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Research article

Stability of delay Hopfield neural networks with generalized proportional Riemann-Liouville fractional derivative

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Abstract: The general delay Hopfield neural network is studied. It is considered the case of time-varying delay, continuously distributed delays, time varying coefficients and a special type of a Riemann-Liouville fractional derivative (GPRLFD) with an exponential kernel. The presence of delays and GPRLFD in the model require two special types of initial conditions. The applied GPRLFD also required a special definition of the equilibrium of the model. A constant equilibrium of the model is defined. We use Razumikhin method and Lyapunov functions to study stability properties of the equilibrium of the model. We apply Lyapunov functions defined by absolute values as well as quadratic Lyapunov functions. We prove some comparison results for Lyapunov function connected deeply with the applied GPRLFD and use them to obtain exponential bounds of the solutions. These bounds are satisfied for intervals excluding the initial time. Also, the convergence of any solution of the model to the equilibrium at infinity is proved. An example illustrating the importance of our theoretical results is also included.

Keywords: Hopfield neural networks; delays; Riemann-Liouville type fractional derivative; Lyapunov functions; Razumikhin method
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1. Introduction

Stability properties of Hopfield neural networks modeled by various types of derivatives have been studied in the literature. Different behavior of the dynamics of the neurons could be adequately described by the application of various types of derivatives. For example, the fractional derivatives have typical memory properties and they could be successively applied to describe the memory of the dynamics of neurons. We can reffer to [1] for the Caputo fractional derivative, [2] for the conformable fractional derivative, [3] for the ordinary derivatives and time varying delays, [4, 5] for the Caputo fractional derivative and delays. The Riemann-Liouville type of fractional derivatives have a singularity at the initial time point and they can model some abnormalities in the behavior of the neurons. In the case when Riemann-Liouville fractional derivative is applied also several results about the study of stability are published (see, for example, [6–9] for delays, [10] for random impulses). A good review of the neural networks with applied classical fractional derivatives is given in [11].

In the last few decades, many different definitions of fractional integrals and derivatives have been proposed. One of the ways to generalize the classical ones is to use Sonine kernels (see, for example [12]). The general fractional integrals and derivatives of Riemann-Liouville type are introduced and studied by Y. Luchko [13]. Recently, in [14, 15] an integral and two types of derivatives of Caputo and of Riemann-Liouville type (GPRLFD), respectively, are introduced and they are called generalized proportional fractional integral and derivatives. The main characteristic of these derivatives is the exponential kernel in the corresponding integrals. The main disadvantage of these type of integrals and derivatives is the absence of mutually inversebility. One of their advantage is that as partial cases the classical Caputo and Riemann-Liouville fractional derivatives are obtained (for some applications of these derivatives, see for example, [16–21]). In this paper, we will apply GPRLFD to delay differential equations and set up the initial value problem for a model of neural networks with time-variable delays and distributed delays in an appropriate way.

We consider Hopfield neural networks with both-time variable delays and distributed delays, and variable in time coefficients and external inputs. The dynamics of the units is modeled by GPRLFD. The applied fractional derivative leads to some new points in the study of Hopfiled neural network. These new points are connected with the initial conditions and the definition of the equilibrium. In the paper, two types of initial conditions are set up. These initial conditions are connected with the singularity of the applied derivatives at the initial time point which coincides the lower limit of the integral. The physical meaning of these initial conditions is similar to the one of the classical Riemann-Liouville fractional derivative, which is well explained in the books [22–24]. The equilibrium of the model is defined appropriately and its stability properties are studied. This equilibrium is deeply connected with the applied derivative.

Note, there are basically two different approaches in studying stability properties of any type of differential equations with delays- the Lyapunov-Krasovskii functional approach and Razumikhin method by Lyapunov functions. The application of Lyapunov functions leads to practically easier applications of sufficient conditions.

By employing the Razumikhin method and appropriate Lyapunov functions, we obtain several upper exponential bounds of the solutions. The obtained results are valid on intervals excluding the initial time which is a singular point for the solutions. We prove the convergence of the solutions to the equilibrium at infinity. The applied derivative gives us as a partial case of the classical Riemann-Liouville fractional derivative, so all results in this paper are a generalization of the Hopfield model with the classical fractional derivative.

The innovations of this article can be described as follows:

- We propose a generalized Hopfield neural network model with both time-variable delays and distributed delays. The dynamic of the units is described by a special generalization of the Riemann-Liouville fractional derivative.
- We set up the appropriate initial conditions to the studied model.

- We define the equilibrium of the studied model. It depends significantly on the delays and the applied type of the derivative.
- We use two basic types of Lyapunov functions, the ones defined by absolute values and the ones by squares.
- We use modified Razumikhin conditions deeply connected with the presence of the exponential kernel in the applied derivative.
- We obtain two types of exponential bounds of the equilibrium of the model. These bounds are valid only for intervals excluding the initial time point which is singular of the applied derivative.
- We prove sufficient conditions for approaching any solution of the model to the equilibrium.

The structure of the rest of the paper pursues the following scheme. The basic definitions and results of derivatives are given in Section 2. Also, some additional results necessary to the proofs of the main results are provided. In Section 3, in connection with the main study of the corresponding model, we prove some stability properties of the solutions of delay differential equations with GPRLFD. We apply Lyapunov like functions and the modified Razumikhin method. We use the Lyapynov function defined by absolute values and quadratic Lyapunov function. Exponential bounds of the solutions are obtained on interval , excluding the initial time point. The generalized Riemann-Liouville delay Hopfield neural network model is introduced and the initial value problem is set up in Section 4. In Section 5 the main results are established on the basis of a Lyapunov-type analysis and Razumikhin method. Several sufficient conditions for the exponential stability properties of defined equilibrium are obtained. The convergence of any solution to the equilibrium is studied. In Section 6 some illustrative examples are elaborated. Some concluding comments are stated in the last section.

2. Some basic definitions and preliminaries

The most classical fractional integrals and derivatives as well as their basic properties are well presented in the the classical books, see, for example, [22–24]. In the last decades, many generalizations of these fractional derivatives are proposed. Some of them are equivalent to the classical ones, whereas others are generalizations. One of the ways is the exponential factor included in the kernel of the derivative and the integral and they are called tempered fractional integral and derivative (see, for example, their applications to stochastic process [25]). In [14, 15] other types of derivatives and integral have been defined by exponential kernel and named generalized proportional fractional derivatives and integrals, respectively.

Definition 2.1. *[14, 15] The generalized proportional fractional integral (GPFI) of a function* $\upsilon : [0, A] \to \mathbb{R},\ \ A \leq \infty$*, and* $\xi \in (0, 1]$*,* $q \geq 0$*, is defined by*

$$ {}_0\mathcal{I}_t^{q,\xi}\upsilon(t) = \frac{1}{\xi^q\Gamma(q)}\int_0^t e^{\frac{\xi-1}{\xi}(t-s)}(t-s)^{q-1}\,\upsilon(s)\,ds, \quad t \in (0, A]. $$

Definition 2.2. *[14, 15] The generalized proportional Riemann-Liouville fractional derivative (GPRLFD) of a function* $\upsilon : [0, A] \to \mathbb{R},\ \ A \leq \infty$*, and* $\xi \in (0, 1]$*,* $q \in (0, 1)$ *is defined by*

$$ {}_0^{RL}\mathcal{D}_t^{q,\xi}\upsilon(t) = \frac{1}{\xi^{1-q}\Gamma(1-q)}\Big((1-\xi)\int_0^t e^{\frac{\xi-1}{\xi}(t-s)}(t-s)^{-q}\,\upsilon(s)\,ds + \xi\frac{d}{dt}\int_0^t e^{\frac{\xi-1}{\xi}(t-s)}(t-s)^{-q}\,\upsilon(s)\,ds\Big), \quad t \in (0, A]. $$

Define the set of functions

$$C_{q,\xi}([0,A],\mathbb{R}^n)=\{\upsilon:[0,A]\to\mathbb{R}^n:\ \forall t\in(0,A]\ \exists\ {}_{0}^{RL}\mathcal{D}_t^{q,\xi}\upsilon(t)<\infty\}.$$

Lemma 2.1. *[26] Let the function $\upsilon\in C([0,A],\mathbb{R})$, $0<A<\infty$ be Lipshitz, and there exists a point $T\in(0,A]$ such that $\upsilon(T)=0$, and $\upsilon(t)<0$, for $0\le t<T$. Then, if the GPRLFD of υ exists for $t=T$ with $q\in(0,1)$, $\xi\in(0,1]$, then the inequality $({}_{0}^{RL}\mathcal{D}^{q,\xi}\upsilon)(t)|_{t=T}\ge 0$ holds.*

Remark 2.1. *Note the similar result to the one in Lemma 2.1 is proved in [27] for Riemann-Liouville fractional derivatives.*

Lemma 2.2. *[26] Let the function $\nu\in C_{q,\xi}([0,A],\mathbb{R})$, $0<A\le\infty$, and $\nu^2\in C_{q,\xi}([0,A],\mathbb{R})$. Then the inequality ${}_{0}^{RL}\mathcal{D}_t^{q,\xi}\nu^2(t)\le 2\nu(t){}_{0}^{RL}\mathcal{D}_t^{q,\xi}\nu(t)$ for $t\in(0,A]$ holds.*

Similar result is true for the multi-valued functions.

Lemma 2.3. *Let the function $\nu\in C_{q,\xi}([0,A],\mathbb{R}^n):\ \nu=(\nu_1,\nu_2,\dots,\nu_n)$, $0<A\le\infty$, and $\nu_k^2\in C_{q,\xi}([0,A],\mathbb{R})$, $k=1,2,\dots,n$. Then the inequality ${}_{0}^{RL}\mathcal{D}_t^{q,\xi}\sum_{k=1}^{n}\nu_k^2(t)\le 2\sum_{k=1}\nu_k(t){}_{0}^{RL}\mathcal{D}_t^{q,\xi}\nu_k(t)$ for $t\in(0,A]$ holds.*

For the scalar functions we have the following result.

Lemma 2.4. *Let $\nu\in C_{q,\xi}([0,A],\mathbb{R})$, $0<A\le\infty$ be such that for any finite interval $[a,b]\subset[0,A]$ the equation $\nu(t)=0$ has only finite number of solutions.*

Then for any $T\in(0,A]$ such that $\nu(T)\neq 0$ the inequality

$$ {}_{0}^{RL}\mathcal{D}_t^{q,\xi}|\nu(t)|\Big|_{t=T}\le sign(\nu(T))\,{}_{0}^{RL}\mathcal{D}_t^{q,\xi}\nu(t)\Big|_{t=T} \tag{2.1}$$

holds.

Proof. Fix an arbitrary point $T\in(0,A]:\ \nu(T)\neq 0$.Without loss of generality we will assume $\nu(T)>0$ (otherwise consider the function $(-\nu(s)),\ s\in[0,A]$). Then inequality (2.1) is equivalent to

$$ {}_{0}^{RL}\mathcal{D}_t^{q,\xi}|\nu(t)|\Big|_{t=T}-{}_{0}^{RL}\mathcal{D}_t^{q,\xi}\nu(t)\Big|_{t=T}\le 0. \tag{2.2}$$

Consider the cases:

Case 1. Let $\nu(\sigma)\ge 0,\ \sigma\in[0,T]$ and $\nu(T)>0$. Then

$$\begin{aligned}{}_{0}^{RL}\mathcal{D}_t^{q,\xi}|\nu(t)|\Big|_{t=T}&=\frac{1}{\xi^{1-q}\Gamma(1-q)}\Big((1-\xi)\int_0^T e^{\frac{\xi-1}{\xi}(T-\sigma)}\,(T-\sigma)^{-q}\,\nu(\sigma)\,d\sigma\\&\quad+\xi\frac{d}{dT}\int_0^T e^{\frac{\xi-1}{\xi}(T-\sigma)}\,(T-\sigma)^{-q}\,\nu(\sigma)\,d\sigma\Big)\\&=sign(\nu(T))\,{}_{0}^{RL}\mathcal{D}_t^{q,\xi}\nu(t)\Big|_{t=T}.\end{aligned}$$

Case 2. Let there exist points $t_k\in(0,T):\ 0<t_1<t_2<\dots t_m<T$ such that $\nu(s)\le 0$ for $s\in(t_{2k},t_{2k+1}),k=0,1,2,\dots,p$, $\nu(s)\ge 0$ for $s\in(t_{2k+1},t_{2k+2}),k=0,1,\dots,p$, $\nu(t_k)=0,k=1,2,\dots,m$

and $v(T) > 0$. Here we assume without loss of generality that $m = 2p + 1$ and $t_0 = 0, t_{2p+2} = T$. Then

$$\begin{aligned}
{}_0^{RL}\mathcal{D}_t^{q,\xi}|v(t)|\Big|_{t=T} &= \frac{1}{\xi^{1-q}\Gamma(1-q)}\Big\{-\sum_{k=0}^{p}\Big((1-\xi)\int_{t_{2k}}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}(T-\sigma)^{-q}v(\sigma)\,d\sigma \\
&\quad + \xi\frac{d}{dT}\int_{t_{2k}}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}(T-\sigma)^{-q}v(\sigma)\,d\sigma\Big) \\
&\quad + \sum_{k=0}^{p}\Big((1-\xi)\int_{t_{2k+1}}^{t_{2k+2}} e^{\frac{\xi-1}{\xi}(t-s)}(T-\sigma)^{-q}v(\sigma)\,d\sigma \\
&\quad + \frac{d}{dT}\int_{t_{2k+1}}^{t_{2k+2}} e^{\frac{\xi-1}{\xi}(T-\sigma)}(T-\sigma)^{-q}v(\sigma)\,d\sigma\Big)\Big\} \\
&= {}_0^{RL}\mathcal{D}_t^{q,\xi}v(t)\Big|_{t=T} - \frac{2}{\xi^{1-q}\Gamma(1-q)}\sum_{k=0}^{p}\Big((1-\xi)\int_{t_{2k}}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}(t-s)^{-q}v(\sigma)\,d\sigma \\
&\quad + \xi\frac{d}{dT}\int_{t_{2k}}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}(T-\sigma)^{-q}v(\sigma)\,d\sigma\Big).
\end{aligned} \tag{2.3}$$

Applying that $v(s) \le 0,\ s \in (t_{2k}, t_{2k+1})$ we have

$$\begin{aligned}
&(1-\xi)\int_{t_{2k}}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}(T-\sigma)^{-q}v(\sigma)\,d\sigma + \xi\frac{d}{dT}\int_{t_{2k}}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}(T-\sigma)^{-q}v(\sigma)\,d\sigma \\
&= (1-\xi)\int_{t_{2k}}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}(T-\sigma)^{-q}v(\sigma)\,d\sigma \\
&\quad + \xi\int_{t_{2k}}^{t_{2k+1}} \frac{\xi-1}{\xi}e^{\frac{\xi-1}{\xi}(T-\sigma)}(T-\sigma)^{-q}v(s)\,ds - q\xi\int_{2k}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}(T-\sigma)^{-q-1}v(\sigma)\,d\sigma \\
&= -q\xi\int_{2k}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}(T-\sigma)^{-q-1}v(\sigma)\,d\sigma \ge 0.
\end{aligned} \tag{2.4}$$

From (2.3) and (2.4) it follows

$$ {}_0^{RL}\mathcal{D}_t^{q,\xi}|v(t)|\Big|_{t=T} \le {}_0^{RL}\mathcal{D}_t^{q,\xi}v(t)\Big|_{t=T} = sign\ v(T){}_0^{RL}\mathcal{D}_t^{q,\xi}v(t)\Big|_{t=T}.$$

Case 3. Let there exists points $t_k \in (0, T) : 0 < t_1 < t_2 < \dots t_m < T$ such that $v(s) \ge 0$ for $s \in (t_{2k}, t_{2k+1}), k = 0, 1, 2, \dots, p$, $v(s) \le 0$ for $s \in (t_{2k+1}, t_{2k+2}), k = 0, 1, \dots, p-1$, $v(t_k) = 0, k = 1, 2, \dots, m$

and $v(T) > 0$. Here we assume without loss of generality that $m = 2p$ and $t_0 = 0, t_{2p+1} = T$. Then

$$
\begin{aligned}
{}_0^{RL}\mathcal{D}_t^{q,\xi}|v(t)|\Big|_{t=T} &= \frac{1}{\xi^{1-q}\Gamma(1-q)}\Big\{ -\sum_{k=0}^{p}\Big((1-\xi)\int_{t_{2k}}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}\,(T-\sigma)^{-q}\,v(\sigma)\,d\sigma \\
&\quad + \xi\frac{d}{dT}\int_{t_{2k}}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}\,(T-\sigma)^{-q}\,v(\sigma)\,d\sigma\Big) \\
&\quad + \sum_{k=0}^{p-1}\Big((1-\xi)\int_{t_{2k+1}}^{t_{2k+2}} e^{\frac{\xi-1}{\xi}(T-\sigma)}\,(T-\sigma)^{-q}\,v(\sigma)\,d\sigma \\
&\quad + \frac{d}{dT}\int_{t_{2k+1}}^{t_{2k+2}} e^{\frac{\xi-1}{\xi}(T-\sigma)}\,(T-\sigma)^{-q}\,v(\sigma)\,d\sigma\Big)\Big\} \\
&= {}_0^{RL}\mathcal{D}_t^{q,\xi}v(t)\Big|_{t=T} - \frac{2}{\xi^{1-q}\Gamma(1-q)}\sum_{k=0}^{p}\Big((1-\xi)\int_{t_{2k}}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}\,(T-\sigma)^{-q}\,v(\sigma)\,d\sigma \\
&\quad + \xi\frac{d}{dT}\int_{t_{2k}}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}\,(T-\sigma)^{-q}\,v(\sigma)\,d\sigma\Big).
\end{aligned}
\tag{2.5}
$$

Applying that $v(s) \geq 0,\ s \in (t_{2k}, t_{2k+1})$ we have

$$
\begin{aligned}
&(1-\xi)\int_{t_{2k}}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}\,(T-\sigma)^{-q}\,v(\sigma)\,d\sigma + \xi\frac{d}{dT}\int_{t_{2k}}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}\,(T-\sigma)^{-q}\,v(\sigma)\,d\sigma \\
&= (1-\xi)\int_{t_{2k}}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}\,(T-\sigma)^{-q}\,v(\sigma)\,d\sigma \\
&\quad + \xi\int_{t_{2k}}^{t_{2k+1}} \frac{\xi-1}{\xi}e^{\frac{\xi-1}{\xi}(T-\sigma)}\,(T-\sigma)^{-q}\,v(\sigma)\,d\sigma - q\xi\int_{2k}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}\,(T-\sigma)^{-q-1}\,v(\sigma)\,d\sigma \\
&= -q\xi\int_{2k}^{t_{2k+1}} e^{\frac{\xi-1}{\xi}(T-\sigma)}\,(T-\sigma)^{-q-1}\,v(\sigma)\,d\sigma \geq 0,\ k = 0, 1, \ldots, p.
\end{aligned}
\tag{2.6}
$$

From (2.5) and (2.6) it follows

$$
{}_0^{RL}\mathcal{D}_t^{q,\xi}|v(t)|\Big|_{t=T} \leq {}_0^{RL}\mathcal{D}_t^{q,\xi}v(t)\Big|_{t=T} = sign\ v(t){}_0^{RL}\mathcal{D}_t^{q,\xi}v(T)\Big|_{t=T}.
$$

□

Remark 2.2. *From Lemma 2.4 it follows that if* $v \in C_{q,\xi}([0, A], \mathbb{R}))$ *then* $|v| \in C_{q,\xi}([0.A], \mathbb{R}))$.

For the multivalued functions we have the following result.

Lemma 2.5. *Let* $v \in C_{q,\xi}([0, A], \mathbb{R}^n)\ :\ v = (v_1, v_2, \ldots, v_n)$*,* $0 < A \leq \infty$ *be such that for any finite interval* $[a, b] \subset [0, A]$ *the equations* $v_k(t) = 0$ *has only finite number of solutions* $(k = 1, 2, \ldots, n)$.

Then for any $T \in (0, A]$ *such that* $v(T) \neq 0$ *the inequality*

$$
{}_0^{RL}\mathcal{D}_t^{q,\xi}\sum_{k=1}^{n}|v_k(t)|\Big|_{t=T} \leq \sum_{k=1}^{n} sign(v_k(T))\,{}_0^{RL}\mathcal{D}_t^{q,\xi}v_k(t)\Big|_{t=T}
\tag{2.7}
$$

holds.

Remark 2.3. *If $\xi = 1$ then Definitions 2.1 and 2.2 are reduced to the classical Riemann-Liouville fractional integral (RLFI)*

$$ {}_0I_t^q \upsilon(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\sigma)^{q-1} \upsilon(\sigma)\, d\sigma, \quad t \in (0, A], \tag{2.8}$$

and Riemann-Liouville fractional derivative (RLFD)

$$ {}_0^{RL}D_t^q \upsilon(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-\sigma)^{-q} \upsilon(\sigma)\, d\sigma, \quad t \in (0, A]. \tag{2.9}$$

In connection with the initial conditions of the differential equations with GPRLFD we will use the following result.

Lemma 2.6. *(Lemma 2 [16]) Let $\xi \in (0,1]$, $q \in (0,1)$ and $y \in C([0,b], \mathbb{R})$.*

(i) There exist a limit $\lim_{t\to 0+}\left(e^{\frac{1-\xi}{\xi}t} t^{1-q} y(t)\right) = c < \infty$. Then ${}_0\mathcal{I}_t^{1-q,\xi} y(t)|_{t=0+} = c\frac{\Gamma(q)}{\xi^{1-q}}$;

(ii) Let ${}_0\mathcal{I}_t^{1-q,\xi} y(t)|_{t=0+} = b < \infty$. If there exists the limit $lim_{t\to 0+}\left(e^{\frac{1-\xi}{\xi}t} t^{1-q} y(t,x)\right)$, then $\lim_{t\to 0+}\left(e^{\frac{1-\xi}{\xi}t} t^{1-q} y(t)\right) = \frac{b\xi^{1-q}}{\Gamma(q)}$.

Proposition 2.1. *[14] For $\xi \in (0,1]$, $q \in (0,1)$ we have*

$$\begin{aligned} &{}_0^{RL}\mathcal{D}_t^{q,\xi}\left(e^{\frac{\xi-1}{\xi}t} t^{q-1}\right) = 0, \quad t > 0, \\ &{}_0^{RL}\mathcal{D}_t^{q,\xi}\left(e^{\frac{\xi-1}{\xi}t}\right) = \frac{1}{\xi^q \Gamma(1-q)} e^{\frac{\xi-1}{\xi}t} t^{-q}, \quad t > 0. \end{aligned} \tag{2.10}$$

3. Some results for delay differential equations with GPRLFD

Consider the delay nonlinear differential equation with GPRLFD

$$ {}_0^{RL}\mathcal{D}_t^{q,\xi} y(t) = f(t, y_t) \ \text{ for } t > 0, \tag{3.1}$$

with initial conditions

$$\begin{aligned} &y(t) = \phi(t), \ \text{ for } t \in [-\tau, 0), \\ &\lim_{t\to 0+}\left[e^{\frac{1-\xi}{\xi}t} t^{1-q} y(t)\right] = \frac{\phi(0)\xi^{1-q}}{\Gamma(q)}, \end{aligned} \tag{3.2}$$

where $q \in (0,1)$, $\xi \in (0,1]$, $y_t(\sigma) = y(t+\sigma)$ for $\sigma \in [-\tau, 0]$, the initial function $\phi : [-\tau, 0] \to \mathbb{R}^n$ and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$.

Remark 3.1. *The initial condition* (3.2) *is equivalent to (see Lemma 2.1)*

$$\begin{aligned} &y(t) = \phi(t), \ \textit{for } t \in [-\tau, 0), \\ &{}_a\mathcal{I}_t^{1-q,\xi} y(t)|_{t=0+} = \phi(0). \end{aligned} \tag{3.3}$$

We will assume that the initial value problems (3.1), (3.2), or its equivalent (3.1), (3.3), has a solution $y(t;\phi) \in C_{q,\xi}([0,\infty), \mathbb{R}^n)$ for any initial function $\phi \in C([-\tau, 0], \mathbb{R}^n)$.

Remark 3.2. *For any vector* $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ *we will use the norm* $||\xi||$. *It could be* $||\xi||_1 = \sum_{i=1}^n |\xi_i|$ *or* $||\xi||_2 = \sqrt{\sum_{i=1}^n \xi_i^2}$.

We will use the norm in $C([-\tau, 0], \mathbb{R}^n)$, defined by $||\phi||_0 = \max_{t\in[-\tau,0]} ||\phi(t)||$ where $||.||$ is a norm in $\mathbb{R}^n$.

Definition 3.1. *Let* $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$. *We will say that the function* $V(x) : \Delta \to \mathbb{R}_+$ *is from the class* $\Lambda(\Delta)$ *if it is continuous in* Δ *and locally Lipschitzian.*

Theorem 3.1. *Let there exists a function* $V \in \Lambda(\mathbb{R}^n)$ *such that*

(i) there exists a function $a \in \mathcal{K}$ *such that* $a(||\xi||) \leq V(\xi)$ *for* $\xi \in \mathbb{R}^n$;

(ii) For any solution $y \in C_{q,\xi}([0, \infty), \mathbb{R}^n)$ *of* (3.1),(3.2) *the following conditions are satisfied:*

- *there exist an increasing function* $g \in C([0, \infty), \mathbb{R}) :\ g(0) = 0$ *such that the inequality*

$$e^{\frac{1-\xi}{\xi}t}t^{1-q}V(y(t))|_{t=0+} = \lim_{t\to 0+} e^{\frac{1-\xi}{\xi}t}t^{1-q}V(y(t)) \leq g(||\phi(0)||)$$

holds;

- *for any point* $T > 0$ *such that*

$$e^{\frac{1-\xi}{\xi}(T+\sigma)}(T+\sigma)^{1-q}V(y(T+\sigma)) < e^{\frac{1-\xi}{\xi}t}T^{1-q}V(y(T)) \quad for \ \ \sigma \in (-\min\{T, \tau\}, 0),$$

the GPRLFD ${}_0^{RL}\mathcal{D}_t^{q,\xi}V(y(t))\Big|_{t=T}$ *exists and the inequality*

$${}_0^{RL}\mathcal{D}_t^{q,\xi}V(y(t))\Big|_{t=T} < 0$$

holds.

Then, there exists a point $T_q > 0$ *such that for any solution of (3.1), (3.2) the inequality*

$$||x(t)|| < a^{-1}\left(g(||\phi||_0)e^{\frac{\xi-1}{\xi}t}\right) \ \ for\ t > T_q$$

holds.

Proof. Let $x(t)$ be a solution of (3.1), (3.2) with the initial function $\phi \in C([-\tau, 0], \mathbb{R}^n)$.

From condition (ii) we get

$$e^{\frac{1-\xi}{\xi}\sigma}\sigma^{1-q}V(x(\sigma))|_{\sigma=0+} = \lim_{\sigma\to 0+} e^{\frac{1-\xi}{\xi}\sigma}\sigma^{1-q}V(x(\sigma)) \leq g\left(||\phi(0)||\right) \leq g(||\phi||_0). \tag{3.4}$$

Therefore, there exists a number $\delta > 0$ such that

$$V(x(t)) \leq g\left(||\phi||_0\right) < e^{\frac{\xi-1}{\xi}t}t^{q-1}g(||\phi||_0), \ \ \text{for } t \in (0, \delta). \tag{3.5}$$

Define the function $H(t) = g(||\phi||_0)e^{\frac{\xi-1}{\xi}t}t^{q-1} \in C_{q,\xi}([0, \infty), \mathbb{R}_+)$ and $\lim_{t\to\infty} H(t) = 0$. Then, there exists $T_\xi > 0$ such that $t^{q-1} < 1$ for $t > T_\xi$ and thus

$$H(t) < g(||\phi||_0)e^{\frac{\xi-1}{\xi}t} \ \ \text{for } t > T_\xi. \tag{3.6}$$

Consider the function $m(t) = V(x(t)) \in C_{q,\xi}([0,\infty),\mathbb{R}_+)$.
We will prove that

$$m(t) < H(t), \quad t > 0. \tag{3.7}$$

Assume inequality (3.7) is not true for all $t > 0$. Then, there exists a point $\eta \geq \delta > 0$ such that

$$m(\eta) = H(\eta), \text{ and } m(t) < H(t), \ t \in (0,\eta). \tag{3.8}$$

Thus, $m(t) - H(t) \in C_{q,\xi}([0,\eta],\mathbb{R})$. According to Lemma 2.1 with $T = \eta, \nu(t) \equiv m(t) - H(t)$, the inequality ${}_0^{RL}\mathcal{D}_t^{q,\xi}\Big(m(t) - H(t)\Big)|_{t=\eta} \geq 0$ holds. From Proposition 2.1, we get ${}_0^{RL}\mathcal{D}^{q,\xi}(e^{\frac{\xi-1}{\xi}t}t^{q-1}) = 0$ and therefore,

$${}_0^{RL}\mathcal{D}_t^{q,\xi}m(t)|_{t=\eta} = {}_0^{RL}\mathcal{D}_t^{q,\xi}\Big(m(t) - H(t)\Big)|_{t=\eta} \geq 0. \tag{3.9}$$

Case 1. Let $\eta > \tau$. Then, $\min\{\eta,\tau\} = \tau$. From (3.8), it follows that

$$e^{\frac{1-\xi}{\xi}t}t^{1-q}m(t) < \varepsilon = e^{\frac{1-\xi}{\xi}\eta}\eta^{1-q}m(\eta), \quad t \in (0,\eta)$$

or

$$e^{\frac{1-\xi}{\xi}(t+\sigma)}(\eta+\sigma)^{1-q}V(\xi+\sigma, x(\eta+\sigma)) < e^{\frac{1-\xi}{\xi}\eta}\eta^{1-q}V(\eta, x(\eta)), \quad \sigma \in (-\tau, 0).$$

According to condition 2(iii) the inequality

$${}_0^{RL}\mathcal{D}_t^{q,\xi}V(t,x(t))|_{t=\eta} < 0 \tag{3.10}$$

holds.

The inequality (3.10) contradicts (3.9).

Case 2. Let $\eta \leq \tau$. Then, $\min\{\eta,\tau\} = \xi$. From (3.8), it follows that

$$e^{\frac{1-\xi}{\xi}\eta}\eta^{1-q}m(\eta) = \varepsilon > e^{\frac{1-\xi}{\xi}t}t^{1-q}m(t), \quad t \in (0,\eta),$$

or

$$e^{\frac{1-\xi}{\xi}(\eta+\sigma)}(\eta+\sigma)^{1-q}V(\eta+\sigma, x(\eta+\sigma)) < e^{\frac{1-\xi}{\xi}\eta}\eta^{1-q}m(\eta) = e^{\frac{1-\xi}{\xi}\eta}\eta^{1-q}V(\eta, x(\eta))$$

for $\sigma \in (-\xi, 0)$. Similar to Case 1 we obtain a contradiction.

From inequalities (3.6), (3.7) and condition (i), it follows that

$$a(\|x(t)\|) \leq V(x(t)) = m(t) < H(t) < g(\|\phi\|_0)e^{\frac{\xi-1}{\xi}t} \text{ for } t > T_q. \tag{3.11}$$

It proves the claim of Theorem 3.1. □

Using that for any fixed $T \geq 0$ the function $Q(\sigma) = e^{\frac{1-\xi}{\xi}(T+\sigma)}(T+\sigma)^{1-q}$ is an increasing function for $\sigma \in (-\min\{T,\tau\}, 0)$ we obtain the following result.

Corollary 3.1. *Let the condition(i) of Theorem 3.1 and*

(ii) For any solution $y \in C_{q,\xi}([0,\infty),\mathbb{R}^n)$ of (3.1),(3.2) the following conditions are satisfied:*

- *there exist an increasing function $g \in C([0,\infty),\mathbb{R})$: $g(0) = 0$ such that the inequality*

$$e^{\frac{1-\xi}{\xi}t}t^{1-q}V(y(t))|_{t=0+} = \lim_{t\to 0+} e^{\frac{1-\xi}{\xi}t}t^{1-q}V(y(t)) \leq g(\|\phi(0)\|)$$

holds;

- *for any point* $T > 0$ *such that*

$$V(y(T+\sigma)) < V(y(T)) \quad \textit{for} \quad \sigma \in (-\min\{T,\tau\}, 0),$$

the GPRLFD ${}_0^{RL}\mathcal{D}_t^{q,\xi} V(y(t))\Big|_{t=T}$ *exists and the inequality*

$${}_0^{RL}\mathcal{D}_t^{q,\xi} V(y(t))\Big|_{t=T} < 0$$

holds.

Then any solution of (3.1), (3.2) satisfies the inequality

$$\|x(t)\| < a^{-1}\left(g(\|\phi\|_0)e^{\frac{\xi-1}{\xi}t}\right) \quad \textit{for } t > T_q.$$

In the case the Lyapunov function is defined by absolute values, we obtain the following result:

Corollary 3.2. *Let for any solution* $y \in C_{q,\xi}([0,\infty),\mathbb{R}^n)$, $y = (y_1, y_2, \ldots, y_n)$, *of* (3.1), (3.2) *and for any point* $T > 0$ *such that*

$$|y_i(T+\sigma)| < |y_i(T)|, \quad \textit{for } \sigma \in (-\min\{T,\tau\}, 0), \; i = 1, 2, \ldots, n, \tag{3.12}$$

the inequality

$${}_0^{RL}\mathcal{D}_t^{q,\xi} \sum_{i=1}^{n} |y_i(t)|\Big|_{t=T} < 0 \tag{3.13}$$

holds.

Then, there exists a point $T_q > 0$ *such that*

$$\sum_{i=1}^{n} |y_i(t)| < \frac{\sum_{i=1}^{n} \max_{t\in[\tau,0]} |\phi_i(t)| \, \xi^{1-q}}{\Gamma(q)} e^{\frac{\xi-1}{\xi}t} \; \textit{for } t > T_q.$$

Proof. For the point T such that the inequality (3.12) hold it is fulfilled $y(T) \neq 0$ and we could apply Lemma 2.4. The proof follows from Theorem 3.1 by taking the Lyapunov function $V(x) = \sum_{i=1}^{n} |x_i|$, using the norm $\|\cdot\|_1$ and applying $\lim_{t\to 0+} e^{\frac{1-\xi}{\xi}t} t^{1-q}|y_i(t)| = \frac{|\phi_i(0)|\xi^{1-q}}{\Gamma(q)}$, i.e., $g(u) = \frac{u\,\xi^{1-q}}{\Gamma(q)}$, $a(u) \equiv u$. □

In the case the Lyapunov function is a quadratic one, we obtain the following result.

Corollary 3.3. *Let for any solution* $y \in C_{q,\xi}([0,\infty),\mathbb{R}^n)$ *of* (3.1), (3.2), $y^T y \in C_{q,\xi}([0,\infty),\mathbb{R}^n)$ *holds and*

- *there exist an increasing function* $g \in C([0,\infty),\mathbb{R})$: $g(0) = 0$ *such that the inequality*

$$\lim_{t\to 0+} e^{\frac{1-\xi}{\xi}t} t^{1-q} y^T(t)y(t) \le g(\|\phi(0)\|_2)$$

holds;

- *for any point* $\mathcal{T} > 0$ *such that*

$$y^T(\mathcal{T}+\sigma)y(\mathcal{T}+\sigma) < y^T(\mathcal{T})y(\mathcal{T}) \quad \textit{for} \quad \sigma \in (-\min\{\mathcal{T},\tau\}, 0), \tag{3.14}$$

the GPRLFD ${}_0^{RL}\mathcal{D}_t^{q,\xi} y^T(t)y(t)\Big|_{t=\mathcal{T}}$ *exists and the inequality*

$${}_0^{RL}\mathcal{D}_t^{q,\xi} y^T(t)y(t)\Big|_{t=\mathcal{T}} < 0$$

holds.

Then, there exists $T_q > 0$ *such that*

$$\|y(t)\|_2 < \sqrt{g\left(\max_{t\in[-\tau,0]} \|\phi\|_2\right) e^{\frac{\xi-1}{2\xi}t}} \quad \textit{for } t > T_q.$$

The proof follows from Theorem 3.1 by taking the Lyapunov function $V(x) = \sum_{i=1}^n x_i^2$ and using the norm $\|\cdot\|_2$, i.e., $a(u) \equiv u^2$.

4. Model description of neural networks by GPRLFD

We will consider the general model of Hopfield neural network with the GPRLFD and time-varying delays and distributed delays

$$\begin{aligned} {}_0^{RL}\mathcal{D}_t^{q,\xi} u_i(t) = &-A_i(t)u_i(t) + \sum_{k=1}^n a_{i,k}(t) f_k(u_k(t)) + \sum_{k=1}^n b_{i,k}(t) g_k(u_k(t-\kappa(t))) \\ &+ \sum_{k=1}^n c_{i,k}(t) \int_{t-\Theta(t)}^t h_k(u_k(s))ds + I_i(t), \quad t > 0, \quad i = 1, 2, \dots, n, \end{aligned} \tag{4.1}$$

where $u_i(t), i = 1, 2, \dots, n$ are the state variables of i-th neuron at time $t > 0$, $a_{ij}(t), b_{ij}(t), c_{ij}(t)$ represent the strengths of the neuron interconnection at time t (assuming they are time changeable), n is the number of units in the neural network, $q \in (0, 1)$, $\xi \in (0, 1]$, $f_j(u)$, $g_j(u)$ and $h_j(u)$ are the activation functions of the j-th neuron, $\kappa(t)$ is the time varying delay and $\Theta(t)$ is the length of interval of the distributed delay with $0 \le \kappa(t) \le \kappa$, $0 \le \Theta(t) \le \Theta$ and $\tau = \max\{\kappa, \Theta\}$, $I_i(t)$ are the external inputs at time t.

The initial time interval is $[-\tau, 0]$. The applied GPRLFD leads to a singularity of the solutions at the initial time 0. It requires appropriate definition of the initial conditions. Some authors (see for example [28] for RLFD) used the integral of the type ${}_0\mathcal{I}_t^{1-q}u(t)$ for $t \in [-\tau, 0]$ in the initial condition but according to Eq (2.8) the RLFI is defined for t greater than the lower limit, which is 0 in our case.

We will use the initial conditions (3.2) or their equivalent (3.3).

We use the assumptions:

A1. The function $A_i \in C(\mathbb{R}, [\mu_i, \infty))$ where μ_i, $i = 1, 2, \dots, n$, are positive constants.

A2. The activation functions $f_i, g_i, h_i \in C(\mathbb{R}, \mathbb{R})$ are Lipschitz with constants $\gamma_i, \beta_i, \delta_i$, $i = 1, 2, \dots, n$ respectively, i.e.,

$$|f_i(x) - f_i(y)| \le \gamma_i |x - y|, \quad |g_i(x) - g_i(y)| \le \beta_i |x - y|, \quad |h_i(x) - h_i(y)| \le \delta_i |x - y|, \quad x, y \in \mathbb{R}.$$

A3. The functions $a_{ij}, b_{ij}, c_{ij} \in C([0, \infty), \mathbb{R})$, $i, j = 1, 2, \dots, n$.

4.1. Equilibrium of the model

We define the equilibrium as a constant vector.

We also need to use the equality (obtained by CAS Wolfram Mathematica 13.2)

$$({}_a\mathcal{D}^{q,\xi}1)(t) = \xi^q e^{\frac{\xi-1}{\xi}t} \left({}_a^{RL}D_t^q \left(e^{\frac{1-\xi}{\xi}t}\right)\right) = (1-\xi)^q \left(1 - \frac{\Gamma(-q, \frac{1-\xi}{\xi}t)}{\Gamma(-q)}\right), \tag{4.2}$$

where $\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt$ is the upper incomplete gamma function. It is clear that $\lim_{t\to 0}\frac{\Gamma(-q,\frac{1-\xi}{\xi}t)}{\Gamma(-q)}=\infty$ and $\lim_{t\to\infty}\frac{\Gamma(-q,\frac{1-\xi}{\xi}t)}{\Gamma(-q)}=0$ for $q\in(0,1)$ and $\xi\in(0,1]$.

Definition 4.1. *The constant vector $V^*=(C_1,C_2,\dots,C_n)$ is called an equilibrium of* (4.1) *if the equalities*

$$\begin{aligned}C_i(1-\xi)^q\left(1-\frac{\Gamma(-q,\frac{1-\xi}{\xi}t)}{\Gamma(-q)}\right)&=-A_i(t)C_i+\sum_{k=1}^n\Big(a_{i,k}(t)f_k(C_k)+b_{i,k}(t)g_k(C_k)\\&\quad+c_{i,k}(t)\Theta(t)h_k(C_k)\Big)+I_i(t),\ \textit{for}\ \ t\geq 0,\ \ i=1,2\dots,n\end{aligned}\tag{4.3}$$

hold.

Remark 4.1. *If $f_i(0)=g_i(0)=h_i(0)=0,\ i=1,2,\dots,n$ and there is no external input, then the model* (4.1) *has a zero equilibrium.*

Remark 4.2. *In most cases in the literature the Hopfield neural networks are studied with constant coefficients a_{ik},b_{ik},c_{ik} and constant external inputs I_i. In the case when RLFD or GPRLFD is applied and all coefficients are constants, the only equilibrium is zero. It is not the case when at least one of the coefficients is variable in time.*

Let V^* be an equilibrium of (4.1). Change the variables $v_i(t)=u_i(t)-C_i,\ t\geq 0$, in system (4.1). Then applying (4.2) and (4.3) we get

$$\begin{aligned}{}_0^{RL}\mathcal{D}_t^{q,\xi}v_i(t)&=-A_i(t)v_i(t)+\sum_{k=1}^n a_{i,k}(t)F_k(v_k(t))+\sum_{k=1}^n b_{i,k}(t)G_k(v_k(t-\xi(t)))\\&\quad+\sum_{k=1}^n c_{i,k}(t)\int_{t-\Theta(t)}^t H_k(v_k(s))ds,\quad t>0,\quad i=1,2,\dots,n,\end{aligned}\tag{4.4}$$

with initial conditions (3.2) or their equivalent (3.3), where

$$F_i(x)=f_i(x+C_i)-f_i(C_i),\ G_i(x)=g_i(x+C_i)-g_i(C_i),,\ H_i(x)=h_i(x+C_i)-h_i(C_i).$$

Note the system (4.4) has a zero solution (with zero initial function).

Remark 4.3. *If assumption A2 is satisfied then*

$$|F_i(x)|\leq\gamma_i|x|,\quad |G_i(x)|\leq\beta_i|x|,\quad |H_i(x)|\leq\delta_i|x|,\quad x,y\in\mathbb{R}.$$

5. Stability of the model by Lyapunov functions and Razumikhin method

Note that the singularity of the solutions of the differential equations with GPRLFD and RLFD at the initial time requires this point to be excluded and to be obtained some stability properties on an interval without the initial time. It is totally different than the case of Caputo type fractional derivative or derivative with an integer order. Some authors apply RLFD but do not exclude the initial time and fact that for order $\gamma\in(0,1)$. The expressions $t^{-\gamma}$ and $t^{\gamma-1}$ are not bounded for points enough close to the initial time 0 (see, for example, [6,29,30]). The main concepts of stability for differential equations with RLFD are discussed and studied in [31].

In connection with the applied GPRLFD we need to define the exponential stability of the equilibrium on an interval excluding the initial time 0.

Definition 5.1. *The constant equilibrium V^* of* (4.1) *is called* exponentially stable in time *if there exists a point $T > 0$, a constant $\lambda > 0$ and an increasing function $\Xi \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that any solution $y(t)$ of* (4.1), (3.2) *satisfies*

$$\|y(t) - V^*\| \leq \Xi\left(\|\phi\|_0\right) e^{\lambda \frac{\xi-1}{\xi} t}, \ t \geq T.$$

5.1. Lyapunov functions defined by absolute values

Theorem 5.1. *Let the assumptions A1–A3 be satisfied and:*

(1) The fractional order neural networks (4.1) *has an equilibrium $V^* = (C_1, C_2, \dots, C_n)$.*
(2) Any solution $u \in C_{q,\xi}([0,\infty), \mathbb{R}^n)$, $u = (u_1, u_2, \dots, u_n)$, of (4.1), (3.2) *is such that for any finite interval $[a, b] \subset [0, \infty)$ and any $k = 1, 2, \dots, n$ the equation $u_k(t) = 0$ has a finite number of solutions (eventually zero).*
(3) For all $i = 1, 2, \dots, n$ and $t \geq 0$ the inequalities

$$\sum_{k=1}^{n} \left(|a_{k,i}(t)|\gamma_i + \beta \max_{i=1,2\dots,n} |b_{k,i}(t)| + \tau\delta \max_{i=1,2,\dots,n} |c_{k,i}(t)| \right) < \mu_i$$

hold where $\beta = \max_{i=1,2,\dots,n} \beta_i$ and $\delta = \max_{i=1,2,\dots,n} \delta_i$.

Then, the equilibrium V^* is exponentially stable in time, i.e., there exists a point $T_q > 0$ such that for any solution $u \in C_{q,\xi}([0,\infty), \mathbb{R}^n)$, $u = (u_1, u_2, \dots, u_n)$, of (4.1), (3.2) *the inequality*

$$\sum_{i=1}^{n} |u_i(t) - C_i| < \frac{\sum_{i=1}^{n} \max_{t\in[\tau,0]} |\phi_i(t)| \, \xi^{1-q}}{\Gamma(q)} e^{\frac{\xi-1}{\xi} t} \text{ for } t > T_q$$

holds.

Proof. Consider the Lyapunov function $V(x) = \sum_{i=1}^{n} |x_i|, \ x \in \mathbb{R}^n$.

Let $u(t) \in C_{q,\xi}([0,\infty), \mathbb{R}^n)$, $u = (u_1, u_2, \dots, u_n)$, be a solution of (4.1), (3.2). Let $v_i(t) = u_i(t) - C_i$, $i = 1, 2, \dots, n$. Then $v \in C_{q,\xi}([0,\infty), \mathbb{R}^n)$ is a solution of (4.4), (3.2).

Let point $t > 0$ be such that $|v_i(t + \sigma)| < |v_i(t)|$ for $\sigma \in [-\min\{\tau, t\}, 0]$, $i = 1, 2, \dots, n$. Thus $v_i(t) \neq 0$.

Apply Lemma 2.4, assumptions A1–A3 and we get

$$\begin{aligned}
&{}_0^{RL}\mathcal{D}_t^{q,\xi}|\nu_i(t)| \leq sign(\nu_i(t)){}_0^{RL}\mathcal{D}_t^{q,\xi}\nu_i(t)\\
&= -sign(\nu_i(t))A_i(t)\nu_i(t) + \sum_{k=1}^{n} a_{i,k}(t)sign(\nu_i(t))F_k(\nu_k(t))\\
&\quad + \sum_{k=1}^{n} b_{i,k}(t)sign(\nu_i(t))G_k(\nu_k(t-\xi(t))) + \sum_{k=1}^{n} c_{i,k}(t)\int_{t-\Theta(t)}^{t} sign(\nu_i(t))H_k(\nu_k(s))ds \\
&\leq -\mu_i|\nu_i(t)| + \sum_{k=1}^{n} |a_{i,k}(t)|\gamma_k|\nu_k(t)|\\
&\quad + \sum_{k=1}^{n} |b_{i,k}(t)|\beta_k|\nu_k(t-\xi(t))| + \sum_{k=1}^{n} |c_{i,k}(t)|\int_{t-\Theta(t)}^{t}\delta_k|\nu_k(s)|ds.
\end{aligned} \tag{5.1}$$

From inequality (5.1) we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}{}_0^{RL}\mathcal{D}_t^{q,\xi}|\nu_i(t)| \leq -\sum_{i=1}^{n}\mu_i|\nu_i(t)| + \sum_{i=1}^{n}\sum_{k=1}^{n}|a_{i,k}(t)|\gamma_k|\nu_k(t)|\\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{n}|b_{i,k}(t)|\beta_k|\nu_k(t-\xi(t))| + \sum_{i=1}^{n}\sum_{k=1}^{n}|c_{i,k}(t)|\int_{t-\Theta(t)}^{t}\delta_k|\nu_k(s)|ds\\
&\leq -\sum_{i=1}^{n}\mu_i|\nu_i(t)| + \sum_{i=1}^{n}\sum_{k=1}^{n}|a_{k,i}(t)|\gamma_i|\nu_i(t)|\\
&\quad + \sum_{i=1}^{n}|\nu_i(t-\xi(t))|\sum_{k=1}^{n}|b_{k,i}(t)|\beta_i + \int_{t-\Theta(t)}^{t}\sum_{i=1}^{n}|\nu_i(s)|ds\sum_{k=1}^{n}|c_{k,i}(t)|\delta_i \\
&\leq -\sum_{i=1}^{n}\mu_i|\nu_i(t)| + \sum_{i=1}^{n}|\nu_i(t)|\sum_{k=1}^{n}|a_{k,i}(t)|\delta_i + \sum_{i=1}^{n}|\nu_i(t)|\beta\sum_{k=1}^{n}-\max_{i=1,2\dots,n}|b_{k,i}(t)|\\
&\quad + \left(\sum_{i=1}^{n}\int_{t-\Theta(t)}^{t}|\nu_i(s)|ds\right)\delta\sum_{k=1}^{n}\max_{i=1,2,\dots,n}|c_{k,i}(t)|\\
&\leq \sum_{i=1}^{n}\left(-\mu_i + \sum_{k=1}^{n}\left(|a_{k,i}(t)|\gamma_i + \beta\max_{i=1,2\dots,n}|b_{k,i}(t)| + \tau\delta\max_{i=1,2,\dots,n}|c_{k,i}(t)|\right)\right)|\nu_i(t)|\\
&\leq 0.
\end{aligned} \tag{5.2}$$

From Corollary 3.2 with $y(t) \equiv \nu(t)$ the claim of Theorem 5.1 follows. □

Corollary 5.1. *Let the conditions of Theorem 5.1 be satisfied. Then any solution* $u \in C_{q,\xi}([0,\infty),\mathbb{R}^n)$ *of* (4.1), (3.2) *satisfies* $\lim_{t\to\infty} u_i(t) = C_i,\ i = 1,2,\dots,n$, *i.e., any solution of the model* (4.1) *approaches the equilibrium at infinity.*

Remark 5.1. *Note the stability of neural network model with RLFD is studied in [6]. It is applied Lyapunov functional and the stability of zero equilibrium is studied. In this paper , first we are using a model with more general GPRLFD, a nonzero equilibrium is defined and its stability by Lyapunov functions and Razumikhin condition are applied. The sufficient conditions in [6] are more restrictive.*

Remark 5.2. *The Hopfield neural network models with ordinary derivative and delays are studied by many authors and different sufficient conditions for various types of stability are obtained (see, for example, [32,33]). When GPRLFD is applied, first, the initial condition has to be defined in a different way, second the equilibrium is defined differently than in the case of ordinary derivative and finally, the special type of stability excluding the initil time point has to be considered. Also, in this paper Razumikhin method combined with Lyapunov functions and the proved inequalities for GPRLFD of the Lyapunov functions is applied. This leads to different types of sufficient conditions.*

5.2. Quadratic Lyapunov functions

Note that if a function is RL integrable (or generalized proportional RL integrable) it is not necessarily to be squared RL integrable (or squared generalized proportional RL integrable).

Example 5.1. *Let* $\xi = 0.5, q = 0.3$ *and* $v(t) = t^{0.8}$. *Then the integrals* $\int_0^1 e^{-(1-s)}\,(1-s)^{-0.3}\,s^{-0.8}\,ds$ *and* $\int_0^1 (1-s)^{-0.3}\,s^{-0.8}\,ds$ *exist but the corresponding integrals of the squared function* $u^2(t) = t^{-1.6}$, *i.e.,* $\int_0^1 e^{-(1-s)}\,(1-s)^{-0.3}\,s^{-1.6}\,ds$ *and* $\int_0^1 (1-s)^{-0.3}\,s^{-1.6}\,ds$, *do not exist.*

It requires in the application of the squared Lyapunov function as well the product $u^T(t)u(t)$ for the solution $u(t)$ of the model (4.1) to be assumed that any solution is squared RL integrable (or squared generalized proportional RL integrable) on the whole time interval of consideration. It is a huge restriction about the solutions of the model (for example, see Theorems 3.1, 3.2 [34] where Lyapunov functional V_1 is used without assuming about squared integrability of the solution).

Now we will aplly quadratic Lyapunov functions to study the stability properties of the model (4.1), (3.2).

Theorem 5.2. *Let the assumptions A1–A4 be satisfied and:*

(1) The fractional neural model (4.1) *has an equilibrium* $V^* = (C_1, C_2, \dots, C_n)$.
(2) Any solution $u \in C_{q,\xi}([0,\infty),\mathbb{R}^n)$ *is squared fractional integrable, i.e,* $u^T u \in C_{q,\xi}([0,\infty),\mathbb{R}^n)$.
(3) For all $i = 1, 2, \dots, n$ *and* $t \geq 0$ *the inequalities*

$$\sum_{k=1}^{n}\Big\{\Big(\max_{i=1,2,\dots,n}|a_{i,k}(t)| + \max_{i=1,2,\dots,n}|a_{k,i}(t)|\Big)\gamma + \Big(\max_{i=1,2,\dots,n}|b_{i,k}(t)| + \max_{i=1,2,\dots,n}|b_{k,i}(t)|\Big)\beta + \Big(\max_{i=1,2,\dots,n}|c_{i,k}(t)| + \max_{i=1,2,\dots,n}|c_{k,i}(t)|\Big)\tau\delta\Big\} < 2\mu_i,$$

hold where $\gamma = \max_{i=1,2,\dots,n}\gamma_i$, $\beta = \max_{i=1,2,\dots,n}\beta_i$, *and* $\delta = \max_{i=1,2,\dots,n}\delta_i$.

Then, the equilibrium V^* *is exponentially stable in time, i.e, there exists a point* $T_q > 0$ *such that any solution* $u \in C_{q,\xi}([0,\infty),\mathbb{R}^n)$ *of* (4.1),(3.2) *satisfies the inequality*

$$\|u(t) - V^*\|_2 < \frac{\sum_{i=1}^{n}\max_{t\in[\tau,0]}|\phi_i(t)|\,\xi^{1-q}}{\Gamma(q)}e^{\frac{\xi-1}{\xi}t}\ \textit{for}\ t > T_q.$$

Proof. Consider the Lyapunov function $V(x) = 0.5x^T x = 0.5\sum_{i=1}^{n} x_i^2,\ x \in \mathbb{R}^n,\ x = (x_1, x_2, \dots, x_n)$.

Let $u(t) \in C_{q,\xi}([0,\infty),\mathbb{R}^n)$, be a solution of (4.1), (3.2). Consider the system (2.1) with initial conditions (3.2). We will study the stability of its zero solution. Let $v_i(t) = u_i(t) - C_i,\ i = 1, 2, \dots, n$. Then $v \in C_{q,\xi}([0,\infty),\mathbb{R}^n)$ is a solution of (2.1), (3.2) and ${}_0^{RL}\mathcal{D}_t^{q,\xi}V(v(t)) = 0.5\sum_{i=1}^{n}{}_0^{RL}\mathcal{D}_t^{q,\xi}v_i^2(t),\ t > 0.$

Apply Lemma 2.2, assumptions A1–A3, inequality $ab \le 0.5a^2 + 0.5b^2$ and we obtain

$$\begin{aligned}
{}_{0}^{RL}\mathcal{D}_t^{q,\xi} V(\nu(t)) \le &- \sum_{i=1}^{n} \mu_i \nu_i^2(t) + 0.5 \sum_{i=1}^{n} \nu_i^2(t) \sum_{k=1}^{n} |a_{i,k}(t)|\gamma_k + 0.5 \sum_{i=1}^{n} \sum_{k=1}^{n} |a_{i,k}(t)|\gamma_k \nu_k^2(t) \\
&+ 0.5 \sum_{i=1}^{n} \nu_i^2(t) \sum_{k=1}^{n} |b_{i,k}(t)|\beta_k + 0.5 \sum_{i=1}^{n} \sum_{k=1}^{n} |b_{i,k}(t)|\beta_k \nu_k^2(t-\xi(t)) \\
&+ 0.5 \sum_{i=1}^{n} \sum_{k=1}^{n} |c_{i,k}(t)| \nu_i^2(t)\Theta(t)\delta_k + 0.5 \sum_{i=1}^{n} \sum_{k=1}^{n} |c_{i,k}(t)| \int_{t-\Theta(t)}^{t} \delta_k \nu_k^2(s)ds \\
\le &- \sum_{i=1}^{n} \mu_i \nu_i^2(t) + 0.5 \sum_{i=1}^{n} \nu_i^2(t) \sum_{k=1}^{n} \max_{i=1,2,\dots,n} |a_{i,k}(t)|\gamma + 0.5 \sum_{i=1}^{n} \nu_i^2(t) \sum_{k=1}^{n} \max_{i=1,2,\dots,n} |a_{k,i}(t)|\gamma \\
&+ 0.5 \sum_{i=1}^{n} \nu_i^2(t) \sum_{k=1}^{n} \max_{i=1,2,\dots,n} |b_{i,k}(t)|\beta + 0.5 \sum_{i=1}^{n} \nu_i^2(t-\xi(t)) \sum_{k=1}^{n} \max_{i=1,2,\dots,n} |b_{k,i}(t)|\beta \\
&+ 0.5 \sum_{i=1}^{n} \nu_i^2(t) \sum_{k=1}^{n} \max_{i=1,2,\dots,n} |c_{i,k}(t)|\tau\delta + 0.5 \int_{t-\Theta(t)}^{t} \sum_{i=1}^{n} \nu_i^2(s)ds \sum_{k=1}^{n} \max_{i=1,2,\dots,n} |c_{k,i}(t)|\delta \\
\le &\, 0.5 \sum_{i=1}^{n} \Big(-2\mu_i + \sum_{k=1}^{n} \Big(\max_{i=1,2,\dots,n} |a_{i,k}(t)|\gamma + \max_{i=1,2,\dots,n} |a_{k,i}(t)|\gamma \\
&+ \max_{i=1,2,\dots,n} |b_{i,k}(t)|\beta + \max_{i=1,2,\dots,n} |b_{k,i}(t)|\beta + \max_{i=1,2,\dots,n} |c_{i,k}(t)|\tau\delta + \tau \max_{i=1,2,\dots,n} |c_{k,i}(t)|\delta \Big) \Big) \nu_i^2(t) \\
< &\, 0.
\end{aligned}$$

From Corollary 3.3 the claim of Theorem 5.1 follows. □

Corollary 5.2. *Let the conditions of Theorem 5.2 be satisfied. Then any solution* $u \in C_{q,\xi}([0,\infty),\mathbb{R}^n)$ *of* (4.1), (3.2) *satisfies* $\lim_{t\to\infty} u_i(t) = C_i,\ i = 1,2,\dots,n$, *i.e., any solution of the model* (4.1) *approaches the equilibrium at infinity.*

6. Applications

Example 6.1. *Consider following neural networks with three neurons with the GPRLFD:*

$$\begin{aligned}
{}_{0}^{RL}\mathcal{D}_t^{0.3,0.5} u_i(t) = &-A_i(t)u_i(t) + \sum_{k=1}^{3} a_{i,k}(t) f_k(u_k(t)) + \sum_{k=1}^{3} b_{i,k}(t) g_k(u_k(t-2)) \\
&+ \sum_{k=1}^{3} c_{i,k}(t) \int_{t-e^{-t}}^{t} h_k(u_k(s))ds + I_i(t), \quad t > 0, \quad i = 1,2,3,
\end{aligned} \tag{6.1}$$

with $\xi = 0.5,\ q = 0.3,\ \xi(t) = 2,\ \Theta(t) = e^{-t},\ \tau = 2$, *coefficients* $A_1(t) = 2,\ A_2(t) = 0.25e^t,\ \ A_3(t) = *1 + 0.5e^t$, *and therefore,* $\mu_1 = 2,\ \mu_2 = 0.25,\ \mu_3 = 1.5$.

The activation functions $f_1(x) = g_1(x) = h_1(x) = \frac{x}{1+e^{-x}}$ *are the Swish functions with constants* $\gamma_1 = \beta_1 = \delta_1 = 1.1$, $f_2(x) = g_2(x) = h_2(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, *are the tanh functions with constants* $\gamma_2 = \beta_2 = \delta_2 = 1$, *and* $f_3(x) = g_3(x) = h_3(x) = 0.5(|x+1| - |x-1|)$ *with* $\gamma_3 = \beta_3 = \delta_3 = 1$, *the external inputs are given by*

$$I_1(t) = 2 + 0.5^{0.3}\left(1 - \frac{\Gamma(-0.3,t)}{\Gamma(-0.3)}\right),\ I_2(t) = 0.1e^{-t}\frac{e}{1+e},\ \ I_3(t) = 0.001\frac{e}{1+e},$$

and the matrices of strengths of interconnections

$$\{a_{i,k}(t)\} = \begin{bmatrix} 0.01 & 0 & 0.02 \\ 0.01e^{-t} & 0.01 & 0 \\ 0 & \frac{0.01t}{1+t} & 0 \end{bmatrix}, \quad \{b_{i,k}(t)\} = \begin{bmatrix} -0.01 & 0.1 & 0.1 \\ -0.1e^{-t} & 0 & 0.1e^{-2t} \\ 0 & 0.001\sin(t) & 0 \end{bmatrix},$$

$$\{c_{i,k}(t)\} = \begin{bmatrix} 0 & 0.01 & -0.01e^{-t} \\ -0.01 & \frac{0.0022t}{1+t} & 0 \\ -0.001 & 0 & 0.005e^{-t} \end{bmatrix},$$

with $\sum_{k=1}^{3} \max_{i=1,2,3} |b_{k,i}(t)| = 0.1 + 0.1e^{-t} + 0.05\sin(t) \le 0.25$ *and* $\sum_{k=1}^{3} \max_{i=1,2,3} |c_{k,i}(t)| = 0.01 + 0.01 + 0.005e^{-t} \le 0.025$.

Then the inequalities

$$\begin{aligned}
&\sum_{k=1}^{3} \Big(|a_{k,1}(t)|1.1 + 1.1 \max_{i=1,2,3} |b_{k,i}(t)| + 2.2 \max_{i=1,2,3} |c_{k,i}(t)| \Big) \\
&= 0.02(1.1) + 0.25(1.1) + 0.025(2.2) = 0.352 < \mu_1 = 2, \\
&\sum_{k=1}^{3} \Big(|a_{k,2}(t)| + 1.1 \max_{i=1,2,3} |b_{k,i}(t)| + 2.2 \max_{i=1,2,3} |c_{k,i}(t)| \Big) \le 0.02 + 0.33 = 0.35 < \mu_2 = 0.36, \\
&\sum_{k=1}^{3} \Big(|a_{k,3}(t)| + 1.1 \max_{i=1,2,3} |b_{k,i}(t)| + 2.2 \max_{i=1,2,3} |c_{k,i}(t)| \Big) \le 0.02 + 0.33 = 0.35 < \mu_i = 0.4,
\end{aligned}$$

are satisfied, i.e. condition 2 of Theorem 5.1 is satisfied.

The model (6.1) *has an equilibrium* $V^* = (1, 0, 0)$ *because* $f_1(1) = g_1(1) = h_1(1) = \frac{e}{1+e}$, $f_2(0) = g_2(0) = h_2(0) = 0$, $f_3(0) = g_3(0) = h_3(0) = 0$ *and the equalities*

$$\begin{aligned}
&0.5^{0.3}\left(1 - \frac{\Gamma(-0.3, t)}{\Gamma(-0.3)}\right) = -A_1(t) + \left(a_{1,1}(t) + b_{1,1}(t) + c_{1,1}(t)e^{-t}\right) \frac{e}{1+e} + I_1(t), \\
&0 = \left(a_{2,1}(t) + b_{2,1}(t) + c_{2,1}(t)e^{-t}\right) \frac{e}{1+e} + I_2(t), \\
&0 = \left(a_{3,1}(t) + b_{3,1}(t) + c_{3,1}(t)e^{-t}\right) \frac{e}{1+e} + I_3(t), \qquad \textit{for } \ t \ge 0
\end{aligned} \tag{6.2}$$

hold.

According to Theorem 5.1 the equilibrium of (6.1) *is exponentially stable in time, i.e., every solution* $(u_1(\cdot), u_2(\cdot), u_3(\cdot))$ *of* (6.1) *is satisfying the inequality*

$$|u_1(t) - 1| + |u_2(t)| + |u_3(t)| \le \sum_{i=1}^{3} \max_{t \in [-2,0]} |\phi_i(t)| \frac{0.5^{0.7} e^{-t}}{\Gamma(0.3)}, \quad t > T = 1,$$

where $T : t^{-0.7} = 1$.

At the same time the condition 4 of Theorem 5.2 is not satisfied because for $i = 2$ *we have*

$$\begin{aligned}
&\sum_{k=1}^{3} \Big\{ \Big(\max_{i=1,2,3} |a_{i,k}(t)| + \max_{i=1,2,3} |a_{k,i}(t)| \Big) 1.1 + \Big(\max_{i=1,2,3} |b_{i,k}(t)| + \max_{i=1,2,3} |b_{k,i}(t)| \Big) 1.1 \\
&+ \Big(\max_{i=1,2,3} |c_{i,k}(t)| + \max_{i=1,2,3} |c_{k,i}(t)| \Big) 2.2 \Big\} \\
&\le (0.04 + 0.04)1.1 + (0.25 + 0.3)1.1 + (0.025 + 0.025)2.2 = 0.803 \not< 2\mu_2 = 0.72,
\end{aligned}$$

and we could not apply the claim of Theorem 5.2 to the model (6.1) *to obtain the estimate for the norm* $||.||_2$.

7. Conclusions

Our aim of the paper is to study Hopfield neural networks with both the variable delay and distributed delay. An important aspect of our study is we consider the general case of variable in time coefficients and external inputs. The dynamic of the units is modeled by the GPRLFD. This derivative is applied to model the behavior with an anomalies at the initial time point. An exponential type of stability is defined and this stability excludes the initial time because of the singularity of the solutions at the initial time. Lyapunov functions and the Razumikhin method are applied. Both types of Lyapunov functions, absolute values and quadratic function, are used. Theoretical results are illustrated with an example.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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