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Three Essays on Assortment Planning in Omni-Channel Retail Supply Chains

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Three Essays on Assortment Planning in Omni-Channel Retail Supply Chains

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Abstract

In omni-channel retail systems, comprising an online sales website and brick-and-mortar (physical) stores, a physical store typically faces limited shelf-space capacity, while capacity is not an issue for the online channel. Consequently, a crucial aspect of such retail systems is to choose a subset of products present online for showcasing in the physical store (i.e., assortment planning).

In my first research stream, I investigate the omni-channel assortment problem when product returns are allowed. Assortment decisions influence product returns, as showcased products provide information to online shoppers who visit the physical store. Therefore, although product returns can be a factor for profit loss, effective assortment planning can mitigate the returns' adverse impact and optimize profitability. My results indicate that even with sufficient capacity, showcasing all products in the physical store may not be optimal. Additionally, retailers generally fare better when customers undervalue hidden attribute levels.

In my second research stream, I explore a decentralized retail supply chain (RSC) comprising an online channel managed by a manufacturer setting wholesale prices, and an independent retailer managing the physical store and making assortment decisions. As a benchmark, I examine a centralized setting where both channels are under a central authority aiming to maximize overall profit. My findings show fundamental differences in optimal centralized and decentralized assortments, indicating inefficiency in the decentralized approach. I propose scope contracts for coordination, wherein the manufacturer offers discounts on wholesale prices for products with specific attribute levels, incentivizing the retailer to adopt the centralized assortment. The scope contracts ensure both parties' profitability and coordinate the RSC.

In the third stream, I suppose that the magnitude of inaccuracy in online assessment of products due to the lack of physical encounter is unknown to the RSC parties, and they make decisions with asymmetric information. I investigate the assortment and wholesale price decisions along with profit regrets. My findings under the decentralized setting indicate that while both parties cannot fare better simultaneously, each party can be advantaged under certain conditions. Under the centralized setting, when supposing accurate online assessments, showcasing an assortment of the highest utility attribute levels possibly minimizes system-wide regret.

Preface

This thesis is an original work by the author. No part of this thesis has been previously published.

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First and foremost, I would like to express my gratitude to my supervisor Dr. Osman Alp who inspired me throughout my Ph.D. journey. I highly appreciate all the time he devoted to my growth as a scholar. Furthermore, I am thankful for the support of my supervisory committee members, Dr. Marco Bijvank and Dr. Alireza Sabouri, whom I learned a lot from in the past years.

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I dedicate this work to my love, Masoumeh, to my caring parents, and to my brothers, Ali and Mahdi.

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As you start to walk on the way, the way appears...

- Rumi

Chapter 1

Introduction

This thesis is structured around three research streams focusing on the domain of assortment planning within the context of omni-channel retailing. The aim of this introductory chapter is to provide an overview of the thesis, emphasizing the significance of this area of research, and motivating the chosen topic. Each of the three research streams tackles a distinct research problem but is built upon a shared foundation in terms of background and modeling approach.

The first section of this chapter will concentrate on explaining the concept of an omni-channel retail environment and the general settings that have been uniformly applied across all three research streams. This will provide readers with a comprehensive understanding of the broader context in which the subsequent research is conducted.

Subsequently, the chapter will delve into a more detailed discussion of each specific research stream, addressing the research inquiries related to assortment planning in the omni-channel retailing context. By doing so, the thesis aims to explore various aspects of assortment planning within this retail landscape and shed light on the particular challenges and opportunities that arise.

1.1 Introduction to Omni-Channel Assortment Planning

Omni-channel retailing, a prevalent approach in contemporary retail practices, involves operating both internet-based sales channels and brick-and-mortar (physical) stores. This strategy has become increasingly popular due to its potential to enhance market share and boost retailers' profitability (Bell et al. 2018). In such retail systems, physical stores often confront the challenge of limited showcase capacity, resulting in a constrained selection of products available for customers. Conversely, the online sales channel enjoys the advantage of virtually unlimited product variety, as it sources directly from a central warehouse or the manufacturer's location. Consequently, a crucial deliberation in omni-channel retailing revolves around determining the optimal selection of products to be showcased in the physical store (i.e., the assortment decision).

Products can consist of various 'attributes', such as color and material, and each attribute can be further characterized by multiple 'levels' – for example, blue and red for the color attribute. According to Dzyabura and Jagabathula (2018), the appeal of a product to a customer, also known as its utility, can be measured by summing up the utility of its individual attribute levels. In the context of omni-channel retailing, products can be categorized as ones with digital or non-digital attributes. The former refers to products for which physical encounter and trying out are not crucial to make an informed evaluation of their utilities. For example, song collections or books can be considered as such products. On the other hand, the latter refers to products for which touching and trying out are essential in the informed evaluation of their utilities. A wide range of products such as apparel, sunglasses, and handbags are considered as products with non-digital attributes, which are the main focus of this thesis. For these products, items showcased in a physical store are accurately evaluated by customers; while, items with attribute levels that are not showcased (but browsed only online) may be inaccurately evaluated.

The inaccuracy in the customers' online evaluation of products and attribute levels will influence their shopping experiences. For example, if a customer overvalues the utility of a product on a computer screen (i.e., the measured utility is greater than the utility that the customer would have obtained if they evaluated the product physically), they will be more likely to purchase the overvalued product compared to the case the customer browses the product in-store with physical evaluation and encounter. Similarly, if the customer undervalues the utility of the product on a screen, then they will be less likely to purchase it.

Over- or under-valuation of a product utility at the time of purchase will also influence the customer's (dis)satisfaction with the product after purchase, which impacts their decision for keeping or returning the product. If a product was purchased online with an overvaluation at the time of purchase, it will disappoint the customer upon receipt because it does not meet the desired expectations. Consequently, the product will be more likely to be returned, in contrast to cases where the customer purchased the product with an accurate evaluation. Likewise, a product purchased with undervaluation will gratify the customer upon receipt because it exceeds their initial expectation. Thus, the product will be less likely to be returned.

The assortment decision for the physical store impacts customers' purchase and keep-or-return decisions in the online channel as well. This influence arises because products showcased in the physical store may share attribute levels with online-only products, providing partially accurate utility information for these products. For instance, consider that a red product with a specific style might be available in-store, but the red color with the style that the customer wants may not be available in-store. In this situation, the customer can accurately evaluate the utility of the red color, but not the utility of the desired style. As a result, the customer will evaluate the overall utility of the desired product inaccurately (partially accurately) in this example. This is specifically crucial for high-value products, where customers prefer to visit the physical store to obtain accurate utility information of the available products and their attribute levels before making a purchase, whether online or from the store (Park et al. 2021). Therefore, in this thesis, we explore the assortment planning problem for the physical store in an omni-channel retail system, while explicitly studying its impact on customers' purchase and keep-or-return decisions in the online channel.

Consumer choice models are typically used for modeling customers' purchase and keep-or-return decisions. In this thesis, we employ the most widely used choice model, the multinomial logit (MNL) for modeling these decisions. The MNL is a utility based model in which each customer takes the action that provides the highest utility for them (Dzyabura and Jagabathula 2018, Ben-Akiva et al. 1985).

1.2 Research Stream 1: Omni-Channel Assortment Planning in Presence of Product Returns

In modern retailing, a common marketing strategy is to allow product returns with limited inconvenience or hassle-free. However, the costs associated with product returns, such as transportation, repackaging, and refurbishing, can accumulate significantly. According to the National Retail Federation of the U.S.A., returns accounted for $761 billion in lost profit in 2021, representing approximately 16.6% of the total value of retail profit. Particularly, within the online segment, 20.8% of purchases have been returned (NRF 2021). One of the reasons for product returns is that customers may not fully experience and accurately evaluate the products before making a purchase (Alptekinoğlu and Grasas 2014, YouGov 2021). As discussed in Section 1.1, 'inaccurate evaluation' is an inherent part of online shopping for products with non-digital attributes.

For retailers operating both physical and online stores, showcasing products in the physical store serves as a means of providing customers with accurate information of the products and their attribute levels (Mantrala et al. 2009). The product information revealed at the physical store can help alleviate the uncertainties that customers might encounter in their evaluations of products (Bell et al. 2014). Therefore, in this research stream, we aim to study the assortment problem in the described omni-channel retail system while product returns are explicitly modeled and their impacts on profitability are studied.

Customers may prefer purchasing from online sales websites or physical stores. In this stream, we assume that customers' shopping preferences have matured and converged such that some prefer shopping with physical encounter and trying out the products, potentially from a limited selection showcased in a physical store. Other customers may opt for the convenience of purchasing from the comfort of their homes and appreciate a wider variety of products available online. These preferences translate into accurate or inaccurate evaluations of the product utilities in the offline and online channels, respectively. In both shopping experiences, it is possible that a customer's subsequent evaluation of a purchased product, such as when they start using it at home or upon receiving it, may not align with the initial evaluation during the purchase. This can cause customer dissatisfaction, which is one of the primary reasons for returns.

We show that an increase in product returns does not necessarily result in a decrease in profit, as higher product returns, in some cases, can be an outcome of greater sales. Moreover, even if the sufficient shelf-space capacity is provided in the store, it may not be optimal to showcase the full variety of products, as it can be possible that not revealing accurate information of some attribute levels can yield a higher overall profit. We also indicate that on average, retailers can attain a greater profit if the attribute levels that are not showcased in the physical store are slightly undervalued by customers.

1.3 Research Stream 2: Assortment Planning Coordination in Omni-Channel Retail Supply Chains

In the second research stream, we study the described system in Section 1.1 as a retail supply chain (RSC). In this RSC, we assume that the online sales channel is the manufacturer's direct sales medium, and the physical store is independently managed by a retailer, for example, a franchise or an independent department store that sells manufacturer's products. Unlike the first research stream, we assume that customers do not have a specific preference between the online channel and the physical store. This allows to capture the competition between the sales channels for higher profits. So, we consider that customers visit the online channel and the physical store and purchase a product that they evaluate as the highest utility product across both channels. An example of such practice is that customers visit the physical store and evaluate the products available while browsing the complete collection available online using their smartphones or a computer screen provided in the store.

In this decentralized RSC, the manufacturer and the retailer are independent parties with conflicting interests, each striving to maximize their own benefits. In a Stackelberg game scenario, the manufacturer establishes the wholesale price to be charged to the retailer for their products, and the retailer selects its assortment decisions accordingly. Evidently, the retailer chooses an assortment that maximizes its own profit. However, as described earlier, the assortment decision's interaction with the purchase and keep-or-return decisions impacts the manufacturer's profit (arising from purchases and returns). As a result, the retailer's assortment decision may not be desired for the manufacturer or the entire system. Therefore, the decentralized assortment decisions may result in inefficiencies in RSC operations. As a benchmark that optimizes the profit of the entire RSC, we consider a centralized setting in which a central authority oversees both sales channels and makes the assortment decision to maximize the total profit of the system, encompassing the retailer's and the manufacturer's profits (Chaharsooghi and Heydari 2010). The inefficiency observed in the decentralized system can be resolved by incentivizing the retailer to showcase the centralized assortment.

We propose "*scope contracts*" as a coordination mechanism that can eliminate the inefficiency of the decentralized structure while achieving a mutually beneficial outcome for both the retailer and the manufacturer. In the scope contract, the manufacturer offers discounts on the wholesale price of the products based on a predefined scope of their attribute levels. The specified scope and discount rates within this contract lead the retailer to ordering the optimal centralized assortment, ensuring that the retailer benefits more from this decision compared to any other assortment, including that of the decentralized setting. The designed contract also guarantees an increase in the manufacturer's expected profit in comparison to the original decentralized adversarial setting.

1.4 Research Stream 3: Omni-Channel Assortment Planning with Asymmetric Information

According to the preceding discussions in Sections 1.2 and 1.3, as well as the findings of the previous studies such as Dzyabura and Jagabathula (2018) and Lo and Topaloglu (2022), the inaccuracy in customers' online evaluation of product utilities is a critical factor that significantly influences the assortment decisions for the physical store. The optimal assortment decision can substantially change based on whether customers accurately evaluate, overvalue, or undervalue the product utilities in their online evaluations. Considering this inaccuracy is essential while making the assortment decision to ensure the most effective and profitable selection of products for the physical store. Therefore, it is crucial to obtain a reliable estimation of this inaccuracy component to ensure effective management of omni-channel retail operations. However, estimating the inaccuracy parameters is not a straightforward or trivial task and might demand substantial efforts that can be both challenging and costly. For instance, devising surveys or lab/field experiments may be necessary to gain insights into customers' behavior as they comparatively evaluate online and offline products. Additionally, the customer profile can vary between different product segments, and the design, appearance, and the interface of the online website also directly influence the inaccuracies. Consequently, these inaccuracies are case specific and can differ across different environments. Therefore, the RSC parties make their decisions based on their own best guesses of the inaccuracy parameters.

It might be reasonable to expect that the party operating the online sales website would conduct the research to understand customer behavior and assess the inaccuracy. However, even if this party obtains full information, they may choose not to share it with the physical store operators. As a result, the challenges faced in practice regarding the estimation of the inaccuracy magnitude revolve around the potential misalignment between expectations and reality, as well as the information asymmetry between the parties managing the physical stores and the online sales channels.

In this research stream, we study the assortment and wholesale price decisions under the described information asymmetry in the RSC. Our results indicate that under the decentralized setting, while both the retailer and the manufacturer cannot fare better at the same time when there is a deviation between the obtained estimate of inaccuracy and its true value, each can be better off in certain conditions. Under the centralized setting, deviations are not desired and never result in higher profitability. Furthermore, when an estimation of zero inaccuracy is obtained, the RSC can avoid maximum regrets by showcasing an assortment that includes the highest utility attribute levels.

Chapter 2

Assortment Planning in Omni-Channel Retailing Under Product Returns and Showcase Capacity

Abstract. In this paper, we investigate the assortment planning decisions of a retailer that operates an online sales channel and a brick-and-mortar store simultaneously. We consider the impact of product returns explicitly, which is a norm in modern retailing but also a factor for lost profit. Assortment decisions impact product returns as the showcased products reveal information to the online shoppers who visit the physical store before their purchase. We model the purchase and keep-or-return decisions through a multinomial logit choice model and derive the expected profit function of the retailer for any given assortment selection. By using analytical and numerical results, we show that (i) the incremental value of showcase capacity is non-monotonic, (ii) an increase in returns does not necessarily mean a decrease in profit, (iii) retailers are generally better off if the hidden attribute levels are undervalued rather than correctly or over-valued, (iv) even if there is enough shelf-space capacity to showcase all products in the offline channel, it is never optimal to fully utilize the capacity under price differentiation per attribute level, (v) under no price differentiation per attribute level and if the hidden attribute levels are undervalued (overvalued), retailers benefit from showcasing some (none) of the attribute levels that are distinctively preferred to others, and (vi) under generous refund policy, retailers should reveal a limited number of undervalued levels whereas their action depends on the size of the online market for the overvalued levels. We also provide a greedy heuristic algorithm that has excellent performance.

2.1 Introduction

A common sales pitch in modern retailing is to allow product returns with no or limited hassle, even though costs related to product returns such as transportation, repackaging, and refurbishing can quickly pile up. According to the National Retail Federation of the U.S.A., returns account for \$368 billion of lost profit in 2018, about 10% of the whole value of retail profit (NRF 2018). These are the potential profit that could have been obtained, but are lost due to product returns. One of the reasons for product returns is that customers cannot fully experience (i.e. cannot accurately evaluate) the products before purchasing (Alptekinoğlu and Grasas 2014, YouGov 2021). 'Inaccurate evaluation' is an inherent part of online shopping for products with non-digital attributes — attributes that require physical evaluation (such as touching, feeling, trying out, etc.) for informed purchasing. For retailers that operate with physical and online stores alongside each other, the set of products that are showcased in the physical store is a means of providing customers with information of products and their attributes (Mantrala et al. 2009). The product information revealed at the physical store can mitigate the uncertainties that customers might have in evaluating those products (Bell et al. 2014).

In this study, we consider an omni-channel retailer that sells products with non-digital attributes. A typical product that such a retailer carries consists of several attributes such as color and size, and each attribute might have several different 'levels' such as blue and yellow for the color attribute and small, medium and large for the size attribute. Due to capacity constraints and operational costs, such retailers may not be able to showcase the full variety of their products in their physical store. However, capacity is typically not an issue for the online sales channel.

Customers can have different shopping preferences. Some prefer shopping with touching, feeling, and trying out the products, potentially from a limited selection showcased in a physical store (offline channel). Other customers may prefer the convenience of buying from home and from a wider variety of products available online. It should be noted that the shopping preference of customers between online and offline channels can vary for different types of products or retailers. For example, according to a StatCan (2022), the share of online sales in clothing retail in Canada was %23.1 in 2021; while this value for electronics and appliances was %35.9 and for Furniture was %13. These preferences translate into consumer choices that predicate on an accurate or inaccurate evaluation of the product utilities in the offline and online channels, respectively. In both shopping experiences, it might be likely that a customer's subsequent evaluation of a purchased product (for example, later at home when the customer starts using the product or receives the product) does not align with the initial evaluation while purchasing. This can cause customer dissatisfaction, which is one of the main reasons for product returns.

A common approach in modeling the consumer choice in literature is to assume that each product and shopping experience (offline or online) has a specific "utility" for a customer and that customers make purchasing decisions by maximizing the "utility" that they obtain from their purchase (Dzyabura and Jagabathula 2018, Ben-Akiva et al. 1985). The overall utility of a product is naturally a function of the utility of the levels of its individual attributes and determines the attractiveness of a product for a customer. A customer's utility will differ from product to product based on the attribute levels of the product and the customer's preferences.

In this paper, an omni-channel retailing system, including one online profit channel and one brick-and-mortar store, is investigated. In the online channel, all possible products are offered. The problem is to determine which products to showcase in the offline channel under showcase capacity (i.e, assortment decision) so that the retailer can maximize their profit by considering potential product returns due to customer dissatisfaction. For example, Warby Parker, the American giant sunglasses retailer makes this decision for its retail stores, while all the variety of its sunglasses are present on its website (Bell et al. 2018). The assortment decision in the physical store affects the purchasing decisions of the customers who visit the store, before making their purchase either directly from the store or the online channel. The focus of our study is on this segment of customers and this shopping experience, which often resembles shopping for products with high value (Park et al. 2021). Without loss of generality, customers who purchase their products online without visiting the physical store are out of the scope of this study.

In this setting, the physical store plays the role of both a profit channel and a showcase that reveals information about products and attributes to facilitate online purchases. As our major contribution to the literature of assortment planning in omni-channels, we consider the possibility of returning products as an option for customers in both profit channels. Dzyabura and Jagabathula (2018) consider a similar problem without product returns and show that when the price of products are equal, the retailer should hide a subset of attribute levels that are overvalued in customers' online evaluations and showcase attribute levels that are undervalued. We show that when product returns are allowed, this result does not necessarily hold. We also provide a framework for the case in which prices are unequal. Our results indicate that identifying a set of products and attribute levels to showcase in the physical store is complicated, and selecting an assortment only based on whether an attribute level is undervalued or overvalued will no longer hold true. This motivates our study to embed returns in the assortment decision and provide new insights.

We can summarize the implications of our results as follows:

- It is not always beneficial to showcase a greater variety of products even if there is freely available shelf-space.

- As a first step to assortment planning, retailers should understand their customers' perceptions of unavailable products through the attribute levels that are not showcased. If those hidden attribute levels are undervalued by the customers, it is likely that retailers would benefit from this situation. In such cases, they should not engage in any marketing effort to change this situation because they will make more profit compared to a case where hidden attribute levels are overvalued. For example, designing an improved web browsing experience that will lead to accurate or overvalued evaluations is not beneficial for retailers. Similarly, any marketing effort that exaggerates the utility of the hidden attribute levels is not a preferred action. On the other hand, if the hidden attribute levels are overvalued, then marketing efforts to change the perception could be valuable.

- For products that have price differentiation based on different attribute levels, retailers should never showcase the full variety of attribute levels and should prioritize showcasing more valuable attribute levels if the hidden levels are undervalued. However, retailers may choose to showcase all these attribute levels if they are overvalued. In both of these cases, the capacity should not be fully utilized if it allows to showcase all variety of products.

- For products for which the price is isolated from the attribute levels, retailers may be better off showcasing all attribute levels and utilizing all available capacity. Furthermore, when the utilities of hidden attribute levels are undervalued, retailers obtain a greater profit if one or more attribute levels have a distinctively higher utility than others, whereas, when hidden attribute levels are overvalued, retailers have a higher profit if the utilities of all attribute levels are close to each other and not distinctively different.

- Finally, we investigate the impact of problem parameters on the optimal decision and profitability of the retailers. We show that retailers should showcase a less diverse assortment if they provide a greater refund fraction and/or an easier return process when hidden attribute levels are undervalued. We also show that when hidden attribute levels are overvalued, the impact of the refund fraction or the difficulty of the return process on the variety of these levels depends on the size of online and offline segments. If OnSs are more than OfSs, then for a greater refund fraction or an easier return process, the retailers should showcase a more diverse assortment, and if OfSs are more, they should showcase a less diverse assortment.

The remainder of this paper is organized as follows. In Section 2, we review the relevant literature. In Section 3, we describe the problem environment. In Section 4, the modeling of the problem is presented with some analytical results. Section 5 is for numerical studies and sensitivity analyses. Finally, we conclude the paper in Section 6.

2.2 Literature Review

Assortment planning in a single profit channel has received a lot of attention in the literature. This problem is the same as product line optimization in the marketing literature, which is defined as a problem for a manufacturer to select a set of products to offer, to maximize profit or market share. Kohli and Krishnamurti (1987) investigate this problem for a manufacturer in a multi-attribute environment and show that the problem is NP-complete. They propose a dynamic-programming heuristic method to solve the problem. Several studies provide other heuristic solution algorithms for this problem, including Kohli and Krishnamurti (1989) with a shortest-path based algorithm, Balakrishnan and Jacob (1996) and Fruchter et al. (2006) with a genetic algorithm, and Dobson and Kalish (1988) with a two-step algorithm based on priority among products.

Similar to product line optimization in the marketing literature, several studies investigate the assortment problem in a single channel setting, in the operations management literature. See Kök et al. (2008) for a literature review of the assortment problem for a wide range of customer choice models. Ryzin and Mahajan (1999) formulate an unconstrained version of the assortment problem. Davis et al. (2013) investigate the capacitated version of the assortment problem for a retailer. The aim is to maximize the expected profit, given a set of uni-modular constraints. Also, Désir et al. (2014) study the capacitated assortment problem and provide a fully polynomial time-approximation scheme to solve the problem. Rusmevichientong et al. (2014) explore the assortment optimization problem with random choice parameters. The randomness in the choice model comes from the fact that there are multiple market segments, and each segment can have different preferences for products. The goal is to maximize the expected profit for a representative customer across all segments.

In this paper, we use multinomial logit (MNL) to model customers' choice behavior. MNL is the most frequently used customer choice model in the literature and is a utility-based model. In this model, each customer visiting the store associates a utility with each item. The associated utility consists of a deterministic and a stochastic part. The stochastic part is modeled as an error term with the Gumbel distribution. Using the properties of the Gumbel distribution, the choice probability of each product is obtained (Ben-Akiva et al. 1985, Anderson et al. 1992, Kök et al. 2008). Besides MNL, there are other choice models, including nested logit (Gallego and Topaloglu 2014, Davis et al. 2013, Feldman and Topaloglu 2015), d-level nested logit (Li et al. 2015), mixed logit (Rusmevichientong et al. 2014), and locational choice (Gaur and Honhon

2006) model.

In an MNL choice model, there are two different approaches to estimating the utility of a product. The first approach is to associate a utility to a product as a whole, based on its attractiveness. Many studies, including Ryzin and Mahajan (1999) and Rusmevichientong et al. (2010), use this approach. The second approach is a standard multi-attribute utility model that associates a utility to a product, based on its attributes, as a summation of the part-worth utilities of its attributes (see, e.g., Green and Rao 1971; Green and Srinivasan 1990; Hoch et al. 1999; and Dzyabura and Jagabathula 2018). Van Herpen and Pieters (2002) argue in favor of the attribute-based assortment planning, which we also adopt in this study, versus the product-based approach.

Assortment planning in omni-channel systems is a relatively new topic. According to Yrjölä et al. (2018), retailers that operate several sales channels may offer different product assortments on each channel and different from the optimal assortment under a single channel. Ye et al. (2018) explore the barriers to transitioning from single profit channels to omni-channels and discuss difficulties in assortment decisions. Rooderkerk and Kök (2019) provide a literature review for omni-channel assortment planning. Gallino and Moreno (2018), Bell et al. (2018), and Bell et al. (2014) investigate the operations of omni-channel systems in terms of product selection and showcasing, via empirical models or logical arguments. Park et al. (2021) investigate an omni-channel assortment problem in which in-person visiting of products and gathering information result in more confident purchases for customers. However, this study does not follow a specific choice model to capture customers' behavior. To the best of our knowledge, Dzyabura and Jagabathula (2018) are the first to propose a model to optimize the assortment decision in a brick-and-mortar store for a retailer that operates an online sales website as well, in which all products are available. Lo and Topaloglu (2022) consider a similar setting where they study omni-channel assortment with a features tree that represents product features. These studies do not allow for product returns.

Assortment decisions and return policies are usually modeled separately in the literature, as the former are strategic decisions, and the latter are operational ones (Stock et al. 2006, Olavson and Fry 2006). Return policies are a fundamental part of operational costs in retailing, and their inclusion complicates the assortment decision even more (Kök et al. 2008, Ramdas 2003). Nageswaran et al. (2020) investigate the return policies in an omni-channel setting, jointly with price decisions, and explore when full refund policies are better and when a return fee should be charged. Alptekinoğlu and Grasas (2014) are the first to address the assortment decision when returns are included. They investigate the assortment decision for a retailer with a single profit channel, in which returns are included. This study uses MNL and nested logit to model customers' purchase and keep-or-return decisions. There are two main differences between this study and our work. First, we consider an omni-channel system, rather than a single channel. Second, we use an attribute-based utility approach rather than a product-based approach.

2.3 Problem Environment

Consider an omni-channel retailer that operates with one physical store (offline channel) — or multiple stores with identical showcase capacity — and an internet-based sales store (online channel). The retailer sells products with one or more non-digital attributes, such as sneakers with different styles; purses with different sizes, shapes, and materials; or dresses with variety of styles and colors. The total number of combinations of the levels of attributes of such products may yield an abundant number of different items; it would not be feasible to showcase all of them in the physical store, but it could be feasible to make all of them available in the online channel.

Since omni-channel retailing has been a common practice for many years, we assume that customers' shopping preferences have matured and converged to the following three categories. The first group of customers only prefer to purchase from the online channel without visiting the physical store. We exclude them from our analysis without loss of generality because their choices would not affect the assortment decision. An α fraction of the remaining customers are the Offline Shoppers (OfSs) who prefer only buying from the physical store from an available selection of items, where they can physically (and accurately) evaluate the products before purchase. Such customers may or may not visit the online channel but they end up buying from the physical store with full information of the attribute levels due to physical touch. The remaining $1-\alpha$ fraction of customers are Online Shoppers (OnSs) who visit the offline channel to garner information about the products but prefer to shop from the online channel eventually, with the convenience of shopping from home and from a wider variety of products, potentially without physically (and accurately) evaluating all the attributes of the product that they purchase (Dzyabura and Jagabathula 2018). It is also possible that any given type of customer ends up not purchasing a product.

Independent of which channel is used, a customer's purchase decision is based on the utility received from the available products. The customer will purchase a product with the maximum positive utility. If the highest utility is non-positive, then no purchase will take place. An OfS's purchase predicates on an "accurate" utility evaluation (because the customer has the opportunity of physically touching, feeling, and trying out the product before making the purchase decision), whereas OnSs make purchase decisions based on potentially "inaccurate" utility evaluations. OnSs visit the physical store and collect accurate utility information from the attribute levels of available products. Then, they visit the online channel to browse other products. The evaluation of the products whose attribute levels were not available at the physical store is likely to be inaccurate. This inaccuracy may cause OnSs to 'overvalue' or 'undervalue' the utility of the products they browse on a computer screen. If the utility of a product is overvalued (undervalued), it will be more (less) likely to be purchased.

Note that both types of customers would have the option of returning their products. Customers decide whether to keep or return their purchased products based on comparing the utility of keeping a product and the utility of returning it. An OfS's purchase decision is based on the accurate utility evaluation at the time of purchase. An OnS's purchase decision may be based on an inaccurate evaluation at the time of purchase, but they will have the opportunity to accurately evaluate the product when the product is delivered. Given this, if a product was overvalued in the online channel at the time of purchase, it would have been more likely to be purchased, but it is also more likely to be returned after purchase because the product would not be as desired when received. We call this the 'disappointment' factor in utility evaluation. On the other hand, if a product was undervalued, it would have been less likely to be purchased, but it is also less likely to be returned because the product would be more desired when received, which we call this the 'gratification' factor. Therefore, the utility obtained from keeping a product for OnSs includes the utility of the product and also the disappointment/gratification factor; while, for OfSs, it will be just the utility of the product. The utility that a customer obtains from returning a product includes several factors such as the disutility of the return process (e.g., due to making an extra trip to the store or a post office) and a potential nonrefundable part of the product price upon return. We assume that the returned products can be re-sold at the same price in the original channel that they were purchased from, but this can easily be relaxed in our model.

Determining the subset of products to showcase in the physical store (i.e., the assortment decision) to maximize the total profit constitutes the foundation of the problem that we analyze in this paper. Our main contribution is to introduce product returns in this setting and to explicitly examine the impact of inaccuracy in customers' online evaluations on profitability. We note that the assortment decision impacts customers' purchases and keep-or-return decisions in both sales channels. The set of products showcased in the physical store determines which attribute levels (and as a result, products) are evaluated accurately in the online channel and which ones are potentially evaluated inaccurately. The retailer's profit in this system is a function of what customers have purchased and what they have returned.

2.4 Modelling Approach

In this section, we propose our main model. All notation is introduced whenever used, but a summary is also available in Table 2.1.

Table 2.1: Summary of Notations

Notation	Definition
C	Capacity of the physical store
$d_{k,l(k)}$	Inaccuracy in evaluating level $l(k)$ of attribute k online
$D_{x\|M}$	Inaccuracy in evaluating product x online, given M
k	Index for attributes
K	Total number of attributes
$l(k)$	Level index in attribute k
$L(k)$	Set of all possible levels for attribute k
M	Decision variable; assortment in the physical store
$P^f_{x\|M}$	Purchasing probability for product x in offline channel, given M
$P^n_{x\|M}$	Purchasing probability for product x in online channel, given M
r	Disutility in returning a product ($r < 0$)
R^f_x	Returning probability for product x purchased offline
$R^n_{x\|M}$	Probability for product x returned online, given M
$S(k\|M)$	Set of levels of attribute k showcased given M
$\widetilde{u}_{k,l(k)}$	Part-worth utility of level $l(k)$ of attribute k
U_x	Utility of product x physically evaluated
$U^n_{x\|M}$	Utility of product x evaluated online, given M
x_k	Level of attribute k in product x
X	Universal set of all products
α	Proportion of offline shoppers ($0 \leq \alpha \leq 1$)
β_1	Price sensitivity of utility ($\beta_1 \leq 0$)
β_2	Refund sensitivity of utility ($\beta_2 \leq 0$)
β_3	Combined price and refund sensitivities of utility ($\beta_3 \leq 0$)
γ	Fraction of money refunded upon return ($0 \leq \gamma \leq 1$)
ε_x	Error term in the utility evaluation of product x
ϕ	Disutility of return sensitivity in utility at the time of purchase ($\phi \geq 0$)
μ	Homoscedasticty of the population under study in purchase decision
μ'	Homoscedasticty of the population under study in keep-or-return decision
ω	Disappointment/gratification sensitivity of inaccuracy in utility ($\omega \geq 0$)
$\pi_{k,l(k)}$	Part-worth price of level $l(k)$ of attribute k
Π	Retailer's profit function
Π_f	Physical store's contribution to retailer's profit
Π_n	Online channel's contribution to retailer's profit

2.4.1 Utility Model

Consider a product type with K non-digital attributes, and let $L(k)$ be the set of levels of attribute $k \in \{1, 2, ..., K\}$. For example, if a product is differentiated by the Color and Size attributes, then $K = 2$. If $k = 1$ corresponds to the Color attribute, then $L(1)$ can be the set {Blue, Yellow}. All combinations of all levels of all attributes constitute the universal set of product, X, which is the set of all varieties of the product type that the retailer can sell.

Following the utility model of Dzyabura and Jagabathula (2018), we define the utility of a product as the summation of the part-worth utilities of its attribute levels. Let $\widetilde{u}_{k,l(k)} \geq 0$ be the part-worth utility of level $l(k) \in L(k)$ of attribute k, which is the base utility of that attribute level for a customer who physically evaluates the product by touching, trying out, etc. Similarly, each attribute level also contributes to the price of the product through $\pi_{k,l(k)}$, which is the part-worth price of level $l(k) \in L(k)$. Moreover, the refund fraction of the price upon a product return (indicated by γ) and also the difficulty of the return process (indicated by r) affect the utility of a product. A greater refund fraction results in a higher utility of a product. Similarly, if the return process is easier and not very challenging, the utility will be higher as well. Given this setting, the utility conceived for a product x when physically evaluated by a customer at the time of purchase is

$$U_x = \widetilde{U}_x + \beta_1 \pi_x + \beta_2 (1-\gamma)\pi_x + \phi r + \varepsilon_x$$

where

$$\widetilde{U}_x = \sum_{k=1}^{K} \sum_{l(k)\in L(k)} \widetilde{u}_{k,l(k)} \cdot 1_{\{x_k = l(k)\}}, \text{ and} \tag{2.1}$$

$$\pi_x = \sum_{k=1}^{K} \sum_{l(k)\in L(k)} \pi_{k,l(k)} \cdot 1_{\{x_k = l(k)\}},$$

x_k is the level of attribute k in product x, $1_{\{x_k = l(k)\}}$ is an indicator function that is equal to 1 if $x_k = l(k)$ and 0 otherwise, $\beta_1 (\leq 0)$ is the price sensitivity of utility, $\beta_2 (\leq 0)$ is the refund sensitivity of utility, $r(\leq 0)$ is the disutility of the return process, $\phi(\geq 0)$ is the disutility of return sensitivity in the utility of product x at the time of purchase, and ε_x is the error term accounting for unobserved components in composing the utility of a product. The error term indicates that although the average utility of a product is constant, the realization for each customer might be different. Re-writing U_x, we get

$$U_x = \widetilde{U}_x + (\beta_1 + \beta_2(1-\gamma))\pi_x + \phi r + \varepsilon_x. \tag{2.2}$$

Defining $\beta_3 = \beta_1 + \beta_2(1-\gamma)$ and $\overline{U}_x = \widetilde{U}_x + \beta_3 \pi_x + \phi r$, we can write (2.2) as

$$U_x = \overline{U}_x + \varepsilon_x. \tag{2.3}$$

As explained in Section 3, evaluations of a product attribute in the online channel (via a computer screen) and the physical store (in-person) can differ. Let $d_{k,l(k)}$ denote this difference (i.e., the magnitude of inaccuracy in evaluation) for level $l(k)$ of attribute k. Then, the part-worth utility of an attribute level that is only available online can be represented by $\widetilde{u}^n_{k,l(k)} = \widetilde{u}_{k,l(k)} + d_{k,l(k)}$. Note that $d_{k,l(k)}$ can be positive, zero, or negative. Positive values correspond to 'overvaluation' of the attribute level by the customer, and negative values correspond to 'undervaluation'. If the customer overvalues a level of an attribute that is not available in the physical store, products that include that level will be more likely to be purchased, because

$\widetilde{u}^n_{k,l(k)} > \widetilde{u}_{k,l(k)}$ and vice versa.

Let $M \subseteq X$ be the subset of products that are showcased in the physical store. Suppose that the retailer can showcase at most C items, which is the showcase capacity of the physical store. Hence, we should have $|M| \leq C$. Let $S(k|M)$ be the set of all levels of attribute $k \in K$ that are present in at least one of the products in M; i.e., $S(k|M) = \bigcup_{x\in M}\{x_k\}$. A product itself might not be offered in the physical store, but some of its attribute levels can be present there, which means that not all of this product's attribute levels are inaccurately evaluated. Therefore, the utility of a product evaluated online consists of the sum of the part-worth utility of its attribute levels that are in set $S(k|M)$ (showcased in the physical store) and of those that are not in set $S(k|M)$ (not showcased). Consequently, for any such product y, the utility can be written as

$$\begin{aligned} U^n_{y|M} \quad = \quad & \sum_{k=1}^{K} \Bigg[\sum_{l(k)\in S(k|M)} (\widetilde{u}_{k,l(k)} + \beta_3 \pi_{k,l(k)}) \cdot 1_{\{y_k = l(k)\}} \\ & + \sum_{l(k)\notin S(k|M)} (\widetilde{u}_{k,l(k)} + d_{k,l(k)} + \beta_3 \pi_{k,l(k)}) \cdot 1_{\{y_k = l(k)\}} \Bigg] + \phi r + \varepsilon_y. \end{aligned} \tag{2.4}$$

Let the total inaccuracy accrued when evaluating the product y online be defined as $D_{y|M} = \sum_{k=1}^{K}\sum_{l(k)\notin S(k|M)} d_{k,l(k)} \cdot 1_{\{y_k=l(k)\}}$. Then, we can rewrite (2.4) as $U^n_{y|M} = \widetilde{U}_y + D_{y|M} + \beta_3\pi_y + \phi r + \varepsilon_y$. Considering that $\overline{U}_y = \widetilde{U}_y + \beta_3\pi_y + \phi r$, we have

$$U^n_{y|M} = \overline{U}_y + D_{y|M} + \varepsilon_y \tag{2.5}$$

We assume that the idiosyncratic error terms ε_x in (2.3) and ε_y in (2.5) follow the standard logit assumption and are independent and identically Gumbel distributed with mean zero and $1/\mu$ scale parameter (variance $\mu^2\pi^2/6$) (Dzyabura and Jagabathula 2018, Anderson et al. 1992, Ben-Akiva et al. 1985), where $\mu > 0$ is a positive scalar, denoting the homoscedasticity in the population of study. The purchasing experience of customers in either of the profit channels includes two phases. In the first phase, customers decide whether to purchase a product or not based on their evaluations of its utility. In the second phase, customers can decide to return or keep their purchased product after they receive and accurately asses it if purchased online, and after experiencing it for a while if purchased offline. The model of this experience consists of two multinomial logit processes, each corresponding to one of the phases. The decision variable for the retailer is to select a set of products M to showcase in the physical store from the universal products set X so that its total profit including sales and returns from both channels is maximized.

2.4.2 Phase 1: Purchase Decisions

A customer in either of the channels purchases a product with the highest utility among the offered products, if that utility is positive. Using properties of the multinomial logit (MNL) model in making purchase decisions, the probabilities that a product x is preferred over any product y based on its utility in the physical store and the online channel are

$$\begin{array}{lll} Pr\left\{U_x > U_y, U_x > 0\right\} & \forall x, y \in M, x \neq y & \text{(a)} \\ Pr\left\{U^n_{x|M} > U^n_{y|M}, U^n_{x|M} > 0\right\} & \forall x, y \in X, x \neq y & \text{(b)} \end{array} \qquad (2.6)$$

respectively. Substituting (2.3) and (2.5) in (2.6a) and (2.6b) respectively, we have

$$\begin{array}{rll} Pr\left\{\overline{U}_x + \varepsilon_x > \overline{U}_y + \varepsilon_y, \overline{U}_x + \varepsilon_x > 0\right\} & \forall x, y \in M, x \neq y & \text{(a)} \\ Pr\left\{\overline{U}_x + D_{x|M} + \varepsilon_x > \overline{U}_y + D_{y|M} + \varepsilon_y, \overline{U}_x + D_{x|M} + \varepsilon_x > 0\right\} & \forall x, y \in X, x \neq y & \text{(b)} \end{array}. \qquad (2.7)$$

In (2.7), ε_x and ε_y are the error terms defined in (2.1) for products x and y at the time of purchase. We can rewrite (2.7) as

$$\begin{array}{rll} Pr\left\{\varepsilon_y - \varepsilon_x < \overline{U}_x - \overline{U}_y, \overline{U}_x + \varepsilon_x > 0\right\} & \forall x, y \in M, x \neq y & \text{(a)} \\ Pr\left\{\varepsilon_y - \varepsilon_x < \overline{U}_x + D_{x|M} - \overline{U}_y - D_{y|M}, \overline{U}_x + D_{x|M} + \varepsilon_x > 0\right\} & \forall x, y \in X, x \neq y & \text{(b)} \end{array}. \qquad (2.8)$$

In (2.8), we know that $\varepsilon_y - \varepsilon_x$ has a Logistic distribution (Ben-Akiva et al. 1985). Let $P_{x|M}$ and $P^n_{x|M}$ be the probability that an OfS purchases a product $x \in M$ and an OnS purchases a product $x \in X$, respectively, given that set M is showcased in the physical store. Therefore, by applying the properties of the cumulative distribution function of Logistic distribution and using the expressions in (2.6), we have the following:

$$\begin{array}{rclll} P_{x|M} & = & \frac{e^{\overline{U}_x/\mu}}{1+\sum_{y\in M} e^{\overline{U}_y/\mu}} & \forall x \in M \cup \{0\} & \text{(a)} \\ P^n_{x|M} & = & \frac{e^{(\overline{U}_x + D_{x|M})/\mu}}{1+\sum_{y\in X} e^{(\overline{U}_y + D_{y|M})/\mu}} & \forall x \in X \cup \{0\} & \text{(b)} \end{array}. \qquad (2.9)$$

In (2.9a) and (2.9b), the fictitious product indicated by "0" corresponds to the utility of no purchase decision. The union of each products set and 0 indicates that if the highest utility is not positive, no purchase takes place (Dzyabura and Jagabathula 2018, Blanchet et al. 2016).

2.4.3 Phase 2: Keep-or-Return Decisions

Customers in either of the channels make purchase decisions based on the realization of product utilities at the time of purchase. Next, customers decide whether to keep or return their purchased products. By considering the disutility of return r and the non-refundable portion of the product price, according to Alptekinoğlu and Grasas (2014), an OfS will return their purchased product x if

$$\widetilde{U}_x + \beta_1 \pi_x + \varepsilon_x + \varepsilon_{\text{keep}|x} < r + \beta_1 (1-\gamma)\pi_x + \varepsilon_x + \varepsilon_{\text{return}|x}. \tag{2.10}$$

In (2.10), the left-hand side is the utility that an OfS will obtain by keeping their purchased product x. After the customer decides to keep the product, the refund sensitivity and disability of return sensitivity will be eliminated from the utility and the only components left will be the part-worth utility of attribute levels in the product and the price sensitivity in utility. The right-hand side is the utility in case the customer returns product x, which includes the disutility of return and the money that will be lost due to the non-refundable portion of the price. Similar to the purchase decision, we assume that $\varepsilon_{\text{keep}|x}$ and $\varepsilon_{\text{return}|x}$ are to address the uncertainty in return-or-keep decisions and are i.i.d. Gumbel distribution with zero mean and scale parameter of μ'.

When OnSs receive their purchased product, they get the opportunity to observe the accurate utility of the product, and the inaccuracy in their evaluation at the time of purchase, $D_{x|M}$, will be resolved. If the product was overvalued at the time of purchase ($D_{x|M} > 0$), there will be a disappointment factor which would amplify the likelihood of return but if the product was originally undervalued ($D_{x|M} < 0$), there will be a gratification effect which will amplify the likelihood of keep. By considering these effects, an OnS's return condition for a purchased product x can be expressed as

$$\widetilde{U}_x + \beta_1 \pi_x - \omega D_{x|M} + \varepsilon_x + \varepsilon_{\text{keep}|x} < r + \beta_1 (1-\gamma)\pi + \varepsilon_x + \varepsilon_{\text{return}|x}. \tag{2.11}$$

In (2.11), ω (≥ 0) reflects the disappointment/gratification sensitivity originating from the revealed inaccuracy in the utility of the product. Given this, if a product has been purchased with overvaluation ($D_{x|M} > 0$), the utility of keeping that product will be smaller because the customer will be disappointed to receive a product that is not as desired as they expected; however, if the product was purchased with undervaluation ($D_{x|M} < 0$), the utility of keeping it will be greater because the customer will be gratified to receive a product better than expected. In the literature, Shulman et al. (2009) model the product returns as comparing two utility values (expected and actual) with no uncertainty, in which disutility of return and the non-refundable portion of the price are also considered. Moreover, Alptekinoğlu and Grasas (2014) consider the same factors in return decision and use a nested logit model to tackle product returns. They include uncertainty in the purchase decision and in the keep-or-return decision. Our keep-or-return decision model is along the lines of this literature.

Let R_x^f be the probability of product $x \in M$ being returned, given that it is purchased from the physical store. Then, considering μ' as the homoscedasticity in customers' keep-or-return decision, by using the properties of Gumbel distribution and (2.10), we can write

$$\begin{aligned} R_x^f &= Pr\{\widetilde{U}_x + \beta_1\pi_x + \varepsilon_x + \varepsilon_{\text{keep}|x} < r + \beta_1(1-\gamma)\pi_x + \varepsilon_x + \varepsilon_{\text{return}|x}\} \\ &= Pr\{\varepsilon_{\text{keep}|x} - \varepsilon_{\text{return}|x} < -\widetilde{U}_x + r - \beta_1\gamma\pi_x\} \\ &= \frac{1}{1 + e^{(\widetilde{U}_x - r + \beta_1\gamma\pi_x)/\mu'}}. \end{aligned} \tag{2.12}$$

Similarly, let $R_{x|M}^n$ be the probability of product $x \in X$ being returned, given that it is purchased from the online channel. Then

$$\begin{aligned} R_{x|M}^n &= Pr\{\widetilde{U}_x + \beta_1\pi_x - \omega D_{x|M} + \varepsilon_x + \varepsilon_{\text{keep}|x} < r + \beta_1(1-\gamma)\pi_x + \varepsilon_x + \varepsilon_{\text{return}|x}\} \\ &= Pr\{\varepsilon_{\text{keep}|x} - \varepsilon_{\text{return}|x} < -\widetilde{U}_x + \omega D_{x|M} + r - \beta_1\gamma\pi_x\} \\ &= \frac{1}{1 + e^{(\widetilde{U}_x - \omega D_{x|M} - r + \beta_1\gamma\pi_x)/\mu'}}. \end{aligned} \tag{2.13}$$

Moreover, let K_x^f ($K_{x|M}^n$) be the probability of product $x \in M$ ($x \in X$ given M) being kept, given that it is purchased from the physical store (online channel). Then, we have

$$\begin{aligned} K_x^f = 1 - R_x^f \quad &= \frac{1}{1 + e^{(-\widetilde{U}_x + r - \beta_1\gamma\pi_x)/\mu'}}, \\ K_{x|M}^n = 1 - R_{x|M}^n &= \frac{1}{1 + e^{(-\widetilde{U}_x + \omega D_{x|M} + r - \beta_1\gamma\pi_x)/\mu'}}. \end{aligned} \tag{2.14}$$

Before delving into the retailer's profit function and its analysis, we present a proposition that shows the impact of product utilities and prices on customers' return decisions.

Proposition 2.1. *Suppose that there are two products x and y such that $\widetilde{U}_x \geq \widetilde{U}_y$ and $\pi_x \geq \pi_y$. Also, suppose that $\gamma \neq 0$ and either $x, y \in M$ or $D_{x|M} = D_{y|M}$. Then*

(i) if $\widetilde{U}_x = \widetilde{U}_y$ and $\pi_x = \pi_y$, x and y are equally likely to be returned,

(ii) if $\widetilde{U}_x = \widetilde{U}_y$ and $\pi_x > \pi_y$, product x is more likely to be returned,.

(iii) if $\widetilde{U}_x > \widetilde{U}_y$ and $\pi_x = \pi_y$, product y is more likely to be returned,

(iv) if $\widetilde{U}_x > \widetilde{U}_y$ *and* $\pi_x > \pi_y$*, product* x *is more likely to be returned if* $\gamma > \frac{\widetilde{U}_x - \widetilde{U}_y}{\beta_1(\pi_y - \pi_x)}$*, and vice versa.*

Proposition 2.1 states that when two products have equal utilities, the more expensive product is more likely to be returned. This is because if customers return either of these two items, they forego an equal utility by not keeping the item, but they get a greater refund. When the utility of a product is greater than the other and prices are equal, customers are more likely to return the product with smaller utility, because they forego a smaller utility. The last criterion states that when the product with greater utility is also more expensive, then for relatively high values of γ, the greater utility product is more likely to be returned, and vice versa. Specifically, if the price of the greater utility product is considerably higher than the price of the other product, it results in a greater return probability for that product.

2.4.4 Retailer's Objective Function

The decision-maker (retailer) intends to find an optimal assortment plan within the showcase capacity limits (i.e., $|M| \leq C$) such that its profit across both channels is maximized. The profit is composed of the revenue obtained through selling the products and the loss due to returned sales. Therefore, considering that α fraction of customers are OfSs, and the remaining $1-\alpha$ fraction are OnSs, the retailer's profit function will be

$$\begin{aligned}\Pi(M) \quad = \quad & \alpha\left(\sum_{x\in M}\pi_x \cdot Pr\{x \text{ purchased} \cap x \text{ kept}\} + \sum_{x\in M}(1-\gamma)\pi_x \cdot Pr\{x \text{ purchased} \cap x \text{ returned}\}\right) \\ & +(1-\alpha)\left(\sum_{x\in X}\pi_x \cdot Pr\{x \text{ purchased} \cap x \text{ kept}\}\right. \\ & \left. + \sum_{x\in X}(1-\gamma)\pi_x \cdot Pr\{x \text{ purchased} \cap x \text{ returned}\}\right) \qquad (2.15)\end{aligned}$$

By using the conditional probability formula, we have

$$\begin{aligned}Pr\{x \text{ purchased} \cap x \text{ returned}\} &= Pr\{x \text{ returned} \mid x \text{ purchased}\} \cdot Pr\{x \text{ purchased}\} \\ &= \begin{cases} P^f_{x|M} \cdot R^f_x & \forall x \in M \\ P^n_{x|M} \cdot R^n_{x|M} & \forall x \in X \end{cases}, \qquad (2.16)\end{aligned}$$

$$\begin{aligned}Pr\{x \text{ purchased} \cap x \text{ kept}\} &= Pr\{x \text{ kept} \mid x \text{ purchased}\} \cdot Pr\{x \text{ purchased}\} \\ &= \begin{cases} P^f_{x|M} \cdot K^f_x & \forall x \in M \\ P^n_{x|M} \cdot K^n_{x|M} & \forall x \in X \end{cases}. \qquad (2.17)\end{aligned}$$

Substituting (2.16) and (2.17) into the profit function in (2.15), we get

$$\Pi(M) = \alpha \left(\sum_{x \in M} \pi_x P^f_{x|M} \cdot K^f_x + \sum_{x \in M} (1-\gamma) \pi_x P^f_{x|M} \cdot R^f_x \right) + (1-\alpha) \left(\sum_{x \in X} \pi_x P^n_{x|M} \cdot K^n_{x|M} + \sum_{x \in X} (1-\gamma) \pi_x P^n_{x|M} \cdot R^n_{x|M} \right).$$

Then, the retailer aims to solve the following Omnichannel Assortment Problem with Returns (OCAPwR):

$$\begin{array}{lll} \text{OCAPwR:} & \text{Max}_{M \subseteq X} & \Pi(M) \\ & \text{s.to} & |M| \leq C. \end{array}$$

Unlike the approach of Dzyabura and Jagabathula (2018), OCAPwR cannot be solved in the product attributes space, due to the capacity constraint in our problem. Dzyabura and Jagabathula (2018) propose the problem to be solved by selecting a set of attribute levels and then showcasing items generated by the Cartesian product of the selected attribute levels. However, given the capacity constraint, showcasing all the items in the Cartesian product of the selected attribute levels may not be feasible. Therefore, we propose to solve OCAPwR with complete enumeration in the product space. This is a combinatorial problem that is NP-hard (Dzyabura and Jagabathula 2018). In Section 2.4.7, we propose a greedy heuristic algorithm that is shown to perform excellently.

2.4.5 Should Available Capacity Always be Utilized?

The capacity constraint imposed by C restricts the number of items showcased and impacts the profit obtained. Extra capacity can provide a retailer with more freedom when it comes to making an assortment decision, but, typically, extra capacity comes with a cost. An extra capacity is utilized, if the marginal profit obtained by utilizing it is more valuable than the cost of it. In OCAPwR, we do not introduce the cost of capacity but solve the constrained problem for a given value of C. One may expect that $|M| = C$ in the optimal solution to OCAPwR; however, we show that this is not necessarily true in the following Proposition.

Proposition 2.2. *For an arbitrary C, let M_C be the optimal assortment to showcase. Suppose that an extra capacity is provided in the showroom such that the shelf-space becomes $C+1$. For a specific M_{C+1}, let $\Pi(M_{C+1}) > \Pi(M'), \forall M' \in X$ such that $|M'| = |M_{C+1}| = C+1$. Then, the extra shelf-space capacity* *provided in the showcase should be unutilized only if*

$$\frac{\alpha}{1-\alpha} < \frac{\left[\sum_{x\in X}\pi_x P^n_{x|M_{C+1}}(1-\gamma R^n_{x|M_{C+1}}) - \sum_{x\in X}\pi_x P^n_{y|M_C}(1-\gamma R^n_{x|M_C})\right]}{\left[\sum_{x\in M_C}\pi_x P^f_{x|M_C}(1-\gamma R^f_x) - \sum_{x\in M_{C+1}}\pi_x P^f_{x|M_{C+1}}(1-\gamma R^f_x)\right]}. \tag{2.18}$$

The fraction on the right-hand side of (2.18) can be positive or negative. If it is negative, then the condition can not hold and the showcase capacity should be fully utilized. However, if it is positive, then the condition may hold for certain parameter values and additional capacity should not be utilized to showcase an extra product even if this extra capacity is costless. Note that as α (proportion of OfSs) decreases, it is more likely for the condition to hold. This is because showcasing more products to OfSs becomes less critical for higher profitability. We also numerically investigate this relationship in Section 2.5.2.

2.4.6 Overvaluation Vs. Undervaluation

Customers may overvalue or undervalue the utility of products for which they do not physically evaluate the attribute levels. In this section, we analyze the conditions under which the retailer prefers customers whether they overvalue or undervalue the utility of products that are not showcased. The retailers can use this information for their benefit through design choices at their online sales channels.

For this analysis, we first characterize the profit function Π in terms of the inaccuracy present for a special case where $|M| = 1$. Suppose that there are two products x and y. Let $d = D_{y|\{x\}}$. We first rewrite the profit function Π as a function of d while introducing the same argument to the purchasing, returning, and keeping probabilities (cf. (2.9b), (2.13), and (2.14)) as follows:

$$\begin{aligned}\overline{\Pi}(d) = {} & \alpha\pi_x P^f_{x|\{x\}}\left(K^f_x + (1-\gamma)R^f_x\right) + (1-\alpha)\pi_x P^n_{x|\{x\}}(d)\left(K^n_x + (1-\gamma)R^n_x\right) \\ & + (1-\alpha)\pi_y P^n_{y|\{x\}}(d)\left(K^n_{y|\{x\}}(d) + (1-\gamma)R^n_{y|\{x\}}(d)\right)\end{aligned}. \tag{2.19}$$

Because $M = \{x\}$, we have $K^n_{x|\{x\}} = K^f_{x|\{x\}}$ and $R^n_{x|\{x\}} = R^f_{x|\{x\}}$. Therefore, we can simplify (2.19) as

$$\overline{\Pi}(d) = \pi_x(1-\gamma R^f_x)\left(\alpha P^f_{x|\{x\}} + (1-\alpha)P^n_{x|\{x\}}(d)\right) + (1-\alpha)\pi_y\left(1-\gamma R^n_{y|\{x\}}(d)\right)P^n_{y|\{x\}}(d).$$

The following theorem proves that $\overline{\Pi}(d)$ is a unimodal function and characterizes the ranges of d that maximize this function.

Theorem 2.1. *Let $X = \{x, y\}$ and $M = \{x\}$. Suppose that $\pi_x = \pi_y$ and, for simplicity, $\mu = \mu' = 1$. Then, $\overline{\Pi}(d)$ is unimodal. Moreover, let d^* be such that $\overline{\Pi}(d^*) \geq \overline{\Pi}(d)\ \forall d : -\infty < d < \infty$. Then,*

$$
\begin{array}{rcll}
 & d^* & \to \infty & \text{if} \quad \overline{U}_x \leq A1 \\
0 \leq & d^* & < \infty & \text{if} \quad A1 < \overline{U}_x \leq A2 \\
-\infty < & d^* & < 0 & \text{if} \quad \max\{A1, A2\} \leq \overline{U}_x
\end{array}
$$

where $A1 = ln\left(\frac{(1+a_1 z)(1-\gamma+\gamma e^{\overline{U}_y}/a_2 z)}{\gamma}\right)$, $A2 = ln\left((\frac{1+a_1 z}{\gamma})\frac{(1-\gamma)(2a_2 z+a_2^2 z^2)-z a_2 \gamma e^{\overline{U}_y}+1}{2a_2 z+a_2^2 z^2-a_1 z}\right)$, $a_1 = e^{-\widetilde{U}_x}$, $a_2 = e^{-\widetilde{U}_y}$, *and* $z = e^{(r-\beta_1 \gamma \pi_x)}$.

This theorem shows that the retailer can benefit from inaccurate assessment of the hidden attribute levels by their customers, for a small problem instance with two products and one showcase capacity. If the utility of the showcased product ($\overline{U}_x$) is relatively small, the retailer would prefer that customers highly overvalue the utility of the other product y in the online channel, which falls into the first range. Since $\overline{U}_x$ is small, overvaluation of $\overline{U}_y$ results in more sales in the online channel. In this case, even if product y is returned, the retailer can benefit from the non-refundable part of the price. When the utility of the showcased is relatively higher, then the retailer would prefer customers to overvalue the other product y to some extent, which means that the retailer can benefit from selling both products. However, when the utility of the showcased product x is even higher so that it falls into the third region, the retailer would prefer customers to undervalue the utility of the other product y. In this situation, product x is already attractive to customers in both channels, and the retailer earns profit from it. Selling product y can bring profit to the retailer as well; but, the retailer would prefer undervaluation, so that if product y is purchased, it will be more likely to be kept. Our numerical analysis reveals that the main result of this theorem also holds for general problem instances so that retailers may prefer overvaluation or undervaluation of the hidden attributes rather than them being accurately assessed.

It should be noted that the retailer cannot determine the value of d, because it is an exogenous variable inherent to customers' perceptions. Nevertheless, the retailer may be able to influence the inaccuracy in customers' evaluations via some marketing tactics. For example, online shoppers may perceive products differently on different choice of pictures or website designs. In this way, the retailer may be able to lead customers towards overvaluation or undervaluation, whichever direction is desirable for them. We further elaborate on this aspect in the Conclusion section.

2.4.7 Greedy Heuristic Algorithm

Suppose that an optimal assortment has been found for a given showcase capacity in a problem instance but there is an opportunity to increase the showcase capacity by one. If the problem is resolved from scratch in this situation, the best decision may contain different products compared to the assortment that is already showcased. This is because, with more capacity, the retailer can choose to provide more accurate information by showcasing a higher variety and combination of attribute levels. This can result in a different combination of products showcased. In practical applications, retailers may be interested in exploiting the extra capacity by adding a product, without changing the already showcased assortment, even though the resulting set may not be optimal. In this section, we exploit this 'greedy' policy, albeit suboptimal, which may be easy to implement in practice as well as easy to compute. Proposition 2.3 provides a framework for this heuristic policy.

Proposition 2.3. *Suppose that an arbitrary set $M \subset X$ is selected for showcasing, and $\pi_i = \pi_j, \forall i, j \in X$. Also suppose that $D_{i|M} = D_{j|M}, \forall i, j \in X \setminus M$. Consider that $M^{'} = M + \{x\}$ and $M^{''} = M + \{y\}, \forall x, y \in X \setminus M$. In selecting one more product between x and y to add to set M, product x is selected over y if the following condition holds:*

$$\frac{1-\gamma R_x^f}{1-\gamma R_{x|M''}^n}\left(\frac{\alpha}{1-\alpha}(\sum_{i\in M'} P_{i|M'}^f - \sum_{j\in M''} P_{j|M''}^f) + (\sum_{i\in M'} P_{i|M'}^n - \sum_{j\in M''} P_{j|M''}^n)\right) \geq \left(P_{x|M''}^n - P_{y|M'}^n\right).$$

Given that OCAPwR is NP-hard and cannot be solved to find the optimal solution in polynomial time, we propose a greedy heuristic algorithm along the lines of Proposition 2.3, that iterates over all C values and adds the most profitable product to the assortment, if any. The proposed algorithm starts from $C = 1$ and provides the best marginal solution, as the showcase capacity is increased one by one until $C = |X|$. At $C = 1$, the algorithm calculates the expected profit obtained by showcasing each product in set X and selects the product that generates the highest profit to showcase. This solution is kept as the optimal solution for $C = 1$ and is kept in the assortment in all of the subsequent iterations. Then, the algorithm sets $C = 2$ and seeks another product from the remaining ones, to maximize the profit of showcases with the already selected product, when $C = 1$. The algorithm proceeds with this greedy approach until all showcase capacity values are iterated. Note that the algorithm does not select an item if adding an item to the assortment does not increase the overall profit at that iteration (cf. Proposition 2.2). In this situation, for the next iteration, the algorithm considers the best item that it could choose if it were to fully utilize the capacity. As defined above, M_C is the optimal assortment when capacity is C, and let M'_C is the optimal assortment when capacity is C but under the condition that the showcase capacity must be fully utilized. Then, Algorithm 1 presents the proposed greedy heuristic.

Similar greedy policies have been widely used in assortment planning literature, e.g., see Nemhauser and Wolsey (1978), Talluri and Van Ryzin (2004), and Désir et al. (2014). The performance of this algorithm

Algorithm 1 Greedy heuristic algorithm for selecting showcase set M

$C \leftarrow 0$ ▷ Initialize variables
$M_C \leftarrow \varnothing$
$M'_C \leftarrow \varnothing$
while $C < |X|$ **do** ▷ Iterate over all possible values of C
    $C \leftarrow C + 1$
    for all $x \in X \setminus M'_{C-1}$ **do** ▷ Calculate profit for adding each product to M'_{C-1}
        $\Pi_x \leftarrow \Pi(M'_{C-1} \cup \{x\})$
    end for
    $M'_C \leftarrow M'_{C-1} \cup \{\arg\max_{x \in X \setminus M'_{C-1}} \Pi_x\}$ ▷ Find addition with maximum profit
    if $\Pi(M'_C) > \Pi(M_{C-1})$ **then** ▷ Check if utilizing extra capacity is optimal
        $M_C \leftarrow M'_C$
    else
        $M_C \leftarrow M_{C-1}$
    end if
end while

is investigated in Section 2.5.5, which indicates that this algorithm captures the optimal decision with very small error.

2.5 Numerical Studies

This section consists of five parts. We (i) investigate the structure of the optimal solution, (ii) analyze the effect of problem parameters on showcase variety and capacity utilization, (iii) explore the impact of inaccuracy on the optimal profit, (iv) provide a sensitivity analysis for problem parameters, and (v) investigate the performance of the proposed greedy heuristic algorithm.

For our numerical test bed, we consider a representative product type with three attributes, i.e. $K = 3$. We assume that the attributes have three, two, and three levels, respectively, i.e., $L(1) = \{1, 2, 3\}$, $L(2) = \{1, 2\}$, and $L(3) = \{1, 2, 3\}$. Because a product is a combination of these three attributes, there are $3 \times 2 \times 3 = 18$ possible unique products, i.e., $|X| = 18$. To isolate the effect of attribute levels, we fix the part-worth utility of the levels of Attribute 1 (A1) and Attribute 2 (A2) (see Table 2.2) and change only the part-worth utility of the levels of Attribute 3 (A3). For A3, we consider five different utility scenarios with an average utility of 1 in each scenario (see Table 2.3). The scenarios differ based on the distribution of magnitudes. The first three scenarios are symmetric with wide, medium, and narrow ranges and are denoted by SyW, SyM, and SyN, respectively and the fourth and fifth scenarios are right and left skewed, denoted by SkR and SkL, respectively. In SyW and SkR scenarios, there is one attribute level that is distinctively more preferred than the others whereas in the other scenarios two or three attribute levels are closer to each other in terms of desirability by the customers. Regarding the pricing structure, we define two scenarios by fixing the part-worth price values of A1 and A2 to 30. The first scenario is 'Equal Prices', in which the part-worth price of all levels of A3 is 40 (i.e., attribute levels contribute equally to the overall price of the product), and in the 'Unequal Prices' scenario we consider 50, 55, and 60 as the part-worth prices of levels 1, 2, and 3 of A3, respectively (i.e., products with Level 3 of A3 are the highest priced). We let $d_{1,l(1)} = d_{2,l(2)} = 0 \ \forall \ l(1) \in L(1)$ and $\forall \ l(2) \in L(2)$ (i.e., A1 and A2 are accurately evaluated); and $d_{3,l(3)} \in \{-0.6, -0.4, -0.2, 0, 0.2, 0.4, 0.6\} \ \forall \ l(3) \in L(3)$ such that $d_{3,1} = d_{3,2} = d_{3,3}$ (i.e., magnitude of inaccuracy is the same for all levels of A3). Finally, we let $\alpha \in \{0.25, 0.5, 0.75\}$, $r \in \{-0.8, -0.5, -0.2\}$, $\mu \in \{0.5, 1, 1.5\}$, $\mu' = \mu$, $\gamma \in \{0.2, 0.4, 0.6, 0.8\}$, $\beta_1 = -0.013$, $\beta_2 = -0.01$, $\phi = 0.4$, and $\omega = 0.5$. Given this setup, we examine 7560 distinct problem instances for each value of showcase capacity $C \in \{1, 2, ..., 18\}$.

Table 2.2: Part-worth Utilities of Levels of Attributes 1 and 2

Attribute k	1			2	
Levels $l(k)$	1	2	3	1	2
$\widetilde{u}_{k,l(k)}$	0.8	1	1.2	0.8	1.2

Table 2.3: Part-worth Utilities of Level of Attribute 3 under Different Scenarios

Scenarios	Levels		
	1	2	3
SyW	0.4	1	1.6
SyM	0.6	1	1.4
SyN	0.8	1	1.2
SkR	0.4	0.6	2
SkL	0.4	1.2	1.4

2.5.1 Analysis of the Optimal Solution

Table 2.4 presents the details of the optimal solution for a problem instance with $\alpha = 0.25$, $r = -0.2$, $\mu = \mu' = 1.5$, $\gamma = 0.6$, and $d_{3,l(3)} = -0.2 \ \forall l(3) \in L(3)$, scenario SyM, and unequal prices. For each possible showcase capacity $C \in \{1, 2, ..., 18\}$, this table lists the optimal number of showcased products ($|M^*|$), the attribute levels showcased (revealed), the optimal profit (Π^*), and the total refunds for returned products sold in the offline (fR) and online (nR) channel. The numbers in the parenthesis in each row under the Π^* column indicate the marginal percentage profit increase with respect to the showcase capacity of the previous row.

Along the lines of Proposition 2.2, we observe in this problem instance that utilizing an extra capacity to showcase one more product does not necessarily bring a benefit. For example, all available showcase capacity is utilized fully when $C \leq 6$, but when $C = 7$, the optimal number of products to be shown remains at six, leaving one available showcase capacity idle. To understand the reasoning behind this, we forced the model

Table 2.4: An Example Optimal Solution for a Problem Instance with SyM Scenario, Unequal Prices, $\alpha = 0.25$, $r = -0.2$, $\gamma = 0.6$, $d_{3,l(3)} = -0.2, \forall l(3) \in L(3)$, and $\mu = \mu' = 1.5$

		Attribute Levels Showcased							
C	$\lvert M^* \rvert$	A1	A2	A3	Π^* (%↑)	Π_f	Π_n	fR	nR
1	1	3	2	3	97.16 (-)	21.06	76.10	1.56	8.27
2	2	2, 3	2	3	99.79 (2.71%)	23.69	76.10	1.86	8.27
3	3	1, 2, 3	2	3	100.75 (0.96%)	24.65	76.10	2.05	8.27
4	4	1, 2, 3	1, 2	3	101.31 (0.60%)	25.21	76.10	2.17	8.27
5	5	1, 2, 3	1, 2	3	101.60 (0.28%)	25.50	76.10	2.28	8.27
6-7	6	1, 2, 3	1, 2	3	101.74 (0.14%)	25.64	76.10	2.40	8.27
8-18	8	1, 2, 3	1, 2	2, 3	101.77 (0.03%)	25.76	76.01	2.44	8.44

to showcase seven products when $C = 7$ and analyzed the sales and return dynamics. When $C = 6$, it is optimal to showcase six products by revealing all levels of A1 and A2 and only Level 3 of A3. If we force the model to fully utilize the capacity when $C = 7$, Level 2 of A3 is revealed by showcasing the 7th product which is cheaper than the previously selected six products. Adding this product to the variety in the physical store decreases the purchase probability of the previously selected six products, because it steals part of their share from sales. However, it increases the overall purchase probability of products in the physical store, and in this case, slightly increases the profit in this channel. In this problem instance, all products are undervalued. Therefore, introducing a new level of A3 in the physical store will cause all products with Level 2 of A3 in the online store to be evaluated more accurately, resulting in an increase in the purchase probability of these products and more sales; however, the return probability increases as well by revealing undervalued attribute levels (if kept undervalued, sold products are less likely to be returned). In this particular case, the increase in returns dominates the increase in sales, which leads to a slight decrease in this channel's overall profit. In aggregate, because the population of OnSs is three times greater than OfSs ($\alpha = 0.25$), the decrease in the profit of the online channel will be greater than the increase in the profit of the physical store, which leads to not utilizing the extra capacity provided. For $C = 8$, all showcase capacity is again fully utilized, but any extra capacity does not bring in any benefit for $C \geq 9$ for the retailer. Forcing the model to utilize the extra capacity when going from $C = 8$ to 9 will result in adding one more product with Level 2 of A3. In this case, although the total purchase probability increases, a great part of the purchase probability for products with Level 3 of A3, which are more expensive, are assigned to products with Level 2 of this attribute, which decreases the profit of the physical store. The profit of the online channel will not change because this new addition reveals no additional information to the OnSs. We observe that Level 1 of A3 is never shown, which attributes to the cheapest products in the portfolio.

We also observe from Table 2.4 that optimal profit (Π^*) is non-decreasing in showcase capacity, which is expected. However, there are no consistent diminishing returns in the marginal profit as one more showcase capacity is made available, even though we observe a general trend towards this direction. This can be better explained considering cases where one more showcase capacity is made available but it is not utilized (e.g., in Table 2.4, when C increases from 6 to 7). In such cases, the marginal profit of adding one more showcase capacity is zero. However, when one more capacity is added again, and this time it is utilized (e.g., when C increases from 7 to 8 in Table 2.4), the marginal profit of this addition is positive and greater than the previous increase in capacity.

The fR and nR values given in Table 2.4 correspond to the 'lost' profit due to returns by offline and online customers, respectively. nR values are higher than fR, because $\alpha = 0.25$ in this problem instance, which is the fraction of OfSs. Also, as expected, returns are non-decreasing as the showcase capacity increases. Showcasing more levels in the physical store (i.e., revealing more information about the products) increases returns in the online channel. This is because the hidden attributes are undervalued in this example which implies less returns after sales, but revealing them will increase the returns (an opposite effect would have been observed if hidden attributes were overvalued). The value of nR is unchanged for $1 \leq C \leq 7$, because the showcased level of A3 does not change. At $C = 8$, two more products are shown with the introduction of Level 2 of A3 for the first time and the value of nR increases. For $C \geq 8$, the nR value does not change as the levels showcased do not change from this point on.

2.5.2 Analysis of Showcase Variety and Capacity Utilization

Let $\mathcal{L}$ be the number of different levels of A3 showcased (revealed) in the optimal solution for a problem instance. Since there are three levels of A3 in our test bed, $\mathcal{L}$ can take a value of 1, 2, or 3, meaning that either one, two, or three levels are showcased (and hence two, one, or zero levels of A3 are hidden) in the optimal assortment solution, respectively. One of the challenges of the retailer is to decide whether to showcase and reveal a limited variety of products (i.e., $\mathcal{L} = 1$ or $\mathcal{L} = 2$) or full variety ($\mathcal{L} = 3$). Recall that A1 and A2 have 3 and 2 levels, respectively, which are always accurately evaluated by the customers, and there are 18 different products that can be showcased. If only one level of A3 is revealed at the optimal solution ($\mathcal{L} = 1$), then up to $3 \times 2 \times 1 = 6$ products can be showcased as the capacity permits; if two levels are revealed ($\mathcal{L} = 2$) then up to $3 \times 2 \times 2 = 12$ products can be showcased; and if all three are revealed ($\mathcal{L} = 3$) then up to 18 products can be showcased. The retailer should also decide to what extent they need to allocate capacity to the desired product variety, as it may not always be optimal to utilize the capacity fully (see Proposition 2.2). We use $\mathcal{C}$ to denote whether available capacity is fully utilized or not. Supposing that $C = 18$, if $\mathcal{L}$ is equal to 1 or 2, then $\mathcal{C} = 0$ and if $\mathcal{L} = 3$ then $\mathcal{C} = 0$ or $\mathcal{C} = 1$ where 0 denotes that not all of the available capacity is utilized and 1 denotes the otherwise. In this section, we attempt to generate managerial insights regarding the values of $\mathcal{L}$ and $\mathcal{C}$ at the optimal solution based on different values of problem parameters. To eliminate the effect of the limit on capacity, we conduct the analysis under no capacity constraint (i.e, $C = 18$). As discussed in the previous section, the decision of showcasing a certain variety of attribute levels is a trade-off between sales and returns in both channels, which defines the retailer's profit. Our analyses indicate that the effect of parameters on the optimal assortment varies with respect to whether attribute levels contribute to the overall prices (Equal vs Unequal Price scenarios) and are over- or under-valued.

In Tables 2.5 - 2.8, we present the average number of the levels of A3 revealed (Columns $\overline{\mathcal{L}}$) and the fraction of cases for which available capacity should be fully utilized (Columns $\overline{\mathcal{C}}$) at the optimal solution for different values of α and γ as well as for four utility scenarios (two symmetric and two skewed). For example, a value of $\overline{\mathcal{L}} = 3$ means that all three levels of A3 have always been revealed in the optimal solution in all problem instances with the corresponding α and γ values. A value of $\overline{\mathcal{L}} = 2.48$ means that although there are instances in which all three levels of A3 should be revealed, there are also other problem instances in which 1 or 2 levels should be revealed under the corresponding α and γ values. Similarly, $\overline{\mathcal{C}} = 0$ means that in all problem instances in that category, available capacity has not been fully utilized; whereas $\overline{\mathcal{C}} = 1$ means that it is fully utilized. Any decimal value indicates the fraction of cases with $\mathcal{C} = 1$ among the problem instances with the corresponding α and γ values.

Equal Prices and Undervaluation

Table 2.5 presents the $\overline{\mathcal{L}}$ and $\overline{\mathcal{C}}$ values for problem instances where prices are equal and all A3 levels are undervalued. Smaller values of γ correspond to lower refunds given to customers upon a product return; hence returns are less detrimental for the retailer under such values and the retailer pays more attention to increasing sales. Since A3 levels are undervalued, showcasing a greater variety of A3 eliminates the inaccuracy from more products in the online channel, which results in higher sales. Therefore, as γ decreases, $\overline{\mathcal{L}}$ is non-decreasing meaning that the retailer tends to showcase a greater variety of attribute levels as the returns from increased sales is not too detrimental (lower refund). As γ increases (higher refund), the impact of returns on the retailer's profit becomes more drastic and the retailer would prefer to avoid returns. Therefore, the retailer tends to showcase only a subset of attribute levels and utilizes the capacity less. For all problem instances in our test bed with $\gamma = 0.2$, all three levels of A3 are showcased ($\overline{\mathcal{L}} = 3$) and all available capacity is utilized in SyW, SyN, and SkL scenarios ($\overline{\mathcal{C}} = 1$) whereas full capacity is used in most of the instances in SkR scenario ($\overline{\mathcal{C}} = 0.78$). This also implies that there can be problem instances in which the full variety of A3 levels are showcased ($\mathcal{L} = 3$) by not utilizing the available capacity fully ($\mathcal{C} = 0$).

SyW and SyN are symmetric utility scenarios where the utilities are dispersed wider in SyW (0.4/1/1.6) and narrower in SyN (0.8/1/1.2). Hence there is one level in SyW that has a distinctively higher utility

Table 2.5: Effect of α and γ on Showcase Variety and Capacity Utilization under Equal Prices and Utility Undervaluation for SyW, SyN, SkR, and SkL Utility Scenarios

	SyW Utility Scenario								**SyN Utility Scenario**							
γ	0.2		0.4		0.6		0.8		0.2		0.4		0.6		0.8	
α	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$
0.25	3	1	2.48	0.11	2	0	1.19	0	3	1	3	0.89	2.85	0.11	2.22	0
0.5	3	1	2.81	0.11	2.11	0	1.59	0	3	1	3	0.89	3	0.11	2.63	0
0.75	3	1	2.93	0.11	2.19	0	1.81	0	3	1	3	0.89	3	0.11	3	0
	SkR Utility Scenario								**SkL Utility Scenario**							
α	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$
0.25	3	0.78	1.93	0	1	0	1	0	3	1	2.59	0.11	2.11	0	2	0
0.5	3	0.78	2.19	0	1.11	0	1	0	3	1	2.81	0.11	2.11	0	2	0
0.75	3	0.78	2.56	0	1.15	0	1	0	3	1	3	0.11	2.22	0	2	0

value than other levels. Under higher values of γ, the retailer prefers to decrease product returns, therefore, it would be better off showcasing a smaller variety of A3 and showcase only the level that has distinctively higher utility in SyW to increase sales, whereas in SyN, retailer benefits from including more levels to obtain higher sales. A similar effect is also observed when we compare SkR (0.4/0.6/2) and SkL (0.4/1.2/1.6). SkR has one level that has distinctively higher utility whereas SkL has two levels with high utility compared to the third one. The retailer is better off showcasing the singled-out level with the highest utility in SkR whereas is better off showcasing the two levels with relatively high utilities in SkL.

As α increases, a greater portion of customers becomes OfSs. As a result, the retailer prefers to showcase a greater variety of levels of A3 compared to smaller α values, to increase sales in the physical store. Since all hidden A3 levels are undervalued, this may lead to higher returns and less profit in the online channel; however, as α grows, the operations of the online channel fade out. Therefore, the retailer prefers to showcase a greater variety of A3.

We also investigate the optimal assortment structure based on r. We observe that as the magnitude of r increases (i.e., the return process becomes harder), the retailer pays more attention to increasing sales; hence, it tends to showcase a greater variety of A3 to eliminate the undervaluation in product utilities and obtain more sales. This is similar to the effect of γ stated earlier; however, γ turns out to be more impactful than r.

Equal Prices and Overvaluation

Table 2.6 presents the $\overline{\mathcal{L}}$ and $\overline{\mathcal{C}}$ values for problem instances where prices are equal and all A3 levels are overvalued. We first observe that the $\overline{\mathcal{L}}$ values are closer to 3 (showcasing all A3 levels) compared to the values in Table 2.5 for the undervaluation case. The retailer is generally better off if OnSs undervalue the hidden attribute levels (see Section 2.5.3). Therefore, when attribute levels are overvalued, the retailer prefers to reveal more compared to the case with undervaluation. The impact of γ on $\overline{\mathcal{L}}$ depends on the value of α. If there are more OnSs than OfSs (lower values of α) then $\overline{\mathcal{L}}$ increases as γ increases (higher refund). This is because the retailer prefers to eliminate the overvaluation from OnSs' purchases; otherwise, returns will be higher and since γ is also high, it results in higher loss of profit. On the other hand, $\overline{\mathcal{L}}$ decreases as γ increases for higher values of α. In this case, more customers are OfSs who purchase with accurate utility evaluations. So, the retailer prefers to showcase a more limited variety of levels of A3 that have higher utility and less return likelihood. It should be noted that this may result in less profit in the online channel, but as α increases, the contribution of the online channel fades out. Similar to the undervaluation case, it is preferable to utilize the available capacity fully in more problem instances under lower γ values compared to higher values.

Table 2.6: Effect of α and γ on Showcase Variety and Capacity Utilization under Equal Prices and Utility Overvaluation for SyW, SyN, SkR, and SkL Utility Scenarios

	SyW Utility Scenario								**SyN Utility Scenario**							
γ	0.2		0.4		0.6		0.8		0.2		0.4		0.6		0.8	
α	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$
0.25	2.44	0.59	2.74	0	2.93	0	3	0	2.63	0.63	2.74	0.63	2.85	0.04	3	0
0.5	2.85	0.85	2.96	0.07	3	0	3	0	2.89	0.89	2.96	0.85	3	0.11	3	0
0.75	3	1	3	0.11	3	0	2.85	0	3	1	3	0.89	3	0.11	3	0
	SkR Utility Scenario								**SkL Utility Scenario**							
α	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$
0.25	2.48	0.52	2.89	0	3	0	3	0	2.63	0.63	2.74	0	2.85	0	3	0
0.5	2.81	0.67	2.96	0	3	0	3	0	2.89	0.89	2.96	0.07	3	0	3	0
0.75	3	0.78	3	0	2.81	0	2.41	0	3	1	3	0.11	3	0	2.93	0

Regarding the impact of α, the general trend is the same as that of the undervaluation case; variety of showcased products increases as α increases. However, under extreme values of γ (as the retailer approaches full refund) $\overline{\mathcal{L}}$ is non-increasing because elevated refund obligations makes the retailer prefer to showcase levels of A3 that are less likely to be returned rather than the full variety. The effect of r depends on the value of α. If there are more OnSs (lower values of α), then as the magnitude of r increases, the retailer tends to showcase a smaller variety of A3 to benefit from the increased sales at the online channel due to overvaluation of product utilities and lower returns as returns become more difficult to the customer. However, when there are more OfSs (larger values of α), the contribution of the physical store sales become the dominant factor; so as the magnitude of r increases, the retailer prefers to showcase a greater variety of A3 to sell more in the physical store.

Unequal Prices and Undervaluation

Table 2.7 presents the $\overline{\mathcal{L}}$ and $\overline{\mathcal{C}}$ values for problem instances where prices are unequal and all A3 levels are undervalued. In this case, the effect of changes in γ and r values on $\overline{\mathcal{L}}$ are the same as described in Section 2.5.2. However, in all problem instances, only a subset of A3 is showcased (i.e., $\overline{\mathcal{L}} \leq 2$) and the capacity is never fully utilized (i.e., $\overline{\mathcal{C}} = 0$). In this case, levels of A3 with higher utilities are preferred to others considerably because not only do they have higher sales and smaller return probabilities, but they are more expensive and profitable. Hence, the retailer is reluctant to showcase levels with smaller utilities. Furthermore, the effect of changes in α value on $\overline{\mathcal{L}}$ is also similar to that in Section 2.5.2 with some exceptions.

Table 2.7: Effect of α and γ on Showcase Variety and Capacity Utilization under Unequal Prices and Utility Undervaluation for SyW, SyN, SkR, and SkL Utility Scenarios

	SyW Utility Scenario								**SyN Utility Scenario**							
γ	0.2		0.4		0.6		0.8		0.2		0.4		0.6		0.8	
α	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$
0.25	1.67	0	1.56	0	1.04	0	1	0	2	0	1.85	0	1.63	0	1.26	0
0.5	1.67	0	1.56	0	1.15	0	1	0	2	0	1.81	0	1.67	0	1.56	0
0.75	1.67	0	1.56	0	1.26	0	1.04	0	2	0	1.78	0	1.67	0	1.67	0
	SkR Utility Scenario								**SkL Utility Scenario**							
α	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$
0.25	1.44	0	1	0	1	0	1	0	1.78	0	1.67	0	1.56	0	1.18	0
0.5	1.44	0	1	0	1	0	1	0	1.78	0	1.67	0	1.63	0	1.41	0
0.75	1.44	0	1	0	1	0	1	0	1.78	0	1.67	0	1.67	0	1.56	0

Unequal Prices and Overvaluation

Table 2.8 presents the $\overline{\mathcal{L}}$ and $\overline{\mathcal{C}}$ values for problem instances where prices are unequal and all A3 levels are overvalued. In all problem instances in this case, the capacity is never fully utilized (i.e., $\overline{\mathcal{C}} = 0$). The effect of changes in γ and r values on $\overline{\mathcal{L}}$ highly depend on the value of α, similar to the equal price and overvaluation case.

As α increases, the retailer becomes more interested in increasing the profit obtained from the physical store. Therefore, the retailer tends to showcase products with higher utilities which are also less likely to be returned and more profitable. As a result, $\overline{\mathcal{L}}$ is non-increasing in α. Although this statement generally holds, there are instances in which the opposite behavior is observed. Specifically, in Table 2.8, in SkL scenario for $\gamma = 0.6$, an increase in α from 0.25 to 0.5 results in an increase in $\overline{\mathcal{L}}$. In this special situation, since the utility values of levels in SyN are close to each other and less differentiable compared to other scenarios, the retailer may want to showcase a greater variety to increase sales while return probabilities are not very

Table 2.8: Effect of α and γ on Showcase Variety and Capacity Utilization under Unequal Prices and Utility Overvaluation for SyW, SyN, SkR, and SkL Utility Scenarios

	SyW Utility Scenario								SyN Utility Scenario							
γ	0.2		0.4		0.6		0.8		0.2		0.4		0.6		0.8	
α	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$
0.25	2.93	0	3	0	3	0	3	0	2.63	0	2.67	0	2.70	0	2.85	0
0.5	2.89	0	2.89	0	2.81	0	2.89	0	2.85	0	2.67	0	2.70	0	2.85	0
0.75	2.48	0	2.44	0	2.22	0	2.07	0	2.52	0	2.22	0	2.15	0	2.15	0
	SkR Utility Scenario								SkL Utility Scenario							
α	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$	$\overline{\mathcal{L}}$	$\overline{\mathcal{C}}$
0.25	2.93	0	3	0	3	0	3	0	2.74	0	2.89	0	2.89	0	2.93	0
0.5	2.85	0	2.78	0	2.78	0	2.78	0	2.67	0	2.96	0	2.96	0	2.89	0
0.75	2.33	0	2.04	0	1.89	0	1.89	0	2.33	0	2.26	0	2.22	0	2.22	0

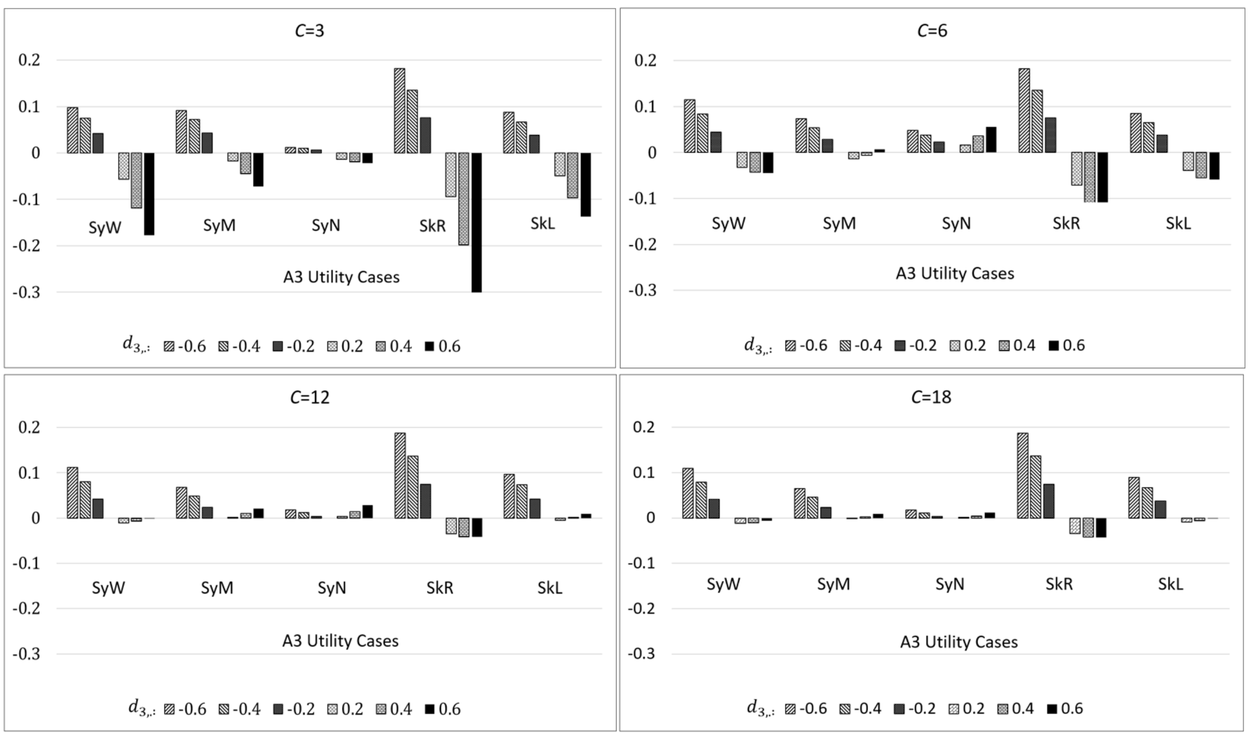
different. But when α gets larger, $\overline{\mathcal{L}}$ decreases because in this situation a greater proportion of customers are OfSs and even small differences in the value of return probabilities are important.

2.5.3 Impact of Inaccuracy on Optimal Profit

In this section, we investigate the impact of inaccuracy in customers' evaluations of the products with hidden attribute levels on the overall profit of the retailer. Even though the evaluations of the customers are inherent to their self-assessment and perception, the retailers may have some influence on this perception through design and showcase decisions — we elaborate on potential actions in the Conclusion section. Figure 2.1 shows the average percentage profit deviation from the perfect information case ($d_{3,\cdot} = 0$) for different utility scenarios and for $C \in \{3, 6, 12, 18\}$, under the Equal Price scenario. We first observe that the retailer makes more profit if their customers undervalue the hidden attribute levels rather than assessing them accurately, and the opposite is true if hidden attribute levels are overvalued, on the average. In our numerical test bed with 68,040 instances for each of the Equal and Unequal Price scenarios, a negative $d_{3,\cdot}$ value (undervalued hidden attribute levels) produced the highest profit in 70. 82% of the instances under Equal Prices and 93. 41% of the instances under Unequal Prices. The percentage benefit increases, as $d_{3,\cdot}$ decreases within $d_{3,\cdot} \in \{0.6, ..., -0.6\}$. For C equal to 12 and 18 in Figure 2.1, the deviation is smaller compared to when C is equal to 3 and 6 for $d_{3,\cdot} > 0$, because most or all of the attribute levels are showcased in these values of C in the optimal solution.

In Figure 2.1, we also observe that the magnitude of the average profit deviation changes for different utility scenarios. In terms of the absolute difference, scenarios with the highest profits change according to whether attribute levels are undervalued or overvalued. In the case of undervaluation, we observe that the asymmetric scenario SkR provides the highest profit, which is the one with a singled-out attribute level that is much more desired than the others. The next highest profit is observed under the SyW scenario, which has a similar structure. Within the symmetric scenarios, we observe that the profit increases as the range of the utility levels increases (from SyN to SyW). These observations indicate that the retailer is better off if a few of the attribute levels are distinctively more desired than the other attribute levels. On the other hand, in the case of overvaluation, the exact opposite trend holds. In this case, SkR and SyW scenarios provide the lowest profits, respectively, and other utility scenarios generate higher profits. This implies that, in case of undervaluation, as the utility of attribute levels are closer and there is not a considerable difference, the retailer can obtain a higher profit.

Figure 2.1: % Average profit deviation from profit of $d_{3,\cdot} = 0$ for different values of $d_{3,\cdot}$, with respect to A3 utility cases, for equal prices

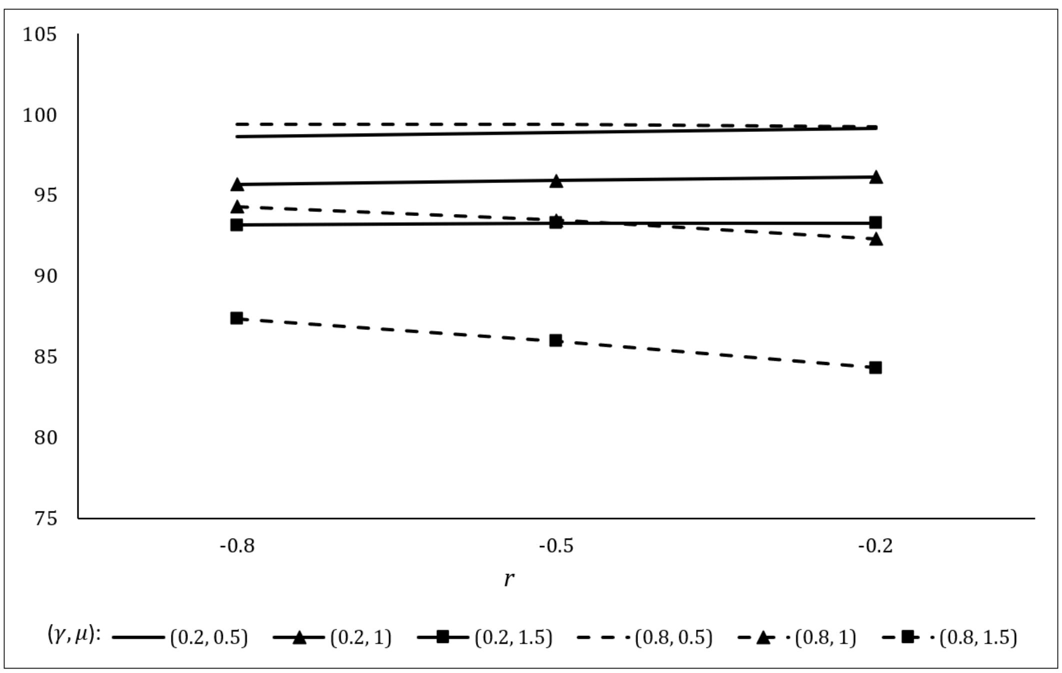


2.5.4 Joint Sensitivity Analysis of γ, μ, and r

In this section, we provide a joint sensitivity analysis for parameters γ (the refund fraction of product price), μ (the homoscedasticty of the customer population) and r (the disutility of return). Note that γ is a strategic decision to be set by the retailer, μ is a given exogenous variable that will be different for different businesses and sectors, and r is also an exogenous parameter inherent to the customers and originates from factors such as making a trip to a post office or the physical store to return a purchased product or repackaging the product at home. This analysis is based on the experiments of the equal prices scenario, and the results hold for the unequal prices as well.

Figure 2.2 displays the average profit with respect to γ and μ for different values of r. For a smaller μ, the population under study is more homoscedastic, i.e., customers' behavior is more similar and less variable. Therefore, the model captures customers' behavior with less variability, and, as a result, the average profit is higher compared to a greater μ. Moreover, under higher values of γ (here 0.8), a smaller disutility of return (the magnitude of r) results in a decrease in the average profit. If the return process is easier and most of the price will be refunded upon a product return, even though there will be a tendency for higher sales with these settings, the loss due to returns increases considerably under large variability. We also note that the magnitude of the decrease in profit is greater for higher μ. This is because, a greater μ always has an adverse effect on profitability. On the contrary, for a small value of γ (here 0.2), a smaller disutility of return results in an increase in profit, since although the return process is easier, only a small portion of the price will be refunded. Considering that an easier return process means higher utility, sales increase and overall profit increases as well. Also, the magnitude of the increase in profit is greater for a smaller μ.

Figure 2.2: Average profit with respect to γ and μ for different values of r in equal prices

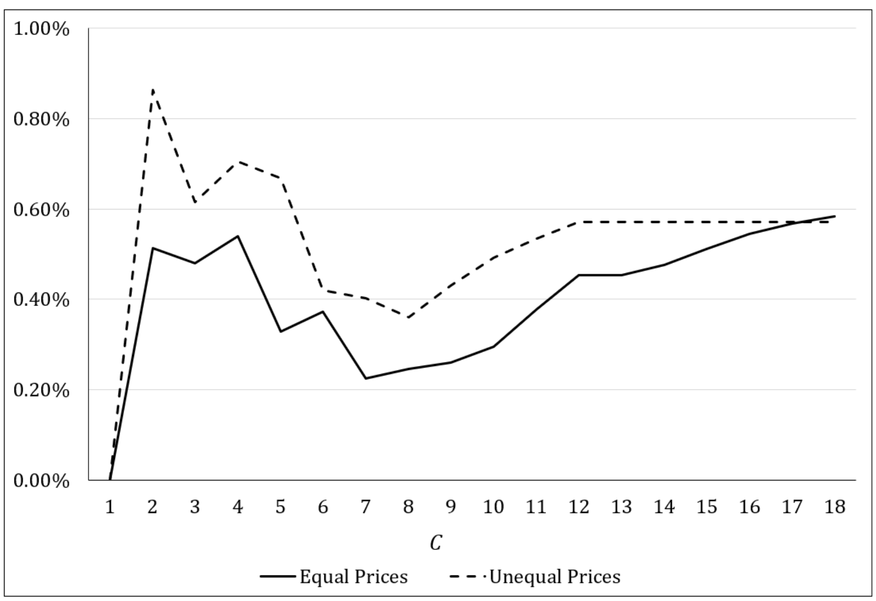


2.5.5 Performance of the Greedy Heuristic Algorithm

Given that OCAPwR is NP-Hard, complete enumeration is not an efficient method to solve the problem in practice. Finding the optimal assortment plan for each possible capacity takes about 20 seconds for an item with three attributes and $|X| = 18$ using a computer with a Core(TM) i7-10700 CPU @ 2.90 GHz processor. Since this duration will increase exponentially with the number of attributes, and the retailers will typically have several items on which to make assortment decisions, we propose a heuristic algorithm to find reasonable, feasible solutions (see Algorithm 1 in Section 2.4.7). The heuristic runs in a greedy manner to select the next item to showcase, as the showcase capacity increases one by one.

To analyze its performance, we extend our test bed such that $d_{1,\cdot}, d_{2,\cdot} \in \{-0.6, -0.4, -0.2, 0, 0.2, 0.4, 0.6\}$. We limit our analysis in this part to the SyW, SkR, and SkL utility scenarios of A3. Therefore, our extended test bed in this section includes 55,566 problem instances for each of the equal price and unequal price cases. Given that we solve the problem for every C from 1 to 18, this results in 1,000,188 distinct problem instances. We compare the results of the complete enumeration and the algorithm. Using complete enumeration to find the optimal solutions for all problem instances in the test bed takes 705,622 seconds (196.01 hours) in the unequal prices scenario and 702,720 seconds (195.20 hours) in the equal prices scenario. The proposed greedy algorithm takes 990.12 seconds (0.28 hours) for unequal prices and 973.36 seconds (0.27 hours) for equal prices.

Figure 2.3: Maximum % of heuristic profit deviation from optimal profit for each capacity



For both scenarios of equal and unequal prices, the average percentage deviation of profit obtained by the greedy algorithm is negligible for all capacities. The highest average percentage deviation in both scenarios occurs when $C = 18$, and the value is 0.0084 % in equal prices and 0.0080 % in unequal prices. However, there are several cases for which the heuristic has an inferior performance. Figure 2.3 presents the maximum percentage deviation of the profit obtained by the greedy heuristic, compared to the optimal profit for equal and unequal price scenarios. As can be seen, the maximum percentage of deviations is 0.86% in unequal prices and 0.58% in equal prices, meaning that in the worst case, the greedy solution is an approximation that is as close to the optimal solution as 99.14%. We observe that the heuristic performs better under higher α values or if the hidden attribute levels are accurately or with small inaccuracy evaluated (either overvaluation or undervaluation). For the small values of γ, the deviation is greater than γ values closer to 1. However, the best performance is when γ has a value in between and a bit closer to 1. The deviation is the smallest under the asymmetric scenario SkR and the greatest under SkL.

2.6 Conclusion

In this paper, we tackle the assortment planning problem for an omni-channel retailer that runs an online sales channel along with a physical store and customers can return their products after purchase. The physical store may bear showcase capacity, whereas the capacity is not an issue for the online channel. Our model applies to high-value products with non-digital attributes that customers prefer to visit the physical store to experience the product, before purchasing either directly from the store or from the online channel. Therefore, the showcased assortment in the physical store acts as a medium to reveal information for the online shoppers, which will lead to more informed purchasing decisions and might affect the post-purchase keep-or-return decisions of the customers. We explicitly consider this relationship and develop a model to select an assortment plan that maximizes the overall profit of the retailer, including the losses due to product returns.

We generate several managerial insights for retailers. One interesting result is that retailers should not necessarily fully utilize their showcase capacity, even if there is no cost incurred for using this available capacity. Therefore, retailers should be diligent in checking all possibilities when making decisions. This check is even more crucial when a retailer offers multiple product lines. In this situation, products compete for the total space available in the store, and the retailer should carefully find an optimal capacity for each product line.

One may expect that retailers would prefer their customers to perfectly evaluate the attribute levels that they cannot experience (i.e., that are not showcased) in the store. This expectation is not true based on an analytical result for a special case and an extensive numerical analysis. In 82% of the problem instances in our numerical test bed, the retailers make the highest profit if the hidden attribute levels are undervalued by the online customers. Hence, in these situations, there is no value for retailers to invest in advanced web interfaces that make the online shopping experience of the customers more realistic on a computer screen. Note that this is counter intuitive because online customers will have less tendency to purchase the products with hidden attribute levels. Retailer is beneficial in this situation because undervaluation also leads to less returns. On the other hand, if the customers are overvaluing the hidden attribute levels, the retailer is likely to benefit from web interfaces that lead to more accurate evaluations. In general, the retailers should avoid situations in which a hidden attribute level stands out among the showcased attribute levels with increased unrealistic expectations and exaggerations. This action will lead to overvaluation of such items and, hence, will hurt the retailer when product returns are also considered.

Retailers may consider having computers in their stores for customers to be able to browse their online sales website for items that are not present in the store. In line with the discussion above, retailers should investigate the impact of having such a mechanism in their stores. If this in-store computer browsing leads customers to undervalue the utility of the hidden attribute levels, then retailers should take advantage of inserting such computers in their showrooms.

Full refund return policies are common in modern omni-channel retailing practices. This strategic decision can have positive effect on retailers' sales. Our results indicate that when a full-refund policy is employed, the retailers should showcase a limited variety of undervalued attribute levels to obtain a higher profit. If the hidden attributes are overvalued, then the retailer should showcase a large variety of the product if the majority of the retailer's customers are online shoppers and should showcase a limited variety if the majority prefer to shop from the physical store. We also observe a similar strategy when it comes to the difficulty of the return process. That is, if the return process is easy, the retailers should showcase a limited variety of undervalued attribute levels. When the hidden attribute levels are overvalued, the retailers should showcase a large variety of the product if the majority of the customers are online shoppers, and vice versa.

We observe that the retailer is better off if the variability of the uncertainty in customer preferences is lower and if the disutility of return for the customers is higher. In addition, when hidden attribute levels are undervalued, the retailers make more profit if there is a prevalent attribute level that is preferred significantly higher than the others. Hence, products which provide less differentiation between the different levels of an attribute provide lower benefit to the retailer compared to the products for which a few attribute levels are much more desired than the others. On the contrary, when hidden attribute levels are overvalued, the retailers make more profit if there is not an one or more levels that are significantly preferred to others.

Retailers may interpret the decrease in product returns as an increase in profit, because returns correspond to lost sales. However, our analyses indicate that an increase in returns does not necessarily lead to a decrease in the profit. There is an intricate interaction between the increased or decreased sales and the corresponding decreased or increased product returns for which our approach should be implemented to find the best assortment decisions. We note that some of our recommendations are based on the extensive numerical analysis that we conducted. Under situations where the choice of parameters in our test bed does not reflect the business environment, the directions for the optimal actions should be found by solving the problem with the exact problem parameters. We proposed a greedy heuristic algorithm with excellent performance that can be used to solve the problem within a short amount of time. As a potential line of research for future work, one might investigate an omni-channel retail setting with multiple physical stores. It is expected that, if physical stores operate with different showcase capacities, then store selection by customers affects their purchasing decisions in both physical and online channels. Moreover, we used the MNL consumer choice model in this study. Although this consumer choice model is widely used in the literature, it has its limitations, such as the Independence of Irrelevant Alternatives (IIA) property. Recent modeling approaches, such as Markov-chain based models, eliminate this assumption and can provide a direction for future research.

Chapter 3

Scope Contracts to Coordinate Assortment Planning in Omni-Channel Retail Supply Chains

Abstract. We consider operations of an omni-channel retail supply chain (RSC) entailing an online sales website as the manufacturer's sales medium and a physical store as an independent retailer selling manufacturer's products. While it is possible to showcase all product variety online, the physical store may have limited showcase capacity that allows showcasing only a limited product variety. Through a Stackelberg game, we investigate the manufacturer's wholesale price decision to be charged to the retailer, and the retailer's assortment decision, both to maximize their profits independently. As a benchmark, we explore the RSC in a centralized setting where the aim is to maximize the total profit as a whole. Our results indicate that the decentralized setting is inefficient, resulting in a lower total profit compared to the centralized setting. To address this inefficiency and coordinate the RSC, we propose a *scope contract* that offers discounts on wholesale prices based on specific attribute levels. This contract incentivizes the parties to make their decision as if the RSC were centralized, by ensuring their profitability. Our results indicate that (i) the optimal decentralized assortment balances showcasing the highest utility products with a high variety of overvalued (low variety of undervalued) attribute levels, (ii) the optimal wholesale price of products should be set at the highest value that allows the retailer to operate its store, (iii) in the centralized setting, selecting inaccurately assessed attribute levels determines the optimal assortment and there can be multiple such assortments, (iv) there can be multiple discount rates in the proposed *scope contract* that coordinate the RSC, each resulting in a different profit distribution, and (v) isolating the RSC from outside competition, the profit functions are unimodel with respect to the retail price.

3.1 Introduction

Omni-channel retailing, which refers to the coexistence of an internet-based sales channel and brick-and-mortar (physical) stores, is a common practice in modern retailing. It increases market share and provides retailers with the opportunity for more profitability (Bell et al. 2018). In such systems, physical stores generally have limited showcase capacity so that customers can choose only from a limited variety of available products; while the online sales channel is practically not limited in product variety because of being directly supplied from a central warehouse or the manufacturer's location. Therefore, a primary decision for omni-channel retailers such as Ray-Ban and Sport Chek is to determine the product variety to showcase in their physical stores given their limited capacity, i.e., the assortment problem.

Products can be comprised of several 'attributes' such as color and material, and each attribute can have several 'levels' such as blue and red for the color attribute. According to Dzyabura and Jagabathula (2018), the utility of a product for a customer (i.e., the attractiveness of that product to the customer) can be measured as the summation of the utility of its attribute levels. One characteristic of omni-channel retailing is the asymmetry in the customers' shopping experience for products with non-digital attributes. These are products such as clothes and sunglasses for which physical assessment and trying out provide accurate information of their utility. For these types of products, items showcased in a physical store are accurately assessed by the customers; whereas, items with attribute levels that are not showcased (but browsed only online) may be inaccurately assessed.

The asymmetry in the customers' shopping experience will result in different purchase likelihoods. For example, if a customer overvalues the utility of a product on a computer screen (i.e., the assessed utility is greater than the utility that the customer would have obtained if they assessed the product physically), they will be more likely to purchase the product compared to the case the customer browses the product in-store with physical touch and feel. Similarly, if the customer undervalues the utility of the product on a screen, then the purchase likelihood will be lower. Over- or under-valuation at the time of purchase will also influence the extent of the customer's (dis)satisfaction with the product after purchase, which impacts their keep-or-return decision for the product. A product that was purchased with overvaluation in the online channel at the time of purchase will disappoint the customer when received because it is not as desired as expected. Therefore, the product will be more likely to be returned compared to the case when the customer purchased the product with an accurate assessment. Similarly, a product purchased with undervaluation will gratify the customer when received because it turns out to be better than expected. Therefore, the product will be less likely to be returned.

The assortment decision for the physical store impacts customers' purchase and keep-or-return decisions in the online channel as well, because products showcased in the physical store can have attribute levels in common with online-only products and can provide partially accurate utility information for these products. For example, a red product with a certain style might be available in-store but the red color of the style that the customer wants may not be available in-store. In such cases, the customer can assess the utility of the red color accurately, but not the utility of the style desired. Hence, the customer will assess the full utility of the desired product inaccurately (partially accurately) in this example. This is specifically crucial for high-value products that customers would prefer to visit the physical store to gather accurate utility information of available products and their attribute levels before purchasing either online or in the store (Park et al. 2021).

In this paper, we investigate the operations of an omni-channel retail supply chain (RSC) that consists of an online sales channel run by the manufacturer as its direct sales medium and a physical store (or several stores with identical showcase capacities), for example, a franchise or an independent department store (the retailer) that sells manufacturer's products. For instance, Ray-Ban, the giant sunglasses manufacturer sells its products online, while various authorized retailers and department stores like Hudson's Bay also carry and sell its sunglasses. Without loss of generality, we assume that all customers visit both the online channel and the physical store and purchase a product that they assess as the highest positive utility across both channels[1]. If the highest utility is non-positive for a customer, no purchase will take place. An example of such practice is that customers visit the physical store and assess the products available while browsing the complete collection available online using their smartphones or a computer screen provided in the store. We assume that the physical store is a showroom in which a number of products are showcased without keeping any inventories[2]. If a customer decides to purchase an item from the showcased products in the physical store, the retailer manages to order the product from the manufacturer for a wholesale price and delivers it to the customer. On the other hand, if a customer decides to purchase a product available only online, the manufacturer receives the order on its website and delivers the product to the customer. Our model also applies to a physical store that operates with inventories. For such situations, the required assumptions for our models to be valid are (i) the inventory management decisions are not coupled with the assortment decisions, (ii) shortages are backordered without extra cost, and (iii) excess inventory at the retailer can be returned to the manufacturer at no cost.

[1] There might be customers who do not visit both channels and only buy from the online store. Such customers are out of the scope of this study since the assortment decision at the physical store does not affect their purchasing decisions.

[2] Referring to Bell et al. (2018), "even a zero-inventory store (which provides informational but not fulfillment capabilities) increases demand and operational efficiency in its trading area. Given that offline showrooms are much less costly to operate than conventional stores and maintain the benefits of centralized fulfillment, they provide a very appealing growth option to online-first firms."

The manufacturer and the retailer are independent entities with an adversarial relationship seeking to maximize their own benefit independently. In this 'decentralized' structure, the manufacturer sets the wholesale price to be charged to the retailer for their products, and the retailer makes its assortment decisions accordingly. Not to create any market segmentation and to isolate the problem from market dynamics, we assume that the retail price of products is fixed in both sales channels. Naturally, the retailer chooses an assortment that maximizes its own expected profit. Due to the interaction of the assortment decision with the purchase and keep-or-return decisions as explained above, the decision made by the retailer affects the expected profit of the manufacturer (resulting from purchases and returns). Consequently, the retailer's assortment decision may not yield the highest possible expected profit for the manufacturer or the whole system. Therefore, the assortment decision under this decentralized setting can lead to inefficiencies in RSC operations. As a benchmark that yields the maximum possible expected profit for the whole system, we consider a centralized RSC in which a central authority manages both sales channels and makes the assortment decision to maximize the total expected profit of the system (including the retailer's and the manufacturer's expected profits) (Chaharsooghi and Heydari 2010). The only decision to make in the centralized structure is the assortment decision for the physical store, and the wholesale price is no more relevant. The aforementioned inefficiency in the decentralized system can be eliminated if the retailer could be incentivized to choose the optimal assortment of the centralized system.

In this paper, we propose a contracting mechanism that can eliminate the inefficiency of the decentralized structure and yield a win-win situation for the retailer and the manufacturer. In specific, we propose a "*scope contract*" designed by the manufacturer that incentivizes the retailer to choose the optimal assortment of the centralized setting. In a *scope contract*, the manufacturer offers discounts on the wholesale price of the products based on a given scope of their attribute levels. The scope and the discount rates defined in this contract lead the retailer to order the optimal assortment of the centralized structure and yet assure that the retailer is better off compared to any other assortment decision (including that of the decentralized setting). The designed contract also assures that the manufacturer increases its expected profit compared to the original decentralized adversarial setting. Scope contracts are commonly used in practice in traditional supply chain settings. For example, a bearing manufacturer typically produces thousands of different bearings supplied to spare parts resellers. In order to increase the sales volume with a buyer (reseller), the manufacturer (as the supplier of the product) can design an "economies of scope" contract by offering discounts to the buyer when a higher variety of products are purchased in one transaction. In this way, the manufacturer can guarantee a higher volume of business with this buyer and retain them to seek other suppliers for different varieties.

Our main contribution to the literature in this paper is to propose a coordination mechanism through *scope contract*s for omni-channel RSCs. To accomplish this, we first propose closed-form expressions to characterize the assortment and wholesale price decisions under decentralized and centralized settings. We show that in a specific case where product returns are not allowed when the utility of products are overvalued (undervalued), the optimal assortment under the decentralized structure is a trade-off between showcasing the highest utility products and showcasing the highest possible (the most limited) variety of the overvalued (undervalued) attribute levels; whereas under the centralized structure, the priority is to showcase the most limited (the highest possible) variety of the overvalued (undervalued) attribute levels. In the general case where product returns are allowed, the optimal assortments are more complex because the marginal profits obtained by selling products can be different. This being said, we observe substantial differences in optimal decentralized and centralized assortments as well. By using these results, we show that *scope contract*s are instrumental in coordinating the RSC by determining a set of discount rates on the wholesale price of all the products that contain a given "scope" of attribute levels, which are the levels that exist in the optimal assortment of the centralized setting. We also show that the profit allocation between the two parties under this mechanism depends on the discount rate set in the contract and that the retailer receives a greater share of the additional profit under higher discount rates, and vice versa. Moreover, although the discount rates are not necessarily equal for all products, we show that a single parameter *scope contract* in which the discount rates are equal for the entire desired scope defined in the contract is equivalent to the original contract and guarantees the same profit allocation between the manufacturer and the retailer.

The remainder of this paper is organized as follows. In Section 3.2, we review the relevant literature. In Section 3.3, we propose the utility model and study customers' purchase and keep-or-return decisions. In Section 3.4, the dynamics of the decentralized and centralized decision settings are discussed and modeled. In Section 3.5, we characterize the optimal decisions under each setting. In Section 3.6, we propose our coordination mechanism and show that it is capable of fully coordinating the RSC. Finally, Section 3.7 is for numerical demonstration, and we conclude the paper in Section 3.8.

3.2 Literature Review

The primary step in assortment planning is to understand customers' decision-making behavior and to model it. This is because, after all, the goal is to optimize the profit obtained via selling products to customers. For this, usually 'consumer choice models' from the interface of operations management and marketing literature are used. The most commonly used choice model, which we also use in this study, is the multinomial logit (MNL) model. In this model, each customer assigns a utility to each product as its attractiveness. The assigned utility consists of a deterministic and a stochastic part, where the stochastic part is modeled as an error term with the Gumbel distribution. Then, the choice probability of each product is determined using properties of this distribution (Ben-Akiva et al. 1985, Anderson et al. 1992). Other choice models used in the literature include nested logit (Gallego and Topaloglu 2014, Davis et al. 2013), d-level nested logit (Li et al. 2015), mixed logit (Rusmevichientong et al. 2014), and locational choice (Gaur and Honhon 2006) models. Blanchet et al. (2016) develop a Markov chain model which is shown equivalent to MNL and a good approximation to other models.

Assortment planning in single-channel retailing where a retailer decides on a selection of products for showcasing in its store has received considerable attention in the literature. See (Kök et al. 2015) for a literature review on this research stream. However, assortment planning in omni-channel retailing is a relatively new topic, albeit it has received attention recently. Yrjölä et al. (2018) discuss that each channel can have a different assortment in an omni-channel system, and also different from that in single-channel systems. Ye et al. (2018) discovers barriers in transitioning from single-channel retailing to omni-channel, which includes assortment planning decisions. There are also empirical and discussion-based studies on omni-channel assortment planning (Gallino and Moreno 2018, Bell et al. 2018, 2014). Rooderkerk and Kök (2019) provide a literature review on this topic and discuss that coordination of different aspects of assortment decisions across different channels is inevitable for a convenient experience for customers.

To the best of our knowledge, Dzyabura and Jagabathula (2018) are the first to propose an optimization model for assortment planning in omni-channel retailing. Lo and Topaloglu (2022) consider a similar setting where they study omni-channel assortment with a features tree that represents product features. These studies consider products in terms of their attributes and in a centralized setting with fixed online and offline customer segments (i.e., no channel substitution by customers). Hense and Hübner (2022) study omni-channel assortment planning and corresponding inventory for online and physical stores with stochastic and independent demand models. They consider in-channel and cross-channel substitutions by customers and model the problem as an integer program. Schäfer et al. (2023) discuss that the effects of assortment on demand in brick-and-mortar stores, web-shops, and across channels can significantly influence retailers' profitability. They come up with an assortment planning model to capture these effects. Note that all the mentioned studies investigate omni-channel assortment planning in a centralized setting and none of them allow for product returns.

Product returns are a norm in modern retailing and for customer satisfaction. However, returns are usually modeled independently of assortment decisions as they are operational, but assortment planning is a strategic decision (Stock et al. 2006, Olavson and Fry 2006). Alptekinoğlu and Grasas (2014) model product returns along with assortment planning for a retailer operating a single-sales channel. To our knowledge, our first research stream in chapter 2 is the first to include product returns in omni-channel assortment planning. To indicate the importance of information provided by showcasing products, we consider a disappointment/gratification factor for products available only online and may be purchased with inaccurate utility assessments. The same approach is used in our study.

Coordination contracts are used in SCs as mechanisms for coordinating the business relationship between two or more independent parties. Once there is a conflict between the benefits of parties, these mechanisms can be designed effectively to provide a framework for cooperation in a way that all parties are better off. Coordination contracts are extensively discussed in the SC literature. See Govindan et al. (2013) for a comprehensive review of this literature. The classic coordination contracts that set the precedent for other proposed contracts include the wholesale price (Lariviere and Porteus 2001), the two-part tariff (Laffont and Tirole 1993, Rey and Vergé 2004), the buyback (Padmanabhan and Png 1997, Kanda et al. 2008) and similarly, the backup (Eppen and Iyer 1997, Kobbacy et al. 2011), the revenue-sharing (Cachon and Lariviere 2005) and the two-way revenue-sharing (for dual-channel SCs) (Xu et al. 2014), the sales rebate (Taylor 2002), and the quantity discount (Rubin et al. 1983, Shi and Su 2004) contracts. The design of coordination contracts is a fast-evolving topic and many studies have been carried out in the literature that are built upon these basic contracts. For example, contracts in multi-channel SCs and those under information asymmetry (Zhan et al. 2019, Vosooghidizaji et al. 2020, Aslani and Heydari 2019, De Giovanni 2017, Li et al. 2012). Most of these contracts provide wholesale price discounts or other incentives to decrease the risk and motivate the retailer to place larger order sizes in an SC with one product. However, in SCs with multiple products where assortment planning is a decision, these classic contracts are no longer instrumental.

The coordination of SCs where assortment planning is the decision to make is rarely addressed in the literature. Cachon and Kök (2007) study assortment decisions of several product categories (considered as different retailers) where each customer purchases from multiple categories. They show that independently made assortment decisions by retailers lead to inefficiency. They propose a basket profit metric which, although does not fully coordinate the system, achieves near-centralized profitability. Aydın and Hausman (2009) consider a single channel RSC with a manufacturer and a retailer where the retailer selects a subset of the manufacturer's products for showcasing in its store. To achieve coordination, the manufacturer considers a prepayment fee for every product offered by the retailer in excess of a certain target level. Note that both these works coordinate the assortment decisions in a single-channel RSC. Consequently, they measure the impact of their mechanisms on the manufacturer only through wholesale price payments it receives from the retailer; while the retailer's showcase as a means of providing information to customers is overlooked.

The *scope contract* for coordination that we introduce in this study is essentially originated from the concept of "economies of scope". Panzar and Willig (1975) and Teece (1980) define economies of scope as when the cost of joint production of two or more outputs (e.g., products) is less than the cost of producing each output separately. This is different from economies of scale when producing more outputs per setup is less costly. As per the definition, we can suppose that the ”diseconomies of scope” exist when the cost of joint production of two or more outputs is greater than that of producing each output separately. In this paper, by transferring the concepts of economies and dis-economies of scope to the variety of attribute levels in the assortment that the retailer selects, we devise the *scope contract* that coordinates the RSC.

3.3 Modelling Approach

In this section, we first introduce the utility model that we use for the products sold in the physical store and the online channel. Then, we develop models for customers' purchase and keep-or-return decision. We define the notation whenever introduced but we also provide all the notation in Table 3.1 for reference.

3.3.1 Utility Model

Consider a product with non-digital attributes. Let A be the set of attributes of this product, $K = |A|$ be the total number of attributes, $k \in A$ be a specific attribute, and $L(k)$ be the set of all levels of attribute k. For example, $A = \{color, material, style\}$ can be the set of attributes of a product with $K = 3$ and $L(k) = \{\text{black, blue, silver}\}$ can be the set of levels for attribute $k =$ "*color*". One level from each attribute will constitute a unique product and all possible such combinations will form the universal set of products X, where $N = |X| = \prod_{\{k \in A\}} |L(k)|$.

We adopt the utility model of Dzyabura and Jagabathula (2018) in which it is assumed that the product utility is the summation of the part-worth utilities of its attribute levels. Let $\widetilde{u}_{k,l(k)}$ be the part-worth utility of level $l(k) \in L(k)$ of attribute $k \in A$ assessed through physical encounter; touching, seeing, or trying out. Then, the 'attribute utility' of a product $x \in X$ (the utility of the product that is associated only to the attributes of the product) can be written as

$$\widetilde{U}_x = \sum_{k \in K} \sum_{l(k) \in L(k)} \widetilde{u}_{k,l(k)} \cdot 1_{\{x_k = l(k)\}} \tag{3.1}$$

where x_k is the level of attribute k in product x and $1_{\{x_k = l(k)\}}$ is an indicator function which is equal to 1 if the level of attribute k in product x is equal to $l(k)$, and 0 otherwise. In addition to the attribute utility given in (3.1), customers can also factor in the price of the product, π_x, the disutility of the return process (i.e., how difficult it would be to return the product), r, and the refundable portion of the price upon return, γ while deciding whether to purchase the product x or not (Alptekinoğlu and Grasas 2014). Consequently, the 'product utility' of $x \in X$ becomes

Table 3.1: Summary of Notations

Notation	Definition
A	Set of all attributes in the product type
C	Capacity of the physical store
$d_{k,l(k)}$	Inaccuracy in assessing level $l(k)$ of attribute k online
$D_{x\|M}$	Inaccuracy in assessing product x online, given M
k	Index for attributes
K	Total number of attributes
$l(k)$	Level index in attribute k
$L(k)$	Set of all possible levels for attribute k
M	Decision variable; assortment in the physical store
N	Total number of products
$P^f_{x\|M}$	Purchase probability for product $x \in M$, given M
$P^n_{x\|M}$	Purchase probability for product $x \in X \setminus M$, given M
$\mathcal{P}^r_x$	Marginal profit obtained by a product x sold in the physical store
$\mathcal{P}^m_x$	Marginal profit obtained by a product x sold in the online website
Q_C	Set of potential optimal assortments in capacity C
Q''_C	Set of potential optimal selections of levels of attribute $\Bbbk$ in capacity C
r	Disutility of the return process ($r < 0$)
R^f_x	Return probability for product $x \in M$
$R^n_{x\|M}$	Return probability for product $x \in X \setminus M$, given M
$S(k)$	Set of levels of attribute k showcased
$\widetilde{u}_{k,l(k)}$	Part-worth utility of level $l(k)$ of attribute k
$\overline{U}_{xM}$	Utility of a product $x \in X$
U_x	Effective utility of product x physically assessed
$U^n_{x\|M}$	Effective utility of product x assessed online, given M
d_k	Inaccuracy in assessing levels of attribute $k, \forall k \in A$ online
$\Bbbk$	The attribute considered to be inaccurately assessed ($d_{\Bbbk} \neq 0$)
v	Value of selling a returned product in a second market
w_x	Wholesale price of product x
x_k	Level of attribute k in product x
X	Universal set of all products
α_x	Discount ratio on the wholesale price of product $x, \forall x \in M^*_{\mathcal{C}}$ in *scope contract*
β_1	Price sensitivity of utility ($\beta_1 \leq 0$)
β_2	Refund sensitivity of utility ($\beta_2 \leq 0$)
β_3	Combined price and refund sensitivities of utility ($\beta_3 \leq 0$)
γ	Fraction of money refunded upon return ($0 \leq \gamma \leq 1$)
ε_x	Error term in the utility assessment of product x
ϕ	Disutility of return sensitivity in utility at the time of purchase ($\phi \geq 0$)
μ	Homoscedasticty of the population under study in purchase decision
μ'	Homoscedasticty of the population under study in keep-or-return decision
ω	Disappointment/gratification sensitivity of inaccuracy in utility ($\omega \geq 0$)
π_x	Retail price of product x
$\pi_{k,l(k)}$	Part-worth retail price of level $l(k)$ of attribute k
$\Pi^T_{\mathcal{C}}$	Total RSC's expected profit under the Centralized/Decentralized structure
$\Pi^m_{\mathcal{D}}$	Manufacturer's expected profit under the Decentralized structure
$\Pi^r_{\mathcal{D}}$	Retailer's expected profit under the decentralised structure
Ω	Retailer's opportunity cost
$[n]$	The product that has the nth highest product utility
$[n]_{k:i}$	The nth highest utility product out of the ones of which kth attribute's level is i

$$\overline{U}_x = \widetilde{U}_x + \beta_1 \pi_x + \beta_2 (1-\gamma)\pi_x + \phi r, \tag{3.2}$$

where β_1 is the price sensitivity of utility, β_2 is the sensitivity of utility to the non-refundable fraction of the price, and ϕ is the sensitivity of utility to the difficulty (disutility) of the return process. Note that $\beta_1 \leq 0$ to reflect an inverse effect with the price of the product, $\beta_2 \leq 0$ to reflect the negative impact of a higher non-refundable portion of the price, and $\phi \geq 0$ since the disutility of the return process is defined as $r \leq 0$. The product utility given by (3.2) is an expected value for an average customer, but because of differences in individual preferences, each customer can have a different realization of the product utility. Therefore the utility of a product x that is showcased (available in the physical store) can be written as

$$U_x = \overline{U}_x + \varepsilon_x \tag{3.3}$$

where ε_x is the error term accounting for the unobserved components that are not caught by the utility model. These idiosyncratic error terms in the utility of products are assumed to be independent and identically distributed (i.i.d) of a Gumbel distribution with mean zero and scale parameter $1/\mu$ – a standard assumption in MNL models, see Kök et al. (2015), Anderson et al. (1992). Here, μ is a positive scalar representing the homoscedasticity of the population such that a larger value of μ reflects a more heterogeneous population.

In the omnichannel retail setting that we consider, the customers who end up purchasing from the online channel also visit the physical store to collect information about the products. Hence, those customers would have an accurate knowledge of the part-worth utilities, $\widetilde{u}_{k,l(k)}$, if the level $l(k)$ of attribute k is present in one of the showcased products (available in the store) but would have an inaccurate knowledge of the part-worth utilities of the attribute levels that are not showcased. Therefore, the utility of the products that are available only online depends on the assortment decision implemented in the physical store. Let $d_{k,l(k)}$ be the magnitude of this inaccuracy in assessing the level $l(k)$ of attribute k if that level is not showcased. Then, the part-worth utility of a level $l(k)$ of attribute k which is not showcased is given by

$$\widetilde{u}_{k,l(k)} + d_{k,l(k)} \tag{3.4}$$

where $d_{k,l(k)} > 0$ if the customers overvalue the level $l(k)$ of attribute k, $\forall l(k) \in K, \forall k \in A$, in the online channel, and $d_{k,l(k)} < 0$ if they undervalue it.

Let $M \subseteq X$ be an assortment of products selected for showcasing in the physical store. Let $S(k|M)$ be the set of all levels of attribute k that are present in at least one of the products available in M, i.e., $S(k|M) = \bigcup_{x \in M}\{x_k\}$, and let $S'(k|M) = L(k) \setminus S(k|M)$ be the set of levels of attribute k that are not available in any of the products in M. In other words, $S(k|M)$ is the set of levels of k that are showcased and $S'(k|M)$ is the set of levels of it that are not showcased. Note that a product itself may not be showcased in the physical store, but some of its attribute levels can be present in $S(k|M), \forall k \in A$, which means that some of this product's attribute levels are accurately assessed. Hence, the utility of an online-only product consists of the sum of the part-worth utilities of the attribute levels that are present in set $S(k|M), \forall k \in A$ and those that are in set $S'(k|M), \forall k \in A$ with an inaccuracy adjustment as stated in (3.4). Consequently, we can write the utility of a product $y \in X \setminus M$ for any given assortment M as

$$\begin{aligned} U_{y|M} &= \sum_{k \in K} \left[\sum_{l(k) \in S(k|M)} \widetilde{u}_{k,l(k)} \cdot 1_{\{x_k = l(k)\}} + \sum_{l(k) \in S'(k|M)} \left(\widetilde{u}_{k,l(k)} + d_{k,l(k)} \right) \cdot 1_{\{y_k = l(k)\}} \right] \\ &\quad + \beta_1 \pi_x + \beta_2 (1 - \gamma) + \phi r + \varepsilon_y \end{aligned} \tag{3.5}$$

If we let

$$D_{y|M} = \sum_{k \in K} \sum_{l(k) \in S'(k|M)} d_{k,l(k)} \cdot 1_{\{y_k = l(k)\}}$$

then

$$\begin{aligned} U_{y|M} &= \sum_{k \in K} \left[\sum_{l(k) \in S(k|M)} \widetilde{u}_{k,l(k)} \cdot 1_{\{x_k = l(k)\}} + \sum_{l(k) \in S'(k|M)} \widetilde{u}_{k,l(k)} \cdot 1_{\{y_k = l(k)\}} \right. \\ &\quad \left. + \sum_{l(k) \in S'(k|M)} d_{k,l(k)} \cdot 1_{\{y_k = l(k)\}} \right] + \beta_1 \pi_x + \beta_2 (1 - \gamma) + \phi r + \varepsilon_y \\ &= \sum_{k \in K} \sum_{l(k) \in L(k)} \widetilde{u}_{k,l(k)} \cdot 1_{\{x_k = l(k)\}} + D_{y|M} + \beta_1 \pi_x + \beta_2 (1 - \gamma) + \phi r + \varepsilon_y \\ &= \overline{U}_y + D_{y|M} + \varepsilon_y \end{aligned} \tag{3.6}$$

The utility functions U_x given by (3.3) and $U_{y|M}$ given by (3.6) represent the utility of the products that are showcased in the physical store and that are available online only, respectively. By the definition of the set $S'(k|M)$ and for any product $x \in M$, we note that $1_{\{x_k = l(k)\}} = 0$ for all $l(k) \in S'(k|M)$. Hence, $D_{x|M} = 0$ and

$$U_{x|M} = \overline{U}_x + D_{x|M} + \varepsilon_x = \overline{U}_x + \varepsilon_x = U_x$$

for any product $x \in M$. For brevity, we use the notation $U_{x|M}$ to denote the utility of any product $x \in X$

in the remainder of this section.

3.3.2 Customers' Purchasing Experience

Purchasing experience of customers consists of two subsequent stages. In the first stage, customers decide whether to purchase or not based on their utility assessment of products across both channels. In the second stage, customers decide whether to keep or return their purchased items after using or experiencing the product post-purchase. Customers' decisions are usually modeled using consumer choice models from the interface of operations management and marketing literature. In these models, customers' decisions are represented by probability measures that reflect their likelihood of that decision. In this study, we use the multinomial logit (MNL) choice model, as one of the most commonly used choice models in the literature (Ben-Akiva et al. 1985, Anderson et al. 1992). Since customers' purchase and keep-or-return decisions subsequently occur at two stages, the problem consists of two MNL choice models each corresponding to one of these stages.

Stage 1: Customers' Purchase Decisions

A customer purchases the product that has the highest utility for them across both channels if this highest value is positive. If that product is available in the physical store, then the customer purchases it from the store, otherwise orders from the online sales website. The probability that a product x is preferred over any other product for purchasing when the assortment selected in the physical store is M can be written as

$$Pr\{U_{x|M} > U_{y|M}, U_{x|M} > 0\} \quad \forall x, y \in X. \tag{3.7}$$

By using (3.6), (3.7) can be rewritten and reorganized as

$$\begin{aligned} Pr\{U_{x|M} > U_{y|M}, U_{x|M} > 0\} &= Pr\{\overline{U}_x + D_{x|M} + \varepsilon_x > \overline{U}_y + D_{y|M} + \varepsilon_y, \overline{U}_x + D_{x|M} + \varepsilon_x > 0\} \\ &= Pr\{\varepsilon_y - \varepsilon_x < \overline{U}_x + D_{x|M} - \overline{U}_y - D_{y|M}, \overline{U}_x + D_{x|M} + \varepsilon_x > 0\} \end{aligned}$$

Note that if ε_x and ε_y are i.i.d. of Gumbel distribution with zero mean and scale parameter $1/\mu$, then $\varepsilon_y - \varepsilon_x$ follows a Logistics distribution with zero mean and scale parameter $1/\mu$ (Ben-Akiva et al. 1985, Anderson et al. 1992). To estimate the profits of the retailer and the manufacturer in the problem under concern, we need to differentiate the purchase probability of a product sold by the retailer, denoted by $P^r_{x|M}$ for a product $x \in M \cup \{0\}$ and the purchase probability of a product sold online by the manufacturer, denoted by

$P^m_{x|M}$ for a product $x \in X \setminus M \cup \{0\}$, where the set $\{0\}$ corresponds to the no purchase probability. Using the properties of the cumulative distribution function of the Logistics distribution and noting that $D_{x|M} = 0$ for a product $x \in M$, we have

$$
\begin{aligned}
P^r_{x|M} &= \frac{e^{(\overline{U}_x + D_{x|M})/\mu}}{1 + \sum_{y \in X} e^{(\overline{U}_y + D_{y|M})/\mu}} \\
&= \frac{e^{\overline{U}_x/\mu}}{1 + \sum_{y \in M} e^{\overline{U}_y/\mu} + \sum_{y \in X \setminus M} e^{(\overline{U}_y + D_{y|M})/\mu}} \quad \forall x \in M \cup \{0\} \\
P^m_{x|M} &= \frac{e^{(\overline{U}_x + D_{x|M})/\mu}}{1 + \sum_{y \in X} e^{(\overline{U}_y + D_{y|M})/\mu}} \\
&= \frac{e^{(\overline{U}_x + D_{x|M})/\mu}}{1 + \sum_{y \in M} e^{\overline{U}_y/\mu} + \sum_{y \in X \setminus M} e^{(\overline{U}_y + D_{y|M})/\mu}} \quad \forall x \in X \setminus M \cup \{0\}.
\end{aligned}
$$

Stage 2: Customers' Keep-or-Return Decisions

After purchasing a product, each customer can decide whether to keep or return their product. A product that is purchased from the physical store has been purchased with accurate utility assessment at the time of purchase (i.e., $U_x = \overline{U}_x + \varepsilon_x$). The utility of keeping the product includes the utility that the customer obtains from the product itself whereas the utility of returning the product includes the disutility of the return process and the effect of the non-refundable fraction of the price. Hence, a product x purchased from the retailer's physical store will be returned if the utility of returning it is greater than the utility of keeping it, which is given by the following condition:

$$\widetilde{U}_x + \beta_1 \pi_x + \varepsilon_x + \varepsilon_{\text{keep}|x} < r + \beta_1(1-\gamma)\pi_x + \varepsilon_x + \varepsilon_{\text{return}|x}, \tag{3.8}$$

where the left-hand side is the utility of keeping the product and the right-hand side is the utility of returning it. It should be noted in the keeping utility that the refund sensitivity and disutility of return sensitivity are no longer included in the utility, and the only components included are the part-worth utility of attribute levels in the product and the price sensitivity. Here, $\varepsilon_{\text{keep}|x}$ and $\varepsilon_{\text{return}|x}$ are error terms to address the uncertainty in keeping and returning utilities, respectively, and we assume they are i.i.d of the Gumbel distribution with zero mean and scale parameter of μ'.

A product x that is purchased from the online sales website may be purchased with inaccurate utility assessment at the time of purchase (i.e., $U_{x|M} = \overline{U}_x + D_{x|M} + \varepsilon_x$). In this situation, if the product has been purchased with overvalued assessment ($D_{x|M} > 0$) at the time of purchase, when it is delivered, the customer will be "disappointed" with their purchase, because the product is not as desired as expected, and the customer will be more likely to return the product. But, if the product was purchased with undervalued assessment ($D_{x|M} < 0$), the customer will be "gratified" because the product turns out better than expected, and the customer will be more likely to keep the product. Hence, the utility of keeping the product includes the utility that the customer obtains from the product itself and the disappointment/gratification component as well. The utility of returning the product will be the same as a product purchased from the physical store. Therefore, a product x purchased from the manufacturer will be returned if

$$\widetilde{U}_x + \beta_1 \pi_x - \omega D_{x|M} + \varepsilon_x + \varepsilon_{\text{keep}|x} < r + \beta_1(1-\gamma)\pi + \varepsilon_x + \varepsilon_{\text{return}|x}. \tag{3.9}$$

In (3.9), ω (≥ 0) reflects the disappointment/gratification sensitivity in the utility of the product. Similar to the physical store case, $\varepsilon_{\text{keep}|x}$ and $\varepsilon_{\text{return}|x}$ are error terms to address the uncertainty in keeping and returning utilities, respectively, which are assumed i.i.d of the Gumbel distribution with zero mean and scale parameter of μ'.

Shulman et al. (2009) model the product returns by considering the difference between expected and actual utility assessments of a product with no uncertainty in utility values. They include the disutility of return and the non-refundable part of the product price in their return function. Also, Alptekinoğlu and Grasas (2014) consider the same factors in tackling the return decision. They use a nested logit model in which uncertainties in making the purchase and keep-or-return decisions are considered. Along these lines of literature, we develop our keep-or-return model for the purchased products.

Let R^r_x indicate the return probability of product $x \in M$ that is purchased from the retailer. Using (3.8), we can write

$$\begin{aligned} R^r_x &= Pr\{\widetilde{U}_x + \beta_1\pi_x + \varepsilon_x + \varepsilon_{\text{keep}|x} < r + \beta_1(1-\gamma)\pi_x + \varepsilon_x + \varepsilon_{\text{return}|x}\} \\ &= Pr\{\varepsilon_{\text{keep}|x} - \varepsilon_{\text{return}|x} < -\widetilde{U}_x + r - \beta_1\gamma\pi_x\} \\ &= \frac{1}{1 + e^{(\widetilde{U}_x - r + \beta_1\gamma\pi_x)/\mu'}}. \end{aligned} \tag{3.10}$$

Similarly, let $R^m_{x|M}$ indicate the return probability of product $x \in X \setminus M$ that is purchased from the online channel. By using (3.9), we can write

$$
\begin{aligned}
R^m_{x|M} &= Pr\{\widetilde{U}_x + \beta_1\pi_x - \omega D_{x|M} + \varepsilon_x + \varepsilon_{\text{keep}|x} < r + \beta_1(1-\gamma)\pi_x + \varepsilon_x + \varepsilon_{\text{return}|x}\} \\
&= Pr\{\varepsilon_{\text{keep}|x} - \varepsilon_{\text{return}|x} < -\widetilde{U}_x + \omega D_{x|M} + r - \beta_1\gamma\pi_x\} \\
&= \frac{1}{1 + e^{(\widetilde{U}_x - \omega D_{x|M} - r + \beta_1\gamma\pi_x)/\mu'}}. \qquad (3.11)
\end{aligned}
$$

Let K^r_x be the keep probability of product $x \in M$ that it is purchased from the retailer, and $K^m_{x|M}$ be this probability of product $x \in X \setminus M$ that it is purchased from the online sales channel of the manufacturer. Then,

$$
\begin{aligned}
K^r_x = 1 - R^r_x &= \frac{1}{1 + e^{(-\widetilde{U}_x + r - \beta_1\gamma\pi_x)/\mu'}}, \\
K^m_{x|M} = 1 - R^m_{x|M} &= \frac{1}{1 + e^{(-\widetilde{U}_x + \omega D_{x|M} + r - \beta_1\gamma\pi_x)/\mu'}}. \qquad (3.12)
\end{aligned}
$$

3.4 Decision Making Processes in Retail Supply Chains

In this section, we summarize the decision process and the dynamics in a retail supply chain when it is operated in a decentralized (Section 3.4.1) or centralized (Section 3.4.2) fashion. In a decentralized setting, the retailer and the manufacturer act independently by aiming to maximize their own profit. The manufacturer sets the wholesale price of its products to be offered to the retailer and the retailer makes the assortment decision for its physical store based on the quoted wholesale prices. In a centralized setting, the assortment decision at the physical store is made by a central authority with the aim of maximizing the overall profit of the chain. Assortment is the only decision to be made; the wholesale price is not relevant in this setting.

3.4.1 Decentralized Setting

To find the optimal assortment and wholesale price values, we use a Stackelberg game, where the manufacturer is the leader and the retailer is the follower. In specific, the manufacturer sets the wholesale price for their products and the retailer decides its assortment according to this price. Within the game dynamics, the manufacturer anticipates the retailer's assortment decision for a given wholesale price, w, and then, sets the optimal wholesale price (w^*) that maximizes its own profit. We first assume that the wholesale and the retail prices of all products are equal; so that $w = w_x$ and $\pi = \pi_x, \forall x \in X$. In Section 3.6, we relax this assumption and let the wholesale prices be different and develop a coordination mechanism for the RSC.

Under the decentralized setting, the sequence of events is as follows:

1. The manufacturer decides the wholesale price of its products w.
2. The retailer decides the assortment of products M in its store, based on the value of w.
3. Customers visit the online website and the physical store and make purchase decisions.
4. Customers decide whether to keep or return their purchased products.
5. Returned products will be sold in a secondary market for a reduced value of v by the channel it was purchased from.

We first analyze the retailer's problem to find the optimal assortment M for a given w. The retailer's expected profit function can be written as

$$\Pi^r_{\mathcal{D}}(M|w) = \sum_{x \in M} \left[(\pi - w) P^r_{x|M} K^r_x + ((1-\gamma)\pi + v - w) \, P^r_{x|M} R^r_x \right] \tag{3.13}$$

where the first term refers to the expected profit obtained from the products that are sold at the retailer's physical store and are kept post-purchase, and the second term refers to the profit obtained from the products that are sold but returned. In both terms in (3.13), the value of the wholesale price is deducted from the marginal profit gained from selling a product, as the retailer needs to fulfill the customers' purchase by ordering them from the manufacturer for the unit cost of w. Noting that $K^r_x = 1 - R^r_x$ and defining $\mathcal{P}^r_x = (1 - R^r_x)\pi + R^r_x \left[(1-\gamma)\pi + v\right] = \pi - R^r_x(\gamma\pi - v), \forall x \in M$ in (3.13) as the marginal profit obtained from a product sold in the physical store, the retailer's expected profit can be re-written as

$$\Pi^r_{\mathcal{D}}(M|w) = \sum_{x \in M} P^r_{x|M} \left(\mathcal{P}^r_x - w \right). \tag{3.14}$$

Consequently, the retailer solves

$$\begin{aligned} &\max_{M \subset X} \Pi^r_{\mathcal{D}}(M|w) \\ &\text{s. to} \quad |M| \le C \end{aligned} \tag{3.15}$$

to find the optimal M where C is the showcase capacity at the retailer. Let $M^*(w)$ be the assortment decision that solves (3.15).

Anticipating the retailer's assortment decision for a given w, $M^*(w)$, the manufacturer can estimate their expected profit for any given value of w, $\Pi^m_{\mathcal{D}}$, as

$$\begin{aligned}\Pi^m_{\mathcal{D}}(w) \quad = \quad & \sum_{x\in X\setminus M^*(w)} \pi P^m_{x|M^*(w)} K^m_{x|M^*(w)} + \sum_{x\in X\setminus M^*(w)} \left[(1-\gamma)\pi + v\right] P^m_{x|M^*(w)} R^n_{x|M^*(w)} \\ & + \sum_{x\in M^*(w)} w P^r_{x|M^*(w)} \end{aligned} \tag{3.16}$$

where the first term refers to the profit obtained from products that are purchased from the manufacturer's online channel and kept by the customers post-purchase, the second term is the profit obtained from products that are purchased from the online channel and returned, and the last term is the profit obtained from selling the products to the retailer for the wholesale price, as the retailer orders products to fulfill the purchases by customers in the physical store. Noting that $K^m_{x|M^*(w)} = 1 - R^m_{x|M^*(w)}$ and defining $\mathcal{P}^m_{x|M^*(w)} = (1-R^m_{x|M^*(w)})\pi + R^m_{x|M^*(w)}\left[(1-\gamma)\pi + v\right] = \pi - R^m_{x|M^*(w)}(\gamma\pi - v), \forall x \in X \setminus M^*(w)$ in (3.16) as the marginal profit obtained from a product sold in the online sales website, the manufacturer's expected profit can be rewritten as

$$\Pi^m_{\mathcal{D}}(w) = \sum_{x\in X\setminus M^*(w)} P^m_{x|M^*(w)} \mathcal{P}^m_{x|M^*(w)} + \sum_{x\in M^*(w)} w P^r_{x|M^*(w)}. \tag{3.17}$$

Consequently, the manufacturer solves the following problem to find the optimal wholesale price, w:

$$\max_w \Pi^m_{\mathcal{D}}(w). \tag{3.18}$$

In the following, we propose two lemmas that characterize the set of potential optimal assortments from different perspectives. Let M^* be the optimal assortment of a retailer with showcase capacity C in a decentralized setting. To avoid any speculative problem parameters, we assume that $\gamma > v/\pi$ so that a "sold and not returned" product is more profitable than a "sold and returned" product for the manufacturer and the retailer.

Lemma 3.1. *In the decentralized setting, M^* contains the highest utility product from each of the selected levels in $S(k|M^*)$, and the remaining $C - |S(k|M^*)|$ products in M^* are the ones with the highest utility values among all the remaining products consisting of the attributes in $S(k|M^*)$.*

Lemma 3.1 characterizes the products in an optimal assortment of the retailer in the decentralized setting. The retailer showcases the products with the highest utility values by picking at least the highest utility product from each of the selected levels. As we discuss later in Sections 3.5.1 and 3.5.2, the optimal variety of the products in the optimal assortment (number of distinct levels of an attribute to showcase) is a function of whether the hidden attributes are over- or under-valued.

Lemma 3.2. *For an attribute k, let $d_{k,l(k)} = d_k, \forall l(k) \in L(k)$. In the decentralized setting, suppose that the retailer prefers to showcase ζ levels of attribute k so that $|S(k|M^*)| = \zeta$. If product returns are not allowed, it is optimal for the retailer to select ζ levels of k with the highest $\widetilde{u}_{k,l(k)}$ values for any value of d_k. If product returns are allowed, the same result is also true for $d_k > 0$ and it is true for $d_k < 0$ only when $\zeta = 1$.*

Lemma 3.2 characterizes the levels of attribute $\Bbbk$ present in the optimal assortment if the optimal number of distinct attribute levels to showcase, ζ, is known. This result coupled with Lemma 3.1 will provide a basis for the analytical results of the decentralized setting under different scenarios.

3.4.2 Centralized Setting

In the centralized setting, the central authority manages the RSC so that the total expected profit is maximized. This function is the sum of the expected profit functions of the retailer and the manufacturer in (3.14) and (3.17):

$$\Pi^T_{\mathcal{C}}(M) = \sum_{x \in X \setminus M} P^m_{x|M} \mathcal{P}^m_{x|M} + \sum_{x \in M} P^r_{x|M} \mathcal{P}^r_x. \tag{3.19}$$

Hence, the central authority solves the following problem:

$$\begin{aligned} \max_{M \subset X} \ & \Pi^T_{\mathcal{C}}(M) \\ \text{s. to} \quad & |M| \leq C \end{aligned} . \tag{3.20}$$

Lemma 3.3 below provides a basis for choosing optimal assortments in the centralized setting.

Lemma 3.3. *Suppose that $d_{k,l(k)} = d_k, \forall k \in A, l(k) \in L(k)$. In the centralized setting, selecting a subset of levels of attributes that are inaccurately assessed in the online channel (i.e., attributes with $d_k \neq 0, \forall k \in A$) suffices to determine optimal assortments. Any arbitrary assortment representing the selected levels of such attributes is optimal.*

According to Lemma 3.3, the set of levels of inaccurately assessed attributes showcased in the physical store dictates the inaccuracy present in the online channel. Hence, once a level of an inaccurately assessed attribute is selected for showcasing, it does not matter which product that contains the chosen level is showcased, because once a product with this level is showcased, that attribute level will be accurately assessed in the online channel, too. This implies that although the optimal selection of levels of inaccurately assessed attributes can be unique, there may exist multiple optimal assortments of products under the centralized structure. Note that this result is true due to the nature of the centralized setting in which the aim is to maximize the overall total profit rather than the individual profits of the manufacturer and the retailer.

Lemma 3.4. *Under the centralized setting, suppose that $d_k = 0$ $\forall k \in A, l(k) \in L(k)$. Then any arbitrary assortment M is optimal.*

Lemma 3.4 refers to a specific case where there is no inaccuracy in the assessment of products and attribute levels in the online channel and hence, M does not provide any additional information. Given that the RSC is centralized, any arbitrary assortment at the physical store yields the same total expected profit.

3.5 Optimal Decisions

In this section, we investigate the optimal decisions of the RSC under both the decentralized and centralized settings. To accentuate the importance of considering product returns on profitability and show how they influence optimal assortment decisions, we study the RSC under two scenarios: when returns are not allowed (RNA), and when returns are allowed (RA). Note that RNA is a special case of the models given in Sections 3.4.1 and 3.4.2 and can be obtained by setting $\beta_2 = 0$ and $\phi = 0$ in the product utility function given in (3.2). Moreover, $R^m_{x|M} = R^r_x = 0$ and we have $\mathcal{P}^m_{x|M} = \mathcal{P}^r_x = \pi$ in the expected profit functions in (3.14), (3.17), and (3.19) when returns are not allowed. This means that the marginal profit obtained from selling a product in either of the channels is its retail price.

Under the RNA scenario, the focus should only be on increasing sales by choosing products with a higher purchase probability, which guarantees to maximize the overall profit in the physical store (the whole RSC) under the decentralized (centralized) setting. However, under the RA scenario, increasing sales does not necessarily result in higher profitability, because the marginal profit obtained from selling different products is influenced by potential product returns. Therefore, the assortment should be selected to compromise the revenues and losses from sales and returns in RA.

For the analysis of the problem environment, we assume that $d_{k,l(k)} = d_k, \forall l(k) \in L(k), \forall k \in A$, i.e., all levels of an attribute are equally inaccurately assessed in the online channel and there is only one attribute, say $\mathbb{k}$, that is inaccurately assessed so that $d_{\mathbb{k}} \neq 0, \forall l(\mathbb{k}) \in L(\mathbb{k})$ and $d_k = 0, \forall k \in A : k \neq \mathbb{k}$. The latter assumption is made to facilitate the analysis by containing the effect of the inaccuracy in levels of one attribute. We define $[n]$ as the product with the n^{th} highest utility among all the products in X. Given this, products can be denoted by $[1], [2], ..., [N]$, with $[1]$ and $[N]$ being the products having the highest and lowest $\overline{U}_x$ values, respectively. For example, [3] denotes the third-highest utility product. According to this, we let $[n]_k$ indicate the level of attribute k in the n^{th} highest utility product. For example $[n]_{\mathbb{k}} = 2$ means that the

n^{th} highest utility product consists of level 2 of attribute $\Bbbk$. We also define $[\lambda]_{\Bbbk:l(\Bbbk)}$ as the λ^{th} highest utility product that includes level $l(\Bbbk)$ of attribute $\Bbbk$. For example, $[4]_{\Bbbk:2}$ is the fourth highest utility product that includes level 2 of attribute $\Bbbk$. Finally, let the levels of an attribute $k \in A$ be numbered as $1, 2, ..., |L(k)|$, with 1 and $|L(k)|$ being the levels with the highest and lowest attribute utility values, $\widetilde{u}_{k,l(k)}, \forall l(k) \in k$, respectively.

3.5.1 Scenario RNA: When returns are not allowed

Decentralized Setting

Inaccuracy in the online assessment of attribute levels, either overvaluation or undervaluation, is a critical factor in determining the optimal assortment of products in the physical store. In either case, the retailer has a tendency to showcase higher utility products (cf. Lemmas 3.1 and 3.2). However, if the hidden attributes are overvalued (i.e., $d_{\Bbbk} > 0$), the retailer would also prefer to showcase a higher variety of these attribute levels because overvaluation leads to a larger utility assessment of the hidden (not-showcased) attribute levels, which means that the probability of sales for the online items with hidden attributes will be higher than normal. The retailer can avoid this by showcasing a higher variety and increase the sales in their store. Nevertheless, since showcasing an assortment with a high variety of levels of $\Bbbk$ may not necessarily result in showcasing the highest utility products, there is a trade-off between the two strategies (showcasing the highest utility vs a higher variety) to be resolved to identify the optimal assortment. For example if $C = 2$, the retailer can possibly select among a number of products. If the two highest utility products do not have the same levels of $\Bbbk$ (i.e., $[1]_{\Bbbk} \neq [2]_{\Bbbk}$), the optimal assortment includes these two products, as it results in showcasing both the highest utility products and the highest possible variety of $\Bbbk$. However, if $[1]_{\Bbbk} = [2]_{\Bbbk}$, the retailer should compare showcasing these two products as one option with showcasing product [1] and the highest utility product with the second highest part-worth utility level of $\Bbbk$ (i.e., $[1]_{\Bbbk:2}$) as an option in which two products with different levels of $\Bbbk$ are included. Similarly, when $C = 3$, if $[1]_{\Bbbk} \neq [2]_{\Bbbk} \neq [3]_{\Bbbk}$, the optimal assortment is to showcase these three products. However, if $[1]_{\Bbbk} = [2]_{\Bbbk} = [3]_{\Bbbk}$, then given Lemmas 3.1 and 3.2, the retailer should compare three potential assortments including {[1], [2], [3]}, {[1], [2], $[1]_{\Bbbk:2}$}, and {[1], $[1]_{\Bbbk:2}$, $[1]_{\Bbbk:3}$}. Also, if $[1]_{\Bbbk} = [2]_{\Bbbk} \neq [3]_{\Bbbk}$ (or $[1]_{\Bbbk} = [3]_{\Bbbk} \neq [2]_{\Bbbk}$), there are two potential assortments including {[1], [2], [3]} and {[1], [3], $[1]_{\Bbbk:3}$} (or {[1], [2], $[1]_{\Bbbk:3}$)}. Similarly, for greater values of C, all potential assortments given that the retailer prefers both a high variety of levels of $\Bbbk$ and the highest utility products should be compared to find the optimal assortment.

If the hidden attribute levels are undervalued (i.e., $d_{\Bbbk} < 0$), the retailer would prefer to select an assortment that reveals only a limited number of levels of $\Bbbk$, because undervaluation leads to lower utility assessments of the products with the hidden (not-showcased) attribute levels and as a result, the showcased products are more likely to be purchased. Since showcasing an assortment of products with a limited variety of levels of $\Bbbk$ may not necessarily result in showcasing the highest utility products, the trade-off between the benefits of these two strategies (showcasing higher utility products vs lower variety) should be resolved. For example when $C = 2$, if $[1]_{\Bbbk} = [2]_{\Bbbk}$, the optimal assortment includes these two products, as it results in showcasing both the highest utility products and the smallest possible variety of $\Bbbk$. However, if $[1]_{\Bbbk} \neq [2]_{\Bbbk}$, the retailer should compare showcasing these two products as one option with showcasing product [1] and the second highest utility product that has the same level of $\Bbbk$ as [1] (i.e., $[2]_{\Bbbk:1}$). When $C = 3$, if $[1]_{\Bbbk} = [2]_{\Bbbk} = [3]_{\Bbbk}$, the optimal assortment is to showcase these three products. However, if products $[1]_{\Bbbk} \neq [2]_{\Bbbk} \neq [3]_{\Bbbk}$, given Lemmas 3.1 and 3.2, the retailer should compare three potential assortments including {[1],[2], [3]}, {[1], [2], $[2]_{\Bbbk:1}$}, and {[1], $[2]_{\Bbbk:1}$, $[3]_{\Bbbk:1}$}. Also, if $[1]_{\Bbbk} = [2]_{\Bbbk} \neq [3]_{\Bbbk}$ (or $[1]_{\Bbbk} = [3]_{\Bbbk} \neq [2]_{\Bbbk}$), there are two potential assortments to compare including {[1],[2], [3]} and {[1], [2], $[3]_{\Bbbk:1}$} (or {[1], $[2]_{\Bbbk:1}$, $[3]_{\Bbbk:1}$}). Similarly, for any $C > 3$, all potential assortments given that the retailer prefers both a small variety of levels of $\Bbbk$ and the highest utility products should be compared to find the optimal assortment.

In what follows, we characterize the optimal assortment decision for the retailer under the decentralized setting and RNA scenario, which is mainly predicated on resolving the trade-off between utility and variety as explained above. We first define a feasibility set denoted by Q_C of which each element is a tuple of C products that can be showcased by the retailer (i.e., a feasible solution to problem (3.15)). This set is constructed according to the following procedure so that its elements can be an optimal solution based on Lemmas 3.1 and 3.2.

Procedure 1. Construction of Set Q_C

1. $Q_c = \varnothing$
2. For $\zeta = 1, 2, ..., \min\{C, |L(\Bbbk)|\}$
   (a) Create an empty tuple of size C denoted by $\mathcal{T}$
   (b) Fill the first ζ elements of $\mathcal{T}$ by the products $[1]_{\Bbbk:1}, [1]_{\Bbbk:2}, ..., [1]_{\Bbbk:\zeta}$
   (c) Fill the remaining $C-\zeta$ elements of $\mathcal{T}$ by that many products that have the highest product utilities out of the remaining products that contains one of attribute levels $1, 2, ..., \zeta$
   (d) $Q_C = Q_C \cup \mathcal{T}$

In the above procedure, ζ is the counter for possible number of distinct levels of attribute $\Bbbk$, corresponding to the possible variety of products in the selected assortment. For each possible value of ζ, step 2b picks the highest utility product from each attribute level selected. Note that since the attribute levels are numbered as 1, 2,... $|L(\Bbbk)|$ in the descending order of their part-worth utilities, the first ζ levels are guaranteed to have the highest part-worth utilities. Step 2c selects from the remaining products with one of the first ζ levels by picking the ones with the highest product utility values as much as the remaining capacity. This procedure guarantees that the set Q_C contains all possible feasible solutions that satisfy the properties laid out by Lemmas 3.1 and 3.2.

Proposition 3.1. *Suppose that the RSC has a decentralized setting and product returns are not allowed. Let Q_C be a set of feasible solutions constructed according to Procedure 1. For any value of $d_\Bbbk$, the retailer's optimal assortment is given by*

If $C = 1$: $\quad M^*_{\mathfrak{D}} = \{[1]\}, \quad S(k|M^*_{\mathfrak{D}}) = \{1\}, \forall k \in A$

If $2 \leq C \leq \prod_{k\in A, k\neq\Bbbk} |L(k)| \times (|L(\Bbbk)| - 1)$:

$$M^*_{\mathfrak{D}} = arg_{t_1\in Q_C}\{T_{t_1,t_2}e^{d_\Bbbk} > T^{'}_{t_1,t_2}\ \forall t_2 \in Q_C, t_2 \neq t_1\}, \quad S(k|M^*_{\mathfrak{D}}) = \bigcup_{x\in M^*_{\mathfrak{D}}}\{x_k\}, \forall k \in A$$

If $C > \prod_{k\in A, k\neq\Bbbk} |L(k)| \times (|L(\Bbbk)| - 1)$:

$$M^*_{\mathfrak{D}} = \{[1], [2], ..., [C]\}, \quad S(\Bbbk|M^*_{\mathfrak{D}}) = L(\Bbbk), \quad S(k|M^*_{\mathfrak{D}}) = \bigcup_{x\in M^*_{\mathfrak{D}}}\{x_k\}, \forall k \in A, k \neq \Bbbk,$$

where

$$T_{t_1,t_2} = (E + F)\textstyle\sum_{i\in t_1} e^{\overline{U}_i} - (E + H)\sum_{i\in t_2} e^{\overline{U}_i},$$
$$T'_{t_1,t_2} = (G + F)\textstyle\sum_{i\in t_2} e^{\overline{U}_i} - (G + H)\sum_{i\in t_1} e^{\overline{U}_i}.$$

$E = \sum_{i\in X, i_\Bbbk\notin S(\Bbbk|t_1)\cup S(\Bbbk|t_2)} e^{\overline{U}_i}$, $F = \sum_{i\in X, i_\Bbbk\in(S(\Bbbk|t_1)\setminus S(\Bbbk|t_2))} e^{\overline{U}_i}$, $H = \sum_{i\in X, i_\Bbbk\in(S(\Bbbk|t_2)\setminus S(\Bbbk|t_1))} e^{\overline{U}_i}$, *and* $G = 1 + \sum_{i\in X, i_\Bbbk\in(S(\Bbbk|t_1)\cap S(\Bbbk|t_2))} e^{\overline{U}_i}$.

The optimal assortment in each case is decided by comparing the expected profits resulting from each possible assortment in Q_C of that case. The terms T_{t_1,t_2} and T'_{t_1,t_2} are derived by comparing the expected profit functions of showcasing assortments t_1 and t_2. If $C = 1$ and $C > \prod_{k\in A, k\neq\Bbbk} |L(k)| \times (|L(\Bbbk)| - 1)$, Proposition 3.1 results in the same assortments under both $d_\Bbbk < 0$ and $d_\Bbbk > 0$. If $C = 1$, the optimal assortment includes product [1], i.e., the product with the highest utility. For all capacities C such that $C > \prod_{k\in A, k\neq\Bbbk} |L(k)| \times (|L(\Bbbk)| - 1)$, it is inevitable to showcase all levels of attribute $\Bbbk$. Therefore, there will be no inaccuracy remaining in the online channel. At these capacities, the products with the highest utilities are selected for showcasing. For $2 \leq C \leq \prod_{k\in A, k\neq\Bbbk} |L(k)| \times (|L(\Bbbk)| - 1)$, however, Proposition 3.1 can yield different assortments for $d_\Bbbk > 0$ and $d_\Bbbk < 0$. At these capacities, the condition $T_{t_1,t_2}e^{d_\Bbbk} > T'_{t_1,t_2}$ determines which assortment in Q_C of each case is optimal. Identifying the optimal assortment by using Proposition 3.1 requires at most $|L(\Bbbk)| - 1$ comparisons since the cardinality of set Q_C is $\min\{C, |L(\Bbbk|\}$.

It should be noted in Proposition 3.1 that there are specific cases where showcasing the highest utility products and showcasing the highest possible (the most limited) variety of $\Bbbk$ for $d_{\Bbbk} > 0$ ($d_{\Bbbk} < 0$) result in the same assortment. For example, at $C = 2$, if $[1]_{\Bbbk} = [2]_{\Bbbk}$ ($[1]_{\Bbbk} \neq [2]_{\Bbbk}$), the assortment consisting of these products is selected as optimal. Also, at $C = 3$, if $[1]_{\Bbbk} = [2]_{\Bbbk} = [3]_{\Bbbk}$ ($[1]_{\Bbbk} \neq [2]_{\Bbbk} \neq [3]_{\Bbbk}$), the assortment consisting of these products is optimal. These specific cases can be extended for greater C values as well. In the following, Corollaries 3.1 and 3.2 generalize these examples for any C values, for $d_{\Bbbk} > 0$ and $d_{\Bbbk} < 0$, respectively.

Corollary 3.1. *Suppose that* $d_{\Bbbk} > 0$. *Let* $x_{\Bbbk} \neq y_{\Bbbk}, \forall x, y \in \{[1], [2], ..., [|L(\Bbbk)|]\}, x \neq y$. *Then the optimal assortment in an arbitrary* C *is to showcase the highest utility products; i.e.,* $M^*_{\mathfrak{D}} = \{[1], [2], ..., [C]\}$, *and* $S(k|M^*_{\mathfrak{D}}) = \bigcup_{x \in M^*_{\mathfrak{D}}} \{x_k\}, \forall k \in A$.

Corollary 3.1 states that if all the products ranked among the $|L(\Bbbk)|$ highest utility products consist of different levels of attribute $\Bbbk$, then showcasing the highest utility products for any $1 \leq C \leq |L(\Bbbk)|$ results in showcasing the highest possible variety of $\Bbbk$ as well, which is desired. For $|L(\Bbbk)| < C$, the optimal assortment again includes just showcasing the highest utility products, because it already showcases all the different levels of $\Bbbk$.

Corollary 3.2. *Suppose that* $d_{\Bbbk} < 0$. *Let* $\Gamma = \prod_{k \in A, k \neq \Bbbk} |L(k)|$ *indicate the number of products in* X *with each level of attribute* $\Bbbk$. *Also, for products* $x, y \in \{[m\Gamma+1], [m\Gamma+2], ..., [(m+1)\Gamma]\}$, *and* $m \in \{0, 1, ..., K-1\}$, *let* $x_{\Bbbk} = y_{\Bbbk}$. *Then, the optimal assortment in an arbitrary* C *is to showcase the highest utility products;* $M^*_D = \{[1], [2], ..., [C]\}$, *and* $S(k|M^*_{\mathfrak{D}}) = \bigcup_{x \in M^*_{\mathfrak{D}}} \{x_k\}, \forall k \in A$.

Corollary 3.2 states that if the utility of level 1 of $\Bbbk$ is sufficiently higher than other levels that all the products consisting of this level compose the list of the Γ highest utility products, then showcasing the highest utility products for any $1 \leq C \leq \Gamma$ also results in showcasing the most limited variety of levels of $\Bbbk$, which is desired. Next, if the utility of level 2 of $\Bbbk$ is so higher than the remaining levels of $\Bbbk$ that all the products consisting of this level compose the list of the next Γ highest utility products, then for any $1 \leq C \leq 2\Gamma$, it is optimal to showcase the highest utility products. Similarly, if the described pattern holds for all the other levels of $\Bbbk$ until the Γ lowest utility products all consist of the lowest utility level of $\Bbbk$, then it is optimal to showcase the highest utility products at any C.

After characterizing the optimal assortment decision in the physical store, the manufacturer, as the leader of the game, uses the decision of the retailer to maximize its own profit by finding the optimal wholesale price w. Let Ω be the retailer's 'opportunity cost', such that if its profit is less than Ω fraction of its sales, it will no longer operate the physical store. Given this, Proposition 3.2 states the optimal value of w for a selected M as the following.

Proposition 3.2. *Suppose that the retailer has an opportunity cost of* $0 < \Omega < 1$*. Then, the optimal wholesale price of a product* x *is* $w^*_{\mathfrak{D}} = (1-\Omega)\pi, \forall x \in X$*.*

Proposition 3.2 states that the manufacturer should set the optimal value of w equal to the maximum value that the retailer is willing to pay as the wholesale price of the products and still run its physical store.

Centralized Setting

In this setting, a central authority runs the RSC in a way to maximize the system-wide expected profit. The set of showcased levels of attribute $\mathbb{k}$ in the physical store dictates the level of inaccuracy (and the purchase probability) in the online channel, which is the main and only factor that influences the RSC's total expected profit. Due to Lemma 3.3, once a level of attribute $\mathbb{k}$ is selected to showcase, it does not matter which product that contains the chosen level is showcased. In the following, Propositions 3.3 and 3.4 determine the optimal assortments of products $M^*_{\mathfrak{C}}$ and their corresponding attribute levels, when hidden attribute levels are overvalued ($d_{\mathbb{k}} > 0$) and undervalued ($d_{\mathbb{k}} < 0$), respectively.

Proposition 3.3. *Suppose that* $d_{\mathbb{k}} > 0$, $\Gamma = \prod_{k \in A, k \neq \mathbb{k}} |L(k)|$, *product returns are not allowed, and the RSC has a centralized setting. Let levels of each attribute be numbered as* $1, 2, ..., |L(k)|, \forall k \in A$ *with* 1 *and* $|L(k)|$ *being the highest and lowest utility levels, respectively. For any* $m = 1, 2, ..., K$*, we have:*

$$
\begin{aligned}
For (m-1)\Gamma \leq C < m\Gamma: \quad S^*_{\mathfrak{C}}(\mathbb{k}) &= \{|L(\mathbb{k})| - m + 1, |L(\mathbb{k})| - m + 2, ..., |L(\mathbb{k})|\}, \\
M^*_{\mathfrak{C}} &= \{C \text{ arbitrary products} \mid S^*_{\mathfrak{C}}(\mathbb{k})\}, \\
S^*_{\mathfrak{C}}(k) &= \bigcup_{x \in M^*_{\mathfrak{C}}} \{x_k\}, \forall k \in A, k \neq \mathbb{k}.
\end{aligned}
$$

When $d_{\mathbb{k}} > 0$, it is preferred to showcase the smallest possible variety of levels of attribute $\mathbb{k}$, because showcasing more levels of $\mathbb{k}$ leads to accurate utility assessments of products in the online channel, which results in a smaller total purchase probability of products. Given this, levels of $\mathbb{k}$ should be hidden as much as possible, and new attribute levels should be revealed as the capacity forces to do so. It is also preferred to select the levels with lowest part-worth utility values, since eliminating the inaccuracy from these levels leads to a greater total purchase probability compared to any other combination of the same number of levels. Once the level(s) of $\mathbb{k}$ are selected, the optimal assortment does not depend on which specific product(s) consisting of those levels are displayed, and hence, any arbitrary set of products that represents $S^*_{\mathfrak{C}}(\mathbb{k})$ could be showcased (Lemma 3.3).

Proposition 3.4. *Suppose that* $d_{\mathbb{k}} < 0$*, product returns are not allowed, and the RSC has a centralized* *setting. Then*

$$\begin{aligned} \textit{If } C \leq |L(\Bbbk)|: \quad S^*_{\mathfrak{C}}(\Bbbk) &= \{1, 2, ..., C\}, \\ M^*_{\mathfrak{C}} &= \{C \textit{ arbitrary products} \mid S^*_{\mathfrak{C}}(\Bbbk)\}, \\ S^*_{\mathfrak{C}}(k) &= \bigcup_{x \in M^*_{\mathfrak{C}}} \{x_k\}, \forall k \in A, k \neq \Bbbk. \end{aligned}$$

$$\begin{aligned} \textit{If } C > |L(\Bbbk)|: \quad S^*_{\mathfrak{C}}(\Bbbk) &= L(\Bbbk), \\ M^*_{\mathfrak{C}} &= \{C \textit{ arbitrary products} \mid S^*_{\mathfrak{C}}(\Bbbk)\}, \\ S^*_{\mathfrak{C}}(k) &= \bigcup_{x \in M^*_{\mathfrak{C}}} \{x_k\}, \forall k \in A, k \neq \Bbbk. \end{aligned}$$

When $d_{\Bbbk} < 0$, it is optimal for the RSC to showcase the highest possible variety of levels of $\Bbbk$ because it will eliminate as much inaccuracy as possible from the online-only products, which results in an increase in the total purchase probability. If $C \leq |L(\Bbbk)|$, the optimal assortment is to include C highest part-worth utility levels of $\Bbbk$ in $S^*_{\mathfrak{C}}(\Bbbk)$. If $C > |L(\Bbbk)|$, all levels of attribute $\Bbbk$ should be included in $S^*_{\mathfrak{C}}(\Bbbk)$. Similar to the previous case, an arbitrary set of products that represents $S^*_{\mathfrak{C}}(\Bbbk)$ can be showcased.

3.5.2 Scenario RA: When returns are allowed

Decentralized Setting

Lemma 3.1 holds for the RA scenario as well for any value of $d_{\Bbbk}$, but Lemma 3.2 holds only for $d_{\Bbbk} > 0$ for any value of C. Therefore, the set Q_C constructed by Procedure 1 is still instrumental in identifying the optimal assortment when $d_{\Bbbk} > 0$, since the retailer still prefers to showcase a high variety of levels of $\Bbbk$ and also the highest utility products in its physical store. In this case, not only do the highest utility products have higher purchase probabilities compared to the products with lower utilities, but they are also less likely to be returned.

Similar to the RNA scenario for $d_{\Bbbk} < 0$, the retailer prefers to showcase higher utility products from each selected level of $\Bbbk$ (Lemma 3.1), but a smaller variety of these levels. Lemma 3.2 does not hold for $d_{\Bbbk} < 0$ case, unless the retailer is determined to showcase only one level of attribute $\Bbbk$. Therefore, in the general situation, the retailer may or may not prefer to showcase products consisting of the levels with higher part-worth utility values. Consequently, we define a new feasibility set denoted by Q'_C which still picks the products with the higher product utility values but also includes the levels with lower part-worth utility values as stated below.

Procedure 2. Construction of Set Q'_C

1. $Q'_C = \varnothing$

2. For $\zeta = 1, 2, ..., \min\{C, |L(\Bbbk)|\}$

   (a) Let $COMB$ be the set of ζ attribute levels consisting of all combinations "Choose ζ out of $\min\{C, |L(\Bbbk)|\}$"

   (b) For $AL \in COMB$

      i. Create an empty tuple $\mathcal{T}$ of size C

      ii. Fill the first ζ elements of $\mathcal{T}$ by products $[1]_{\Bbbk:1}, [1]_{\Bbbk:2}, ..., [1]_{\Bbbk:\zeta}$

      iii. Pick the $C - \zeta$ highest utility products that have one of the attribute levels in AL and fill the remaining $C - \zeta$ elements of $\mathcal{T}$ with these products

      iv. $Q'_C = Q'_C \cup \mathcal{T}$

Proposition 3.5 below characterizes the optimal assortment of products and attribute levels when levels of $\Bbbk$ are either overvalued or undervalued in the online channel. For both cases, Proposition 3.5 provides a comparison term for comparing potential assortments in Q_C (for $d_\Bbbk > 0$) and Q'_C (for $d_\Bbbk < 0$), to select the optimal one.

Proposition 3.5. *Let the levels of each attribute be numbered as $1, 2, ..., |L(k)|, \forall k \in A$. Suppose that the RSC has a decentralized setting and product returns are allowed. Let Q_C (Q'_C) be the set of all the potential optimal assortments for $d_\Bbbk > 0$ ($d_\Bbbk < 0$) when the showcase capacity is C. Then the retailer's optimal assortment is given by*

If $C = 1$: $\quad M^*_{\mathcal{D}} = \{[1]\}, \quad S(k|M^*_{\mathcal{D}}) = \{1\}, \forall k \in A$

If $2 \leq C \leq \prod_{k \in A, k \neq \Bbbk} |L(k)| \times (|L(\Bbbk)| - 1)$:

For $d_\Bbbk > 0$:

$$M^*_{\mathcal{D}} = arg_{t_1 \in Q_C}\{\mathcal{T}_{t_1,t_2} e^{d_\Bbbk} > \mathcal{T}'_{t_1,t_2} \, \forall t_2 \in Q_C, t_2 \neq t_1\}, \quad S(k|M^*_{\mathcal{D}}) = \bigcup_{x \in M^*_{\mathcal{D}}} \{x_k\}, \forall k \in A$$

For $d_\Bbbk < 0$:

$$M^*_{\mathcal{D}} = arg_{t_1 \in Q'_C}\{\mathcal{T}_{t_1,t_2} e^{d_\Bbbk} > \mathcal{T}'_{t_1,t_2} \, \forall t_2 \in Q'_C, t_2 \neq t_1\}, \quad S(k|M^*_{\mathcal{D}}) = \bigcup_{x \in M^*_{\mathcal{D}}} \{x_k\}, \forall k \in A$$

If $C > \prod_{k \in A, k \neq \Bbbk} |L(k)| \times (|L(\Bbbk)| - 1)$:

$$M^*_{\mathcal{D}} = \{[1], [2], ..., [C]\}, \quad S(\Bbbk|M^*_{\mathcal{D}}) = L(\Bbbk), \quad S(k|M^*_{\mathcal{D}}) = \bigcup_{x \in M^*_{\mathcal{D}}} \{x_k\}, \forall k \in A, k \neq \Bbbk,$$

where

$$\mathcal{T}_{t_1,t_2} = (E+F)\sum_{i\in t_1}(\mathcal{P}_i^r - w)e^{\overline{U}_i} - (E+H)\sum_{i\in t_2}(\mathcal{P}_i^r - w)e^{\overline{U}_i},$$
$$\mathcal{T}'_{t_1,t_2} = (G+F)\sum_{i\in t_2}(\mathcal{P}_i^r - w)e^{\overline{U}_i} - (G+H)\sum_{i\in t_1}(\mathcal{P}_i^r - w)e^{\overline{U}_i}.$$

$E = \sum_{i\in X, i_{\Bbbk}\notin S(\Bbbk|t_1)\cup S(\Bbbk|t_2)} e^{\overline{U}_i}$, $F = \sum_{i\in X, i_{\Bbbk}\in(S(\Bbbk|t_1)\setminus S(\Bbbk|t_2))} e^{\overline{U}_i}$, $H = \sum_{i\in X, i_{\Bbbk}\in(S(\Bbbk|t_2)\setminus S(\Bbbk|t_1))} e^{\overline{U}_i}$, *and* $G = 1 + \sum_{i\in X, i_{\Bbbk}\in(S(\Bbbk|t_1)\cap S(\Bbbk|t_2))} e^{\overline{U}_i}$.

Proposition 3.5 can be interpreted the same as Proposition 3.1. Note that in Proposition 3.5, the difference between the cases $d_{\Bbbk} > 0$ or $d_{\Bbbk} < 0$ is the set Q_C or Q'_C when searching for t_1. When $d_{\Bbbk} < 0$, identifying the optimal assortment by using Proposition 3.5 requires at most $\sum_{i=1}^{|L(\Bbbk)|}\binom{|L(\Bbbk)|}{i} - 1$ comparisons since the cardinality of set Q'_C is such that all selections of 1 to $\min\{C, |L(\Bbbk|\}$ levels of attribute $\Bbbk$ should be considered. Note that Q'_C includes more potential assortments compared to Q_C.

Proposition 3.6 below provides the optimal wholesale price decision of the manufacturer, that is obtained by knowing the retailer's assortment decision that follows it.

Proposition 3.6. *Given the retailer's opportunity cost of* $0 < \Omega < 1$, *the optimal wholesale price of a product* x *is*

$$w^*_{\mathcal{D}} = \frac{(1-\Omega)\sum_{x\in M^*_{\mathcal{D}}} P^f_{x|M^*_{\mathcal{D}}}\mathcal{P}^r_x}{\sum_{x\in M^*_{\mathcal{D}}} P^f_{x|M^*_{\mathcal{D}}}}.$$

Proposition 3.6 can be interpreted like Proposition 3.2.

Centralized Setting

Lemmas 3.3 holds under the RA scenario and states that selecting only a subset of levels of inaccurately assessed attributes is sufficient to determine the optimal assortment. Therefore, we need to select a subset of levels of attribute $\Bbbk$ for showcasing. The complexity of finding the optimal solution will be a function of the order of the number of distinct attribute levels, rather than the total number of unique products. Let Q''_C be the feasibility set that includes all the selections of levels of attribute $\Bbbk$ that can be possibly the optimal selection and should be compared. Note that unlike the sets Q_C and Q'_C in Procedures 1 and 2 that included assortments of products, set Q''_C includes the feasible selections of levels of attribute $\Bbbk$.

Procedure 3. Construction of Set Q''_C

1. $Q''_C = \varnothing$

2. For $\zeta = 1, 2, ..., \min\{C, |L(\Bbbk)|\}$

   (a) Let $COMB$ be the set of ζ attribute levels consisting of all combinations "Choose ζ out of $\min\{C, |L(\Bbbk)|\}$"

   (b) For $AL \in COMB$, $Q''_C = Q''_C \cup AL$

In the following, Proposition 3.7 characterizes the optimal selection of attribute levels for showcasing under the centralized setting.

Proposition 3.7. *Suppose that the RSC has a centralized setting and product returns are allowed. Let Q''_C be the set of all the potential optimal selections of levels of attribute $\Bbbk$ when the showcase capacity is C. Then the retailer's optimal assortment is given by*

If $1 \le C \le \prod_{k\in A, k\neq \Bbbk} |L(k)| \times (|L(\Bbbk)| - 1)$*:*

$$S^*_{\mathcal{C}}(\Bbbk) = arg_{t_1 \in Q''_C}\{\mathbb{T}_{t_1,t_2} e^{d_\Bbbk} + \mathbb{T}'_{t_1,t_2} e^{2d_\Bbbk} > \mathbb{T}''_{t_1,t_2}, \forall t_2 \in Q''_C, t_2 \neq t_1\},$$
$$M^*_{\mathcal{C}} = \{C \textit{ arbitrary products} \mid S^*_{\mathcal{C}}(\Bbbk)\},$$
$$S^*_{\mathcal{C}}(k) = \bigcup_{x\in M^*_{\mathcal{C}}} \{x_k\}, \forall k \in A, k \neq \Bbbk.$$

If $C > \prod_{k\in A, k\neq \Bbbk} |L(k)| \times (|L(\Bbbk)| - 1)$*:*

$$S^*_{\mathcal{C}}(\Bbbk) = L(\Bbbk),$$
$$M^*_{\mathcal{C}} = \{C \textit{ arbitrary products} \mid S^*_{\mathcal{C}}(\Bbbk)\},$$
$$S^*_{\mathcal{C}}(k) = \bigcup_{x\in M^*_{\mathcal{C}}} \{x_k\}, \forall k \in A, k \neq \Bbbk.$$

where

$$\mathbb{T}_{t_1,t_2} = (G+H)\sum_{i_\Bbbk \notin t_1} \mathcal{P}^m_{i|M} e^{\overline{U}_i} + (E+F)\sum_{i_\Bbbk \in t_1} \mathcal{P}^r_i e^{\overline{U}_i} - (G+F)\sum_{i_\Bbbk \notin t_2} \mathcal{P}^m_{i|M} e^{\overline{U}_i} - (E+H)\sum_{i_\Bbbk \in t_2} \mathcal{P}^r_i e^{\overline{U}_i},$$
$$\mathbb{T}'_{t_1,t_2} = (E+F)\sum_{i_\Bbbk \notin t_1} \mathcal{P}^m_{i|M} e^{\overline{U}_i} - (E+H)\sum_{i_\Bbbk \notin t_2} \mathcal{P}^m_{i|M} e^{\overline{U}_i},$$
$$\mathbb{T}''_{t_1,t_2} = (G+F)\sum_{i_\Bbbk \in t_2} \mathcal{P}^r_i e^{\overline{U}_i} - (G+H)\sum_{i_\Bbbk \in t_1} \mathcal{P}^r_i e^{\overline{U}_i},$$

$E = \sum_{i\in X, i_\Bbbk \notin (t_1 \cup t_2)} e^{\overline{U}_i}$, $F = \sum_{i\in X, i_\Bbbk \in (t_1 \setminus t_2)} e^{\overline{U}_i}$, $H = \sum_{i\in X, i_\Bbbk \in (t_2 \setminus t_1)} e^{\overline{U}_i}$, *and* $G = 1 + \sum_{i\in X, i_\Bbbk \in (t_1 \cap t_2)} e^{\overline{U}_i}$.

In Preposition 3.7, when $1 \le C \le \prod_{k\in A, k\neq \Bbbk} |L(k)| \times (|L(\Bbbk)| - 1)$, the optimal selection of levels of attribute $\Bbbk$ can be found by comparing the potential such selections in Q''_C. The comparison term in this situation is to compare the expected profits of showcasing levels in set t_1 and t_2. When $C > \prod_{k\in A, k\neq \Bbbk} |L(k)| \times (|L(\Bbbk)| - 1)$, it becomes inevitable to showcase the full variety of levels of $\Bbbk$.

Note that identifying the optimal selection of levels of $\Bbbk$ requires at most $\sum_{i=1}^{|L(\Bbbk)|}\binom{|L(\Bbbk)|}{i}-1$ comparisons since the cardinality of set Q''_C is such that all selections of 1 to $\min\{C,|L(\Bbbk|\}$ levels of attribute $\Bbbk$ should be considered. Therefore, the complexity of set Q''_C is the same as Q'_C.

3.6 *Scope Contract*s for Coordination

As it is well documented in the supply chain literature, managing supply chains in a centralized fashion may increase the total profit generated in the chain compared to the summation of the individual profits of each entity when managed in a decentralized fashion (Govindan et al. 2013). Aligned with these results, we note that the optimal assortment decisions and their corresponding variety of levels of $\Bbbk$ under the centralized setting can be considerably different from that under the decentralized setting in the problem environment that we consider. In specific, it is possible that the solutions to problems (3.15) and (3.20) do not align, making the decentralized setting inefficient so that $\Pi^m_{\mathcal{D}}(w^*_{\mathcal{D}})+\Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|w^*_{\mathcal{D}}) < \Pi^T_{\mathcal{C}}(M^*_{\mathcal{C}})$. The inefficiency here is given by $\Pi^T_{\mathcal{C}}(M^*_{\mathcal{C}})-\Pi^m_{\mathcal{D}}(w^*_{\mathcal{D}})-\Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|w^*_{\mathcal{D}})$, which is the squandered profit due to not coordinating the decisions across the supply chain. In this section, we propose mechanisms that can be implemented by the manufacturer to coordinate the assortment decisions made by the retailer under the decentralized setting so that the inefficiency can be eliminated and the maximum possible profit can be generated across the chain to create a win-win situation for both parties.

First we note that the alignment of the optimal attribute levels in $M^*_{\mathcal{D}}$ and $M^*_{\mathcal{C}}$ (i.e., attaining $S(\Bbbk|M^*_{\mathcal{D}}) = S^*_{\mathcal{C}}(\Bbbk)$) is sufficient to coordinate the chain (and hence eliminate the inefficiency) even if $M^*_{\mathcal{D}} \neq M^*_{\mathcal{C}}$, because the optimal selection of the showcased attribute levels implies the remaining inaccuracy in the online assessment of products. Hence, if it turns out that $S(\Bbbk|M^*_{\mathcal{D}}) = S^*_{\mathcal{C}}(\Bbbk)$ for a problem instance, then $\Pi^m_{\mathcal{D}}(w^*_{\mathcal{D}})+\Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|w^*_{\mathcal{D}}) = \Pi^T_{\mathcal{C}}(M^*_{\mathcal{C}})$ with no inefficiency. If $S(\Bbbk|M^*_{\mathcal{D}}) \neq S^*_{\mathcal{C}}(\Bbbk)$, then the RSC is run inefficiently as the selected assortment by the retailer is $M^*_{\mathcal{D}}$ but not $M^*_{\mathcal{C}}$. The retailer does not showcase $M^*_{\mathcal{C}}$ because it would mean implementing a suboptimal assortment by sacrificing from their own profit. If the retailer can be induced to showcase the variety of attribute $\Bbbk$ given by $S^*_{\mathcal{C}}(\Bbbk)$, the extra profit generated (i.e., $\Pi^T_{\mathcal{C}}(M^*_{\mathcal{C}})-\Pi^m_{\mathcal{D}}(w^*_{\mathcal{D}})-\Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|w^*_{\mathcal{D}})$) can be distributed between the manufacturer and the retailer such that the retailer's loss of profit is compensated at a minimum and the supply chain can be coordinated. To this end, we propose *scope contract*s as a mechanism that can be instrumental in accomplishing the desired coordination.

Definition 3.1. *A scope contract, $\mathcal{SC}$, is defined as a contract that incentivizes the buyers to purchase a* *certain variety of products in the same transaction. It is given by*

$$\mathcal{SC} = [\alpha_1, \alpha_2, ..., \alpha_{|L(k)|}]$$

where α_j such that $0 \leq \alpha_j \leq 1$ is the discount rate applied on all products that contain the level j of the attribute k. If $\alpha_j = 0$ then the products that contain level j are sold at the regular price. An $\mathcal{SC}$ endorses Economies of Scope if there are greater number of instances with $\alpha_j \neq 0$ so that buyers are incentivized to purchase a larger variety of products whereas an $\mathcal{SC}$ endorses Diseconomies of Scope if there are fewer number of instances with $\alpha_j \neq 0$ so that buyers are incentivized to purchase a smaller variety of products.

For an arbitrary M_C^*, let $\Phi = 1 + \sum_{x \in M_C^*} e^{\overline{U}_x} + \sum_{x \in X \setminus M_C^*} e^{\overline{U}_x + D_{x|M_C^*}}$, $G_1 = \sum_{x \in X \setminus M_C^*} \mathcal{P}_{x|M}^m e^{\overline{U}_x + D_{x|M_C^*}}$, and $G_2 = \sum_{x \in M_C^*} \mathcal{P}_x^r e^{\overline{U}_x}$. Theorem 3.1 below proves that the *scope contract* is instrumental in coordinating the omni-channel RSC considered in this paper.

Theorem 3.1. *Let $S_{\mathcal{C}}^*(\Bbbk)$ be the levels of attribute $\Bbbk$ in $M_{\mathcal{C}}^*$ (solution to (3.20)). Let $\vec{\alpha} = [\alpha_1, \alpha_2, ..., \alpha_{|L(\Bbbk)|}]$ and $\vec{\beta} = [\beta_1, \beta_2, ..., \beta_{|L(\Bbbk)|}]$ be such that*

$$\{\alpha_j = 0; \beta_j = 0\} \ \ \text{if } j \notin S_{\mathcal{C}}^*(\Bbbk),$$

$$\left\{ \frac{\Phi\big(\Pi_{\mathcal{D}}^m(w_{\mathcal{D}}^*) - G_1\big)}{w_{\mathcal{D}}^*} \leq \sum_{x \in M_{\mathcal{C}}^*, x_{\Bbbk} = j} (1 - \alpha_j) e^{\overline{U}_x} \leq \frac{\Phi\big(G_2 - \Pi_{\mathcal{D}}^r(M_{\mathcal{D}}^* | w_{\mathcal{D}}^*)\big)}{w_{\mathcal{D}}^*}; \beta_j = 1 \right\} \ \ \text{if } j \in S_{\mathcal{C}}^*(\Bbbk). \qquad (3.21)$$

If $G_2 - \Pi_{\mathcal{D}}^r(M_{\mathcal{D}}^* | w_{\mathcal{D}}^*) \geq 0$, i.e., non-negative upper-bound for (3.21), then $\mathcal{SC} = \vec{\alpha}$ coordinates the RSC. Otherwise, $\mathcal{SC} = [\vec{\beta}, \mathcal{L}]$ coordinates the RSC where $\mathcal{L}$ is a lump-sum payment made by the manufacturer to the retailer such that

$$\Pi_{\mathcal{D}}^r(M_{\mathcal{D}}^* | w_{\mathcal{D}}^*) - \Pi_{\mathcal{SC}}^r(M_{\mathcal{C}}^* | w_{\mathcal{D}}^*, \vec{\beta}) \leq \mathcal{L} \leq \Pi_{\mathcal{SC}}^m(w_{\mathcal{D}}^*, \vec{\beta}) - \Pi_{\mathcal{D}}^m(w_{\mathcal{D}}^*)$$

where

$$\Pi_{\mathcal{SC}}^r(M_{\mathcal{C}}^* | w_{\mathcal{D}}^*, \vec{\beta}) = \sum_{x \in M_{\mathcal{C}}^*, x_{\Bbbk} = j} P_{x|M_{\mathcal{C}}^*}^r (\mathcal{P}_x^r - (1 - \beta_j) w_{\mathcal{D}}^*) \ \ \text{and}$$

$$\Pi_{\mathcal{SC}}^m(w_{\mathcal{D}}^*, \vec{\beta}) = \sum_{x \in X \setminus M_{\mathcal{C}}^*} P_{x|M_{\mathcal{C}}^*}^m \mathcal{P}_{x|M_{\mathcal{C}}^*}^m + \sum_{x \in M_{\mathcal{C}}^*, x_{\Bbbk} = j} (1 - \beta_j) w_{\mathcal{D}}^* P_{x|M_{\mathcal{C}}^*}^r$$

are the retailer's and the manufacturer's expected profit functions under $\mathcal{SC} = \vec{\beta}$, respectively.

The *scope contract* in Theorem 3.1 incentivizes the retailer to showcase $S_{\mathcal{C}}^*(\Bbbk)$ by providing discounts on all the products consisting of level j of $\Bbbk, \forall j \in S_{\mathcal{C}}^*(\Bbbk)$. According to Lemma 3.3, showcasing $S_{\mathcal{C}}^*(\Bbbk)$ is sufficient to achieve the highest possible system-wide expected profit (i.e., the centralized expected profit), and there can be multiple assortments $M^*_{\mathcal{C}}$ that all represent $S^*_{\mathcal{C}}(\mathbb{k})$. However, the specific $M^*_{\mathcal{C}}$ selected by the retailer impacts its own and the manufacturer's expected profits, and not all of the optimal assortments result in the same profit distribution for the parties. Hence, as a rational party, the retailer selects the $M^*_{\mathcal{C}}$ that yields the highest expected profit for itself. Among the potentially multiple $M^*_{\mathcal{C}}$ sets, we define $M^*_{\mathcal{SC}}$ as the assortment that provides the retailer with the highest expected profit. Note that by knowing $S^*_{\mathcal{C}}(\mathbb{k})$, $M^*_{\mathcal{SC}}$ can easily be identified via Lemma 3.1. By knowing $M^*_{\mathcal{SC}}$, the manufacturer can substitute this set into $M^*_{\mathcal{C}}$ in Theorem 3.1, as the assortment that the retailer would eventually choose under the *scope contract*, to create the contract and propose the discount rates. Below, as a summary, we propose a framework for designing and implementing a *scope contract* that coordinates the RSC.

Procedure 4. $\mathcal{SC}$ Design Process

Step 1: Find $S^*_{\mathcal{C}}(\mathbb{k})$.

Step 2: Find $M^*_{\mathcal{SC}}(\mathbb{k})$ by using Lemma 3.1.

Step 3: Using $M^*_{\mathcal{SC}}(\mathbb{k})$, find $\vec{\alpha}$ and $\vec{\beta}$ such that

$$\{\alpha_j = 0; \beta_j = 0\} \;\; if\; j \notin S^*_{\mathcal{C}}(\mathbb{k}),$$

$$\left\{ \frac{\Phi\big(\Pi^m_{\mathcal{D}}(w^*_{\mathcal{D}}) - G_1\big)}{w^*_{\mathcal{D}}} \leq \sum_{x \in M^*_{\mathcal{SC}}, x_{\mathbb{k}} = j} (1-\alpha_j) e^{\overline{U}_x} \leq \frac{\Phi\big(G_2 - \Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|w^*_{\mathcal{D}})\big)}{w^*_{\mathcal{D}}}; \beta_j = 1 \right\} \;\; if\; j \in S^*_{\mathcal{C}}(\mathbb{k}).$$

Step 4: If $G_2 - \Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|w^*_{\mathcal{D}}) \geq 0$ then offer $\mathcal{SC} = \vec{\alpha}$ to the retailer. Otherwise, offer $\mathcal{SC} = \vec{\beta}$ with a lump sum payment of $\mathcal{L}$ such that $\Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|w^*_{\mathcal{D}}) - \Pi^r_{\mathcal{SC}}(M^*_{\mathcal{C}}|w^*_{\mathcal{D}}, \vec{\beta}) \leq \mathcal{L} \leq \Pi^m_{\mathcal{SC}}(w^*_{\mathcal{D}}, \vec{\beta}) - \Pi^m_{\mathcal{D}}(w^*_{\mathcal{D}})$.

For each $j \in S^*_{\mathcal{C}}(\mathbb{k})$, there may be infinitely many α_j that satisfy (3.21) for an $\mathcal{SC}$ that coordinates the RSC. Although any of these values would successfully coordinate the RSC, different α_j values can result in different profit distributions between the manufacturer and the retailer. In Section 3.7, we investigate the impact of selecting different discount rates on profit distributions between the parties. The exact value of each α_j within the valid range can be determined though negotiation powers of the parties. For the exposition and implementation in practice, several different discount rates for different attribute levels would pose a challenge to the manufacturer. Theorem 3.2 below shows that for any $\mathcal{SC}$, there is a single discount rate that can be used instead of all $\alpha_j, \forall j \in S^*_{\mathcal{C}}(\mathbb{k})$ to coordinate the RSC and to ensure the same profit distribution between the parties as the original contract.

Theorem 3.2. *Suppose that* $G_2 - \Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|w^*_{\mathcal{D}}) \geq 0$ *and* $\mathcal{SC} = \vec{\alpha} = [\alpha_1, \alpha_2, ..., \alpha_{|L(\mathbb{k})|}]$ *coordinates the RSC.*

Let $\overline{\alpha} = 1 - \frac{\sum_{x \in M^*_{\mathcal{SC}}, x_{\Bbbk}=j}(1-\alpha_j)e^{\overline{U}_x}}{\sum_{x \in M^*_{\mathcal{SC}}} e^{\overline{U}_x}}$. *Construct a new discount vector* $\vec{\alpha}' = [\alpha'_1, \alpha'_2, ..., \alpha'_{|L(\Bbbk)|}]$ *such that* $\alpha'_j = \overline{\alpha}$ *if* $j \in S^*_{\mathcal{C}}(\Bbbk)$ *and* $\alpha' = \alpha_j = 0$ *if* $j \notin S^*_{\mathcal{C}}(\Bbbk)$. *Then,* $\mathcal{SC} = \vec{\alpha}'$ *coordinates the RSC with a single discount factor.*

Theorem 3.2 proves the existence of a single parameter *scope contract* that coordinates the RSC with all the benefits of the original $\mathcal{SC}$ being preserved. Given this, we can construct a *scope contract* with one discount rate applicable to all $j \in S^*_{\mathcal{C}}(\Bbbk)$. Consider the discount vector $\vec{\alpha}' = [\alpha'_1, \alpha'_2, ..., \alpha'_{|L(\Bbbk)|}]$ such that $\alpha'_j = \alpha'$ if $j \in S^*_{\mathcal{C}}(\Bbbk)$ and $\alpha'_j = 0$ if $j \notin S^*_{\mathcal{C}}(\Bbbk)$. Then, $\mathcal{SC} = \vec{\alpha}'$ coordinates the RSC with a single discount factor.

3.6.1 Discussion

To facilitate the process of finding the optimal assortments in the physical store in Section 3.5, we assumed that $d_{\Bbbk} \neq 0, \forall l(\Bbbk) \in L(\Bbbk)$ and $d_k = 0, \forall k \in A : k \neq \Bbbk$. This assumption states that there is only one attribute in the product under concern that is inaccurately assessed in the online channel. Suppose now that there are two attributes $\Bbbk$ and $\Bbbk'$ that are inaccurately assessed (i.e., $d_{\Bbbk} \neq 0, d_{\Bbbk'} \neq 0$). When both $\Bbbk$ and $\Bbbk'$ are inaccurately assessed in the same directions meaning that they are both overvalued or undervalued (i.e., $d_{\Bbbk} > 0$ and $d_{\Bbbk'} > 0$ or $d_{\Bbbk} < 0$ and $d_{\Bbbk'} < 0$), the results regarding the optimal assortment decisions in Section 3.5 still hold. When the inaccuracies in levels of $\Bbbk$ and $\Bbbk'$ are in different directions (i.e., $\Bbbk > 0, \Bbbk' < 0$ or $\Bbbk < 0, \Bbbk' > 0$), although the results in Section 3.5 are applicable to $\Bbbk$ and $\Bbbk'$ individually, finding the optimal assortments can involve higher complexity as the inaccuracies are in different directions and also the overall inaccuracy $d_{\Bbbk} + d_{\Bbbk'}$ can be positive or negative which means products are overvalued or undervalued, respectively. In these situations, complete enumeration or a greedy heuristic algorithm similar to the one proposed in Chapter 2 can be needed in finding the optimal assortment decisions.

Suppose that there are two attributes that are both either overvalued or undervalued. Let $S^*(\Bbbk)$ and $S^*(\Bbbk')$ be the optimal levels of $\Bbbk$ and $\Bbbk'$ to showcase in the centralized setting. We define $\alpha_{i,j}$ as the discount rate on all the products containing level i of attribute $\Bbbk$ and level j of attribute $\Bbbk'$. Then, the *scope contract* defined in Theorem 3.1 can be modified so that $\mathcal{SC} = \vec{\alpha} = [\alpha_{11}, \alpha_{21}, ..., \alpha_{|L(\Bbbk)|,|L(\Bbbk')|}]$ such that $\alpha_{i,j} = 0, \forall i \notin S^*(\Bbbk)$ or $\forall j \notin S^*(\Bbbk')$ would still coordinate the RSC with a modification of the conditions given in the Theorem 3.1. Note than for all products containing a level $i \in S^*(\Bbbk)$ but a level of $j \notin S^*(\Bbbk')$ (or vice versa), we consider $\alpha_{i,j} = 0$ because their level of attribute $\Bbbk'$ is not present in $S^*(\Bbbk')$. We numerically demonstrate this setup in Section 3.7.4.

In this part, we interpret the proposed *scope contract* for the RNA scenario. In the RNA scenario, when $d_{\Bbbk} < 0$, the optimal assortment in the centralized setting is to showcase the highest possible variety of levels of attribute $\Bbbk$ given C. However, in the decentralized setting, it can be to showcase only a limited variety of levels of $\Bbbk$, which results in $S(\Bbbk|M^*_{\mathcal{D}}) \leq S^*_{\mathcal{C}}(\Bbbk)$. Therefore, by switching from $M^*_{\mathcal{D}}$ to $M^*_{\mathcal{SC}}$, the variety of showcased levels of $\Bbbk$ increases. In this case, the *scope contract* can be called an "Economies of Scope" that increases the showcased variety of levels of $\Bbbk$. On the other hand, when $d_{\Bbbk} > 0$, the optimal centralized and decentralized assortments are such that $S(\Bbbk|M^*_{\mathcal{D}}) \geq S^*_{\mathcal{C}}(\Bbbk)$. So, by switching from $M^*_{\mathcal{D}}$ to $M^*_{\mathcal{SC}}$, the variety of showcased levels of $\Bbbk$ decreases. In this case, the *scope contract* can be called a "Dis-economies of Scope" contract. In the RA scenario, we may encounter $S(\Bbbk|M^*_{\mathcal{D}}) \geq S^*_{\mathcal{C}}(\Bbbk)$ or $S(\Bbbk|M^*_{\mathcal{D}}) \leq S^*_{\mathcal{C}}(\Bbbk)$, in both $d_{\Bbbk} < 0$ and $d_{\Bbbk} > 0$.

3.7 Numerical Demonstration

In this section, we numerically demonstrate an implementation of *scope contract*s, including finding the optimal assortment and wholesale price decisions, determining the inefficiency in the RSC, devising and implementing the *scope contract* for coordination, and profit distribution between the manufacturer and the retailer.

3.7.1 Test Bed

Suppose that the product under concern in the RSC consists of 3 attributes, $K = 3$, where attributes 1, 2, and 3 (A1, A2, and A3) include three, two, and three different levels, respectively, making up $3 \times 2 \times 3 = 18$ combinations of different attribute levels, each corresponding to a unique item. The part-wroth utility of levels of attributes are presented in Table 3.2. We assume that $d_1 = d_2 = 0$ and $d_3 \in \{-0.6, 0.6, 0.8\}$ so that the attribute $\Bbbk = 3$ is the one that is inaccurately assessed. Finally, we let $\pi = 100$, $v = 30$, $\Omega = 0.5$, $r = -0.8$, $\gamma = 0.6$, $\mu = \mu' = 1$, $\beta_1 = -0.013$, $\beta_2 = -0.01$, $\omega = 0.5$, and $\phi = 0.4$. The showcase capacity in the physical store is assumed to be $C = 6$, which corresponds to a capacity that can hold one third of the product portfolio (six out of 18) and the remaining 12 products are available only online. Table 3.3 presents the attribute levels in each product listed in the descending order of their product utilities, $\overline{U}_x$, in columns.

Table 3.2: Part-worth Utilities of Levels of Attributes 1, 2, and 3

Attribute k	1			2		3		
Levels $l(k)$	1	2	3	1	2	1	2	3
$\widetilde{u}_{k,l(k)}$	0.4	0.25	0.1	0.3	0.2	0.35	0.25	0.15

Table 3.3: Products based on $\overline{U}_x$ and their corresponding attribute levels. $[n]$ represents the product which has the nth highest product utility.

Product	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]	[14]	[15]	[16]	[17]	[18]
A1 level	1	1	1	2	1	1	2	2	1	3	2	2	3	3	2	3	3	3
A2 level	1	2	1	1	2	1	1	2	2	1	1	2	1	2	2	1	2	2
A3 level	1	1	2	1	2	3	2	1	3	1	3	2	2	1	3	3	2	3

3.7.2 Optimal Decisions in Decentralized and Centralized Settings

We first compare and contrast the optimal decisions under decentralized and centralized settings. Table 3.4 presents the attribute levels revealed (S^*) in the optimal assortment (M^*) decision under both settings, and the optimal wholesale price decisions in the decentralized setting for $d_3 = -0.6$ and $d_3 = 0.6$ when product returns are allowed.

Table 3.4: Comparison of Centralized and Decentralized Settings under RA scenario

	$d_3 = -0.6$				$d_3 = 0.6$			
	M^*	$S^*(3)$	$w^*_{\mathcal{D}}$	Π^T	M^*	$S^*(3)$	$w^*_{\mathcal{D}}$	Π^T
Decentralized	[1],[2],[4],[8],[10],[14]	1	45.61	71.98	[1] – [6]	1 – 3	45.81	75.81
Centralized	*	1 – 3	–	75.81	[6],[9],[11],[15],[16],[18]	3	–	79.29

*Any six of [1] to [18] by selecting at least one of Levels 1 to 3

As can be observed from Table 3.4, when $d_3 = -0.6$, a specific set of six products should be showcased (see M^* column) by revealing only attribute level 3 under the decentralized setting whereas any six products can be showcased by revealing all attribute levels under the centralized setting. The profit of the retailer and the manufacturer under the decentralized setting are 53.57 and 18.41, respectively, totaling up to 71.98 whereas the system wide total profit is 75.81, showing the inefficiency of the decentralized setting as the total profit can be increased by 3.83 which corresponds to 5.3%.

When $d_3 = 0.6$, the revealed attribute levels change from all to level 1 as we switch from the decentralized to the centralized setting. The optimal assortments are also completely different as expected. The total inefficiency in this case corresponds to a 4.6% increase in profit in the centralized setting compared to the decentralized one.

3.7.3 Optimality of the *Scope Contract* for Coordination

In this section, we demonstrate how the inefficiencies reported above can be eliminated with a *scope contract* designed with Procedure 4.

Scope Contract when $d_3 = -0.6$

According to Table 3.4, manufacturer should design a *scope contract* that will induce the retailer to reveal the levels in $S^*_{\mathcal{C}}(3) = \{1, 2, 3\}$. By implementing Lemma 3.1, the manufacturer can anticipate that the retailer would select an assortment of the highest utility products given by $M^*_{\mathcal{SC}} = \{[1], [2], ..., [6]\}$. The next step is to find $\mathcal{SC} = \vec{\alpha}$ by explicitly writing (3.21). Finding the values of Φ, G_1, and G_2, and substituting them into (3.21), we have:

$$1.06 \leq \sum_{x \in M^*_{\mathcal{SC}}} (1-\alpha_j) e^{\overline{U}_x} \leq 1.57. \tag{3.22}$$

Since the upper-bound is greater than zero, $\mathcal{SC} = \vec{\alpha}$ coordinates the RSC. Explicitly writing (3.22) with $M^*_{\mathcal{SC}}$, we get:

$$1.06 \leq (1-\alpha_1)(e^{\overline{U}_{[1]}} + e^{\overline{U}_{[2]}} + e^{\overline{U}_{[4]}}) + (1-\alpha_2)(e^{\overline{U}_{[3]}} + e^{\overline{U}_{[5]}}) + (1-\alpha_3)(e^{\overline{U}_{[6]}}) \leq 1.57.$$

Therefore, any $\vec{\alpha} = [\alpha_1, \alpha_2, \alpha_3]$ that satisfies the following will coordinate the RSC.

$$0.44 \leq 1.05\alpha_1 + 0.65\alpha_2 + 0.31\alpha_3 \leq 0.95.$$

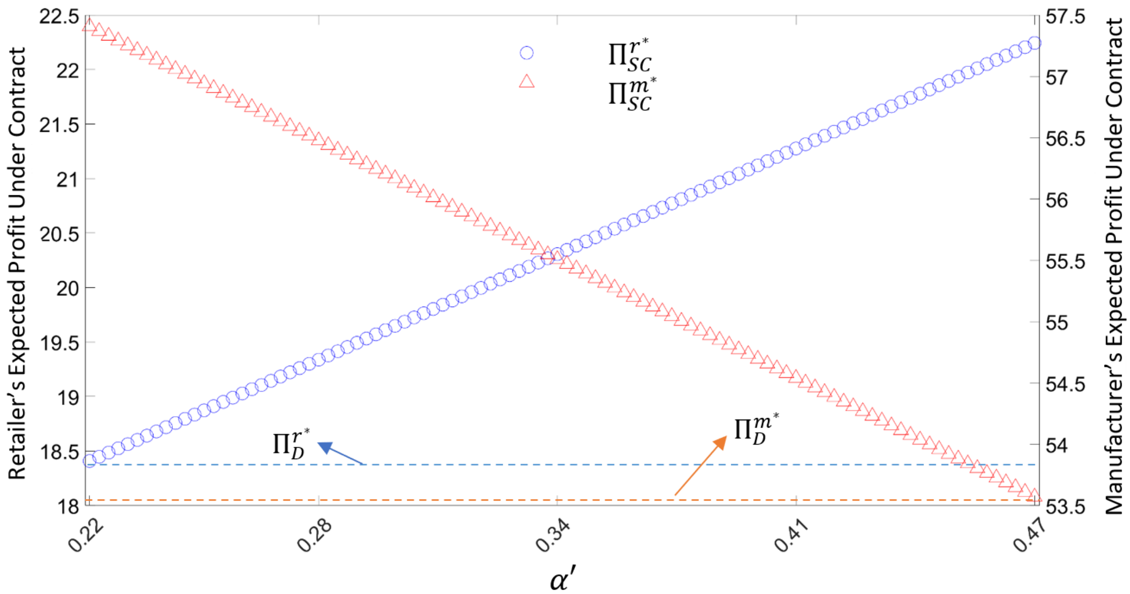
For example, one valid $\mathcal{SC} = \vec{\alpha}$ is $\vec{\alpha} = [0.5, 0.4, 0.3]$. By Theorem 3.2, it can be found that $\overline{\alpha} = 0.44$ and hence $\mathcal{SC} = \vec{\alpha}' = [0.44, 0.44, 0.44]$ also coordinates the RSC, with the same profit distribution between the manufacturer and the retailer as $\vec{\alpha} = [0.3, 0.4, 0.5]$. Furthermore, this can be transformed into a single parameter contract by letting $\alpha_1 = \alpha_2 = \alpha_3 = \alpha'$, which results in $0.22 \leq \alpha' \leq 0.47$. Although any discount rate in $0.22 \leq \alpha' \leq 0.47$ is valid and coordinates the RSC, different α' values result in different distributions of the additional profit gain between the manufacturer and the retailer.

Figure 3.1 shows the profit distribution between the manufacturer and the retailer for different values of α' in the $0.22 \leq \alpha' \leq 0.47$ range. In this figure, neither the retailer nor the manufacturer is worse off compared to their optimal decentralized expected profits. One end of the range of α' determines the retailer's minimum expected profit under contract (i.e., equal to its expected profit in the decentralized structure), and the other end of the range of α' determines that of the manufacturer. When α' gets its least value (i.e., $\alpha' = 0.22$), the manufacturer gains its highest possible expected profit under the contract while the retailer is also not worse off compared to its decentralized expected profit. Similarly, when α' gets its highest value (i.e., $\alpha' = 0.47$), the retailer gains its highest possible expected profit under the contract while the manufacturer is not worse off compared to its decentralized expected profit.

Scope Contract **when** $d_3 = 0.6$

According to Table 3.4, manufacturer should design a *scope contract* that will induce the retailer to reveal only attribute level 1 since $S^*_{\mathcal{C}}(3) = \{3\}$. The target optimal assortment by the manufacturer should be $M^*_{\mathcal{SC}} = \{[6], [9], [11], [15], [16], [18]\}$. The next step is to find $\mathcal{SC} = \vec{\alpha}$ by explicitly writing (3.21). Finding the

Figure 3.1: Profit Distribution between Manufacturer and Retailer Given $0.22 \leq \alpha' \leq 0.47$, when $d_3 = -0.6$



values of Φ, G_1, and G_2, and substituting them into (3.21), we have:

$$-0.65 \leq \sum_{x \in M^*_{\mathcal{SC}}} (1 - \alpha_3) e^{\overline{U}_x} \leq 0.04. \tag{3.23}$$

Since the upper-bound is greater than zero, $\mathcal{SC} = \vec{\alpha}$ coordinates the RSC. Explicitly writing (3.22) with $M^*_{\mathcal{SC}}$, we get:

$$-0.65 \leq (1 - \alpha_3)(e^{\overline{U}_{[6]}} + e^{\overline{U}_{[9]}} + e^{\overline{U}_{[11]}} + e^{\overline{U}_{[15]}} + e^{\overline{U}_{[16]}} + e^{\overline{U}_{[18]}}) \leq 0.04.$$

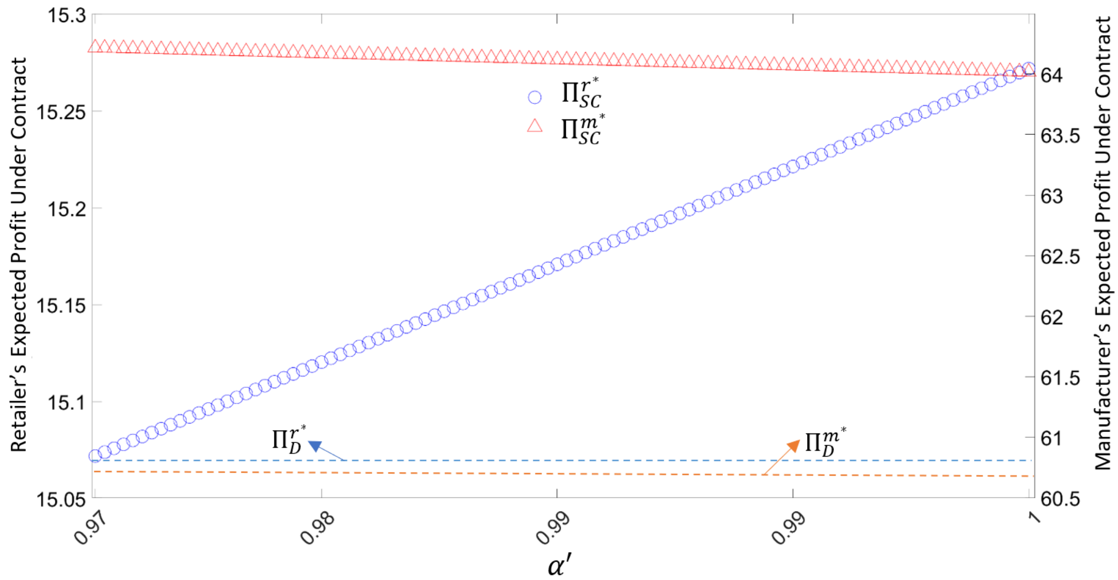
Therefore, any $\vec{\alpha} = [0, 0, \alpha_3]$ that satisfies the following will coordinate the RSC:

$$0.97 \leq \alpha_3 \leq 1.42.$$

Note that the upper-bound is not binding; so we have $0.97 \leq \alpha_3 \leq 1$. Since there is only one discount rate in this contract, we do not need to implement Theorem 3.2 in this case, and the corresponding single parameter version of the contract is also $0.97 \leq \alpha' \leq 1$

In the following, Figure 3.2 indicates the profit distribution between the RSC parties when $d_3 = 0.6$ for the $0.97 \leq \alpha' \leq 1$ range. The same analysis as Figure 3.1 holds here as well. In Figure 3.2, while an α' value less than 0.97 will worsen the retailer compared to the decentralized setting, there is no value of α' that will worsen the manufacturer, because such α' should be greater than 1.42.

Figure 3.2: Profit Distribution between Manufacturer and Retailer Given $0.97 \leq \alpha' \leq 1$, when $d_3 = 0.6$



***Scope Contract* for Coordination when** $d_3 = 0.8$

When $d_3 = -0.6$ or $d_3 = 0.6$, a *scope contract* with $\mathcal{SC} = \vec{\alpha}$ can be found to coordinate the RSC without the need of a lump sum payment by the manufacturer to the retailer. When $d_3 = 0.8$, this is not the case. First, we note that $M_{\mathcal{D}}^* = \{[1], [2], [3], [4], [5], [6]\}$ with revealing all levels of attribute 3 so that $S(3|M_{\mathcal{D}}^*) = \{1, 2, 3\}$. Similar to the $d = 0.6$ case, the optimal assortment in the centralized setting is $M_{\mathcal{C}}^* = \{[6], [9], [11], [15], [16], [18]\}$ with revealing only level 3, $S_{\mathcal{C}}^* = \{3\}$. The inefficiency is 80.08-60.74-15.07 = 4.27 which corresponds to 5.6% increase in the centralized setting from the decentralized setting. In this case, the inequality given by (3.21) turns out to be

$$-1.41 \leq \sum_{x \in M_{\mathcal{SC}}^*} (1 - \alpha_3) e^{\overline{U}_x} \leq -0.43 \tag{3.24}$$

not letting for any feasible α_1 value. Hence, we design a $\mathcal{SC} = [\vec{\beta}, \mathcal{L}]$ contract with $\vec{\beta} = [0, 0, 1]$ and an $\mathcal{L}$ value that satisfies

$$\begin{aligned} 15.07 - 13.17 \leq \quad & \mathcal{L} \quad \leq 66.91 - 60.74 \\ 1.9 \leq \quad & \mathcal{L} \quad \leq 6.17. \end{aligned}$$

Note in this problem instance that $w_{\mathcal{D}}^* = 45.81$, $\Pi_{\mathcal{D}}^r(M_{\mathcal{D}}^*|w_{\mathcal{D}}^*) = 15.07$, $\Pi_{\mathcal{SC}}^r(M_{\mathcal{C}}^*|w_{\mathcal{D}}^*, \vec{\alpha}) = 13.17$, $\Pi_{\mathcal{SC}}^m(w_{\mathcal{D}}^*, \vec{\beta}) = 66.91$, and $\Pi_{\mathcal{D}}^m(w_{\mathcal{D}}^*) = 60.74$.

3.7.4 *Scope Contract*s with Multiple Inaccurately Assessed Attributes

When there is inaccuracy in assessing levels of more than one attribute, the optimal decisions may differ compared to the case where only levels of one of the attributes are inaccurately assessed. In this section, we investigate the operations of the RSC when levels of attributes 2 and 3 are simultaneously inaccurately assessed. Let $d_2 = 0.6$ and $d_3 = -0.6$, while the values of all other parameters are the same as described at the beginning of Section 3.7. In the following, Table 3.5 shows the optimal decisions of the RSC in the decentralized and centralized settings.

Table 3.5: Optimal Solutions for the Representative Examples under RA Scenario, when $d_2 = 0.6$ and $d_3 = -0.6$

Decentralized								**Centralized**		
$\Pi_{\mathcal{D}}^{m*}$	$\Pi_{\mathcal{D}}^{r*}$	$\Pi_{\mathcal{D}}^{m*}+\Pi_{\mathcal{D}}^{r*}$	$S(1\|M_{\mathcal{D}}^*)$	$S(2\|M_{\mathcal{D}}^*)$	$S(3\|M_{\mathcal{D}}^*)$	$M_{\mathcal{D}}^*$	$w_{\mathcal{D}}^*$	$\Pi_{\mathcal{C}}^{T*}$	$S_{\mathcal{C}}^*(2)$	$S_{\mathcal{C}}^*(3)$
53.57	18.41	71.98	1-3	1-2	1	[1],[2],[4],[8],[10],[14]	45.61	78.66	2	1-3

In Table 3.5, $S(2|M_{\mathcal{D}}^*) = \{1, 2\}$ and $S(3|M_{\mathcal{D}}^*) = \{1\}$. Whereas, $S_{\mathcal{C}}^*(2) = \{2\}$ and $S_{\mathcal{C}}^*(3) = \{1, 2, 3\}$. Hence, coordinating this RSC means decreasing the scope of attribute 2 and increasing the scope of attribute 3. Given $S_{\mathcal{C}}^*(2)$ and $S_{\mathcal{C}}^*(3)$, there are multiple optimal assortments (and as a result, multiple $S(1|M_{\mathcal{C}}^*)$ sets). Given Lemma 3.1, among all $M_{\mathcal{C}}^*$ sets, we have $M_{\mathcal{SC}}^* = \{[2], [5], [8], [9], [12], [14]\}$.

To achieve coordination in this problem instance, a *scope contract* should be devised such that both the inaccurately assessed attributes are considered in proposing the discounts. Let $\mathcal{SC} = \vec{\alpha} = [\alpha_{2,1}, \alpha_{2,2}, \alpha_{2,3}]$, where $\alpha_{2,1}$ is the discount rate on products consisting of level 2 of attribute 2 and level 1 of attribute 3, $\alpha_{2,2}$ is the that on products consisting of level 2 of attribute 2 and level 2 of attribute 3, and $\alpha_{2,3}$ is that on products with level 2 of attribute 2 and level 3 of attribute 3. Therefore, the condition of the *scope contract* can be written as

$$-1.09 \leq (1 - \alpha_{2,1})(e^{\overline{U}_{[2]}} + e^{\overline{U}_{[8]}} + e^{\overline{U}_{[14]}}) + (1 - \alpha_{2,2})(e^{\overline{U}_{[5]}} + e^{\overline{U}_{[12]}})(1 - \alpha_{2,3})e^{\overline{U}_{[9]}} \leq 0.12,$$

Since the upper-bound is non-negative, $\mathcal{SC} = \vec{\alpha}$ coordinates the RSC. Simplifying the inequality, we get a non-binding upper-bound. Therefore, we have

$$0.93 \leq 0.89\alpha_{2,1} + 0.58\alpha_{2,2} + 0.28\alpha_{2,3} \leq 1.$$

Letting $\alpha_{2,1} = \alpha_{2,2} = \alpha_{2,3} = \alpha'$, the corresponding single parameter contract can also be written. Hence, the proposed *scope contract* in this paper as a coordination mechanism is not limited to cases where only levels of one attribute are inaccurately assessed. This contract can be simply written for cases where an arbitrary number of attribute levels are inaccurately assessed with arbitrary values of inaccuracies for each of them.

3.7.5 Analysis of the Impact of Retail Price

As stated in Section 3.4, we assume that the retail price π is fixed. This assumption is meant to isolate the results and insights from the market dynamics and the potential competition between different retail supply chains of competing products. This price can be considered as the equilibrium price for the product under concern in such competitive environments, but there might still be some room for price adjustments for increasing the total profit of the chain. In this section, we investigate the impact of changes in π on the profitability of the RSC members and the whole retail system. It should be noted that we assume the changes in the retail price do not affect the chain's potential customer base; however, they can affect the purchase and return probability of products, and hence, the overall observed demand. A greater π, for example, can lead to smaller purchase probabilities and higher return probabilities. On the other hand, it can be expected to increase the profit margin obtained by selling each product for the RSC as a whole. Therefore, the trade-off between the decreased number of sales and the increased profit margins determines whether greater or smaller prices can be desired, while the competition effect with other similar products in the market is neglected.

For the numerical experiment presented in this section, suppose that we have a menu of retail prices from which a π can be selected to sell the products to the customers so that $\pi \in \{60, 80, 100, 120, 140, 160, 180, 200, 220, 240, 260\}$. We assume that $d_3 \in \{-0.6, 0.6\}$ and $\gamma = 0.75$ (note that the γ value should be high enough to be valid due to the required condition $\gamma > v/\pi$). To explicitly observe the effect of π on purchase probabilities and returns, we let the price sensitivity of utility take any of the $\beta_1 \in \{-0.013, -0.018, -0.024\}$ and the part-worth utility of levels of attribute 1 be 0.5, 0.3, and 0.1, attribute 2 be 0.35 and 0.15, and attribute 3 be 0.45, 0.25, and 0.05, respectively. The values of all other parameters are the same as Section 3.7.1. The following results are stated for when $d_3 = -0.6$, but they all hold for $d_3 = 0.6$ as well.

Figure 3.3 shows the changes in $\Pi_{\mathfrak{C}}^{T^*}$ with respect to different π and β_1 values under the centralized setting. Recall that β_1 is the price sensitivity of product utilities. As can be observed, $\Pi_{\mathfrak{C}}^{T^*}$ has a unimodal structure (not necessarily concave) in an increasing-decreasing pattern. Note that for extreme values of the retail price, the total expected profit approaches zero. This general trend states that an increase in price initially improves profitability because although it decreases the purchase probability and increases returns, the increase in profit margin is substantial and results in higher profit. However, once the retail price gets larger, the decrease in the purchase probability and increase in returns dominate the increase in the marginal profit, and overall the total expected profit decreases. It should be noted in Figure 3.3 that when the magnitude of β_1 is greater, the impact of the decrease in the purchase probability and increase in returns dominates the increase in marginal profit at smaller prices. This is because in this case, product utilities are more sensitive to the price.

Figure 3.3: $\Pi_{\mathfrak{C}}^{T^*}$ values with respect to π and β_1 under the centralized setting when $d_3 = -0.6$

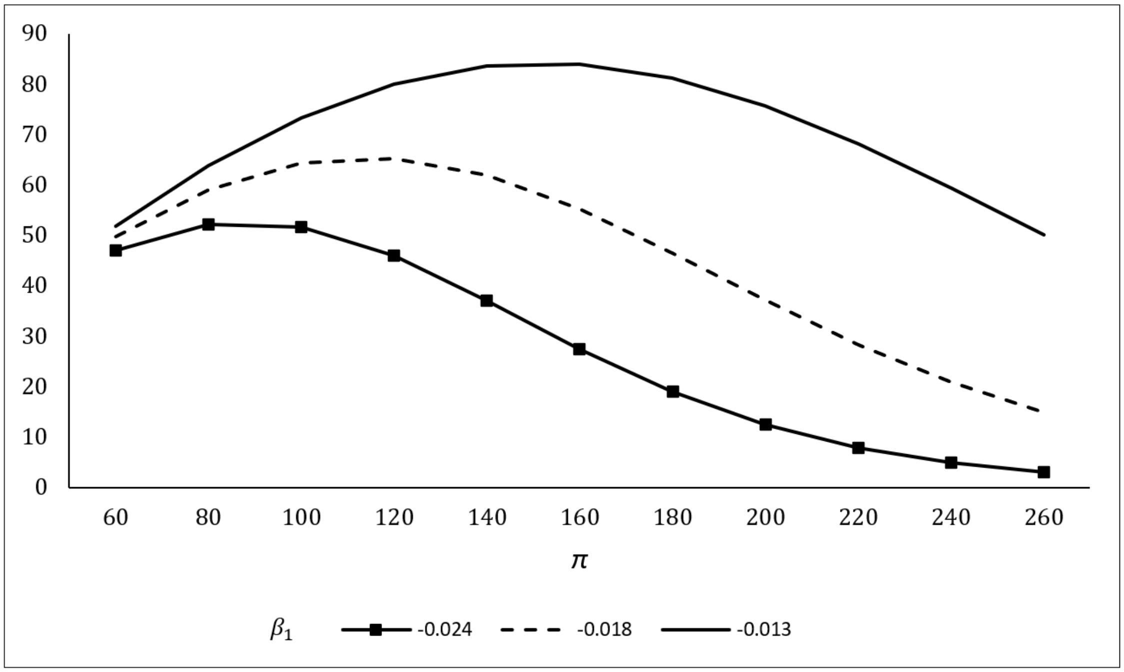


Table 3.6 indicates the $w_{\mathfrak{D}}^*$ values with respect to changes in π and β_1 under the decentralized setting. As expected, an increase in π results in an increase in $w_{\mathfrak{D}}^*$. Given the retailer's opportunity cost Ω, a higher π translates into a greater marginal profit from each product and allows for more flexibility regarding $w_{\mathfrak{D}}^*$ by the retailer. On the contrary, greater β_1 values lead to smaller purchase probabilities and higher returns, which overall yields smaller profits. Hence, given Ω, the $w_{\mathfrak{D}}^*$ that the retailer can tolerate will be smaller.

Table 3.6: $w_{\mathfrak{D}}^*$ values with respect to π and β_1 under the decentralized setting when $d_3 = -0.6$

β_1	-0.024	-0.018	-0.013
π		$w_{\mathfrak{D}}^*$	
60	33.11	33.61	33.98
80	40.75	42.23	43.33
100	46.74	49.69	51.94
120	51.14	55.92	59.74
140	54.62	60.90	66.66
160	56.63	64.71	72.68
180	57.98	68.17	77.77
200	59.05	70.16	81.98
220	60.10	71.54	86.30
240	61.31	72.55	89.00
260	62.75	73.39	91.04

Figure 3.4 below shows the changes in the total expected profit of the RSC (i.e., $\Pi_{\mathfrak{D}}^{m^*} + \Pi_{\mathfrak{D}}^{r^*}$) with respect to different π and β_1 values under the decentralized setting. This figure can be interpreted the same as

we explained Figure 3.3. It should be noted that the $\Pi_{\mathfrak{D}}^{m^*} + \Pi_{\mathfrak{D}}^{r^*}$ values in Figure 3.4 are smaller than the corresponding $\Pi_{\mathfrak{C}}^{T^*}$ values in Figure 3.3, which denotes the inefficiency of the decentralized setting. In Figure 3.5, we depict the changes in the manufacturer's and retailer's expected profits individually. As can be observed, the same trend similar to Figure 3.4 holds individually as well. It should be noted that the manufacturer's average profit stands at a higher level for any π and β_1. This is in part because the store capacity is assumed to be $C = 6$, which results in the manufacturer selling more products than the retailer. Note that at extreme prices in which the total RSC expected profit approaches zero, the difference between the manufacturer's and the retailer's expected profits shrinks.

Figure 3.4: $\Pi_{\mathfrak{D}}^{m^*} + \Pi_{\mathfrak{D}}^{r^*}$ values with respect to π and β_1 under the centralized setting when $d_3 = -0.6$

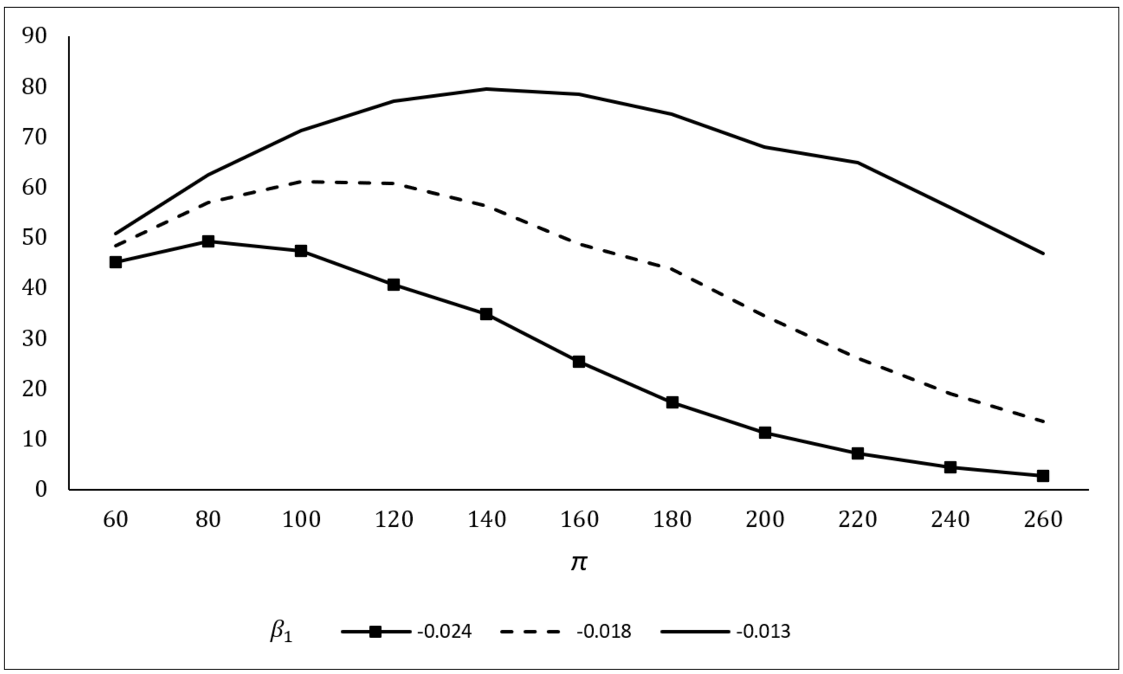
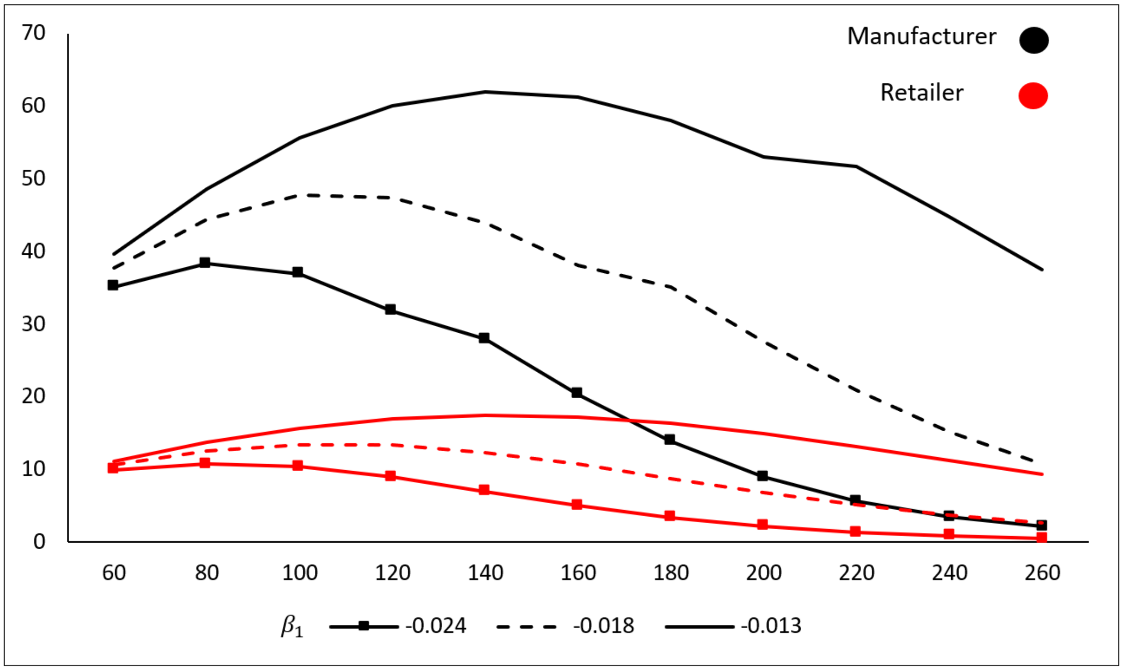


Figure 3.6 indicates the percentage of inefficiency in the total expected profit of the RSC under the decentralized setting compared to the centralized setting. According to this figure, the inefficiency for all β_1 values shows similar behavior with respect to the retail price. First, the inefficiency increases up to its maximum value. Then, it shows a decreasing trend after which, it increases again. Note that the stated behavior takes place at smaller prices for greater β_1 values since a greater β_1 translates into a greater impact from π on product utilities.

According to Propositions 3.5 and 3.7, the utility of products is crucial in the optimal assortment of products and attribute levels. The changes in π directly impact the utility of products. Therefore, the assortment of products and attribute levels may be subject to change with alterations in π. In the following, Table 3.7 shows the impact of changes in retail price on the optimal assortment when $\beta_1 = -0.024$ under

Figure 3.5: $\Pi_{\mathcal{D}}^{m^*}$ and $\Pi_{\mathcal{D}}^{r^*}$ values with respect to π and β_1 under the centralized setting when $d_3 = -0.6$



the centralized and decentralized settings for $d = -0.6$ and $d = 0.6$. We observe that the optimal assortment under the decentralized setting when $d = -0.6$ can include a greater variety of levels of attribute 3 as π increases. Although we do not see any changes in the selected levels under the centralized setting for this problem instance, the assortment can obviously change for different problem instances for different π values. We do not observe any change in the assortment for $d = 0.6$ because only level 3 of attribute 3 is showcased in this case and it remains optimal to only showcase this level as π increases.

Table 3.7: Optimal assortment decisions with respect to π when $\beta_1 = -0.024$

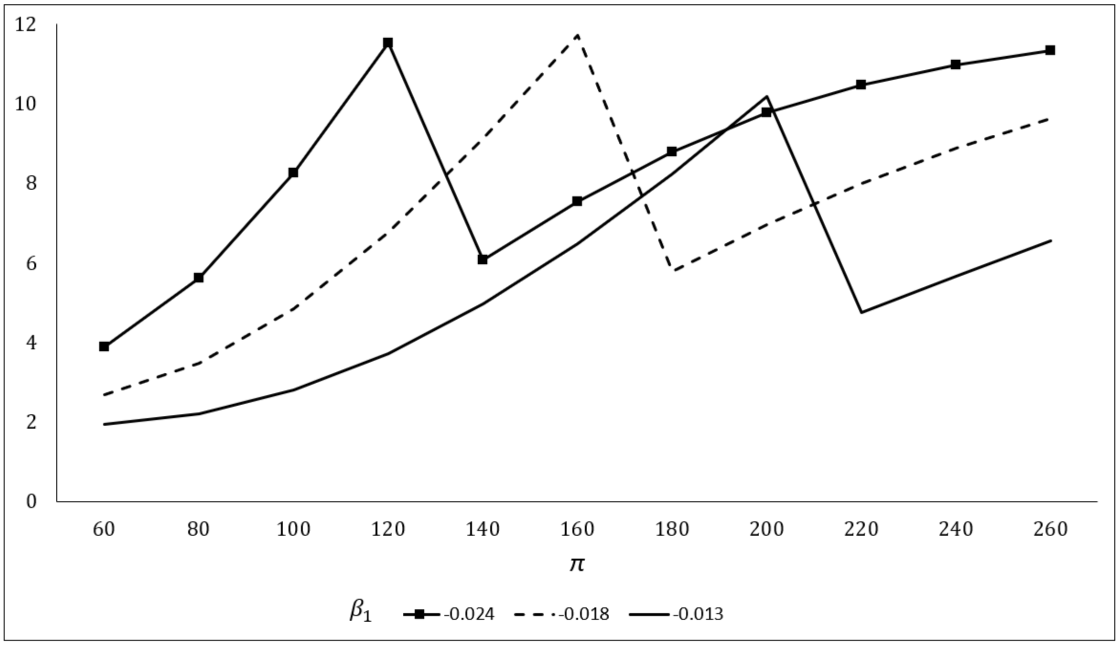
	π	60		100		180		260	
d_3	setting	S^*	M^*	S^*	M^*	S^*	M^*	S^*	M^*
-0.6	$\mathcal{C}$	1-3	*	1-3	*	1-3	*	1-3	*
	$\mathcal{D}$	1	[1]-[3],[5],[6],[9]	1	[1]-[3],[5],[6],[9]	1-2	[1]-[6]	1-2	[1]-[6],[7]
0.6	$\mathcal{C}$	3	**	3	**	3	**	3	**
	$\mathcal{D}$	1-3	[1]-[5],[8]	1-3	[1]-[5],[8]	1-3	[1]-[5],[8]	1-3	[1]-[4],[7],[8]

*Any six of [1] to [18] by selecting at least one of Levels 1 to 3
**All six products with Level 1 of attribute 3

3.8 Conclusion

In this paper, we investigate the operations of an omni-channel retail supply chain (RSC) consisting of an online sale website and a physical store. We study two decision settings including decentralized and centralized. In the decentralized setting, the manufacturer owns the online channel as its direct sales medium,

Figure 3.6: % inefficiency in the RSC profit with respect to π and β_1 under the centralized setting when $d_3 = -0.6$



and the retailer sells the manufacturer's products in its physical store, both trying to maximize their own profits independently. Through a Stackelberg game, the manufacturer decides the wholesale price of its products to be charged to the retailer, and consequently, the retailer selects an assortment of manufacturer's products for showcasing. In the centralized setting, both sales channels are considered to be run by a central authority that aims to maximize the system-wide profit by an assortment decision for the physical store.

The manufacturer should charge the retailer the highest wholesale price that the retailer can bear with while operating its store. This depends on the retailer's opportunity cost; i.e., the amount of profit that it expects to attain as a fraction of its sales; otherwise, it would invest its capital elsewhere. While assortment decisions can be complex, we propose analytical results shown to be effective in reducing the complexity of the problem considerably.

When product returns are not allowed (RNA), under the decentralized setting, if customers overvalue the hidden attribute levels (i.e., not showcased in the physical store), the retailer should select an assortment that balances the benefits of showcasing the highest utility products and a high variety of such levels. If customers undervalue the hidden attribute levels, this will be a trade-off between showcasing the highest utility products and a limited variety of attribute levels. Under the centralized setting, the assortment decisions can be fundamentally different as it is optimal to showcase the most limited variety of hidden attribute levels that are overvalued and the highest possible variety of undervalued levels.

When product returns are allowed (RA), under the decentralized setting, the same strategy as RNA holds if the hidden attribute levels are overvalued. However, if the hidden levels are undervalued, it can be optimal to showcase lower utility levels as well. This means that in this case, for example, products with unpopular colors can be candidates for showcasing as well; on the contrary to the RNA case where these products would not have a chance to appear in the assortment. Also, under the centralized setting, any selection of attribute levels can be a potential optimal selection, and there can be multiple such assortments.

The differences among assortment decisions under the decentralized and centralized settings in both scenarios can lead to higher total profitability under the centralized setting, which indicates the inefficiency of the decentralized assortment decisions. To incentivize the retailer to switch to the centralized assortment, we propose *scope contract*s as a coordination mechanism in which the manufacturer offers discounts on the wholesale price of products with attribute levels that it prefers the retailer to showcase (for example, products with certain colors). By adjusting the discount rates, the manufacturer increases the attractiveness of showcasing such products for the retailer. This contract is widely applicable for the coordination of assortment planning problems, and it resembles the "economies of scope" and "dis-economies of scope" in the literature.

In the *scope contract*, if the manufacturer prefers a great variety of attribute levels to be showcased by the retailer, designing the contract can be difficult since several discount rates should be adjusted. To address this issue, we show that a *scope contract* in which the discount rates are equal can coordinate the RSC while the benefits of the contract with several rates are preserved. Manufacturers can implement this contract with minimum hassle to improve the overall profitability of their RSC.

In this study, we suppose that the retail price of products are at equilibrium with competition with other retail supply chains selling similar products. Isolating the RSC under concern from the outside competition in the market, we analyze the changes in the retail price. Our results indicate that the expected profit of the RSC is unimodal, implying that prices that are too high or too small can result in a loss of profit. Specifically, if the price is extremely high, the demand for the products will approach zero resulting in zero expected profit. The best price can be determined based on the price sensitivity of utility.

As a future study, this paper can be extended to a situation where the physical store carries an inventory of products along with the showcase. Carrying the right amount of each showcased product is a crucial key to profitability. Note that in this problem, once a product goes out of stock, the purchase probability of all products can be impacted. Moreover, this problem can be investigated in a multi-period sales horizon where the assortment and inventory decisions can be made for each period. The decisions for each period will have an influence on the other sales periods, as excess inventory or shortages can be carried over to the next periods.

Chapter 4

Assortment Planning in Omni-Channel Retail Supply Chains Under Information Asymmetry

Abstract. In omni-channel retail supply chains (RSCs), online assessment of products with non-digital attributes can be inaccurate and different from physical stores. These are products like apparel for which physical assessment and trying out provide accurate information about their utility. It is known in the literature that the aforementioned inaccuracy in customers' online assessments plays a crucial role in the assortment of products that retailers decide to showcase in their physical stores, because physical stores are not only a sales medium but also means of providing accurate information for customers who visit them. However, the inaccuracies may not be known to RSC decision makers, resulting in an asymmetric information situation where they need to make an assortment decision based on their best estimates. We investigate a RSC consisting of an online sales website and a physical store under a decentralized setting where the sales channels are independently managed, and a centralized setting where both channels are managed by a central authority. Based on our numerical study, we observe that if an estimation of inaccuracy results in the same assortment as the true value of inaccuracy, there will be no regret in profitability in both settings. Under the decentralized setting, each party can also fare better under certain conditions, but both cannot fare better at the same time. Under the centralized setting, the RSC is never better off with an imprecise estimation of the inaccuracy. Furthermore, if an estimation of zero inaccuracy is obtained, the RSC is likely to minimize its regret by showcasing an assortment that contains the highest utility attribute levels.

4.1 Introduction

Omni-channel retailing, a widely adopted practice in contemporary retail, involves the co-presence of both online sales channels and physical stores. This strategy contributes to expanding market share and offers retailers opportunities for increased profitability (Bell et al. 2018). In such retail systems, the online sales channel enjoys the advantage of virtually unlimited product variety, as it sources directly from central warehouses or manufacturers. On the other hand, physical stores often face capacity constraints in showcasing products, resulting in customers having access to a limited variety. Consequently, omni-channel retailers encounter a crucial decision that revolves around determining the assortment of products to be displayed in physical stores, which is commonly known as the assortment problem.

Products possess various attributes, such as color and material, each with multiple levels, like blue and red for the color attribute. According to Dzyabura and Jagabathula (2018), the customer's perceived utility or attractiveness of a product can be evaluated by summing up the utilities associated with its attribute levels. One distinctive feature of omni-channel retailing is the difference in shopping experience for products that have non-digital attributes. These are items such as clothing and sunglasses where physical assessment and trying them out provide reliable information about their utility. In the case of these products, customers can accurately assess the items displayed in physical stores. However, items with attribute levels that are not showcased in the store but only browsed online may be subject to 'inaccurate' assessments.

The (in)accuracy in customers' assessments of product utilities have a significant impact on their purchase likelihoods. When a customer assigns a higher utility to a product based on its online representation compared to what they would have assessed physically (i.e., overvaluation), they are more inclined to purchase the product. Conversely, if the customer undervalues the product based on its online presentation, the purchase likelihood will be lower. The degree of inaccuracy during the purchase phase also influences customer's satisfaction with the product, subsequently affecting their decision to keep or return it. If a product was purchased online with an overvaluation, the customer may be disappointed upon receiving it as it doesn't meet their expectations. In such cases, the likelihood of returning the product is higher compared to when the customer made an accurate assessment. Conversely, if the product was undervalued during the online purchase, the customer may be pleasantly gratified by its quality upon receipt. As a result, the likelihood of returning such a product decreases.

The assortment decision made for the physical store has implications not only for customers' purchasing decisions in-store but also for those when shopping online. This is because products showcased in the physical store may share certain attribute levels with online-only products, providing partial accurate utility information for these items. For instance, a red product of a specific style may be available in-store, but the desired red color for the preferred style might not be in stock. Thus, the customer can accurately assess the utility of the red color but not the utility of the desired style available only online. Consequently, the customer's assessment of the overall utility of the desired product will be partially accurate, as illustrated by this example. Assortment planning becomes particularly significant for high-value products, where customers prefer to visit the physical store to gather precise utility information about the available products and their attribute levels before making a purchase, either online or in-store (Park et al. 2021).

Based on the preceding discussion as well as the findings of Dzyabura and Jagabathula (2018), Lo and Topaloglu (2022), the inaccuracy in customers' assessments of product utilities in the online sales website plays a crucial role in making an optimal assortment decision for the physical store. The optimal assortment decision can significantly change based on whether products are accurately assessed, overvalued, or undervalued in online assessments, and it is crucial to take the inaccuracy into account while making this decision. Therefore, a trustworthy estimation of this inaccuracy component is imperative for effective management of omni-channel retail operations. However, estimating the inaccuracy parameters is not straightforward or trivial. It might require substantial efforts which might be challenging and costly. For example, surveys or lab/field experiments can be devised to understand the customers behaviour in their relative assessment of the online and offline products. The customer profile might be different from one segment of products to another. The design, look, and the interface of the online sales channel would also directly impact whether the customers over- or under-value the hidden attributes. For instance, for sunglasses or furniture, customers may have different perception of the utility, and the inaccuracy in each customer's assessment can be different. This can be different even for the same RSC with a another product. Therefore, such inaccuracies would be case specific and change from an environment to another. In certain cases, the online sales channel may be owned and operated independent of the physical stores. In those settings, each party's operational decisions would be predicated on their own best guesses of the inaccuracy parameters. It might be reasonable to expect the party which operates the online sales channels to conduct the research in understanding the customer behavior. For example, through sampling and comparing the utilities that customers assign to certain group of products when physically assessed and when observed online. Even if this party has it in full, this information may not be shared with the physical store operators. In summary, the challenges that are faced in practice regarding the estimation of the magnitude of the inaccuracy are the potential misalignment of the expectation and the reality and information asymmetry between the parties that operate the physical stores and the online sales channels.

In this paper, we examine the impact of the challenges arising from having incomplete or asymmetric information about the assessment inaccuracy in a retail supply chain setting, which is composed of an online sales channel, operated directly by a manufacturer, and a physical store (a retailer) that sells the manufacturer's products. We assume that all customers have access to both the online channel and the physical store, and their purchase decisions are based on selecting the product with the highest positive utility across both channels[1]. If a customer does not find any product with positive utility, no purchase occurs. For instance, customers may visit the physical store to assess the available products while also browsing the complete collection online using their smartphones or a computer screen provided in the store. The physical store serves as a showcase, displaying an assortment of products without maintaining inventory. If a customer chooses to purchase an item from the showcased products in the physical store, the retailer places an order with the manufacturer at a wholesale price and arranges for the product to be delivered to the customer. On the other hand, if a customer decides to purchase a product that is only available online, the order is placed through the manufacturer's website, and the product is delivered directly to the customer. The manufacturer sells the products to the retailer at a wholesale price. Based on this price and their estimation of the inaccuracy in the assessment of the customers of the hidden attributes, the retailer makes the assortment decision, which is determining the products to showcase in their store, to maximize their own profit. The manufacturer can anticipate the action that the retailer would take (i.e., the assortment decision) and set a wholesale price accordingly so that they maximize their own profit. In this process, the manufacturer needs to estimate the profit of the retailer and itself under different configurations and the magnitude of the inaccuracy parameters should be factored in these estimations.

In a decentralized setting of this RSC, the inaccuracy parameter should be estimated by each party separately as they are independent entities. In this paper, we investigate the impact of making erroneous estimations of this parameter and explore whether being optimistic or pessimistic could be a better strategy when the decision maker is dubious about the true value of this parameter. We analyze this situation under the possibility of both parties not knowing the true value (in a decentralized and centralized setting) and one party knowing the true value without sharing it with the other party (in a decentralized setting). In specific, we define a regret function for the retailer, the manufacturer, and the whole RSC as the difference between their expected profit if they knew the value of inaccuracy and their expected profit when an estimate is obtained. We investigate which type of estimation of the inaccuracy by the retailer and the manufacturer (i.e., supposing customers would over- or under-value or accurately assess utilities) can lead to smaller regret for them in the decentralized setting, and that for the whole RSC in the centralized setting. The main contribution of this paper to the literature is to explore information asymmetry in the context of omni-channel retailing. This work also addresses asymmetry about a crucial parameter (i.e., the inaccuracy in

[1]This study focuses on customers who visit both channels and primarily considers high-value products, as customers tend to gather as much information as possible before making a purchase. For other product types, this study is applicable to the segment of customers that visit both channels.

online assessments) in an omni-channel RSC that has not been previously studies.

4.2 Literature Review

The initial step in assortment planning involves understanding customers' decision-making behavior and representing it. This is crucial since the objective is to optimize the profit obtained from selling products to customers. Typically, 'consumer choice models' are draw upon to model the customers' behavior. In this paper, we adopt the multinomial logit (MNL) as the most widely used choice model. In MNL, each customer corresponds a utility to each product, which consists of a deterministic and a stochastic part, where the stochastic part is modeled as an error term with the Gumbel distribution. Then, the choice probability of each product is determined using properties of this distribution (Ben-Akiva et al. 1985, Anderson et al. 1992). Other choice models employed in the literature contain nested logit (Gallego and Topaloglu 2014, Davis et al. 2013), d-level nested logit (Li et al. 2015), mixed logit (Rusmevichientong et al. 2014), locational (Gaur and Honhon 2006), and Markov chain-based (Blanchet et al. 2016) choice models.

Assortment planning in single-channel retailing, where a retailer determines the product selection for its physical store, has been extensively studied in the literature. Refer to the literature review by Kök et al. (2015) for an overview of this research stream. Nonetheless, assortment planning in the context of omni-channel retailing is a relatively an emerging topic, although it has gained attention recently. Ye et al. (2018) discuss the challenges that retailers face when, including more complex assortment planning, expanding their operations across multiple channels. Several empirical and discussion-based studies have explored omni-channel assortment planning (Bell et al. 2014, 2018, Gallino and Moreno 2018). Rooderkerk and Kök (2019) provide a literature review specifically focused on omni-channel assortment planning.

To our knowledge, the study by Dzyabura and Jagabathula (2018) is the first to propose an optimization model specifically for assortment planning in omni-channel retailing. In a similar vein, Lo and Topaloglu (2022) examined omni-channel assortment planning using a features tree for product features. Hense and Hübner (2022) investigate omni-channel assortment planning and corresponding inventory management, incorporating stochastic and independent demand models. Schäfer et al. (2023) highlight the significant influence of assortment on demand in brick-and-mortar stores, web-shops, and across channels, ultimately impacting retailer profitability. These studies on omni-channel assortment planning assume all problem parameters affecting the assortment decisions are known a priori.

According to Ha (2001), each party in an SC aims to optimize its own profit and does not wish to reveal its private information unless a sufficient incentive is provided. Information asymmetry and its effects on decision-making are widely studied in different realms of SC management. One approach to information asymmetry in SC decision-making is to consider one sales season and try to optimize the operations by obtaining a probability distribution as an estimate for the value of unknown parameters. Studies in this approach mainly address the demand uncertainty for a product in an RSC when pricing or ordering decisions are explored (Yue and Raghunathan 2007, Mishra and Prasad 2004, Yue et al. 2006, Arcelus et al. 2008, Akan et al. 2011, Yan and Pei 2011, Raj et al. 2021). Shen et al. (2019) and Vosooghidizaji et al. (2020) provide reviews on SC contracting and SC coordination with information asymmetry, respectively. Also, Arcelus et al. (2007) consider a setting where a retailer's risk preference is unknown to its supplier, and Çakanyıldırım et al. (2012) study a setting where a supplier's unit production cost is privately known to itself. Another approach to information asymmetry is to study the problem over multiple sales seasons and update information about the unknown parameters. For example, Aviv (2001) studies a two-stage SC consisting of a retailer and a supplier, where parties update their forecasts of future demand periodically.

In the domain of assortment planning, studies with full information are typically modeled as static problems (Talluri and Van Ryzin 2004, Désir et al. 2014, Bultez and Naert 1988). However, in the case that problem parameters are not known a priori, problems are usually modeled as dynamic where over the course of time, the optimal decisions are improved through updating information about the unknown parameter (Caro and Gallien 2007, Rusmevichientong et al. 2010, Agrawal et al. 2019, Chen et al. 2020, Bernstein et al. 2019). However, all these studies consider a single-channel RSC where the parameter considered unknown is customers' preferences among products. With the advent of omni-channel RSCs, inaccuracy in customers' assessment of products in the online channel is a real obstacle (and opportunity) that retailers encounter. In this study, we investigate omni-channel RSCs in a single sales season where inaccuracies in online assessments of customers are unknown.

4.3 Modelling Approach

In this section, we first introduce the utility model that we use for the products sold in the physical store and the online channel. Then, we develop models for customers' purchase and keep-or-return decision.

4.3.1 Utility Model

Consider a product with non-digital attributes. Let A be the set of attributes of this product, $K = |A|$ be the total number of attributes, $k \in A$ be a specific attribute, and $L(k)$ be the set of all levels of attribute k. For example, $A = \{color, material, style\}$ can be the set of attributes of a product with $K = 3$ and $L(k) = \{\text{black, blue, silver}\}$ can be the set of levels for attribute $k =$ "*color*". One level from each attribute will constitute a unique product and all possible such combinations will form the universal set of products

X, where $N = |X| = \prod_{\{k \in A\}} |L(k)|$.

We adopt the utility model of Dzyabura and Jagabathula (2018) in which it is assumed that the product utility is the summation of the part-worth utilities of its attribute levels. Let $\widetilde{u}_{k,l(k)}$ be the part-worth utility of level $l(k) \in L(k)$ of attribute $k \in A$ assessed through physical encounter; touching, seeing, or trying out. Then, the 'attribute utility' of a product $x \in X$ (the utility of the product that is associated only to the attributes of the product) can be written as

$$\widetilde{U}_x = \sum_{k \in K} \sum_{l(k) \in L(k)} \widetilde{u}_{k,l(k)} \cdot 1_{\{x_k = l(k)\}} \tag{4.1}$$

where x_k is the level of attribute k in product x and $1_{\{x_k = l(k)\}}$ is an indicator function which is equal to 1 if the level of attribute k in product x is equal to $l(k)$, and 0 otherwise. In addition to the attribute utility given in (4.1), customers can also factor in the price of the product, π_x, the disutility of the return process (i.e., how difficult it would be to return the product), r, and the refundable portion of the price upon return, γ while deciding whether to purchase the product x or not (Alptekinoğlu and Grasas 2014). Consequently, the 'product utility' of $x \in X$ becomes

$$\overline{U}_x = \widetilde{U}_x + \beta_1 \pi_x + \beta_2 (1 - \gamma) \pi_x + \phi r, \tag{4.2}$$

where β_1 is the price sensitivity of utility, β_2 is the sensitivity of utility to the non-refundable fraction of the price, and ϕ is the sensitivity of utility to the difficulty (disutility) of the return process. Note that $\beta_1 \leq 0$ to reflect an inverse effect with the price of the product, $\beta_2 \leq 0$ to reflect the negative impact of a higher non-refundable portion of the price, and $\phi \geq 0$ since the disutility of the return process is defined as $r \leq 0$. The product utility given by (4.2) is an expected value for an average customer, but because of differences in individual preferences, each customer can have a different realization of the product utility. Therefore the utility of a product x that is showcased (available in the physical store) can be written as

$$U_x = \overline{U} + \varepsilon_x \tag{4.3}$$

where ε_x is the error term accounting for the unobserved components that are not caught by the utility model. These idiosyncratic error terms in the utility of products are assumed to be independent and identically distributed (i.i.d) of a Gumbel distribution with mean zero and scale parameter $1/\mu$ – a standard assumption in MNL models, see Kök et al. (2015), Anderson et al. (1992). Here, μ is a positive scalar representing the homoscedasticity of the population such that a larger value of μ reflects a more heterogeneous population.

In the omni-channel retail setting that we consider, the customers who end up purchasing from the online channel also visit the physical store to collect information about the products. Hence, those customers would have an accurate knowledge of the part-worth utilities, $\widetilde{u}_{k,l(k)}$, if the level $l(k)$ of attribute k is present in one of the showcased products (available in the store) but would have an inaccurate knowledge of the part-worth utilities of the attribute levels that are not showcased. Therefore, the utility of the products that are available only online depends on the assortment decision implemented in the physical store. Let $d_{k,l(k)}$ be the true value of inaccuracy in customers' assessments of level $l(k)$ of attribute k if that level is not showcased. Then, the part-worth utility of a level $l(k)$ of attribute k which is not showcased is given by

$$\widetilde{u}_{k,l(k)} + d_{k,l(k)} \tag{4.4}$$

where $d_{k,l(k)} > 0$ if the customers overvalue the level $l(k)$ of attribute k, $\forall l(k) \in K, \forall k \in A$, in the online channel, and $d_{k,l(k)} < 0$ if they undervalue it. In this paper, we assume that this parameter (its true value) may not be known by the decision maker, and we analyze the impact of deviations in estimations.

Let $M \subseteq X$ be an assortment of products selected for showcasing in the physical store. Let $S(k|M)$ be the set of all levels of attribute k that are present in at least one of the products available in M, i.e., $S(k|M) = \bigcup_{x \in M}\{x_k\}$, and let $S'(k|M) = L(k) \setminus S(k|M)$ be the set of levels of attribute k that are not available in any of the products in M.

By letting,

$$D_{y|M} = \sum_{k \in K} \sum_{l(k) \in S'(k|M)} d_{k,l(k)} \cdot 1_{\{y_k = l(k)\}}$$

and following the derivation in Chapter 2,

$$\begin{aligned} U_{y|M} &= \sum_{k \in K} \sum_{l(k) \in L(k)} \widetilde{u}_{k,l(k)} \cdot 1_{\{x_k = l(k)\}} + D_{y|M} + \beta_1 \pi_x + \beta_2 (1 - \gamma) + \phi r + \varepsilon_y \\ &= \overline{U}_y + D_{y|M} + \varepsilon_y. \end{aligned} \tag{4.5}$$

4.3.2 Customers' Purchasing Experience

The purchasing experience of the customers follow the same logic explained in Chapter 2: Customers decide whether to purchase or not based on the product utility given by (4.5) through the MNL model. If the product is purchased from the physical store with accurate assessment, then the keep-or-return decision is predicated on how the product utility compares to the disutility of return. If purchased from the online channel, then in addition to the disutility, it is predicated on whether the customer had over- or under-valued the product during purchase. Consequently, the purchase probability of a product x given an assortment M at the retailer and the manufacturer, $P^r_{x|M}$ and $P^m_{x|M}$, are given by

$$
\begin{aligned}
P^r_{x|M} &= \frac{e^{(\overline{U}_x+D_{x|M})/\mu}}{1+\sum_{y\in X} e^{(\overline{U}_y+D_{y|M})/\mu}} \\
&= \frac{e^{\overline{U}_x/\mu}}{1+\sum_{y\in M} e^{\overline{U}_y/\mu}+\sum_{y\in X\setminus M} e^{(\overline{U}_y+D_{y|M})/\mu}} \quad \forall x \in M\cup\{0\} \\
P^m_{x|M} &= \frac{e^{(\overline{U}_x+D_{x|M})/\mu}}{1+\sum_{y\in X} e^{(\overline{U}_y+D_{y|M})/\mu}} \\
&= \frac{e^{(\overline{U}_x+D_{x|M})/\mu}}{1+\sum_{y\in M} e^{\overline{U}_y/\mu}+\sum_{y\in X\setminus M} e^{(\overline{U}_y+D_{y|M})/\mu}} \quad \forall x \in X\setminus M\cup\{0\}.
\end{aligned}
$$

Similarly, the keep-or-return probabilities can be written as follows. Let R^r_x indicate the return probability of product $x \in M$ that is purchased from the retailer. Then,

$$
\begin{aligned}
R^r_x &= Pr\{\widetilde{U}_x+\beta_1\pi_x+\varepsilon_x+\varepsilon_{\text{keep}|x} < r+\beta_1(1-\gamma)\pi_x+\varepsilon_x+\varepsilon_{\text{return}|x}\} \\
&= Pr\{\varepsilon_{\text{keep}|x}-\varepsilon_{\text{return}|x} < -\widetilde{U}_x+r-\beta_1\gamma\pi_x\} \\
&= \frac{1}{1+e^{(\widetilde{U}_x-r+\beta_1\gamma\pi_x)/\mu'}}. && (4.6)
\end{aligned}
$$

Let $R^m_{x|M}$ indicate the return probability of product $x \in X\setminus M$ that is purchased from the online channel. Then,

$$
\begin{aligned}
R^m_{x|M} &= Pr\{\widetilde{U}_x+\beta_1\pi_x-\omega D_{x|M}+\varepsilon_x+\varepsilon_{\text{keep}|x} < r+\beta_1(1-\gamma)\pi_x+\varepsilon_x+\varepsilon_{\text{return}|x}\} \\
&= Pr\{\varepsilon_{\text{keep}|x}-\varepsilon_{\text{return}|x} < -\widetilde{U}_x+\omega D_{x|M}+r-\beta_1\gamma\pi_x\} \\
&= \frac{1}{1+e^{(\widetilde{U}_x-\omega D_{x|M}-r+\beta_1\gamma\pi_x)/\mu'}}. && (4.7)
\end{aligned}
$$

Let K^r_x be the keep probability of product $x \in M$ that it is purchased from the retailer, and $K^m_{x|M}$ be this probability of product $x \in X\setminus M$ that it is purchased from the online sales channel of the manufacturer. Then,

$$K_x^r = 1 - R_x^r \quad = \frac{1}{1 + e^{(-\widetilde{U}_x + r - \beta_1 \gamma \pi_x)/\mu'}},$$
$$K_{x|M}^m = 1 - R_{x|M}^m = \frac{1}{1 + e^{(-\widetilde{U}_x + \omega D_{x|M} + r - \beta_1 \gamma \pi_x)/\mu'}}. \tag{4.8}$$

In concluding this section, we note that each of the measures presented above is a function of the inaccuracy parameter $D_{x|M} = \sum_{k \in K} \sum_{l(k) \in S'(k|M)} d_{k,l(k)} \cdot 1_{\{x_k = l(k)\}}$. Also note that these probability measures are the key components of the expected profit function as given in the next section. Hence, whether the estimated magnitude of the inaccuracy term, $d_{k,l(k)}$, aligns with the true magnitude that transpires in practice plays a key role in making profound decisions by the decision makers.

4.4 Expected Profit and Regret Functions

In this section, we derive the expected profit functions under the decentralized and centralized settings and define a regret function of not knowing the true value of the inaccuracy parameter. We assume that $d_{k,l(k)} = d_k \forall l(k) \in L(k), \forall k \in A$, i.e., all levels of an attribute are equally inaccurately assessed in the online channel, and we let $\mathbb{d} = (d_1, d_2, ..., d_K)$ be the vector of true values of the inaccuracies for all levels of all attributes in $A = \{1, 2, ..., K\}$. Let $\hat{\mathbb{d}}^m = (\hat{d}_1^m, \hat{d}_2^m, ..., \hat{d}_K^m)$ and $\hat{\mathbb{d}}^r = (\hat{d}_1^r, \hat{d}_2^r, ..., \hat{d}_K^r)$ be the estimated inaccuracy vector by the manufacturer and the retailer, respectively. In other words, these vectors are their best guesses of the inaccuracy parameters d_k for all $k \in A$. Naturally, the decision makers of the manufacturer and the retailer use their own best guesses when making their operational decisions.

4.4.1 Decentralized Setting

In a decentralized setting, the manufacturer quotes the wholesale price w, and the retailer decides which products to purchase and showcase in their store. In this setting, for any given value of w and the inaccuracy vector $\mathbb{d}$, the retailer's expected profit function can be stated as (see Chapter 2)

$$\Pi_{\mathcal{D}}^r(M|w, \mathbb{d}) = \sum_{x \in M} \left[(\pi - w) P_{x|M}^r K_x^r + ((1 - \gamma)\pi + v - w) \, P_{x|M}^r R_x^r \right] \tag{4.9}$$

where the first term refers to the expected profit obtained from the products sold at the retailer's physical store and are kept post-purchase, and the second term refers to the profit obtained from the products that are sold but returned. Recall that the probability measures P, R, and K are also functions of the inaccuracy vector $\mathbb{d}$ in the above equation. Noting that $K_x^r = 1 - R_x^r$ and defining $\mathcal{P}_x^r = (1 - R_x^r)\pi + R_x^r \left[(1 - \gamma)\pi + v\right] =$

$\pi - R_x^r(\gamma\pi - v), \forall x \in M$ in (4.9) as the marginal profit obtained from a product sold in the physical store, the retailer's expected profit can be re-written as

$$\Pi_{\mathcal{D}}^r(M|w, \mathbb{d}) = \sum_{x \in M} P_{x|M}^r \left(\mathcal{P}_x^r - w\right). \tag{4.10}$$

Consequently, by using their best estimate $\hat{\mathbb{d}}^r$ for the inaccuracy vector $\mathbb{d}$, the retailer solves

$$\begin{aligned} &\max_{M \subset X} \Pi_{\mathcal{D}}^r(M|w, \hat{\mathbb{d}}^r) \\ &\text{s. to} \quad |M| \leq C \end{aligned} \tag{4.11}$$

to find the optimal M where C is the showcase capacity at the retailer. Let $M(w, \hat{\mathbb{d}}^r)$ be the assortment decision that solves (4.11). This action can be anticipated by the manufacturer for any given w value, but the best guesses of inaccuracy vector made by the retailer and the manufacturer may not align. In such cases (if $\hat{\mathbb{d}}^r \neq \hat{\mathbb{d}}^m$), the manufacturer will anticipate that the retailer would order $M(w, \hat{\mathbb{d}}^m)$ whereas they would order $M(w, \hat{\mathbb{d}}^r)$ and it is possible that these two assortment decisions are not the same. Nevertheless, the manufacturer estimates their expected profit for any given value of w as

$$\begin{aligned} \Pi_{\mathcal{D}}^m(w|\hat{\mathbb{d}}^m) \quad = \quad & \sum_{x \in X \setminus M(w, \hat{\mathbb{d}}^m)} \pi P_{x|M(w,\hat{\mathbb{d}}^m)}^m K_{x|M(w,\hat{\mathbb{d}}^m)}^m \\ & + \sum_{x \in X \setminus M(w, \hat{\mathbb{d}}^m)} \left[(1-\gamma)\pi + v\right] P_{x|M(w,\hat{\mathbb{d}}^m)}^m R_{x|M(w,\hat{\mathbb{d}}^m)}^n + \sum_{x \in M(w, \hat{\mathbb{d}}^m)} w P_{x|M(w,\hat{\mathbb{d}}^m)}^r \end{aligned} \tag{4.12}$$

where the first term refers to the profit obtained from products that are purchased from the manufacturer's online channel and kept by the customers post-purchase, the second term is the profit obtained from products that are purchased from the online channel and returned, and the last term is the profit obtained from selling the products to the retailer for the wholesale price, as the retailer orders products to fulfill the purchases by customers in the physical store. Noting that $K_{x|M(w,\hat{\mathbb{d}}^m)}^m = 1 - R_{x|M(w,\hat{\mathbb{d}}^m)}^m$ and defining $\mathcal{P}_{x|M(w,\hat{\mathbb{d}}^m)}^m = (1 - R_{x|M(w,\hat{\mathbb{d}}^m)}^m)\pi + R_{x|M(w,\hat{\mathbb{d}}^m)}^m \left[(1-\gamma)\pi + v\right] = \pi - R_{x|M(w,\hat{\mathbb{d}}^m)}^m(\gamma\pi - v), \forall x \in X \setminus M(w, \hat{\mathbb{d}}^m)$ in (4.12) as the marginal profit obtained from a product sold in the online sales website, the manufacturer's expected profit can be rewritten as

$$\Pi_{\mathcal{D}}^m(w|\hat{\mathbb{d}}^m) = \sum_{x \in X \setminus M(w, \hat{\mathbb{d}}^m)} P_{x|M(w,\hat{\mathbb{d}}^m)}^m \mathcal{P}_{x|M(w,\hat{\mathbb{d}}^m)}^m + \sum_{x \in M(w, \hat{\mathbb{d}}^m)} w P_{x|M(w,\hat{\mathbb{d}}^m)}^r. \tag{4.13}$$

Consequently, the manufacturer solves the following problem to find the optimal wholesale price, w:

$$\max_{w} \Pi^{m}_{\mathbb{D}}(w|\hat{\mathbb{d}}^{m}). \tag{4.14}$$

In the previous chapter, in Proposition 3.6, we showed that the determined wholesale price depends on the assortment showcased in the physical store and the retailer's opportunity cost; i.e., the fraction of its sales that the retailer wishes to be its profit. Our results denoted that the wholesale price will be set to the greatest value that the manufacturer believes the retailer is willing to pay and operate its store, and we formulated this decision.

If there is no asymmetry between the estimations and the true values, then all parties would have used the inaccuracy vector $\mathbb{d}$ in identifying their respective optimal actions. Let $M^{*}_{\mathbb{D}}$ and $w^{*}_{\mathbb{D}}$ be the optimal assortment and the wholesale price under this ideal situation. Since asymmetry can exist in practice, the optimal solutions might differ from the ideal situation and from one party to the other. Let $\hat{M}^{r}_{\mathbb{D}}$ be the optimal assortment that the retailer would order with $\hat{\mathbb{d}}^{r}$ and let $\hat{w}^{r}_{\mathbb{D}}$ be the maximum wholesale price that they would be willing to pay in this situation. Similarly, let $\hat{M}^{m}_{\mathbb{D}}$ be the optimal assortment found with $\hat{\mathbb{d}}^{m}$ that the manufacturer anticipates that the retailer will order and $\hat{w}^{m}_{\mathbb{D}}$ be the optimal wholesale price quoted by the manufacturer.

After these parameters are resolved by each party independently, the manufacturer announces $\hat{w}^{m}_{\mathbb{D}}$. If it is greater than $\hat{w}^{r}_{\mathbb{D}}$, the retailer decides not to order from this manufacturer because the charged price is greater than the maximum value that it can pay. Therefore, the manufacturer operates its online sales channel as the only sales medium for this particular product family. Whereas, if $\hat{w}^{m}_{\mathbb{D}} \leq \hat{w}^{r}_{\mathbb{D}}$, the retailer accepts the quoted wholesale price and showcases its assortment. In this situation, the final wholesale price and assortment decisions under which the retail system operates are $\hat{w}^{m}_{\mathbb{D}}$ and $\hat{M}^{r}_{\mathbb{D}}$.

Note that if the retailer exits the market, there will be no medium to reveal information about products' utilities and attribute levels; so customers have to purchase with inaccurate assessments. Moreover, all the customers who could have ended up purchasing in either of the channels now will have to purchase from the online website.

Depending on the level of the information asymmetry, expectations of each party may not be met when the customers starts browsing and buying from the showcased assortment and the online sales channel, and then potentially returning their products as they are disappointed. We introduce the following notation to denote the expected profit of each party under their best estimate of the inaccuracy vector and under the true inaccuracy vector:

$$\begin{aligned}
\hat{\Pi}^r_{\mathcal{D}} &= \Pi^r_{\mathcal{D}}(\hat{M}^r_{\mathcal{D}}|\hat{w}^m_{\mathcal{D}}, \hat{\mathbb{d}}^r) \\
\Pi^r_{\mathcal{D}}{}^* &= \Pi^r_{\mathcal{D}}(M^*|w^*, \mathbb{d}) \\
\hat{\Pi}^m_{\mathcal{D}} &= \Pi^m_{\mathcal{D}}(\hat{w}^m_{\mathcal{D}}, \hat{\mathbb{d}}^m) \\
\Pi^{m*}_{\mathcal{D}} &= \Pi^m_{\mathcal{D}}(w^*, \mathbb{d})
\end{aligned}$$

In the above expressions, $\hat{w}^m_{\mathcal{D}}$ is the solution to (4.14) with $\hat{\mathbb{d}}^m$, and $\hat{M}^r_{\mathcal{D}}$ is the optimal assortment found by solving (4.11) with $\hat{w}^m_{\mathcal{D}}$ and $\hat{\mathbb{d}}^r$. Obviously, $\Pi^r_{\mathcal{D}}{}^*$ and $\Pi^{m*}_{\mathcal{D}}$ are the ideal expected profits of the retailer and the manufacturer if there was no information asymmetry.

We define $\mathcal{R}^r_{\mathcal{D}}(\hat{\mathbb{d}}^r, \hat{\mathbb{d}}^m, \mathbb{d})$ as the retailer's regret function when the true inaccuracies $\mathbb{d}$ are estimated by the retailer as $\hat{\mathbb{d}}^r$ and by the manufacturer as $\hat{\mathbb{d}}^m$. $\mathcal{R}^r_{\mathcal{D}}(\hat{\mathbb{d}}^r, \hat{\mathbb{d}}^m, \mathbb{d})$ is the disparity between the retailers optimal expected profit that it would have achieved if $\mathbb{d}$ was known and the expected profit that it obtains as a result of $\hat{\mathbb{d}}^r$ and $\hat{\mathbb{d}}^m$. Similarly, the manufacturer's regret function is denoted by $\mathcal{R}^m_{\mathcal{D}}(\hat{\mathbb{d}}^r, \hat{\mathbb{d}}^m, \mathbb{d})$, and the total RSC regret function is given by $\mathcal{R}^T_{\mathcal{D}}(\hat{\mathbb{d}}^r, \hat{\mathbb{d}}^m, \mathbb{d})$. Therefore, according to the definition, we have

$$\mathcal{R}^r_{\mathcal{D}}(\hat{\mathbb{d}}^r, \hat{\mathbb{d}}^m, \mathbb{d}) = \Pi^r_{\mathcal{D}}{}^* - \hat{\Pi}^r_{\mathcal{D}} \tag{4.15a}$$

$$\mathcal{R}^m_{\mathcal{D}}(\hat{\mathbb{d}}^r, \hat{\mathbb{d}}^m, \mathbb{d}) = \Pi^{m*}_{\mathcal{D}} - \hat{\Pi}^m_{\mathcal{D}} \tag{4.15b}$$

$$\mathcal{R}^T_{\mathcal{D}}(\hat{\mathbb{d}}^r, \hat{\mathbb{d}}^m, \mathbb{d}) = \Pi^r_{\mathcal{D}}{}^* + \Pi^{m*}_{\mathcal{D}} - \hat{\Pi}^r_{\mathcal{D}} - \hat{\Pi}^m_{\mathcal{D}}. \tag{4.15c}$$

Consequently, we let $\mathcal{PR}^r_{\mathcal{D}}(\hat{\mathbb{d}}^r, \hat{\mathbb{d}}^m, \mathbb{d})$, $\mathcal{PR}^m_{\mathcal{D}}(\hat{\mathbb{d}}^r, \hat{\mathbb{d}}^m, \mathbb{d})$, and $\mathcal{PR}^T_{\mathcal{D}}(\hat{\mathbb{d}}^r, \hat{\mathbb{d}}^m, \mathbb{d})$ be the percentage of regret for the retailer and the manufacturer, respectively, defined as the percentage of deviation from their optimal profit if the true value of inaccuracies, i.e., $\mathbb{d}$, were known. Therefore

$$\mathcal{PR}^r_{\mathcal{D}}(\hat{\mathbb{d}}^r, \hat{\mathbb{d}}^m, \mathbb{d}) = \frac{(\Pi^r_{\mathcal{D}}{}^* - \hat{\Pi}^r_{\mathcal{D}}) \times 100}{\Pi^r_{\mathcal{D}}{}^*} \tag{4.16a}$$

$$\mathcal{PR}^m_{\mathcal{D}}(\hat{\mathbb{d}}^r, \hat{\mathbb{d}}^m, \mathbb{d}) = \frac{(\Pi^{m*}_{\mathcal{D}} - \hat{\Pi}^m_{\mathcal{D}}) \times 100}{\Pi^{m*}_{\mathcal{D}}} \tag{4.16b}$$

$$\mathcal{PR}^T_{\mathcal{D}}(\hat{\mathbb{d}}^r, \hat{\mathbb{d}}^m, \mathbb{d}) = \frac{(\Pi^T_{\mathcal{D}}{}^* - \hat{\Pi}^T_{\mathcal{D}}) \times 100}{\Pi^T_{\mathcal{D}}{}^*}. \tag{4.16c}$$

Proposition 4.1 below states a specific situation where there is no inaccuracy in customers' online assessments of products.

Proposition 4.1. *Suppose that $d_k = 0, \forall k \in A$. Under the decentralized setting, if $\hat{w}^r_{\mathcal{D}} \geq \hat{w}^m_{\mathcal{D}}$, for any* *arbitrary* $\hat{\mathbb{d}}^r$ *and* $\hat{\mathbb{d}}^m$*, we have* $\Pi_{\mathcal{D}}^{r\,*} + \Pi_{\mathcal{D}}^{m\,*} = \hat{\Pi}_{\mathcal{D}}^r + \hat{\Pi}_{\mathcal{D}}^m$.

Proposition 4.1 states that if $d_k = 0, \forall k \in A$ and the manufacturer quotes a wholesale price that the retailer can bear with (i.e., is willing to pay), then independent of what estimation of the inaccuracies the retailer and the manufacturer make, the total expected profit of the RSC is the same as that if $d_k = 0, \forall k \in A$ were known. However, given $\hat{M}_{\mathcal{D}}^r$, the retailer and the manufacturer sell certain products that may not align with $M_{\mathcal{D}}^*$, resulting in possibly different individual expected profits compared to those if $d_k = 0, \forall k \in A$ were known; i.e., although we have $\Pi_{\mathcal{D}}^{r\,*} + \Pi_{\mathcal{D}}^{m\,*} = \hat{\Pi}_{\mathcal{D}}^r + \hat{\Pi}_{\mathcal{D}}^m$, we may encounter $\hat{\Pi}_{\mathcal{D}}^r \neq \Pi_{\mathcal{D}}^{r\,*}$ and $\hat{\Pi}_{\mathcal{D}}^m \neq \Pi_{\mathcal{D}}^{m\,*}$.

4.4.2 Centralized Setting

In the centralized setting, there is a central authority that manages the RSC in a way to maximize the total expected profit. This function is the sum of the expected profit functions of the retailer and the manufacturer in (4.10) and (4.13):

$$\Pi_{\mathcal{C}}^T(M|\mathbb{d}) = \sum_{x \in X \setminus M} P_{x|M}^m \mathcal{P}_{x|M}^m + \sum_{x \in M} P_{x|M}^r \mathcal{P}_x^r. \tag{4.17}$$

Hence, the central authority solves the following problem:

$$\begin{aligned} &\max_{M \subset X} \Pi_{\mathcal{C}}^T(M|\mathbb{d}) \\ &\text{s. to} \quad |M| \leq C \end{aligned} . \tag{4.18}$$

Let $M_{\mathcal{C}}^*$ be the optimal assortment that solves this problem under the true values of the inaccuracy vector, $\mathbb{d}$ and $\Pi_{\mathcal{C}}^{T^*}$ be the total system-wide optimal expected profit. Similar to the decentralized case, we define the best guess of the central authority for the inaccuracy vector as $\hat{\mathbb{d}}$. Let $\hat{M}_{\mathcal{C}}$ be the optimal assortment showcased and $\hat{\Pi}_{\mathcal{C}}^T$ be the total expected profit, both under $\hat{\mathbb{d}}$.

Similar to the decentralized setting, we define $\mathcal{R}_{\mathcal{C}}^T(\hat{\mathbb{d}}, \mathbb{d})$ as the RSC's regret function when the true inaccuracies $\mathbb{d}$ are estimated by $\hat{\mathbb{d}}$. $\mathcal{R}_{\mathcal{C}}^T(\hat{\mathbb{d}}, \mathbb{d})$ is the disparity between the total RSC's optimal expected profit that would have been achieved if $\mathbb{d}$ was known and the expected profit that is obtained as a result of $\hat{\mathbb{d}}$. Given this definition, we have

$$\mathcal{R}_{\mathcal{C}}^T(\hat{\mathbb{d}}, \mathbb{d}) = \Pi_{\mathcal{C}}^{T^*} - \hat{\Pi}_{\mathcal{C}}^T \tag{4.19}$$

Moreover, we also let $\mathcal{PR}_{\mathcal{C}}^T(\hat{\mathbb{d}}, \mathbb{d})$ be the percentage of regret for the total RSC profit, defined as the percentage of deviation from the optimal total profit if the true value of inaccuracies, i.e., $\mathbb{d}$, was known. Hence

$$\mathcal{PR}_{\mathcal{C}}^T(\hat{\mathbb{d}}, \mathbb{d}) = \frac{(\Pi_{\mathcal{C}}^{T^*} - \hat{\Pi}_{\mathcal{C}}^T) \times 100}{\Pi_{\mathcal{C}}^{T^*}}. \tag{4.20}$$

Note that in the centralized setting, the profits in both sales channels are obtained by the central authority as the total expected profit of the system. Therefore, it does not matter if a specific product is showcased and sold in the physical store or in the online channel. Rather, the critical factor is the showcased levels of attributes that are supposed to be inaccurately assessed on the online website. In the following, Proposition 4.2 describes the optimal assortment in this situation.

Proposition 4.2. *In the centralized setting, selecting a subset of levels of attributes that are estimated to be inaccurately assessed in the online channel (i.e., attributes with $\hat{d}_k \neq 0, \forall k \in A$) suffices to determine optimal assortment. Any arbitrary assortment of products (and hence, attributes with $\hat{d}_k = 0, \forall k \in A$) representing the selected levels of such attributes is optimal.*

Proposition 4.2 indicates that under the centralized setting, determining $S_{\mathbb{C}}(k), \forall k \in A$ is sufficient to have an optimal assortment. Therefore, there may be multiple assortments under this setting.

Proposition 4.3. *Under the centralized setting:*

(i) Suppose that $d_k = 0, \forall k \in A$. For any estimate of inaccuracy $\hat{\mathbb{d}}$, any arbitrary assortment is optimal and results in the same total expected profit.

(ii) When an estimate of $\hat{\mathbb{d}} = 0$ is obtained, any selection of attribute levels showcased in the physical store is considered optimal by the central authority. However, these assortments can result in different $\hat{\Pi}^T_{\mathbb{C}}(\hat{M}_{\mathbb{C}})$ and $\mathcal{PR}^T_{\mathbb{C}}(\hat{\mathbb{d}}, \mathbb{d})$, depending on the true value of inaccuracies, i.e., $\mathbb{d}$.

In part (i) of Proposition 4.3, note that there is no inaccuracy present in customers' assessment of products on the online channel. Therefore, no specific assortment reveals additional information to the customers. Since we face the centralized setting, profits from both channels are obtained by the central authority; hence, any assortment yields the same total expected profit. Part (ii) of Proposition 4.3 states that if $\hat{\mathbb{d}} = 0$ is the obtained estimate of inaccuracies, under the centralized setting, according to Proposition 4.2, any assortment is considered optimal by the central authority. Note that the central authority makes its assortment decision based on its assumed estimate $\hat{\mathbb{d}} = 0$; therefore, based on this estimation, any assortment is considered optimal. However, depending on the assortment that is eventually showcased, and given the true inaccuracies, the expected profit and regret can be different.

4.5 Numerical Studies and Managerial Insights

In this section, we conduct numerical analysis to investigate the expected profits and regret values under the centralized setting (Section 4.5.2) and the decentralized setting (Section 4.5.3).

4.5.1 Test Bed

Suppose that the product under concern in the RSC consists of 3 attributes, $K = 3$, where attributes 1, 2, and 3 include three, two, and three different levels, respectively, making up $3 \times 2 \times 3 = 18$ combinations of different attribute levels, each corresponding to a unique item. The part-worth utility of levels of attributes are presented in Table 4.1. We assume that $\mathbb{d} = (0, 0, d_3)$ where $d_3 \in \{-1, -0.8, -0.6, -0.4, -0.2, 0, 0.2, 0.4, 0.6, 0.8, 1\}$ and $\hat{\mathbb{d}} = (0, 0, \hat{d}_3)$ where $\hat{d}_3$ can take any of the possible values of d_3 as the best estimate of the decision maker. Finally, we let $\pi = 100$, $v = 30$, $\Omega = 0.4$, $r = -0.8$, $\gamma = 0.6$, $\mu = \mu' = 1$, $\beta_1 = -0.013$, $\beta_2 = -0.01$, $\omega = 0.5$, and $\phi = 0.4$. We consider three different showcase capacities in the physical store as $C \in \{3, 6, 12\}$ such that they refer to small, medium, and large showrooms, respectively. Table 4.2 presents the attribute levels in each product listed in the descending order of their product utilities, $\overline{U}_x$, in columns.

Table 4.1: Part-worth utilities of levels of attributes 1, 2, and 3

Attribute k	1			2		3		
Levels $l(k)$	1	2	3	1	2	1	2	3
$\widetilde{u}_{k,l(k)}$	0.1	0.3	0.5	0.15	0.35	0.05	0.25	0.45

Table 4.2: Products based on $\overline{U}_x$ and their corresponding attribute levels. $[n]$ represents the product which has the nth highest product utility.

Product	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]	[14]	[15]	[16]	[17]	[18]
A1 level	3	3	3	2	3	3	2	2	3	1	2	2	1	1	2	1	1	1
A2 level	2	1	2	2	1	2	2	1	1	2	2	1	2	1	1	2	1	1
A3 level	3	3	2	3	2	1	2	3	1	3	1	2	2	3	1	1	2	1

4.5.2 Centralized Setting

Optimal Decisions When d_3 is Known

Table 4.3 below presents the optimal $S^*_{\mathbb{C}}(3)$ and $\Pi^{T^*}_{\mathbb{C}}$ for each of the small, medium, and large capacities, when d_3 is known. Note that given Proposition 4.2, determining $S^*_{\mathbb{C}}(3)$ is sufficient since attribute 3 is the only attribute with inaccurately assessed levels. Given this, after identifying $S^*_{\mathbb{C}}(3)$, any assortment of products selected given C representing this set is optimal; i.e., an optimal $M^*_{\mathbb{C}}$.

In Table 4.3, the central authority turns out to be willing to showcase all levels of A3 when $d_3 < 0$ so that no inaccuracy will be present in the customers' online assessment of products. Therefore, in all $C = 3$, $C = 6$, and $C = 12$, we have $S^*_{\mathbb{C}}(3) = \{1, 2, 3\}$. One implication of this decision is that since all inaccuracy is eliminated from the online channel, $\Pi^{T^*}_{\mathbb{C}}$ values are all equal for $d_3 < 0$. Note that for $d_3 < 0$, $\Pi^{T^*}_{\mathbb{C}}$ values are equal to that of $d_3 = 0$. This is also because $S^*_{\mathbb{C}}(3) = \{1, 2, 3\}$ includes all the levels of A3 in the showcase;

Table 4.3: Optimal solutions for examples in the test bed under centralized setting when d_3 is Known*

		d_3										
		-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
$C=3$	$S^*_{\mathcal{C}}(3)$	1-3	1-3	1-3	1-3	1-3	**	1	1	1	1	1
	$\Pi^{T^*}_{\mathcal{C}}$	77.09	77.09	77.09	77.09	77.09	77.09	78.44	79.61	80.58	81.36	81.93
$C=6$	$S^*_{\mathcal{C}}(3)$	1-3	1-3	1-3	1-3	1-3	**	1	1	1	1	1
	$\Pi^{T^*}_{\mathcal{C}}$	77.09	77.09	77.09	77.09	77.09	77.09	78.44	79.61	80.58	81.36	81.93
$C=12$	$S^*_{\mathcal{C}}(3)$	1-3	1-3	1-3	1-3	1-3	**	1-2	1-2	1-2	1-2	1-2
	$\Pi^{T^*}_{\mathcal{C}}$	77.09	77.09	77.09	77.09	77.09	77.09	77.92	78.73	79.50	80.20	80.81

*Under the centralized setting, any C products that represent $S^*_{\mathcal{C}}(3)$ is an optimal $M^*_{\mathcal{C}}$

**Any selections of levels of attribute 3 is an optimal $S^*_{\mathcal{C}}(3)$, since there is no inaccuracy

therefore, no inaccuracy will be present in the online sales website. Hence, $\Pi^{T^*}_{\mathcal{C}}$ in all these cases is the same as $d_3 = 0$.

Furthermore, when $d_3 > 0$, the central authority is reluctant to showcase a great variety of levels of A3, unless capacity forces it to do so. Therefore, at $C = 3$ and $C = 6$, it only showcases one level of A3. However, at $C = 12$ the capacity forces to showcase a greater variety, although it is not desired by the central authority; Thus, one more level of A3 is included in the showcase, $\Pi^{T^*}_{\mathcal{C}}$ decreases. Note that since $S^*_{\mathcal{C}}(3)$ for $C = 3$ and $C = 6$ is the same when $d_3 > 0$, the value of $\Pi^{T^*}_{\mathcal{C}}$ at each d_3 also the same. However, at $C = 12$, because one additional level is included in $S^*_{\mathcal{C}}(3)$ compared to that at $C = 3$ and $C = 6$, and it was undesired, $\Pi^{T^*}_{\mathcal{C}}$ is smaller.

Decisions and Regret Analysis When d_3 is Unknown

When d_3 is not known to the central authority, it will be estimated as $\hat{d}_3$. Note that the selected levels of attribute 3 in Table 4.3 also hold for this section, albeit they will be obtained based on an estimation of the inaccuracy, i.e., $\hat{d}_3$, and therefore, the expected profit values may not be the same as Table 4.3. This is, the central authority makes an estimation of the true value of inaccuracy, and then based on this estimation, it selects the levels of attribute 3 that would optimize the total expected profit if the estimate was correct. Therefore, as can be inferred, although the decisions in Table 4.3 hold for any estimation $\hat{d}_3$, the $\Pi^{T^*}_{\mathcal{C}}$ values may not hold true, since the value of d_3 may be different than $\hat{d}_3$. In this case, an assortment selected based on $\hat{d}_3$ may not be actually optimal when $d_3 \neq \hat{d}_3$.

Figures 4.1, 4.2, and 4.3 below indicate the $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3)$ for all combinations of $\hat{d}_3$ and d_3, with color codes. Regarding the color codes, $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3) = 0$ indicates that the regret percentage in the total expected profit when d_3 is estimated by $\hat{d}_3$ is zero, and we use color "*white*" to describe this situation. As the value of $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3)$ increases (i.e., there is greater regret through the obtained estimation), the color will become a darker shade of "*red*".

Figure 4.1: $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3)$ for different combinations of $\hat{d}_3$ and d_3 when $C = 3$, under centralized setting

	d_3										
$\hat{d}_3$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
-1	0	0	0	0	0	0	1.72	3.17	4.34	5.25	5.91
-0.8	0	0	0	0	0	0	1.72	3.17	4.34	5.25	5.91
-0.6	0	0	0	0	0	0	1.72	3.17	4.34	5.25	5.91
-0.4	0	0	0	0	0	0	1.72	3.17	4.34	5.25	5.91
-0.2	0	0	0	0	0	0	1.72	3.17	4.34	5.25	5.91
0.2	10.85	8.58	6.30	4.07	1.96	0	0	0	0	0	0
0.4	10.85	8.58	6.30	4.07	1.96	0	0	0	0	0	0
0.6	10.85	8.58	6.30	4.07	1.96	0	0	0	0	0	0
0.8	10.85	8.58	6.30	4.07	1.96	0	0	0	0	0	0
1	10.85	8.58	6.30	4.07	1.96	0	0	0	0	0	0

Figure 4.2: $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3)$ for different combinations of $\hat{d}_3$ and d_3 when $C = 6$, under centralized setting

	d_3										
$\hat{d}_3$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
-1	0	0	0	0	0	0	1.72	3.17	4.34	5.25	5.91
-0.8	0	0	0	0	0	0	1.72	3.17	4.34	5.25	5.91
-0.6	0	0	0	0	0	0	1.72	3.17	4.34	5.25	5.91
-0.4	0	0	0	0	0	0	1.72	3.17	4.34	5.25	5.91
-0.2	0	0	0	0	0	0	1.72	3.17	4.34	5.25	5.91
0.2	10.85	8.58	6.30	4.07	1.96	0	0	0	0	0	0
0.4	10.85	8.58	6.30	4.07	1.96	0	0	0	0	0	0
0.6	10.85	8.58	6.30	4.07	1.96	0	0	0	0	0	0
0.8	10.85	8.58	6.30	4.07	1.96	0	0	0	0	0	0
1	10.85	8.58	6.30	4.07	1.96	0	0	0	0	0	0

Figure 4.3: $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3)$ for different combinations of $\hat{d}_3$ and d_3 when $C = 12$, under centralized setting

	d_3										
$\hat{d}_3$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
-1	0	0	0	0	0	0	1.07	2.09	3.04	3.88	4.61
-0.8	0	0	0	0	0	0	1.07	2.09	3.04	3.88	4.61
-0.6	0	0	0	0	0	0	1.07	2.09	3.04	3.88	4.61
-0.4	0	0	0	0	0	0	1.07	2.09	3.04	3.88	4.61
-0.2	0	0	0	0	0	0	1.07	2.09	3.04	3.88	4.61
0.2	4.85	4.01	3.10	2.11	1.07	0	0	0	0	0	0
0.4	4.85	4.01	3.10	2.11	1.07	0	0	0	0	0	0
0.6	4.85	4.01	3.10	2.11	1.07	0	0	0	0	0	0
0.8	4.85	4.01	3.10	2.11	1.07	0	0	0	0	0	0
1	4.85	4.01	3.10	2.11	1.07	0	0	0	0	0	0

In Figures 4.1 to 4.3, if $\hat{d}_3 = d_3$ (i.e., the estimation of the true value of inaccuracy is exactly correct), then $\mathcal{PR}^T_{\mathcal{C}}(d_3, \hat{d}_3) = 0$. If the resulting showcased attribute levels based on an obtained estimation of inaccuracy is the same as the optimal selection if the true inaccuracy was known (i.e., $\hat{S}_{\mathcal{C}}(3) = S^*_{\mathcal{C}}$), then $\mathcal{PR}^T_{\mathcal{C}}(d_3, \hat{d}_3) = 0$. In Table 4.3, for all $d_3 > 0$ and for all $d_3 < 0$, for each of $C = 3, 6, 12$, the selected levels of A3 are the same. Therefore, in Figures 4.1 to 4.3, when $d_3 > 0$ and $\hat{d}_3 > 0$, or $d_3 < 0$ and $\hat{d}_3 < 0$, we also have $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3) = 0$. Moreover, according to Proposition 4.3, when $d_3 = 0$, any estimation obtained by the central authority results in $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3) = 0$, because the showcased levels do not reveal any additional information about products and attribute levels.

If the resulting showcased attribute levels based on an obtained estimation is different than the optimal selection when the true inaccuracy was known (i.e., $\hat{S}_{\mathcal{C}}(3) \neq S^*_{\mathcal{C}}$), then $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3) > 0$. According to Table 4.3, $\hat{S}_{\mathcal{C}}(3)$ for $\hat{d}_3 > 0$ and $\hat{d}_3 < 0$ are different. Therefore, in Figures 4.1 to 4.3, estimating the true value of inaccuracy with $\hat{d}_3 > 0$ while $d_3 < 0$ (or vice versa), results in different showcased levels of attribute 3, and therefore results in regret. Note that under the centralized setting, the RSC cannot fare better when an inaccurate estimation is obtained.

Comparing the regret values for pairs of $(\hat{d}_3, d_3)$ in Figures 4.1 to 4.3, for a greater C, we observe that $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3)$ is non-increasing. When $d_3 < 0$, Table 4.3 indicates that the optimal decision is to showcase the full variety of levels of attribute 3; but, if $\hat{d}_3 > 0$, the estimation suggests showcasing only a limited variety. As C increases, although the central authority would like to showcase only a limited variety based on its estimation, at some C it becomes inevitable to showcase a greater variety of levels of attribute 3, resulting in a similar showcased variety to the actual optimal assortment. Whereas, when $d_3 > 0$, Table 4.3 indicates that the optimal decision is to showcase only a limited variety of attribute 3; but if $\hat{d}_3 < 0$, the estimation suggests showcasing the full variety. In this situation, a larger C can mean that the optimal variety of levels of attribute 3 is inevitable to include more levels and becomes more similar to that of estimation. Therefore, as C increases, $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3)$ is non-increasing.

Generally, RSCs may be interested in obtaining an estimation of inaccuracy that results in smaller regret for them. This can provide insights and directions in minimizing the potential regret. Our analysis indicate that given specific circumstances, $\hat{d}_3 < 0$ or $\hat{d}_3 > 0$ can be estimated with minimum regret. In all Figures 4.1 to 4.3, both the maximum and the average $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3)$ values when a $\hat{d}_3 < 0$ is obtained is smaller than when $\hat{d}_3 > 0$. To observe this in the mentioned figures, consider a $\hat{d}_3 < 0$ and observe its row, which indicates $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3)$ for different d_3 values. This implies that in this problem instance, the RSC is on average better off with smaller regret when a negative estimation of inaccuracy is obtained. However, this result is specific to this problem instance. Our analysis shows that there exist problem instance where the opposite is true. For example, consider the stated test bed in Section 4.5.1 with $\gamma = 0.98$ and the part-wroth utility of levels of A3 as 0.05, 0.35, and 0.65, respectively. Figure 4.4 below shows the $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3)$ values in this problem instance when $C = 3$. As can be observed, both the maximum and the average $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3)$ values when a $\hat{d}_3 > 0$ is obtained is smaller than when $\hat{d}_3 < 0$.

One crucial case in the regret analysis for the centralized setting is when an estimation of $\hat{d}_3 = 0$ is

Figure 4.4: $\mathcal{PR}_{\mathcal{C}}^{T}(\hat{d}_3, d_3)$ for different combinations of $\hat{d}_3$ and d_3 when $C = 3$ under centralized setting, with $\gamma = 0.98$ and A3 levels 0.05/0.35/0.65

	d_3										
$\hat{d}_3$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
-1	0	0	0	0.02	0.07	0	0.87	1.70	2.49	3.28	4.66
-0.8	0	0	0	0.02	0.07	0	0.87	1.70	2.49	3.28	4.66
-0.6	0	0	0	0.02	0.07	0	0.87	1.70	2.49	3.28	4.66
-0.4	0.72	0.39	0.13	0	0	0	1.04	2.12	3.21	4.33	6.01
-0.2	0.72	0.39	0.13	0	0	0	1.04	2.12	3.21	4.33	6.01
0.2	4.31	3.51	2.66	1.80	0.96	0	0	0	0	0.04	0.74
0.4	4.31	3.51	2.66	1.80	0.96	0	0	0	0	0.04	0.74
0.6	4.31	3.51	2.66	1.80	0.96	0	0	0	0	0.04	0.74
0.8	0.70	0.72	0.69	0.59	0.42	0	0.36	0.49	0.36	0	0
1	0.70	0.72	0.69	0.59	0.42	0	0.36	0.49	0.36	0	0

obtained. According to Proposition 4.3, when $\hat{d}_3 = 0$, any selected levels of attribute 3 is considered optimal by the central authority. However, the specific levels that are decided to be showcased considerably impact the resulting regret. Once the optimal selection based on the true value of inaccuracy is to showcase a greater (or a limited) variety, the best decision is to showcase this variety. However, d_3 is not known to the RSC. In this part, we aim to explore the cases where $\hat{d}_3 = 0$ is the estimate for d_3, to indicate which variety of levels of attribute 3 results in a smaller regret. For this, we consider all the possible $S_{\mathcal{C}}(3)$ given C. Given this, at $C = 3$ and 6, $S_{\mathcal{C}}(3)$ can be any possible selections including only one level (which can be either of levels 1, 2, or 3), two levels (either levels 1 and 2, levels 1 and 3, or levels 2 and 3), and showcasing all three levels of attribute 3. However, at $C = 12$, given that there are only six products consisting of each level of attribute 3, at least two levels must be selected, resulting in showcasing two levels (either levels 1 and 2, levels 1 and 3, or levels 2 and 3), and showcasing all three levels of attribute 3. In the following, Tables 4.4 and 4.5 show $\mathcal{PR}_{\mathcal{C}}^{T}(\hat{d}_3, d_3)$ values when $\hat{d}_3 = 0$ is adopted but d_3 can be any arbitrary value in the defined test bed, for each possible $S_{\mathcal{C}}(3)$.

Table 4.4: $\mathcal{PR}_{\mathcal{C}}^{T}(\hat{d}_3, d_3)$ when $\hat{d}_3 = 0$ for all the possible $S_{\mathcal{C}}(3)$ when $C = 3, 6$

	d_3												
$S_{\mathcal{C}}(3)$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1	Max.	Avg.
1	10.85	8.58	6.30	4.07	1.96	0	0	0	0	0	0	10.85	2.89
2	9.10	7.25	5.37	3.50	1.70	0	0.20	0.35	0.46	0.53	0.58	9.10	2.64
3	7.10	5.71	4.27	2.81	1.37	0	0.47	0.84	1.11	1.31	1.45	7.10	2.40
1, 2	4.85	4.01	3.10	2.11	1.07	0	0.66	1.10	1.34	1.42	1.37	4.85	1.91
1, 3	3.40	2.83	2.19	1.50	0.77	0	0.95	1.66	2.13	2.41	2.53	3.40	1.85
2, 3	2.37	1.97	1.53	1.05	0.54	0	1.18	2.10	2.77	3.23	3.51	3.51	1.84
1, 2, 3	0	0	0	0	0	0	1.72	3.17	4.34	5.25	5.91	5.91	1.85

Table 4.5: $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3)$ when $\hat{d}_3 = 0$ for all the possible $S_{\mathcal{C}}(3)$ when $C = 12$

	d_3												
$S_{\mathcal{C}}(3)$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1	Max.	Avg.
1, 2	4.85	4.01	3.10	2.11	1.07	0	0	0	0	0	0	4.85	1.38
1, 3	3.40	2.83	2.19	1.50	0.77	0	0.29	0.56	0.80	1.01	1.18	3.40	1.32
2, 3	2.37	1.97	1.53	1.05	0.54	0	0.52	1.01	1.45	1.84	2.17	2.37	1.31
1, 2, 3	0	0	0	0	0	0	1.07	2.09	3.04	3.88	4.61	4.61	1.33

In Tables 4.4 and 4.5, the last two columns reflect the maximum and average $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3)$ when selecting its corresponding $S_{\mathcal{C}}(3)$. Given these values, in Table 4.4, the smallest maximum $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3)$ associates with $S_{\mathcal{C}}(3) = \{1, 3\}$, and the smallest average $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3)$ pertains to $S_{\mathcal{C}}(3) = \{2, 3\}$, albeit the latter is very close to that for $S_{\mathcal{C}}(3) = \{1, 3\}$ and $S_{\mathcal{C}}(3) = \{1, 2, 3\}$. In Table 4.5, both the smallest maximum and the smallest average $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3)$ pertain to $S_{\mathcal{C}}(3) = \{2, 3\}$, although the smallest average is also very close to that for $S_{\mathcal{C}}(3) = \{1, 3\}$ and $S_{\mathcal{C}}(3) = \{1, 2, 3\}$. We also carry out the same analysis for the problem instance with $\gamma = 0.98$ and the part-worth utility of levels of A3 as 0.05, 0.3, and 0.65, at $C = 3$. In the following, Table 4.6 indicates the regret analysis for this example. As can be observed, in this case, showcasing the full variety of levels of A3 can achieve both the smallest maximum and average regret. In all Tables 4.4, 4.5, and 4.6, the desired set of levels include the level with the highest part-wroth utility.

Table 4.6: $\mathcal{PR}^T_{\mathcal{C}}(\hat{d}_3, d_3)$ when $\hat{d}_3 = 0$ for all the possible $S_{\mathcal{C}}(3)$ when $C = 3$, with $\gamma = 0.98$ and A3 levels 0.05/0.35/0.65

	d_3												
$S_{\mathcal{C}}(3)$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1	Max.	Avg.
1	6.58	4.86	3.32	2.00	0.94	0	0.21	0.57	1.04	1.60	2.80	6.58	2.18
2	4.13	3.04	2.07	1.25	0.60	0	0.49	1.07	1.71	2.41	3.73	4.13	1.86
3	0.72	0.39	0.13	0	0	0	1.04	2.12	3.21	4.33	6.01	6.01	1.63
1, 2	4.31	3.51	2.66	1.80	0.96	0	0	0	0	0.04	0.74	4.31	1.28
1, 3	1.46	1.20	0.91	0.64	0.38	0	0.56	1.10	1.60	2.11	3.24	3.24	1.20
2, 3	0	0	0	0.02	0.07	0	0.87	1.70	2.49	3.28	4.66	4.66	1.19
1, 2, 3	0.70	0.72	0.69	0.59	0.42	0	0.36	0.49	0.36	0	0	0.72	0.39

4.5.3 Decentralized Setting

Under the decentralized setting, we consider two information asymmetry scenarios. In the first scenario, MDRD, neither the manufacturer nor the retailer knows d_3. However, we suppose that their estimates are equal, i.e., $\hat{d}^r_3 = \hat{d}^m_3 = \hat{d}_3$. This scenario may resemble a situation where the parties cooperate or one party shares their estimation with the other. In the second scenario, MKRD, the manufacturer possesses private information about the inaccuracy (i.e., the manufacturer knows d_3), but it does not share this information with the retailer. Hence, the retailer estimates d_3 as $\hat{d}_3^r$. We use $\mathcal{PR}_{\mathcal{D}}^r(\hat{d}_3, d_3)$, $\mathcal{PR}_{\mathcal{D}}^m(\hat{d}_3, d_3)$, and $\mathcal{PR}_{\mathcal{D}}^T(\hat{d}_3, d_3)$ in MDRD where $\hat{d}_3$ denotes the cooperatively obtained estimation. Also, in MKRD, we use $\mathcal{PR}_{\mathcal{D}}^r(\hat{d}_3^r, d_3)$, $\mathcal{PR}_{\mathcal{D}}^m(\hat{d}_3^r, d_3)$, and $\mathcal{PR}_{\mathcal{D}}^T(\hat{d}_3^r, d_3)$ where the retailer obtains $\hat{d}_3^r$ as its estimation while the manufacturer knows d_3.

Optimal Decisions When d_3 is Known

Table 4.7 below indicates the optimal decisions for each of the small, medium, and large capacities when it is assumed that d_3 is known to both the manufacturer and the retailer.

Table 4.7: Optimal solutions for the examples in the test bed under decentralized setting when d_3 is known

	$C = 3$							
d_3	$\Pi_{\mathcal{D}}^{m*}$	$\Pi_{\mathcal{D}}^{r*}$	$\Pi_{\mathcal{D}}^{m*}+\Pi_{\mathcal{D}}^{r*}$	$S(1\|M_{\mathcal{D}}^*)$	$S(2\|M_{\mathcal{D}}^*)$	$S(3\|M_{\mathcal{D}}^*)$	$M_{\mathcal{D}}^*$	$w_{\mathcal{D}}^*$
-1	60.78	10.83	71.61	2-3	1-2	3	[1],[2],[4]	55.81
-0.8	62.47	10.21	72.68	2-3	1-2	3	[1],[2],[4]	55.81
-0.6	64.25	9.55	73.80	2-3	1-2	3	[1],[2],[4]	55.81
-0.4	66.08	8.84	74.92	2-3	1-2	3	[1],[2],[4]	55.81
-0.2	67.92	8.11	76.03	2-3	1-2	3	[1],[2],[4]	55.81
0	69.73	7.37	77.09	2-3	1-2	3	[1],[2],[4]	55.81
0.2	70.50	7.01	77.51	2-3	2	2-3	[1],[3],[4]	55.81
0.4	70.15	6.93	77.08	3	2	1-3	[1],[3],[6]	55.64
0.6	70.15	6.93	77.08	3	2	1-3	[1],[3],[6]	55.64
0.8	70.15	6.93	77.08	3	2	1-3	[1],[3],[6]	55.64
1	70.15	6.93	77.08	3	2	1-3	[1],[3],[6]	55.64
	$C = 6$							
-1	53.18	18.43	71.61	1-3	1-2	3	[1],[2],[4],[8],[10],[14]	55.33
-0.8	55.31	17.37	72.68	1-3	1-2	3	[1],[2],[4],[8],[10],[14]	55.33
-0.6	57.56	16.24	73.80	1-3	1-2	3	[1],[2],[4],[8],[10],[14]	55.33
-0.4	59.88	15.04	74.92	1-3	1-2	3	[1],[2],[4],[8],[10],[14]	55.33
-0.2	62.77	13.90	76.67	1-3	1-2	2-3	[1],[2],[3],[4],[5],[10]	55.51
0	63.76	13.32	77.08	1-3	1-2	2-3	[1],[2],[3],[4],[8],[10]	55.51
0.2	63.76	13.32	77.08	1-3	1-2	1-3	[1],[2],[3],[4],[6],[10]	55.51
0.4	63.76	13.32	77.08	1-3	1-2	1-3	[1],[2],[3],[4],[6],[10]	55.51
0.6	63.76	13.32	77.08	1-3	1-2	1-3	[1],[2],[3],[4],[6],[10]	55.51
0.8	63.76	13.32	77.08	1-3	1-2	1-3	[1],[2],[3],[4],[6],[10]	55.51
1	63.76	13.32	77.08	1-3	1-2	1-3	[1],[2],[3],[4],[6],[10]	55.51
	$C = 12$							
-1	48.79	26.47	75.26	1-3	1-2	2-3	[1]-[5],[7],[8],[10],[12],[13],[14],[17]	55
-0.8	49.66	25.91	75.57	1-3	1-2	2-3	[1]-[5],[7],[8],[10],[12],[13],[14],[17]	55
-0.6	50.65	25.26	75.91	1-3	1-2	2-3	[1]-[5],[7],[8],[10],[12],[13],[14],[17]	55
-0.4	51.78	24.50	76.27	1-3	1-2	2-3	[1]-[5],[7],[8],[10],[12],[13],[14],[17]	55
-0.2	53.04	23.64	76.67	1-3	1-2	2-3	[1]-[5],[7],[8],[10],[12],[13],[14],[17]	55
0	53.78	23.30	77.08	1-3	1-2	1-3	[1]-[8],[10],[12]-[14]	55.01
0.2	53.78	23.30	77.08	1-3	1-2	1-3	[1]-[8],[10],[12]-[14]	55.01
0.4	53.78	23.30	77.08	1-3	1-2	1-3	[1]-[8],[10],[12]-[14]	55.01
0.6	53.78	23.30	77.08	1-3	1-2	1-3	[1]-[8],[10],[12]-[14]	55.01
0.8	53.78	23.30	77.08	1-3	1-2	1-3	[1]-[8],[10],[12]-[14]	55.01
1	53.78	23.30	77.08	1-3	1-2	1-3	[1]-[8],[10],[12]-[14]	55.01

Note that in Table 4.7, the retailer decides its assortment to maximize its own expected profit function, and the manufacturer sets its wholesale price to the maximum value that the retailer would pay, given

$\Omega = 0.4$. Hence, unlike the centralized setting, other than $S(3|M^*_{\mathcal{D}})$, the specific products selected, i.e., $M^*_{\mathcal{D}}$ determines each party's and also the total expected profits. In this table, for the same d_3, as C increases, the retailer's expected profit is increasing, which is expected since the retailer can showcase more products and obtain a higher profit. In specific, as can be observed, the ratio of store showcase size (i.e., C) is roughly reflected on the ratio of ${\Pi^r_{\mathcal{D}}}^*$ values. Whereas, the manufacturer's and the total RSC's expected profits do not show a consistent behavior; i.e., they can increase or decrease because, under the decentralized setting, the assortment is determined by the retailer to optimize its own expected profit. Furthermore, the variety of showcased levels of attribute 3, as the attribute inaccurately assessed in the online sales channel, is greater when $d_3 > 0$ compared to when $d_3 < 0$.

Decisions and Regret Analysis in MDRD

In the following, Figures 4.5 to 4.7 show $\mathcal{PR}^r_{\mathcal{D}}(\hat{d}_3, d_3)$, $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}_3, d_3)$, and $\mathcal{PR}^T_{\mathcal{D}}(\hat{d}_3, d_3)$ for any combination of $\hat{d}_3$ and d_3. It should be noted that since both the manufacturer and the retailer obtain the same estimation of d_3 in this scenario, the wholesale price determined by the manufacturer is always the same as the highest price that the retailer is willing to pay. Therefore, the retailer never prefers to withdraw from the market.

When the selected assortments given Table 4.7 are the same for a d_3 and its estimation $\hat{d}_3$, there is no regret for either of the parties because resulting expected profits will be the same as if the estimate was precise. This can explain the $\mathcal{PR}^r_{\mathcal{D}}(\hat{d}_3, d_3) = 0$, $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}_3, d_3) = 0$, and $\mathcal{PR}^T_{\mathcal{D}}(\hat{d}_3, d_3) = 0$ values in Figures 4.5 to 4.7. Moreover, when $d_3 = 0$, different estimations can yield $\mathcal{PR}^r_{\mathcal{D}}(\hat{d}_3, d_3) < 0$ and $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}_3, d_3) > 0$; however, we always have $\mathcal{PR}^T_{\mathcal{D}}(\hat{d}_3, d_3) = 0$ because similar to the centralized setting, there is no inaccuracy in the online assessments, and so, any assortment results in the same total expected profit for the RSC.

Under the MDRD scenario, the RSC's total expected profit under an imprecise estimation of inaccuracy can be higher, lower, or the same as the total expected profit if the true value of inaccuracy was known. If the estimated assortment includes the attribute levels that are preferred under the centralized setting, the total expected profit will be higher. If the estimated assortment includes the levels that are not desired under the centralized setting, then the total expected profit will be lower. Moreover, if the estimated assortment is the same as the optimal decentralized assortment, the expected total profits will be the same. However, with an imprecise estimation, it is not possible that both parties fare better at the same time. This will resemble a quasi-coordinated RSC, but since no adjustment is provided (i.e., a coordination mechanism), one of the parties is always worse off and the other is better off. Depending on the decisions that the retailer and the manufacturer make, a number of possibilities can happen in terms of the value of regrets. In the following, we elaborate on these possibilities.

First, the retailer and the manufacturer can obtain their optimal expected profits, such that $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}_3, d_3) =$

Figure 4.5: $\mathcal{PR}(\hat{d}_3, d_3)$ for different combinations of $\hat{d}_3$ and d_3 when $C = 3$, under the MDRD scenario

	$\hat{d}_3$ \ d_3	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
Retailer	-1	0	0	0	0	0	0	5.54	14.89	24.93	34.39	43.14
	-0.8	0	0	0	0	0	0	5.54	14.89	24.93	34.39	43.14
	-0.6	0	0	0	0	0	0	5.54	14.89	24.93	34.39	43.14
	-0.4	0	0	0	0	0	0	5.54	14.89	24.93	34.39	43.14
	-0.2	0	0	0	0	0	0	5.54	14.89	24.93	34.39	43.14
	0	0	0	0	0	0	0	5.54	14.89	24.93	34.39	43.14
	0.2	20.54	17.52	13.99	9.92	5.26	0	0	4.41	10.47	16.90	23.61
	0.4	36.01	32.14	27.40	21.62	14.55	5.92	1.18	0	0	0	0
	0.6	36.01	32.14	27.40	21.62	14.55	5.92	1.18	0	0	0	0
	0.8	36.01	32.14	27.40	21.62	14.55	5.92	1.18	0	0	0	0
	1	36.01	32.14	27.40	21.62	14.55	5.92	1.18	0	0	0	0
Manufacturer	-1	0	0	0	0	0	0	-1.34	-4.11	-6.17	-7.96	-9.47
	-0.8	0	0	0	0	0	0	-1.34	-4.11	-6.17	-7.96	-9.47
	-0.6	0	0	0	0	0	0	-1.34	-4.11	-6.17	-7.96	-9.47
	-0.4	0	0	0	0	0	0	-1.34	-4.11	-6.17	-7.96	-9.47
	-0.2	0	0	0	0	0	0	-1.34	-4.11	-6.17	-7.96	-9.47
	0	0	0	0	0	0	0	-1.34	-4.11	-6.17	-7.96	-9.47
	0.2	-9.67	-7.48	-5.36	-3.38	-1.57	0	0	-1.65	-2.83	-4.00	-5.14
	0.4	-15.43	-12.30	-9.19	-6.17	-3.30	-0.63	0.49	0	0	0	0
	0.6	-15.43	-12.30	-9.19	-6.17	-3.30	-0.63	0.49	0	0	0	0
	0.8	-15.43	-12.30	-9.19	-6.17	-3.30	-0.63	0.49	0	0	0	0
	1	-15.43	-12.30	-9.19	-6.17	-3.30	-0.63	0.49	0	0	0	0
Total RSC	-1	0	0	0	0	0	0	-0.71	-2.40	-3.37	-4.15	-4.74
	-0.8	0	0	0	0	0	0	-0.71	-2.40	-3.37	-4.15	-4.74
	-0.6	0	0	0	0	0	0	-0.71	-2.40	-3.37	-4.15	-4.74
	-0.4	0	0	0	0	0	0	-0.71	-2.40	-3.37	-4.15	-4.74
	-0.2	0	0	0	0	0	0	-0.71	-2.40	-3.37	-4.15	-4.74
	0	0	0	0	0	0	0	-0.71	-2.40	-3.37	-4.15	-4.74
	0.2	-5.10	-3.97	-2.86	-1.81	-0.85	0	0	-1.10	-1.63	-2.13	-2.56
	0.4	-7.65	-6.06	-4.46	-2.89	-1.39	0	0.55	0	0	0	0
	0.6	-7.65	-6.06	-4.46	-2.89	-1.39	0	0.55	0	0	0	0
	0.8	-7.65	-6.06	-4.46	-2.89	-1.39	0	0.55	0	0	0	0
	1	-7.65	-6.06	-4.46	-2.89	-1.39	0	0.55	0	0	0	0

and $\mathcal{PR}^r_{\mathcal{D}}(\hat{d}_3, d_3) = 0$. This case happens if the $\hat{M}_{\mathcal{D}}$ (and consequently the $\hat{w}_{\mathcal{D}}$) with the estimation $\hat{d}_3$ is the same as $M^*_{\mathcal{D}}$ if d_3 was known. For example, when $\hat{d}_3 = -0.8$ and $d_3 = -0.6$, given Table 4.7, we have $\hat{M}_{\mathcal{D}} = M^*_{\mathcal{D}}$ and $\hat{w}_{\mathcal{D}} = w^*_{\mathcal{D}}$. Therefore, for $\hat{d}_3 = -0.8$ and $d_3 = -0.6$, we have $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}_3, d_3) =$ and $\mathcal{PR}^r_{\mathcal{D}}(\hat{d}_3, d_3) = 0$ in Figures 4.5 to 4.7.

Second, the manufacturer is better off while the retailer is worse off, such that $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}_3, d_3) > 0$ and $\mathcal{PR}^r_{\mathcal{D}}(\hat{d}_3, d_3) < 0$. This case can happen when the imprecise $\hat{d}_3$ leads to sub-optimal assortment and wholesale price decisions; i.e., $\hat{M}_{\mathcal{D}} \neq M^*_{\mathcal{D}}$ and $\hat{w}_{\mathcal{D}} \neq w^*_{\mathcal{D}}$. In such situations, if the difference between $\hat{w}_{\mathcal{D}}$ and $w^*_{\mathcal{D}}$ is negligible, but $\hat{M}_{\mathcal{D}}$ is considerably different from $M^*_{\mathcal{D}}$, then the retailer's loss due to showcasing a sub-optimal assortment cannot be compensated with the sub-optimal wholesale price, even if $\hat{w}_{\mathcal{D}} < w^*_{\mathcal{D}}$. However, the manufacturer can benefit from the retailer's sub-optimal assortment because it can include products that

Figure 4.6: $\mathcal{PR}(\hat{d}_3, d_3)$ for different combinations of $\hat{d}_3$ and d_3 when $C = 6$, under the MDRD scenario

		d_3										
	$\hat{d}_3$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
Retailer	-1	0	0	0	0	0.71	5.94	15.41	24.68	33.57	41.94	49.68
	-0.8	0	0	0	0	0.71	5.94	15.41	24.68	33.57	41.94	49.68
	-0.6	0	0	0	0	0.71	5.94	15.41	24.68	33.57	41.94	49.68
	-0.4	0	0	0	0	0.71	5.94	15.41	24.68	33.57	41.94	49.68
	-0.2	15.53	12.31	8.56	4.23	0	0	4.80	10.07	15.77	21.82	28.13
	0	15.53	12.31	8.56	4.23	0	0	4.80	10.07	15.77	21.82	28.13
	0.2	27.69	23.31	17.96	11.42	4.13	0	0	0	0	0	0
	0.4	27.69	23.31	17.96	11.42	4.13	0	0	0	0	0	0
	0.6	27.69	23.31	17.96	11.42	4.13	0	0	0	0	0	0
	0.8	27.69	23.31	17.96	11.42	4.13	0	0	0	0	0	0
	1	27.69	23.31	17.96	11.42	4.13	0	0	0	0	0	0
Manufacturer	-1	0	0	0	0	0.87	-1.24	-4.76	-8.06	-11.09	-13.78	-16.11
	-0.8	0	0	0	0	0.87	-1.24	-4.76	-8.06	-11.09	-13.78	-16.11
	-0.6	0	0	0	0	0.87	-1.24	-4.76	-8.06	-11.09	-13.78	-16.11
	-0.4	0	0	0	0	0.87	-1.24	-4.76	-8.06	-11.09	-13.78	-16.11
	-0.2	-12.25	-9.08	-6.08	-3.32	0	0	-1.67	-3.44	-5.27	-7.13	-8.97
	0	-12.25	-9.08	-6.08	-3.32	0	0	-1.67	-3.44	-5.27	-7.13	-8.97
	0.2	-19.89	-15.28	-10.78	-6.49	-1.58	0	0	0	0	0	0
	0.4	-19.89	-15.28	-10.78	-6.49	-1.58	0	0	0	0	0	0
	0.6	-19.89	-15.28	-10.78	-6.49	-1.58	0	0	0	0	0	0
	0.8	-19.89	-15.28	-10.78	-6.49	-1.58	0	0	0	0	0	0
	1	-19.89	-15.28	-10.78	-6.49	-1.58	0	0	0	0	0	0
Total RSC	-1	0	0	0	0	0.84	0	-1.27	-2.40	-3.37	-4.15	-4.74
	-0.8	0	0	0	0	0.84	0	-1.27	-2.40	-3.37	-4.15	-4.74
	-0.6	0	0	0	0	0.84	0	-1.27	-2.40	-3.37	-4.15	-4.74
	-0.4	0	0	0	0	0.84	0	-1.27	-2.40	-3.37	-4.15	-4.74
	-0.2	-5.10	-3.97	-2.86	-1.81	0	0	-0.55	-1.10	-1.63	-2.13	-2.56
	0	-5.10	-3.97	-2.86	-1.81	0	0	-0.55	-1.10	-1.63	-2.13	-2.56
	0.2	-7.65	-6.06	-4.46	-2.89	-0.54	0	0	0	0	0	0
	0.4	-7.65	-6.06	-4.46	-2.89	-0.54	0	0	0	0	0	0
	0.6	-7.65	-6.06	-4.46	-2.89	-0.54	0	0	0	0	0	0
	0.8	-7.65	-6.06	-4.46	-2.89	-0.54	0	0	0	0	0	0
	1	-7.65	-6.06	-4.46	-2.89	-0.54	0	0	0	0	0	0

benefit the manufacturer compared to the retailer's optimal assortment where the manufacturer's benefit was completely overlooked. For example, when $\hat{d}_3 = -0.4$ and $d_3 = 1$, given Table 4.7, the difference in wholesale prices are minimal. However, the assortments are considerably different, and under the estimation, the assortment is the same as the optimal centralized assortment, which is preferred by the manufacturer. Hence, in Figures 4.5 to 4.7, $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}_3, d_3) > 0$ and $\mathcal{PR}^r_{\mathcal{D}}(\hat{d}_3, d_3) < 0$.

Third, both the retailer and the manufacturer are worse off, such that $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}_3, d_3) < 0$ and $\mathcal{PR}^r_{\mathcal{D}}(\hat{d}_3, d_3) < 0$. In this case, due to an imprecise estimation of d_3, we have $\hat{w}_{\mathcal{D}} \neq w^*_{\mathcal{D}}$, and the sub-optimal assortment $\hat{M}_{\mathcal{D}}$ is neither desired by the retailer nor the manufacturer. Therefore, neither of the parties can benefit from such situation. For instance, when $\hat{d}_3 = 0.6$ and $d_3 = 0.2$ when $C = 3$, given Table 4.7, the wholesale prices and assortments are different. Moreover, the assortment under estimation is also different from the

Figure 4.7: $\mathcal{PR}(\hat{d}_3, d_3)$ for different combinations of $\hat{d}_3$ and d_3 when $C = 12$, under the MDRD scenario

		d_3										
	$\hat{d}_3$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
Retailer	-1	0	0	0	0	0	2.76	7.43	12.55	18.09	23.98	30.11
	-0.8	0	0	0	0	0	2.76	7.43	12.55	18.09	23.98	30.11
	-0.6	0	0	0	0	0	2.76	7.43	12.55	18.09	23.98	30.11
	-0.4	0	0	0	0	0	2.76	7.43	12.55	18.09	23.98	30.11
	-0.2	0	0	0	0	0	2.76	7.43	12.55	18.09	23.98	30.11
	0	11.97	10.06	7.73	4.89	1.41	0	0	0	0	0	0
	0.2	11.97	10.06	7.73	4.89	1.41	0	0	0	0	0	0
	0.4	11.97	10.06	7.73	4.89	1.41	0	0	0	0	0	0
	0.6	11.97	10.06	7.73	4.89	1.41	0	0	0	0	0	0
	0.8	11.97	10.06	7.73	4.89	1.41	0	0	0	0	0	0
	1	11.97	10.06	7.73	4.89	1.41	0	0	0	0	0	0
Manufacturer	-1	0	0	0	0	0.00	-1.19	-4.01	-7.02	-10.18	-13.44	-16.71
	-0.8	0	0	0	0	0.00	-1.19	-4.01	-7.02	-10.18	-13.44	-16.71
	-0.6	0	0	0	0	0.00	-1.19	-4.01	-7.02	-10.18	-13.44	-16.71
	-0.4	0	0	0	0	0.00	-1.19	-4.01	-7.02	-10.18	-13.44	-16.71
	-0.2	0	0	0	0	0	-1.19	-4.01	-7.02	-10.18	-13.44	-16.71
	0	-10.24	-8.31	-6.19	-3.88	-1.41	0	0	0	0	0	0
	0.2	-10.24	-8.31	-6.19	-3.88	-1.41	0	0	0	0	0	0
	0.4	-10.24	-8.31	-6.19	-3.88	-1.41	0	0	0	0	0	0
	0.6	-10.24	-8.31	-6.19	-3.88	-1.41	0	0	0	0	0	0
	0.8	-10.24	-8.31	-6.19	-3.88	-1.41	0	0	0	0	0	0
	1	-10.24	-8.31	-6.19	-3.88	-1.41	0	0	0	0	0	0
Total RSC	-1	0	0	0	0	0	0	-0.55	-1.10	-1.63	-2.13	-2.56
	-0.8	0	0	0	0	0	0	-0.55	-1.10	-1.63	-2.13	-2.56
	-0.6	0	0	0	0	0	0	-0.55	-1.10	-1.63	-2.13	-2.56
	-0.4	0	0	0	0	0	0	-0.55	-1.10	-1.63	-2.13	-2.56
	-0.2	0	0	0	0	0	0	-0.55	-1.10	-1.63	-2.13	-2.56
	0	-2.42	-2.01	-1.56	-1.06	-0.54	0	0	0	0	0	0
	0.2	-2.42	-2.01	-1.56	-1.06	-0.54	0	0	0	0	0	0
	0.4	-2.42	-2.01	-1.56	-1.06	-0.54	0	0	0	0	0	0
	0.6	-2.42	-2.01	-1.56	-1.06	-0.54	0	0	0	0	0	0
	0.8	-2.42	-2.01	-1.56	-1.06	-0.54	0	0	0	0	0	0
	1	-2.42	-2.01	-1.56	-1.06	-0.54	0	0	0	0	0	0

centralized assortment when $d_3 = 0.2$, which is undesired by the manufacturer as well. Therefore, in Figure 4.5, we have $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}_3, d_3) < 0$ and $\mathcal{PR}^r_{\mathcal{D}}(\hat{d}_3, d_3) < 0$.

Fourth, the retailer is better off while the manufacturer is worse off, such that $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}_3, d_3) < 0$ and $\mathcal{PR}^r_{\mathcal{D}}(\hat{d}_3, d_3) > 0$. Unlike the second case, in this case, despite $\hat{M}_{\mathcal{D}} \neq M^*_{\mathcal{D}}$, $\hat{w}_{\mathcal{D}}$ is considerably smaller than $w^*_{\mathcal{D}}$. Hence, the retailer can benefit from low wholesale prices. We do not observe this case in any of the Figures 4.5 to 4.7. Considering a problem instance with $\gamma = 0.98$ and A3 attribute levels 0.05, 0.35, and 0.65, $\mathcal{PR}^r_{\mathcal{D}}(\hat{d}_3, d_3)$ for $C = 6$ will be as in Figure 4.8. As can be observed, when $\hat{d}_3 = -0.8$ and $d_3 = -0.2$, the retailer fares better while the manufacturer is worse off.

Figure 4.8: $\mathcal{PR}(\hat{d}_3, d_3)$ for different combinations of $\hat{d}_3$ and d_3 when $C = 6$, $\gamma = 0.98$, and A3 levels 0.05/0.35/0.65, under the MDRD scenario

		d_3										
	$\hat{d}_3$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
Retailer	-1	0	0	0	0	-0.83	4.71	11.71	21.40	30.69	39.43	47.52
	-0.8	0	0	0	0	-0.83	4.71	11.71	21.40	30.69	39.43	47.52
	-0.6	0	0	0	0	-0.83	4.71	11.71	21.40	30.69	39.43	47.52
	-0.4	0	0	0	0	-0.83	4.71	11.71	21.40	30.69	39.43	47.52
	-0.2	17.43	14.16	10.33	5.90	0	0	1.65	6.83	12.47	18.50	24.82
	0	17.43	14.16	10.33	5.90	0	0	1.65	6.83	12.47	18.50	24.82
	0.2	30.80	26.59	21.46	15.19	6.76	2.96	0	0	0	0	0
	0.4	30.80	26.59	21.46	15.19	6.76	2.96	0	0	0	0	0
	0.6	30.80	26.59	21.46	15.19	6.76	2.96	0	0	0	0	0
	0.8	30.80	26.59	21.46	15.19	6.76	2.96	0	0	0	0	0
	1	30.80	26.59	21.46	15.19	6.76	2.96	0	0	0	0	0
Manufacturer	-1	0	0	0	0	0.12	-1.12	-1.85	-2.90	-3.54	-3.74	-3.53
	-0.8	0	0	0	0	0.12	-1.12	-1.85	-2.90	-3.54	-3.74	-3.53
	-0.6	0	0	0	0	0.12	-1.12	-1.85	-2.90	-3.54	-3.74	-3.53
	-0.4	0	0	0	0	0.12	-1.12	-1.85	-2.90	-3.54	-3.74	-3.53
	-0.2	-7.45	-5.34	-3.38	-1.63	0	0	0.26	-0.07	-0.24	-0.22	0.03
	0	-7.45	-5.34	-3.38	-1.63	0	0	0.26	-0.07	-0.24	-0.22	0.03
	0.2	-11.43	-8.59	-5.93	-3.50	-1.24	-0.71	0	0	0	0	0
	0.4	-11.43	-8.59	-5.93	-3.50	-1.24	-0.71	0	0	0	0	0
	0.6	-11.43	-8.59	-5.93	-3.50	-1.24	-0.71	0	0	0	0	0
	0.8	-11.43	-8.59	-5.93	-3.50	-1.24	-0.71	0	0	0	0	0
	1	-11.43	-8.59	-5.93	-3.50	-1.24	-0.71	0	0	0	0	0
Total RSC	-1	0	0	0	0	-0.07	0	0.69	1.64	2.86	4.33	6.01
	-0.8	0	0	0	0	-0.07	0	0.69	1.64	2.86	4.33	6.01
	-0.6	0	0	0	0	-0.07	0	0.69	1.64	2.86	4.33	6.01
	-0.4	0	0	0	0	-0.07	0	0.69	1.64	2.86	4.33	6.01
	-0.2	-0.73	-0.39	-0.13	0.02	0	0	0.52	1.22	2.13	3.28	4.66
	0	-0.73	-0.39	-0.13	0.02	0	0	0.52	1.22	2.13	3.28	4.66
	0.2	-0.02	0.34	0.56	0.59	0.35	0	0	0	0	0	0
	0.4	-0.02	0.34	0.56	0.59	0.35	0	0	0	0	0	0
	0.6	-0.02	0.34	0.56	0.59	0.35	0	0	0	0	0	0
	0.8	-0.02	0.34	0.56	0.59	0.35	0	0	0	0	0	0
	1	-0.02	0.34	0.56	0.59	0.35	0	0	0	0	0	0

Decisions and Regret Analysis in MKRD

In this scenario, the manufacturer privately possesses information about the true value of inaccuracy, i.e., d_3. However, the retailer does not know this value and needs to estimate it. So, the manufacturer, based on d_3, determines its wholesale price (i.e., $w^*_{\mathcal{D}}$) according to its expectation of the retailer's assortment (i.e., $M^*_{\mathcal{D}}$) and quotes this price to the retailer. On the other hand, the retailer, based on $\hat{d}^r_3$, determines its assortment $\hat{M}^r_{\mathcal{D}}$ and the maximum wholesale price that it can pay (i.e., $\hat{w}^r_{\mathcal{D}}$). In this situation, if $w^*_{\mathcal{D}} \leq \hat{w}^r_{\mathcal{D}}$, the retailer accepts the quote and the transaction will be completed. However, if $w^*_{\mathcal{D}} > \hat{w}^r_{\mathcal{D}}$, the retailer does not accept the quoted price and decides to withdraw from the market. In this situation, the only sales channel is the manufacturer's online website. Figures 4.9 to 4.11 indicate $\mathcal{PR}^r_{\mathcal{D}}(\hat{d}^r_3, d_3)$, $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}^r_3, d_3)$, and $\mathcal{PR}^T_{\mathcal{D}}(\hat{d}^r_3, d_3)$ for any pair of $(\hat{d}_3^r, d_3)$ for the test bed described in Section 4.5.1. We investigate this scenario in two cases including $w_{\mathcal{D}}^* > \hat{w}_{\mathcal{D}}^r$ where the retailer withdraws from the market, and $w_{\mathcal{D}}^* \leq \hat{w}_{\mathcal{D}}^r$ where it operates its store.

Figure 4.9: $\mathcal{PR}(\hat{d}_3, d_3)$ for different combinations of $\hat{d}_3$ and d_3 when $C = 3$, under the MDRD scenario

		d_3										
	$\hat{d}_3^r$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
Retailer	-1	0	0	0	0	0	0	5.54	14.49	24.58	34.08	42.87
	-0.8	0	0	0	0	0	0	5.54	14.49	24.58	34.08	42.87
	-0.6	0	0	0	0	0	0	5.54	14.49	24.58	34.08	42.87
	-0.4	0	0	0	0	0	0	5.54	14.49	24.58	34.08	42.87
	-0.2	0	0	0	0	0	0	5.54	14.49	24.58	34.08	42.87
	0	0	0	0	0	0	0	5.54	14.49	24.58	34.08	42.87
	0.2	20.54	17.52	13.99	9.92	5.26	0	0	3.97	10.05	16.51	23.25
	0.4	NaN	NaN	NaN	NaN	NaN	NaN	NaN	0	0	0	0
	0.6	NaN	NaN	NaN	NaN	NaN	NaN	NaN	0	0	0	0
	0.8	NaN	NaN	NaN	NaN	NaN	NaN	NaN	0	0	0	0
	1	NaN	NaN	NaN	NaN	NaN	NaN	NaN	0	0	0	0
Manufacturer	-1	0	0	0	0	0	0	-1.34	-4.07	-6.13	-7.93	-9.45
	-0.8	0	0	0	0	0	0	-1.34	-4.07	-6.13	-7.93	-9.45
	-0.6	0	0	0	0	0	0	-1.34	-4.07	-6.13	-7.93	-9.45
	-0.4	0	0	0	0	0	0	-1.34	-4.07	-6.13	-7.93	-9.45
	-0.2	0	0	0	0	0	0	-1.34	-4.07	-6.13	-7.93	-9.45
	0	0	0	0	0	0	0	-1.34	-4.07	-6.13	-7.93	-9.45
	0.2	-9.67	-7.48	-5.36	-3.38	-1.57	0	0	-1.60	-2.79	-3.97	-5.10
	0.4	-3.29	-6.35	-8.49	-9.83	-10.48	-10.57	-11.71	0	0	0	0
	0.6	-3.29	-6.35	-8.49	-9.83	-10.48	-10.57	-11.71	0	0	0	0
	0.8	-3.29	-6.35	-8.49	-9.83	-10.48	-10.57	-11.71	0	0	0	0
	1	-3.29	-6.35	-8.49	-9.83	-10.48	-10.57	-11.71	0	0	0	0
Total RSC	-1	0	0	0	0	0	0	-0.71	-2.40	-3.37	-4.15	-4.74
	-0.8	0	0	0	0	0	0	-0.71	-2.40	-3.37	-4.15	-4.74
	-0.6	0	0	0	0	0	0	-0.71	-2.40	-3.37	-4.15	-4.74
	-0.4	0	0	0	0	0	0	-0.71	-2.40	-3.37	-4.15	-4.74
	-0.2	0	0	0	0	0	0	-0.71	-2.40	-3.37	-4.15	-4.74
	0	0	0	0	0	0	0	-0.71	-2.40	-3.37	-4.15	-4.74
	0.2	-5.10	-3.97	-2.86	-1.81	-0.85	0	0	-1.10	-1.63	-2.13	-2.56
	0.4	12.33	8.59	5.54	3.13	1.30	0	-1.61	0	0	0	0
	0.6	12.33	8.59	5.54	3.13	1.30	0	-1.61	0	0	0	0
	0.8	12.33	8.59	5.54	3.13	1.30	0	-1.61	0	0	0	0
	1	12.33	8.59	5.54	3.13	1.30	0	-1.61	0	0	0	0

I. **Analysis of MKRD When $w_{\mathcal{D}}^* > \hat{w}_{\mathcal{D}}^r$**

In this case, the retailer withdraws from the market and, in Figures 4.9 to 4.11, we use "NaN" as the retailer's percentage of regret. In this case, we assume that all the customers who would have purchased from the store, now make purchase decisions in the online channel. Therefore, the manufacturer can sell products to more customers. On the other hand, the potential assortment that could have been showcased in the physical store was a means for the manufacturer to provide information to the customers. In this situation, if showcasing products and attribute levels could positively affect the manufacturer's expected profit, the withdrawal of the retailer from the market can negatively impact the manufacturer. Given these influences,

Figure 4.10: $\mathcal{PR}(\hat{d}_3, d_3)$ for different combinations of $\hat{d}_3$ and d_3 when $C = 6$, under the MDRD scenario

	$\hat{d}_3^r$ \ d_3	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
Retailer	-1	0	0	0	0	NaN	NaN	NaN	NaN	NaN	NaN	NaN
	-0.8	0	0	0	0	NaN	NaN	NaN	NaN	NaN	NaN	NaN
	-0.6	0	0	0	0	NaN	NaN	NaN	NaN	NaN	NaN	NaN
	-0.4	0	0	0	0	NaN	NaN	NaN	NaN	NaN	NaN	NaN
	-0.2	15.12	11.88	8.12	3.76	0	0	4.80	10.07	15.77	21.82	28.13
	0	15.12	11.88	8.12	3.76	0	0	4.80	10.07	15.77	21.82	28.13
	0.2	27.34	22.94	17.56	10.99	4.13	0	0	0	0	0	0
	0.4	27.34	22.94	17.56	10.99	4.13	0	0	0	0	0	0
	0.6	27.34	22.94	17.56	10.99	4.13	0	0	0	0	0	0
	0.8	27.34	22.94	17.56	10.99	4.13	0	0	0	0	0	0
	1	27.34	22.94	17.56	10.99	4.13	0	0	0	0	0	0
Manufacturer	-1	0	0	0	0	-19.54	-20.90	-23.52	-25.59	-27.15	-28.28	-29.04
	-0.8	0	0	0	0	-19.54	-20.90	-23.52	-25.59	-27.15	-28.28	-29.04
	-0.6	0	0	0	0	-19.54	-20.90	-23.52	-25.59	-27.15	-28.28	-29.04
	-0.4	0	0	0	0	-19.54	-20.90	-23.52	-25.59	-27.15	-28.28	-29.04
	-0.2	-12.10	-8.95	-5.95	-3.21	0	0	-1.67	-3.44	-5.27	-7.13	-8.97
	0	-12.10	-8.95	-5.95	-3.21	0	0	-1.67	-3.44	-5.27	-7.13	-8.97
	0.2	-19.77	-15.16	-10.67	-6.38	-1.58	0	0	0	0	0	0
	0.4	-19.77	-15.16	-10.67	-6.38	-1.58	0	0	0	0	0	0
	0.6	-19.77	-15.16	-10.67	-6.38	-1.58	0	0	0	0	0	0
	0.8	-19.77	-15.16	-10.67	-6.38	-1.58	0	0	0	0	0	0
	1	-19.77	-15.16	-10.67	-6.38	-1.58	0	0	0	0	0	0
Total RSC	-1	0	0	0	0	2.13	0.00	-2.17	-3.88	-5.18	-6.11	-6.73
	-0.8	0	0	0	0	2.13	0.00	-2.17	-3.88	-5.18	-6.11	-6.73
	-0.6	0	0	0	0	2.13	0.00	-2.17	-3.88	-5.18	-6.11	-6.73
	-0.4	0	0	0	0	2.13	0.00	-2.17	-3.88	-5.18	-6.11	-6.73
	-0.2	-5.10	-3.97	-2.86	-1.81	0	0	-0.55	-1.10	-1.63	-2.13	-2.56
	0	-5.10	-3.97	-2.86	-1.81	0	0	-0.55	-1.10	-1.63	-2.13	-2.56
	0.2	-7.65	-6.06	-4.46	-2.89	-0.54	0	0	0.00	0.00	0.00	0.00
	0.4	-7.65	-6.06	-4.46	-2.89	-0.54	0	0	0	0	0	0
	0.6	-7.65	-6.06	-4.46	-2.89	-0.54	0	0	0	0	0	0
	0.8	-7.65	-6.06	-4.46	-2.89	-0.54	0	0	0	0	0	0
	1	-7.65	-6.06	-4.46	-2.89	-0.54	0	0	0	0	0	0

two possibilities can take place to the manufacturer's expected profit that we investigate in the following.

First, the manufacturer obtains a greater expected profit compared to its expected profit if the retailer knew d_3. This case happens either when the manufacturer would have preferred the retailer to showcase no or a limited variety of attribute levels (so that with its withdrawal, the manufacturer is better off), or when the increase in the number of customers who purchase online dominates the useful information that the retailer could have provided but now is lost. For example, when $\hat{d}_3^r = 1$, and $d_3 = -0.2$ at $C = 3$, Table 4.7 indicates that $S^*_{\mathcal{D}} = \{3\}$ while $\hat{S}^r_{\mathcal{D}} = \{1, 2, 3\}$. Therefore, the manufacturer expected the retailer to showcase a limited variety of A3 while the retailer would have showcased a great variety if it operated the store. In such case, the retailer's withdrawal benefits the manufacturer and in Figure 4.9, we have $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}_3^r, d_3) < 0$.

Second, the manufacturer obtains a smaller expected profit compared to its expected profit when the

Figure 4.11: $\mathcal{PR}(\hat{d}_3, d_3)$ for different combinations of $\hat{d}_3$ and d_3 when $C = 12$, under the MDRD scenario

		d_3										
	$\hat{d}^r_3$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
Retailer	-1	0	0	0	0	0	NaN	NaN	NaN	NaN	NaN	NaN
	-0.8	0	0	0	0	0	NaN	NaN	NaN	NaN	NaN	NaN
	-0.6	0	0	0	0	0	NaN	NaN	NaN	NaN	NaN	NaN
	-0.4	0	0	0	0	0	NaN	NaN	NaN	NaN	NaN	NaN
	-0.2	0	0	0	0	0	NaN	NaN	NaN	NaN	NaN	NaN
	0	11.78	9.87	7.53	4.68	1.20	0	0	0	0	0	0
	0.2	11.78	9.87	7.53	4.68	1.20	0	0	0	0	0	0
	0.4	11.78	9.87	7.53	4.68	1.20	0	0	0	0	0	0
	0.6	11.78	9.87	7.53	4.68	1.20	0	0	0	0	0	0
	0.8	11.78	9.87	7.53	4.68	1.20	0	0	0	0	0	0
	1	11.78	9.87	7.53	4.68	1.20	0	0	0	0	0	0
Manufacturer	-1	0	0	0	0	0	-43.33	-46.43	-48.89	-50.75	-52.09	-52.98
	-0.8	0	0	0	0	0	-43.33	-46.43	-48.89	-50.75	-52.09	-52.98
	-0.6	0	0	0	0	0	-43.33	-46.43	-48.89	-50.75	-52.09	-52.98
	-0.4	0	0	0	0	0	-43.33	-46.43	-48.89	-50.75	-52.09	-52.98
	-0.2	0	0	0	0	0	-43.33	-46.43	-48.89	-50.75	-52.09	-52.98
	0	-10.13	-8.21	-6.09	-3.78	-1.32	0	0	0	0	0	0
	0.2	-10.13	-8.21	-6.09	-3.78	-1.32	0	0	0	0	0	0
	0.4	-10.13	-8.21	-6.09	-3.78	-1.32	0	0	0	0	0	0
	0.6	-10.13	-8.21	-6.09	-3.78	-1.32	0	0	0	0	0	0
	0.8	-10.13	-8.21	-6.09	-3.78	-1.32	0	0	0	0	0	0
	1	-10.13	-8.21	-6.09	-3.78	-1.32	0	0	0	0	0	0
Total RSC	-1	0	0	0	0	0	0	-2.17	-3.88	-5.18	-6.11	-6.73
	-0.8	0	0	0	0	0	0	-2.17	-3.88	-5.18	-6.11	-6.73
	-0.6	0	0	0	0	0	0	-2.17	-3.88	-5.18	-6.11	-6.73
	-0.4	0	0	0	0	0	0	-2.17	-3.88	-5.18	-6.11	-6.73
	-0.2	0	0	0	0	0	0	-2.17	-3.88	-5.18	-6.11	-6.73
	0	-2.42	-2.01	-1.56	-1.06	-0.54	0	0	0	0	0	0
	0.2	-2.42	-2.01	-1.56	-1.06	-0.54	0	0	0	0	0	0
	0.4	-2.42	-2.01	-1.56	-1.06	-0.54	0	0	0	0	0	0
	0.6	-2.42	-2.01	-1.56	-1.06	-0.54	0	0	0	0	0	0
	0.8	-2.42	-2.01	-1.56	-1.06	-0.54	0	0	0	0	0	0
	1	-2.42	-2.01	-1.56	-1.06	-0.54	0	0	0	0	0	0

retailer also knew d_3. This case happens when the manufacturer benefits if the retailer would have showcased a great variety levels of A3. However, the retailer's withdrawal eliminated this benefit for the manufacturer, such that the increase in the number of customer purchasing online cannot compensate for the value of the eliminated information that could have been revealed by the store. Hence, we have $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}^r_3, d_3) > 0$.

It should be noted that given whether the manufacturer benefits from the retailer's withdrawal or not, and its magnitude, the total RSC expected profit can change accordingly.

II. **Analysis of MKRD When** $w^*_{\mathcal{D}} < \hat{w}^r_{\mathcal{D}}$

In this case, the retailer operates its store with $w^*_{\mathcal{D}}$ and $\hat{M}^r_{\mathcal{D}}$. But it cannot fare better compared to when it knew d_3, because the manufacturer makes optimal decisions while the retailer can showcase sub-optimal assortments. Under this case, there can be a number of possibilities that we explore.

First, both the retailer and the manufacturer obtain their optimal expected profits, such that $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}_3, d_3) = 0$ and $\mathcal{PR}^r_{\mathcal{D}}(\hat{d}_3, d_3) = 0$. This case takes place when $\hat{M}^r_{\mathcal{D}} = M^*_{\mathcal{D}}$. For example, when $\hat{d}^r_3 = 0.8$ and $d_3 = 0.4$, according to Table 4.3, this situation holds for all C values observed. Therefore, in Figures 4.9 to 4.11, we have $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}_3, d_3) =$ and $\mathcal{PR}^r_{\mathcal{D}}(\hat{d}_3, d_3) = 0$.

Second, the manufacturer is better off but the retailer is worse off, such that $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}_3, d_3) > 0$ and $\mathcal{PR}^r_{\mathcal{D}}(\hat{d}_3, d_3) < 0$. This case happens when $\hat{M}^r_{\mathcal{D}} \neq M^*_{\mathcal{D}}$, but $\hat{M}^r_{\mathcal{D}}$ includes attribute levels preferred by the manufacturer (e.g., the centralized assortment). For instance, when $\hat{d}^r_3 = 0.6$ and $d_3 = -1$ at $C = 12$, according to Table 4.3, $\hat{M}^r_{\mathcal{D}}$ represents levels 1, 2, and 3 of A3, while $S^*_{\mathcal{C}}$ also includes the same levels. Therefore, the retailer's imprecise estimation of inaccuracy is desired by the manufacturer; so $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}_3, d_3) > 0$.

Third, both the retailer and the manufacturer are worse off, such that $\mathcal{PR}^m_{\mathcal{D}}(\hat{d}_3, d_3) < 0$ and $\mathcal{PR}^r_{\mathcal{D}}(\hat{d}_3, d_3) < 0$. Unlike the second case, this situation happens if $\hat{M}^r_{\mathcal{D}}$ is also undesired by the manufacturer. In the following, Figure 4.12 denotes $\mathcal{PR}(\hat{d}_3, d_3)$ values for $C = 6$ with $\gamma = 0.98$ and levels of A3 as 0.05, 0.35, and 0.65. As can be observed, when $\hat{d}^r_3 = -0.2$ and $d_3 = 0.2$, both the retailer and manufacturer are worse off based on the argument provided.

4.6 Conclusion

In this paper, we investigate the operations of an omni-channel retail supply chain (RSC) containing an online sale website and a physical store. Due to the limited showcase capacity, an assortment decision is needed to determine the subset of products to be made available in the physical store. For products with non-digital attributes like apparel, for which customers' online assessment may be inaccurate and different from their physical assessment, understanding this inaccuracy is critical in making the assortment decision. We consider a situation where information about the inaccuracy is not available and estimations should be adopted for it. Under the decentralized setting, the physical store is assumed to be an independent retailer that sells a manufacturer's products, while the manufacturer runs its own online sales channel, both to maximize their own profits. In this setting, the retailer makes an assortment decision for its store, and the manufacturer decides on the wholesale price to be charged to the retailer. Under the centralized setting, both channels are run by a central authority who endeavors to maximize the total profit of the RSC.

Under both settings, if the resulting assortment with an estimation of the inaccuracy is the same as the assortment of the true inaccuracy, there will be no regrets. Under the decentralized setting, we investigate two scenarios. In the first scenario, we assume that none of the retailer and the manufacturer knows the true value of inaccuracy (MDRD), but they cooperatively obtain an estimation. In this situation, while both

Figure 4.12: $\mathcal{PR}(\hat{d}_3, d_3)$ for different combinations of $\hat{d}_3$ and d_3 when $C = 6$, $\gamma = 0.98$, and A3 levels 0.05/0.35/0.65, under the MKRD scenariog

		d_3										
	$\hat{d}_3$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
Retailer	-1	0	0	0	0	NaN	NaN	NaN	NaN	NaN	NaN	NaN
	-0.8	0	0	0	0	NaN	NaN	NaN	NaN	NaN	NaN	NaN
	-0.6	0	0	0	0	NaN	NaN	NaN	NaN	NaN	NaN	NaN
	-0.4	0	0	0	0	NaN	NaN	NaN	NaN	NaN	NaN	NaN
	-0.2	16.60	13.29	9.43	4.96	0	0	1.06	6.28	11.95	18.01	24.37
	0	16.60	13.29	9.43	4.96	0	0	1.06	6.28	11.95	18.01	24.37
	0.2	30.52	26.29	21.14	14.84	NaN	NaN	0	0	0	0	0
	0.4	30.52	26.29	21.14	14.84	NaN	NaN	0	0	0	0	0
	0.6	30.52	26.29	21.14	14.84	NaN	NaN	0	0	0	0	0
	0.8	30.52	26.29	21.14	14.84	NaN	NaN	0	0	0	0	0
	1	30.52	26.29	21.14	14.84	NaN	NaN	0	0	0	0	0
Manufacturer	-1	0	0	0	0	-39.36	-42.40	-44.46	-46.89	-48.72	-50.04	-50.92
	-0.8	0	0	0	0	-39.36	-42.40	-44.46	-46.89	-48.72	-50.04	-50.92
	-0.6	0	0	0	0	-39.36	-42.40	-44.46	-46.89	-48.72	-50.04	-50.92
	-0.4	0	0	0	0	-39.36	-42.40	-44.46	-46.89	-48.72	-50.04	-50.92
	-0.2	-7.14	-5.04	-3.10	-1.36	0	0	0.39	0.06	-0.12	-0.11	0.13
	0	-7.14	-5.04	-3.10	-1.36	0	0	0.39	0.06	-0.12	-0.11	0.13
	0.2	-11.33	-8.49	-5.83	-3.40	-39.36	-42.40	0	0	0	0	0
	0.4	-11.33	-8.49	-5.83	-3.40	-39.36	-42.40	0	0	0	0	0
	0.6	-11.33	-8.49	-5.83	-3.40	-39.36	-42.40	0	0	0	0	0
	0.8	-11.33	-8.49	-5.83	-3.40	-39.36	-42.40	0	0	0	0	0
	1	-11.33	-8.49	-5.83	-3.40	-39.36	-42.40	0	0	0	0	0
Total RSC	-1	0	0	0	0	-11.52	-14.969	-17.46	-19.43	-20.92	-21.99	-22.71
	-0.8	0	0	0	0	-11.52	-14.969	-17.46	-19.43	-20.92	-21.99	-22.71
	-0.6	0	0	0	0	-11.52	-14.969	-17.46	-19.43	-20.92	-21.99	-22.71
	-0.4	0	0	0	0	-11.52	-14.969	-17.46	-19.43	-20.92	-21.99	-22.71
	-0.2	-0.73	-0.39	-0.13	0.02	0	0	0.52	1.22	2.13	3.28	4.66
	0	-0.73	-0.39	-0.13	0.02	0	0	0.52	1.22	2.13	3.28	4.66
	0.2	-0.02	0.34	0.56	0.59	-11.52	-14.969	0	0	0	0	0
	0.4	-0.02	0.34	0.56	0.59	-11.52	-14.969	0	0	0	0	0
	0.6	-0.02	0.34	0.56	0.59	-11.52	-14.969	0	0	0	0	0
	0.8	-0.02	0.34	0.56	0.59	-11.52	-14.969	0	0	0	0	0
	1	-0.02	0.34	0.56	0.59	-11.52	-14.969	0	0	0	0	0

parties can not fare better at the same time with an imprecise estimation, each can be better off under certain conditions. In the second scenario, we assume that the manufacturer possesses private information about the true value of inaccuracy but does not share it with the retailer (MKRD). In this scenario, the retailer cannot be better off compared to when it also knows the true value of inaccuracy; however, the manufacturer can be more profitable if the retailer imprecisely estimates the inaccuracy. Hence, the manufacturer does not share its private information with the retailer, unless an incentive is provided.

Under the decentralized setting, we show that if the true value of inaccuracy is zero (i.e., customer accurately assess products and attribute levels online), any arbitrary estimations by the manufacturer and the retailer result in the same total RSC expected profit, while this may not hold for the manufacturer and the retailer individually.

Under the centralized setting, no imprecise assessment of inaccuracy can result in higher expected profit for the RSC; hence, the best case is when the estimated assortment includes the same attribute levels if the true value of inaccuracy was known, which implies zero regret. For an imprecise assortment, the value of regret increases as the disparity between the true value of inaccuracy and its estimation gets larger. If customers' online assessment is in fact the same as their physical assessment, then any arbitrary assortment results in the same expected profit.

A specific case under the centralized setting is when it is assumed that customers accurately assess products and attribute levels online (i.e., a zero estimation of inaccuracy is obtained). In this case, the central authority supposes that any assortment yields the same expected profit. However, due to the possible error in the estimation, this may not take place in practice. We show that the RSC can avoid substantial regrets if an assortment including the highest utility attribute levels is showcased.

One potential extension to this study is to consider the retailer's understanding of the manufacturer's announced wholesale price in cases when the manufacturer knows the true value of inaccuracy but does not share it with the retailer. When the manufacturer quotes its wholesale price to the retailer, if it is different from the price that the retailer is willing to pay, it can be a "signal" to the retailer that its estimation is not precise. This can help the retailer to obtain a better understanding of the inaccuracy. For example, through a backward process, the retailer may be able to find the true value of inaccuracy (or at least a better estimation of it) based on the price quoted by the manufacturer.

Chapter 5

Conclusion

Omni-channel retailing, comprising brick-and-mortar (physical) stores and online sales websites, is a common practice in modern retailing. While capacity may not be an issue for an online sales channel, a physical store may have a limited shelf-space capacity. Hence, one primary question in such retail systems is to select an assortment of products to be made available in the store. Our study applies to high-value products with non-digital attributes that customers prefer to visit the physical store to experience products, before purchasing either directly from the store or from the online channel. Thus, the selected assortment of products influences the purchase and keep-or-return decisions of both in-store and online customers. This is because through strategic assortment selection, retailers can provide customers with accurate utility information of showcased products and their attribute levels, resulting in more informed purchase and keep-or-return decisions.

In our first study, we address the omni-channel assortment planning problem when product returns are allowed. We explicitly model product returns and study their impacts on profitability of the retail system. We find that retailers should not necessarily fully utilize their showcase capacity, even if sufficient capacity is available. Also, retailers may not necessarily attain a higher profit as product returns decrease. Although product returns usually are associated with profit loss, it is shown that an increase in returns may originate from an increase in sales, and overall can result in higher profit. In addition, our findings suggest that when operating under a full-refund policy, commonly seen in modern retail, retailers should offer a small variety of undervalued attribute levels to maximize profits. Alternatively, if the hidden attributes are overvalued, retailers should offer a high variety of the products when their customer base predominantly consists of online shoppers. Conversely, they should limit the variety of showcased products if most customers prefer to shop in physical stores.

Our results in the first study indicate that the retailers generally fare better if not showcased attribute levels are undervalued in the online channel, rather than accurately or over-valued. This unintuitive result implies marketing strategies in their website design. If hidden attribute levels would be undervalued, retailers should avoid tools that lead to more accurate evaluations. However, if hidden attribute levels would be overvalued, retailers may employ web interfaces that eliminate the inaccuracy as much as possible.

In our second study, we address the coordination problem of a decentralized retail supply chain (RSC) containing a manufacturer that operates its online sales website making wholesale price decisions, and a retailer that independently runs a physical store deciding the assortment of manufacturer's products for its store. We show that the assortment decisions under this decentralized setting should balance the benefits of showcasing the highest utility products and the highest (smallest) variety of overvalued (undervalued) attribute levels. Out findings indicate that the assortment decisions under the decentralized fashion are inefficient compared to those under a centralized setting. So, we propose scope contracts for coordination and eliminating the inefficiency. In our devised contract, the manufacturer offers discounts on the wholesale price of products that consist of certain attribute levels showcasing which benefits the whole RSC. The contract is shown to be instrumental in coordinating the RSC under various situations.

Moreover, for limited-hassle design and implementation, we propose a specific version of the contract where the discount factors are the same for all the products with the desired attribute levels. The manufacturer (or any other type of supplier in a supplier-buyer framework) can employ this contract and attain the highest possible profitability for their supply chains. In the scope contracts, we show that there may exist multiple discount factors that all guarantee coordination of the RSC. However, different discount factors can yield different profit distributions between the manufacturer and the retailer. We show that greater discounts increase the profit share of the retailer and vice versa. The specific discount rates can be determined through the negotiation power of the parties.

In our third study, we tackle the omni-channel assortment planning when inaccuracy in customers' online evaluations of products and attribute levels are not known, and the RSC parties should obtain estimations for these parameters to make their assortment and wholesale price decisions. Our analyses under the decentralized setting indicate that the retailer and the manufacturer never fare better at the same time if an imprecise estimation is obtained. However, they can be better off separately under certain conditions. In a specific case when the manufacturer privately possesses information about the true value of inaccuracy, it can benefit from this private information, but the retailer is never better off. Therefore, it is crucial for the retailer to strive for more information regarding the inaccuracies. However, the manufacturer may not share their private information with the retailer, unless an incentive is provided from the retailer.

Under the centralized setting, RSCs are not better off with imprecise estimations of the inaccuracies. Furthermore, RSCs may estimate the inaccuracies to be zero; i.e., supposing that customers accurately evaluate products in the online channel. However, since there can be potentially errors in these estimations, they should generally showcase an assortment of products that represent the highest utility attribute levels in the physical store for less regret.

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Appendix A

Appendix: Proofs of Chapter 2

Proof. Proposition 2.1.

Suppose that $x, y \in M$; so, they are accurately evaluated by customers in the online channel ($D_{x|M} = D_{y|M} = 0$). Therefore, the return probability of these products once purchased online are equal to their offline return probability; i.e., $R^n_{x|M} = R^f_x$ and $R^n_{y|M} = R^f_y$. Given (2.12), $(i), (ii)$, and (iii) in Proposition 2.1 can be easily observed. For (iv), to have $R^f_x > R^f_y$, we can write

$$\frac{1}{1 + e^{(\widetilde{U}_x - r + \beta_1 \gamma \pi_x)/\mu'}} > \frac{1}{1 + e^{(\widetilde{U}_y - r + \beta_1 \gamma \pi_y)/\mu'}}.$$

Simplifying A, we get

$$e^{(\widetilde{U}_y - r + \beta_1 \gamma \pi_y)/\mu'} > e^{(\widetilde{U}_x - r + \beta_1 \gamma \pi_x)/\mu'},$$

which can be further simplified and result in in $(\widetilde{U}_x - \widetilde{U}_y)/\beta_1(\pi_y - \pi_x) < \gamma$.

If $x, y \in X \setminus M$ and $D_{x|M} = D_{y|M}$, the same argument holds for comparison of $R^n_{x|M}$ and $R^n_{y|M}$ given (2.13). □

Proof. Proposition 2.2.

For the extra capacity to be left unutilized, we need $\Pi(M_{C+1}) < \Pi(M_C)$. Noticing that $K^{f/n}_{x|M} = 1 - R^{f/n}_{x|M}$, M_{C+1} and M_C can be written as the following, respectively:

$$\Pi(M_{C+1}) = \alpha \sum_{x \in M_{C+1}} \pi_x P^f_{x|M_{C+1}}(1 - \gamma R^f_x) + (1 - \alpha) \sum_{x \in X} \pi_x P^n_{x|M_{C+1}}(1 - \gamma R^n_{x|M_{C+1}}),$$

$$\Pi(M_C) = \alpha \sum_{x \in M_C} \pi_x P^f_{x|M_C}(1 - \gamma R^f_x) + (1 - \alpha) \sum_{x \in X} \pi_x P^n_{x|M_C}(1 - \gamma R^n_{x|M_C}).$$

Substituting $\Pi(M_{C+1})$ and $\Pi(M_C)$ into $\Pi(M_{C+1}) < \Pi(M_C)$, and simplifying, we can get (2.18). □

Proof. Theorem 2.1.

The retailer's profit as a function of d, $d = D_{y|x}$, is shown in (2.19). Simplifying and writing (2.19) explicitly, we get:

$$\begin{aligned}\overline{\Pi}(d) =&\pi_x \left(1 - \frac{\gamma}{1+e^{(-r-\beta(1-\gamma)\pi_x)/\mu}}\right)\left(\alpha\frac{e^{\overline{U}_x/\mu}}{1+e^{\overline{U}_x/\mu}} + (1-\alpha)\frac{e^{\overline{U}_x/\mu}}{1+e^{\overline{U}_x/\mu}+e^{(\overline{U}_y+d)/\mu}}\right)+ \\ &(1-\alpha)\pi_y\left(1-\frac{\gamma}{1+e^{(-d-r-\beta(1-\gamma)\pi_x)/\mu}}\right)\frac{e^{(\overline{U}_y+d)/\mu}}{1+e^{\overline{U}_x/\mu}+e^{(\overline{U}_y+d)/\mu}}\end{aligned}. \tag{A.1}$$

Suppose that $\alpha\pi_x(1-\gamma)\frac{e^{\overline{U}_x}}{1+e^{\overline{U}_x}} = A$, $\alpha\pi_x\gamma\frac{e^{\overline{U}_x}}{1+e^{\overline{U}_x}} = G$, $(1-\alpha)\pi_x(1-\gamma)e^{\overline{U}_x} = B$, $(1-\alpha)\pi_x\gamma e^{\overline{U}_x} = H$, $e^{(r-\beta_1\gamma\pi_x)} = z$, $1+e^{\overline{U}_x} = C$, $e^{\overline{U}_y} = b$, $(1-\alpha)\pi_y(1-\gamma)e^{\overline{U}_y} = E$, $(1-\alpha)\pi_y\gamma e^{\overline{U}_y} = f$, $a_1 = e^{-\widetilde{U}_x}$, and $a_2 = e^{-\widetilde{U}_y}$. Given these, the profit function in (A.1) can be written as the following:

$$\overline{\Pi}(d) = A + \frac{G}{1+a_1z} + \left(\frac{B+Ee^d}{C+be^d}\right) + \frac{H}{(1+a_1z)(C+be^d)} + \frac{fe^d}{(1+a_2ze^d)(C+be^d)}.$$

Taking the first derivative of $\overline{\Pi}(d)$ with respect to $\mathbf{x}$ and simplifying it, we get:

$$\frac{\partial\overline{\Pi}(d)}{\partial d} = \frac{fe^d(C-a_2bze^{2d})}{(1+a_2ze^d)^2(C+be^d)^2} - \frac{e^d(bH-(1+a_1z)E)}{(1+a_1z)(C+be^d)^2}.$$

Extremums (if any) of $\Pi(d)$ take place at $\partial\Pi(d)/\partial d = 0$. This is to solve the following equation for finding d:

$$\frac{fe^d(C-a_2bze^{2d})}{(1+a_2ze^d)^2(C+be^d)^2} - \frac{e^d(bH-(1+a_1z)E)}{(1+a_1z)(C+be^d)^2} = 0. \tag{A.2}$$

By simplifying (A.2), we have:

$$\frac{f(C-a_2bze^{2d})}{(1+a_2ze^d)^2} = \frac{bH-(1+a_1z)E}{1+a_1z}.$$

This can be more simplified as the following:

$$f(C-a_2bze^{2d})(1+a_1z) = (bH-(1+a_1z)E)(1+a_2ze^d)^2. \tag{A.3}$$

Substituting the value of the defined parameters into (A.3) and considering $t = e^{\overline{U}_y}\left(1+a_1z(1+\gamma e^{\overline{U}_x})\right)$, $h = 2a_2ze^{\overline{U}_y}\left[(1+a_1z)(1-\gamma)-\gamma e^{\overline{U}_x}\right]$, and $q = \frac{ha_2z}{2} - (1+a_1z)za_2\gamma e^{2\overline{U}_y}$, (A.3) can be re-written as the following:

$$\Phi(d) = qe^{2d} + he^{d} + t = 0.$$

$\lim_{d\to-\infty} \Phi(d) = t > 0$; which means that $\Phi(d)$ starts from a positive value. Also, taking derivative of $\Phi(d)$ with respect to d, we have:

$$\frac{\partial \Phi(d)}{\partial d} = e^{d}(2qe^{d} + h).$$

Given that $q = \frac{ha_2 z}{2} - (1 + a_1 z)za_2\gamma e^{2\overline{U}_y}$, only three cases are possible for values of q and h: 1. $q \geq 0, h > 0$; 2. $q < 0, h \geq 0$; and 3. $q < 0, h < 0$.

1. $q \geq 0, h > 0$

For this case, $\frac{\partial \Phi(d)}{\partial d} \geq 0$. Therefore, $\Phi(d)$ is an increasing function, and it can never be zero because $\lim_{d\to-\infty} \Phi(d) = t > 0$. So, there is no root for $\Phi(d) = 0$.

2. $q < 0, h \geq 0$

In this case, $\Phi(d)$ is an increasing function for $d \leq ln\left(\frac{-h}{2q}\right)$, and a decreasing function for $d \geq ln\left(\frac{-h}{2q}\right)$. Therefore, since $\lim_{d\to-\infty} \Phi(d) = t > 0$, $\Phi(d)$ becomes equal to zero only once. As a result, there is one root for $\Phi(d) = 0$.

3. $q < 0, h < 0$

In this case, $\Phi(d)$ is a decreasing function, since $\frac{\partial \Phi(d)}{\partial d} = e^{d}(2qe^{d}+h) \leq 0$. Therefore, because $\lim_{d\to-\infty} \Phi(d) = t > 0$, $\Phi(d)$ becomes equal to zero only once. This means that there is one root for $\Phi(d) = 0$.

Given the above cases, when $q \geq 0$, there is no root for $\Phi(d) = 0$. However, when $q < 0$, there is one root for $\Phi(d) = 0$. Since $\Phi(d)$ is a quadratic function, this root is the maximum point of $\overline{\Pi}(d)$; which means that $\overline{\Pi}(d)$ is unimodal. We investigate these cases separately in the following.

1. $q \geq 0$

In this case, since there is no maxima for the profit function, the maximum profit takes place either when $d \to +\infty$ or $d \to -\infty$. The first derivative of $\Pi(d)$ with respect to d at $d \to -\infty$ is:

$$\lim_{d\to-\infty} \frac{\partial \overline{\Pi}(d)}{\partial d} = \lim_{d\to-\infty} \left(\frac{fe^{d}(C - bze^{2d})}{(1 + ze^{d})^2(C + be^{d})^2} - \frac{e^{d}(bH - (1+z)E)}{(1+z)(C + be^{d})^2} \right) \geq 0. \tag{A.4}$$

As a result, $\Pi(d)$ is increasing at $d \to -\infty$. Given this, for $q \geq 0$, the maximum profit happens at $d \to +\infty$.

2. $q < 0$

In this case, there is one maxima for the profit function. To find this maxima, we need to solve $\frac{\partial \Pi(d)}{\partial d} = 0$. Given (A.4), $\frac{\partial \Pi(d)}{\partial d}$ is a quadratic function, so it can potentially have two roots, as the following:

$$d_1 = ln\left(\frac{-h-\sqrt{h^2-4qt}}{2q}\right),$$

$$d_2 = ln\left(\frac{-h+\sqrt{h^2-4qt}}{2q}\right).$$

Earlier, we showed that that $\frac{\partial \overline{\Pi}(d)}{\partial d} = 0$ has only one root. Therefore, only one of d_1 or d_2 can be acceptable. The acceptable root is the one that returns a positive value inside the natural logarithm function. Since $q < 0$ in the denominator, the numerator of the acceptable root must be negative. Because $-h - \sqrt{h^2-4qt} \leq -h+\sqrt{h^2-4qt}$, d_1 is the acceptable root for $\frac{\partial \overline{\Pi}(d)}{\partial d} = 0$, and the maxima for $\overline{\Pi}(d)$. To have a positive maxima ($d \geq 0$), we need to have $\frac{-h-\sqrt{h^2-4qt}}{2q} \geq 1$. Since $q < 0$, we can write this inequality as the following:

$$-\sqrt{h^2-4qt} \leq 2q + h. \tag{A.5}$$

In (A.5), the right-hand side is positive if $h > -2q$ and negative or zero if $h \leq -2q$.

2.1. $h > -2q$

In this case, given that whether $\sqrt{h^2-4qt}$ is greater than or less than or equal to $|2q+h|$ we can have

2.1.1. $\sqrt{h^2-4qt} \geq |2q+h|$

In this situation, (A.5) can be simplified as $-t \leq q+h$.

2.1.2. $\sqrt{h^2-4qt} < |2q+h|$

In this situation, (A.5) can be simplified as $-t > q+h$.

As can be observed, in case of 2.1, both $-t \leq q+h$ and $-t \geq q+h$ result in a positive maxima. Therefore, in this case, the maxima is always positive.

2.2. $h \leq -2q$

In this situation, (A.5) can be simplified as $-t \leq q+h$. Therefore, if $-t > q+h$, then the root is positive; otherwise, the root is negative.

Therefore, to wrap up, the maximum profit takes place at:

$$\begin{cases} d \to \infty & if\ q \geq 0 \\ 0 \leq d < \infty & if\ q < 0, h > -2q \\ 0 \leq d < \infty & if\ q < 0, h \leq -2q, -t \leq q+h \\ -\infty < d < 0 & if\ q < 0, h \leq -2q, -t > q+h \end{cases}. \tag{A.6}$$

The conditions in (A.6) can be explicitly written as the following:

$$q < 0 \equiv \overline{U}_x > ln\left(\frac{(1+a_1z)(1-\gamma+\gamma e^{\overline{U}_y}/a_2z)}{\gamma}\right),$$

$$h \leq -2q \equiv \overline{U}_x \geq ln\left(\frac{(1+a_1z)(1-\gamma-e^{\overline{U}_y})}{\gamma}\right),$$

and

$$-t > q+h \equiv \overline{U}_x > ln\left((\frac{1+a_1z}{\gamma})\frac{(1-\gamma)(2a_2z+a_2^2z^2)-za_2\gamma e^{\overline{U}_y}+1}{2a_2z+a_2^2z^2-a_1z}\right).$$

We define $A1 = ln\left(\frac{(1+a_1z)(1-\gamma+\gamma e^{\overline{U}_y}/a_2z)}{\gamma}\right)$, $A2 = ln\left((\frac{1+a_1z}{\gamma})\frac{(1-\gamma)(2a_2z+a_2^2z^2)-za_2\gamma e^{\overline{U}_y}+1}{2a_2z+a_2^2z^2-a_1z}\right)$, and $A3 = ln\left(\frac{(1+a_1z)(1-\gamma-e^{\overline{U}_y})}{\gamma}\right)$. Because $A1 > A2$, $q < 0$ and $h > -2q$ in (A.6) is impossible to happen and $q < 0$ implies $h \leq -2q$. Therefore, (A.6) can be simplified as

$$\begin{cases} d \to \infty & if\ q \geq 0 \\ 0 \leq d < \infty & if\ q < 0, -t \leq q+h \\ -\infty < d < 0 & if\ q < 0, -t > q+h \end{cases}. \tag{A.7}$$

substituting the equivalents conditions found earlier for the criterion in (A.7) and considering that $A1$ can be greater than, equal, or smaller that $A2$, we get

$$\begin{cases} d \to \infty & if\ \overline{U}_x \leq A1 \\ 0 \leq d < \infty & if\ A1 < \overline{U}_x \leq A2 \\ -\infty < d < 0 & if\ max\{A1, A2\} \leq \overline{U}_x \end{cases}.$$

□

Proof. Proposition 2.3.

The profit function of showcasing set M' and that of showcasing set M'' can be written in simplified forms similar to (22) as the following, respectively:

$$\Pi(M^{'}) = \alpha\Bigg((1-\gamma R_x^f)\sum_{i\in M'}\pi_i P^f_{i|M'}\Bigg) + (1-\alpha)\Bigg(P^n_{y|M'}(1-\gamma R^n_{y|M'}) + (1-\gamma R_x^f)\sum_{i\in M'}\pi_i P^n_{i|M'}\Bigg),$$

$$\Pi(M^{''}) = \alpha\Bigg((1-\gamma R_y^f)\sum_{j\in M''}\pi_j P^f_{j|M''}\Bigg) + (1-\alpha)\Bigg(P^n_{x|M''}(1-\gamma R^n_{x|M''}) + (1-\gamma R_y^f)\sum_{j\in M''}\pi_j P^n_{j|M''}\Bigg).$$

Because $\pi_x = \pi_y$, we have $R^f_x = R^f_y$. Moreover, since it is assumed that $D_{x|M} = D_{y|M}$, we can conclude that $D_{x|M''} = D_{y|M'}$. Hence $R^n_{y|M'} = R^n_{x|M''}$. For M' to be preferred to M'' for showcasing, (i.e., for x to be selected over y), we should have $\Pi(M^{'}) \geq \Pi(M^{''})$. As a result, we get:

$$\frac{1-\gamma R^f_x}{1-\gamma R^n_{x|M''}}\left(\frac{\alpha}{1-\alpha}(\sum_{i\in M'} P^f_{i|M'} - \sum_{j\in M''} P^f_{j|M''}) + (\sum_{i\in M'} P^n_{i|M'} - \sum_{j\in M''} P^n_{j|M''})\right) \geq \left(P^n_{x|M''} - P^n_{y|M'}\right).$$

□

Appendix B

Appendix: Proofs of Chapter 3

Proof. Lemma 3.1.

For an arbitrary attribute k, suppose that arbitrarily $S(k|M*) = \{1, 2, ..., m\}$. Without loss of generality, let the levels of all attributes except k be accurately assessed by customers. First, for $C = m$, we show that selecting the highest utility product with each level in $S(k|M^*)$ yields a higher expected profit for the retailer compared to any other selection of products. for this, we define $M_C^* = \{[1]_{k:1}, [1]_{k:2}, ..., [1]_{k:m}\}$, and an arbitrary $M_C' = \{[1]_{k:1}, [1]_{k:2}, ..., [1]_{k:t-1}, [2]_{k:t}, [1]_{k:t+1}, ..., [1]_{k:m}\}$. Then, we can write the retailer's expected profit function for M_C^* and M_C' as the following, respectively:

$$\Pi_{\mathcal{D}}^r(M_C^*) = \frac{\left(\mathcal{P}_{[1]_{k:1}}^r - w\right) e^{\overline{U}_{[1]_{k:1}}} + ... + \left(\mathcal{P}_{[1]_{k:t}}^r - w\right) e^{\overline{U}_{[1]_{k:t}}} + ... + \left(\mathcal{P}_{[1]_{k:m}}^r - w\right) e^{\overline{U}_{[1]_{k:m}}}}{1 + \sum_{x \in X, x_k = 1} e^{\overline{U}_x} + ... + \sum_{x \in X, x_k = m} e^{\overline{U}_x} + \sum_{x \in X, x_k \notin S(k|M^*)} e^{\overline{U}_x + d_{k,x_k}}}, \tag{B.1}$$

$$\Pi_{\mathcal{D}}^r(M_C') = \frac{\left(\mathcal{P}_{[1]_{k:1}}^r - w\right) e^{\overline{U}_{[1]_{k:1}}} + ... + \left(\mathcal{P}_{[2]_{k:t}}^r - w\right) e^{\overline{U}_{[2]_{k:t}}} + ... + \left(\mathcal{P}_{[1]_{k:m}}^r - w\right) e^{\overline{U}_{[1]_{k:m}}}}{1 + \sum_{x \in X, x_k = 1} e^{\overline{U}_x} + ... + \sum_{x \in X, x_k = m} e^{\overline{U}_x} + \sum_{x \in X, x_k \notin S(k|M^*)} e^{\overline{U}_x + d_{k,x_k}}}. \tag{B.2}$$

The denominator in (B.1) and (B.2) are the same, since the both represent $S(k|M^*)$. Therefore, to show $\Pi_{\mathcal{D}}^r(M_C^*) > \Pi_{\mathcal{D}}^r(M_C')$, we need to compare the numerators. So, we have

$$\begin{gathered}\left(\mathcal{P}_{[1]_{k:1}}^r - w\right) e^{\overline{U}_{[1]_{k:1}}} + ... + \left(\mathcal{P}_{[1]_{k:t}}^r - w\right) e^{\overline{U}_{[1]_{k:t}}} + ... + \left(\mathcal{P}_{[1]_{k:m}}^r - w\right) e^{\overline{U}_{[1]_{k:m}}} > \\ \left(\mathcal{P}_{[1]_{k:1}}^r - w\right) e^{\overline{U}_{[1]_{k:1}}} + ... + \left(\mathcal{P}_{[2]_{k:t}}^r - w\right) e^{\overline{U}_{[2]_{k:t}}} + ... + \left(\mathcal{P}_{[1]_{k:m}}^r - w\right) e^{\overline{U}_{[1]_{k:m}}}.\end{gathered}$$

Simplifying this inequality, we get $\left(\mathcal{P}_{[1]_{k:t}}^r - w\right) e^{\overline{U}_{[1]_{k:t}}} > \left(\mathcal{P}_{[2]_{k:t}}^r - w\right) e^{\overline{U}_{[2]_{k:t}}}$ which always holds since $\overline{U}_{[1]_{k:t}} > \overline{U}_{[2]_{k:t}}$ and also $\mathcal{P}_{[1]_{k:t}}^r > \mathcal{P}_{[2]_{k:t}}^r$. For any other M', the same comparison can be carried out which yields the same result.

Next, we suppose that $C > m$. Here, we aim to show that other than $[1]_{k:i}, \forall i \in S(k|M^*)$, the remaining capacity $C - |S(k|M^*)|$ should be filled with the remaining highest utility products that include one of the levels of k included in $S(k|M^*)$. Let $M^*_{C-|S(k|M^*)|}$. Without loss of generality, let $M'_{C-|S(k|M^*)|}$ be a set of $C - |S(k|M^*)|$ products with the levels in $S(k|M^*)$ other than $[1]_{k:i}, \forall i \in S(k|M^*)$ that differs from $M_{C-|S(k|M^*)|}$ in at least one element. Defining $M^* = M^*_C \cup M^*_{C-|S(k|M^*)|}$ and $M' = M^*_C \cup M'_{C-|S(k|M')|}$, for showing $\Pi^r_{\mathcal{D}}(M^*) > \Pi^r_{\mathcal{D}}(M')$, similar to the previous step, we need to compare the numerators of these two retailer's expected profit functions. By explicitly writing $\Pi^r_{\mathcal{D}}(M^*)$ and $\Pi^r_{\mathcal{D}}(M')$, we get

$$\sum_{i \in S(k|M^*)} \left(\mathcal{P}^r_{[1]_{k:i}} - w\right) e^{\overline{U}_{[1]_{k:i}}} + \sum_{x \in M^*_{C-|S(k|M^*)|}} \left(\mathcal{P}^r_x - w\right) e^{\overline{U}_x} >$$
$$\sum_{i \in S(k|M^*)} \left(\mathcal{P}^r_{[1]_{k:i}} - w\right) e^{\overline{U}_{[1]_{k:i}}} + \sum_{x \in M'_{C-|S(k|M^*)|}} \left(\mathcal{P}^r_x - w\right) e^{\overline{U}_x}.$$

Simplifying this inequality, we have

$$\sum_{x \in M^*_{C-|S(k|M^*)|}} \left(\mathcal{P}^r_x - w\right) e^{\overline{U}_x} > \sum_{x \in M'_{C-|S(k|M^*)|}} \left(\mathcal{P}^r_x - w\right) e^{\overline{U}_x},$$

which always holds because products in $M^*_{C-|S(k|M^*)|}$ are the highest utility products other than $[1]_{k:i}, \forall i \in S(k|M^*)$ that include one of the levels of k included in $S(k|M^*)$, while at least one of the products in products in $M^*_{C-|S(k|M^*)|}$ is not among these highest utility products other than $[1]_{k:i}, \forall i \in S(k|M^*)$. □

Proof. Lemma 3.2.

Let levels of k be numbered as 1 to $|L(k)|$ with 1 being the highest utility level and $|L(k)$ be the lowest utility level. To facilitate the proof process, assume that each level of an attribute k is present in one product. Note that this does not reduce the generality of the proof, since if there are more than one product with each level of k, when a number of these levels are selected for showcasing, the specific products can be determined using Lemma 3.1. We begin the proof with the generic case (RA) and then break it down to specific cases stated in the lemma. Given $|S(k|M^*)| = \zeta$ and the setting above, we assume that $M^* = \{[1]_{k:1}, [1]_{k:2}, ..., [1]_{k:\zeta}\}$. Also, we define a set $M' = \{[1]_{k:1}, [1]_{k:2}, ..., [1]_{k:t-1}, [1]_{k:t+1}, ..., [1]_{k:\zeta}, [1]_{k:t'}\}, \zeta < t' < |(L(k)|$ which differs from M^* in one element. The retailer's expected profit function by showcasing each of these sets are, respectively:

$$\Pi^r_{\mathcal{D}}(M^*) = \frac{\left(\mathcal{P}^r_{[1]_{k:1}} - w\right) e^{\overline{U}_{[1]_{k:1}}} + ... + \left(\mathcal{P}^r_{[1]_{k:\zeta}} - w\right) e^{\overline{U}_{[1]_{k:\zeta}}}}{1 + e^{\overline{U}_{[1]_{k:1}}} + ... + e^{\overline{U}_{[1]_{k:\zeta}}} + e^{\overline{U}_{[1]_{k:t'}} + d_k} + \sum_{x \notin M^*, x \neq [1]_{k:t'}} e^{\overline{U}_x + d_k}}, \tag{B.3}$$

$$\Pi^r_{\mathcal{D}}(M') = \frac{\left(\mathcal{P}^r_{[1]_{k:1}} - w\right) e^{\overline{U}_{[1]_{k:1}}} + ... + \left(\mathcal{P}^r_{[1]_{k:\zeta}} - w\right) e^{\overline{U}_{[1]_{k:\zeta}}} - \left(\mathcal{P}^r_{[1]_{k:t}} - w\right) e^{\overline{U}_{[1]_{k:t}}} + \left(\mathcal{P}^r_{[1]_{k:t'}} - w\right) e^{\overline{U}_{[1]_{k:t'}}}}{1 + e^{\overline{U}_{[1]_{k:1}}} + ... + e^{\overline{U}_{[1]_{k:\zeta}}} - e^{\overline{U}_{[1]_{k:t}}} + e^{\overline{U}_{[1]_{k:t}}+d_k} + e^{\overline{U}_{[1]_{k:t'}}} + \sum_{x \notin M', x \neq [1]_{k:t}} e^{\overline{U}_x + d_k}}. \quad \text{(B.4)}$$

Let $\mathcal{H} = \left(\mathcal{P}^r_{[1]_{k:1}} - w\right) e^{\overline{U}_{[1]_{k:1}}} + ... + \left(\mathcal{P}^r_{[1]_{k:\zeta}} - w\right) e^{\overline{U}_{[1]_{k:\zeta}}} - \left(\mathcal{P}^r_{[1]_{k:t}} - w\right) e^{\overline{U}_{[1]_{k:t}}}$ and $\mathcal{G} = 1 + e^{\overline{U}_{[1]_{k:1}}} + ... + e^{\overline{U}_{[1]_{k:\zeta}}} - e^{\overline{U}_{[1]_{k:t}}} + \sum_{x \notin M^*, x \neq [1]_{k:t'}} e^{\overline{U}_x + d_k}$. Then, (B.3) and (B.4) can be written as

$$\Pi^r_{\mathcal{D}}(M^*) = \frac{\mathcal{H} + \left(\mathcal{P}^r_{[1]_{k:t}} - w\right) e^{\overline{U}_{[1]_{k:t}}}}{\mathcal{G} + e^{\overline{U}_{[1]_{k:t}}} + e^{\overline{U}_{[1]_{k:t'}}+d_k}}, \quad \text{(B.5)}$$

$$\Pi^r_{\mathcal{D}}(M') = \frac{\mathcal{H} + \left(\mathcal{P}^r_{[1]_{k:t'}} - w\right) e^{\overline{U}_{[1]_{k:t'}}}}{\mathcal{G} + e^{\overline{U}_{[1]_{k:t}}+d_k} + e^{\overline{U}_{[1]_{k:t'}}}}. \quad \text{(B.6)}$$

Using (B.5) and (B.6) in $\Pi^r_{\mathcal{D}}(M^*) > \Pi^r_{\mathcal{D}}(M')$, we will get

$$\begin{aligned}
&\mathcal{G}\left[\left(\mathcal{P}^r_{[1]_{k:t}} - w\right) e^{\overline{U}_{[1]_{k:t}}} - \left(\mathcal{P}^r_{[1]_{k:t'}} - w\right) e^{\overline{U}_{[1]_{k:t'}}}\right] + \mathcal{H}\left[e^{\overline{U}_{[1]_{k:t}}+d_k} - e^{\overline{U}_{[1]_{k:t'}}+d_k}\right] + \\
&\mathcal{H}\left[e^{\overline{U}_{[1]_{k:t'}}} - e^{\overline{U}_{[1]_{k:t}}}\right] + \left[\left(\mathcal{P}^r_{[1]_{k:t}} - w\right) e^{\overline{U}_{[1]_{k:t}}} e^{\overline{U}_{[1]_{k:t}}+d_k} - \left(\mathcal{P}^r_{[1]_{k:t'}} - w\right) e^{\overline{U}_{[1]_{k:t'}}} e^{\overline{U}_{[1]_{k:t'}}+d_k}\right] + \\
&\left[\left(\mathcal{P}^r_{[1]_{k:t}} - w\right) e^{\overline{U}_{[1]_{k:t}}} e^{\overline{U}_{[1]_{k:t'}}} - \left(\mathcal{P}^r_{[1]_{k:t'}} - w\right) e^{\overline{U}_{[1]_{k:t'}}} e^{\overline{U}_{[1]_{k:t}}}\right] > 0,
\end{aligned}$$

which can be further simplified as the following

$$\begin{aligned}
&\mathcal{G}\left[\left(\mathcal{P}^r_{[1]_{k:t}} - w\right) e^{\overline{U}_{[1]_{k:t}}} - \left(\mathcal{P}^r_{[1]_{k:t'}} - w\right) e^{\overline{U}_{[1]_{k:t'}}}\right] + \mathcal{H}\left(e^{d_k} - 1\right)\left[e^{\overline{U}_{[1]_{k:t}}} - e^{\overline{U}_{[1]_{k:t'}}}\right] + \\
&\left(\mathcal{P}^r_{[1]_{k:t}} - w\right) e^{\overline{U}_{[1]_{k:t}}} \left(e^{\overline{U}_{[1]_{k:t}}+d_k} + e^{\overline{U}_{[1]_{k:t'}}}\right) - \left(\mathcal{P}^r_{[1]_{k:t'}} - w\right) e^{\overline{U}_{[1]_{k:t'}}} \left(e^{\overline{U}_{[1]_{k:t'}}+d_k} + e^{\overline{U}_{[1]_{k:t}}}\right) > 0.
\end{aligned}$$

The first term in this inequality is always positive. However, the signs of the combination of the second, third, and fourth terms depend on the value of d_k. If $d_k > 0$, then these terms are also always positive and the inequality holds. However, when $d_k < 0$, depending on the magnitude of d_k they can be positive or negative, which means that the inequality may or may not hold. Therefore, in this case, it may not be necessarily optimal to showcase the highest utility levels of k for showcasing. Note that if $\zeta = 1$ (i.e., only one level of k is to be showcased), then $\mathcal{H} = 0$. Then, in this specific case, the inequality holds even for $d_k < 0$. Under the RNA scenario, since product returns are not allowed, we have $\mathcal{P}^r_x = \pi$. substituting this into (B.3) and (B.4) and simplifying, it can be observed that $\Pi^r_{\mathcal{D}}(M^*) > \Pi^r_{\mathcal{D}}(M')$ holds for both $d_k > 0$ and $d_k < 0$. Note that for any M' that differs from M^* in an arbitrary number of elements, the same proof can be conducted. □

Proof. Lemma 3.3.

Under the centralized setting, the profits through selling products in both channels are obtained by the central authority. Therefore, it does not matter which channel sells a specific products. However, showcasing products that reveal information about the inaccurately assessed attribute levels is important. As a result, once a level of an inaccurately assessed attribute is selected, any product consisting of this level can be showcased.

For an arbitrary attribute k, suppose that arbitrarily $S(k|M^*) = \{1, 2, ..., m\}$. Without loss of generality, let the levels of all attributes except k be accurately assessed by customers. Here we show that any arbitrary M that represents $S(k|M^*)$ is an optimal assortment. Suppose that M_1 and M_2 are two assortments that represent $S(k|M^*)$ but differ in at least one element. We can write the total RSC expected profit for M_1 and M_2 as the following, respectively

$$\Pi^T_{\mathcal{C}}(M_1) = \frac{\sum_{x\in X, x_k\in S(k|M^*)} \mathcal{P}^r_x e^{\overline{U}_x} + \sum_{x\in X, x_k\notin S(k|M^*)} \mathcal{P}^m_x e^{\overline{U}_x+d_k}}{1+\sum_{x\in X, x_k\in S(k|M^*)} e^{\overline{U}_x} + \sum_{x\in X, x_k\notin S(k|M^*)} e^{\overline{U}_x+d_k}}, \tag{B.7}$$

$$\Pi^T_{\mathcal{C}}(M_2) = \frac{\sum_{x\in X, x_k\in S(k|M^*)} \mathcal{P}^r_x e^{\overline{U}_x} + \sum_{x\in X, x_k\notin S(k|M^*)} \mathcal{P}^m_x e^{\overline{U}_x+d_k}}{1+\sum_{x\in X, x_k\in S(k|M^*)} e^{\overline{U}_x} + \sum_{x\in X, x_k\notin S(k|M^*)} e^{\overline{U}_x+d_k}}. \tag{B.8}$$

As can be observed, $\Pi^T_{\mathcal{C}}(M_1) = \Pi^T_{\mathcal{C}}(M_2)$ because the only important factor is $S(k|M^*)$ which is represented by both M_1 and M_2. Similarly, if levels of more than one attribute are inaccurately assessed, the same analysis can be conducted. Note that when levels of all attributes are inaccurately assessed, it will be important which specific set of products are showcased, because in such cases, each not-showcased product hides specific inaccuracy information from customers. □

Proof. Lemma 3.4.

In Lemma 3.3, it is discussed that only selecting a subset of levels of inaccurately assessed attributes, independent of which products that consist of these attributes are selected, determines the optimal assortment. This Lemma is a specific case of Lemma 3.3. Here, since levels of all attributes are accurately assessed by customers in the online channel, any arbitrary selection of attribute levels and products is considered an optimal assortment. □

Proof. Proposition 3.1

Note that this proposition represents the RNA scenario. Hence, both Lemmas 3.1 and 3.2 hold for both $d_{\Bbbk} > 0$ and $d_{\Bbbk} < 0$. When $C = 1$, one product should be selected for showcasing. According to Lemmas 3.1 and 3.2, this product should represent the highest part-wroth utility level of attribute $\Bbbk$ and the highest utility product with this level, which results in showcasing product [1]. When $C > \prod_{k\in A, k\neq \Bbbk} |L(k)| \times (|L(\Bbbk)| - 1)$, the capacity is so large that the retailer becomes inevitable to showcase the full variety of levels of $\Bbbk$ (i.e., $L(\Bbbk)$). In this situation, given Lemma 3.1, the optimal decision is to showcase the highest utility products.

When $2 \leq C \leq \prod_{k \in A, k \neq \Bbbk} |L(k)| \times (|L(\Bbbk)| - 1)$, the potential optimal assortments in Q_C obtained through Procedure 1 should be compared. Here, we derive the comparison term for such capacities. Let $t_1, t_2 \in Q_C$. Then, the retailer's expected profit function by showcasing t_1 and t_2 can be written as

$$\Pi^r_{\mathcal{D}}(t_1) = \frac{\sum_{i \in t_1} (\pi - w) e^{\overline{U}_i}}{1 + \sum_{i \in X, i_k \in S(\Bbbk|t_1)} e^{\overline{U}_i} + \sum_{i \in X, i_k \notin S(\Bbbk|t_1)} e^{\overline{U}_i + d_\Bbbk}}, \tag{B.9}$$

$$\Pi^r_{\mathcal{D}}(t_2) = \frac{\sum_{i \in t_2} (\pi - w) e^{\overline{U}_i}}{1 + \sum_{i \in X, i_k \in S(\Bbbk|t_2)} e^{\overline{U}_i} + \sum_{i \in X, i_k \notin S(\Bbbk|t_2)} e^{\overline{U}_i + d_\Bbbk}}. \tag{B.10}$$

Using E, F, H, and G defined in the proposition, we have

$$\Pi^r_{\mathcal{D}}(t_1) = \frac{\sum_{i \in t_1} (\pi - w) e^{\overline{U}_i}}{G + F + (E + H) e^{d_\Bbbk}}, \tag{B.11}$$

$$\Pi^r_{\mathcal{D}}(t_2) = \frac{\sum_{i \in t_2} (\pi - w) e^{\overline{U}_i}}{G + H + (E + F) e^{d_\Bbbk}}. \tag{B.12}$$

For t_1 to be preferred over t_2, we must have $\Pi^r_{\mathcal{D}}(t_1) > \Pi^r_{\mathcal{D}}(t_2)$. Writing and simplifying this condition using (B.9) and (B.10), we get

$$e^{d_\Bbbk} \left[(E + F) \sum_{i \in t_1} e^{\overline{U}_i} - (E + H) \sum_{i \in t_2} e^{\overline{U}_i} \right] > \left[(G + F) \sum_{i \in t_2} e^{\overline{U}_i} - (G + H) \sum_{i \in t_1} e^{\overline{U}_i} \right].$$

Letting $(E+F) \sum_{i \in t_1} e^{\overline{U}_i} - (E+H) \sum_{i \in t_2} e^{\overline{U}_i} = T_{t_1,t_2}$ and $(G+F) \sum_{i \in t_2} e^{\overline{U}_i} - (G+H) \sum_{i \in t_1} e^{\overline{U}_i} = T'_{t_1,t_2}$, Proposition 3.1 will be proven. □

Proof. Corollary 3.1

As it is explained in Section 3.5.1, when $d_\Bbbk > 0$, the retailer's optimal assortment balances the benefits of showcasing the highest utility products and showcasing the highest possible variety of levels of $\Bbbk$ based on Lemma 3.2, as both strategies are desired but may work against each other (i.e., it is possible that showcasing the highest utility products does not result in showcasing the highest possible variety, and vice versa). If the $|L(\Bbbk)|$ highest utility products all consist of different levels of $\Bbbk$, then for $C \leq |L(\Bbbk)|$, showcasing the highest products in fact results in showcasing the highest possible variety of levels of $\Bbbk$. Similarly, for $C > |L(\Bbbk)|$, also showcasing the C highest utility products is optimal. □

Proof. Corollary 3.2

As discussed in Section 3.5.1, when $d_\Bbbk < 0$, the retailer's optimal assortment balances the benefits of showcasing the highest utility products and showcasing the most limited variety of levels of $\Bbbk$ based on Lemma 3.2. If the $\prod_{k \in A, k \neq \Bbbk} |L(k)|$ highest utility products all consist of the same level of $\Bbbk$, then for

$C \leq \prod_{k\in A, k\neq \Bbbk} |L(k)|$, showcasing the highest products results in showcasing only one level of $\Bbbk$ (the most possible limited variety), which is desired. Moreover, if all the next $\prod_{k\in A,k\neq \Bbbk} |L(k)|$ highest utility products also consist of one level of $\Bbbk$, then for $|L(\Bbbk)| < C \leq 2\times|L(\Bbbk)|$, also showcasing the C highest utility products is optimal because it result in showcasing only two levels of $\Bbbk$ (the most possible limited variety). Similarly, Lemma 3.2 can be argued for greater C values. $\square$

Proof. Proposition 3.2.

The retailers expected profit function in the RNA scenario is indicated by (3.14) when $\mathcal{P}^r_x = \pi$. We can write its expected sales function by taking out the wholesale price paid for each products that the retailer sells. Therefore the retailer's expected sales function $\sum_{x\in M} \pi P^r_{x|M}$. Considering the retailer's opportunity cost, it operates the physical store only if

$$\sum_{x\in M} P^r_{x|M} (\pi - w) \geq \Omega \sum_{x\in M} \pi P^r_{x|M}.$$

This can be written as $\pi(1-\Omega)\sum_{x\in M} P^r_{x|M} \geq w \sum_{x\in M} P^r_{x|M}$, which results in $w \leq \pi(1-\Omega)$. The manufacturer sets the wholesale price to the maximum value that the retailer is willing to pay; therefore, $w^* = (1-\Omega)\pi, \forall x \in X$. $\square$

Proof. Proposition 3.3.

Proposition 3.3 states that when $d_\Bbbk > 0$, under the centralized setting, it is optimal to showcase the most limited variety of the lowest part-worth utility levels of attribute $\Bbbk$. Once $S^*_{\mathcal{C}}(\Bbbk)$ is determined, then given Lemma 3.3, any product assortments that represents this set is an optimal assortment. To facilitate the proof process, without loss of generality we assume that each level of $\Bbbk$ is represented by one product. Here, we first show that (i) showcasing m levels of $\Bbbk$ with the smallest part-worth utilities is preferred to any other m levels. Next, we show that (ii) showcasing m levels of the lowest part-wroth utility levels of $\Bbbk$ is preferred over showcasing more levels of $\Bbbk$ with the lowest part-wroth utilities.

(i) Let $S^*_{\mathcal{C}}(\Bbbk) = \{|L(\Bbbk)| - m + 1, ..., |L(\Bbbk)|\}$ and $S^{'}_{\mathcal{C}}(\Bbbk) = \{t', |L(\Bbbk)| - m + 1, t - 1, t + 1, ..., |L(\Bbbk)|\}, t' < |L(\Bbbk)| - m + 1$. Note that $S^*_{\mathcal{C}}(\Bbbk)$ includes the m lowest utility levels of $\Bbbk$, and $S^{'}_{\mathcal{C}}(\Bbbk)$ differs from $S^*_{\mathcal{C}}(\Bbbk)$ in one element. Also, suppose that M^* is the assortment of products representing $S^*_{\mathcal{C}}(\Bbbk)$ and M' is the assortment of products representing $S^{'}_{\mathcal{C}}(\Bbbk)$.

$$\Pi^T_{\mathcal{C}}(M^*) = \frac{\sum_{i=1}^{|L(\Bbbk)|-m} \pi e^{\overline{U}_{[1]_{k:i}} + d_\Bbbk} + \sum_{i=|L(\Bbbk)|-m+1}^{|L(\Bbbk)|} \pi e^{\overline{U}_{[1]_{k:i}}}}{1 + \sum_{i=1}^{|L(\Bbbk)|-m} e^{\overline{U}_{[1]_{k:i}} + d_\Bbbk} + \sum_{i=|L(\Bbbk)|-m+1}^{|L(\Bbbk)|} e^{\overline{U}_{[1]_{k:i}}}}, \tag{B.13}$$

$$\Pi_{\mathbb{C}}^{T}(M') = \frac{\sum_{i=1,i\neq t'}^{|L(\Bbbk)|-m} \pi e^{\overline{U}_{[1]_{k:i}}+d_{\Bbbk}} + \sum_{i=|L(\Bbbk)|-m+1,i\neq t}^{|L(\Bbbk)|} \pi e^{\overline{U}_{[1]_{k:i}}} + \pi e^{\overline{U}_{[1]_{k:t'}}} + \pi e^{\overline{U}_{[1]_{k:t}}+d_{\Bbbk}}}{1+\sum_{i=1,i\neq t'}^{|L(\Bbbk)|-m} e^{\overline{U}_{[1]_{k:i}}+d_{\Bbbk}} + \sum_{i=|L(\Bbbk)|-m+1,i\neq t}^{|L(\Bbbk)|} e^{\overline{U}_{[1]_{k:i}}} + e^{\overline{U}_{[1]_{k:t'}}} + e^{\overline{U}_{[1]_{k:t}}+d_{\Bbbk}}}. \tag{B.14}$$

Supposing $\mathcal{A} = \sum_{i=1,i\neq t'}^{|L(\Bbbk)|-m} e^{\overline{U}_{[1]_{k:i}}+d_{\Bbbk}} + \sum_{i=|L(\Bbbk)|-m+1,i\neq t}^{|L(\Bbbk)|} e^{\overline{U}_{[1]_{k:i}}}$, we can write $\Pi_{\mathbb{C}}^{T}(M^*)$ and $\Pi_{\mathbb{C}}^{T}(M')$ as

$$\Pi_{\mathbb{C}}^{T}(M^*) = \frac{\pi\mathcal{A} + \pi e^{\overline{U}_{[1]_{k:t'}}+d_{\Bbbk}} + \pi e^{\overline{U}_{[1]_{k:t}}}}{1+\mathcal{A} + e^{\overline{U}_{[1]_{k:t'}}+d_{\Bbbk}} + e^{\overline{U}_{[1]_{k:t}}}},$$

$$\Pi_{\mathbb{C}}^{T}(M') = \frac{\pi\mathcal{A} + \pi e^{\overline{U}_{[1]_{k:t'}}} + \pi e^{\overline{U}_{[1]_{k:t}}+d_{\Bbbk}}}{1+\mathcal{A} + e^{\overline{U}_{[1]_{k:t'}}} + e^{\overline{U}_{[1]_{k:t}}+d_{\Bbbk}}}.$$

Since $d_{\Bbbk} > 0$, we know that $e^{\overline{U}_{[1]_{k:t'}}+d_{\Bbbk}} + e^{\overline{U}_{[1]_{k:t}}} > e^{\overline{U}_{[1]_{k:t'}}} + e^{\overline{U}_{[1]_{k:t}}+d_{\Bbbk}}$. Therefore, it can be easily observed that $\Pi_{\mathbb{C}}^{T}(M^*) > \Pi_{\mathbb{C}}^{T}(M')$. Note that if $S_{\mathbb{C}}^{*}(\Bbbk)$ and $S_{\mathbb{C}}^{'}(\Bbbk)$ differ in more than one element, the same proof can be carried out.

(ii) In this step, let $S_{\mathbb{C}}^{'}(\Bbbk) = \{|L(\Bbbk)|-m, |L(\Bbbk)|-m+1, ..., |L(\Bbbk)|\}$. Note that in this case, $S_{\mathbb{C}}^{'}(\Bbbk)$ includes all the levels in $S_{\mathbb{C}}^{*}(\Bbbk)$ plus one more level, i.e., $|L(\Bbbk)|-m$.

$$\Pi_{\mathbb{C}}^{T}(M^*) = \frac{\sum_{i=1}^{|L(\Bbbk)|-m} \pi e^{\overline{U}_{[1]_{k:i}}+d_{\Bbbk}} + \sum_{i=|L(\Bbbk)|-m+1}^{|L(\Bbbk)|} \pi e^{\overline{U}_{[1]_{k:i}}}}{1+\sum_{i=1}^{|L(\Bbbk)|-m} e^{\overline{U}_{[1]_{k:i}}+d_{\Bbbk}} + \sum_{i=|L(\Bbbk)|-m+1}^{|L(\Bbbk)|} e^{\overline{U}_{[1]_{k:i}}}},$$

$$\Pi_{\mathbb{C}}^{T}(M') = \frac{\sum_{i=1}^{|L(\Bbbk)|-m-1} \pi e^{\overline{U}_{[1]_{k:i}}+d_{\Bbbk}} + \sum_{i=|L(\Bbbk)|-m}^{|L(\Bbbk)|} \pi e^{\overline{U}_{[1]_{k:i}}}}{1+\sum_{i=1}^{|L(\Bbbk)|-m-1} e^{\overline{U}_{[1]_{k:i}}+d_{\Bbbk}} + \sum_{i=|L(\Bbbk)|-m}^{|L(\Bbbk)|} e^{\overline{U}_{[1]_{k:i}}}}.$$

Since $d_{\Bbbk} > 0$, we know that $e^{\overline{U}_{[1]_{k:|L(\Bbbk)|-m}}+d_{\Bbbk}} > e^{\overline{U}_{[1]_{k:|L(\Bbbk)|-m}}}$. Hence, it can be easily observed that $\Pi_{\mathbb{C}}^{T}(M^*) > \Pi_{\mathbb{C}}^{T}(M')$. Note that if $S_{\mathbb{C}}^{'}(\Bbbk)$ includes more levels in addition to $S_{\mathbb{C}}^{*}(\Bbbk)$, the same result still holds. $\square$

Proof. Proposition 3.4.

Proposition 3.4 states that when $d_{\Bbbk} < 0$, under the centralized setting, it is optimal to showcase the highest possible variety of the highest part-worth utility levels of attribute $\Bbbk$. Once $S_{\mathbb{C}}^{*}(\Bbbk)$ is determined, then given Lemma 3.3, any product assortments that represents this set is an optimal assortment. To facilitate the proof process, without loss of generality we assume that each level of $\Bbbk$ is represented by one product. Here, we first show that (i) showcasing m levels of $\Bbbk$ with the highest part-worth utilities is preferred to any other m levels. Next, we show that (ii) showcasing m levels of the highest part-wroth utility levels of $\Bbbk$ is preferred over showcasing fewer levels of $\Bbbk$ with the highest part-wroth utilities.

(i) Let $S_{\mathbb{C}}^{*}(\Bbbk) = \{1, 2, ..., m\}$ and $S_{\mathbb{C}}^{'}(\Bbbk) = \{1, 2, ..., t-1, t+1, ..., m, t'\}, t' > m$. Note that $S_{\mathbb{C}}^{*}(\Bbbk)$ includes the m highest utility levels of $\Bbbk$, and $S_{\mathbb{C}}^{'}(\Bbbk)$ differs from $S_{\mathbb{C}}^{*}(\Bbbk)$ in one element. Suppose that M^* is the assortment of products representing $S^*_{\mathcal{C}}(\Bbbk)$ and M' is the assortment of products representing $S'_{\mathcal{C}}(\Bbbk)$.

$$\Pi^T_{\mathcal{C}}(M^*) = \frac{\sum_{i=1}^{m} \pi e^{\overline{U}_{[1]_{k:i}}} + \sum_{i=m+1}^{|L(\Bbbk)|} \pi e^{\overline{U}_{[1]_{k:i}}+d_\Bbbk}}{1+\sum_{i=1}^{m} e^{\overline{U}_{[1]_{k:i}}} + \sum_{i=m+1}^{|L(\Bbbk)|} e^{\overline{U}_{[1]_{k:i}} d_\Bbbk}}, \tag{B.15}$$

$$\Pi^T_{\mathcal{C}}(M') = \frac{\sum_{i=1,i\neq t}^{m} \pi e^{\overline{U}_{[1]_{k:i}}} + \sum_{i=m+1,i\neq t'}^{|L(\Bbbk)|} \pi e^{\overline{U}_{[1]_{k:i}}+d_\Bbbk} + \pi e^{\overline{U}_{[1]_{k:t'}}} + \pi e^{\overline{U}_{[1]_{k:t}}+d_\Bbbk}}{1+\sum_{i=1,i\neq t}^{m} e^{\overline{U}_{[1]_{k:i}}} + \sum_{i=m+1,i\neq t'}^{|L(\Bbbk)|} e^{\overline{U}_{[1]_{k:i}}+d_\Bbbk} + e^{\overline{U}_{[1]_{k:t'}}} + e^{\overline{U}_{[1]_{k:t}}+d_\Bbbk}}. \tag{B.16}$$

Supposing $\mathcal{B} = \sum_{i=1,i\neq t'}^{m} e^{\overline{U}_{[1]_{k:i}}+d_\Bbbk} + \sum_{i=m+1,i\neq t}^{|L(\Bbbk)|} e^{\overline{U}_{[1]_{k:i}}}$, we can write $\Pi^T_{\mathcal{C}}(M^*)$ and $\Pi^T_{\mathcal{C}}(M')$ as

$$\Pi^T_{\mathcal{C}}(M^*) = \frac{\pi\mathcal{B} + \pi e^{\overline{U}_{[1]_{k:t}}} + \pi e^{\overline{U}_{[1]_{k:t'}}+d_\Bbbk}}{1+\mathcal{B}+e^{\overline{U}_{[1]_{k:t}}} + e^{\overline{U}_{[1]_{k:t'}}+d_\Bbbk}},$$

$$\Pi^T_{\mathcal{C}}(M') = \frac{\pi\mathcal{B} + \pi e^{\overline{U}_{[1]_{k:t}}+d_\Bbbk} + \pi e^{\overline{U}_{[1]_{k:t'}}}}{1+\mathcal{B}+e^{\overline{U}_{[1]_{k:t}}+d_\Bbbk} + e^{\overline{U}_{[1]_{k:t'}}}}.$$

Since $d_\Bbbk < 0$, we know that $e^{\overline{U}_{[1]_{k:t}}} + e^{\overline{U}_{[1]_{k:t'}}+d_\Bbbk} > e^{\overline{U}_{[1]_{k:t}}+d_\Bbbk} + e^{\overline{U}_{[1]_{k:t'}}}$. Therefore, it can be observed that $\Pi^T_{\mathcal{C}}(M^*) > \Pi^T_{\mathcal{C}}(M')$. Note that if $S^*_{\mathcal{C}}(\Bbbk)$ and $S'_{\mathcal{C}}(\Bbbk)$ differ in more than one element, the same proof can be carried out.

(ii) In this step, let $S'_{\mathcal{C}}(\Bbbk) = \{1, 2, ..., m-1\}$. Note that in this case, $S'_{\mathcal{C}}(\Bbbk)$ includes all the levels in $S^*_{\mathcal{C}}(\Bbbk)$ except one level, i.e., level m. This can be any other level that has been excluded from $S^*_{\mathcal{C}}(\Bbbk)$.

$$\Pi^T_{\mathcal{C}}(M^*) = \frac{\sum_{i=1}^{m} \pi e^{\overline{U}_{[1]_{k:i}}} + \sum_{i=m+1}^{|L(\Bbbk)|} \pi e^{\overline{U}_{[1]_{k:i}}+d_\Bbbk}}{1+\sum_{i=1}^{m-1} e^{\overline{U}_{[1]_{k:i}}} + \sum_{i=m}^{|L(\Bbbk)|} e^{\overline{U}_{[1]_{k:i}}+d_\Bbbk}},$$

$$\Pi^T_{\mathcal{C}}(M') = \frac{\sum_{i=1}^{m-1} \pi e^{\overline{U}_{[1]_{k:i}}+d_\Bbbk} + \sum_{i=m}^{|L(\Bbbk)|} \pi e^{\overline{U}_{[1]_{k:i}}}}{1+\sum_{i=1}^{m-1} e^{\overline{U}_{[1]_{k:i}}+d_\Bbbk} + \sum_{i=m}^{|L(\Bbbk)|} e^{\overline{U}_{[1]_{k:i}}}}.$$

Since $d_\Bbbk < 0$, we know that $e^{\overline{U}_{[1]_{k:m}}} > e^{\overline{U}_{[1]_{k:m}}+d_\Bbbk}$. Hence, it can be observed that $\Pi^T_{\mathcal{C}}(M^*) > \Pi^T_{\mathcal{C}}(M')$. Note that if $S'_{\mathcal{C}}(\Bbbk)$ includes fewer levels than $S^*_{\mathcal{C}}(\Bbbk)$, the same result holds, too. □

Proof. Proposition 3.5

Note that this proposition represents the RA scenario. Hence, both Lemmas 3.1 and 3.2 hold for $d_\Bbbk > 0$, but only the former holds all the time for $d_\Bbbk < 0$. In both cases, for $C = 1$ and $C > \prod_{k\in A,k\neq\Bbbk} |L(k)| \times (|L(\Bbbk)| - 1)$, the same argument as the proof of Proposition 3.1 holds.

When $2 \leq C \leq \prod_{k\in A,k\neq\Bbbk} |L(k)| \times (|L(\Bbbk)| - 1)$, the potential optimal assortments in Q_C for the $d_\Bbbk > 0$ case and Q'_C for the $d_\Bbbk < 0$ obtained through Procedures 1 and 2 should be compared. Here, we derive the comparison term for such capacities. Let $t_1, t_2 \in Q_C$ or similarly $t_1, t_2 \in Q'_C$. Then, the retailer's expected profit function by showcasing t_1 and t_2 can be written as

$$\Pi^r_{\mathcal{D}}(t_1) = \frac{\sum_{i\in t_1}(\mathcal{P}^r_i - w)e^{\overline{U}_i}}{1 + \sum_{i\in X, i_k\in S(\Bbbk|t_1)} e^{\overline{U}_i} + \sum_{i\in X, i_k\notin S(\Bbbk|t_1)} e^{\overline{U}_i + d_\Bbbk}}, \tag{B.17}$$

$$\Pi^r_{\mathcal{D}}(t_2) = \frac{\sum_{i\in t_2}(\mathcal{P}^r_i - w)e^{\overline{U}_i}}{1 + \sum_{i\in X, i_k\in S(\Bbbk|t_2)} e^{\overline{U}_i} + \sum_{i\in X, i_k\notin S(\Bbbk|t_2)} e^{\overline{U}_i + d_\Bbbk}}. \tag{B.18}$$

Using E, F, H, and G defined in the proposition, we have

$$\Pi^r_{\mathcal{D}}(t_1) = \frac{\sum_{i\in t_1}(\mathcal{P}^r_i - w)e^{\overline{U}_i}}{G + F + (E+H)e^{d_\Bbbk}}, \tag{B.19}$$

$$\Pi^r_{\mathcal{D}}(t_2) = \frac{\sum_{i\in t_2}(\mathcal{P}^r_i - w)e^{\overline{U}_i}}{G + H + (E+F)e^{d_\Bbbk}}. \tag{B.20}$$

For t_1 to be preferred over t_2, we must have $\Pi^r_{\mathcal{D}}(t_1) > \Pi^r_{\mathcal{D}}(t_2)$. Writing and simplifying this condition using (B.19) and (B.20), we get

$$\begin{aligned} e^{d_\Bbbk}\left[(E+F)\sum_{i\in t_1}(\mathcal{P}^r_i - w)e^{\overline{U}_i} - (E+H)\sum_{i\in t_2}(\mathcal{P}^r_i - w)e^{\overline{U}_i}\right] > \\ \left[(G+F)\sum_{i\in t_2}(\mathcal{P}^r_i - w)e^{\overline{U}_i} - (G+H)\sum_{i\in t_1}(\mathcal{P}^r_i - w)e^{\overline{U}_i}\right]. \end{aligned}$$

Letting $(E+F)\sum_{i\in t_1}(\mathcal{P}^r_i - w)e^{\overline{U}_i} - (E+H)\sum_{i\in t_2}(\mathcal{P}^r_i - w)e^{\overline{U}_i} = \mathcal{T}_{t_1,t_2}$ and $(G+F)\sum_{i\in t_2}(\mathcal{P}^r_i - w)e^{\overline{U}_i} - (G+H)\sum_{i\in t_1}(\mathcal{P}^r_i - w)e^{\overline{U}_i} = \mathcal{T}'_{t_1,t_2}$, Proposition 3.5 will be proven. □

Proof. Proposition 3.6.

The retailer's expected profit function is given by (3.14), and its expected sales function can be written by taking out the wholesale price from (3.14). Therefore, the retailer's expected sales function will be $\sum_{x\in M} \mathcal{P}^r_x P^r_{x|M}$. Given the retailer's opportunity cost, Ω, the wholesale price can be determined such that

$$\sum_{x\in M} P^r_{x|M}\left(\mathcal{P}^r_x - w\right) > \Omega \sum_{x\in M} \mathcal{P}^r_x P^r_{x|M}.$$

Simplifying this inequality, we get $w \le \frac{(1-\Omega)\sum_{x\in M} P^f_{x|M}\mathcal{P}^r_x}{\sum_{x\in M} P^f_{x|M}}$. Therefore, the manufacturer sets the wholesale that the retailer could pay given its opportunity cost. So

$$w^* = \frac{(1-\Omega)\sum_{x\in M} P^f_{x|M}\mathcal{P}^r_x}{\sum_{x\in M} P^f_{x|M}}.$$

□

Proof. Theorem 3.1.

Under the $\mathcal{SC} = \vec{\alpha}$, neither of the parties should be worse off compared to their decentralized expected profits. Therefore, we must have

$$\Pi^r_{\mathcal{SC}}(M^*_{\mathcal{C}}|w^*_{\mathcal{D}}, \vec{\alpha}) \geq \Pi^r_D(M^*_{\mathcal{D}}|w^*_{\mathcal{D}}) \tag{B.21a}$$

$$\Pi^m_{\mathcal{SC}}(w^*_{\mathcal{D}}, \vec{\alpha}) \geq \Pi^m_D(w^*_{\mathcal{D}}). \tag{B.21b}$$

Substituting corresponding profit functions into (B.21), we get (3.21). We need the upper-bound of (3.21) to be greater than or equal to its lower-bound. Writing this condition yields $\Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|w^*_{\mathcal{D}})+\Pi^m_{\mathcal{D}}(w^*_{\mathcal{D}}) \leq \Pi^T_{\mathcal{C}}(M^*_{\mathcal{C}})$, which always holds. Moreover, since $\sum_{x\in M^*_{\mathcal{C}}, x_{\Bbbk}=j}(1-\alpha_j)e^{\overline{U}_x}$ obtains a non-negative value, it is necessary for the upper-bound to be non-negative. Otherwise, if the upper-bound is negative, there is no $\mathcal{SC} = \vec{\alpha}$ that can compensate for the retailer's loss. In this situation, by using $\mathcal{SC} = \vec{\beta}$, a one-time payment of $\mathcal{L}$ from the manufacturer to the retailer can compensate its loss (i.e., $\Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|w^*_{\mathcal{D}}) \leq \Pi^r_{\mathcal{D}}(M^*_{\mathcal{C}}|w^*_{\mathcal{D}}, \vec{\beta}) + \mathcal{L}$). The value of this payment should also guarantee that the manufacturer is not worse off either compared to its decentralized profit (i.e., $\Pi^m_{\mathcal{D}}(w^*_{\mathcal{D}}) \leq \Pi^m_{\mathcal{D}}(w^*_{\mathcal{D}}, \vec{\beta}) - \mathcal{L}$). □

Proof. Theorem 3.2.

We need to show that $\Pi^r_{\mathcal{SC}}(M^*_{\mathcal{SC}}|w^*_{\mathcal{D}}, \overline{\alpha}) = \Pi^r_{\mathcal{SC}}(M^*_{\mathcal{SC}}|w^*_{\mathcal{D}}, \vec{\alpha})$ and $\Pi^m_{\mathcal{SC}}(w^*_{\mathcal{D}}, \overline{\alpha}) = \Pi^m_{\mathcal{SC}}(w^*_{\mathcal{D}}, \vec{\alpha})$. Note that if we show that these two equalities hold, there is not need to check whether $\overline{\alpha}$ satisfies the coordination conditions in (3.21). This is because the coordination conditions are used to satisfy (B.21a) and (B.21b), which if $\Pi^r_{\mathcal{SC}}(M^*_{\mathcal{SC}}|w^*_{\mathcal{D}}, \overline{\alpha}) = \Pi^r_{\mathcal{SC}}(M^*_{\mathcal{SC}}|w^*_{\mathcal{D}}, \vec{\alpha})$ and $\Pi^m_{\mathcal{SC}}(w^*_{\mathcal{D}}, \overline{\alpha}) = \Pi^m_{\mathcal{SC}}(w^*_{\mathcal{D}}, \vec{\alpha})$ hold, then $\overline{\alpha}$ is equivalent to $\vec{\alpha}$ that is assumed to satisfy the conditions.

To have $\Pi^r_{\mathcal{SC}}(M^*_{\mathcal{SC}}|w^*_{\mathcal{D}}, \overline{\alpha}) = \Pi^r_{\mathcal{SC}}(M^*_{\mathcal{SC}}|w^*_{\mathcal{D}}, \vec{\alpha})$ and $\Pi^m_{\mathcal{SC}}(w^*_{\mathcal{D}}, \overline{\alpha}) = \Pi^m_{\mathcal{SC}}(w^*_{\mathcal{D}}, \vec{\alpha})$, we can write

$$\sum_{x\in M^*_{\mathcal{SC}}} P^r_{x|M^*_{\mathcal{SC}}} \left(\mathcal{P}^r_x - (1-\overline{\alpha})w^*_{\mathcal{D}}\right) = \sum_{x\in M^*_{\mathcal{SC}}, x_{\Bbbk}=j} P^r_{x|M^*_{\mathcal{SC}}} \left(\mathcal{P}^r_x - (1-\alpha_j)w^*_{\mathcal{D}}\right),$$

$$\sum_{x\in X\setminus M^*_{\mathcal{SC}}} P^m_{x|M^*_{\mathcal{SC}}} \mathcal{P}^m_{x|M^*_{\mathcal{SC}}} + \sum_{x\in M^*_{\mathcal{SC}}} (1-\overline{\alpha})w^*_{\mathcal{D}} P^r_{x|M^*_{\mathcal{SC}}} = \sum_{x\in X\setminus M^*_{\mathcal{SC}}} P^m_{x|M^*_{\mathcal{SC}}} \mathcal{P}^m_{x|M^*_{\mathcal{SC}}} + \sum_{x\in M^*_{\mathcal{SC}}, x_{\Bbbk}=j} (1-\alpha_j)w^*_{\mathcal{D}} P^r_{x|M^*_{\mathcal{SC}}}.$$

Simplifying both equations, we get the same outcomes as the following

$$\sum_{x\in M^*_{\mathcal{SC}}} (1-\overline{\alpha}) P^r_{x|M^*_{\mathcal{SC}}} = \sum_{x\in M^*_{\mathcal{SC}}, x_{\Bbbk}=j} (1-\alpha_j) P^r_{x|M^*_{\mathcal{SC}}}. \tag{B.22}$$

Substituting the value of $\overline{\alpha}$ into (B.22) and simplifying, we get $\frac{\sum_{x\in M^*_{\mathcal{SC}}} e^{\overline{U}_x} \left(\sum_{x\in M^*_{\mathcal{SC}}, x_{\Bbbk}=j} (1-\alpha_j) e^{\overline{U}_x}\right)}{\Phi \sum_{x\in M^*_{\mathcal{SC}}} e^{\overline{U}_x}}$. This can be further simplified as $\sum_{x \in M^*_{\mathbb{S}^{\mathbb{C}}}, x_{\mathbb{k}}=j}(1-\alpha_j)P^r_{x|M^*_{\mathbb{S}^{\mathbb{C}}}}$, which is equal to the right-hand side of (B.22) and completes the proof.

$\square$

Appendix C

Appendix: Proofs of Chapter 4

Proof. Proposition 4.1.

There sum of the retailer's and manufacturer's expected profit functions can be written as the sum of the functions in (4.10) and (4.13). So, we have

$$\hat{\Pi}^r_{\mathcal{D}} + \hat{\Pi}^m_{\mathcal{D}} = \sum_{x \in X \setminus M} P^m_{x|M} \mathcal{P}^m_{x|M} + \sum_{x \in M} P^r_{x|M} \mathcal{P}^r_x.$$

Since $\mathbb{d} = 0$, we have $P^m_{x|M} = P^r_{x|M}$ and $\mathcal{P}^m_{x|M} = \mathcal{P}^r_x, \forall x \in X$. Therefore, for the total expected profit if the RSC, it does not matter if a product is purchased online or from the store. Furthermore, it also does not matter which products or attribute levels are showcased because assortment do not reveal any specific accurate utility information. Note that if $\hat{w}^r_{\mathcal{D}} < \hat{w}^m_{\mathcal{D}}$, the retailer will withdraw from the market and no assortment will be showcased. □

Proof. Proposition 4.2.

This proposition can be shown the same way as Lemma 3.3 in Chapter 3. □

Proof. Proposition 4.3.

(i) This part can be shown the same way as Proposition 4.1.

(ii) This part is an extension of Proposition 4.2. We showed in Proposition 4.2 that only selecting levels of attributes estimated to be inaccurately assessed suffices for assortment planning. When $\hat{\mathbb{d}} = 0$, it is estimated that there is no inaccuracy in any of the attribute levels. Therefore, any selection of levels and products is supposed to be optimal. □