AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(11): 26650–26664.
DOI: 10.3934/math.20231363
Received: 23 June 2023
Revised: 23 August 2023
Accepted: 24 August 2023
Published: 19 September 2023

Research article

The pullback attractor for the 2D g-Navier-Stokes equation with nonlinear damping and time delay

Xiaoxia Wang* **and Jinping Jiang**

College of Mathematics and Computer Science, Yan'an University, Yan'an 716000, China

* **Correspondence:** Email: yd-wxx@163.com.

Abstract: In this article, the global well-posedness of weak solutions for 2D non-autonomous g-Navier-Stokes equations on some bounded domains were investigated by the Faedo-Galerkin method. Then the existence of pullback attractors for 2D g-Navier-Stokes equations with nonlinear damping and time delay was obtained using the method of pullback condition (PC).

Keywords: pullback attractor; g-Navier-Stokes equation; pullback condition; nonliear damping; time delay
Mathematics Subject Classification: 35B41, 37B55, 76D05

1. Introduction

It is well-known that the Navier-Stokes equations are important in fluid mechanics and turbulence. In the last decades, the research of the asymptotic properties of the solution for Navier-Stokes equations has attracted the attention of scholars [1–5]. Especially in the past years, the Navier-Stokes equations with nonlinear damping have been studied [6–9], where the damping comes from the resistance to the motion of the flow. It describes various physical situations such as porous media flow, drag or friction effects and some dissipative mechanisms. In [6], Cai and Jiu considered the following Navier-Stokes equations with damping:

$$\begin{cases} u_t - \mu\Delta u + (u\cdot\nabla)u + \alpha|u|^{\beta-1}u + \nabla p = 0, \ (x,t)\in \mathrm{R}^3\times(0,T), \\ divu = 0, \ (x,t)\in \mathrm{R}^3\times[0,T), \\ u|_{t=0} = u_0, \ x\in \mathrm{R}^3, \\ |u|\to 0, \ |x|\to\infty, \end{cases} \tag{1.1}$$

where $\alpha|u|^{\beta-1}u$ is nonlinear damping and β is damping exponent. For any $\beta \geq 1$, the global weak solutions of the Navier-Stokes equations with damping $\alpha|u|^{\beta-1}u$ ($\alpha > 0$) is obtained, and for any $\frac{7}{2} \leq \beta \leq 5$, the existence and uniqueness of strong solution is proved. Furthermore, the existence and uniqueness of strong solution is proved for any $3 \leq \beta \leq 5$ in [7], the L^2 decay of weak solutions with $\beta \geq \frac{10}{3}$ is studied and the optimal upper bounds of the higher-order derivative of the strong solution is proved in [8]. In recent years, Song et al. researched the following non-autonomous 3D Navier-Stokes equation with nonlinear damping:

$$\begin{aligned} & u_t - \mu\Delta u + (u \cdot \nabla)u + \alpha|u|^{\beta-1}u + \nabla p = f(x,t),\ x \in \Omega,\ t > \tau, \\ & divu = 0,\ x \in \Omega,\ t > \tau, \\ & u|_{t=\tau} = u_\tau,\ x \in \Omega, \\ & |u|_{\partial\Omega} = 0,\ t > \tau. \end{aligned} \tag{1.2}$$

The existence of pullback attractors for the 3D Navier-Stokes equations with damping $\alpha|u|^{\beta-1}u$ ($\alpha > 0, 3 \leq \beta \leq 5$) were proved in [9]. Furthermore, Baranovskii and Artemov investigated the solvability of the steady-state flow model for low-concentrated aqueous polymer solutions with a damping term in a bounded domain under the no-slip boundary condition in [10]. They proved that the obtained solutions of the original problem converged to a solution of the steady-state damped Navier-Stokes system as the relaxation viscosity tends to zero.

The research of the 2D g-Navier-Stokes equations is originated from the 3D Navier-Stokes equations on thin region. Its form is as follows:

$$\begin{aligned} & \frac{\partial u}{\partial t} - \mu\Delta u + (u \cdot \nabla)u + \nabla p = f \ \ in\ \Omega, \\ & \nabla \cdot (gu) = 0 \ \ in\ \Omega, \end{aligned} \tag{1.3}$$

where $g = g(x_1, x_2)$ is a suitable smooth real-valued function defined on $(x_1, x_2) \in \Omega$ and Ω is a suitable bounded domain in R^2. In [11], by the vertical mean operator, the 2D g-Navier-Stokes equations are derived from 3D Navier-Stokes equations. We study the 2D g-Navier-Stokes equations as a small perturbation of the usual Navier-Stokes equations, so we want to understand the Navier-Stokes equations completely by studying the 2D g-Navier-Stokes equations systematically. Therefore, the research on the g-Navier-Stokes equations has theoretical basis and practical significance.

There are many studies on g-Navier-Stokes equations [12–18], such as in [12], where Roh showed the existence of the global attractors for the periodic boundary conditions and proved the semiflows was robust with respect to g. The existence and uniqueness of solutions of g-Navier-Stokes equations were proved on R^2 for n=2,3 in [13]. Moreover, the existence of global solutions and the global attractor for the spatial periodic and Dirichlet boundary conditions were proved and the dimension of the global attractor was estimated in [14]. On the other hand, the global attractor of g-Navier-Stokes equations with linear dampness on R^2 were proved. The estimation of the Hausdorff and Fractal dimensions were also obtained in [15]. We investigated the existence of pullback attractors for the 2D non-autonomous g-Navier-Stokes equations on some bounded domains in [16]. D. T. Quyet proved the existence of pullback attractor in V_g for the continuous process in [17]. Recently, we discussed the uniform attractor of g-Navier-Stokes equations with weak dampnesss and time delay in [18], and the corresponding equations have the following forms:

$$\begin{aligned} & \frac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)u + \alpha u + \nabla p = f(x,t) + h(t, u_t) \ \ on\ (\tau, \infty) \times \Omega, \\ & \nabla \cdot (gu) = 0 \ \ on\ (\tau, \infty) \times \partial\Omega, \\ & u(x,t) = 0 \ \ on\ \tau, \infty) \times \partial\Omega, \\ & u(\tau, x) = u_0(x),\ x \in \Omega. \end{aligned} \tag{1.4}$$

For the equation with the restriction of the forcing term f belonging to translational compacted function space, we proved the existence of the uniform attractor by the method of asymptotic compactness. However, as far as we know, the pullback attractor of g-Navier-Stokes equations with nonlinear damping $\alpha|u|^{\beta-1}u$ and time delay $h(t, u_t)$ have not been studied, so this is the main motivation of our research.

In this article, we will study pullback asymptotic behavior of solution for the g-Navier-Stokes equations which has nonlinear damping and time delay on some bounded domain $\Omega \subset \mathrm{R}^2$, and the usual form as follows:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + c|u|^{\beta-1}u + \nabla p = f(x,t) + h(t,u_t) \ \ in\ \Omega\times(0,\infty), \\ \nabla\cdot(gu) = 0 \ \ in\ \Omega\times(0,\infty), \\ u(x,t) = 0 \ \ on\ \partial\Omega, \\ u(x,0) = u_0(x) \ \ in\ \Omega, \end{cases} \tag{1.5}$$

where $p(x,t) \in \mathrm{R}$ and $u(x,t) \in \mathrm{R}^2$ denote the pressure and the velocity respectively, $\nu > 0$ and $c|u|^{\beta-1}u$ is nonlinear damping, β is the damping exponent, $\beta \geq 1$ and $c > 0$ are constant, $0 < m_0 \leq g = g(x_1, x_2) \leq M_0$, $g = g(x_1, x_2)$ is a real-valued smooth function. When $g = 1$, Eq (1.5) become the usual two dimensional Navier-Stokes equations with nonlinear damping and time delay. $f = f(x,t)$ is the external force, $h(t, u_t)$ is another external force term with time delay, u_t is the function defined by the relation $u_t(\theta) = u(t+\theta), \forall\theta \in (-r, 0)$, $r > 0$ is constant. For the 2D g-Navier-Stokes equations can be seen as a small perturbation of the usual Navier-Stokes equations, so the 2D g-Navier-Stokes equations with nonlinear damping and time delay can be used to describe a certain state of fluid affected by external resistance and historical status. The nonlinear damping term $c|u|^{\beta-1}u$ in the balance of linear momentum realizes an absorption if $c < 0$ and a nonlinear source if $c > 0$.

By the Faedo-Galerkin method in [19,20], we investigate the global well-posedness of weak solutions for 2D non-autonomous g-Navier-Stokes equations with nonlinear damping and time delay in this article. Then, we prove the existence of pullback attractors using θ-cocycle and the method of pullback condition (PC). Compared with [18], the methods and conclusions are completely different, which can be seen as a further study of related issues. On this basis, inspired by [21–23], we can further use the pullback attractor to construct the invariant measures and statistical solutions of 2D g-Navier-Stokes equations and study their statistical solution, invariant sample measures and Liouville type theorem in the future.

The outline of the article is as follows. In the next section, we provide basic definitions and results we use in this article. In Section 3, we prove the global well-posedness of weak solutions and the existence of pullback attractors for 2D non-autonomous g-Navier-Stokes equations with nonlinear damping and time delay. In Section 4, we give some relevant conclusion.

2. Preliminaries

We define $L^2(g) = (L^2(\Omega))^2$ and $H_0^1(g) = (H_0^1(\Omega))^2$, the inner product of $L^2(g)$ is $(u, v) = \int_\Omega u\cdot vgdx$ and inner product of $H_0^1(g)$ is $((u,v)) = \int_\Omega \Sigma_{j=1}^2 \nabla u_j \cdot \nabla v_j gdx$, corresponding norm is $|\cdot| = (\cdot,\cdot)^{1/2}$ and $\|\cdot\| = ((\cdot,\cdot))^{1/2}$ respectively.

Let $M = \{v \in (D(\Omega))^2 : \nabla\cdot gv = 0\ in\ \Omega\}$; $H_g = closure\ of\ M\ in\ L^2(g)$; $V_g = closure\ of\ M\ in\ H_0^1(g)$. Furthermore, H_g is endowed with the inner product and norm of $L^2(g)$, V_g is endowed with the inner product and norm of $H_0^1(g)$, where $D(\Omega)$ is the space of C^∞ functions which have compact support contained in Ω, and $C_{H_g} = C^0([-h, 0]; H_g), C_{V_g} = C^0([-h, 0]; V_g)$.

Let $h : \mathrm{R} \times C_{H_g} \to (L^2(\Omega))^2$ satisfies the following assumptions:

(I) $\forall\ \xi \in C_{H_g}, t \in \mathrm{R} \to h(t, \xi) \in (L^2(\Omega))^2$ is measureable;

(II) $\forall\ t \in \mathrm{R}, h(t, 0) = 0$;

(III) $\exists L_g > 0$, such that $\forall\ t \in \mathrm{R}, \forall \xi, \eta \in C_{H_g}$, there is $|h(t, \xi) - h(t, \eta)| \le L_g \|\xi - \eta\|_{C_{H_g}}$;

(IV) $\exists m_0 \ge 0, C_g > 0, \forall m \in [0, m_0], \tau \le t, u, v \in C^0([\tau - r, t]; H_g)$, such that

$$\int_\tau^t e^{ms} |h(s, u_s) - h(s, v_s)|^2 ds \le C_g^2 \int_{\tau-r}^t e^{ms} |u(s) - v(s)|^2 ds.$$

$\forall\ t \in [\tau, T]$, $\forall u, v \in L^2(\tau - r, T; H_g)$, from (IV), we have

$$\int_\tau^t |h(s, u_s) - h(s, v_s)|^2_{(L^2(\Omega))^2} ds \le C_g^2 \int_{\tau-r}^t |u(s) - v(s)|^2 ds.$$

Since the Poincaré inequality holds on Ω: There exists $\lambda_1 > 0$ such that

$$\int_\Omega \phi^2 g dx \le \frac{1}{\lambda_1} \int_\Omega |\nabla \phi|^2 g dx, \quad \forall \phi \in H_0^1(\Omega), \tag{2.1}$$

then,

$$|u|^2 \le \frac{1}{\lambda_1} \|u\|^2, \quad \forall u \in V_g. \tag{2.2}$$

The g-Laplacian operator is defined as follows:

$$-\Delta_g u = -\frac{1}{g}(\nabla \cdot g \nabla) u = -\Delta u - \frac{1}{g} \nabla g \cdot \nabla u,$$

the first equation of (1.5) can be rewritten as follows:

$$\frac{\partial u}{\partial t} - \nu \Delta_g u + \nu \frac{\nabla g}{g} \cdot \nabla u + (u, \nabla) u + c|u|^{\beta-1} u + \nabla p = f + h(t, u_t). \tag{2.3}$$

A g-orthogonal projection is defined by $P_g : L^2(g) \to H_g$ and g-Stokes operator with $A_g u = -P_g(\frac{1}{g}(\nabla \cdot (g \nabla u)))$. Applying the projection P_g to (1.5), $\forall v \in V_g, \forall t > 0$, we obtain

$$\frac{d}{dt}(u, v) + \nu((u, v)) + b_g(u, u, v) + c(|u|^{\beta-1} u, v) + \nu(Ru, v) = \langle f, v \rangle + \langle h(t, u_t), v \rangle, \tag{2.4}$$

$$u(0) = u_0, \tag{2.5}$$

where $b_g : V_g \times V_g \times V_g \to \mathrm{R}$, and $b_g(u, v, w) = \sum_{i,j=1}^2 \int u_i \frac{\partial v_j}{\partial x} w_j g dx$, $Ru = P_g[\frac{1}{g}(\nabla g \cdot \nabla) u]$, $\forall u \in V_g$. Let $G(u) = P_g F(u)$, $F(u) = c|u|^{\beta-1} u$, then the formulations (2.4) and (2.5) are equivalent to the following equations:

$$\frac{du}{dt} + \nu A_g u + Bu + G(u) + \nu Ru = f + h, \tag{2.6}$$

$$u(0) = u_0, \tag{2.7}$$

where $A_g : V_g \to V_g'$, $\langle A_g u, v \rangle = ((u, v)), \forall u, v \in V_g$, and $B(u) = B(u, u) = P_g(u \cdot \nabla) u$ is a bilinear operator, and $B : V_g \times V_g \to V_g'$ with $\langle B(u, v), w \rangle = b_g(u, v, w), \forall u, v, w \in V_g$.

For any $u, v \in D(A_g)$, $|B(u,v)| \leq C|u|^{1/2}|A_g u|^{1/2}\|v\|$, where C denote positive constants. From [11,12], we have the following inequality:

$$|\varphi|_{L^\infty(\Omega)^2} \leq C\|\varphi\|(1+\ln\frac{|A_g\varphi|^2}{\lambda_1\|\varphi\|^2})^{1/2}, \quad \forall\varphi \in D(A_g), \tag{2.8}$$

$$|B(u,v)| \leq |(u\cdot\nabla)v| \leq |u|_{L^\infty(\Omega)}|\nabla v|, \tag{2.9}$$

$$|B(u,v)| \leq C\|u\|\|v\|(1+\ln\frac{|A_g u|^2}{\lambda_1\|u\|^2})^{1/2}, \tag{2.10}$$

$$\|B(u)\|_{V_g'} \leq c|u|\|u\|, \quad \|Ru\|_{V_g'} \leq \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u\|, \ \ \forall u \in V_g. \tag{2.11}$$

From [3,4,16], we have the following concepts and conclusions.

Let Γ be a nonempty set and we define a family $\{\theta_t\}_{t\in\mathbb{R}}$ of mappings $\theta_t : \Gamma \to \Gamma$ satisfying

(1) $\theta_0\gamma = \gamma$ for all $\gamma \in \Gamma$,

(2) $\theta_t(\theta_\tau\gamma) = \theta_{t+\tau}\gamma$ for all $\gamma \in \Gamma, t, \tau \in \mathrm{R}$,

then the operators θ_t are called the shift operators.

Let X be a metric space, for any $(\gamma, x) \in \Gamma\times X$ and $t, \tau \in \mathrm{R}^+$, $\phi : \mathrm{R}^+\times\Gamma\times X \to X$ is said a θ-cocycle on X if and only if

(1) $\phi(0,\gamma,x) = x$,

(2) $\phi(t+\tau,\gamma,x) = \phi(t,\theta_\tau\gamma,\phi(\tau,\gamma,x))$, where θ_t is the shift operators.

If for all $(t,\gamma) \in \mathrm{R}^+\times\Gamma$, we have the mapping $\phi(t,\gamma,\cdot) : X \to X$ is continuous, then the θ-cocycle ϕ is said to be continuous.

Definition 2.1. [3] A family $\widetilde{A} = \{A(\gamma); \gamma\in\Gamma\} \in \phi$ is said to be pullback $\mathcal{D}$-attractor if it satisfies

(1) $A(\gamma)$ is compact for any $\gamma \in \Gamma$,

(2) $\widetilde{A}$ is pullback $\mathcal{D}$-attracting, i.e.,

$$\lim_{t\to+\infty} dist(\phi(t,\theta_{-t}\gamma, D(\theta_{-t}\gamma)), A(\gamma)) = 0 \ \ for\ all\ \widetilde{D} \in \mathcal{D}, \gamma \in \Gamma,$$

(3) $\widetilde{A}$ is invariant, i.e.,

$$\phi(t,\gamma,A(\gamma)) = A(\theta_t\gamma) \ \ for\ any\ (t,\gamma) \in \mathrm{R}^+\times\Gamma.$$

Definition 2.2. [4] Let ϕ be a θ-cocycle on X. A set $B_0 \subset X$ is said to be uniformly absorbing set for ϕ, if for any $B \in B(X)$ there exists $T_0 = T_0(B) \in \mathrm{R}^+$ such that

$$\phi(t,\gamma,B) \subset B_0 \quad for\ all\ t \geq T_0,\ \gamma \in \Gamma.$$

Theorem 2.1. [4] Let ϕ be a θ-cocycle on X. If ϕ is continuous and possesses a uniformly absorbing set B_0, then ϕ possesses a pullback attractor $\mathcal{A} = \{A_\gamma\}_{\gamma\in\Gamma}$ if and only if it is pullback ω-limit compact.

Definition 2.3. [4] Let ϕ be a θ-cocycle on X. A cocycle ϕ is said to be satisfying pullback condition if for any $\gamma \in \Gamma, B \in B(X)$ and $\varepsilon > 0$, there exist $t_0 = t_0(\gamma, B, \varepsilon) \geq 0$ and a finite dimensional subspace X_1 of X such that

(1) $P(\bigcup_{t\geq t_0}\phi(t,\theta_{-t}(\gamma),B))$ is bounded,

(2) $\|(I-P)(\bigcup_{t\geq t_0}\phi(t,\theta_{-t}(\gamma),x)\| \leq \varepsilon, \ \forall x \in B$,

where $P : X \to X_1$ is a bounded projector.

Theorem 2.2. [4] Let X be a Banach space and let ϕ be a θ-cocycle on X. If ϕ satisfies pullback condition, then ϕ is pullback ω-limit compact. Moreover, let X is a uniformly convex Banach space, then ϕ is pullback ω-limit compact if and only if pullback condition holds true.

We denote the metrizable space of function $f(s) \in X$ with $s \in \mathrm{R}$ by $L^2_{loc}(\mathrm{R}, X)$, where X is locally two-power integrable in the Bochner sense. It is equipped with the local two-power mean convergence topology.

Lemma 2.1. [16] If H_g is Hilbert space and $\{\omega_i\}_{i \in N}$ is orthonormal in H_g, let $f(x,t) \in L^2_{loc}(\mathrm{R}; H_g)$ and there exists a $\sigma > 0$, such that for any $t \in \mathrm{R}$, $\int_{-\infty}^{t} e^{\sigma s} \|f(x,s)\|^2_{H_g} ds < \infty$, then,

$$\lim_{n\to\infty} \int_{-\infty}^{t} e^{\sigma s} \|(I - P_m) f(x,s)\|^2_{H_g} ds = 0, \quad \forall\, t \in \mathrm{R},$$

where $P_m : H_g \to span\{\omega_1, \dots, \omega_n\}$ be an orthogonal projector.

3. Proofs of the main results

In the section, we will prove the well-posedness of the weak solution for 2D g-Navier-Stokes equations with nonlinear damping and time delay by the Faedo-Galerkin method.

Definition 3.1. Let $u_0 \in H_g, f \in L^2_{Loc}(\mathrm{R}; V_g')$, for any $\tau \in \mathrm{R}$, $u \in L^\infty(\tau, T; V_g) \cap L^2(\tau, T; V_g) \cap L^{\beta+1}(\tau, T; L^{\beta+1}(\Omega)), \forall T > \tau$ is called a weak solution of problem (1.5) if it fulfils

$$\frac{d}{dt}u(t) + \nu A_g u(t) + B(u(t)) + c|u|^{\beta-1}u + \nu R(u(t)) = f(x,t) + h(t,u_t) \; on \; \mathcal{D}'(\tau, +\infty; V_g'),$$

$$u(\tau) = u_0.$$

Theorem 3.1. Let $\beta \geq 1$, $f \in L^2_{Loc}(\mathrm{R}; V_g')$, then for every $u_\tau \in V_g$, the Eq (1.5) exist the only weak solution $u(t) = u(t; \tau, u_\tau) \in L^\infty(\tau, T; V_g) \cap L^2(\tau, T; V_g) \cap L^{\beta+1}(\tau, T; L^{\beta+1}(\Omega))$, and $u(t)$ continuously depends on the initial value in V_g.

Proof. Let $\{w_j\}_{j\geq 1}$ be the eigenfunctions of $-\Delta$ on Ω with homogeneous Dirichlet boundary conditions, its corresponding eigenvalues are $0 < \lambda_1 \leq \lambda_2 \leq \dots$, obviously, $\{w_j\}_{j\geq 1} \subset V_g$ forms a Hilbert basis in H_g, given $u_\tau \in V_g$ and $f \in L^2_{Loc}(\mathrm{R}; V_g')$.

For every positive integer $n \geq 1$, we structure the Galerkin approximate solutions as $u_n(t) = u_n(t; T, u_\tau)$. It has the following form:

$$u_n(t; T, u_\tau) = \sum_{j=1}^{n} \gamma_{n,j}(t) w_j,$$

where $\gamma_{n,j}(t)$ is determined from the initial values of the following system of nonlinear ordinary differential equations:

$$\begin{aligned} & (u_n'(t), w_j) + \nu((u_n(t), w_j)) + c(|u_n(t)|^{\beta-1}u_n(t), w_j) + b(u_n(t), u_n(t), w_j) + b(\tfrac{\nabla g}{g}, u_n(t), w_j) \\ = \;& \langle f(x,t), w_j\rangle + \langle h(t,u_t), w_j\rangle, \; t > \tau, \; j = 1, 2, \dots n, \\ & ((u_n(t), w_j)) = ((u_\tau, w_j)), \end{aligned} \tag{3.1}$$

where $\langle\cdot\rangle$ is dual product of V_g and V_g'.

According to the results of the initial value problems of ordinary differential equations, there exists a unique local solution to problem (3.1). In the following, we prove that the time interval of the solution can be extended to $[\tau, \infty)$.

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|_2^2 + \nu\|u_n(t)\|^2 + c|u_n(t)|_{\beta+1}^{\beta+1} + b((\frac{\nabla g}{g}\cdot\nabla)u_n(t), u_n(t)) = \langle f(x,t), u_n(t)\rangle + \langle h(t,u_t), u_n(t)\rangle. \quad (3.2)$$

Using Cauchy's inequality and Young's inequality, we have

$$\langle f(x,t), u_n(t)\rangle \le \|f(x,t)\|_* \cdot \|u_n(t)\| \le \frac{\nu}{2}\|u_n\|^2 + \frac{1}{2\nu}\|f(x,t)\|_*^2, \quad (3.3)$$

where $\|\cdot\|_*$ is norm of V_g'.

$$\langle h(t,u_t), u_n(t)\rangle \le \frac{1}{2C_g}|h(t,u_t)|^2 + \frac{C_g}{2\lambda_1}\|u_n(t)\|^2. \quad (3.4)$$

We take (3.3) and (3.4) into (3.2) to obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|u_n(t)|_2^2 + \nu\|u_n(t)\|^2 + c|u_n(t)|_{\beta+1}^{\beta+1} + b((\frac{\nabla g}{g}\cdot\nabla)u_n(t), u_n(t)) \\ \le\ & \frac{\nu}{2}\|u_n\|^2 + \frac{1}{2\nu}\|f(x,t)\|_*^2 + \frac{1}{2C_g}|h(t,u_t)|^2 + \frac{C_g}{2\lambda_1}\|u_n(t)\|^2, \\ & \frac{d}{dt}|u_n(t)|_2^2 + 2\nu\|u_n(t)\|^2 + 2c|u_n(t)|_{\beta+1}^{\beta+1} + 2b((\frac{\nabla g}{g}\cdot\nabla)u_n(t), u_n(t)) \\ \le\ & \nu\|u_n\|^2 + \frac{1}{\nu}\|f(x,t)\|_*^2 + \frac{1}{C_g}|h(t,u_t)|^2 + \frac{C_g}{\lambda_1}\|u_n(t)\|^2, \end{aligned}$$

$$\begin{aligned} & \frac{d}{dt}|u_n(t)|_2^2 + (\nu - \frac{C_g}{\lambda_1})\|u_n(t)\|^2 + 2c|u_n(t)|_{\beta+1}^{\beta+1} + 2b((\frac{\nabla g}{g}\cdot\nabla)u_n(t), u_n(t)) \\ \le\ & \frac{1}{\nu}\|f(x,t)\|_*^2 + \frac{1}{C_g}|h(t,u_t)|^2, \end{aligned} \quad (3.5)$$

that is

$$\frac{d}{dt}|u_n(t)|_2^2 + (\nu - \frac{C_g}{\lambda_1})\|u_n(t)\|^2 + 2c|u_n(t)|_{\beta+1}^{\beta+1} \le \frac{1}{\nu}\|f(x,t)\|_*^2 + \frac{1}{C_g}|h(t,u_t)|^2 + 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u_n(t)\|^2.$$

$$\frac{d}{dt}|u_n(t)|_2^2 + \nu(1 - \frac{C_g}{\nu\lambda_1} - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}})\|u_n(t)\|^2 + 2c|u_n(t)|_{\beta+1}^{\beta+1} \le \frac{1}{\nu}\|f(x,t)\|_*^2 + \frac{1}{C_g}|h(t,u_t)|^2. \quad (3.6)$$

By integrating (3.6) from τ to t, we can obtain

$$\begin{aligned} & |u_n(t)|^2 + \nu(1 - \frac{C_g}{\nu\lambda_1} - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}})\int_\tau^t \|u_n(s)\|^2 ds + 2c\int_\tau^t |u_n(s)|_{\beta+1}^{\beta+1} ds \\ \le\ & |u_n(\tau)|^2 + \frac{1}{\nu}\int_\tau^t \|f(x,s)\|_*^2 ds + \frac{1}{C_g}\int_\tau^t |h(s,u_s)|^2 ds. \end{aligned}$$

For any $T > 0$ and $\beta \ge 1$, we have

$$\begin{aligned} & \sup_{\tau\le t\le T}(|u_n(t)|^2) + \nu(1 - \frac{C_g}{\nu\lambda_1} - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}})\int_\tau^t \|u_n(s)\|^2 ds + 2c\int_\tau^t |u_n(s)|_{\beta+1}^{\beta+1} ds \\ \le\ & |u_n(\tau)|^2 + \frac{1}{\nu}\int_\tau^t \|f(x,s)\|_*^2 ds + \frac{1}{C_g}\int_\tau^t |h(s,u_s)|^2 ds \le C, \end{aligned}$$

then we can obtain that

$$\{u_n(t)\} \textit{ is bounded in } L^\infty(\tau, T; V_g), \quad (3.7)$$

$$\{u_n(t)\} \textit{ is bounded in } L^2(\tau, T; V_g), \quad (3.8)$$

and $\{u_n(t)\}$ is bounded in $L^{\beta+1}(\tau,T;L^{\beta+1}(\Omega))$. So $u_n(t)\in L^\infty(\tau,T;V_g)$, therefore $B(u_n(t))\in L^\infty(\tau,T;V_g^{'})$, $|u_n(t)|^{\beta-1}u_n(t)\in L^{\beta+1}(\tau,T;L^{\beta+1}(\Omega))$. As a result,

$$\frac{d}{dt}\langle u_n(t),v\rangle=\langle f(x,t)+h(t,u_t)-c|u_n(t)|^{\beta-1}u_n(t)-\nu Au_n(t)-B(u_n(t))-\nu R(u_n(t)),v\rangle,\quad \forall v\in V_g.$$

Since $\{u_n^{'}(t)\}$ is bounded in $L^2(\tau,T;V_g)$, then there exists a subsequence in $\{u_n(t)\}$, it still denoted by $\{u_n(t)\}$, we have $u_n(t)\in L^2(\tau,T;V_g)$ and $u_n^{'}(t)\in L^2(\tau,T;V_g)$ such that

(i) $u_n(t)\to u(t)$ is weakly * convergent in $L^\infty(\tau,T;V_g)$;

(ii) $u_n(t)\to u(t)$ is weakly convergent in $L^2(\tau,T;V_g)$;

(iii) $|u_n(t)|^{\beta-1}u_n(t)\to\xi$ is weakly convergent in $L^{\beta+1}(\tau,T;L^{\beta+1}(\Omega))$;

(iv) $u_n^{'}(t)\to u^{'}(t)$ is weakly convergent in $L^2(\tau,T;V_g)$;

(v) $u_n(t)\to u(t)$ is strongly convergent in $L^2(\tau,T;H_g)$;

(vi) $u_n(t)\to u(t),\ \ a\ e\ (x,t)\in\Omega\times[\tau,T]$.

From Lemma 1.3 of [24], we can see $\xi=|u|^{\beta-1}u$, since $\bigcup_{n\in N^+}Span\{w_1,w_2,\cdots,w_n\}$ is denseness in V_g, taking the limit $n\to\infty$ on both sides of (3.1), we can obtain that u is a weak solution of (1.5).

In the following, the solution is proved to be unique and continuously dependent on initial values. Let u_1 and u_2 be two weak solutions of (1.5) corresponding to the initial values $u_{1\tau},u_{2\tau}\in V_g$, we take $u=u_1-u_2$, from (2.6) we obtain

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}|u|^2+\nu\|u\|^2+c(|u_1|^{\beta-1}u_1-|u_2|^{\beta-1}u_2,u)+\nu(Ru,u)\\ =\ &\langle B(u_2)-B(u_1),u\rangle+\langle h(t,u_{2t})-h(t,u_{1t}),u\rangle.\end{aligned}\tag{3.9}$$

Using Hölder inequality and Sobolev embedding theorem, we obtain

$$\begin{aligned}&(|u_1|^{\beta-1}u_1-|u_2|^{\beta-1}u_2,u)=\int_\Omega(|u_1|^{\beta-1}u_1-|u_2|^{\beta-1}u_2)(u_1-u_2)dx\\ \geq\ &\int_\Omega(|u_1|^{\beta+1}-|u_1|^{\beta}|u_2|-|u_2|^{\beta}u_1+|u_2|^{\beta+1})dx\\ =\ &\int_\Omega(|u_1|^{\beta}-|u_2|^{\beta})(|u_1|-|u_2|)dx\geq 0.\end{aligned}\tag{3.10}$$

We have

$$\begin{aligned}&|\langle B(u_2)-B(u_1),u\rangle|=|\langle B(u_2,u_2-u_1)-B(u_1-u_2,u_1),u\rangle|\\ \leq\ &C_1\|u_2\|\|u_2-u_1\|\|u\|+C_1\|u_1-u_2\|\|u_1\|\|u\|\\ =\ &C_1\|u\|^2(\|u_1+\|u_2\|)\\ \leq\ &C_1\|u\|^2,\end{aligned}\tag{3.11}$$

where $C_1>0$ is any constant.

$$\begin{aligned}|\langle h(t,u_{2t})-h(t,u_{1t}),u\rangle|\ &\leq\int_0^t|h(s,u_{2s})-h(s,u_{1s})|\cdot|u(s)|ds\leq L_g\|u_t\|_{C_{H_g}}\cdot|u(t)|\\ &\leq\tfrac{\nu\lambda_1}{4}|u(t)|^2+\tfrac{L_g}{2\lambda_1}\|u_t\|^2_{C_{H_g}}\leq\tfrac{\nu}{4}\|u(t)\|^2+\tfrac{L_g}{2\lambda_1}\|u_t\|^2_{C_{H_g}},\end{aligned}$$

$$\nu|(Ru,u)|\leq\nu\frac{\|\nabla g\|_\infty}{m_0\lambda_1^{1/2}}\|u\||u|\leq\frac{\nu\|\nabla g\|_\infty}{2m_0\lambda_1^{1/2}}(\|u\|^2+|u|^2)=\xi(\|u\|^2+|u|^2),\tag{3.12}$$

where $\xi=\frac{\nu\|\nabla g\|_\infty}{2m_0\lambda_1^{1/2}}$, so

$$\frac{1}{2}\frac{d}{dt}|u|^2+\nu\|u\|^2\leq C_1\|u\|^2+\xi(\|u\|^2+|u|^2)+\frac{\nu}{4}\|u(t)\|^2+\frac{L_g}{2\lambda_1}\|u_t\|^2_{C_{H_g}},$$

$$\frac{d}{dt}|u|^2 + (2\nu - 2C_1 - 2\xi - \frac{\nu}{2})\|u\|^2 \le 2\xi|u|^2 + \frac{L_g}{2\lambda_1}\|u_t\|^2_{C_{H_g}}.$$

Let $2\nu - 2C_1 - 2\xi - \frac{\nu}{2} > 0$, then

$$|u(t)|^2 \le 2\xi \int_0^t |u(s)|^2 ds + \frac{L_g}{\lambda_1}\int_0^t \|u_s\|^2_{C_{H_g}} ds.$$

Since $u(s) = 0$ for $s \le 0$, we take the maximum in $[0, t]$ for any $t \in [0, T]$, and we obtain

$$\|u_t\|^2_{C_{H_g}} \le (2\xi + \frac{L_g}{\lambda_1})\int_0^t \|u_s\|^2_{c_{H_g}} ds.$$

We can obtain that the uniqueness of the solution holds after applying the Gronwall inequality.

In the following, we will prove the existence of pullback attractor for (1.5). First, we will prove the existence of pullback absorbing sets.

Lemma 3.1. Let $f \in L^2_{loc}(\mathrm{R}, H_g)$, $|f|^2_b = \sup_{t\in\mathrm{R}} \int_t^{t+1} |f(s)|^2 ds < \infty$, $|h|^2_b = \sup_{t\in\mathrm{R}} \int_t^{t+1} |h(s, u_s)|^2 ds < \infty$, suppose $u(x,t) = u(t;\tau,u_\tau) \in L^\infty(\tau, T; V_g) \cap L^2(\tau, T; V_g) \cap L^{\beta+1}(\tau, T; L^{\beta+1}(\Omega))$ be a weak solution of Eq (1.5). Let $\sigma = \nu\lambda_1$, for any $t \ge \tau$, then

$$|u(t)|^2 \le |u_0|^2 e^{-\sigma\gamma_0(t-\tau)} + R_1^2,$$

where $R_1^2 = \frac{1}{\sigma(1-e^{-\sigma\gamma_0})}(|f|^2_b + |h|^2_b)$ and $\gamma_0 = \frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}} - 1 + \frac{C_g}{\nu\lambda_1}$.

Proof. Let $f \in L^2_{loc}(\mathrm{R}, H_g)$ and $|f|^2_b = \sup_{t\in\mathrm{R}} \int_t^{t+1} |f(s)|^2 ds < \infty$, $|h|^2_b = \sup_{t\in\mathrm{R}} \int_t^{t+1} |h(s, u_s)|^2 ds < \infty$. Let $u(x,t)$ be a weak solution of Eq (1.5), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u|^2 &= \langle u', u\rangle = \langle f + h - \nu A_g u - Bu - c|u|^{\beta-1}u - \nu Ru, u\rangle \\ &= \langle f, u\rangle + \langle h, u\rangle - \nu\|u\|^2 - b_g(u,u,u) - c|u|^{\beta+1}_{\beta+1} - \nu((\tfrac{1}{g}\nabla g\cdot\nabla)u, u),\end{aligned}$$

then,

$$\frac{d}{dt}|u|^2 + 2\nu\|u\|^2 + 2c|u|^{\beta+1}_{\beta+1} = 2\langle f, u\rangle + 2\langle h, u\rangle - 2\nu((\frac{\nabla g}{g}\cdot\nabla)u, u).$$

So

$$\begin{aligned}\frac{d}{dt}|u|^2 + 2\nu\|u\|^2 + 2c|u|^{\beta+1}_{\beta+1} &\le \frac{|f|^2}{\nu\lambda_1} + \nu\lambda_1|u|^2 + \frac{1}{C_g}|h(t,u_t)|^2 + C_g|u|^2 + 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u\|^2 \\ &\le \frac{|f|^2}{\nu\lambda_1} + \nu\|u\|^2 + \frac{1}{C_g}|h(t,u_t)|^2 + C_g\frac{\|u\|^2}{\lambda_1} + 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u\|^2,\end{aligned}$$

then,

$$\frac{d}{dt}|u|^2 + (\nu - \frac{C_g}{\lambda_1})\|u\|^2 + 2c|u|^{\beta+1}_{\beta+1} \le \frac{|f|^2}{\nu\lambda_1} + \frac{1}{C_g}|h(t,u_t)|^2 + 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u\|^2.$$

Hence,

$$\frac{d}{dt}|u|^2 \le \nu\gamma_0\|u\|^2 + \frac{|f|^2}{\nu\lambda_1} + \frac{1}{C_g}|h(t,u_t)|^2,$$

where $\gamma_0 = 2\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} - 1 + \frac{C_g}{\nu\lambda_1} > 0$ for sufficiently small $|\nabla g|_\infty$. So

$$\frac{d}{dt}|u|^2 \le \nu\lambda_1\gamma_0|u|^2 + \frac{|f|^2}{\nu\lambda_1} + \frac{1}{C_g}|h(t,u_t)|^2.$$

Let $\sigma = \nu\lambda_1$, we have

$$\begin{aligned}
|u(t)|^2 &\le |u_0|^2 e^{\sigma\gamma_0(\tau-t)} + \tfrac{1}{\sigma}\int_\tau^t e^{-\sigma\gamma_0(t-r)}|f(r)|^2dr + \tfrac{1}{C_g}\int_\tau^t e^{-\sigma\gamma_0(t-r)}|h(r,u_r)|^2dr \\
&\le |u_0|^2 e^{\sigma\gamma_0(\tau-t)} + \tfrac{1}{\sigma}[\int_{t-1}^t e^{-\sigma\gamma_0(t-r)}|f(r)|^2dr + \int_{t-2}^{t-1} e^{-\sigma\gamma_0(t-r)}|f(r)|^2dr + \cdots] \\
&\quad + \tfrac{1}{C_g}[\int_{t-1}^t e^{-\sigma\gamma_0(t-r)}|h(r,u_r)|^2dr + \int_{t-2}^{t-1} e^{-\sigma\gamma_0(t-r)}|h(r,u_r)|^2dr + \cdots] \\
&\le |u_0|^2 e^{\sigma\gamma_0(\tau-t)} + \tfrac{1}{\sigma}(1 + e^{-\sigma\gamma_0} + e^{-2\sigma\gamma_0} + \cdots)\sup_{t\in R}\int_t^{t+1}|f(r)|^2dr \\
&\quad + \tfrac{1}{C_g}(1 + e^{-\sigma\gamma_0} + e^{-2\sigma\gamma_0} + \cdots)\sup_{t\in R}\int_t^{t+1}|h(r,u_r)|^2dr \\
&\le |u_0|^2 e^{\sigma\gamma_0(\tau-t)} + R_1^2,
\end{aligned}$$

where $R_1^2 = \frac{1}{\sigma(1-e^{-\sigma\gamma_0})}(|f|_b^2 + |h|_b^2)$.

For any $f \in L^2_{loc}(R, H_g)$, $|f|_b^2 = |f_0|_b^2$, we have the uniformly absorbing set

$$B_0 = \{u \in H_g \| u| \le 2R_1^2 = \rho_0^2\}$$

in H_g.

Lemma 3.2. Let $f \in L^2_{loc}(R, H_g)$, $|f|_b^2 = \sup_{t\in R}\int_t^{t+1}|f(s)|^2ds < \infty$, $|h|_b^2 = \sup_{t\in R}\int_t^{t+1}|h(s,u_s)|^2ds < \infty$, $u_0(x) \in H_g$, suppose

$$u(x,t) \in L^\infty(\tau,T;V_g) \cap L^2(\tau,T;V_g) \cap L^{\beta+1}(\tau,T;L^{\beta+1}(\Omega)), u'(x,t) \in L^2_{loc}(R_\tau;H_g)\ (\forall t > 0)$$

is a strong solution of (1.5), for any $t \ge \tau$, then

$$\|u(t)\|^2 \le \|u(\tau)\|^2 e^{\gamma(\tau-t)} + \frac{1}{\nu}(1 - e^{-\gamma})^{-1}(|f|_b^2 + |h|_b^2),$$

where $\gamma = \lambda(\nu - C_g - \frac{2\nu|\nabla g|_\infty}{m_0\lambda_0^{1/2}})$.

Proof. We suppose $u(x,t)$ be a strong solution of (1.5), multiplying (2.6) by $A_g u$ and we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu|A_g u|^2 + (Bu, A_g u) + (c|u|^{\beta-1}u, A_g u) = (f, A_g u) + (h, A_g u) - \nu(Ru, A_g u).$$

Then,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu|A_g u|^2 + c\int_\Omega |u|^{\beta-1}|\nabla u|^2dx + \frac{c(\beta-1)}{4}\int_\Omega |u^{\beta-3}\|\nabla|u|^2|^2dx \\
&= (f, A_g u) + (h, A_g u) - \nu(Ru, A_g u), \\
&\frac{d}{dt}\|u\|^2 + 2\nu|A_g u|^2 + 2c\int_\Omega |u|^{\beta-1}|\nabla u|^2dx + \frac{c(\beta-1)}{2}\int_\Omega |u^{\beta-3}\|\nabla|u|^2|^2dx \\
&= 2(f, A_g u) + 2(h, A_g u) - 2\nu(Ru, A_g u).
\end{aligned}$$

So

$$\begin{aligned}
&\tfrac{d}{dt}\|u\|^2 + 2\nu|A_g u|^2 + 2c\int_\Omega |u|^{\beta-1}|\nabla u|^2dx + \tfrac{c(\beta-1)}{2}\int_\Omega |u^{\beta-3}\|\nabla|u|^2|^2dx \\
&\le \tfrac{1}{\nu}|f|^2 + \nu|A_g u|^2 + \tfrac{1}{C_g}|h(t,u_t)|^2 + C_g|A_g u|^2 + \tfrac{2\nu|\nabla g|_\infty}{m_0}\|u\||A_g u| \\
&\le \tfrac{1}{\nu}|f|^2 + (\nu - C_g)|A_g u|^2 + \tfrac{1}{C_g}|h(t,u_t)|^2 + \tfrac{2\nu|\nabla g|_\infty}{m_0\lambda_0^{1/2}}|A_g u|^2.
\end{aligned}$$

Since

$$2c\int_{\Omega}|u|^{\beta-1}|\nabla u|^2dx+\frac{c(\beta-1)}{2}\int_{\Omega}|u|^{\beta-3}\|\nabla|u|^2|^2dx\geq 0,$$

we deduce

$$\frac{d}{dt}\|u\|^2+(\nu-C_g-\frac{2\nu|\nabla g|_\infty}{m_0\lambda_0^{1/2}})|A_gu|^2\leq\frac{1}{\nu}|f|^2+\frac{1}{C_g}|h(t,u_t)|^2,$$

$$\frac{d}{dt}\|u\|^2+\lambda(\nu-C_g-\frac{2\nu|\nabla g|_\infty}{m_0\lambda_0^{1/2}})\|u\|^2\leq\frac{1}{\nu}|f|^2+\frac{1}{C_g}|h(t,u_t)|^2.$$

Then we have

$$\frac{d}{dt}\|u\|^2+\gamma\|u\|^2\leq\frac{1}{\nu}|f|^2+\frac{1}{C_g}|h(t,u_t)|^2,$$

where

$$\gamma=\lambda(\nu-C_g-\frac{2\nu|\nabla g|_\infty}{m_0\lambda_0^{1/2}})>0.$$

Using Gronwall's inequality, we deduce

$$\begin{aligned}\|u\|^2&\leq\|u(\tau)\|^2e^{\gamma(\tau-t)}+\tfrac{1}{\nu}\int_\tau^t e^{-\gamma(t-r)}|f|^2dr+\tfrac{1}{C_g}\int_\tau^t e^{-\gamma(t-r)}|h(r,u_r)|^2dr\\&\leq\|u(\tau)\|^2e^{\gamma(\tau-t)}+\tfrac{1}{\nu}[\int_{t-1}^t e^{-\gamma(t-r)}|f|^2dr+\int_{t-2}^{t-1}e^{-\gamma(t-r)}|f|^2dr+\cdots]\\&\quad+\tfrac{1}{C_g}[\int_{t-1}^t e^{-\gamma(t-r)}|h(r,u_r)|^2dr+\int_{t-2}^{t-1}e^{-\gamma(t-r)}|h(r,u_r)|^2dr+\cdots],\end{aligned}$$

$$\begin{aligned}\|u\|^2&\leq\|u(\tau)\|^2e^{\gamma(\tau-t)}+\tfrac{1}{\nu}(1+e^{-\gamma}+e^{-2\gamma}+\cdots)\sup_{t\in\mathrm{R}}\int_t^{t+1}|f|^2dr\\&\quad+\tfrac{1}{C_g}(1+e^{-\gamma}+e^{-2\gamma}+\cdots)\sup_{t\in\mathrm{R}}\int_t^{t+1}|h(r,u_r)|^2dr\\&\leq\|u(\tau)\|^2e^{\gamma(\tau-t)}+\tfrac{1}{\nu}(1-e^{-\gamma})^{-1}(|f|_b^2+|h|_b^2).\end{aligned}$$

Let

$$B_1=\bigcup_{f\in\Gamma}\bigcup_{t>t_0+1}\phi(t_0+1,f,h,B_0),$$

then B_1 is bound and B_1 is the uniformly absorbing set in V_g.

Theorem 3.2. Let $f\in L^2_{loc}(\mathrm{R},H_g)$, $|f|_b^2=\sup_{t\in\mathrm{R}}\int_t^{t+1}|f(s)|^2ds<\infty$, $|h|_b^2=\sup_{t\in\mathrm{R}}\int_t^{t+1}|h(s,u_s)|^2ds<\infty$, then the cocycle $\{\phi(t,\gamma,x)\}$ corresponding to Eq (1.5) possesses a compact pullback attractor.

Proof. The following we will prove that cocycle $\{\phi(t,\gamma,x)\}$ satisfies pullback condition in V_g. As $(A_g)^{-1}$ is continuous compact in H_g, we can use spectral theory, there is a sequence $\{\lambda_j\}_{j=1}^\infty$, $0\leq\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_i\leq\cdots\leq\lambda_j\to\infty,\ as\ j\to\infty$, and a family of $\{\omega_j\}_{j=1}^\infty$ of $D(A_g)$, they are orthonormal in H_g and $A_g\omega_j=\lambda_j\omega_j,\ \ \forall j\in\mathrm{N}$. We suppose $V_m=span\{\omega_1,\omega_2,\ldots,\omega_m\}$ in V_g, $P_m:V_g\to V_m$ is orthogonal projector.

For all $u\in D(A_g)$, we set $u=P_mu+(I-P_m)u=u_1+u_2$, and multiply the first equation of (2.6) by A_gu_2 in H_g, then we obtain

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\|u_2\|^2+\nu|A_gu_2|^2+(B(u),A_gu_2)+(G(u),A_gu_2)+\nu(Ru,A_gu_2)\\=\ &(f,A_gu_2)+(h,A_gu_2).\end{aligned}\tag{3.13}$$

We deduce

$$
\begin{aligned}
|(B(u),A_gu_2)| &\le |(B(u_1,u_1+u_2),A_gu_2)|+|(B(u_2,u_1+u_2),A_gu_2)|\\
&\le cL^{1/2}\|u_1\||A_gu_2|(\|u_1\|+\|u_2\|)+c|u_2|^{1/2}|A_gu_2|^{3/2}(\|u_1\|+\|u_2\|)\\
&\le \frac{\nu}{4}|A_gu_2|^2+\frac{c}{\nu}\rho_1^4L+\frac{c}{\nu^3}\rho_0^2\rho_1^4,\quad t\ge t_0+1,
\end{aligned}
$$

where $|A_gu_1|^2\le\lambda_m\|u_1\|^2$ and $L=1+\log\frac{\lambda_{m+1}}{\lambda_1}$, $\|F(u)\|^2=c^2|u|^{2\beta-2}\|u\|^2\le c^2\rho_0^{2\beta-2}\rho_1^2=r_0^2$.

$$
|(Ru,-\Delta u_2)|\le\frac{|\nabla g|_\infty}{m_0}\|u\|\cdot|A_gu_2|\le\frac{|\nabla g|_\infty}{m_0}(\frac{|A_gu_2|^2}{2}+2\|u\|^2)\le\frac{|\nabla g|_\infty}{m_0}(\frac{|A_gu_2|^2}{2}+2\rho_1^2),
$$

and

$$
\begin{aligned}
(f,A_gu_2)&\le\frac{2|f|^2}{\nu}+\frac{\nu|A_gu_2|^2}{8},\\
(G(u),A_gu_2)&\le\frac{2}{\nu}\|F(u)\|^2+\frac{\nu}{8}|A_gu_2|^2\le\frac{2r_0^2}{\nu}+\frac{\nu}{8}|A_gu_2|^2,\\
(h,A_gu)&\le\frac{1}{2C_g}|h(t,u_t)|^2+\frac{C_g}{2}|A_gu|^2.
\end{aligned}
$$

From (3.13), we have

$$
\begin{aligned}
&\tfrac{d}{dt}\|u_2\|^2+2\nu|A_gu_2|^2\\
&\le 2(f,A_gu_2)+2(h,A_gu_2)-2(B(u),A_gu_2)-2(G(u),A_gu_2)-2\nu(Ru,A_gu_2)\\
&\le \tfrac{4|f|^2}{\nu}+\tfrac{\nu|A_gu_2|^2}{4}+\tfrac{1}{C_g}|h(t,u_t)|^2+C_g|A_gu|^2+\tfrac{\nu}{2}|A_gu_2|^2+\tfrac{2c}{\nu}\rho_1^4L+\tfrac{2c}{\nu^3}\rho_0^2\rho_1^4+\tfrac{2\nu|\nabla g|_\infty}{m_0}(\tfrac{|A_gu_2|^2}{2}+2\rho_1^2)\\
&= \tfrac{4|f|^2}{\nu}+\tfrac{3\nu|A_gu_2|^2}{4}+\tfrac{1}{C_g}|h(t,u_t)|^2+C_g|A_gu|^2+\tfrac{\nu|\nabla g|_\infty}{m_0}|A_gu_2|^2+\tfrac{2c}{\nu}\rho_1^4L+\tfrac{2c}{\nu^3}\rho_0^2\rho_1^4+\tfrac{4\nu|\nabla g|_\infty}{m_0}\rho_1^2.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\tfrac{d}{dt}\|u_2\|^2+\nu(\tfrac{5}{4}-\tfrac{C_g}{\nu}-\tfrac{|\nabla g|_\infty}{m_0})|A_gu_2|^2 &\le \tfrac{4|f|^2}{\nu}+\tfrac{1}{C_g}|h(t,u_t)|^2+\tfrac{2c}{\nu}\rho_1^4L+\tfrac{2c}{\nu^3}\rho_0^2\rho_1^4+\tfrac{4\nu|\nabla g|_\infty}{m_0}\rho_1^2,\\
\tfrac{d}{dt}\|u_2\|^2+\nu(\tfrac{5}{4}-\tfrac{C_g}{\nu}-\tfrac{|\nabla g|_\infty}{m_0})|A_gu_2|^2 &\le 2c(\tfrac{2}{c\nu}|(I-P_m)f|^2+\tfrac{1}{\nu}\rho_1^4L+\tfrac{1}{\nu^3}\rho_0^2\rho_1^4+\tfrac{2\nu|\nabla g|_\infty}{cm_0}\rho_1^2)+\tfrac{1}{C_g}|h(t,u_t)|^2.
\end{aligned}
$$

We set $\xi=\frac{5}{4}-\frac{C_g}{\nu}-\frac{|\nabla g|_\infty}{m_0}>0$, then

$$
\frac{d}{dt}\|u_2\|^2+\nu\lambda_{m+1}\xi\|u_2\|^2\le 2c(\frac{2}{c\nu}|(I-P_m)f|^2+\frac{1}{\nu}\rho_1^4L+\frac{1}{\nu^3}\rho_0^2\rho_1^4+\frac{2\nu|\nabla g|_\infty}{cm_0}\rho_1^2)+\frac{1}{C_g}|h(t,u_t)|^2.
$$

By Gronwall lemma, we deduce

$$
\begin{aligned}
\|u_2\|^2 \;\le\; & \|u_2(t_0+1)\|^2e^{\nu\lambda_{m+1}\xi(t_0+1-t)}+\int_{t_0+1}^t e^{\nu\lambda_{m+1}\xi(r-t)}[2c(\tfrac{1}{c\nu}|(I-P_m)f|^2+\tfrac{1}{\nu}\rho_1^4L\\
&+\tfrac{1}{\nu^3}\rho_0^2\rho_1^4+\tfrac{2|\nabla g|_\infty}{c\nu m_0}\rho_1^2)+\tfrac{1}{C_g}|h(r,u_r)|^2]dr\\
=\;& \|u_2(t_0+1)\|^2e^{\nu\lambda_{m+1}\xi(t_0+1-t)}+2c(\tfrac{1}{\nu}\rho_1^4L+\tfrac{1}{\nu^3}\rho_0^2\rho_1^4+\tfrac{2|\nabla g|_\infty}{c\nu m_0}\rho_1^2)\int_{t_0+1}^t e^{\nu\lambda_{m+1}\xi(r-t)}dr\\
&+\tfrac{2}{\nu}\int_{t_0+1}^t e^{\nu\lambda_{m+1}\xi(r-t)}|(I-P_m)f|^2dr+\tfrac{1}{C_g}\int_{t_0+1}^t e^{\nu\lambda_{m+1}\xi(r-t)}|h(r,u_r)|^2dr\\
=\;& \|u_2(t_0+1)\|^2e^{\nu\lambda_{m+1}\xi(t_0+1-t)}+\tfrac{2c}{\nu^2\lambda_{m+1}\xi}(\rho_1^4L+\tfrac{\rho_0^2\rho_1^4}{\nu^2}+\tfrac{2|\nabla g|_\infty}{cm_0}\rho_1^2)\\
&+\tfrac{2}{\nu}\int_{t_0+1}^t e^{\nu\lambda_{m+1}\xi(r-t)}|(I-P_m)f|^2dr+\tfrac{1}{C_g}\int_{t_0+1}^t e^{\nu\lambda_{m+1}\xi(r-t)}|h(r,u_r)|^2dr.
\end{aligned}
$$

From Lemma 2.1 and (IV), for any $\varepsilon > 0$, when $m+1$ sufficiently large, then,

$$\frac{2}{\nu}\int_{t_0+1}^{t} e^{\nu\lambda_{m+1}\xi(r-t)}|(I-P_m)f|^2 dr \le \frac{\varepsilon}{4},$$

$$\frac{1}{C_g}\int_{t_0+1}^{t} e^{\nu\lambda_{m+1}\xi(r-t)}|h(r,u_r)|^2 dr \le \frac{\varepsilon}{4},$$

$$\frac{2c}{\nu^2\lambda_{m+1}\xi}(\rho_1^4 L + \frac{\rho_0^2\rho_1^4}{\nu^2} + \frac{2|\nabla g|_\infty}{cm_0}\rho_1^2) \le \frac{\varepsilon}{4}.$$

Let $t_2 = t_0 + 1 + \frac{1}{\nu\lambda_{m+1}\xi \ln\frac{3\rho_1^2}{\varepsilon}}$, then $t \ge t_2$, we obtain

$$\|u_2(t_0+1)\|^2 e^{\nu\lambda_{m+1}\xi((t_0+1)-t)} \le \rho_1^2 e^{\nu\lambda_{m+1}\xi((t_0+1)-t)} \le \frac{\varepsilon}{4}.$$

Then $\forall t \ge t_2$, $\|u_2(t)\|^2 \le \varepsilon$, from Theorems 2.1 and 2.2, that is, $\{\phi(t,\gamma,x)\}$ has satisfied pullback condition in V_g, then the Eq (1.5) possesses a compact pullback attractor.

4. Conclusions

In this article, we show how to deal with the nonlinear dampness $c|u|^{\beta-1}u$ ($\beta \ge 1$) and time delay $h(t,u_t)$ to obtain the existence of pullback attractor of the 2D g-Navier-Stokes equation. The calculation process is more complicated due to nonlinear damping and time delay. When we prove the existence of pullback absorbing sets, we must suppose that $h(t,u_t)$ satisfies $|h|_b^2 = \sup_{t\in\mathbb{R}}\int_t^{t+1}|h(s,u_s)|^2 ds < \infty$, this condition is also required to hold in the process of proving asymptotic compactness, we find that the pullback absorbing sets exist in H_g when $|\nabla g|_\infty > \frac{m_0\lambda_1^{1/2}}{2}(1+\frac{C_g}{\nu\lambda_1})$, and exist in V_g when $0 < |\nabla g|_\infty < \frac{m_0\lambda_0^{1/2}}{2}(\nu - C_g)$. We prove the existence of pullback attractor by the method of pullback condition when $0 < |\nabla g|_\infty < \frac{m_0}{4\nu}(5\nu - 4g)$. The conclusions of this article are innovative and will further promote the research of 3D Navier-Stokes equations.

Obviously, it is necessary to analyze the connection between Navier-Stokes equations and g-Navier-Stokes equations. To obtain more research results for the study of g-Navier-Stokes equations in future research, we may consider that the pullback asymptotic behavior of solutions for 2D g-Navier-Stokes equations with nonlinear dampness and time delay on the unbounded domain. On the other hand, it is well-known that the invariant measures and statistical solutions have been proven to be very useful in the understanding of turbulence in the case of Navier-Stokes equations. The main reason is that the measurements of several aspects of turbulent flows are actually measurements of time-average quantities. Using the method in [21,22], we will construct a family of Borel invariant probability measures on the pullback attractor of 2D nonautonomous g-Navier-Stokes flow in a bounded domain and investigate the relationship between invariant measures and statistical solutions of this system.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (NSFC) (Grant No. 12261090).

Conflict of interest

The authors declare no any conflicts of interest.

References

1. Y. G. Sinai, K. M. Khanin, Renormalization group method in the theory of dynamical systems, *Int. J. Modern Phys. B*, **2** (1988), 147–165. https://doi.org/10.1142/S0217979288000123
2. F. Abergel, Attractor for a Navier-Stokes flow in an unbounded domain, *Math. Model. Numer. Anal.*, **23** (1989), 359–370. https://doi.org/10.1051/m2an/1989230303591
3. T. Caraballo, G. Łukaszewicz, J. Real, Pullback attractors for asymptotically compact non-autonomous dynamical systems, *Nonlinear Anal.*, **64** (2006), 484–498. https://doi.org/10.1016/j.na.2005.03.111
4. Y. J. Wang, C. K. Zhong, S. F. Zhou, Pullback attractors of nonautonomous dynamical systems, *Discrete Cont. Dyn. Syst.*, **16** (2006), 587–614. https://doi.org/10.3934/dcds.2006.16.587
5. C. Boldrighini, S. Frigio, P. Maponi, A. Pellegrinotti, Y. G. Sinai, An antisymmetric solution of the 3D incompressible Navier-Stokes equations with "Tornado-Like" behavior, *J. Exp. Theor. Phys.*, **131** (2020), 356–360. https://doi.org/10.1134/S1063776120060023
6. X. J. Cai, Q. S. Jiu, Weak and strong solutions for the incompressible Navier-Stokes equations with damping, *J. Math. Anal. Appl.*, **343** (2008), 799–809. https://doi.org/10.1016/j.jmaa.2008.01.041
7. Z. J. Zhang, X. L. Wu, M. Lu, On the uniqueness of strong solution to the incompressible Navier-Stokes equations with damping, *J. Math. Anal. Appl.*, **377** (2011), 414–419. https://doi.org/10.1016/j.jmaa.2010.11.019
8. Y. Jia, X. W. Zhang, B. Q. Dong, The asymptotic behavior of solutions to three-dimensional Navier-Stokes equations with nonlinear damping, *Nonlinear Anal. Real World Appl.*, **12** (2011), 1736–1747. https://doi.org/10.1016/j.nonrwa.2010.11.006
9. X. L. Song, F. Liang, J. H. Wu, Pullback D-attractors for the three-dimensional Navier-Stokes equations with nonlinear damping, *Bound. Value Probl.*, **2016** (2016), 1–15. https://doi.org/10.1186/s13661-016-0654-z
10. E. S. Baranovskii, M. A. Artemov, Model for aqueous polymer solutions with damping term: Solvability and vanishing relaxation limit, *Polymers*, **14** (2022), 1–17. https://doi.org/10.3390/polym14183789
11. J. Roh, *g-Navier-Stokes equations*, University of Minnesota, 2001.
12. J. Roh, Dynamics of the g-Navier-stokes equations, *J. Differ. Equ.*, **211** (2005), 452–484. https://doi.org/10.1016/j.jde.2004.08.016

13. H. O. Bae, J. Roh, Existence of solutions of the g-Navier-Stokes equations, *Taiwanese J. Math.*, **8** (2004), 85–102. https://doi.org/10.11650/twjm/1500558459

14. M. Kwak, H. Kwean, J. Roh, The dimension of attractor of the 2D g-Navier-Stokes equations, *J. Math. Anal. Appl.*, **315** (2006), 436–461. https://doi.org/10.1016/j.jmaa.2005.04.050

15. J. P. Jiang, Y. R. Hou, The global attractor of g-Navier-Stokes equations with linear dampness on R^2, *Appl. Math. Comput.*, **215** (2009), 1068–1076. https://doi.org/10.1016/j.amc.2009.06.035

16. J. P. Jiang, Y. R. Hou, Pullback attractor of 2D non-autonomous g-Navier-Stokes equations on some bounded domains, *Appl. Math. Mech.*, **31** (2010), 697–708. https://doi.org/10.1007/s10483-010-1304-x

17. D. T. Quyet, Pullback attractors for strong solutions of 2D non-autonomous g-Navier-Stokes equations, *Acta Math. Vietnam.*, **40** (2015), 637–651. https://doi.org/10.1007/s40306-014-0073-0

18. X. X. Wang, J. P. Jiang, The long-time behavior of 2D nonautonomous g-Navier-Stokes equations with weak dampness and time delay, *J. Funct. Spaces*, **2022** (2022), 1–11. https://doi.org/10.1155/2022/2034264

19. M. Kaya, A. O. Celebi, Existence of weak solutions of the g-Kelvin-Voigt equation, *Math. Comput. Model.*, **49** (2009), 497–504. https://doi.org/10.1016/j.mcm.2008.03.005

20. J. K. Hale, *Asymptotic behaviour of dissipative dynamical systems*, Providence, RI: American Mathematical Society, 1988.

21. G. Łukaszewicz, Pullback attractors and statistical solutions for 2-D Navier-Stokes equations, *Discrete Cont. Dyn. Systs. B*, **9** (2008), 643–659. https://doi.org/10.3934/dcdsb.2008.9.643

22. C. D. Zhao, L. Yang, Pullback attractors and invariant measures for the non-autonomous globally modified Navier-Stokes equations, *Commun. Math. Sci.*, **15** (2017), 1565–1580. https://doi.org/10.4310/cms.2017.v15.n6.a4

23. C. D. Zhao, T. Caraballo, G. Łukaszewicz, Statistical solution and Liouville type theorem for the Klein-Gordon-Schrödinger equations, *J. Differ. Equ.*, **281** (2021), 1–32. https://doi.org/10.1016/j.jde.2021.01.039

24. J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non-lineaires*, Paris: Dunod, 1969.

AIMS Press