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Kernel-based Estimation of Ageing Intensity Function: Properties and Applications

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Abstract

The notion of ageing plays an important role in reliability and survival analysis as it is an inherent property of all systems and products. Jiang, Ji, and Xiao (2003) proposed a new quantitative measure, known as ageing intensity (AI) function, an alternative measure to study the ageing pattern of probability models. In this paper, we propose a non-parametric estimator for ageing intensity function. Asymptotic properties of the estimator are established under suitable regularity conditions. A set of simulation studies are carried out based on various probability models to examine the performance of estimator and to establish its efficiency over the classical estimator. The usefulness of the estimator is also examined through a real data set.

Keywords: ageing intensity function, Kernel density estimation, mean squared error.

1. Introduction

Let X be a non-negative random variable representing lifetime of a living organism or component/system with an absolutely continuous cumulative distribution function $F(.)$, survival function $\bar{F}(.) = 1 - F(.)$ and hazard function $h(.) = \frac{f(.)}{\bar{F}(.)}$. Then AI function of X is defined as the ratio of hazard rate to a baseline hazard rate. When the baseline hazard rate is average hazard rate $\frac{1}{t}\int_0^t h(u)du$, the AI function is defined by

$$L(t) = \frac{h(t)}{\frac{1}{t}\int_0^t h(u)du} = -\frac{t\ f(t)}{\bar{F}(t)\log(\bar{F}(t))} = \frac{t\ h(t)}{H(t)}, \tag{1}$$

where $H(.)$ denote the cumulative hazard rate function. The larger the value of $L(.)$, the stronger the tendency of ageing of the random variable X. Also, $L(t) = c$, for $t > 0$, c being a constant, characterizes the family of Weibull distribution with shape parameter c. It is to be noted that the hazard rate function uniquely determines the AI function but not conversely.

$$L(t) = \frac{h(t)}{\frac{1}{t}\int_0^t h(u)du} \Leftrightarrow \frac{L(t)}{t} = \frac{d}{dt}\log\left[\int_0^t h(u)du\right] \Leftrightarrow h(t) \propto \frac{d}{dt}\left\{exp\left[\int_0^t \frac{L(u)}{u}du\right]\right\}. \tag{2}$$

Jiang *et al.* (2003) introduced AI function as a quantitative measure in determining the ageing behaviour of a lifetime random variable. Jiang *et al.* (2003) proposed a general theoretic framework to optimal burn-in problem when the hazard rate of a product is unimodal. It includes three main components. First, any unimodal hazard rate can be viewed as either quasi-decreasing, quasi-constant, or quasi-increasing. Secondly, in the ageing tendency analysis, AI function reveals the relationship between the ageing property and the model parameters. Thirdly, the AI function provide two critical values of the model parameters, indicating the partitions between different ageing features. Nanda, Bhattacharjee, and Alam (2007) proposed an ageing intensity ordering and studied its closure properties under different reliability operations, for the formation of k-out-of-n system, and increasing transformations. Sunoj and Rasin (2018) have introduced a quantile-based ageing intensity function and obtained various ageing properties and ordering relationships. Recently, a family of generalized ageing intensity functions of univariate absolutely continuous lifetime random variables has been studied by Szymkowiak (2019). For more results and generalizations of AI function, one can refer to Bhattacharjee, Nanda, and Misra (2013a,b), Misra and Bhattacharjee (2018), Szymkowiak (2018, 2020), Sunoj, Nair, Nanda, and Rasin (2020), and the references therein.

In the present paper, we propose a non-parametric kernel estimator for ageing intensity function for complete sample in the identically and independently distributed case and numerically establish that it is efficient than the classical empirical estimator. The paper is organized as follows. In Section 2, we propose a non-parametric estimator for $L_X(t)$ and obtained its Bias and Mean Squared Error (MSE). In Section 3, the asymptotic properties of the estimator are studied under suitable regularity conditions. In Section 4, simulation studies are is carried out using different probability models to validate the efficiency of the estimator and to examine the usefulness of the estimator in real situations.

2. Non-parametric estimation of $L(t)$

Let $\{X_i; 1 \leq i \leq n\}$ be a sequence of identical and independent random variables representing lifetimes for n components or devices. $X_i's$ have a common absolutely continuous distribution function $F(.)$, probability density function $f(.) = F'(.)$, survival function $\bar{F}(.) = 1 - F(.)$, hazard rate function $h(.)$ and cumulative hazard rate $H(.)$. Nanda *et al.* (2007) defined a classical (empirical) estimator for AI function. Here we define the classical estimator as follows,

2.1. Classical estimator

For failure data, let N units be put to test at $t = 0$. Further, let the number of units having survived at ordered times t_j be $N_s(t_j)$; $j = 0, 1, 2, \ldots, k$. Then a classical estimate for $L_X(t)$, for $t > 0$, is

$$L_n(t) = \frac{-t\left\{N_s\left(t_j\right) - N_s\left(t_j + \Delta t_j\right)\right\}}{N_s\left(t_j\right)\Delta t_j \log \frac{N_s(t_j)}{N}}, \quad \text{for } t_j < t < t_j + \Delta t_j. \tag{3}$$

Szymkowiak (2018) has also analyzed $L_n(t)$ through generated data, real complete data and censored data sets. Further, a kernel-based estimation of AI function was initially proposed by Misra and Bhattacharjee (2018) through a case study of bone marrow transplantation data of leukemia patients in which certain observations censored. The estimator proposed by Misra and Bhattacharjee (2018) made use of plug-in estimator for hazard rate of Nelson-Aalen, without studying the asymptotic properties of the estimator. Motivated with these, in the present study we propose a non-parametric kernel-based estimator for AI function in the complete sample case and study its asymptotic properties.

2.2. Kernel-based estimator of $L(t)$

From equation (1), $L(t)$ is the ratio of two functions of t. Using Chen, Hsu, and Liaw (2009) for the iid case, a new non-parametric estimator of $L(t)$ denoted as $L_n^*(t)$ by combining the kernel estimators of $h(t)$ and $H(t)$, denoted respectively $h_n(t)$ and $H_n(t)$ becomes,

$$L_n^*(t) = \frac{t\ h_n(t)}{H_n(t)}. \tag{4}$$

Here, $h_n(.)$ is the kernel density estimator of $h(.)$ (see Roussas (1989)) and $H_n(.)$ is the corresponding estimator of cumulative hazard function $H(.)$, and $h_n(.)$ is obtained from

$$h_n(t) = \frac{f_n(t)}{\bar{F}_n(t)} \tag{5}$$

where $f_n(t)$ is the kernel density function of $f(.)$, defined by,

$$f_n(t) = \frac{1}{n\ b}\sum_{i=1}^{n} K\left(\frac{t - X_i}{b}\right). \tag{6}$$

with kernel function $K(.)$ and band width b, and $\bar{F}_n(.) = 1 - F_n(.)$ is the empirical survival function, with the empirical distribution function

$$F_n(t) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq t). \tag{7}$$

Then,

$$H_n(t) = \int_0^t h_n(u)du, \tag{8}$$

is an integral estimator of $H(t)$. Applying Taylor series expansion to $\frac{1}{H_n(t)}$ in (4), we obtain

$$\frac{1}{H_n(t)} = \frac{1}{H(t)} - \frac{(H_n(t) - H(t))}{(H(t))^2} + R_n, \tag{9}$$

where $R_n = R_n(t) = 2\int_0^1 \frac{(1-\tau)[H_n(t) - H(t)]}{[H(t) + \tau\,(H_n(t) - H(t))]^3]} d\tau$. By using equation (5), (8) & (9), the estimator $L_n^*(t)$ in (4) becomes,

$$L_n^*(t) = \frac{t\ h_n(t)}{H_n(t)} = \frac{t\ h_n(t)}{H(t)} - \frac{(H_n(t) - H(t))\,t\ h_n(t)}{H(t)^2} + R_n. \tag{10}$$

3. Asymptotic properties

In this section, we prove the consistency, asymptotic normality of $L_n^*(t)$.

Theorem 3.1. *The Bias and MSE of $L_n^*(.)$ are respectively obtained as*

$$\begin{aligned} Bias(L_n^*(t)) &= \frac{t\ Bias(h_n(t))}{H(t)} - \frac{t\ Cov\,(H_n(t), h_n(t))}{[H(t)]^2} - \frac{t\ h(t) Bias(H_n(t))}{[H(t)]^2} \\ &\quad + t\ E\,((h_n(t) - h(t))R_n) + t\ h(t)E(R_n) \end{aligned} \tag{11}$$

and

$$\begin{aligned} MSE(L_n^*(t)) &= \frac{t^2\ MSE(h_n(t))}{[H(t)]^2} - \frac{t^2\ E[(h_n(t) - h(t))^2\ (H_n(t) - H(t))^2]}{[H(t)]^4} \\ &\quad - \frac{t^2\ h(t)^2}{[H(t)]^4} MSE(H_n(t)) + t^2\ E[(h_n(t) - h(t))^2\ R_n^2] + t^2\ h(t)^2\ ER_n^2 \end{aligned} \tag{12}$$

Proof: Using (1) and (10), we obtain

$$\begin{aligned} L_n^*(t) - L(t) &= \frac{t\ (h_n(t) - h(t))}{H(t)} - \frac{t\ (H_n(t) - H(t))\ h_n(t)}{[H(t)]^2} + t\ h_n(t)\ R_n \\ &= \frac{t\ (h_n(t) - h(t))}{H(t)} - \frac{t\ (H_n(t) - H(t))\ (h_n(t) - h(t))}{[H(t)]^2} \\ &\quad - \frac{t\ (H_n(t) - H(t))\ h(t)}{[H(t)]^2} + t\ (h_n(t) - h(t))\ R_n + t\ h(t)\ R_n. \end{aligned} \tag{13}$$

From (13), we get

$$\begin{aligned} Bias(L_n^*(t)) &= E\left(L_n^*(t) - L(t)\right) \\ &= \frac{t\ E(h_n(t) - h(t))}{H(t)} - \frac{tE(H_n(t) - H(t))(h_n(t) - h(t))}{[H(t)]^2} \\ &\quad - \frac{t\ E(H_n(t) - H(t))h(t)}{[H(t)]^2} + t\ E[(h_n(t) - h(t))\ R_n] + t\ h(t)\ E(R_n) \\ &= \frac{t\ Bias(h_n(t))}{H(t)} - \frac{t\ Cov\left(H_n(t), h_n(t)\right)}{[H(t)]^2} - \frac{t\ h(t)\ Bias(H_n(t))}{[H(t)]^2} \\ &\quad + t\ E\left((h_n(t) - h(t))\ R_n\right) + t\ h(t)\ E(R_n). \end{aligned}$$

Since observations are assumed to be independent, we have

$$\begin{aligned} MSE(L_n^*(t)) &= E(L_n^*(t) - L(t))^2 \\ &= \frac{t^2\ E[(h_n(t) - h(t))]^2}{[H(t)]^2} - \frac{t^2\ E[(h_n(t) - h(t))^2\ (H_n(t) - H(t))^2]}{[H(t)]^4} \\ &\quad - \frac{t^2\ h(t)^2\ E(H_n(t) - H(t))^2}{[H(t)]^4} + t^2\ E[(h_n(t) - h(t))^2\ R_n^2] + t^2\ h(t)^2\ E(R_n^2) \\ &= \frac{t^2\ MSE(h_n(t))}{[H(t)]^2} - \frac{t^2\ E[(h_n(t) - h(t))^2\ (H_n(t) - H(t))^2]}{[H(t)]^4} \\ &\quad - \frac{t^2\ h(t)^2}{[H(t)]^4}\ MSE(H_n(t)) + t^2\ E[(h_n(t) - h(t))^2\ \ R_n^2] + t^2\ h(t)^2\ E(R_n^2). \end{aligned}$$

□

In the following theorem we prove the week consistency of $L_n^*(.)$.

Theorem 3.2. *Assume that as* $h \to 0$, $n \to \infty$ *we have* $nh \to \infty$ *and* $\frac{1}{nh} \to 0$. *Then* $Bias\left(L_n^*(t)\right) \to 0$ *and* $MSE\left(L_n^*(t)\right) \to 0$.

Proof: From (Rao (2014), p.184) and Chen *et al.* (2009) we have,

$$E|R_n|^j \le \frac{2^j}{[H(t)]^{3-j}}\ E[H_n(t) - H(t)]^{2j} + O(x^j h^j)e^{(-c_{nh})}. \tag{14}$$

For $j = 1, 2$,

$$\begin{aligned} E|R_n| &\le \frac{2}{[H(t)]^2}\ E[H_n(t) - H(t)]^2 + O(n_h)e^{(-c_{nh})} \\ &= \frac{2}{[H(t)]^3}\ MSE(H_n(t)) + O(n_h)e^{(-c_{nh})} \end{aligned}$$

and

$$E|R_n|^2 \le \frac{4}{[H(t)]}\ E[H_n(t) - H(t)]^4 + O(x^2h^2)e^{(-c_{nh})}.$$

By Cauchy-Schwartz inequality, we have the following inequalities

$$\begin{aligned} E[(h_n(t) - h(t))\ (H_n(t) - H(t))] &\le E(h_n(t) - h(t))\ E(H_n(t) - H(t)) \\ &= Bias(h_n(t))\ Bias(H_n(t)), \end{aligned}$$

$$E[(h_n(t) - h(t)R_n] \leq E(h_n(t) - h(t))E(R_n) = Bias(h_n(t))E(R_n),$$

$$\begin{aligned} E[(h_n(t) - h(t))\ (H_n(t) - H(t))]^2 &\leq E(h_n(t) - h(t))^2\ E(H_n(t) - H(t))^2 \\ &\leq MSE(h_n(t))\ MSE(H_n(t))^2, \end{aligned}$$

$$E[(h_n(t) - h(t)\ R_n]^2 \leq E(h_n(t) - h(t)^2\ E(R_n)^2 = MSE(h_n(t))\ E(R_n)^2,$$

$$E[(H_n(t) - H(t)\ R_n]^2 \leq E(H_n(t) - H(t)^2\ E(R_n)^2 = MSE(H_n(t))\ E(R_n)^2,$$

and

$$Cov(h_n(t), H_n(t) \leq Bias(h_n(t))\ Bias(H_n(t)).$$

Then (11) becomes,

$$\begin{aligned} Bias(L_n^*(t)) &\approx \frac{t\ Bias(h_n(t))}{H(t)} - \frac{t\ Bias(H_n(t))\ Bias(h_n(t))}{[H(t)]^2} - \frac{t\ h(t)\ Bias(H_n(t))}{[H(t)]^2} \\ &\quad + t\ Bias(h_n(t))\ E(R_n) + t\ h(t)\ E(R_n). \end{aligned} \tag{15}$$

Under the regularity conditions stated in the statement (Theorem 3.2) and from Chen *et al.* (2009) and Rao (2014), $E(R_n)$ and $E(R_n^2)$ are becomes negligible and we have $h_n(t)$ and $H_n(t)$ are consistent estimators for $h(t)$ and $H(t)$ respectively. Hence the Bias $(L_n(t)) \to 0$, That is, $L_n(t)$ is one of asymptotically unbiased estimators for $L(t)$.

In a similar manner, (12) becomes

$$\begin{aligned} MSE(L_n^*(t)) &\approx \frac{t^2\ MSE(h_n(t))}{[H(t)]^2} - \frac{t^2\ MSE(h_n(t))\ MSE(H_n(t))}{[H(t)]^4} \\ &\quad - \frac{t^2\ h(t)^2}{[H(t)]^4} MSE(H_n(t)) + t^2\ MSE(h_n(t))\ R_n^2 + t^2\ h(t)^2\ E(R_n^2), \end{aligned} \tag{16}$$

by Chen *et al.* (2009), MSE $(L_n(t)) \to 0$. □

Definition 3.1. A sequence θ_n of estimators is integratedly consistent in quadratic mean if the mean integrated squared error (MISE) tends to zero for every $\theta \in \Theta$, a family of univariate θ's, that is,

$$\lim_{n\to\infty} E \int [\theta_n(t) - \theta(t)]^2 dt = 0; \qquad t > 0,$$

(see Rao (2014)).

The selection of bandwidth is also a key factor in density estimation. There are many optimization techniques available in literature to choose most appropriate bandwidth corresponding to a given kernel. The most common optimality criterion used to optimize the bandwidth h is by minimizing the expected risk function, also termed the mean integrated squared error (MISE). For $L_n^*(t)$, for some fixed $t > 0$, it becomes

$$\text{MISE}\,(L_n^*(t)) = \text{E}\,||L_n^*(t) - L(t)||^2 = \text{E}\left[\int (L_n^*(t) - L(t))^2\ dt\right]. \tag{17}$$

Theorem 3.3. *Under the assumptions of Theorem 3.2, $L_n^*(t)$ is intergratedly consistent in quadratic mean. That is, MISE of $L_n^*(t)$ (for some fixed $t > 0$) tends to zero as $n \to \infty$.*

$$\lim_{n\to\infty} \text{E}\left[\int (L_n^*(t) - L(t))^2\ dt\right] = 0. \tag{18}$$

Proof: Under weak assumptions on f and K, (f is the, generally unknown, real density function), $MISE(b) = AMISE(b) + o(1/(nb) + b^4)$ where $AMISE$ is the asymptotic $MISE$ which consists of the two leading terms

$$\text{AMISE}(b) = \frac{R(K)}{nb} + \frac{1}{4}\, m_2(K)^2\, b^4\, R(f''),$$

where $R(g) = \displaystyle\int g(t)^2\, dt$ for a function g, $m_2(K) = \displaystyle\int t^2 K(t)\, dt$ and f'' is the second derivative of f. The minimum of this $AMISE$ is the solution to this differential equation. □

Next we prove the asymptotic normality of $L_n^*(t)$.

Theorem 3.4. *Let* $b \to 0$, $n \to \infty$, $nb \to \infty$ *and* $\frac{1}{nb} \to 0$. *Then,* $\sqrt{nb}\,(L_n^*(t) - L(t)) \mapsto N(\mu, \sigma^2)$.

Proof: Applying Chen *et al.* (2009) for the iid observations, we have

$$\sqrt{nb}\left(\frac{t\ h_n(t)}{H_n(t)} - \frac{t\ h(t)}{H(t)}\right) = \frac{t}{H_n(t)}\left(\sqrt{nb}(h_n(t) - h(t))\right) - \frac{t\ h(t)}{H(t)\ H_n(t)}\left(\sqrt{nb}(H_n(t) - H(t))\right).$$

Since both $h_n(t)$ and $H_n(t)$ are asymptotically normal, and using the Remark 4 in Chen *et al.* (2009), the proof is complete. □

4. Numerical study

In this section, we investigate the performance of $L_n^*(t)$ using simulation and real data sets.

4.1. Simulated study

Let $(X_1, X_2, \ldots, X_n)$ be a random sample of size n taken from an exponential distribution with mean 0.5. We have carried out a series of $m = 1000$ simulations each of sample of sizes $n = 30, 50$ and 100, and estimated $L(t)$ for a fixed t as $(L_1(t), L_2(t), \ldots, L_m(t))$. Then we compute the Bias and MSE for each of these sample sizes. We consider 9 percentiles $P_i's$, of t, which has exactly i out of 10 observations are below P_i and $10-i$ out of 10 is greater than P_i. For estimation purpose we choose the Gaussian symmetric kernel $K(\text{u}) = \frac{1}{\sqrt{2\pi}} \exp\left(-(\frac{u}{\sqrt{2}})^2\right)$ and the Silverman's rule ($b = 1.06\ \sigma\ n^{-1/5}$) has been applied to determine bandwidth b. Table 1 provides the Bias and MSE (in parentheses) of $L_n(t)$ and $L_n^*(t)$ and Relative Efficiency (RE) which is $RE(L_n^*(t), L_n(t)) = \frac{MSE(L_n(t))}{MSE(L_n^*(t))}$ for three different values of $n = 30, 50$ and 100. It is evident from Table 1 that the proposed estimator fits well with respect of Bias and MSE and both these values decrease for increase in sample size, approves the validity of the estimator. Figures 1 and 2 also justify our findings. Finally we check the asymptotic normality for each $(P_i,\ i = 1, 2, \ldots, 9)$ based on $Q - Q$ plot (see, Figure 3). It is evident to say that the nature of normality as sample size increases.

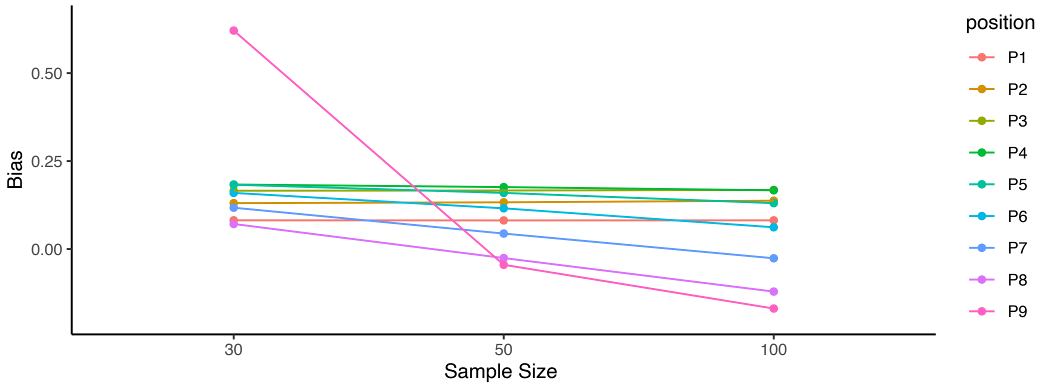

Figure 1: Bias of $L_n^*(\cdot)$ for simulated exponential data

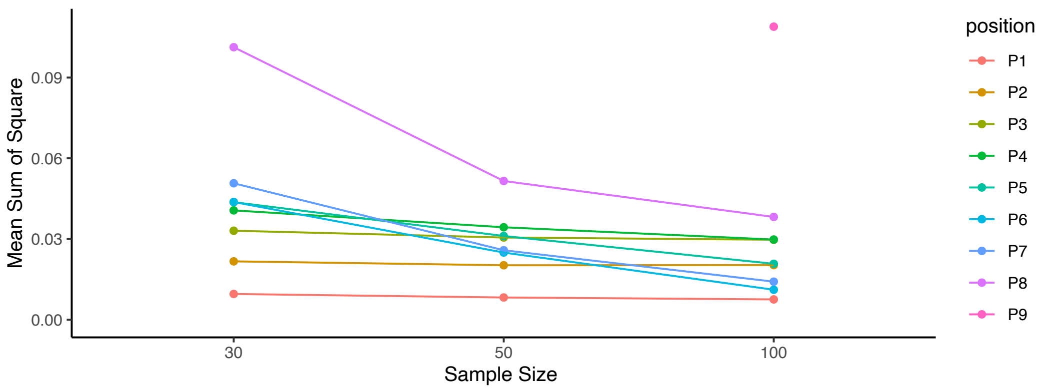

Figure 2: MSE of $L_n^*(\cdot)$ for simulated exponential data

Table 1: Bias and MSE of $L_n(\cdot)$ and $L_n^*(\cdot)$ of exponential distribution

n	Position	Empirical Estimator ($L_n(\cdot)$)	Kernel Estimator ($L_n^*(\cdot)$)	RE
30	P1	$-5.577\times10^{+00}$ $(3.117\times10^{+01})$	3.461×10^{-01} (8.834×10^{-01})	$3.53\times10^{+01}$
	P2	$-2.330\times10^{+00}$ $(9.796\times10^{+00})$	2.795×10^{-01} (6.498×10^{-01})	$1.51\times10^{+01}$
	P3	-9.502×10^{-01} $(6.034\times10^{+00})$	2.125×10^{-01} (6.166×10^{-01})	$9.79\times10^{+00}$
	P4	-4.130×10^{-01} $(5.784\times10^{+00})$	1.797×10^{-01} (5.481×10^{-01})	$1.06\times10^{+01}$
	P5	-5.334×10^{-01} $(6.472\times10^{+00})$	1.671×10^{-01} (5.318×10^{-01})	$1.22\times10^{+01}$
	P6	-7.122×10^{-01} $(7.479\times10^{+00})$	8.181×10^{-02} (4.667×10^{-01})	$1.60\times10^{+01}$
	P7	$-1.063\times10^{+00}$ $(9.297\times10^{+00})$	-1.425×10^{-02} (6.453×10^{-01})	$1.44\times10^{+01}$
	P8	$-1.444\times10^{+00}$ $(1.151\times10^{+01})$	-2.381×10^{-01} (7.992×10^{-01})	$1.44\times10^{+01}$
	P9	$-1.856\times10^{+00}$ $(1.366\times10^{+01})$	$-1.183\times10^{+00}$ $(3.143\times10^{+00})$	$4.35\times10^{+00}$
50	P1	——— (———)	1.866×10^{-01} (3.613×10^{-01})	———
	P2	-3.857×10^{-01} (1.721×10^{-01})	2.138×10^{-01} (2.120×10^{-01})	8.12×10^{-01}
	P3	3.024×10^{-01} (7.915×10^{-01})	2.668×10^{-01} (1.202×10^{-01})	$6.58\times10^{+00}$
	P4	-2.768×10^{-02} (6.672×10^{-01})	2.723×10^{-01} (1.185×10^{-01})	$5.63\times10^{+00}$
	P5	-2.727×10^{-01} $(1.175\times10^{+00})$	-2.827×10^{-03} (7.330×10^{-02})	$1.60\times10^{+01}$
	P6	-5.311×10^{-01} $(1.499\times10^{+00})$	-6.821×10^{-02} (8.868×10^{-02})	$1.69\times10^{+01}$
	P7	-7.479×10^{-01} $(1.896\times10^{+00})$	-4.092×10^{-01} (3.878×10^{-01})	$4.89\times10^{+00}$
	P8	$-1.161\times10^{+00}$ $(2.562\times10^{+00})$	8.120×10^{-02} (2.788×10^{-01})	$9.19\times10^{+00}$
	P9	$-1.769\times10^{+00}$ $(5.323\times10^{+00})$	-3.138×10^{-01} (7.659×10^{-01})	$6.95\times10^{+00}$
100	P1	——— (———)	7.756×10^{-01} (6.297×10^{-01})	———
	P2	-7.223×10^{-01} (5.217×10^{-01})	9.217×10^{-01} (8.855×10^{-01})	5.89×10^{-01}
	P3	3.013×10^{-01} (1.001×10^{-01})	5.212×10^{-01} (2.877×10^{-01})	3.48×10^{-01}
	P4	1.345×10^{-01} (4.800×10^{-02})	1.264×10^{-01} (2.620×10^{-02})	$1.83\times10^{+00}$
	P5	-1.668×10^{-01} (6.056×10^{-02})	9.040×10^{-03} (1.079×10^{-02})	$5.61\times10^{+00}$
	P6	-4.433×10^{-01} (2.321×10^{-01})	-4.032×10^{-02} (1.314×10^{-02})	$1.77\times10^{+01}$
	P7	-8.172×10^{-01} (7.072×10^{-01})	1.245×10^{-01} (2.848×10^{-02})	$2.48\times10^{+01}$
	P8	$-1.033\times10^{+00}$ $(1.107\times10^{+00})$	-5.709×10^{-01} (3.538×10^{-01})	$3.13\times10^{+00}$
	P9	$-1.433\times10^{+00}$ $(2.098\times10^{+00})$	$-1.046\times10^{+00}$ $(1.147\times10^{+00})$	$1.83\times10^{+00}$

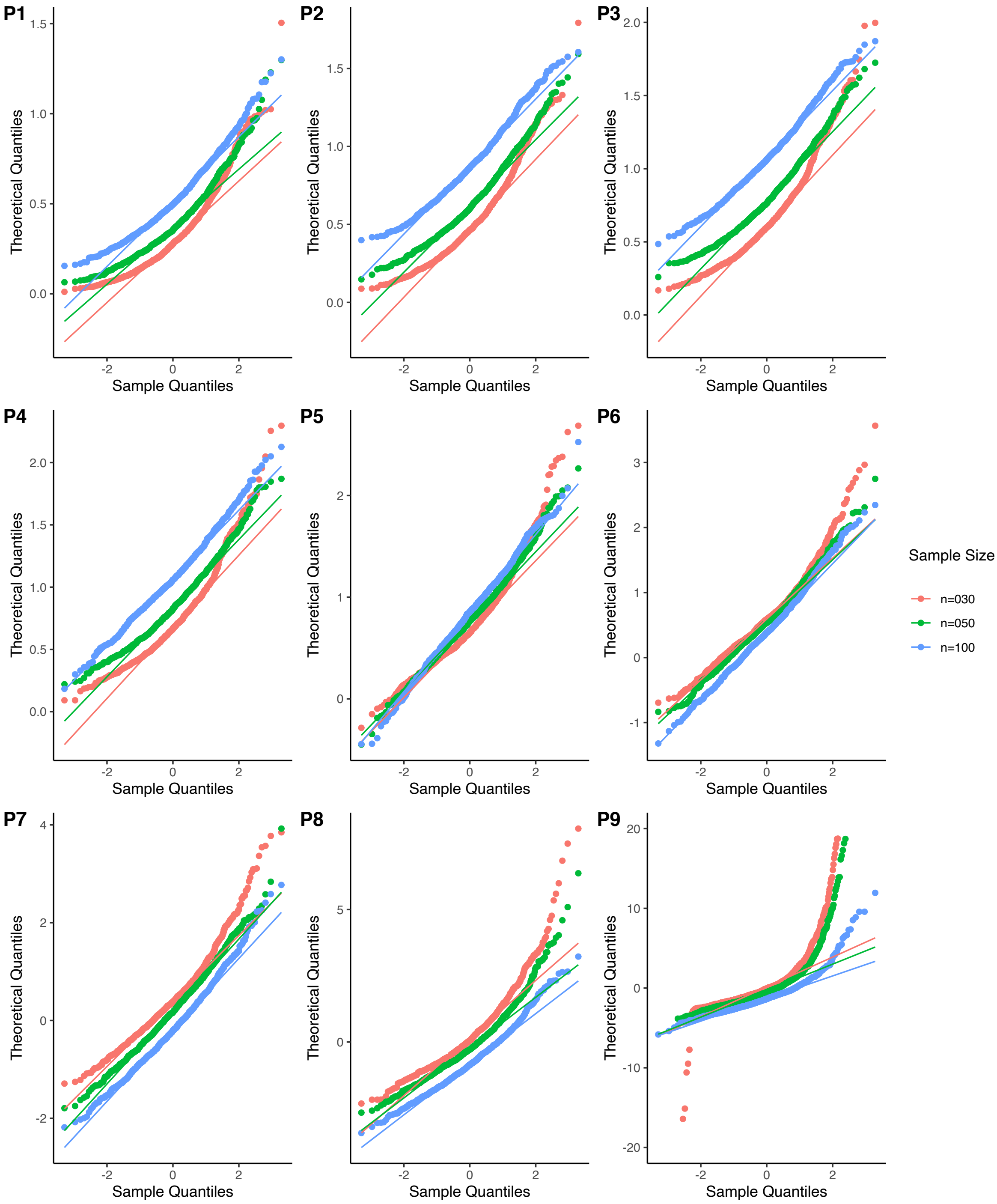

Figure 3: $Q-Q$ Plot for $L_n^*(\cdot)$ of simulated exponential data

To check further on the validity and performance of $L_n^*(\cdot)$ over $L_n(\cdot)$, we simulate 1000 datasets having sample sizes $n = 30, 50$ & 100, from the following life distributions viz., Weibull with shape and scale parameters 4 and 2; Gompertz with $\gamma = \alpha = 2$, Makeham distribution with $\alpha = 2$, $\beta = 3$ & $\lambda = 4$, and inverse Lomax with $\alpha = 2$ & $\beta = 3$.

Example 4.1. Weibull Distribution: Suppose X follows a Weibull distribution $(W(\alpha, \beta))$ for $t \in (0, \infty)$ and $\alpha, \beta > 0$. Then the survival function, hazard rate and the AI function are given by

$$\bar{F}_X(t) = \exp\left\{-\alpha t^{\beta}\right\}, \tag{19}$$
$$h_X(t) = \alpha\beta\, t^{\beta-1},$$
$$L_X(t) = \beta. \tag{20}$$

Example 4.2. Gompertz Distribution: Suppose X follows a Gompertz distribution $(G(\alpha, \gamma))$ for $t \in (0, \infty)$ and $\alpha, \gamma > 0$. Then the survival function, hazard rate (Gompertz (1825)) and the AI function (Szymkowiak (2018)) are given by

$$\bar{F}_X(t) = \exp\left\{-\gamma[\exp(\alpha\, t) - 1]\right\}, \tag{21}$$
$$h_X(t) = \alpha\gamma \exp(\alpha\, t),$$
$$L_X(t) = \frac{\alpha t \exp(\alpha\, t)}{\exp(\alpha\, t) - 1}. \tag{22}$$

Example 4.3. Makeham Distribution: Consider a Makeham distribution $(M(\alpha, \beta, \Lambda))$ if for $t \in (0, \infty)$, $\alpha > 0$ and $\beta, \Lambda \geq 0$, such that $\beta + \Lambda > 0$. The survival function, hazard rate (Lai and Xie (2006) and Jodrá (2009)) and the AI function (Szymkowiak (2018)) are given by

$$\bar{F}_X(t) = \exp\left\{-\lambda t - \frac{\beta}{\alpha}[\exp(\alpha\, t) - 1]\right\}, \tag{23}$$
$$h_X(t) = \lambda + \beta \exp(\alpha\, t),$$
$$L_X(t) = \frac{\alpha\, t\, [\lambda + \beta\, \exp(\alpha\, t)]}{\alpha\, \lambda\, t + [\exp(\alpha\, t) - 1]}. \tag{24}$$

Example 4.4. Inverse Lomax Distribution: When X follows Inverse Lomax distribution $(IL(\alpha, \beta))$ if for $t \in (0, \infty)$, $\alpha > 0$ and $\beta, > 0$, then the survival function, hazard rate (Kleiber and Kotz (2003) and ZeinEldin, Ahsan ul Haq, Hashmi, and Elsehety (2020)) and the AI function are given by

$$\bar{F}_X(t) = 1 - \left(1 + \frac{\beta}{t}\right)^{-\alpha}, \tag{25}$$
$$h_X(t) = \left(\frac{\alpha\, \beta}{t^2}\right) \frac{\left(1 + \frac{\beta}{t}\right)^{-\alpha-1}}{1 - \left(1 + \frac{\beta}{t}\right)^{-\alpha}},$$
$$L_X(t) = \frac{\alpha\, \beta\, \left(1 + \frac{\beta}{t}\right)^{-\alpha-1}}{t\left(1 - \left(1 + \frac{\beta}{t}\right)^{-\alpha}\right) \log\left(1 - \left(1 + \frac{\beta}{t}\right)^{-\alpha}\right)}. \tag{26}$$

Tables 2, 3, 4 & 5 provide the Bias and MSE of $L_n(\cdot)$ and $L_n^*(\cdot)$ of Weibull, Gompertz, Makeham and inverse Lomax distributions and give a comparison of performance of two estimators. Further, the plot and tables given in Figure 4 and Table 2 to 5 explain that in most cases, the RE of $L_n^*(\cdot)$ over $L_n(\cdot)$ is greater than unity, shows that $L_n^*(\cdot)$ outperforms $L_n(\cdot)$. Also, the MSE of $L_n(\cdot)$ is much more scattered than $L_n^*(\cdot)$ as evident from Figures 5 and Table 2 to 5 corresponding to Weibull, Gompertz, Makeham and inverse Lomax distributions. This enable us to conclude that $L_n^*(\cdot)$ performs better than $L_n(\cdot)$ in terms of Bias and MSE.

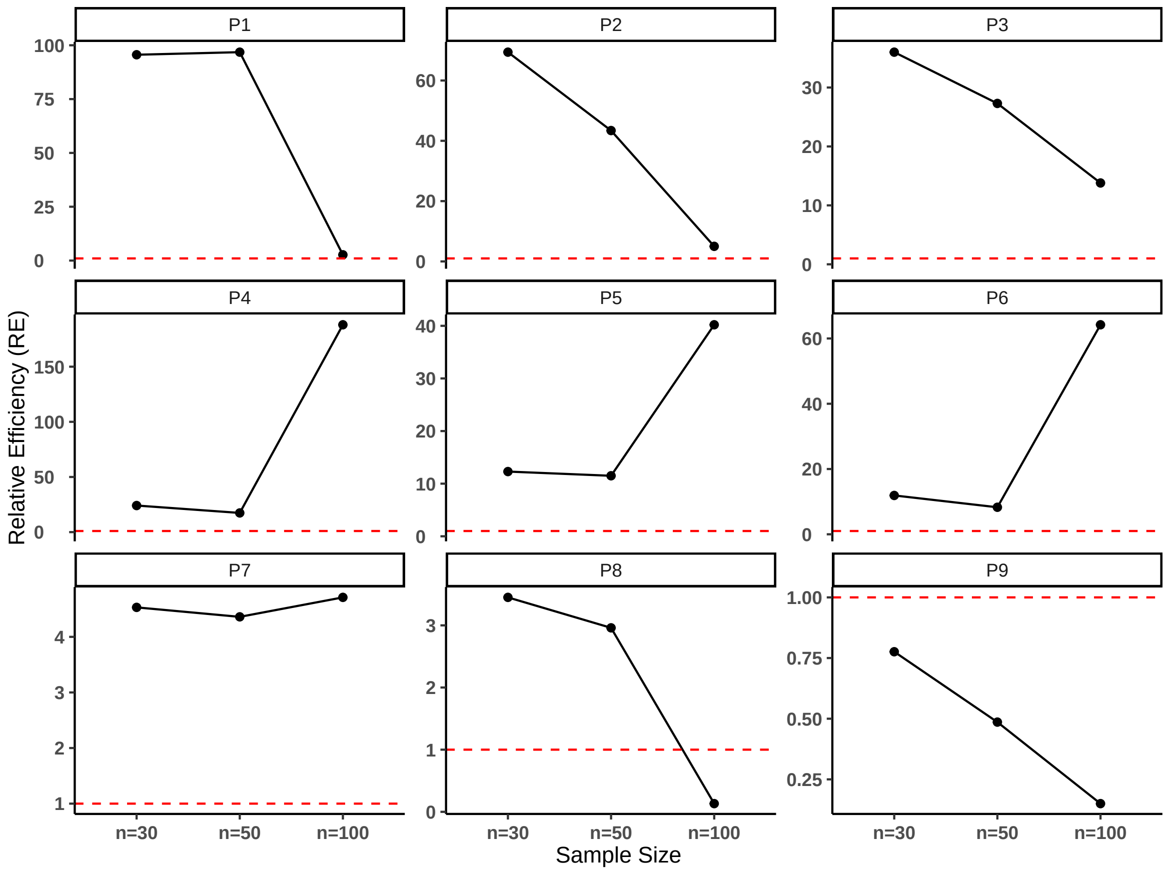

Figure 4: RE of $L_n^*(\cdot)$ over $L_n(\cdot)$ for Weibull Distribution

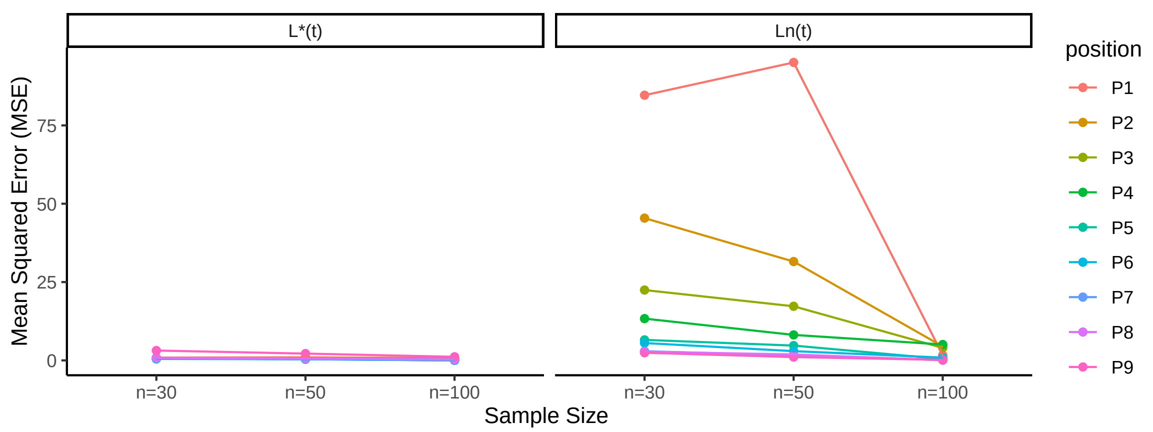

Figure 5: MSE of $L_n^*(\cdot)$ and $L_n(\cdot)$ for Weibull Distribution

Table 2: Bias and MSE of $L_n(\cdot)$ and $L_n^*(\cdot)$ of Weibull Distribution

n	Position	Empirical Estimator ($L_n(\cdot)$)	Kernel Estimator ($L_n^*(\cdot)$)	RE
30	P1	-6.030 $\times 10^{+00}$ (8.463 $\times 10^{+01}$)	3.459 $\times 10^{-01}$ (8.851 $\times 10^{-01}$)	9.56 $\times 10^{+01}$
	P2	-4.354 $\times 10^{+00}$ (4.540 $\times 10^{+01}$)	2.789 $\times 10^{-01}$ (6.547 $\times 10^{-01}$)	6.94 $\times 10^{+01}$
	P3	-2.494 $\times 10^{+00}$ (2.245 $\times 10^{+01}$)	2.150 $\times 10^{-01}$ (6.234 $\times 10^{-01}$)	3.60 $\times 10^{+01}$
	P4	-1.984 $\times 10^{+00}$ (1.333 $\times 10^{+01}$)	1.820 $\times 10^{-01}$ (5.530 $\times 10^{-01}$)	2.41 $\times 10^{+01}$
	P5	-1.150 $\times 10^{+00}$ (6.538 $\times 10^{+00}$)	1.668 $\times 10^{-01}$ (5.332 $\times 10^{-01}$)	1.23 $\times 10^{+01}$
	P6	-9.628 $\times 10^{-01}$ (5.544 $\times 10^{+00}$)	8.079 $\times 10^{-02}$ (4.646 $\times 10^{-01}$)	1.19 $\times 10^{+01}$
	P7	-3.938 $\times 10^{-01}$ (2.921 $\times 10^{+00}$)	-1.859 $\times 10^{-02}$ (6.449 $\times 10^{-01}$)	4.53 $\times 10^{+00}$
	P8	-2.239 $\times 10^{-01}$ (2.766 $\times 10^{+00}$)	-2.438 $\times 10^{-01}$ (8.017 $\times 10^{-01}$)	3.45 $\times 10^{+00}$
	P9	-1.779 $\times 10^{-01}$ (2.435 $\times 10^{+00}$)	-1.178 $\times 10^{+00}$ (3.138 $\times 10^{+00}$)	7.76 $\times 10^{-01}$
50	P1	-5.706 $\times 10^{+00}$ (9.509 $\times 10^{+01}$)	4.025 $\times 10^{-01}$ (9.823 $\times 10^{-01}$)	9.68 $\times 10^{+01}$
	P2	-2.948 $\times 10^{+00}$ (3.154 $\times 10^{+01}$)	3.350 $\times 10^{-01}$ (7.261 $\times 10^{-01}$)	4.34 $\times 10^{+01}$
	P3	-1.753 $\times 10^{+00}$ (1.727 $\times 10^{+01}$)	3.253 $\times 10^{-01}$ (6.325 $\times 10^{-01}$)	2.73 $\times 10^{+01}$
	P4	-1.705 $\times 10^{+00}$ (8.132 $\times 10^{+00}$)	2.590 $\times 10^{-01}$ (4.671 $\times 10^{-01}$)	1.74 $\times 10^{+01}$
	P5	-1.012 $\times 10^{+00}$ (4.725 $\times 10^{+00}$)	2.158 $\times 10^{-01}$ (4.118 $\times 10^{-01}$)	1.15 $\times 10^{+01}$
	P6	-4.689 $\times 10^{-01}$ (2.954 $\times 10^{+00}$)	1.441 $\times 10^{-01}$ (3.568 $\times 10^{-01}$)	8.28 $\times 10^{+00}$
	P7	-2.506 $\times 10^{-01}$ (1.590 $\times 10^{+00}$)	3.969 $\times 10^{-02}$ (3.648 $\times 10^{-01}$)	4.36 $\times 10^{+00}$
	P8	-6.447 $\times 10^{-02}$ (1.885 $\times 10^{+00}$)	-2.617 $\times 10^{-01}$ (6.363 $\times 10^{-01}$)	2.96 $\times 10^{+00}$
	P9	-1.536 $\times 10^{-01}$ (1.053 $\times 10^{+00}$)	-9.787 $\times 10^{-01}$ (2.166 $\times 10^{+00}$)	4.86 $\times 10^{-01}$
100	P1	-7.553 $\times 10^{-01}$ (1.686 $\times 10^{+00}$)	7.793 $\times 10^{-01}$ (6.319 $\times 10^{-01}$)	2.67 $\times 10^{+00}$
	P2	-2.034 $\times 10^{+00}$ (4.440 $\times 10^{+00}$)	9.241 $\times 10^{-01}$ (8.880 $\times 10^{-01}$)	5.00 $\times 10^{+00}$
	P3	-1.950 $\times 10^{+00}$ (3.986 $\times 10^{+00}$)	5.229 $\times 10^{-01}$ (2.896 $\times 10^{-01}$)	1.38 $\times 10^{+01}$
	P4	-2.219 $\times 10^{+00}$ (5.074 $\times 10^{+00}$)	1.275 $\times 10^{-01}$ (2.694 $\times 10^{-02}$)	1.88 $\times 10^{+02}$
	P5	-6.275 $\times 10^{-01}$ (4.417 $\times 10^{-01}$)	8.683 $\times 10^{-03}$ (1.100 $\times 10^{-02}$)	4.02 $\times 10^{+01}$
	P6	-9.058 $\times 10^{-01}$ (8.880 $\times 10^{-01}$)	-4.111 $\times 10^{-02}$ (1.384 $\times 10^{-02}$)	6.42 $\times 10^{+01}$
	P7	-3.066 $\times 10^{-01}$ (1.399 $\times 10^{-01}$)	1.233 $\times 10^{-01}$ (2.970 $\times 10^{-02}$)	4.71 $\times 10^{+00}$
	P8	-2.114 $\times 10^{-02}$ (4.707 $\times 10^{-02}$)	-5.728 $\times 10^{-01}$ (3.570 $\times 10^{-01}$)	1.32 $\times 10^{-01}$
	P9	-3.279 $\times 10^{-01}$ (1.719 $\times 10^{-01}$)	-1.046 $\times 10^{+00}$ (1.148 $\times 10^{+00}$)	1.50 $\times 10^{-01}$

Table 3: Bias and MSE of $L_n(\cdot)$ and $L_n^*(\cdot)$ of Gompertz Distribution

n	Position	Empirical Estimator ($L_n(\cdot)$)	Kernel Estimator ($L_n^*(\cdot)$)	RE
30	P1	-1.612 $\times 10^{+00}$ (4.394 $\times 10^{+00}$)	-1.985 $\times 10^{-01}$ (5.246 $\times 10^{-02}$)	8.38 $\times 10^{+01}$
	P2	-1.200 $\times 10^{+00}$ (3.064 $\times 10^{+00}$)	-2.727 $\times 10^{-01}$ (8.865 $\times 10^{-02}$)	3.46 $\times 10^{+01}$
	P3	-1.210 $\times 10^{+00}$ (3.613 $\times 10^{+00}$)	-3.186 $\times 10^{-01}$ (1.223 $\times 10^{-01}$)	2.95 $\times 10^{+01}$
	P4	-6.515 $\times 10^{-01}$ (1.380 $\times 10^{+00}$)	-3.140 $\times 10^{-01}$ (1.308 $\times 10^{-01}$)	1.06 $\times 10^{+01}$
	P5	-5.979 $\times 10^{-01}$ (1.055 $\times 10^{+00}$)	-3.357 $\times 10^{-01}$ (1.555 $\times 10^{-01}$)	6.79 $\times 10^{+00}$
	P6	-4.481 $\times 10^{-01}$ (7.413 $\times 10^{-01}$)	-3.387 $\times 10^{-01}$ (1.729 $\times 10^{-01}$)	4.29 $\times 10^{+00}$
	P7	-3.561 $\times 10^{-01}$ (6.981 $\times 10^{-01}$)	-3.826 $\times 10^{-01}$ (2.400 $\times 10^{-01}$)	2.91 $\times 10^{+00}$
	P8	-4.438 $\times 10^{-01}$ (8.598 $\times 10^{-01}$)	-3.681 $\times 10^{-01}$ (3.606 $\times 10^{-01}$)	2.38 $\times 10^{+00}$
	P9	-4.304 $\times 10^{-01}$ (8.982 $\times 10^{-01}$)	-7.422 $\times 10^{-01}$ (1.052 $\times 10^{+00}$)	8.54 $\times 10^{-01}$
50	P1	-2.183 $\times 10^{+00}$ (6.918 $\times 10^{+00}$)	-1.661 $\times 10^{-01}$ (3.143 $\times 10^{-02}$)	2.20 $\times 10^{+02}$
	P2	-9.718 $\times 10^{-01}$ (3.362 $\times 10^{+00}$)	-2.364 $\times 10^{-01}$ (6.153 $\times 10^{-02}$)	5.46 $\times 10^{+01}$
	P3	-5.433 $\times 10^{-01}$ (1.219 $\times 10^{+00}$)	-2.606 $\times 10^{-01}$ (7.595 $\times 10^{-02}$)	1.60 $\times 10^{+01}$
	P4	-2.507 $\times 10^{-01}$ (3.598 $\times 10^{-01}$)	-2.345 $\times 10^{-01}$ (6.850 $\times 10^{-02}$)	5.25 $\times 10^{+00}$
	P5	-2.699 $\times 10^{-01}$ (3.860 $\times 10^{-01}$)	-2.320 $\times 10^{-01}$ (7.451 $\times 10^{-02}$)	5.18 $\times 10^{+00}$
	P6	-3.021 $\times 10^{-01}$ (4.470 $\times 10^{-01}$)	-2.044 $\times 10^{-01}$ (8.502 $\times 10^{-02}$)	5.26 $\times 10^{+00}$
	P7	-2.835 $\times 10^{-01}$ (5.822 $\times 10^{-01}$)	-2.110 $\times 10^{-01}$ (1.266 $\times 10^{-01}$)	4.60 $\times 10^{+00}$
	P8	-2.428 $\times 10^{-01}$ (4.005 $\times 10^{-01}$)	-2.205 $\times 10^{-01}$ (1.895 $\times 10^{-01}$)	2.11 $\times 10^{+00}$
	P9	-1.160 $\times 10^{-01}$ (6.129 $\times 10^{-01}$)	-4.615 $\times 10^{-01}$ (5.894 $\times 10^{-01}$)	1.04 $\times 10^{+00}$
100	P1	-1.095 $\times 10^{+00}$ (2.006 $\times 10^{+00}$)	-1.793 $\times 10^{-01}$ (3.460 $\times 10^{-02}$)	5.80 $\times 10^{+01}$
	P2	-4.053 $\times 10^{-01}$ (5.544 $\times 10^{-01}$)	-2.550 $\times 10^{-01}$ (6.824 $\times 10^{-02}$)	8.12 $\times 10^{+00}$
	P3	-4.182 $\times 10^{-01}$ (2.825 $\times 10^{-01}$)	-2.646 $\times 10^{-01}$ (7.463 $\times 10^{-02}$)	3.79 $\times 10^{+00}$
	P4	-4.436 $\times 10^{-01}$ (4.594 $\times 10^{-01}$)	-2.445 $\times 10^{-01}$ (7.284 $\times 10^{-02}$)	6.31 $\times 10^{+00}$
	P5	-6.264 $\times 10^{-02}$ (1.535 $\times 10^{-01}$)	-2.405 $\times 10^{-01}$ (8.248 $\times 10^{-02}$)	1.86 $\times 10^{+00}$
	P6	-2.467 $\times 10^{-01}$ (2.592 $\times 10^{-01}$)	-2.046 $\times 10^{-01}$ (6.427 $\times 10^{-02}$)	4.03 $\times 10^{+00}$
	P7	-2.032 $\times 10^{-01}$ (2.043 $\times 10^{-01}$)	-2.467 $\times 10^{-01}$ (1.033 $\times 10^{-01}$)	1.98 $\times 10^{+00}$
	P8	-5.022 $\times 10^{-02}$ (1.964 $\times 10^{-01}$)	-1.117 $\times 10^{-01}$ (5.574 $\times 10^{-02}$)	3.52 $\times 10^{+00}$
	P9	2.386 $\times 10^{-01}$ (1.695 $\times 10^{-01}$)	-2.051 $\times 10^{-01}$ (1.845 $\times 10^{-01}$)	9.19 $\times 10^{-01}$

Table 4: Bias and MSE of $L_n(\cdot)$ and $L_n^*(\cdot)$ of Makeham Distribution

n	Position	Empirical Estimator ($L_n(\cdot)$)	Kernel Estimator ($L_n^*(\cdot)$)	RE
30	P1	-2.959 $\times 10^{+00}$ (8.754 $\times 10^{+00}$)	5.066 $\times 10^{-02}$ (8.629 $\times 10^{-03}$)	1.01 $\times 10^{+03}$
	P2	-1.205 $\times 10^{+00}$ (3.371 $\times 10^{+00}$)	-4.605 $\times 10^{-02}$ (1.023 $\times 10^{-02}$)	3.29 $\times 10^{+02}$
	P3	-6.923 $\times 10^{-01}$ (1.813 $\times 10^{+00}$)	-1.093 $\times 10^{-01}$ (2.177 $\times 10^{-02}$)	8.33 $\times 10^{+01}$
	P4	-2.535 $\times 10^{-01}$ (1.017 $\times 10^{+00}$)	-1.378 $\times 10^{-01}$ (3.669 $\times 10^{-02}$)	2.77 $\times 10^{+01}$
	P5	-2.480 $\times 10^{-01}$ (1.144 $\times 10^{+00}$)	-1.643 $\times 10^{-01}$ (5.138 $\times 10^{-02}$)	2.23 $\times 10^{+01}$
	P6	-2.246 $\times 10^{-01}$ (3.535 $\times 10^{-01}$)	-1.607 $\times 10^{-01}$ (6.480 $\times 10^{-02}$)	5.45 $\times 10^{+00}$
	P7	-5.536 $\times 10^{-02}$ (4.105 $\times 10^{-01}$)	-1.521 $\times 10^{-01}$ (1.029 $\times 10^{-01}$)	3.99 $\times 10^{+00}$
	P8	1.241 $\times 10^{-01}$ (2.912 $\times 10^{-01}$)	-2.167 $\times 10^{-01}$ (1.953 $\times 10^{-01}$)	1.49 $\times 10^{+00}$
	P9	1.681 $\times 10^{-01}$ (3.357 $\times 10^{-01}$)	-3.021 $\times 10^{-01}$ (5.941 $\times 10^{-01}$)	5.65 $\times 10^{-01}$
50	P1	——— (———)	6.477 $\times 10^{-02}$ (6.143 $\times 10^{-03}$)	———
	P2	-6.347 $\times 10^{-01}$ (4.028 $\times 10^{-01}$)	-2.753 $\times 10^{-02}$ (3.958 $\times 10^{-03}$)	1.02 $\times 10^{+02}$
	P3	-2.104 $\times 10^{-01}$ (1.592 $\times 10^{-01}$)	-9.440 $\times 10^{-02}$ (1.251 $\times 10^{-02}$)	1.27 $\times 10^{+01}$
	P4	1.460 $\times 10^{-01}$ (1.556 $\times 10^{-01}$)	-1.111 $\times 10^{-01}$ (1.868 $\times 10^{-02}$)	8.33 $\times 10^{+00}$
	P5	5.457 $\times 10^{-02}$ (1.887 $\times 10^{-01}$)	-1.169 $\times 10^{-01}$ (2.624 $\times 10^{-02}$)	7.19 $\times 10^{+00}$
	P6	-6.794 $\times 10^{-02}$ (1.720 $\times 10^{-01}$)	-7.313 $\times 10^{-02}$ (3.308 $\times 10^{-02}$)	5.20 $\times 10^{+00}$
	P7	2.595 $\times 10^{-01}$ (2.335 $\times 10^{-01}$)	-3.473 $\times 10^{-02}$ (5.564 $\times 10^{-02}$)	4.20 $\times 10^{+00}$
	P8	2.539 $\times 10^{-01}$ (2.354 $\times 10^{-01}$)	2.663 $\times 10^{-02}$ (1.025 $\times 10^{-01}$)	2.30 $\times 10^{+00}$
	P9	3.167 $\times 10^{-01}$ (2.779 $\times 10^{-01}$)	-6.666 $\times 10^{-02}$ (2.755 $\times 10^{-01}$)	1.01 $\times 10^{+00}$
100	P1	——— (———)	3.529 $\times 10^{-02}$ (1.316 $\times 10^{-03}$)	———
	P2	——— (———)	-7.029 $\times 10^{-02}$ (5.067 $\times 10^{-03}$)	———
	P3	-6.232 $\times 10^{-02}$ (2.070 $\times 10^{-02}$)	-1.789 $\times 10^{-01}$ (3.241 $\times 10^{-02}$)	6.39 $\times 10^{-01}$
	P4	1.566 $\times 10^{-01}$ (2.755 $\times 10^{-02}$)	-1.548 $\times 10^{-01}$ (2.437 $\times 10^{-02}$)	1.13 $\times 10^{+00}$
	P5	-1.512 $\times 10^{-01}$ (2.905 $\times 10^{-02}$)	-1.489 $\times 10^{-01}$ (2.292 $\times 10^{-02}$)	1.27 $\times 10^{+00}$
	P6	-5.290 $\times 10^{-01}$ (2.968 $\times 10^{-01}$)	-4.094 $\times 10^{-02}$ (2.641 $\times 10^{-03}$)	1.12 $\times 10^{+02}$
	P7	7.101 $\times 10^{-01}$ (5.165 $\times 10^{-01}$)	-1.157 $\times 10^{-01}$ (1.562 $\times 10^{-02}$)	3.31 $\times 10^{+01}$
	P8	4.266 $\times 10^{-01}$ (1.863 $\times 10^{-01}$)	4.928 $\times 10^{-01}$ (2.523 $\times 10^{-01}$)	7.38 $\times 10^{-01}$
	P9	6.207 $\times 10^{-01}$ (3.895 $\times 10^{-01}$)	3.494 $\times 10^{-01}$ (1.296 $\times 10^{-01}$)	3.01 $\times 10^{+00}$

Table 5: Bias and MSE of $L_n(\cdot)$ and $L_n^*(\cdot)$ of Inverse Lomax Distribution

n	Position	Empirical Estimator ($L_n(\cdot)$)	Kernel Estimator ($L_n^*(\cdot)$)	RE
30	P1	-6.831 $\times 10^{+00}$ (1.086 $\times 10^{+02}$)	-2.464 $\times 10^{+00}$ (6.097 $\times 10^{+00}$)	1.78 $\times 10^{+01}$
	P2	-5.023 $\times 10^{+00}$ (5.168 $\times 10^{+01}$)	-2.372 $\times 10^{+00}$ (5.645 $\times 10^{+00}$)	9.16 $\times 10^{+00}$
	P3	-2.663 $\times 10^{+00}$ (1.698 $\times 10^{+01}$)	-2.303 $\times 10^{+00}$ (5.321 $\times 10^{+00}$)	3.19 $\times 10^{+00}$
	P4	-2.229 $\times 10^{+00}$ (1.870 $\times 10^{+01}$)	-2.253 $\times 10^{+00}$ (5.092 $\times 10^{+00}$)	3.67 $\times 10^{+00}$
	P5	-2.597 $\times 10^{+00}$ (3.244 $\times 10^{+01}$)	-2.247 $\times 10^{+00}$ (5.070 $\times 10^{+00}$)	6.40 $\times 10^{+00}$
	P6	-2.927 $\times 10^{+00}$ (5.864 $\times 10^{+01}$)	-2.241 $\times 10^{+00}$ (5.055 $\times 10^{+00}$)	1.16 $\times 10^{+01}$
	P7	-1.411 $\times 10^{+00}$ (2.093 $\times 10^{+01}$)	-2.224 $\times 10^{+00}$ (4.998 $\times 10^{+00}$)	4.19 $\times 10^{+00}$
	P8	-2.659 $\times 10^{+00}$ (5.425 $\times 10^{+01}$)	-2.222 $\times 10^{+00}$ (5.091 $\times 10^{+00}$)	1.07 $\times 10^{+01}$
	P9	-2.311 $\times 10^{+00}$ (2.306 $\times 10^{+02}$)	-1.892 $\times 10^{+00}$ (4.041 $\times 10^{+00}$)	5.71 $\times 10^{+01}$
50	P1	-6.886 $\times 10^{+00}$ (6.084 $\times 10^{+01}$)	-2.505 $\times 10^{+00}$ (6.285 $\times 10^{+00}$)	9.68 $\times 10^{+00}$
	P2	-4.509 $\times 10^{+00}$ (2.985 $\times 10^{+01}$)	-2.391 $\times 10^{+00}$ (5.720 $\times 10^{+00}$)	5.22 $\times 10^{+00}$
	P3	-2.538 $\times 10^{+00}$ (1.172 $\times 10^{+01}$)	-2.330 $\times 10^{+00}$ (5.436 $\times 10^{+00}$)	2.16 $\times 10^{+00}$
	P4	-2.255 $\times 10^{+00}$ (1.331 $\times 10^{+01}$)	-2.280 $\times 10^{+00}$ (5.207 $\times 10^{+00}$)	2.56 $\times 10^{+00}$
	P5	-2.459 $\times 10^{+00}$ (1.838 $\times 10^{+01}$)	-2.270 $\times 10^{+00}$ (5.164 $\times 10^{+00}$)	3.56 $\times 10^{+00}$
	P6	-2.434 $\times 10^{+00}$ (2.211 $\times 10^{+01}$)	-2.247 $\times 10^{+00}$ (5.070 $\times 10^{+00}$)	4.36 $\times 10^{+00}$
	P7	-2.374 $\times 10^{+00}$ (3.200 $\times 10^{+01}$)	-2.235 $\times 10^{+00}$ (5.032 $\times 10^{+00}$)	6.36 $\times 10^{+00}$
	P8	-3.167 $\times 10^{+00}$ (9.394 $\times 10^{+01}$)	-2.124 $\times 10^{+00}$ (4.628 $\times 10^{+00}$)	2.03 $\times 10^{+01}$
	P9	-1.190 $\times 10^{+00}$ (3.450 $\times 10^{+01}$)	-1.842 $\times 10^{+00}$ (3.754 $\times 10^{+00}$)	9.19 $\times 10^{+00}$
100	P1	-3.836 $\times 10^{+00}$ (1.507 $\times 10^{+01}$)	-2.433 $\times 10^{+00}$ (5.918 $\times 10^{+00}$)	2.55 $\times 10^{+00}$
	P2	-5.388 $\times 10^{+00}$ (2.939 $\times 10^{+01}$)	-2.338 $\times 10^{+00}$ (5.466 $\times 10^{+00}$)	5.38 $\times 10^{+00}$
	P3	-3.028 $\times 10^{+00}$ (9.472 $\times 10^{+00}$)	-2.355 $\times 10^{+00}$ (5.546 $\times 10^{+00}$)	1.71 $\times 10^{+00}$
	P4	-3.256 $\times 10^{+00}$ (1.081 $\times 10^{+01}$)	-2.260 $\times 10^{+00}$ (5.108 $\times 10^{+00}$)	2.12 $\times 10^{+00}$
	P5	-6.219 $\times 10^{+00}$ (3.966 $\times 10^{+01}$)	-2.267 $\times 10^{+00}$ (5.142 $\times 10^{+00}$)	7.71 $\times 10^{+00}$
	P6	-8.244 $\times 10^{-01}$ (1.165 $\times 10^{+00}$)	-2.251 $\times 10^{+00}$ (5.068 $\times 10^{+00}$)	2.30 $\times 10^{-01}$
	P7	-7.157 $\times 10^{-01}$ (7.582 $\times 10^{-01}$)	-2.440 $\times 10^{+00}$ (5.958 $\times 10^{+00}$)	1.27 $\times 10^{-01}$
	P8	-2.350 $\times 10^{+01}$ (5.747 $\times 10^{+02}$)	-2.064 $\times 10^{+00}$ (4.261 $\times 10^{+00}$)	1.35 $\times 10^{+02}$
	P9	-5.116 $\times 10^{-01}$ (6.817 $\times 10^{+00}$)	-1.835 $\times 10^{+00}$ (3.377 $\times 10^{+00}$)	2.02 $\times 10^{+00}$

4.2. Real data: Groove ball bearings

In this section, we have considered a data set consisting of arose in tests on endurance of deep groove ball bearings, which is taken from Kundu and Manglick (2004). The data are the number of million revolutions before failure for each of the 23 ball bearings in the life tests and are given in Table 6. Kundu and Manglick (2004) have also shown that the data set is best fit to Weibull distribution with shape parameter 2.

Table 6: Groove ball bearings data given in Kundu and Manglick (2004)

17.88	28.92	33.00	41.52	42.12	45.60	48.80	51.84	51.96
54.12	55.56	67.80	68.44	68.64	68.88	84.12	93.12	98.64
		105.12	105.84	127.92	128.04	173.40		

The K-S distance between the data for the fitted Weibull distribution is 0.1521 with a p - value 0.63, which further confirm the conclusion of Kundu and Manglick (2004). We use this data set to estimate the AI function using the classical estimator $(L_n(t))$ defined in section 2.1 which is almost identical to Szymkowiak (2018), kernel-based estimators $(L^*(t))$ defined by Szymkowiak (2019) and kernel-based estimator $(L_n^*(t))$ in section 2.2 and are plotted in Figure 6. The theoretical value of AI function for Weibull model with shape parameter 2 is given in the figure for comparison. From the figure it is evident that estimated values of AI functions $L_n(t)$ and $L_n^*(t)$ oscillate around $L_X(t) = 2$, a characteristic property of AI function for Weibull distribution (Jiang *et al.* (2003)), where $L^*(t)$ located near the top. It can also be seen that the estimates based on $L_n^*(t)$ lie more closely around the true value of the parameter, thus reconfirm the conclusions made using the simulation study in Section 4.1 that the proposed estimator $L_n^*(t)$ has a better performance over $L_n(t)$. This allow us to assume that estimators can be used for two purposes; viz., to examine the ageing behaviour of a lifetime random variable and to identify the underlying distribution based on the characteristic property of AI function.

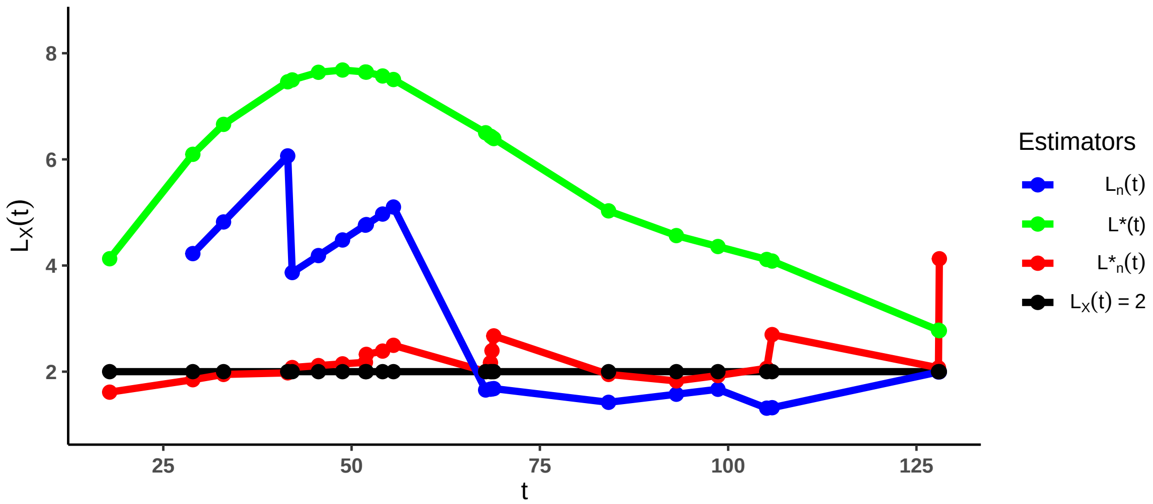

Figure 6: Plots of AI functions based on the data from Kundu and Manglick (2004)

5. Concluding remarks

In this article, we have proposed a non-parametric estimator for AI function based on the kernel density method. We have obtained expressions for Bias and MSE of the proposed estimator and studied its asymptotic behaviours. For numerical illustration using simulated data, we have estimated AI function based on some important life distributions such as exponential, Weibull, Gompertz, Makeham and inverse Lomax models and compared it with the classical estimator of AI function. It has been found that the proposed estimator has better performance compared to the classical estimator in terms of Bias and MSE. A real-life application has also been considered to illustrate and validate the usefulness of the estimator in studying the ageing behaviour of a lifetime random variable and in identifying the underlying probability model based on its characteristic property. We expect that the proposed estimator will be useful for reliability practitioners and theoreticians in modelling and analysis of the ageing behaviour of lifetime data.

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References

Bhattacharjee S, Nanda AK, Misra SK (2013a). "Inequalities Involving Expectations to Characterize Distributions." *Statistics & Probability Letters*, **83**(9), 2113–2118. doi:10.1016/j.spl.2013.05.022.

Bhattacharjee S, Nanda AK, Misra SK (2013b). "Reliability Analysis Using Ageing Intensity Function." *Statistics & Probability Letters*, **83**(5), 1364–1371. doi:10.1016/j.spl.2013.01.016.

Chen SM, Hsu YS, Liaw JT (2009). "On Kernel Estimators of Density Ratio." *Statistics*, **43**(5), 463–479. doi:10.1080/02331880802496399.

Gompertz B (1825). "XXIV. On the Nature of the Function Expressive of the Law of Human Mortality, and on a New Mode of Determining the Value of Life Contingencies. In a Letter to Francis Baily, Esq. FRS &c." *Philosophical Transactions of the Royal Society of London*, (115), 513–583. doi:10.1098/rstl.1825.0026.

Jiang R, Ji P, Xiao X (2003). "Aging Property of Unimodal Failure Rate Models." *Reliability Engineering & System Safety*, **79**(1), 113–116. doi:10.1016/S0951-8320(02)00175-8.

Jodrá P (2009). "A Closed-form Expression for the Quantile Function of the Gompertz–Makeham Distribution." *Mathematics and Computers in Simulation*, **79**(10), 3069–3075. doi:10.1016/j.matcom.2009.02.002.

Kleiber C, Kotz S (2003). *Statistical Size Distributions in Economics and Actuarial Sciences*, volume 470. John Wiley & Sons. ISBN 978-0-471-15064-0. doi:10.1002/0471457175.

Kundu D, Manglick A (2004). "Discriminating between the Weibull and Log-normal Distributions." *Naval Research Logistics*, **51**(6), 893–905. doi:10.1080/02331888.2010.504990.

Lai CD, Xie M (2006). *Stochastic Ageing and Dependence for Reliability*. Springer Science & Business Media. ISBN 978-0-387-29742-2. doi:10.1007/0-387-34232-X.

Misra SK, Bhattacharjee S (2018). "A Case Study of Aging Intensity Function on Censored Data." *Alexandria Engineering Journal*, **57**(4), 3931–3952. doi:10.1016/j.aej.2018.03.009.

Nanda AK, Bhattacharjee S, Alam SS (2007). "Properties of Aging Intensity Function." *Statistics & Probability Letters*, **77**(4), 365–373. doi:10.1016/j.spl.2006.08.002.

Rao BLSP (2014). *Nonparametric Functional Estimation*. Academic press. ISBN 978-0-12-564020-6. doi:10.1016/C2013-0-11326-8.

Roussas GG (1989). "Hazard Rate Estimation under Dependence Conditions." *Journal of Statistical Planning and Inference*, **22**(1), 81–93. doi:10.1016/0378-3758(89)90067-0.

Sunoj SM, Nair NU, Nanda AK, Rasin RS (2020). "Ageing Intensity Function for Conditionally Specified Models." *American Journal of Mathematical and Management Sciences*. doi:10.1080/01966324.2020.1762143.

Sunoj SM, Rasin RS (2018). "A Quantile-based Study on Ageing Intensity Function." *Communications in Statistics-Theory and Methods*, **47**(22), 5474–5484. doi:10.1080/03610926.2017.1395049.

Szymkowiak M (2018). "Characterizations of Distributions through Aging Intensity." *IEEE Transactions on Reliability*, **67**(2), 446–458. doi:10.1109/TR.2018.2817739.

Szymkowiak M (2019). "Measures of Ageing Tendency." *Journal of Applied Probability*, **56**(2), 358–383. doi:10.1017/jpr.2019.28.

Szymkowiak M (2020). "Aging Intensities Vector for Bivariate Absolutely Continuous Distributions." In *Lifetime Analysis by Aging Intensity Functions*, pp. 51–63. Springer. ISBN 978-3-030-12106-8. doi:10.1007/978-3-030-12107-5_4.

ZeinEldin RA, Ahsan ul Haq M, Hashmi S, Elsehety M (2020). "Alpha Power Transformed Inverse Lomax Distribution with Different Methods of Estimation and Applications." *Complexity*, **2020**. doi:10.1155/2020/1860813.

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