization of a zero-mean normal complex random variable obeying the condition $E\left[P_s^{(i)}P_r^{(j)*}\right]=\delta_{ij}\delta_{sr}$. Eq.(9) can be expressed in its real form as [1,8]

$$\mathbf{Q}_j(t)=\sum_{r=1}^{N}\mathbf{\Psi}_j(\omega_r)\sqrt{2\Lambda_j(\omega_r)\Delta\omega}\left(R_r^{(j)}\cos(\omega_r t)+I_r^{(j)}\sin(\omega_r t)\right) \tag{10}$$

being $R_r^{(j)}$ and $I_r^{(j)}$, respectively, the real part and the imaginary part of $P_r^{(j)}$.

The use of Eq.(10) allows to generate samples of the fully coherent vector $\mathbf{Q}_j(t)$ faster than the Shinozuka's formula. In fact, since $P_r^{(j)}$ is normally distributed, a small number of harmonic waves can be considered in the summation and thus the computational burden can be drastically reduced. The time needed for the computation of the process $\mathbf{V}(t)$ can still be significantly reduced taking into account only $j=1$ in Eq.(8). However, if only the first terms of the summation in Eq.(8) is considered, then the variance of the process $V_i(t)$ is not well estimated and thus, to have good results, the variance has to be preserved. To do this, firstly, the exact value of the variance of $V_i(t)$ can be simply calculated as

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$$\sigma_{V_i}^2 = \sum_{r=1}^{N} G_{V_i V_i}(\omega_r)\Delta\omega \tag{11}$$

and then the vector process $\mathbf{V}(t)$ can be approximated as

$$\mathbf{V}(t) \approx \tilde{\mathbf{Q}}_1(t) \tag{12}$$

in which the elements $\tilde{Q}_i(t)$ of the vector $\tilde{\mathbf{Q}}_1(t)$ are defined as

$$\tilde{Q}_i(t) = \frac{1}{\sigma_{Q_{i1}}} \sum_{r=1}^{N} \Psi_{i1}(\omega_r)\sqrt{2\Lambda_1(\omega_r)\Delta\omega}P_r^{(1)}\sqrt{\sigma_{V_i}^2} \tag{13}$$

being $\sqrt{\sigma_{V_i}^2}$ the standard deviation of $V_i(t)$ and $\sigma_{Q_{i1}}$ the standard deviation of the i-th row of the vector $\mathbf{Q}_1(t)$. The vector $\mathbf{Q}_1(t)$ can be simply obtained considering $j=1$ in Eq.(10).

In this way, the computational burden required is drastically reduced since only the first blowing modal shape is considered. Particularly the time required to generate the process $\mathbf{V}(t)$ by using the proposed method can be expressed as a percentage of the time required to generate the process $\mathbf{V}(t)$ considering all the terms in the summation present in Eq.(8). This percentage is labelled as $T_*\%$ and can be expressed in the form

$$T_*\% \approx 100/n\,. \tag{14}$$

From Eq.(14) it is clear that the proposed method allows to drastically reduce the time required to generate a stationary Gaussian multi-variate stochastic process.

Numerical Simulation

To assess the reliability of the proposed method a numerical simulation has been performed considering three different points, $P_1(x_1,y_1,z_1)$; $P_2(x_2,y_2,z_2)$ and $P_3(x_3,y_3,z_3)$, having $x_1=x_2=x_3$, $y_1=y_2=y_3$, $z_1=10$ m, $z_2=20$ m and $z_3=30$ m. The velocity $\bar{V}^{(10)}$ is equal to 22 m/s ,while L_V, σ_V^2 and C_z has been assumed as unitary values. In this simulation 1000 samples of the process $\mathbf{V}(t)$ have been generated. Each sample has a duration of 100 s discretized with a sampling frequency of 100 Hz. The components of PSDs matrix of the process $\mathbf{V}(t)$ have been calculated by using Eq.(3) and are represented in Fig.1.

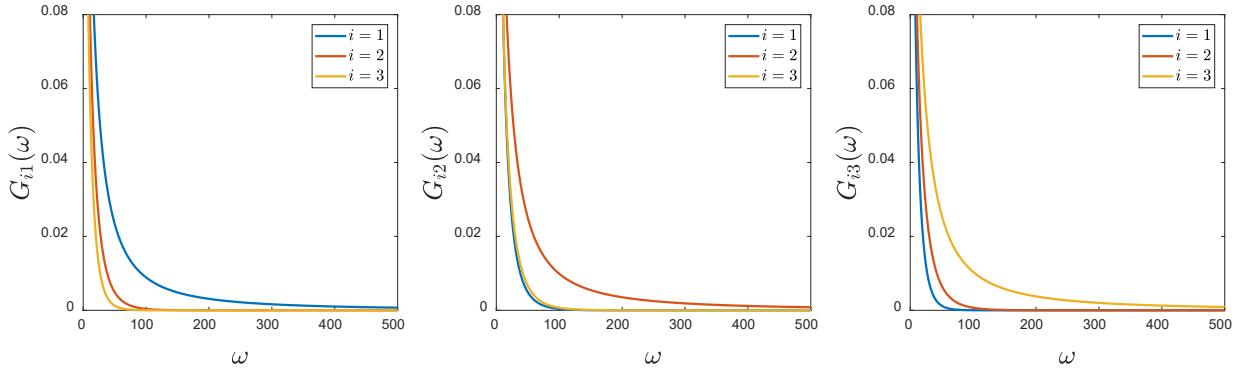

Figure 1 – Components of the PSDs matrix of the process $\mathbf{V}(t)$

From Fig.1 it is clear that the cross-PSDs decrease if the distance between the point increases. Moreover, all the components of the PSDs matrix decrease when the frequency increases.
The eigenvalues and the eigenvectors of the PSDs matrix have been calculated and are reported in Fig.2.

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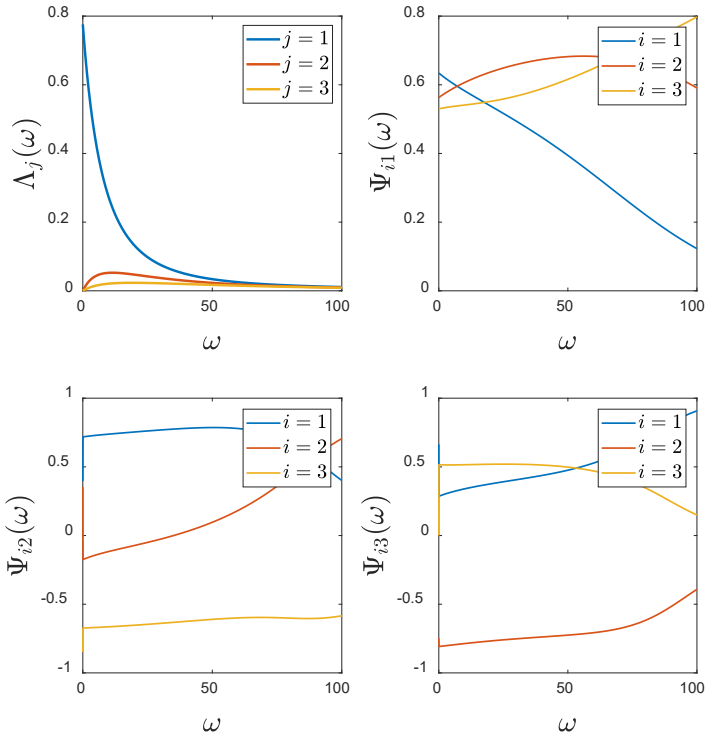

Figure 2 – Eigenvalues ($\Lambda_j(\omega)$) and components of the eigenvectors ($\Psi_{ij}(\omega)$) of $\mathbf{G}_{\mathbf{V}}(\omega)$

From Fig.2 can be observed that the eigenvalues have the same behavior of the PSDs matrix, i.e. they decrease when the frequencies increases. Moreover, from Fig.2, it can be seen that the first eigenvalue gives the major contribution to the total vector process.

Once that the PSDs matrix of the process $\mathbf{V}(t)$ and its eigenvalues and eigenvectors have been calculated, Eqs.(12, 13) have been used to generate the process $\mathbf{V}(t)$.

Since Priestley's formula has already been applied in the literature to reduce the computational burden required to generate a process $\mathbf{V}(t)$ [1,8], the main innovation of this paper is the truncation of the summation in Eq.(8) linked to the way suggested to preserve the variance. For this reason, in this numerical simulation, two processes have been generated considering the same number of waves: the first one has been obtained by using Eqs.(8,10) without any truncation, while the second one has been generated through the use of the proposed method, i.e. by using Eqs.(12,13). In Fig.3 few second of a sample of the two generated processes are reported.

From Fig.3 it is clear that the proposed method leads to good results. Moreover, the discrepancies between the variances of the processes calculated by using Eq.(12) and the variances of the processes calculated by using Eq.(8) are computed as

$$\varepsilon_j\% = 100\frac{\int_0^{t_f}\left|\sigma^2_{V_j}(t)-\sigma^2_{\hat{Q}_j}(t)\right|dt}{\int_0^{t_f}\left|\sigma^2_{V_j}(t)\right|dt} \tag{15}$$

The results obtained from Eq.(15) are reported in Tab.1

Table 1 – Discrepancies between the variances of the processes $V_i(t)$ generated by using Eq.(8) and the variances of the processes $V_i(t)$ generated by using Eq.(12)

$\varepsilon_1\%$	$\varepsilon_2\%$	$\varepsilon_3\%$
3.5464	3.0232	2.5727

From these results it is clear that the proposed method can be considered reliable.

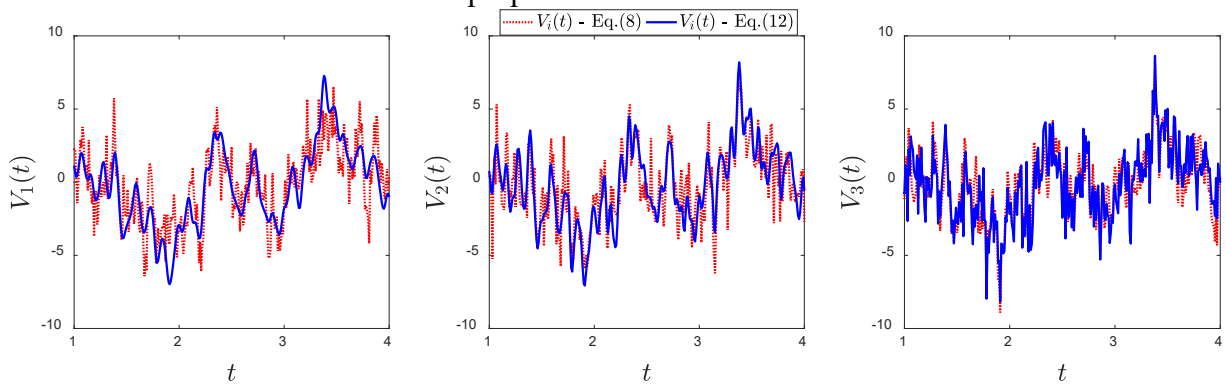

Figure 3 – Comparison between the processes $V_i(t)$ generated by using Eq.(8) and the processes $V_i(t)$ generated by using Eq.(12)

The time taken to generate 1000 samples of the process $\mathbf{v}(t)$ by using Eq.(8) is equal to 176 min, while the time taken to generate the same number of samples by using Eq.(12) is equal to 59.33 min, thus, according to Eq.(14), $T_*\% = 33.71\% \approx 1/n$. Furthermore, the computational burden can still be reduced by exploit the advantages deriving from the use of the Priestley's formula. To this regard, the PSDs in Fig.1 and its eigenvalues reported in Fig.2 can help to choose the cut-off frequency $N\Delta\omega$.

Conclusions

In this paper an innovative method to simulate multi-variate stochastic processes is proposed. It uses the Priestley formulation since it is more performant than the Shinozuka's one for the generation of Gaussian processes. The computational burden required to generate a multi-variate stochastic processes is drastically reduced taking into account only the first blowing mode shape provided that the variance of the process is preserved. The numerical simulations performed show that the proposed method is reliable, thus it can be used to drastically reduce the computational burden required to generate multi-variate stochastic processes.

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