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Research article

Stationary distribution for a three-dimensional stochastic viral infection model with general distributed delay

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Abstract: This work examines a stochastic viral infection model with a general distributed delay. We transform the model with weak kernel case into an equivalent system through the linear chain technique. First, we establish that a global positive solution to the stochastic system exists and is unique. We establish the existence of a stationary distribution of a positive solution under the stochastic condition $R^s > 0$, also referred to as a stationary solution, by building appropriate Lyapunov functions. Finally, numerical simulation is proved to verify our analytical result and reveals the impact of stochastic perturbations on disease transmission.

Keywords: stochastic viral infection model; distributed delay; stationary distribution;

1. Introduction

In the past few decades, there has been a lot of interest in mathematical models of viral dynamics and epidemic dynamics. Since viruses can directly reproduce inside of their hosts, a suitable model can shed light on the dynamics of the viral load population in vivo. In fact, by attacking infected cells, cytotoxic T lymphocytes (CTLs) play a crucial part in antiviral defense in the majority of virus infections. As a result, recent years have seen an enormous quantity of research into the population dynamics of viral infection with CTL response (see [1–4]). On the other hand, Bartholdy et al. [3] and Wodarz et al. [4] found that the turnover of free virus is much faster than that of infected cells, which allowed them to make a quasi-steady-state assumption, that is, the amount of free virus is simply proportional to the number of infected cells. In addition, the most basic models only consider the source of uninfected cells but ignore proliferation of the target cells. Therefore, a reasonable model for the population dynamics of target cells should take logistic proliferation term into consideration. Furthermore, in many biological models, time delay cannot be disregarded. A length of time τ may be required for antigenic stimulation to produce CTLs, and the CTL response at time t may rely on the antigen population at time $t-\tau$. Xie et al. [4] present a model of delayed viral infection with immune response

$$\begin{cases} x'(t) = \lambda - dx(x) - \beta x(t)y(t), \\ y'(t) = \beta x(t)y(t) - ay(t) - py(t)z(t), \\ z'(t) = cy(t-\tau)z(t-\tau) - bz(t), \end{cases} \tag{1.1}$$

where $x(t), y(t)$ and $z(t)$ represent the number of susceptible host cells, viral population and CTLs, respectively. At a rate of λ, susceptible host cells are generated, die at a rate of dx and become infected by the virus at a rate of βxy. According to the lytic effector mechanisms of the CTL response, infected cells die at a rate of ay and are killed by the CTL response at a rate of pyz. The CTL response occurs proportionally to the number of infected cells at a given time $cy(t-z)(t-z)$ and exponentially decays according to its level of activity bz. Additionally, the CTL response time delay is τ.

The dynamical behavior of infectious diseases model with distributed delay has been studied by many researchers (see [5 – 8]). Similar to [5], in this paper, we will mainly consider the following viral infection model with general distribution delay

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda - dx(t) - \beta x(t)y(t), \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \beta x(t)y(t) - ay(t) - py(t)z(t), \\ \frac{\mathrm{d}z}{\mathrm{d}t} = c\int_{-\infty}^{t} F(t-\tau)y(\tau)z(\tau)d\tau - bz(t). \end{cases}$$

The delay kernel $F : [0,\infty) \to [0,\infty)$ takes the form $F(s) = \frac{s^n \alpha^{n+1} e^{-\alpha s}}{n!}$ for constant $\alpha > 0$ and integer $n \geq 0$. The kernel with $n = 0$, i.e., $F(s) = \alpha e^{-\alpha s}$ is called the weak kernel which is the case to be considered in this paper.

However, in the real world, many unavoidable factors will affect the viral infection model. As a result, some authors added white noise to deterministic systems to demonstrate how environmental noise affects infectious disease population dynamics (see [9–12]). Linear perturbation, which is the simplest and most common assumption to introduce stochastic noise into deterministic models, is extensively used for species interactions and disease transmission. Here, we establish the stochastic infection model with distributed delay by taking into consideration the two factors mentioned above.

$$\begin{cases} \mathrm{d}x(t) = \Big[\lambda - dx(t) - \beta x(t)y(t)\Big]\mathrm{d}t + \sigma_1 x(t)\mathrm{d}B_1(t), \\ \mathrm{d}y(t) = \Big[\beta x(t)y(t) - ay(t) - py(t)z(t)\Big]\mathrm{d}t + \sigma_2 y(t)\mathrm{d}B_2(t), \\ \mathrm{d}z(t) = \Big[c\int_{-\infty}^{t} F(t-\tau)y(\tau)z(\tau)d\tau - bz(t)\Big]\mathrm{d}t + \sigma_3 z(t)\mathrm{d}B_3(t). \end{cases} \tag{1.2}$$

In our literature, we will consider weight function is weak kernel form. Let

$$w(t) = \int_{-\infty}^{t} \alpha e^{-\alpha(t-\tau)} y(\tau)z(\tau)d\tau.$$

Based on the linear chain technique, the equations for system (1.2) are transformed as follows

$$\begin{cases} \mathrm{d}x(t) = \Big[\lambda - dx(t) - \beta x(t)y(t)\Big]\mathrm{d}(t) + \sigma_1 x(t)\mathrm{d}B_1(t), \\ \mathrm{d}y(t) = \Big[\beta x(t)y(t) - ay(t) - py(t)z(t)\Big]dt + \sigma_2 y(t)\mathrm{d}B_2(t), \\ \mathrm{d}z(t) = \Big[cw(t) - bz(t)\Big]dt + \sigma_3 z(t)\mathrm{d}B_3(t), \\ \mathrm{d}w(t) = \Big[\alpha y(t)z(t) - \alpha w(t)\Big]\mathrm{d}t. \end{cases} \tag{1.3}$$

For the purpose of later analysis and comparison, we need to introduce the corresponding deterministic system of model (1.3), namely,

$$\begin{cases} \mathrm{d}x(t) = \Big[\lambda - dx(t) - \beta x(t)y(t)\Big]\mathrm{d}t, \\ \mathrm{d}y(t) = \Big[\beta x(t)y(t) - ay(t) - py(t)z(t)\Big]\mathrm{d}t, \\ \mathrm{d}z(t) = \Big[cw(t) - bz(t)\Big]\mathrm{d}t, \\ \mathrm{d}w(t) = \Big[\alpha y(t)z(t) - \alpha w(t)\Big]\mathrm{d}t. \end{cases} \tag{1.4}$$

Using the similar method of Ma [13], the basic reproduction of system (1.4) can be expressed as $R_0 = \lambda\beta/ad$. If $R_0 \le 1$, system (1.4) has an infection-free equilibrium $E_0 = (\frac{\lambda}{d}, 0, 0, 0)$ and is globally asymptotically stable. If $1 < R_0 \le 1 + b\beta/cd$, in addition to the infection-free equilibrium E_0, then system (1.4) has another unique equilibrium $E_1 = (\bar{x}, \bar{y}, \bar{z}, \bar{w}) = (\frac{a}{\beta}, \frac{\beta\lambda - ad}{a\beta}, 0, 0)$ and is globally asymptotically stable. If $R_0 > 1 + \frac{b\beta}{cd}$, in addition E_0 and E_1, then system (1.4) still has another unique infected equilibrium $E_2 = (x^+, y^+, z^+, w^+) = \left(\frac{c\lambda}{cd+b\beta}, \frac{b}{c}, \frac{c\beta\lambda - acd - ab\beta}{cdp + bp\beta}, \frac{b(c\beta\lambda - acd - ab\beta)}{c(cdp + bp\beta)}\right)$.

We shall focus on the existence and uniqueness of a stable distribution of the positive solutions to model (1.3) in this paper. The stability of positive equilibrium state plays a key role in the study of the dynamical behavior of infectious disease systems. Compared with model (1.4), stochastic one (1.3) has no positive equilibrium to investigate its stability. Since stationary distribution means weak stability in stochastic sense, we focus on the existence of stationary distribution for model (1.3). The main effort is to construct the suitable Lyapunov function. As far as we comprehend, it is very challenging to create the proper Lyapunov function for system (1.3). This encourages us to work in this area. The remainder of this essay is structured as follows. The existence and uniqueness of a global beneficial solution to the system (1.3) are demonstrated in Section 2. In Section 3, several suitable Lyapunov functions are constructed to illustrate that the global solution of system (1.3) is stationary.

2. Existence and uniqueness of a global positive solution

Theorem 2.1: For any initial value $(x(0), y(0), z(0), w(0)) \in \mathbb{R}^4_+$, there is a unique solution $(x(t), y(t), z(t), w(t))$ of system (1.3) on $t \ge 0$ and the solution will remain in $\mathbb{R}^4_+$ with probability 1, i.e., $(x(t), y(t), z(t), w(t)) \in \mathbb{R}^4_+$ for $t \ge 0$ almost surely ($a.s.$).

Proof. In light of the similarity with [14], the beginning of the proof is omitted. We only present the key stochastic Lyapunov function.

Define a C^2-function $Q(x, y, z, w)$ by

$$Q(x, y, z, w) = x - c_1 - c_1 \ln \frac{x}{c_1} + y - c_2 - c_2 \ln \frac{y}{c_2} + z - 1 - \ln z + c_3 w - 1 - \ln c_3 w.$$

where c_1, c_2, c_3 are positive constant to be determined later. The nonnegativity of this function can be seen from

$$u - 1 - \ln u \geq 0 \quad \text{for any} \quad u > 0.$$

Using Itô's formula, we get

$$\mathrm{d}Q = LQ\mathrm{d}t + \sigma_1(x - c_1)\mathrm{d}B_1 + \sigma_2(y - c_2)dB_2 + \sigma_3(z - 1)\mathrm{d}B_3,$$

where

$$\begin{aligned}
LQ &= (1 - \frac{c_1}{x})\Big(\lambda - dx - \beta xy\Big) + (1 - \frac{c_2}{y})\Big(\beta xy - ay - pyz\Big) + (1 - \frac{1}{z})\Big(cw - bz\Big) \\
&+ (c_3 - \frac{1}{w})\Big(\alpha yz - \alpha w\Big) + \frac{1}{2}c_1\sigma_1^2 + \frac{1}{2}c_2\sigma_2^2 + \frac{1}{2}\sigma_3^2 \\
&= \lambda - dx - \beta xy + c_1(-\frac{\lambda}{x} + d + \beta y + \frac{1}{2}\sigma_1^2) + \beta xy - ay - pyz + c_2(-\beta x + a + pz + \frac{1}{2}\sigma_2^2) \\
&+ cw - bz - \frac{cw}{z} + b + \frac{1}{2}\sigma_3^2 + c_3(\alpha yz - \alpha w) - \frac{\alpha yz}{w} + \alpha \\
&\leq \lambda + c_1 d + c_1\frac{1}{2}\sigma_1^2 + c_2 a + \frac{1}{2}c_2\sigma_2^2 + b + \frac{1}{2}\sigma_3^2 + \alpha + (c_1\beta - a)y + (c_2 p - b)z + (c_3\alpha - p)yz \\
&+ (c - c_3\alpha)w.
\end{aligned}$$

Let $c_1 = \frac{a}{\beta}$, $c_2 = \frac{b}{p}$, $0 < c_3 \leq \min\{\frac{p}{\alpha}, \frac{c}{\alpha}\}$ such that $c_1\beta - a = 0$, $c_2 p - b = 0$, $c_3\alpha - p \leq 0$, $c - c_3\alpha \leq 0$. Then,

$$LQ \leq \lambda + c_1 d + c_1\frac{1}{2}\sigma_1^2 + c_2 a + \frac{1}{2}c_2\sigma_2^2 + b + \frac{1}{2}\sigma_3^2 + \alpha := k_0.$$

Obviously, k_0 is a positive constant which is independent of x, y, z and w. Hence, we omit the rest of the proof of Theorem 2.1 since it is mostly similar to Wang [14]. This completes the proof.

3. Existence of stationary distribution

We need the following lemma to prove our main result. Consider the integral equation:

$$\mathrm{d}X(t) = X(t_0) + \int_{t_0}^{t} b(s, X(s))\mathrm{d}s + \sum_{n=1}^{m} \int_{t_0}^{t} \sigma_n(s, X(s))\mathrm{d}\beta_n(s). \tag{3.1}$$

Lemma 3.1 ([15]). Suppose that the coefficients of (3.1) are independent of t and satisfy the following conditions for some constant B:

$$\begin{aligned}
|b(s, x) - b(s, y)| + \sum_{n=1}^{m} |\sigma_n(s, x) - \sigma_n(s, y)| &\leq B|x - y|, \\
|b(s, x)| + \sum_{n=1}^{m} |\sigma_n(s, x)| &\leq B(1 + |x|),
\end{aligned} \tag{3.2}$$

in $D_\rho \in \mathbb{R}^d_+$ for every $\rho > 0$, and that there exists a nonnegative C^2-function $V(x)$ in $\mathbb{R}^d_+$ such that

$$LV \leq -1 \ \text{ outside some compact set.} \tag{3.3}$$

Then, system (3.1) has a solution, which is a stationary Markov process.

Here, we present a stationary distribution theorem. Define

$$R^s := \frac{\Lambda}{2} - \frac{8c^2r^2}{\lambda(d-\sigma_1^2)(r-\sigma_2^2)^2}\left[\left(1+\frac{(d-\sigma_1^2)(r-\sigma_2^2)}{2r^2}\right)\sigma_1^2\bar{x} + \frac{\bar{y}}{2}\left(\frac{a}{\beta} + \frac{(d-\sigma_1^2)(r-\sigma_2^2)\bar{y}}{r^2}\right)\sigma_2^2\right],$$

where $\Lambda = c\bar{y} - (b + \frac{\sigma_3^2}{2}) > 0$, $r = d \wedge a$, and we denote $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$.

Theorem 3.1. Assume $R^s > 0$, $d - \sigma_1^2 > 0$ and $r - \sigma_2^2 > 0$. Then there exists a positive solution $(x(t), y(t), z(t), w(t))$ of system (1.3) which is a stationary Markov process.

Proof. We can substitute the global existence of the solutions of model (1.3) for condition (3.2) in Lemma 3.1, based on Remark 5 of Xu et al. [16]. We have established that system (1.3) has a global solution by Theorem 2.1. Thus condition (3.2) is satisfied. We simply need to confirm that condition (3.3) holds. This means that for any $(x, y, z, w) \in \mathbb{R}^4_+ \backslash D_\epsilon$, $LV(x, y, z, w) \leq -1$, we only need to construct a nonnegative C^2-function V and a closed set D_ϵ. As a convenience, we define

$$\begin{aligned}
V_1(x,y,z,w) &= -\ln z - \frac{e_1}{\alpha}\ln w + l\left[\frac{(x-\bar{x})^2}{2} + \frac{a}{\beta}\left(y - \bar{y} - \bar{y}\ln\frac{y}{\bar{y}}\right) + \frac{(d-\sigma_1^2)(r-\sigma_2^2)}{2r^2}\frac{(x-\bar{x}+y-\bar{y})^2}{2}\right.\\
&\quad \left. + \frac{ap\bar{y}}{b\beta}z + \frac{ap\bar{y}c}{\alpha b\beta}w\right]\\
&:= Q_1 + l\left[U_1 + \frac{a}{\beta}U_2 + \frac{(d-\sigma_1^2)(r-\sigma_2^2)}{2r^2}U_3 + Q_3\right]\\
&:= Q_1 + l(Q_2 + Q_3),
\end{aligned}$$

where e_1 is a positive constant to be determined later, $l = \frac{8r^2c^2}{(d-\sigma_1^2)(r-\sigma_2^2)\Lambda}$, $U_1 = \frac{(x-\bar{x})^2}{2}$, $U_2 = y - \bar{y} - \bar{y}\ln\frac{y}{\bar{y}}$, $U_3 = \frac{(x-\bar{x}+y-\bar{y})^2}{2}$, $Q_1 = -\ln z - \frac{e_1}{\alpha}\ln w$, $Q_2 = U_1 + \frac{a}{\beta}U_2 + \frac{(d-\sigma_1^2)(r-\sigma_2^2)}{2r^2}U_3$, $Q_3 = \frac{ap\bar{y}}{b\beta}z + \frac{ap\bar{y}c}{\alpha\beta b}w$.

Since $\lambda - d\bar{x} = \beta\bar{x}\bar{y} = a\bar{y}$, we apply Itô's formula to obtain

$$\begin{aligned}
LU_1 &= (x-\bar{x})\Big[\lambda - \mathrm{d}x - \beta xy\Big] + \frac{1}{2}\sigma_1^2x^2\\
&= (x-\bar{x})\Big[-d(x-\bar{x}) + \beta(\bar{x}\bar{y} - xy)\Big] + \frac{1}{2}\sigma_1^2(x - \bar{x} + \bar{x})^2\\
&= (x-\bar{x})\Big[-d(x-\bar{x}) + \beta\Big(\bar{x}\bar{y} - \bar{x}y + \bar{x}y - xy\Big)\Big] + \frac{1}{2}\sigma_1^2(x-\bar{x}+\bar{x})^2\\
&\leq -d(x-\bar{x})^2 - \beta(x-\bar{x})^2y - \beta(x-\bar{x})(y-\bar{y})\bar{x} + \sigma_1^2\bar{x}^2 + \sigma_1^2(x-\bar{x})^2\\
&\leq -d(x-\bar{x})^2 - a(x-\bar{x})(y-\bar{y}) + \sigma_1^2(x-\bar{x})^2 + \sigma_1^2\bar{x}^2\\
&= -(d-\sigma_1^2)(x-\bar{x})^2 - a(x-\bar{x})(y-\bar{y}) + \sigma_1^2\bar{x}^2,
\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
LU_2 &= (1 - \frac{\bar{y}}{y})(\beta xy - ay - pyz) + \frac{1}{2}\bar{y}\sigma_2^2 \\
&= (y - \bar{y})(\beta x - a - pz) + \frac{1}{2}\sigma_2^2\bar{y} \\
&= (y - \bar{y})(\beta x - \beta\bar{x} + \beta\bar{x} - a - pz) + \frac{1}{2}\sigma_2^2\bar{y} \\
&= (y - \bar{y})\Big(\beta(x - \bar{x}) - pz\Big) + \frac{1}{2}\sigma_2^2\bar{y} \\
&= \beta(x - \bar{x})(y - \bar{y}) - p(y - \bar{y})z + \frac{1}{2}\sigma_2^2\bar{y} \\
&\leq \beta(x - \bar{x})(y - \bar{y}) + p\bar{y}z + \frac{\bar{y}}{2}\sigma_2^2,
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
LU_3 &= \Big(x - \bar{x} + y - \bar{y}\Big)\Big(\lambda - \mathrm{d}x - ay - pyz\Big) + \frac{1}{2}\sigma_1^2x^2 + \frac{1}{2}\sigma_2^2y^2 \\
&= \Big(x - \bar{x} + y - \bar{y}\Big)\Big(\lambda - \mathrm{d}x + \mathrm{d}\bar{x} - \mathrm{d}\bar{x} - ay - pyz\Big) + \frac{1}{2}\sigma_1^2x^2 + \frac{1}{2}\sigma_2^2y^2 \\
&= \Big(x - \bar{x} + y - \bar{y}\Big)\Big(- d(x - \bar{x}) - a(y - \bar{y}) - pyz\Big) + \frac{1}{2}\sigma_1^2\Big(x - \bar{x} + \bar{x}\Big)^2 + \frac{\sigma_2^2}{2}\Big(y - \bar{y} + \bar{y}\Big)^2 \\
&\leq -\Big(d \wedge a\Big)\Big(x - \bar{x} + y - \bar{y}\Big)^2 - p\Big(x - \bar{x} + y - \bar{y}\Big)yz + \sigma_1^2\Big(x - \bar{x}\Big)^2 + \sigma_1^2\bar{x}^2 + \sigma_2^2(y - \bar{y})^2 + \sigma_2^2\bar{y}^2 \\
&= -\Big(d \wedge a\Big)\Big(x - \bar{x}\Big)^2 - \Big(d \wedge a\Big)\Big(y - \bar{y}\Big)^2 - 2\Big(d \wedge a\Big)\Big(x - \bar{x}\Big)\Big(y - \bar{y}\Big) + p\Big(\bar{x} + \bar{y}\Big)yz + \sigma_1^2\Big(x - \bar{x}\Big)^2 \\
&+ \sigma_1^2\bar{x}^2 + \sigma_2^2\Big(y - \bar{y}\Big)^2 + \sigma_2^2\bar{y}^2 \\
&= -(r - \sigma_1^2)\Big(x - \bar{x}\Big)^2 - (r - \sigma_2^2)\Big(y - \bar{y}\Big)^2 - 2r\Big(x - \bar{x}\Big)\Big(y - \bar{y}\Big) + \frac{p(a^2 + \beta\lambda - ad)}{a\beta}\,yz \\
&+ \sigma_1^2\bar{x}^2 + \sigma_2^2\bar{y}^2 \\
&\leq -(r - \sigma_2^2)\Big(y - \bar{y}\Big)^2 + \frac{(r - \sigma_2^2)}{2}\Big(y - \bar{y}\Big)^2 + \frac{2r^2}{r - \sigma_2^2}\Big(x - \bar{x}\Big)^2 + \frac{p(a^2 + \beta\lambda - ad)}{a\beta}\,yz \\
&+ \sigma_1^2\bar{x}^2 + \sigma_2^2\bar{y}^2 \\
&= -\frac{(r - \sigma_2^2)}{2}\Big(y - \bar{y}\Big)^2 + \frac{2r^2}{r - \sigma_2^2}\Big(x - \bar{x}\Big)^2 + \frac{p(a^2 + \beta\lambda - ad)}{a\beta}\,yz + \sigma_1^2\bar{x}^2 + \sigma_2^2\bar{y}^2,
\end{aligned}
\tag{3.6}
$$

where $r = d \wedge a$, we also use the basic inequality $(a + b)^2 \leq 2(a^2 + b^2)$ and Young inequality. It follows from (3.4)–(3.6) that

$$
\begin{aligned}
LQ_2 \leq &-\frac{(d - \sigma_1^2)(r - \sigma_2^2)}{4r^2}(y - \bar{y})^2 + \frac{(d - \sigma_1^2)(r - \sigma_2^2)}{2r^2}\,\frac{p(a^2 + \beta\lambda - ad)}{a\beta}\,yz + \frac{ap\,\bar{y}}{\beta}\,z \\
&+ \Big(1 + \frac{(d - \sigma_1^2)(r - \sigma_2^2)}{2r^2}\Big)\sigma_1^2\bar{x}^2 + \Big(\frac{a}{\beta} + \frac{(d - \sigma_1^2)(r - \sigma_2^2)\bar{y}}{r^2}\Big)\frac{\sigma_2^2\bar{y}}{2},
\end{aligned}
$$

Making use of Itô's formula to Q_3 yields

$$\begin{aligned} LQ_3 &= \frac{ap\bar{y}}{b\beta}(cw - bz) + \frac{ap\bar{y}\,c}{\alpha b\beta}(\alpha yz - \alpha w) \\ &= -\frac{ap\bar{y}}{\beta}z + \frac{ap\bar{y}c}{b\beta}yz. \end{aligned}$$

Therefore,

$$\begin{aligned} L(Q_2 + Q_3) \leq & -\frac{(d-\sigma_1^2)(r-\sigma_2^2)}{4r^2}(y-\bar{y})^2 + \frac{p}{\beta}\Big(\frac{ac\bar{y}}{b} + \frac{(d-\sigma_1^2)(r-\sigma_2^2)(a^2+\beta\lambda-ad)}{2r^2a}\Big)yz \\ & + \Big(1 + \frac{(d-\sigma_1^2)(r-\sigma_2^2)}{2r^2}\Big)\sigma_1^2\bar{x}^2 + \Big(\frac{a}{\beta} + \frac{(d-\sigma_1^2)(r-\sigma_2^2)\bar{y}}{r^2}\Big)\frac{\sigma_2^2\bar{y}}{2}. \end{aligned} \tag{3.7}$$

In addition,

$$\begin{aligned} LQ_1 &= -\frac{cw}{z} - \frac{e_1yz}{w} + e_1 + b + \frac{1}{2}\sigma_3^2 \\ &\leq -2\sqrt{y\,c\,e_1} + e_1 + b + \frac{1}{2}\sigma_3^2 \\ &= -2\sqrt{\bar{y}\,c\,e_1} + e_1 + b + \frac{1}{2}\sigma_3^2 - 2\sqrt{c\,e_1}(\sqrt{y} - \sqrt{\bar{y}}). \end{aligned}$$

Letting $e_1 = c \cdot \bar{y}$, by virtue of Young inequality, one gets

$$\begin{aligned} LQ_1 &\leq -c\bar{y} + b + \frac{1}{2}\sigma_3^2 + \frac{2c\sqrt{\bar{y}}\,|y-\bar{y}|}{\sqrt{y} + \sqrt{\bar{y}}} \\ &\leq -\Lambda + 2c|y-\bar{y}| \\ &\leq -\frac{\Lambda}{2} + \frac{2c^2}{\Lambda}(y-\bar{y})^2. \end{aligned}$$

Together with (3.7), this results in

$$\begin{aligned} LV_1 &\leq -R^s + \Big(\frac{2r^2ac\bar{y}}{b(d-\sigma_1^2)(r-\sigma_2^2)} + \frac{a^2+\beta\lambda-ad}{a}\Big)\frac{4c^2p}{(r-\sigma_2^2)\lambda\beta}yz \\ &= -R^s + qyz, \end{aligned} \tag{3.8}$$

where

$$q = \Big(\frac{2r^2ac\bar{y}}{b(d-\sigma_1^2)(r-\sigma_2^2)} + \frac{a^2+\beta\lambda-ad}{a}\Big)\frac{4c^2p}{(r-\sigma_2^2)\lambda\beta}.$$

Define

$$V_2(x) = -\ln x, \quad V_3(w) = -\ln w.$$

Then, we obtain

$$LV_2 = -\frac{\lambda}{x} + d + \beta y + \frac{1}{2}\sigma_1^2,$$

and

$$LV_3 = -\frac{yz}{w} + \alpha. \tag{3.9}$$

Define

$$V_4(x,y,z,w)=\frac{1}{\theta+2}(x+y+\frac{p}{2c}z+\frac{p}{\alpha}w)^{\theta+2},$$

where θ is a constant satisfying $0<\theta<\min\{\frac{d-\frac{\sigma_1^2}{2}}{d+\frac{\sigma_1^2}{2}},\frac{a-\frac{\sigma_2^2}{2}}{a+\frac{\sigma_2^2}{2}},\frac{b-\frac{\sigma_3^2}{2}}{b+\frac{\sigma_3^2}{2}}\}$. Then,

$$LV_4=\Big(x+y+\frac{p}{2c}z+\frac{p}{\alpha}w\Big)^{\theta+1}\Big(\lambda-dx-ay-\frac{pb}{2c}z-\frac{p}{2}w\Big)$$

$$\begin{aligned}LV_4=&\Big(x+y+\frac{p}{2c}z+\frac{p}{\alpha}w\Big)^{\theta+1}\Big(\lambda-dx-ay-\frac{pb}{2c}z-\frac{p}{2}w\Big)\\&+\frac{\theta+1}{2}\Big(x+y+\frac{p}{2c}z+\frac{p}{\alpha}w\Big)^{\theta}\Big(\sigma_1^2x^2+\sigma_2^2y^2+(\frac{p}{2c})^2\sigma_3^2z^2\Big)\\&\le\lambda\Big(x+y+\frac{p}{2c}z+\frac{p}{\alpha}w\Big)^{\theta+1}-dx^{\theta+2}-ay^{\theta+2}-b(\frac{p}{2c})^{\theta+2}z^{\theta+2}-\frac{1}{2}\frac{p^{\theta+2}}{\alpha^{\theta+1}}w^{\theta+2}\\&+\frac{\theta+1}{2}\Big(x+y+\frac{p}{2c}z+\frac{p}{\alpha}w\Big)^{\theta}\Big(\sigma_1^2x^2+\sigma_2^2y^2+(\frac{p}{2c})^2\sigma_3^2z^2\Big)\\&\le F_1-d\theta x^{\theta+2}-a\theta y^{\theta+2}-b\theta(\frac{p}{2c})^{\theta+2}z^{\theta+2}-\frac{1}{4}\frac{p^{\theta+2}}{\alpha^{\theta+1}}w^{\theta+2},\end{aligned}\tag{3.10}$$

in which

$$\begin{aligned}F_1=\sup_{(x,y,z,w)\in\mathbb{R}_+^4}&\Big\{\lambda\Big(x+y+\frac{p}{2c}z+\frac{p}{\alpha}w\Big)^{\theta+1}-d(1-\theta)x^{\theta+2}-a(1-\theta)y^{\theta+2}-b(1-\theta)(\frac{p}{2c})^{\theta+2}z^{\theta+2}\\&-\frac{1}{4}\frac{p^{\theta+2}}{\alpha^{\theta+1}}w^{\theta+2}+\frac{\theta+1}{2}\Big(x+y+\frac{p}{2c}z+\frac{p}{\alpha}w\Big)^{\theta}\Big(\sigma_1^2x^2+\sigma_2^2y^2+(\frac{p}{2c})^2\sigma_3^2z^2\Big)\Big\}<\infty.\end{aligned}$$

Construct

$$G(x,y,z,w)=MV_1(x,y,z,w)+V_2(x)+V_3(w)+V_4(x,y,z,w),$$

where $M>0$, satisfies

$$-MR^s+F_2\le-2,$$

and

$$F_2=\sup_{y\in\mathbb{R}_+}\Big\{\beta y-\frac{a\theta}{2}y^{\theta+2}+d+\alpha+\frac{1}{2}\sigma_1^2+F_1\Big\}.\tag{3.11}$$

Note that G is a continuous function and $\liminf_{n\to\infty,(x,y,z,w)\in\mathbb{R}_+^4\setminus Q_n}G(x,y,z,w)=+\infty$, where $Q_n=(\frac{1}{n},n)\times(\frac{1}{n},n)\times(\frac{1}{n},n)\times(\frac{1}{n},n)$. Thus, $G(x,y,z,w)$ has a minimum point (x_0,y_0,z_0,w_0) in the interior of $\mathbb{R}_+^4$. Define a nonnegative C^2 -function by

$$V(x,y,z,w)=G(x,y,z,w)-G(x_0,y_0,z_0,w_0)$$

In view of (3.8)–(3.10) and (3.11), we get

$$LV\le-MR^s+M\,q\,y\,z\,-\frac{\lambda}{x}-\frac{yz}{w}-d\theta x^{\theta+2}-\frac{a\theta}{2}y^{\theta+2}-b\theta(\frac{p}{2c})^{\theta+2}z^{\theta+2}-\frac{1}{4}\frac{p^{\theta+2}}{\alpha^{\theta+1}}w^{\theta+2}+F_2,\tag{3.12}$$

One can easily see from (3.12) that, if $y \to 0^+$ or $z \to 0^+$, then

$$LV \le -MR^s + F_2 \le -2;$$

if $x \to 0^+$ or $w \to 0^+$ or $x \to +\infty$ or $y \to +\infty$ or $z \to +\infty$ or $w \to +\infty$, then

$$LV \le -\infty.$$

In other words,

$$LV \le -1 \text{ for any } (x, y, z, w) \in \mathbb{R}_+^4 \backslash D_\epsilon,$$

where $D_\epsilon = \{(x, y, z, w) \in \mathbb{R}_+^4 : \epsilon \le x \le \frac{1}{\epsilon}, \epsilon \le y < \frac{1}{\epsilon}, \epsilon \le z \le \frac{1}{\epsilon}, \epsilon^3 \le w \le \frac{1}{\epsilon^3}\}$ and ϵ is a sufficiently small constant. The proof is completed.

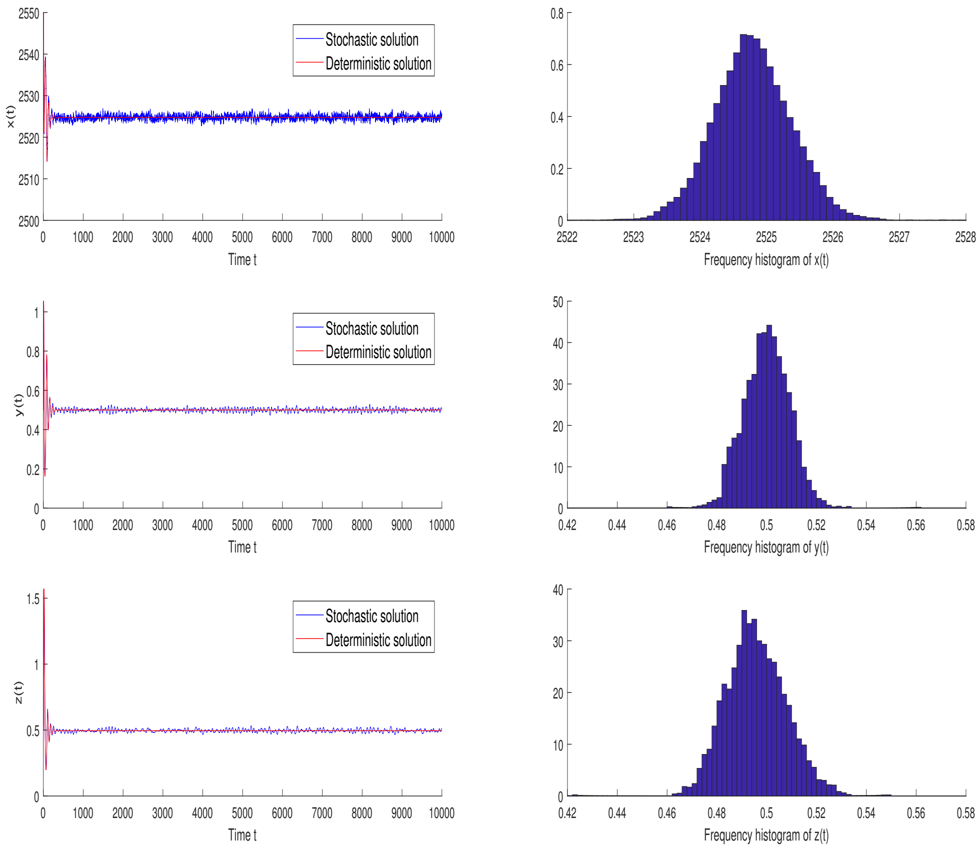

Figure 1. The solution$(x(t), y(t), z(t))$ in system (1.4) and stochastic system (1.3) with the white noise $\sigma_1 = \sigma_2 = \sigma_3 = 0.0001$ are numerically simulated in the left-hand column. The frequency histograms for x, y and z in system (1.3) are displayed in the right-hand column.

Remark 3.1. In the proof of above theorem, the construction of V_1 is one of the difficulties. The term Q_3 in V_1 is is constructed to eliminate $\frac{ap\bar{y}}{\beta}z$ in LQ_2. The item $l(Q_2 + Q_3)$ is used to eliminate $\frac{2c^2}{\Lambda}(y - \bar{y})^2$ in LQ_1.

Remark 3.2 . From the expression of R^s, we can see that if there is no white noise, $R^s > 0$ is equivalent to $R_1 > 1 + \frac{b\beta}{cd}$.

4. Numerical simulations

Using the well-known numerical method of Milstein [17] , we get the discretization equation for system (1.3)

$$\begin{cases} x_{k+1} = x_k + \Big(\lambda - dx_k - \beta x_k y_k\Big)\triangle t + \sigma_1 x_k \sqrt{\triangle t}\eta_{1,k} + \dfrac{\sigma_1^2 x_k}{2}(\eta_{1,k}^2 - 1)\triangle t, \\ y_{k+1} = y_k + \Big(\beta x_k y_k - ay_k - py_k z_k\Big)\triangle t + \sigma_2 y_k \sqrt{\triangle t}\eta_{2,k} + \dfrac{\sigma_2^2 y_k}{2}(\eta_{2,k}^2 - 1)\triangle t, \\ z_{k+1} = z_k + \Big(cw_k - bz_k\Big)\triangle t + \sigma_3 z_k \sqrt{\triangle t}\eta_{3,k} + \dfrac{\sigma_3^2 z_k}{2}(\eta_{3,k}^2 - 1)\triangle t, \\ w_{k+1} = w_k + \Big(\alpha y_k z_k - \alpha w_k\Big)\triangle t. \end{cases}$$

where the time increment $\triangle t > 0$ and $\eta_{i,k}$ $(i = 1, 2, 3)$ are three independent Gaussian random variables which follow the distribution $N(0, 1)$, equivalently, they come from the three independent from each other components of a three dimensional Wiener process with zero mean and variance $\triangle t$, for $k = 1, 2, \cdots$. According to Xie et al. [4], the corresponding biological parameters of system (1.3) are assumed: $\lambda = 255, \alpha = 1, d = 0.1, \beta = 0.002, a = 5, p = 0.1, c = 0.2, b = 0.1, r = d \wedge a = 0.1$, $\sigma_1 = \sigma_2 = \sigma_3 = 0.0001$. The initial condition is $(x_0, y_0, z_0, w_0) = (2600, 0.5, 0.5, 0.25)$. Then, we calculate that $R^s = 0.05 > 0$. Based on Theorems 2.1 and 3.1, we can conclude that system (1.3) admits a global positive stationary solution on $\mathbb{R}_+^4$, see the left-hand figures of Figure 1 and the corresponding histograms of each population can be seen in right-hand column.

5. Conclusions

In this paper, we consider a special kernel function $F(t) = \alpha e^{-\alpha t}$ to investigate the continuous delay effect on the population of stochastic viral infection systems. We derived the sufficient conditions for the existence of stationary distribution by constructing a suitable stochastic Lyapunov function. In addition, we only consider the effect of white noise on the dynamics of the viral infection system with distributed delays. It is interesting to consider the effect of Lévy jumps. Some researchers [18,19] studied the persistence and extinction of the stochastic systems with Lévy jumps. Furthermore, it should be noted that the system may be analytically solved by using the Lie algebra method [20,21]. In our further research, we will study the existence of a unique stationary distribution of the stochastic systems with distributed delay and Lévy jumps. Also, it may be possible to solve the stochastic model via the Lie algebra method.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

No potential conflict of interest.

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