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Research article

Numerical analysis of fractional heat transfer and porous media equations within Caputo-Fabrizio operator

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Abstract: This paper presents a comparative study of two popular analytical methods, namely the Homotopy Perturbation Transform Method (HPTM) and the Adomian Decomposition Transform Method (ADTM), to solve two important fractional partial differential equations, namely the fractional heat transfer and porous media equations. The HPTM uses a perturbation approach to construct an approximate solution, while the ADTM decomposes the solution into a series of functions using the Adomian polynomials. The results obtained by the HPTM and ADTM are compared with the exact solutions, and the performance of both methods is evaluated in terms of accuracy and convergence rate. The numerical results show that both methods are efficient in solving the fractional heat transfer and porous media equations, and the HPTM exhibits slightly better accuracy and convergence rate than the ADTM. Overall, the study provides a valuable insight into the application of the HPTM and ADTM in solving fractional differential equations and highlights their potential for solving complex mathematical models in physics and engineering.

Keywords: Yang transform; fractional heat transfer equation; fractional porous media; Adomian decomposition method; Homotopy perturbation method
Mathematics Subject Classification: 33B15, 34A34, 35A20, 35A22, 44A10

Nomenclature

$\mathfrak{h}$: Independent variable
ς : Time
$\mathfrak{u}(\mathfrak{h}, \varsigma)$: Dependent function representing the physical quantity
δ : Fractional order
$\mathbb{Y}$ Yang transform
$\mathbb{Y}^{-1}$: Inverse Yang transform
ϵ : Perturbation parameter

1. Introduction

The study of transport phenomena in porous media has been a topic of interest in several areas of engineering and science, such as geology, environmental science, and chemical engineering. Porous media are materials that contain voids or pores filled with fluids, which can include gases, liquids, or a combination of both. The behavior of these fluids within the pores is influenced by various factors, including the geometry of the pores, the properties of the fluid, and the interactions between the fluid and the solid matrix [1–5].

Recently, there has been increasing interest in the study of transport phenomena in fractional porous media, where the properties of the medium are described by fractional calculus. Fractional calculus is a generalization of traditional calculus that deals with derivatives and integrals of non-integer order. This mathematical tool has been successfully used to model various physical phenomena that exhibit anomalous behavior, such as diffusion in complex media, viscoelasticity, and fractal geometry [6, 7]. One of the areas where fractional calculus has been particularly successful is in the study of heat transfer in porous media. The fractional heat transfer equation (FHTE) has been proposed as a more accurate and comprehensive model for heat transfer in porous media compared to the traditional heat transfer equation. The FHTE takes into account the effects of non-locality and memory on heat transfer, which are not captured by the traditional model [8–12].

Several studies have shown the potential of fractional calculus in improving our understanding of heat transfer in porous media. For example, investigated the effect of fractional calculus on heat transfer in a porous channel, and showed that the FHTE can accurately predict the temperature distribution in the channel [13]. Another study by compared the performance of traditional and fractional models for heat transfer in a fractured rock mass, and demonstrated that the FHTE provides a better fit to experimental data. In summary, the study of fractional porous media and the use of the FHTE in modeling heat transfer in these media represent promising areas of research that have the potential to improve our understanding of transport phenomena in porous materials [14, 15].

Homotopy perturbation method (HPM) and Adomian decomposition method (ADM) are two popular techniques used in solving nonlinear differential equations [16–18]. The HPM is a powerful analytical method based on constructing a homotopy between the problem at hand and an auxiliary linear problem. The method involves the use of a small parameter that helps to obtain an analytical solution through a series expansion [19, 20]. On the other hand, ADM is an iterative method that decomposes the nonlinear equation into a series of linear subproblems, and then solves them iteratively. Recently, a new method known as Yang's transform has been introduced to solve the nonlinear differential equations. This method is based on the concept of integral transforms and is a combination of HPM and ADM. The Yang transform is known for its accuracy, robustness, and efficiency in solving nonlinear differential equations [21–24].

In this article, we will explore the HPM, ADM, and Yang transform in detail and discuss their applications in solving various types of fractional porous media and heat transfer equation. We will also compare and contrast these methods and highlight their strengths and weaknesses. By the end of this article, readers will have a good understanding of these methods and their potential for solving complex nonlinear problems.

2. Preliminaries

We give the basic definitions which are needed in the rest of paper. For the sake of simplicity, we write the exponential decay kernel as $K(\varsigma, \varrho) = \exp[-\delta(\varsigma - \varrho)/(1 - \delta)]$.

Definition 2.1. If $h(\varsigma) \in \mathbf{H}^1[0, T], T > 0$, then the Caputo Fabrizio (CF) derivative may be expressed as follows [25]:

$$ {}^{CF}D_{\varsigma}^{\delta}[h(\varsigma)] = \frac{N(\delta)}{1-\delta}\int_0^{\varsigma} h'(\varrho)K(\varsigma, \varrho)d\varrho. \tag{2.1}$$

$N(\delta)$ is the normalization function with $N(1) = N(0) = 1$. However, if $h(\varsigma) \notin \mathbf{H}^1[0, T]$, then the above derivative is defined as follows:

$$ {}^{CF}D_{\varsigma}^{\delta}[h(\varsigma)] = \frac{N(\delta)}{1-\delta}\int_0^{\varsigma} [h(\varsigma) - h(\varrho)]K(\varsigma, \varrho)d\varrho. \tag{2.2}$$

Definition 2.2. The definition of Laplace transform for the derivative of the characteristic function, when $N(\delta) = 1$, is given by [25]:

$$ L\left[{}^{CF}D_{\varsigma}^{\delta}[h(\varsigma)]\right] = \frac{\omega L[h(\varsigma)] - h(0)}{\omega + \delta(1-\omega)}. \tag{2.3}$$

Definition 2.3. The given expression is the fractional CF integral [25]

$$ {}^{CF}I_{\varsigma}^{\delta}[h(\varsigma)] = \frac{1-\delta}{N(\delta)}h(\varsigma) + \frac{\delta}{N(\delta)}\int_0^{\varsigma} h(\varrho)d\varrho, \quad \varsigma \geq 0, \quad \delta \in (0, 1]. \tag{2.4}$$

Definition 2.4. The Yang transformation of $h(\varsigma)$ is given as [25]

$$ \mathcal{Y}[h(\varsigma)] = \chi(\omega) = \int_0^{\infty} h(\varsigma)e^{-\frac{\varsigma}{\omega}}d(\varsigma), \quad \varsigma > 0. \tag{2.5}$$

Remark 2.1. The Yang transformation of few term formulaes are given as [25]

$$\begin{aligned} Y[1] =& \omega, \\ Y[\varsigma] =& \omega^2, \\ Y[\varsigma^i] =& \Gamma(i+1)\omega^{i+1}. \end{aligned} \tag{2.6}$$

Lemma 2.1. *Yang-Laplace duality*

Let the Laplace transformation of $h(\varsigma)$ *is* $F(\omega)$, *then* $\chi(\omega) = F(1/\omega)$ *[25].*

Proof. By substituting $\varsigma/\omega = \xi$ into Eq (2.5), we can derive an alternative expression for the Yang transform.

$$L[h(\varsigma)] = \chi(\omega) = \omega \int_0^\infty h(\omega\xi)e^{\xi} d\xi. \ \xi > 0, \tag{2.7}$$

Since $L[h(\varsigma)] = F(\omega)$, this $\Longrightarrow$

$$F(\omega) = L[h(\varsigma)] = \int_0^\infty h(\varsigma)e^{-\omega\varsigma} d\varsigma. \tag{2.8}$$

Put $\varsigma = \xi/\omega$ in (2.8), we get

$$F(\omega) = \frac{1}{\omega}\int_0^\infty h\left(\frac{\xi}{\omega}\right)e^{\xi} d\xi. \tag{2.9}$$

Thus, from Eq (2.7), we achieved

$$F(\omega) = \chi\left(\frac{1}{\omega}\right). \tag{2.10}$$

Also from Eqs (2.5) and (2.8), we have

$$F\left(\frac{1}{\omega}\right) = \chi(\omega). \tag{2.11}$$

The Laplace and Yang transforms are dually connected, as represented by the links (2.10) and (2.11). □

Lemma 2.2. *Let $h(\varsigma)$ be a continue function, then, the Yang transformation CF derivative of $h(\varsigma)$ is defined as [25]*

$$\mathbb{Y}[h(\varsigma)] = \frac{\mathbb{Y}[h(\varsigma)] - \omega h(0)}{1 + \delta(\omega - 1)}. \tag{2.12}$$

Proof. The definition of the Laplace transformation of a fractional CF operator is as follows:

$$L[h(\varsigma)] = \frac{\omega L[h(\varsigma)] - h(0)}{\omega + \delta(1 - \omega)}, \tag{2.13}$$

Additionally, we can establish a relationship between Laplace and Yang properties, expressed as $\chi(\omega) = F(1/\omega)$. To derive this result, we substitute ω with $1/\omega$ in Eq (2.13). This yields:

$$\begin{aligned}\mathbb{Y}[h(\varsigma)] &= \frac{\frac{1}{\omega}\mathbb{Y}[h(\varsigma)] - h(0)}{\frac{1}{\omega} + \delta(1 - \frac{1}{\omega})},\\ \mathbb{Y}[h(\varsigma)] &= \frac{\mathbb{Y}[h(\varsigma)] - \omega h(0)}{1 + \delta(\omega - 1)}.\end{aligned} \tag{2.14}$$

The proof is completed. □

3. General implementation of HPTM

Consider the fractional partial differential equation is given as

$$D_{\varsigma}^{\delta}\mathfrak{u}(\mathfrak{h},\varsigma) = \mathcal{P}_1\mathfrak{u}(\mathfrak{h},\varsigma) + \mathcal{R}_1\mathfrak{u}(\mathfrak{h},\varsigma), \quad 0 < \delta \leq 1, \tag{3.1}$$

with initial condition

$$\mathfrak{u}(\mathfrak{h},0) = \xi(\mathfrak{h}).$$

The notation $D_{\varsigma}^{\delta} = \frac{\partial^{\delta}}{\partial\varsigma^{\delta}}$ represents the fractional Caputo-Fabrizio derivative, while $\mathcal{P}_1$ and $\mathcal{R}_1$ show that linear and nonlinear functions.

Applying the Yang transformation to Eq (3.1), we get

$$\mathbb{Y}[D_{\varsigma}^{\delta}\mathfrak{u}(\mathfrak{h},\varsigma)] = \mathbb{Y}[\mathcal{P}_1\mathfrak{u}(\mathfrak{h},\varsigma) + \mathcal{R}_1\mathfrak{u}(\mathfrak{h},\varsigma)], \tag{3.2}$$

$$\frac{1}{1+\delta(s-1)}\{\mathbb{Y}[\mathfrak{u}(\mathfrak{h},\varsigma)] - s\mathfrak{u}(\mathfrak{h},0)\} = \mathbb{Y}[\mathcal{P}_1\mathfrak{u}(\mathfrak{h},\varsigma) + \mathcal{R}_1\mathfrak{u}(\mathfrak{h},\varsigma)]. \tag{3.3}$$

On simplification, we have

$$\mathbb{Y}[\mathfrak{u}(\mathfrak{h},\varsigma)] = s\mathfrak{u}(\mathfrak{h},0) + [1+\delta(s-1)]\mathbb{Y}[\mathcal{P}_1\mathfrak{u}(\mathfrak{h},\varsigma) + \mathcal{R}_1\mathfrak{u}(\mathfrak{h},\varsigma)]. \tag{3.4}$$

We have, by using the inverse of Yang transform:

$$\mathfrak{u}(\mathfrak{h},\varsigma) = \mathfrak{u}(\mathfrak{h},0) + \mathbb{Y}^{-1}[(1+\delta(s-1))\mathbb{Y}[\mathcal{P}_1\mathfrak{u}(\mathfrak{h},\varsigma) + \mathcal{R}_1\mathfrak{u}(\mathfrak{h},\varsigma)]]. \tag{3.5}$$

The basic solution in a power series can be expressed in terms of the high-performance computing language HPM as follows:

$$\mathfrak{u}(\mathfrak{h},\varsigma) = \sum_{k=0}^{\infty}\epsilon^k\mathfrak{u}_k(\mathfrak{h},\varsigma). \tag{3.6}$$

with homotopy parameter $\epsilon \in [0,1]$.

The nonlinear term reads

$$\mathcal{R}_1\mathfrak{u}(\mathfrak{h},\varsigma) = \sum_{k=0}^{\infty}\epsilon^k H_k(\mathfrak{u}), \tag{3.7}$$

$$H_k(\mathfrak{u}_0,\mathfrak{u}_1,\ldots,\mathfrak{u}_n) = \frac{1}{\Gamma(n+1)}D_{\epsilon}^{k}\left[\mathcal{R}_1\left(\sum_{k=0}^{\infty}\epsilon^i\mathfrak{u}_i\right)\right]_{\epsilon=0}, \tag{3.8}$$

where $D_{\epsilon}^{k} = \frac{\partial^k}{\partial\epsilon^k}$.

Putting Eqs (3.6) and (3.7) into Eq (3.5), we get

$$\sum_{k=0}^{\infty}\epsilon^k\mathfrak{u}_k(\mathfrak{h},\varsigma) = \mathfrak{u}(\mathfrak{h},0) + \epsilon\times\left(\mathbb{Y}^{-1}\left[(1+\delta(s-1))\mathbb{Y}\{\mathcal{P}_1\sum_{k=0}^{\infty}\epsilon^k\mathfrak{u}_k(\mathfrak{h},\varsigma) + \sum_{k=0}^{\infty}\epsilon^k H_k(\mathfrak{u})\}\right]\right). \tag{3.9}$$

By setting the coefficients of ϵ equal to each other, we obtain:

$$
\begin{aligned}
&\epsilon^0 : \mathfrak{u}_0(\mathfrak{h},\varsigma) = \mathfrak{u}(\mathfrak{h},0),\\
&\epsilon^1 : \mathfrak{u}_1(\mathfrak{h},\varsigma) = \mathbb{Y}^{-1}\left[(1+\delta(s-1))\mathbb{Y}(\mathcal{P}_1\mathfrak{u}_0(\mathfrak{h},\varsigma) + H_0(\mathfrak{u}))\right],\\
&\epsilon^2 : \mathfrak{u}_2(\mathfrak{h},\varsigma) = \mathbb{Y}^{-1}\left[(1+\delta(s-1))\mathbb{Y}(\mathcal{P}_1\mathfrak{u}_1(\mathfrak{h},\varsigma) + H_1(\mathfrak{u}))\right],\\
&.\\
&.\\
&.\\
&\epsilon^k : \mathfrak{u}_k(\mathfrak{h},\varsigma) = \mathbb{Y}^{-1}\left[(1+\delta(s-1))\mathbb{Y}(\mathcal{P}_1\mathfrak{u}_{k-1}(\mathfrak{h},\varsigma) + H_{k-1}(\mathfrak{u}))\right],\ \ k>0, k\in N.
\end{aligned}
\tag{3.10}
$$

Ultimately, we express the analytical solution in series form by approximation.

$$
\mathfrak{u}(\mathfrak{h},\varsigma) = \lim_{M\to\infty}\sum_{k=1}^{M}\mathfrak{u}_k(\mathfrak{h},\varsigma). \tag{3.11}
$$

4. General implementation of Yang transform decomposition method

Consider the fractional partial differential equation is given as

$$
D_\varsigma^\delta\mathfrak{u}(\mathfrak{h},\varsigma) = \mathcal{P}_1\mathfrak{u}(\mathfrak{h},\varsigma) + \mathcal{R}_1\mathfrak{u}(\mathfrak{h},\varsigma), 0<\delta\le 1, \tag{4.1}
$$

with initial condition

$$
\mathfrak{u}(\mathfrak{h},0) = \xi(\mathfrak{h}).
$$

The notation $D_\varsigma^\delta = \frac{\partial^\delta}{\partial\varsigma^\delta}$ refers to the fractional Caputo-Fabrizio derivative, while $\mathcal{P}_1$ and $\mathcal{R}_1$ are represented linear and nonlinear functions.

Employing the Yang transform Eq (4.1), we get

$$
\begin{aligned}
&\mathbb{Y}[D_\varsigma^\delta\mathfrak{u}(\mathfrak{h},\varsigma)] = \mathbb{Y}[\mathcal{P}_1\mathfrak{u}(\mathfrak{h},\varsigma) + \mathcal{R}_1\mathfrak{u}(\mathfrak{h},\varsigma)],\\
&\frac{1}{1+\delta(s-1)}\{\mathbb{Y}[\mathfrak{u}(\mathfrak{h},\varsigma)] - s\mathfrak{u}(\mathfrak{h},0)\} = \mathbb{Y}[\mathcal{P}_1\mathfrak{u}(\mathfrak{h},\varsigma) + \mathcal{R}_1\mathfrak{u}(\mathfrak{h},\varsigma)].
\end{aligned}
\tag{4.2}
$$

On simplification, we have

$$
\mathbb{Y}[\mathfrak{u}(\mathfrak{h},\varsigma)] = s\mathfrak{u}(\mathfrak{h},0) + [1+\delta(s-1)]\mathbb{Y}[\mathcal{P}_1\mathfrak{u}(\mathfrak{h},\varsigma) + \mathcal{R}_1\mathfrak{u}(\mathfrak{h},\varsigma)]. \tag{4.3}
$$

By using the inverse of Yang transform, we get

$$
\mathfrak{u}(\mathfrak{h},\varsigma) = \mathfrak{u}(\mathfrak{h},0) + \mathbb{Y}^{-1}[1+\delta(s-1)]\mathbb{Y}[\mathcal{P}_1\mathfrak{u}(\mathfrak{h},\varsigma) + \mathcal{R}_1\mathfrak{u}(\mathfrak{h},\varsigma)]. \tag{4.4}
$$

The series form solution of $\mathfrak{u}(\mathfrak{h},\varsigma)$ reads:

$$
\mathfrak{u}(\mathfrak{h},\varsigma) = \sum_{m=0}^{\infty}\mathfrak{u}_m(\mathfrak{h},\varsigma). \tag{4.5}
$$

The nonlinear term is defined as

$$\mathcal{R}_1 \mathfrak{u}(\mathfrak{h}, \varsigma) = \sum_{m=0}^{\infty} \mathcal{A}_m(\mathfrak{u}). \tag{4.6}$$

with

$$\mathcal{A}_m\Big(\mathfrak{u}_0, \mathfrak{u}_1, \mathfrak{u}_2, \cdots, \mathfrak{u}_m\Big) = \frac{1}{m!}\left[\frac{\partial^m}{\partial \ell^m}\left\{\mathcal{R}_1\left(\sum_{m=0}^{\infty} \ell^m \mathfrak{u}_m\right)\right\}\right]_{\ell=0}, \quad m = 0, 1, 2, \cdots \tag{4.7}$$

Substituting (4.5) and (4.6) into (4.4), we obtain

$$\sum_{m=0}^{\infty} \mathfrak{u}_m(\mathfrak{h}, \varsigma) = \mathfrak{u}(\mathfrak{h}, 0) + \mathbb{Y}^{-1}[1 + \delta(s-1)]\left[\mathbb{Y}\left\{\mathcal{P}_1\left(\sum_{m=0}^{\infty} \mathfrak{u}_m(\mathfrak{h}, \varsigma)\right) + \sum_{m=0}^{\infty} \mathcal{A}_m(\mathfrak{u})\right\}\right]. \tag{4.8}$$

Similarly,

$$\mathfrak{u}_0(\mathfrak{h}, \varsigma) = \mathfrak{u}(\mathfrak{h}, 0), \tag{4.9}$$

$$\mathfrak{u}_1(\mathfrak{h}, \varsigma) = \mathbb{Y}^{-1}\left[(1 + \delta(s-1))\mathbb{Y}\{\mathcal{P}_1(\mathfrak{u}_0) + \mathcal{A}_0\}\right],$$

In general for $m \geq 1$, it can be written as

$$\mathfrak{u}_{m+1}(\mathfrak{h}, \varsigma) = \mathbb{Y}^{-1}\left[(1 + \delta(s-1))\mathbb{Y}\{\mathcal{P}_1(\mathfrak{u}_m) + \mathcal{A}_m\}\right].$$

5. Numerical problems

This section marks a novel application of the Homotopy perturbation transform method and the Adomian decomposition transform method utilizing the Caputo-Fabrizio operator. Our focus lies on solving two significant fractional partial differential equations: the fractional heat transfer equation and the porous media equation.

Example 5.1. We consider the nonlinear heat equation called the porous media equation [26]:

$$\frac{\partial \mathfrak{u}(\mathfrak{h}, \varsigma)}{\partial \varsigma} = \frac{\partial}{\partial \mathfrak{h}}\left(\mathfrak{u}^m(\mathfrak{h}, \varsigma)\frac{\partial \mathfrak{u}(\mathfrak{h}, \varsigma)}{\partial \mathfrak{h}}\right), \tag{5.1}$$

where m is a rational number. This equation often occurs in nonlinear problems of heat and mass transfer, combustion theory and flows in porous media [23]. For instance, it describes unsteady heat transfer in a quiescent medium with the heat diffusivity as a power-law function of a temperature [27].

Let us consider $m = 1$ in Eq (5.1).

Now we consider the fractional porous media equation as [28]:

$$\frac{\partial^\delta \mathfrak{u}(\mathfrak{h}, \varsigma)}{\partial \varsigma^\delta} = \frac{\partial}{\partial \mathfrak{h}}\left(\mathfrak{u}(\mathfrak{h}, \varsigma)\frac{\partial \mathfrak{u}(\mathfrak{h}, \varsigma)}{\partial \mathfrak{h}}\right), \quad 0 < \delta \leq 1, \tag{5.2}$$

with the initial condition

$$\mathfrak{u}(\mathfrak{h}, 0) = \mathfrak{h}.$$

Employing YT to Eq (5.2), we obtain

$$\mathbb{Y}\left(\frac{\partial^{\delta}\mathfrak{u}}{\partial\varsigma^{\delta}}\right)=\mathbb{Y}\left[\frac{\partial}{\partial\mathfrak{h}}\left(\mathfrak{u}(\mathfrak{h},\varsigma)\frac{\partial\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}}\right)\right]. \tag{5.3}$$

On simplification, we have

$$\frac{1}{1+\delta(s-1)}\{\mathbb{Y}[\mathfrak{u}(\mathfrak{h},\varsigma)]-s\mathfrak{u}(\mathfrak{h},0)\}=\mathbb{Y}\left[\frac{\partial}{\partial\mathfrak{h}}\left(\mathfrak{u}(\mathfrak{h},\varsigma)\frac{\partial\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}}\right)\right], \tag{5.4}$$

$$\mathbb{Y}[\mathfrak{u}(\mathfrak{h},\varsigma)]=s\mathfrak{u}(\mathfrak{h},0)+[1+\delta(s-1)]\mathbb{Y}\left[\frac{\partial}{\partial\mathfrak{h}}\left(\mathfrak{u}(\mathfrak{h},\varsigma)\frac{\partial\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}}\right)\right]. \tag{5.5}$$

By employing the inverse of YT, we obtain

$$\begin{aligned}\mathfrak{u}(\mathfrak{h},\varsigma)&=\mathfrak{u}(\mathfrak{h},0)+\mathbb{Y}^{-1}\left[(1+\delta(s-1))\left\{\mathbb{Y}\left[\frac{\partial}{\partial\mathfrak{h}}\left(\mathfrak{u}(\mathfrak{h},\varsigma)\frac{\partial\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}}\right)\right]\right\}\right],\\ \mathfrak{u}(\mathfrak{h},\varsigma)&=\mathfrak{h}+\mathbb{Y}^{-1}\left[(1+\delta(s-1))\left\{\mathbb{Y}\left[\frac{\partial}{\partial\mathfrak{h}}\left(\mathfrak{u}(\mathfrak{h},\varsigma)\frac{\partial\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}}\right)\right]\right\}\right].\end{aligned} \tag{5.6}$$

In terms of HPM, we have

$$\sum_{k=0}^{\infty}\epsilon^{k}\mathfrak{u}_{k}(\mathfrak{h},\varsigma)=\mathfrak{h}+\mathbb{Y}^{-1}\left[(1+\delta(s-1))\mathbb{Y}\left[\frac{\partial}{\partial\mathfrak{h}}\left(\sum_{k=0}^{\infty}\epsilon^{k}H_{k}(\mathfrak{u})\right)\right]\right]. \tag{5.7}$$

Here, the nonlinears term read

$$\begin{aligned}H_0(\mathfrak{u})&=\mathfrak{u}_0\mathfrak{u}_{0\mathfrak{h}},\\ H_1(\mathfrak{u})&=\mathfrak{u}_0\mathfrak{u}_{1\mathfrak{h}}+\mathfrak{u}_1\mathfrak{u}_{0\mathfrak{h}},\\ H_2(\mathfrak{u})&=\mathfrak{u}_0\mathfrak{u}_{2\mathfrak{h}}+\mathfrak{u}_1\mathfrak{u}_{1\mathfrak{h}}+\mathfrak{u}_2\mathfrak{u}_{0\mathfrak{h}},\\ &\vdots\end{aligned}$$

Now by equating the coefficient of ϵ, we have

$$\begin{aligned}\epsilon^0&:\mathfrak{u}_0(\mathfrak{h},\varsigma)=\mathfrak{h},\\ \epsilon^1&:\mathfrak{u}_1(\mathfrak{h},\varsigma)=\mathbb{Y}^{-1}\left[(1+\delta(s-1))\mathbb{Y}\left[\frac{\partial}{\partial\mathfrak{h}}\left(\sum_{k=0}^{\infty}\epsilon^{k}H_{0}(\mathfrak{u})\right)\right]\right]=\left(1+\delta\varsigma-\delta\right),\\ \epsilon^2&:\mathfrak{u}_2(\mathfrak{h},\varsigma)=\mathbb{Y}^{-1}\left[(1+\delta(s-1))\mathbb{Y}\left[\frac{\partial}{\partial\mathfrak{h}}\left(\sum_{k=0}^{\infty}\epsilon^{k}H_{1}(\mathfrak{u})\right)\right]\right]=0,\\ &\vdots\end{aligned}$$

Finally, we approximate the analytical solution in series form as

$$\begin{aligned}\mathfrak{u}(\mathfrak{h},\varsigma)&=\mathfrak{u}_0(\mathfrak{h},\varsigma)+\mathfrak{u}_1(\mathfrak{h},\varsigma)+\mathfrak{u}_2(\mathfrak{h},\varsigma)+\cdots.\\ \mathfrak{u}(\mathfrak{h},\varsigma)&=\mathfrak{h}+\left(1+\delta\varsigma-\delta\right)+\cdots.\end{aligned}$$

If $\delta=1$, then Eq (5.2) can be reduced to the formula:

$$\mathfrak{u}(\mathfrak{h},\varsigma)=\mathfrak{h}+\varsigma.$$

Now apply Yang transform decomposition method

Using the Yang transform by Eq (5.2), we get

$$\mathbb{Y}\left\{\frac{\partial^{\delta}\mathfrak{u}}{\partial\varsigma^{\delta}}\right\} = \mathbb{Y}\left[\frac{\partial}{\partial\mathfrak{h}}\left(\mathfrak{u}(\mathfrak{h},\varsigma)\frac{\partial\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}}\right)\right]. \tag{5.8}$$

On simplification, we have

$$\frac{1}{1+\delta(s-1)}\{\mathbb{Y}[\mathfrak{u}(\mathfrak{h},\varsigma)] - s\mathfrak{u}(\mathfrak{h},0)\} = \mathbb{Y}\left[\frac{\partial}{\partial\mathfrak{h}}\left(\mathfrak{u}(\mathfrak{h},\varsigma)\frac{\partial\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}}\right)\right], \tag{5.9}$$

$$\mathbb{Y}[\mathfrak{u}(\mathfrak{h},\varsigma)] = s\mathfrak{u}(\mathfrak{h},0) + (1+\delta(s-1))\mathbb{Y}\left[\frac{\partial}{\partial\mathfrak{h}}\left(\mathfrak{u}(\mathfrak{h},\varsigma)\frac{\partial\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}}\right)\right]. \tag{5.10}$$

By using the inverse of Yang transform, we obtain

$$\begin{aligned}\mathfrak{u}(\mathfrak{h},\varsigma) &= \mathfrak{u}(\mathfrak{h},0) + \mathbb{Y}^{-1}\left[(1+\delta(s-1))\left\{\mathbb{Y}\left[\frac{\partial}{\partial\mathfrak{h}}\left(\mathfrak{u}(\mathfrak{h},\varsigma)\frac{\partial\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}}\right)\right]\right\}\right],\\ \mathfrak{u}(\mathfrak{h},\varsigma) &= \mathfrak{h} + \mathbb{Y}^{-1}\left[(1+\delta(s-1))\left\{\mathbb{Y}\left[\frac{\partial}{\partial\mathfrak{h}}\left(\mathfrak{u}(\mathfrak{h},\varsigma)\frac{\partial\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}}\right)\right]\right\}\right].\end{aligned} \tag{5.11}$$

The series form solution of $\mathfrak{u}(\mathfrak{h},\varsigma)$ is as:

$$\mathfrak{u}(\mathfrak{h},\varsigma) = \sum_{m=0}^{\infty}\mathfrak{u}_m(\mathfrak{h},\varsigma), \tag{5.12}$$

with $\mathfrak{u}(\mathfrak{h},\varsigma)\frac{\partial\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}} = \sum_{m=0}^{\infty}\mathcal{A}_m$, shows the nonlinear term in terms of Adomian polynomial as, and

$$\begin{aligned}\sum_{m=0}^{\infty}\mathfrak{u}_m(\mathfrak{h},\varsigma) &= \mathfrak{u}(\mathfrak{h},0) + \mathbb{Y}^{-1}\left[(1+\delta(s-1))\left\{\mathbb{Y}\left[\frac{\partial}{\partial\mathfrak{h}}\left(\sum_{m=0}^{\infty}\mathcal{A}_m\right)\right]\right\}\right],\\ \sum_{m=0}^{\infty}\mathfrak{u}_m(\mathfrak{h},\varsigma) &= \mathfrak{h} + \mathbb{Y}^{-1}\left[(1+\delta(s-1))\left\{\mathbb{Y}\left[\frac{\partial}{\partial\mathfrak{h}}\left(\sum_{m=0}^{\infty}\mathcal{A}_m\right)\right]\right\}\right].\end{aligned} \tag{5.13}$$

Here, the nonlinear terms read,

$$\begin{aligned}A_0 &= \mathfrak{u}_0\mathfrak{u}_{0\mathfrak{h}},\\ A_1 &= \mathfrak{u}_0\mathfrak{u}_{1\mathfrak{h}} + \mathfrak{u}_1\mathfrak{u}_{0\mathfrak{h}},\\ A_2 &= \mathfrak{u}_0\mathfrak{u}_{2\mathfrak{h}} + \mathfrak{u}_1\mathfrak{u}_{1\mathfrak{h}} + \mathfrak{u}_2\mathfrak{u}_{0\mathfrak{h}},\\ &\vdots\end{aligned}$$

Similarly,

$$\mathfrak{u}_0(\mathfrak{h},\varsigma) = \mathfrak{h}.$$

On $m = 0$

$$\mathfrak{u}_1(\mathfrak{h},\varsigma) = \Big(1 + \delta\varsigma - \delta\Big).$$

On $m = 1$

$$\mathfrak{u}_2(\mathfrak{h}, \varsigma) = 0.$$

Finally, we approximate the analytical solution in series form as

$$\mathfrak{u}(\mathfrak{h}, \varsigma) = \sum_{m=0}^{\infty} \mathfrak{u}_m(\mathfrak{h}, \varsigma) = \mathfrak{u}_0(\mathfrak{h}, \varsigma) + \mathfrak{u}_1(\mathfrak{h}, \varsigma) + \mathfrak{u}_2(\mathfrak{h}, \varsigma) + \cdots .$$

$$\mathfrak{u}(\mathfrak{h}, \varsigma) = \mathfrak{h} + \Big(1 + \delta\varsigma - \delta\Big) + \cdots .$$

The exact result at $\delta = 1$ is

$$\mathfrak{u}(\mathfrak{h}, \varsigma) = \mathfrak{h} + \varsigma. \tag{5.14}$$

Example 5.2. Assuming that fractional heat transfer equation [28]

$$\frac{\partial^{\delta}\mathfrak{u}(\mathfrak{h}, \varsigma)}{\partial\varsigma^{\delta}} = \frac{\partial^{2}\mathfrak{u}(\mathfrak{h}, \varsigma)}{\partial\mathfrak{h}^{2}} - 2\mathfrak{u}^{3}(\mathfrak{h}, \varsigma), \quad 0 < \delta \leq 1, \tag{5.15}$$

with the initial condition

$$\mathfrak{u}(\mathfrak{h}, 0) = \frac{1 + 2\mathfrak{h}}{\mathfrak{h}^2 + \mathfrak{h} + 1}.$$

Employing the YT (5.15), we get

$$\mathbb{Y}\left(\frac{\partial^{\delta}\mathfrak{u}}{\partial\varsigma^{\delta}}\right) = \mathbb{Y}\left[\frac{\partial^{2}\mathfrak{u}(\mathfrak{h}, \varsigma)}{\partial\mathfrak{h}^{2}} - 2\mathfrak{u}^{3}(\mathfrak{h}, \varsigma)\right]. \tag{5.16}$$

On simplification, we have

$$\frac{1}{1 + \delta(s - 1)}\{\mathbb{Y}[\mathfrak{u}(\mathfrak{h}, \varsigma)] - s\mathfrak{u}(\mathfrak{h}, 0)\} = \mathbb{Y}\left[\frac{\partial^{2}\mathfrak{u}(\mathfrak{h}, \varsigma)}{\partial\mathfrak{h}^{2}} - 2\mathfrak{u}^{3}(\mathfrak{h}, \varsigma)\right], \tag{5.17}$$

$$\mathbb{Y}[\mathfrak{u}(\mathfrak{h}, \varsigma)] = s\mathfrak{u}(\mathfrak{h}, 0) + [1 + \delta(s - 1)]\mathbb{Y}\left[\frac{\partial^{2}\mathfrak{u}(\mathfrak{h}, \varsigma)}{\partial\mathfrak{h}^{2}} - 2\mathfrak{u}^{3}(\mathfrak{h}, \varsigma)\right]. \tag{5.18}$$

By employing the inverse of YT, we obtain

$$\begin{aligned} \mathfrak{u}(\mathfrak{h}, \varsigma) &= \mathfrak{u}(\mathfrak{h}, 0) + \mathbb{Y}^{-1}\left[(1 + \delta(s - 1))\left\{\mathbb{Y}\left[\frac{\partial^{2}\mathfrak{u}(\mathfrak{h}, \varsigma)}{\partial\mathfrak{h}^{2}} - 2\mathfrak{u}^{3}(\mathfrak{h}, \varsigma)\right]\right\}\right], \\ \mathfrak{u}(\mathfrak{h}, \varsigma) &= \frac{1 + 2\mathfrak{h}}{\mathfrak{h}^2 + \mathfrak{h} + 1} + \mathbb{Y}^{-1}\left[(1 + \delta(s - 1))\left\{\mathbb{Y}\left[\frac{\partial^{2}\mathfrak{u}(\mathfrak{h}, \varsigma)}{\partial\mathfrak{h}^{2}} - 2\mathfrak{u}^{3}(\mathfrak{h}, \varsigma)\right]\right\}\right]. \end{aligned} \tag{5.19}$$

In terms of HPM, we have

$$\sum_{k=0}^{\infty} \epsilon^{k}\mathfrak{u}_k(\mathfrak{h}, \varsigma) = \frac{1 + 2\mathfrak{h}}{\mathfrak{h}^2 + \mathfrak{h} + 1} + \left(\mathbb{Y}^{-1}\left[(1 + \delta(s - 1))\mathbb{Y}\left[\frac{\partial^{2}\mathfrak{u}(\mathfrak{h}, \varsigma)}{\partial\mathfrak{h}^{2}} - \sum_{k=0}^{\infty} \epsilon^{k} H_k(\mathfrak{u})\right]\right]\right). \tag{5.20}$$

Here, the nonlinear terms read

$$H_0(\mathfrak{u}) = 2\mathfrak{u}_0^3(\mathfrak{h},\varsigma),$$
$$H_1(\mathfrak{u}) = 6\mathfrak{u}_0^2(\mathfrak{h},\varsigma)\mathfrak{u}_1(\mathfrak{h},\varsigma),$$
$$H_2(\mathfrak{u}) = 6\mathfrak{u}_0(\mathfrak{h},\varsigma)\mathfrak{u}_1^2(\mathfrak{h},\varsigma) + 6\mathfrak{u}_0^2(\mathfrak{h},\varsigma)\mathfrak{u}_2(\mathfrak{h},\varsigma),$$
$$\vdots$$

Now by equating the coefficient of ϵ, we have

$$\epsilon^0 : \mathfrak{u}_0(\mathfrak{h},\varsigma) = \frac{1+2\mathfrak{h}}{\mathfrak{h}^2+\mathfrak{h}+1},$$
$$\epsilon^1 : \mathfrak{u}_1(\mathfrak{h},\varsigma) = \mathbb{Y}^{-1}\left((1+\delta(s-1))\mathbb{Y}\left[\frac{\partial^2\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}^2} - \sum_{k=0}^{\infty}\epsilon^k H_0(\mathfrak{u})\right]\right) = \frac{-6(1+2\mathfrak{h})}{(\mathfrak{h}^2+\mathfrak{h}+1)^2}\Big(1+\delta\varsigma-\delta\Big),$$
$$\epsilon^2 : \mathfrak{u}_2(\mathfrak{h},\varsigma) = \mathbb{Y}^{-1}\left((1+\delta(s-1))\mathbb{Y}\left[\frac{\partial^2\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}^2} - \sum_{k=0}^{\infty}\epsilon^k H_1(\mathfrak{u})\right]\right) = \frac{72(1+2\mathfrak{h})}{(\mathfrak{h}^2+\mathfrak{h}+1)^3}\left(\delta-2\delta+\delta^2\frac{\varsigma^2}{2}-2\delta(\delta-1)\varsigma+1\right),$$
$$\epsilon^3 : \mathfrak{u}_3(\mathfrak{h},\varsigma) = \mathbb{Y}^{-1}\left((1+\delta(s-1))\mathbb{Y}\left[\frac{\partial^2\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}^2} - \sum_{k=0}^{\infty}\epsilon^k H_2(\mathfrak{u})\right]\right) = \left(-\frac{1296(1+2\mathfrak{h})}{(\mathfrak{h}^2+\mathfrak{h}+1)^4}+\frac{432(1+2\mathfrak{h})^3}{(\mathfrak{h}^2+\mathfrak{h}+1)^5}-\right.$$
$$\left.\frac{216(1+2\mathfrak{h})^3}{(\mathfrak{h}^2+\mathfrak{h}+1)^5}\right)\Big(3\delta(-2\delta+1+\delta^2)\varsigma+\frac{\delta^3\varsigma^3}{6}-\frac{3\delta^2(\delta-1)\varsigma^2}{2}+3\delta^2-3\delta+1-\delta^3\Big),$$
$$\vdots$$

Finally, we approximate the analytical solution in series form as

$$\mathfrak{u}(\mathfrak{h},\varsigma) = \mathfrak{u}_0(\mathfrak{h},\varsigma)+\mathfrak{u}_1(\mathfrak{h},\varsigma)+\mathfrak{u}_2(\mathfrak{h},\varsigma)+\mathfrak{u}_3(\mathfrak{h},\varsigma)+\cdots.$$
$$\mathfrak{u}(\mathfrak{h},\varsigma) = \frac{1+2\mathfrak{h}}{\mathfrak{h}^2+\mathfrak{h}+1}+\frac{-6(1+2\mathfrak{h})}{(\mathfrak{h}^2+\mathfrak{h}+1)^2}\Big(1+\delta\varsigma-\delta\Big)+\frac{72(1+2\mathfrak{h})}{(\mathfrak{h}^2+\mathfrak{h}+1)^3}\left(\delta-2\delta+\delta^2\frac{\varsigma^2}{2}-2\delta(\delta-1)\varsigma+1\right)$$
$$+\left(-\frac{1296(1+2\mathfrak{h})}{(\mathfrak{h}^2+\mathfrak{h}+1)^4}+\frac{432(1+2\mathfrak{h})^3}{(\mathfrak{h}^2+\mathfrak{h}+1)^5}-\frac{216(1+2\mathfrak{h})^3}{(\mathfrak{h}^2+\mathfrak{h}+1)^5}\right)\Big(3\delta(-2\delta+1+\delta^2)\varsigma+\frac{\delta^3\varsigma^3}{6}$$
$$\left.-\frac{3\delta^2(\delta-1)\varsigma^2}{2}+3\delta^2-3\delta+1-\delta^3\right)+\cdots.$$

Now we apply Yang transform decomposition method

Employing the YT by Eq (5.15), we obtain

$$\mathbb{Y}\left\{\frac{\partial^\delta\mathfrak{u}}{\partial\varsigma^\delta}\right\} = \mathbb{Y}\left[\frac{\partial^2\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}^2}-2\mathfrak{u}^3(\mathfrak{h},\varsigma)\right]. \tag{5.21}$$

On simplification, we have

$$\frac{1}{1+\delta(s-1)}\{\mathbb{Y}[\mathfrak{u}(\mathfrak{h},\varsigma)]-s\mathfrak{u}(\mathfrak{h},0)\} = \mathbb{Y}\left[\frac{\partial^2\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}^2}-2\mathfrak{u}^3(\mathfrak{h},\varsigma)\right], \tag{5.22}$$

$$\mathbb{Y}[\mathfrak{u}(\mathfrak{h},\varsigma)] = s\mathfrak{u}(\mathfrak{h},0)+(1+\delta(s-1))\mathbb{Y}\left[\frac{\partial^2\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}^2}-2\mathfrak{u}^3(\mathfrak{h},\varsigma)\right]. \tag{5.23}$$

By employing the inverse of YT, we obtain

$$\begin{aligned}\mathfrak{u}(\mathfrak{h},\varsigma) &= \mathfrak{u}(\mathfrak{h},0) + \mathbb{Y}^{-1}\left[(1+\delta(s-1))\left\{\mathbb{Y}\left[\frac{\partial^2\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}^2} - 2\mathfrak{u}^3(\mathfrak{h},\varsigma)\right]\right\}\right],\\ \mathfrak{u}(\mathfrak{h},\varsigma) &= \frac{1+2\mathfrak{h}}{\mathfrak{h}^2+\mathfrak{h}+1} + \mathbb{Y}^{-1}\left[(1+\delta(s-1))\left\{\mathbb{Y}\left[\frac{\partial^2\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}^2} - 2\mathfrak{u}^3(\mathfrak{h},\varsigma)\right]\right\}\right].\end{aligned} \tag{5.24}$$

The series form solution of $\mathfrak{u}(\mathfrak{h},\varsigma)$ is as:

$$\mathfrak{u}(\mathfrak{h},\varsigma) = \sum_{m=0}^{\infty}\mathfrak{u}_m(\mathfrak{h},\varsigma). \tag{5.25}$$

with $2\mathfrak{u}^3(\mathfrak{h},\varsigma) = \sum_{m=0}^{\infty}\mathcal{A}_m$, shows the nonlinear term in terms of Adomian polynomial as, and

$$\begin{aligned}\sum_{m=0}^{\infty}\mathfrak{u}_m(\mathfrak{h},\varsigma) &= \mathfrak{u}(\mathfrak{h},0) - \mathbb{Y}^{-1}\left[(1+\delta(s-1))\left\{\mathbb{Y}\left[\frac{\partial^2\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}^2} - \sum_{m=0}^{\infty}\mathcal{A}_m\right]\right\}\right],\\ \sum_{m=0}^{\infty}\mathfrak{u}_m(\mathfrak{h},\varsigma) &= \frac{1+2\mathfrak{h}}{\mathfrak{h}^2+\mathfrak{h}+1} - \mathbb{Y}^{-1}\left[(1+\delta(s-1))\left\{\mathbb{Y}\left[\frac{\partial^2\mathfrak{u}(\mathfrak{h},\varsigma)}{\partial\mathfrak{h}^2} - \sum_{m=0}^{\infty}\mathcal{A}_m\right]\right\}\right].\end{aligned} \tag{5.26}$$

Here, the nonlinear terms read,

$$\begin{aligned}A_0 &= 2\mathfrak{u}_0^3(\mathfrak{h},\varsigma),\\ A_1 &= 6\mathfrak{u}_0^2(\mathfrak{h},\varsigma)\mathfrak{u}_1(\mathfrak{h},\varsigma),\\ A_2 &= 6\mathfrak{u}_0(\mathfrak{h},\varsigma)\mathfrak{u}_1^2(\mathfrak{h},\varsigma) + 6\mathfrak{u}_0^2(\mathfrak{h},\varsigma)\mathfrak{u}_2(\mathfrak{h},\varsigma),\\ &\vdots\end{aligned}$$

Similarly,

$$\mathfrak{u}_0(\mathfrak{h},\varsigma) = \frac{1+2\mathfrak{h}}{\mathfrak{h}^2+\mathfrak{h}+1}.$$

For $m = 0$

$$\mathfrak{u}_1(\mathfrak{h},\varsigma) = \frac{-6(1+2\mathfrak{h})}{(\mathfrak{h}^2+\mathfrak{h}+1)^2}\Big(1+\delta\varsigma-\delta\Big).$$

For $m = 1$

$$\mathfrak{u}_2(\mathfrak{h},\varsigma) = \frac{72(1+2\mathfrak{h})}{(\mathfrak{h}^2+\mathfrak{h}+1)^3}\left(\delta - 2\delta + \delta^2\frac{\varsigma^2}{2} - 2\delta(\delta-1)\varsigma + 1\right).$$

For $m = 2$

$$\mathfrak{u}_3(\mathfrak{h},\varsigma) = \left(-\frac{1296(1+2\mathfrak{h})}{(\mathfrak{h}^2+\mathfrak{h}+1)^4} + \frac{432(1+2\mathfrak{h})^3}{(\mathfrak{h}^2+\mathfrak{h}+1)^5} - \frac{216(1+2\mathfrak{h})^3}{(\mathfrak{h}^2+\mathfrak{h}+1)^5}\right)\Big(3\delta(-2\delta+1+\delta^2)\varsigma + \frac{\delta^3\varsigma^3}{6} - \frac{3\delta^2(\delta-1)\varsigma^2}{2} + 3\delta^2 - 3\delta + 1 - \delta^3\Big).$$

Finally, we approximate the analytical solution in series form as

$$\mathfrak{u}(\mathfrak{h},\varsigma) = \sum_{m=0}^{\infty}\mathfrak{u}_m(\mathfrak{h},\varsigma) = \mathfrak{u}_0(\mathfrak{h},\varsigma) + \mathfrak{u}_1(\mathfrak{h},\varsigma) + \mathfrak{u}_2(\mathfrak{h},\varsigma) + \mathfrak{u}_3(\mathfrak{h},\varsigma) + \cdots.$$

$$\mathfrak{u}(\mathfrak{h},\varsigma)=\frac{1+2\mathfrak{h}}{\mathfrak{h}^2+\mathfrak{h}+1}+\frac{-6(1+2\mathfrak{h})}{(\mathfrak{h}^2+\mathfrak{h}+1)^2}\Big(1+\delta\varsigma-\delta\Big)+\frac{72(1+2\mathfrak{h})}{(\mathfrak{h}^2+\mathfrak{h}+1)^3}\left(\delta-2\delta+\delta^2\frac{\varsigma^2}{2}-2\delta(\delta-1)\varsigma+1\right)$$
$$+\left(-\frac{1296(1+2\mathfrak{h})}{(\mathfrak{h}^2+\mathfrak{h}+1)^4}+\frac{432(1+2\mathfrak{h})^3}{(\mathfrak{h}^2+\mathfrak{h}+1)^5}-\frac{216(1+2\mathfrak{h})^3}{(\mathfrak{h}^2+\mathfrak{h}+1)^5}\right)\Big(3\delta(-2\delta+1+\delta^2)\varsigma+\frac{\delta^3\varsigma^3}{6}-\frac{3\delta^2(\delta-1)\varsigma^2}{2}$$
$$+3\delta^2-3\delta+1-\delta^3\Big)+\cdots.$$

The result obtained above is the same as the result obtained. If $\delta = 1$, the above can be rearranged as:

$$\mathfrak{u}(\mathfrak{h},\varsigma)=\frac{1+2\mathfrak{h}}{\mathfrak{h}^2+\mathfrak{h}+1}-\frac{6(1+2\mathfrak{h})}{(\mathfrak{h}^2+\mathfrak{h}+1)^2}\varsigma+\frac{36(1+2\mathfrak{h})}{(\mathfrak{h}^2+\mathfrak{h}+1)^3}\varsigma^2-\frac{216(1+2\mathfrak{h})}{(\mathfrak{h}^2+\mathfrak{h}+1)^4}\varsigma^3+\cdots. \quad (5.27)$$

6. Results and discussion

Figure 1 presents a comprehensive comparison of solutions for the fractional porous media equation in Example 5.1. Panel (a) displays the exact solution, showcasing its precise representation. In panel (b), the Yang transform decomposition method (YTDM) solution is depicted, demonstrating its efficacy in approximating the true solution. The Homotopy Perturbation Transform Method (HPTM) solution is showcased in the same panel, marked as (c), highlighting its accuracy and alignment with the exact solution. Notably, panel (c) explores the influence of different fractional orders δ on the solution $\mathfrak{u}(\mathfrak{h},\varsigma)$, providing insights into how changing δ impacts the overall behavior of the solution. Figure 2 extends the analysis to the fractional heat transfer equation of Example 5.2. In panel (a), the YDTM solution is illustrated, emphasizing its ability to approximate the solution profile. Subsequently, panel (b) showcases the HPTM solution, revealing its remarkable agreement with the true solution. Similar to the previous figure, panel (c) presents the impact of varying fractional orders δ on the solution $\mathfrak{u}(\mathfrak{h},\varsigma)$, offering a comprehensive exploration of the parameter's influence on the solution's characteristics. These graphical representations provide valuable insights into the performance and accuracy of the Yang transform method and homotopy perturbation transform method in approximating solutions for fractional porous media and heat transfer equations. The varying effects of fractional order δ on the solutions underscore the nuanced interplay between mathematical techniques and the underlying physical phenomena.

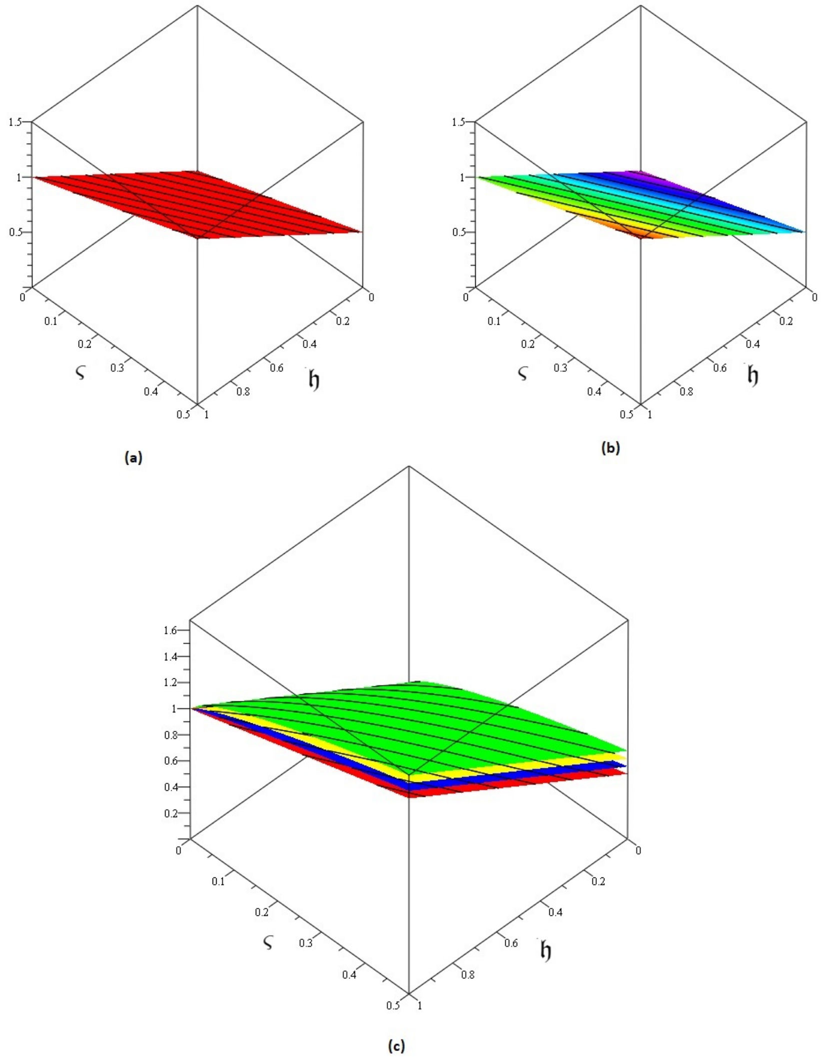

Figure 1. The (a) show that the exact (b) represent YTDM and HPTM (c) different fractional order of δ for $\mathfrak{u}(\mathfrak{h}, \varsigma)$.

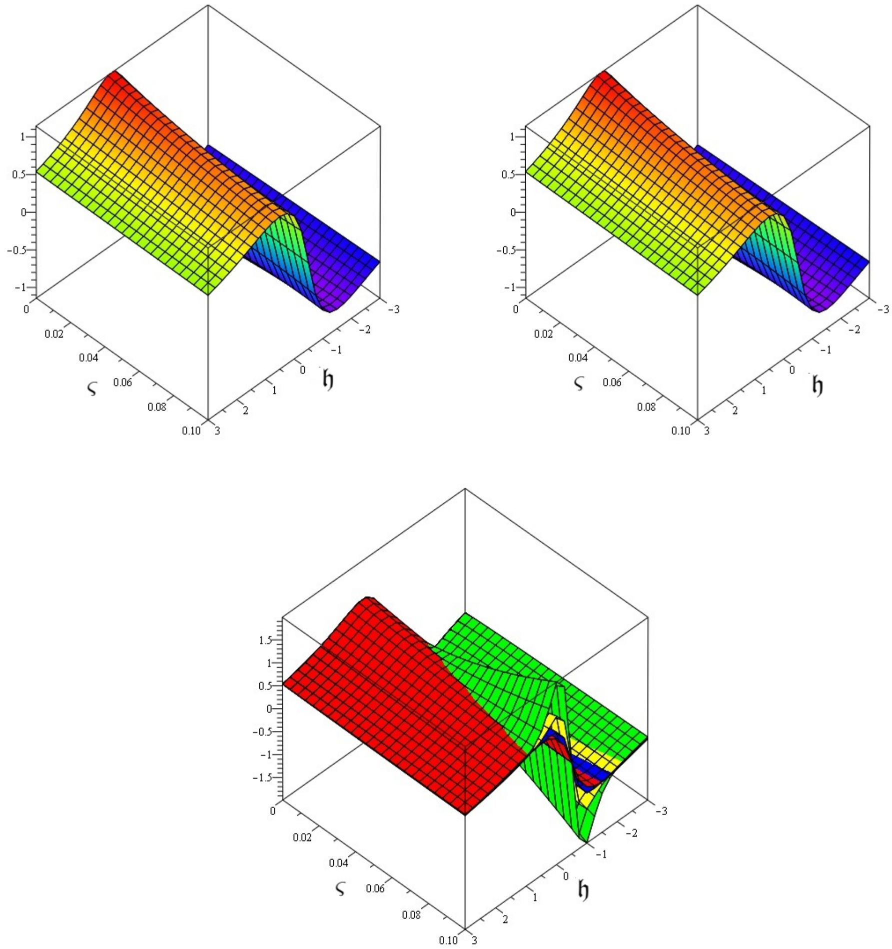

Figure 2. The (a) show that YTDM, (b) represents HPTM and (c) different fractional order of δ for $\mathfrak{u}(\mathfrak{h}, \varsigma)$.

7. Conclusions

In conclusion, the homotopy perturbation transform method (HPTM) and the Yang transform decomposition method (YTDM) are two powerful numerical techniques used to solve fractional partial differential equations (FPDEs) arising in various fields of science and engineering. The application of these methods to fractional porous media and fractional heat transfer equations has shown promising results, with both techniques being able to accurately approximate the solutions of these complex equations. The HPTM and YTDM offer several advantages over traditional numerical methods, including their ability to handle nonlinear and non-homogeneous equations, their simplicity in implementation, and their efficiency in computation. These methods have also been shown to have high convergence rates, which allows for faster and more accurate solutions to be obtained. Overall, the HPTM and YTDM are valuable tools for solving fractional porous media and fractional heat transfer equations, and their continued development and application are expected to lead to significant advancements in the field of fractional calculus and its applications.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interest.

References

1. D. Kumar, J. Singh, K. Tanwar, D. Baleanu, A new fractional exothermic reactions model having constant heat source in porous media with power, exponential and Mittag-Leffler laws, *Int. J. Heat Mass Tran.*, **138** (2019), 1222–1227. https://doi.org/10.1016/j.ijheatmasstransfer.2019.04.094
2. L. M. Yan, Modified homotopy perturbation method coupled with Laplace transform for fractional heat transfer and porous media equations, *Therm. Sci.*, **17** (2013), 1409–1414. https://doi.org/10.2298/TSCI1305409Y
3. Q. Liu, H. Peng, Z. Wang, Convergence to nonlinear diffusion waves for a hyperbolic-parabolic chemotaxis system modelling vasculogenesis, *J. Differ. Equations*, **314** (2022), 251–286. https://doi.org/10.1016/j.jde.2022.01.021
4. H. Jin, Z. Wang, L. Wu, Global dynamics of a three-species spatial food chain model, *J. Differ. Equations*, **333** (2022), 144–183. https://doi.org/10.1016/j.jde.2022.06.007
5. H. Y. Jin, Z. A. Wang, Global stabilization of the full attraction-repulsion Keller-Segel system, *Discrete Cont. Dyn-A*, **40** (2020), 3509–3527. https://doi.org/10.3934/dcds.2020027
6. M. Pan, L. Zheng, F. Liu, C. Liu, X. Chen, A spatial-fractional thermal transport model for nanofluid in porous media, *Appl. Math. Model.*, **53** (2018), 622–634. https://doi.org/10.1016/j.apm.2017.08.026
7. H. Yasmin, N. H. Aljahdaly, A. M. Saeed, R. Shah, Investigating families of soliton solutions for the complex structured coupled fractional Biswas-Arshed model in birefringent fibers using a novel analytical technique, *Fractal Fract.*, **7** (2023), 491. https://doi.org/10.3390/fractalfract7070491
8. F. Liu, Z. Sun, H. Bian, M. Ding, X. Meng, Identification and classification of the flow pattern of hydrogen-air-steam mixture gas under steam condensation, *Int. J. Therm. Sci.*, **183** (2023), 107854. https://doi.org/10.1016/j.ijthermalsci.2022.107854

9. B. Bai, D. Rao, T. Chang, Z. Guo, A nonlinear attachment-detachment model with adsorption hysteresis for suspension-colloidal transport in porous media, *J. Hydrol.*, **578** (2019), 124080. https://doi.org/10.1016/j.jhydrol.2019.124080

10. M. Pan, L. Zheng, F. Liu, X. Zhang, Modeling heat transport in nanofluids with stagnation point flow using fractional calculus, *Appl. Math. Model.*, **40** (2016), 8974–8984. https://doi.org/10.1016/j.apm.2016.05.044

11. M. K. Alaoui, K. Nonlaopon, A. M. Zidan, A. Khan, R. Shah, Analytical investigation of fractional-order Cahn-Hilliard and gardner equations using two novel techniques, *Mathematics*, **10** (2022), 1643. https://doi.org/10.3390/math10101643

12. P. Sunthrayuth, N. H. Aljahdaly, A. Ali, R. Shah, I. Mahariq, A. M. J. Tchalla, ψ-Haar wavelet operational matrix method for fractional relaxation-oscillation equations containing ψ-Caputo fractional derivative, *J. Funct. Space.*, **2021** (2021), 7117064. https://doi.org/10.1155/2021/7117064

13. C. Guo, J. Hu, J. Hao, S. Celikovsky, X. Hu, Fixed-time safe tracking control of uncertain high-order nonlinear pure-feedback systems via unified transformation functions, *Kybernetika*, **59** (2023), 342–364. https://doi.org/10.14736/kyb-2023-3-0342

14. C. Guo, J. Hu, Y. Wu, S. Celikovsky, Non-singular fixed-time tracking control of uncertain nonlinear pure-feedback systems with practical state constraints, *IEEE T. Circuits-I*, **70** (2023), 3746–3758. https://doi.org/10.1109/TCSI.2023.3291700

15. Q. Meng, Q. Ma, Y. Shi, Adaptive fixed-time stabilization for a class of uncertain nonlinear systems, *IEEE T. Automat. Contr.*, 2023, 1–8. https://doi.org/10.1109/TAC.2023.3244151

16. A. S. Alshehry, H. Yasmin, F. Ghani, R. Shah, K. Nonlaopon, Comparative analysis of Advection-Dispersion equations with Atangana-Baleanu fractional derivative, *Symmetry*, **15** (2023), 819. https://doi.org/10.3390/sym15040819

17. M. A. Shah, H. Yasmin, F. Ghani, S. Abdullah, I. Khan, R. Shah, Fuzzy fractional gardner and Cahn-Hilliard equations with the Atangana-Baleanu operator, *Front. Phys.*, **11** (2023), 1169548. https://doi.org/10.3389/fphy.2023.1169548

18. H. Yasmin, A. S. Alshehry, A. Khan, R. Shah, K. Nonlaopon, Numerical analysis of the fractional-order belousov zhabotinsky system, *Symmetry*, **15** (2023), 834. https://doi.org/10.3390/sym15040834

19. N. Iqbal, A. M. Albalahi, M. S. Abdo, W. W. Mohammed, Analytical analysis of fractional-order Newell-Whitehead-Segel equation: A modified homotopy perturbation transform method, *J. Funct. Space.*, **2022** (2022), 3298472. https://doi.org/10.1155/2022/3298472

20. H. K. Jassim, M. G. Mohammed, Natural homotopy perturbation method for solving nonlinear fractional gas dynamics equations, *Int. J. Nonlinear Appl.*, **12** (2021), 812–820. https://doi.org/10.22075/IJNAA.2021.4936

21. Y. J. Yang, S. Q. Wang, Fractional residual method coupled with Adomian decomposition method for solving local fractional differential equations, *Therm. Sci.*, **26** (2022), 2667–2675. https://doi.org/10.2298/TSCI2203667Y

22. M. Alesemi, N. Iqbal, A. A. Hamoud, The analysis of fractional-order proportional delay physical models via a novel transform, *Complexity*, **2022** (2022), 2431533. https://doi.org/10.1155/2022/2431533
23. F. Chen, Q. Q. Liu, Adomian decomposition method combined with padé approximation and laplace transform for solving a model of HIV infection of $CD4^{+}$ T cells, *Discrete Dyn. Nat. Soc.*, **2015** (2015), 584787. https://doi.org/10.1155/2015/584787
24. D. Chen, Q. Wang, Y. Li, H. Zhou, Y. Fan, A general linear free energy relationship for predicting partition coefficients of neutral organic compounds, *Chemosphere*, **247** (2020), 125869. https://doi.org/10.1016/j.chemosphere.2020.125869
25. S. Ahmad, A. Ullah, A. Akgül, M. De la Sen, A novel homotopy perturbation method with applications to nonlinear fractional order KdV and burger equation with exponential-decay kernel, *J. Funct. Space.*, **2021** (2021), 8770488. https://doi.org/10.1155/2021/8770488
26. S. Pamuk, Solution of the porous media equation by Adomian's decomposition method, *Phys. Lett. A*, **344** (2005), 184–188. https://doi.org/10.1016/j.physleta.2005.06.068
27. A. D. Polyanin, V. F. Zaitsev, *Handbook of nonlinear partial differential equations: Exact solutions, methods, and problems*, New York: CRC, 2003. https://doi.org/10.1201/9780203489659
28. D. D. Ganji, A. Sadighi, Application of homotopy-perturbation and variational iteration methods to nonlinear heat transfer and porous media equations, *J. Comput. Appl. Math.*, **207** (2007), 24–34. https://doi.org/10.1016/j.cam.2006.07.030