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Original Paper

Several Methods on the Limit Problem of Integration

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Abstract

If the expression of a limit contains definite integral, we call this kind of limit as the limit of integration. This paper takes the graduate entrance exams of some famous universities in China as examples, and gives several methods to solve the limit problem of integration by comprehensively utilizing the characteristics and operational properties of integral.

Keywords

definite integral, the limit of integration, dichotomy, fitting method, the Riemann lemma

1. Introduction

The limit problem related to integral is often the contents of graduate entrance examinations and mathematical competitions. In this paper, some methods for solving limit problem of integration are given by using various characteristics and operation properties of definite integral.

Let $\{f_n(x)\}_{n\geq 1}$ be a sequence of integrable functions defined on some interval. Three kinds of methods for solving the limit $\lim\limits_{n\to\infty}\int_a^b f_n(x)dx$ are given below.

2. Interval Dichotomy

For any $\varepsilon > 0$, assume that the integral interval $[a,b]$ can be divided into two segments: on one segment, the integrands are uniformly bounded, and the length of such interval is less than ε, on the other integral interval, the integrands converges uniformly to some constant α, then

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \alpha(b-a).$$

The following is some examples.

Example 1.

(1) Verification $\lim_{n\to\infty}\int_0^1 e^{x^n}dx = 1$; (Wuhan University)

(2) Verification $\lim_{n\to\infty}\int_0^{\frac{\pi}{2}}(1-\sin x)^n dx = 0$; (Beijing University of Aeronautics and Astronautics)

(3) Find the limit $\lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\sin^n xdx$. (Lanzhou University)

The above questions can be solved by interval dichotomy. The following is a solution of the graduate entrance examination of Wuhan University.

Solution: Given $0<\varepsilon<\dfrac{\pi}{2}$, then

$$1 \le \int_0^1 e^{x^n}dx = \int_0^{1-\varepsilon} e^{x^n}dx + \int_{1-\varepsilon}^1 e^{x^n}dx \le (1-\varepsilon)e^{(1-\varepsilon)^n} + \varepsilon e.$$

Since $\lim_{n\to\infty} e^{(1-\varepsilon)^n} = 1$, therefore there exists N>0 such that for any n>N, $e^{(1-\varepsilon)^n} < 1+\varepsilon$. So

$$\text{above equation} < (1-\varepsilon)(1+\varepsilon)+\varepsilon e < 1+2e\varepsilon .$$

Therefore the original limit is 1.

A method of the dichotomous intervals that varies with the parameters n is given below:

Example 2. Let f be a continuous function on $[0,1]$. Find the limit

$$\lim_{n\to\infty}\int_0^1 f(\sqrt[n]{x})dx.$$

Solution: Consider the following interval segmentation

$$\int_0^1 f(\sqrt[n]{x})dx = \int_0^{\frac{1}{n}} f(\sqrt[n]{x})dx + \int_{\frac{1}{n}}^1 f(\sqrt[n]{x})dx .$$

Due to the continuity of f, the mean value theorem of integral can be applied to the two integrals on the right side of the above equation, then

Comments on this example: error-prone solutions: do not divide the interval, directly apply the mean value theorem. However in this case the median point $\xi(n)$ changes with parameter n changes, its range is not clear, so we can not get that $\lim_{n\to\infty}\sqrt[n]{\xi(n)}) = 1$.

3. Fitting Method

When some abstract functions and concrete functions appear simultaneously in the integrand, and the abstract functions have some continuity property, then the limit problem of integrals can be solved by fitting method.

Example 3. Assume that f is a continuous function on $[0,1]$. Prove

$$\lim_{n\to\infty}\frac{2}{\pi}\int_0^1\frac{n}{n^2x^2+1}f(x)dx = f(0)$$. (Wuhan University; Fudan University)

The main steps of fitting method are given below (along with a proof of the problem) :

1) Determine the fitting results

In this case, $f(0)$ is the fitting result (the part is actually the abstract function part which contains continuous points).

2) Remove the abstract function of the original integral, and calculate the limit of the integration

$$\lim_{n\to\infty}\frac{2}{\pi}\int_0^1\frac{n}{n^2x^2+1}dx = \lim_{n\to\infty}\frac{2}{\pi}\arctan n = 1 .$$

3) Modify the results and fit

The problem becomes

$$\lim_{n\to\infty}\frac{2}{\pi}\int_0^1\frac{n}{n^2x^2+1}f(x)dx = f(0) = \lim_{n\to\infty}\frac{2}{\pi}\int_0^1\frac{n}{n^2x^2+1}f(0)dx$$

equivalent to

$$\lim_{n\to\infty}\frac{2}{\pi}\int_0^1\frac{n}{n^2x^2+1}\left(f(x)-f(0)\right)dx = 0 .$$

4) Divide the integral interval (apply interval dichotomy method)

Due to the continuity of f, when variable x is close to 0 ,the factor $\left|f(x)-f(0)\right|$ is sufficiently small, when x goes away from 0, the factors $\left\{\frac{n}{n^2x^2+1}\right\}$ converges uniformly to 0, thus we can divided the interval $[0,1]$ into two parts for discussion (if necessary, it can be divided into several intervals for discussion). Specifically, for given $\varepsilon > 0$, by the continuity of f, there exists $0<\delta<1$ and $M>0$, such that for any $0\le x\le\delta$, $\left|f(x)-f(0)\right| < \frac{\pi\varepsilon}{4}$ and $\left|f(x)\right|\le M$, for $0\le x\le 1$. So

$$\left|\frac{2}{\pi}\int_0^1\frac{n}{n^2x^2+1}\left(f(x)-f(0)\right)dx\right| \le \frac{2}{\pi}\int_0^{\delta}\frac{n}{n^2x^2+1}\left|f(x)-f(0)\right|dx$$

$$+\frac{2}{\pi}\int_{1-\delta}^1\frac{n}{n^2x^2+1}\left|f(x)-f(0)\right|dx$$

$$\le \frac{2\varepsilon}{\pi}\int_0^{\delta}\frac{n}{n^2x^2+1}dx + \frac{2M}{\pi}\int_{1-\delta}^1\frac{n}{n^2x^2+1}dx \le \frac{\varepsilon}{2} + \frac{2M}{\pi}\int_{1-\delta}^1\frac{n}{n^2x^2+1}dx .$$

Since $\lim_{n\to\infty}\int_{1-\delta}^1\frac{n}{n^2x^2+1}dx = 0$, then there exists an integer $N>0$ such that for $n\ge N$,

$\int_{1-\delta}^1\frac{n}{n^2x^2+1}dx < \frac{\pi}{4M}\varepsilon$, therefore $\left|\frac{2}{\pi}\int_0^1\frac{n}{n^2x^2+1}\left(f(x)-f(0)\right)dx\right| < \varepsilon$.

This completes the proof.

Comments on this example:

(1) The condition on function f of this example can be weakened to Riemannian integrable function and continuous at $x=0$. One can prove the following example by a similar consideration: Suppose that f is an integrable function on interval $[-1,1]$. Set $\varphi_n(x)=\begin{cases}(1-x)^n, 0\le x\le 1,\\ e^{nx}, -1\le x\le 0\end{cases}$. Verification $\lim_{n\to\infty}\frac{n}{2}\int_0^1 f(x)\varphi_n(x)dx = f(0)$. (Zhejiang University)

(2) This method can also be deal with the limit problem with continuous parameters, for example, if f is a continuous function, one can prove

$$\lim_{h\to 0^+}\frac{2}{\pi}\int_0^1\frac{h}{h^2+x^2}f(x)dx = f(0).$$

4. Using Riemann Lemma

Riemann lemma: If f is an integrable function on $[a,b]$ and g is a periodic function with period T and integrable on $[0,T]$, then

$$\lim_{n\to\infty}\int_a^b f(x)g(nx)dx=\frac{1}{T}\int_a^b f(x)dx\int_0^T g(x)dx.$$

The proof of this lemma can be found in (Pei, 2006). If the limit problem of the integral contains periodic functions, the Riemann lemma can be tried to deal with, for example, the following limit

$$\lim_{n\to\infty}\int_0^\pi\frac{\sin x}{1+3\cos^2 nx}dx=\frac{1}{\pi}\int_0^\pi\frac{1}{1+3\cos^2 x}dx\int_0^\pi\sin xdx=\frac{2}{\pi}\int_0^\pi\frac{1}{1+3\cos^2 x}dx=1.$$

Riemann lemma also gives us a new way to deal with parametric integral problems.

Example 4. Let $h(x)=4[x]-2[2x]+1$, where $[x]$ is the integer part of x and f be an integrable function on $[0,1]$. Try to prove

$$\lim_{n\to\infty}\int_0^1 f(x)h(nx)dx=0. \text{ (Lanzhou University)}$$

Proof. (directly using the Riemann lemma) In fact, it is not difficult to see that $h(x)$ is a periodic function with period 1 and integrable on $[0,1]$, more precisely, $h(x)=\begin{cases}1, & x\in[0,\frac{1}{2})\\ -1, & x\in[\frac{1}{2},1)\end{cases}$. From the Riemann lemma, we get

$$\lim_{n\to\infty}\int_0^1 f(x)h(nx)dx=\int_0^1 f(x)dx\int_0^1 h(x)dx=0.$$

Comments on this example: Riemann lemma is still hold for continuous parameter.

5. Conclusion

This paper introduces three classical methods for dealing with the limit problem of integrals. It is also the commonly used ideas in dealing with problems of real variable function, functional analysis and Fourier analysis in later mathematics courses, in which fitting method has already had the rudiments of identity approximation. The limit problems discussed above still belongs to the limit problem of the series (function), so the usual methods used to discuss the limit of the series (function) are applicable in principle, but the difference is that various properties and algorithms of the integral are fully used here. The types of questions involved in this paper contain many knowledge points and are comprehensive, and the methods adopted are relatively classical, which can be used as reference for the teaching of mathematical analysis and help us to expand mathematical thinking.

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