Contributions to Mathematics
www.shahindp.com/locate/cm

Contrib. Math. **8** (2023) 48–55
DOI: 10.47443/cm.2023.047

Research Article

On the minimum and second-minimum values of degree-based energies for trees

Yanjie Fu, Yubin Gao*

School of Mathematics, North University of China, Taiyuan, Shanxi 030051, P. R. China

(Received: 8 August 2023. Received in revised form: 18 September 2023. Accepted: 2 October 2023. Published online: 10 October 2023.)

Abstract

Let G be a simple graph with the vertex set $V(G)$ and edge set $E(G)$. Let $d_G(v_i)$ be the degree of the vertex $v_i \in V(G)$. A vertex-degree-based topological index (TI) of G is defined as $TI(G) = \sum_{v_iv_j \in E(G)} f(d_G(v_i), d_G(v_j))$, where f is a real function with the property $f(x,y) = f(y,x) \geq 0$. The (i,j)-th element $a_{TI}(i,j)$ of the general extended adjacency matrix $A_{TI}(G)$ of G is defined as $a_{TI}(i,j) = f(d_G(v_i), d_G(v_j))$ if $v_iv_j \in E(G)$ and $a_{TI}(i,j) = 0$ otherwise. The TI energy of G is the sum of the absolute values of the eigenvalues of $A_{TI}(G)$. In this paper, some sufficient conditions for the minimum and second-minimum TI energy of trees are presented. Using the main result, trees with the minimum sum-connectivity Gourava energy and minimum product-connectivity Gourava energy are characterized among all trees on n vertices. Trees with the second-minimum values of the Randić energy, sum-connectivity energy, and sum-connectivity Gourava energy are also characterized among all trees on n vertices.

Keywords: tree (graph); sum-connectivity Gourava energy; product-connectivity Gourava energy; sum-connectivity energy; Randić energy.

2020 Mathematics Subject Classification: 05C05, 05C92.

1. Introduction

All graphs considered in this paper are finite, undirected and simple. Let G be a such graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Denote by $d_G(v_i)$ the degree of the vertex $v_i \in V(G)$. The adjacency matrix of G, denoted by $A(G)$, is an $n \times n$ matrix whose (i,j)-th element a_{ij} is defined as $a_{ij} = 1$ if $v_iv_j \in E(G)$ and $a_{ij} = 0$ otherwise. The characteristic polynomial of $A(G)$, denoted by $\phi_A(G,\lambda) = |\lambda I - A(G)|$, is called the characteristic polynomial of G. The roots of the equation $\phi_A(G,\lambda) = 0$, denoted by $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$, are called the eigenvalues of G. The energy $\varepsilon(G)$ of G is defined [10, 12] as

$$\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i(G)|.$$

There are numerous results on $\varepsilon(G)$, especially on trees [14, 15, 21–23].

In chemical graph theory, various graph invariants, also known as topological indices, are currently being studied. In this paper, we are mainly interested in vertex-degree-based topological indices and energy of graphs. The formal definition of a vertex-degree-based topological index of G is as follows:

$$TI(G) = \sum_{v_iv_j \in E(G)} f(d_G(v_i), d_G(v_j)),$$

where f is a pertinently chosen function with the property $f(x,y) = f(y,x) \geq 0$; some examples of interest are listed below:

Name	$f(x,y)$
Randić index	$\frac{1}{\sqrt{xy}}$
Sum-connectivity index	$\frac{1}{\sqrt{x+y}}$
Sum-connectivity Gourava index	$\frac{1}{\sqrt{x+y+xy}}$
Product-connectivity Gourava index	$\frac{1}{\sqrt{(x+y)xy}}$

Corresponding to every vertex-degree-based topological index TI given above, a matrix $A_{TI}(G) = (a_{TI}(i,j))_{n\times n}$ can be defined (see [20]) such that

$$a_{TI}(i,j) = \begin{cases} f(d_G(v_i), d_G(v_j)), & v_iv_j \in E(G), \\ 0, & v_iv_j \notin E(G). \end{cases}$$

*Corresponding author (ybgao@nuc.edu.cn).

The characteristic polynomial of $A_{TI}(G)$, denoted by $\phi_{TI}(G,\lambda) = |\lambda I - A_{TI}(G)|$, is called the TI characteristic polynomial of G. The roots of the equation $\phi_{TI}(G,\lambda) = 0$, denoted by $f_1(G), f_2(G), \ldots, f_n(G)$, are called TI eigenvalues of G. The TI energy of G is defined [3] as

$$\varepsilon_{TI}(G) = \sum_{i=1}^{n} |f_i(G)|.$$

Recent results on the TI energy can be found in the papers [3,11,20]. Several particular forms of ε_{TI} have already been extensively studied; see [1,2,4–6,25] for various results about the Randić energy, see [7,8] for some results concerning the ABC energy, and also see [9,13,16–19,24] for several results on some other forms of ε_{TI}. In this paper, some sufficient conditions for the minimum and second-minimum TI energy of trees are presented. Using the main result, trees with the minimum sum-connectivity Gourava energy and minimum product-connectivity Gourava energy are characterized among all trees on n vertices. Trees with the second-minimum values of the Randić energy, sum-connectivity energy, and sum-connectivity Gourava energy are also characterized among all trees on n vertices.

2. Main results

Let $T(n)$ be the set of all trees of order n. For $T \in T(n)$, let $M_k(T)$ be the set of all k-matching of T for $1 \leq k \leq \left[\frac{n}{2}\right]$. For $e = uv \in E(T)$ and $\alpha_k = \{e_1, e_2, \ldots, e_k\} \in M_k(T)$, we take $f_T(e) = f_T(uv) = f^2\left(d_T(u), d_T(v)\right)$ and $f_T(\alpha_k) = \prod_{i=1}^{k} f_T(e_i)$. We introduce the following functions associated with $f(x,y)$ that will be used in the sequel.

$$f_1(s,t) = f^2(s+1,2) + (t-3)f^2(2,2) + f^2(1,2) - (t-1)f^2(s+t-1,1), \tag{1}$$

$$f_2(s,t) = f^2(s+1,t-1) + (t-2)f^2(t-1,1) - (t-1)f^2(s+t-1,1), \tag{2}$$

$$f_3(t) = f^2(2,t+1) + tf^2(t+1,1) - (t+1)f^2(1,t+2), \tag{3}$$

$$f_4(s,t) = (t+1)f^2(1,t+2) \times sf^2(1,s+1) - f^2(1,2) \times (t+s)f^2(1,s+t+1), \tag{4}$$

$$f_5(s,t) = (t+1)f^2(1,t+2) + sf^2(1,s+1) + f^2(t+2,s+1) - f^2(2,s+t+1) - f^2(1,2) - (t+s)f^2(1,s+t+1). \tag{5}$$

Lemma 2.1 (see [20]). *Let $T_1, T_2 \in T(n)$. Let*

$$\phi_{TI}(T_1,x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k a_{2k} x^{n-2k}, \quad \phi_{TI}(T_2,x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k b_{2k} x^{n-2k},$$

be the TI characteristic polynomials of T_1, T_2, respectively. If $a_{2k} \geq b_{2k}$ for all $k \geq 1$ and if there is an integer number k such that $a_{2k} > b_{2k}$, then $\varepsilon_{TI}(T_1) > \varepsilon_{TI}(T_2)$.

Consider the tree $T \in T(n)$ shown in Figure 1, where T_1 is a subtree of T with $v_1 \in V(T_1)$, $t \geq 3$, and $d_T(v_1) \geq 2$. Let $T' = T - \{v_2v_3, v_3v_4, \ldots, v_{t-1}v_t\} + \{v_1v_3, v_1v_4, \ldots, v_1v_t\}$. We say that T' is obtained from T by Operation I (as depicted in Figure 1).

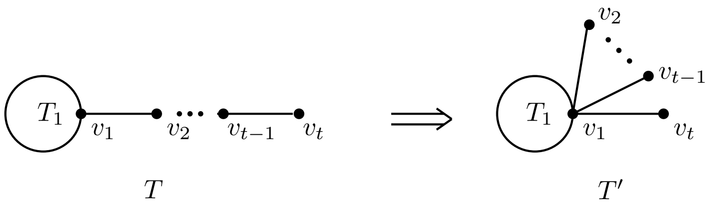

Figure 1: Operation I.

Lemma 2.2. *Let T' be the tree obtained from the tree T by Operation I as shown in Figure 1. If $f(x,y)$ is decreasing with respect to the variable x (with respect to the variable y too, of course) and if $f_1(s,t) \geq 0$ for $t \geq 3$ and $s \geq 1$, then*

$$\varepsilon_{TI}(T') < \varepsilon_{TI}(T).$$

Proof. Let $T, T' \in T(n)$. Let

$$\phi_{TI}(T,x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k a_{2k} x^{n-2k}, \ \phi_{TI}(T',x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k b_{2k} x^{n-2k},$$

be the TI characteristic polynomials of T, T', respectively, where $a_0 = b_0 = 1$. Take $N_T(v_1) = \{v_2, u_1, \ldots, u_s\}$, where $s \geq 1$. Note that $t \geq 3$, $d_T(v_1) = s+1$, $d_{T'}(v_1) = s+t-1$, and $d_T(u_i) = d_{T'}(u_i)$ for $i = 1, 2, \ldots, s$. Then

$$\begin{aligned} a_2 - b_2 &= \sum_{i=1}^{s} f_T(v_1u_i) + \sum_{j=1}^{t-1} f_T(v_jv_{j+1}) - \sum_{i=1}^{s} f_{T'}(v_1u_i) - \sum_{j=2}^{t} f_{T'}(v_1v_j) \\ &> \sum_{j=1}^{t-1} f_T(v_jv_{j+1}) - \sum_{j=2}^{t} f_{T'}(v_1v_j) = f_1(s,t) \geq 0, \end{aligned}$$

that is, $a_2 > b_2$. For $k = 2, \ldots, [\frac{n}{2}]$, we have

$$\begin{aligned} a_{2k} &= \sum_{\alpha_k \in M_k(T)} f_T(\alpha_k) \\ &\geq \sum_{\alpha_k \in M_k(T_1)} f_T(\alpha_k) + \sum_{j=1}^{t-1} f_T(v_jv_{j+1}) \sum_{\alpha_{k-1} \in M_{k-1}(T-v_1)} f_T(\alpha_{k-1}) \\ &= \sum_{\alpha_k \in M_k(T_1)} f_T(\alpha_k) + \left(f^2(s+1,2) + (t-3)f^2(2,2) + f^2(1,2)\right) \sum_{\alpha_k \in M_{k-1}(T-v_1)} f_T(\alpha_{k-1}), \end{aligned}$$

and

$$\begin{aligned} b_{2k} &= \sum_{\alpha_k \in M_k(T')} f_{T'}(\alpha_k) = \sum_{\alpha_k \in M_k(T_1)} f_{T'}(\alpha_k) + \sum_{i=2}^{t} f_{T'}(v_1v_i) \sum_{\alpha_{k-1} \in M_{k-1}(T-v_1)} f_{T'}(\alpha_{k-1}) \\ &\leq \sum_{\alpha_k \in M_k(T_1)} f_T(\alpha_k) + (t-1)f^2(s+t-1,1) \sum_{\alpha_k \in M_{k-1}(T-v_1)} f_T(\alpha_{k-1}), \end{aligned}$$

and so

$$a_{2k} - b_{2k} \geq f_1(s,t) \sum_{\alpha_k \in M_{k-1}(T-v_1)} f_T(\alpha_{k-1}) \geq 0,$$

that is, $a_{2k} \geq b_{2k}$. Now, by Lemma 2.1, the required result holds. □

Consider the tree $T \in T(n)$ shown in Figure 2, where T_1 is a subtree of T with $v_1 \in V(T_1)$, $t \geq 4$ and $d_T(v_1) \geq 2$. Let $T' = T - \{v_2v_3, \ldots, v_2v_t\} + \{v_1v_3, \ldots, v_1v_t\}$. We say that T' is obtained from T by Operation II (as depicted in Figure 2).

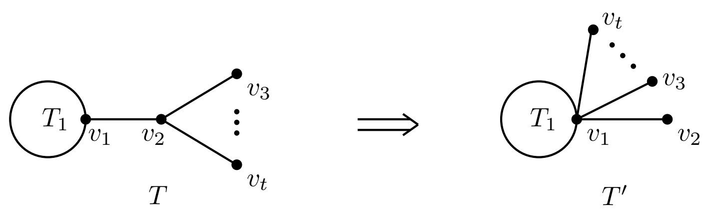

Figure 2: Operation II.

Lemma 2.3. *Let T' be the tree obtained from the tree T by Operation II as shown in Figure 2. If $f(x,y)$ is decreasing with respect to the variable x (with respect to the variable y too, of course) and $f_2(s,t) \geq 0$ for $t \geq 4$ and $s \geq 1$, then*

$$\varepsilon_{TI}(T') < \varepsilon_{TI}(T).$$

Proof. Let $T, T' \in T(n)$. Let

$$\phi_{TI}(T,x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k a_{2k} x^{n-2k},\ \phi_{TI}(T',x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k b_{2k} x^{n-2k},$$

be the TI characteristic polynomials of T, T', respectively, where $a_0 = b_0 = 1$. Take $N_T(v_1) = \{v_2, u_1, \ldots, u_s\}$, where $s \geq 1$. Note that $t \geq 4$, $d_T(v_1) = s+1$, $d_{T'}(v_1) = s+t-1$, and $d_T(u_i) = d_{T'}(u_i)$ for $i = 1, 2, \ldots, s$. Thus

$$a_2 - b_2 > f_T(v_1v_2) + \sum_{j=3}^{t} f_T(v_2v_j) - \sum_{j=2}^{t} f_{T'}(v_1v_j) = f_2(s,t) \geq 0,$$

that is, $a_2 > b_2$. For $k = 2, \ldots, [\frac{n}{2}]$, we have

$$\begin{aligned} a_{2k} &= \sum_{\alpha_k \in M_k(T)} f_T(\alpha_k) \\ &\geq \sum_{\alpha_k \in M_k(T_1)} f_T(\alpha_k) + \left(f_T(v_1v_2) + \sum_{i=3}^{t} f_T(v_2v_i) \right) \sum_{\alpha_{k-1} \in M_{k-1}(T-v_1)} f_T(\alpha_{k-1}) \\ &= \sum_{\alpha_k \in M_k(T_1)} f_T(\alpha_k) + \left(f^2(s+1,t-1) + (t-2) f^2(t-1,1) \right) \sum_{\alpha_{k-1} \in M_{k-1}(T-v_1)} f_T(\alpha_{k-1}), \end{aligned}$$

and

$$\begin{aligned} b_{2k} &= \sum_{\alpha_k \in M_k(T')} f_{T'}(\alpha_k) = \sum_{\alpha_k \in M_k(T_1)} f_{T'}(\alpha_k) + \sum_{i=2}^{t} f_{T'}(v_1v_i) \sum_{\alpha_{k-1} \in M_{k-1}(T-v_1)} f_{T'}(\alpha_{k-1}) \\ &\leq \sum_{\alpha_k \in M_k(T_1)} f_T(\alpha_k) + (t-1) f^2(s+t-1,1) \sum_{\alpha_k \in M_{k-1}(T-v_1)} f_T(\alpha_{k-1}), \end{aligned}$$

and hence

$$a_{2k} - b_{2k} \geq f_2(s,t) \sum_{\alpha_k \in M_{k-1}(T-v_1)} f_T(\alpha_{k-1}) \geq 0,$$

that is, $a_{2k} \geq b_{2k}$. Now, by Lemma 2.1, the desired result holds. □

Consider the tree $T \in T(n)$ as shown in Figure 3, where T_1 is a subtree of T with $v_1 \in V(T_1)$, $t \geq 2$ and $d_T(v_1) \geq 1$. Let $P_s = v_1 v_2 \cdots v_s$ and $T' = T - \{v_{s+2}u_1, \ldots, v_{s+2}, u_t\} + \{v_{s+1}u_1, \ldots, v_{s+1}, u_t\}$. We say that T' is obtained from T by Operation III (as depicted in Figure 3).

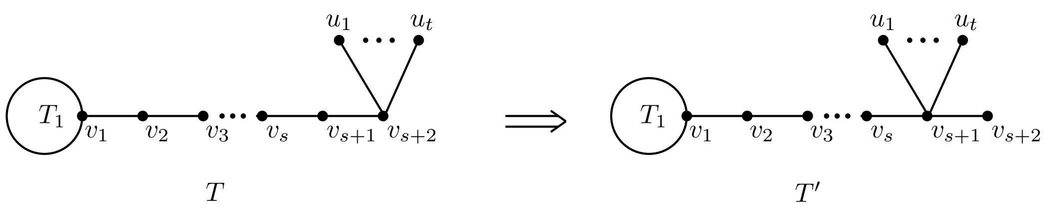

Figure 3: Operation III.

Lemma 2.4. *Let T' be the tree obtained from the tree T by Operation III as shown in Figure 3. If $f(x,y)$ is decreasing with respect to the variable x (with respect to the variable y too, of course) and $f_3(t) \geq 0$ for $t \geq 2$, then $\varepsilon_{TI}(T') < \varepsilon_{TI}(T)$.*

Proof. Take $T, T' \in T(n)$ and let

$$\phi_{TI}(T,x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k a_{2k} x^{n-2k},\ \phi_{TI}(T',x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k b_{2k} x^{n-2k},$$

be the TI characteristic polynomials of T, T', respectively, where $a_0 = b_0 = 1$. Note that $a_2 > b_2$ because

$$a_2 - b_2 = f_T(v_sv_{s+1}) + f_T(v_{s+1}v_{s+2}) + \sum_{i=1}^{t} f_T(v_{s+2}u_i) - f_{T'}(v_sv_{s+1}) - f_{T'}(v_{s+1}v_{s+2}) - \sum_{i=1}^{t} f_{T'}(v_{s+1}u_i) > f_3(t) \geq 0.$$

For $k = 2, \ldots, [\frac{n}{2}]$, we have

$$\begin{aligned} a_{2k} &= \sum_{\alpha_k \in M_k(T)} f_T(\alpha_k) \\ &\geq \sum_{\alpha_k \in M_k(T_1 \cup P_s)} f_T(\alpha_k) + f_T(v_s v_{s+1}) \sum_{\alpha_{k-1} \in M_{k-1}((T_1 \cup P_s) - v_s)} f_T(\alpha_{k-1}) \\ &+ \left(f_T(v_{s+1} v_{s+2}) + \sum_{i=1}^{t} f_T(v_{s+2} u_i) \right) \sum_{\alpha_{k-1} \in M_{k-1}(T_1 \cup P_s)} f_T(\alpha_{k-1}) \\ &= \sum_{\alpha_k \in M_k(T_1 \cup P_s)} f_T(\alpha_k) + f^2\left(d_T(v_s), 2\right) \sum_{\alpha_{k-1} \in M_{k-1}((T_1 \cup P_s) - v_s)} f_T(\alpha_{k-1}) \\ &+ \left(f^2(2, t+1) + t f^2(t+1, 1) \right) \sum_{\alpha_{k-1} \in M_{k-1}(T_1 \cup P_s)} f_T(\alpha_{k-1}), \end{aligned}$$

and

$$\begin{aligned} b_{2k} &= \sum_{\alpha_k \in M_k(T_1 \cup P_s)} f_{T'}(\alpha_k) + f_{T'}(v_s v_{s+1}) \sum_{\alpha_{k-1} \in M_{k-1}((T_1 \cup P_s) - v_s)} f_{T'}(\alpha_{k-1}) \\ &+ \left(f_{T'}(v_{s+1} v_{s+2}) + \sum_{i=1}^{t} f_{T'}(v_{s+1} u_i) \right) \sum_{\alpha_{k-1} \in M_{k-1}(T_1 \cup P_s)} f_{T'}(\alpha_{k-1}) \\ &= \sum_{\alpha_k \in M_k(T_1 \cup P_s)} f_{T'}(\alpha_k) + f^2\left(d_T(v_s), t+2\right) \sum_{\alpha_{k-1} \in M_{k-1}((T_1 \cup P_s) - v_s)} f_{T'}(\alpha_{k-1}) \\ &+ (t+1) f^2(1, t+2) \sum_{\alpha_{k-1} \in M_{k-1}(T_1 \cup P_s)} f_{T'}(\alpha_{k-1}), \end{aligned}$$

and so

$$a_{2k} - b_{2k} = \left(f^2\left(d_T(v_s), 2\right) - f^2\left(d_T(v_s), t+2\right) \right) \sum_{\alpha_{k-1} \in M_{k-1}((T_1 \cup P_s) - v_s)} f_T(\alpha_{k-1}) + f_3(t) \sum_{\alpha_{k-1} \in M_{k-1}(T_1 \cup P_s)} f_T(\alpha_{k-1}) \geq 0,$$

that is, $a_{2k} \geq b_{2k}$. By Lemma 2.1, the lemma holds. □

Consider the tree $T \in T(n)$ as shown in Figure 4, where $t \geq 1$ and $s \geq 2$. Let $T' = T - \{v_2 w_1, \ldots, v_2 w_t\} + \{v_1 w_1, \ldots, v_1 w_t\}$. We say that T' is obtained from T by Operation IV (as depicted in Figure 4).

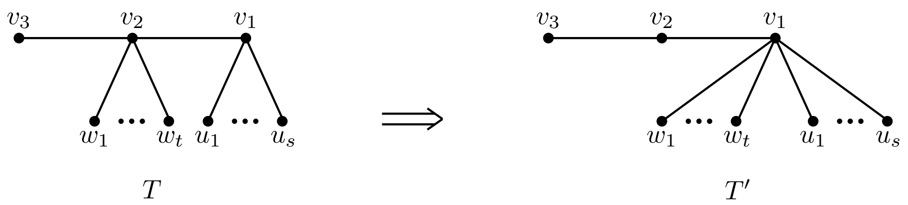

Figure 4: Operation IV.

Lemma 2.5. *Let T' be the tree obtained from the tree T by Operation IV as shown in Figure 4. If $f_4(s,t) > 0$ as well as $f_5(s,t) > 0$ for $t \geq 1$ and $s \geq 2$, then $\varepsilon_{TI}(T') < \varepsilon_{TI}(T)$.*

Proof. Take $T, T' \in T(n)$ and let

$$\phi_{TI}(T, x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k a_{2k} x^{n-2k}, \ \phi_{TI}(T', x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k b_{2k} x^{n-2k},$$

be the TI characteristic polynomials of T, T', respectively, where $a_0 = b_0 = 1$. Note that

$$a_2 - b_2 = \sum_{i=1}^{s} f_T(v_1 u_i) + \sum_{j=1}^{t} f_T(v_2 w_j) + f_T(v_2 v_3) + f_T(v_2 v_1) - \sum_{i=1}^{s} f_{T'}(v_1 u_i) - \sum_{j=1}^{t} f_{T'}(v_1 w_j) - f_{T'}(v_2 v_3) - f_{T'}(v_2 v_1) = f_5(s,t) > 0,$$

$$a_4 - b_4 = \sum_{i=1}^{s} f_T(v_1u_i) \times \left(\sum_{j=1}^{t} f_T(v_2w_j) + f_T(v_2v_3)\right) - \left(\sum_{i=1}^{s} f_{T'}(v_1u_i) + \sum_{j=1}^{t} f_{T'}(v_1w_j)\right) \times f_{T'}(v_2v_3) = f_4(s,t) > 0,$$

and for $k \geq 3$, $a_{2k} = b_{2k} = 0$. By Lemma 2.1, the lemma holds. □

Let S_n be the star of order n. Let S_n^* be the tree formed by attaching a new vertex to a pendent vertex of the star S_{n-1} as depicted in Figure 5.

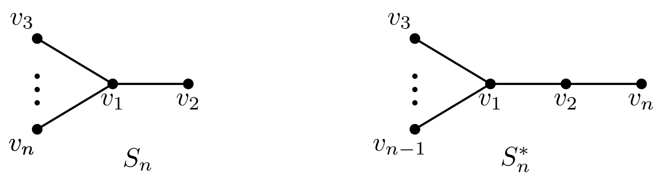

Figure 5: The trees S_n and S_n^*.

Theorem 2.1. *Let $f(x,y)$ be a symmetric real function. Consider the functions $f_1(s,t)$, $f_2(s,t)$, $f_3(t)$, $f_4(s,t)$ and $f_5(s,t)$ defined via Equations* (1)–(5)*. Consider also the following conditions:*

(C1) *$f(x,y)$ is decreasing with respect to the variable x (with respect to the variable y too, of course);*

(C2) *$f_1(s,t) \geq 0$ for $t \geq 3$ and $s \geq 1$;*

(C3) *$f_2(s,t) \geq 0$ for $t \geq 4$ and $s \geq 1$;*

(C4) *$f_3(t) \geq 0$ for $t \geq 2$;*

(C5) *$f_4(s,t) > 0$ and $f_5(s,t) > 0$ for $t \geq 1$ and $s \geq 2$.*

If the conditions (C1)–(C4) *hold, then S_n is the unique tree with the minimum TI energy among all trees of order n. If the condition* (C1)–(C5) *hold, then S_n^* is the unique tree with the second-minimum TI energy among all trees of order n.*

Proof. Suppose that $T \in T(n)\backslash\{S_n\}$ and the conditions (C1)-(C4) hold. We repeatedly perform Operations I–III on T and finally get the star S_n. By Lemmas 2.2-2.4, we have $\varepsilon_{TI}(S_n) < \varepsilon_{TI}(T)$. Therefore, S_n is the unique tree with the minimum TI energy in $T(n)$.

In the following, we assume that $T \in T(n)\backslash\{S_n, S_n^*\}$ and the conditions (C1)-(C5) hold. We will prove the inequality $\varepsilon_{TI}(S_n^*) < \varepsilon_{TI}(T)$. Take $R(T) = \{v \in V(T) \mid d_T(v) \geq 3\}$. We consider two cases.

Case 1. $|R(T)| \leq 1$.
By applying Operation I, we get the graph S_n^* from T. By Lemma 2.2, we have $\varepsilon_{TI}(S_n^*) < \varepsilon_{TI}(T)$.

Case 2. $|R(T)| \geq 2$.
By applying Operations I-IV repeatedly, we get the graph S_n^* from T. By Lemmas 2.2-2.5, the inequality $\varepsilon_{TI}(S_n^*) < \varepsilon_{TI}(T)$ holds. □

3. Applications

Theorem 3.1. *The star S_n is the unique tree with the minimum values of the sum-connectivity Gourava energy and product-connectivity Gourava energy among all trees on n vertices.*

Proof. For each of the two considered energies, it is obvious that the condition (C1) of Theorem 2.1 holds. Hence, we only need to verify that the conditions (C2)–(C4) of Theorem 2.1 hold for the considered energies.

Sum-connectivity Gourava energy. For this energy, $f(x,y) = \frac{1}{\sqrt{x+y+xy}}$ and hence

$$f_1(s,t) = \frac{(30t-42)s^2 + (30t^2 - 127t + 151)s + 50t^2 - 215t + 195}{40(3s+5)(2s+2t-1)} > 0,$$
$$f_2(s,t) = \frac{s(t-2)(2st+2t-1)}{(st+2t-1)(2t-1)(2s+2t-1)} > 0,$$
$$f_3(t) = \frac{4t^2 + 7t}{12t^3 + 68t^2 + 125t + 75} > 0.$$

Product-connectivity Gourava energy. For this energy, $f(x,y) = \frac{1}{\sqrt{(x+y)xy}}$ and hence

$$f_1(s,t) = \frac{(3t-1)s^4 + (6t^2+7t-3)s^3 + (3t^3+20t^2-58t+73)s^2}{48(s+3)(s+1)(s+t)(s+t-1)}$$
$$+ \frac{(12t^3+2t^2-155t+171)s + 9t^3 + 12t^2 - 165t + 144}{48(s+3)(s+1)(s+t)(s+t-1)} > 0,$$

$$f_2(s,t) = \frac{s(t-2)(s^2+2st+t-1)}{t(s+t)(t-1)(s+1)(s+t-1)} > 0,$$

$$f_3(t) = \frac{3t}{2(t+1)(t+2)(t+3)} > 0.$$

This completes the proof. □

Theorem 3.2. *Among all trees on n vertices, the graph S_n^* is the unique tree with the second-minimum value of each of the following energies: Randić energy, sum-connectivity energy, sum-connectivity Gourava energy.*

Proof. For each of the considered energies, the condition (C1) of Theorem 2.1 trivially holds. Thus, we only need to verify that the conditions (C2)–(C5) of Theorem 2.1 hold the considered energies.

Randić energy. For this energy, we have $f(x,y) = \frac{1}{\sqrt{xy}}$ and thus

$$f_1(s,t) = \frac{(t-1)s^2 + (t^2-5t+6)s + t^2 - 4t + 3}{4(s+1)(s+t-1)} > 0,$$

$$f_2(s,t) = \frac{(t-2)s^2}{(t-1)(s+1)(s+t-1)} > 0,$$

$$f_3(t) = \frac{t}{2(t+1)(t+2)} > 0,$$

$$f_4(s,t) = f_5(s,t) = \frac{t(s-1)(s+t+2)}{2(s+1)(t+2)(s+t+1)} > 0.$$

Sum-connectivity energy. For this energy, we have $f(x,y) = \frac{1}{\sqrt{x+y}}$ and thus

$$f_1(s,t) = \frac{(3t-5)s^2 + (3t^2-8t+9)s + t^2 - 39t + 36}{12(s+3)(s+t)} > 0,$$

$$f_2(s,t) = \frac{t-2}{t} - \frac{t-2}{s+t} > 0, \ f_3(t) = \frac{t}{t+2} - \frac{t}{t+3} > 0,$$

$$f_4(s,t) = \frac{2t(s-1)(s+t+3)}{3(s+2)(t+3)(s+t+2)} > 0,$$

$$f_5(s,t) = \frac{2t(s-1)(s+t+5)}{3(s+2)(t+3)(s+t+2)} > 0.$$

Sum-connectivity Gourava energy. For this energy, we have $f(x,y) = \frac{1}{\sqrt{x+y+xy}}$. From the proof of Theorem 3.1, we only need to verify that the condition (C5) of Theorem 2.1 holds. Here,

$$f_4(s,t) = \frac{3t(s-1)(2s+2t+5)}{5(2s+3)(2t+5)(2s+2t+3)} > 0,$$

$$f_5(s,t) > \frac{36t(s-1)}{5(2s+3)(2t+5)(2s+2t+3)(3s+3t+5)(st+3s+2t+5)} > 0.$$

This completes the proof. □

4. Conclusion

In this study, we obtained sufficient conditions for the TI energy of trees to be minimum and second-minimum. Using the main result, we characterized the unique tree with the minimum values of the sum-connectivity Gourava energy and product-connectivity Gourava energy among all trees on n vertices. We also characterized the unique tree with the second-minimum values of the Randić energy, sum-connectivity energy, and sum-connectivity Gourava energy, among all trees on n vertices.

Acknowledgement

The authors would like to thank the anonymous referees very much for their useful comments and valuable suggestions. This work is supported by Shanxi Scholarship Council of China (Grant No. 2022-149).

References

[1] Ş. B. Bozkurt, B. Furtula, I. Gutman, On Randić energy, *Linear Algebra Appl.* **442** (2014) 50–57.

[2] Ş. B. Bozkurt, A. D. Güngör, I. Gutman, A. S. Çevik, Randić matrix and Randić energy, *MATCH Commun. Math. Comput. Chem.* **64** (2010) 239–250.

[3] K. C. Das, I. Gutman, I. Milovanović, E. Milovanović, B. Furtula, Degree-based energies of graphs, *Linear Algebra Appl.* **554** (2018) 185–204.

[4] K. C. Das, I. Gutman, S. Sorgun, On Randić energy, *MATCH Commun. Math. Comput. Chem.* **73** (2015) 81–92.

[5] W. Gao, The Randić energy of generalized double sun, *Czechoslovak Math. J.* **72** (2022) 285–312.

[6] Y. Gao, W. Gao, Y. Shao, The minimal Randić energy of trees with given diameter, *Appl. Math. Comput.* **411** (2021) #126489.

[7] Y. Gao, Y. Shao, The minimum ABC energy of trees, *Linear Algebra Appl.* **577** (2019) 186–203.

[8] M. Ghorbani, X. Li, M. Hakimi-Nezhaad, J. Wang, Bounds on the ABC spectral radius and ABC energy of graphs, *Linear Algebra Appl.* **598** (2020) 145–164.

[9] X. Guo, Y. Gao, Arithmetic-geometric spectral radius and energy of graphs, *MATCH Commun. Math. Comput. Chem.* **83** (2020) 651–660.

[10] I. Gutman, Acyclic systems with extremal Hückel π-electron energy, *Theor. Chim. Acta.* **45** (1997) 79–87.

[11] I. Gutman, J. Monsalve, J. Rada, A relation between a vertex-degree-based topological index and its energy, *Linear Algebra Appl.* **636** (2022) 134–142.

[12] I. Gutman, H. Ramane, Research on graph energies in 2019, *MATCH Commun. Math. Comput. Chem.* **84** (2020) 277–292.

[13] I. Gutman, I. Redžepović, J. Rada, Relating energy and Sombor energy, *Contrib. Math.* **4** (2021) 41–44.

[14] N. Li, S. Li, On the extremal energies of trees, *MATCH Commun. Math. Comput. Chem.* **59** (2008) 291–314.

[15] J. Monsalve, J. Rada, On the maximal energy among orientations of a tree, *Kuwait J. Sci.* **47** (2020) 2–13.

[16] B. A. Rather, M. Imran, Sharp bounds on the Sombor energy of graphs, *MATCH Commun. Math. Comput. Chem.* **88** (2022) 605–624.

[17] B. A. Rather, M. Imran, A note on energy and Sombor energy of graphs, *MATCH Commun. Math. Comput. Chem.* **89** (2023) 467–477.

[18] I. Redžepović, I. Gutman, Comparing energy and Sombor energy – An empirical study, *MATCH Commun. Math. Comput. Chem.* **88** (2022) 133–140.

[19] Y. Shao, Y. Gao, The maximal geometric-arithmetic energy of trees with at most two branched vertices, *Appl. Math. Comput.* **362** (2019) #124528.

[20] Y. Shao, Y. Gao, W. Gao, X. Zhao, Degree-based energies of trees, *Linear Algebra Appl.* **621** (2021) 18–28.

[21] W. Yan, L. Ye, On the minimal energy of trees with a given diameter, *Appl. Math. Lett.* **18** (2005) 1046–1052.

[22] W. Yan, L. Ye, On the maximal energy and the Hosoya index of a type of trees with many pendant vertices, *MATCH Commun. Math. Comput. Chem.* **53** (2005) 449–459.

[23] J. Zhang, B. Zhou, On minimal energies of non-starlike trees with given number of pendent vertices, *MATCH Commun. Math. Comput. Chem.* **62** (2009) 481–490.

[24] X. Zhao, Y. Shao, Y. Gao, The maximal geometric-arithmetic energy of trees, *MATCH Commun. Math. Comput. Chem.* **84** (2020) 363–367.

[25] X. Zhao, Y. Shao, Y. Gao, On Randić energy of coral trees, *MATCH Commun. Math. Comput. Chem.* **88** (2022) 157–170.