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Stochastic model reduction and transport noise in fluid dynamics

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Chapter 1

Introduction

1.1 Motivations

Nowadays, it is widely accepted among the scientific community that a rigorous understanding of turbulence in fluids is one of the most important open problems in Mathematics and Physics.
Suppose we are given a fluid, injected into a pipe at sufficiently high speed. Experience suggests that, as we observe the fluid farther and farther downstream, the state of the fluid becomes more and more independent of its initial conditions at the inlet. The final flow will display universal characteristics, and we use the word *turbulence* as a label for these characteristics.
What are then the properties of a turbulent flow? First of all, following [Pan13], a turbulent flow has irregular, unpredictable, intermittent and self-sustaining fluctuations of velocity in all directions; it displays eddies at several length scales, ranging from that of the entire region where the fluid is turbulent, to one for which viscous forces become so strong that the eddies themselves are destroyed. Also, a turbulent flow is *diffusive*, because turbulent eddies transport fluid parcels across different regions of the fluid, resulting in an effective mixing of the fluid itself; and it is *dissipative*, meaning that kinetic energy is dissipated much faster than usual, due to the steepness of the velocity gradient. Finally, fully developed turbulence (in dimension three) manifest energy cascade from large to small scales [Val06] (the direction is the opposite in dimension two [BE12]); and consequentially, a sustained turbulent flow has a very peculiar energy spectrum, at least across some portion of scales.
However, none of the properties above has been deduced directly from the Navier-Stokes equation, that is the equation governing the evolution of the velocity field of an incompressible fluid:

$$\partial_t U + (U \cdot \nabla)U + \nabla P = \nu \Delta U + F, \qquad \operatorname{div} U = 0.$$

In addition, most flows of real life practical interest are too complex to be amenable to direct numerical simulations. As a consequence, there has been a huge amount of efforts towards *turbulence modelling*, aimed at finding simplified constitutive equations that predict the statistical evolution of turbulent flows. To this purpose, it proves convenient to split the actual velocity field U into *resolved* large-scale, slow-varying and *unresolved* small-scale, fast-varying components

$$U = u + v,$$

and seek closures with respect to the resolved component u only; the effect of the unresolved scales on the resolved ones must therefore be modelled. In the previous example of fluid passing through a pipe, for instance, we may be interested only on the average velocity u of the fluid across the whole pipe, neglecting local fluctuations v that shall remain unresolved.
Besides other approaches, in the last decades many mathematicians have tried to study these problems using tools from stochastic analysis; namely, by modifying the equations adding a noise term, heuristically due to fluctuations of v. Much has been proved, but a key question arises: how to properly choose the noise?

1.1.1 Why do we study stochastic PDEs?

Let us start from a preliminary question: what ultimately is randomness?
Ancient Greeks distinguished between the phenomena that obey natural laws, established once for all, and those unpredictable phenomena they attributed to *chance*, which are not subject to any law. During 18th century, in Europe, the notion of chance was put aside in favour of a totally deterministic point of view on the world: it was believed that a mind infinitely powerful and infinitely well-informed about the present state of the universe - the Laplace's demon - could predict the future using only the laws of classical mechanics. Although some ideal classical system are in fact indeterminate under extremely particular conditions (think of a ball balanced on the tip of a cone), the demon could appreciate the slightest asymmetry in the system and restore determinism. But this conception is not ours anymore: quantum mechanics postulates indeterminacy as a fundamental property of nature. An observer measuring the spin of an electron would measure $+1$ or -1 with certain probabilities, depending on the state of the system; and this indeterminacy cannot reflect any prior unknown condition of the system, as a consequence of various no-hidden variables theorems.
Returning to the initial question, we can definitely say that randomness is *not* chance, as entailed for instance by spin measurements in quantum mechanics. Fluid dynamics is deterministic, and there is no chance involved in the evolution of a system. Rather, we put randomness in the equations to take into account our incomplete knowledge of the system. *Randomness is the name we give to our ignorance.*
The hope is that: *i*) the addition of randomness makes it easier to establish a link between abstract equations of fluid dynamics and phenomenological laws of turbulence, somewhat in the same way statistical mechanics links the microscopic dynamics of molecules in a gas to the macroscopic laws of thermodynamics; *ii*) lack of uniqueness and singularities formation in fluids require very particular conditions that never happen in nature, and noise can reintroduce asymmetries and restore well-posedness (as in the previous example of the balanced ball); and *iii*) numerical simulations of the resolved variable u become more feasible for the reduced stochastic turbulence models, when compared with direct simulation of U solving the Navier-Stokes equation.

1.2 Content of this thesis

In this thesis I have collected a series of results mainly oriented towards the understanding of three intimately connected questions, detailed separately in the present section, and motivated by the discussion above. In order to ease the reading of what follows, I have gathered some frequently used notation in Section 1.4.

1.2.1 Stochastic model reduction

In applications to geophysics and climate studies, especially when one is interested in the simulations of complex turbulent flows like weather forecast, one necessarily has to deal with the fact that limited computational power often implies an under-representation of the real physical processes with spatial or temporal scale smaller than a certain threshold, typically the length of the grid parametrisation and the time discretization step. However, these small-scale processes may have a non-trivial impact on the large-scale ones, and thus it is important to take this impact into the account in order to obtain accurate description of the evolution of the simulated process.
Therefore, the first topic discussed in this thesis (more specifically in Chapter 2) is *stochastic model reduction* of fluid dynamics systems. This line of research finds its theoretical root in the seminal work [MTVE01] by Majda, Timofeyev and Vanden-Eijnden, and it consists in the following. Rewrite the Navier-Stokes equations (or any other equation of fluid dynamics relevance) in abstract form as an evolution equation in a Hilbert space H

$$\dot{U} = AU + B(U,U) + F; \tag{1.1}$$

then, it is assumed that resolved and unresolved variables u, v are identified via some orthogonal decomposition $H = H_d \oplus H_\infty$, so that (1.1) splits in

$$\begin{cases} \dot{u} = A_1^1 u + A_2^1 v + B_{11}^1(u,u) + B_{12}^1(u,v) + B_{21}^1(v,u) + B_{22}^1(v,v) + F^1, \\ \dot{v} = A_1^2 u + A_2^2 v + B_{11}^2(u,u) + B_{12}^2(u,v) + B_{21}^2(v,u) + B_{22}^2(v,v) + F^2. \end{cases} \tag{1.2}$$

At this point, the second equation in (1.2) is replaced by a simplified stochastic equation, in the spirit of turbulence closure (taking into account also a certain degree of scale separation $\epsilon \ll 1$), and convergence of the solution u of the first equation in (1.2) is investigated as $\epsilon \to 0$. It is worth of mention that, for numerical applications, the space H_d corresponding to resolved variable u is often supposed to be finite dimensional: $\dim H_d = d < \infty$, whereas $\dim H_\infty = \infty$ in most cases, since in general (1.1) describes a PDE (although for practical purposes a finite dimensional H_∞ may be sufficient, for instance replacing (1.1) with a sufficiently accurate Galerkin approximation).
In [MTVE01], under suitable rescaling of the coefficients (so that quantities involved are expressed with respect to large-scale coordinates) the simplified model takes the form

$$\begin{cases} \dot{u}^\epsilon = A_1^1 u^\epsilon + A_2^1 v^\epsilon + B_{11}^1(u^\epsilon,u^\epsilon) + B_{12}^1(u^\epsilon,v^\epsilon) + B_{21}^1(v^\epsilon,u^\epsilon) + \epsilon^{1/2} B_{22}^1(v^\epsilon,v^\epsilon) + F^1, \\ \dot{v}^\epsilon = \epsilon^{-1} A_1^2 u^\epsilon + \epsilon^{-1/2} A_2^2 v^\epsilon + \epsilon^{-1} B_{11}^2(u^\epsilon,u^\epsilon) + \epsilon^{-1/2} B_{12}^2(u^\epsilon,v^\epsilon) + \epsilon^{-1/2} B_{21}^2(v^\epsilon,u^\epsilon) \\ \qquad + \epsilon^{-1} f^2 - \epsilon^{-1} v^\epsilon + \epsilon^{-1} Q^{1/2} \dot{W}, \end{cases} \tag{1.3}$$

where $f_t^2 = F^2_{\epsilon^{-1/2}t}$ and $Q^{1/2}\dot{W}$ is a Gaussian noise, white in time and coloured in space, with trace-class covariance matrix Q.
Motivated by this construction, in [AFP21] Sigurd Assing, Franco Flandoli and I have studied the following fast-slow system

$$\begin{cases} \dot{X}^\epsilon = F(t,X^\epsilon) + \sigma(t,X^\epsilon) Y^\epsilon + \epsilon^{1/2} \beta(Y^\epsilon,Y^\epsilon), \\ \dot{Y}^\epsilon = -\epsilon^{-1} Y^\epsilon + \epsilon^{-1} Q^{1/2} \dot{W}, \end{cases} \tag{1.4}$$

where X_0^ϵ equals some deterministic $x_0 \in H_d$, Y^ϵ is stationary, $F : [0,T] \times H_d \to H_d$, $\sigma : [0,T] \times H_d \to \mathcal{L}(H_\infty, H_d)$ for some $T < \infty$, and $\beta : H_\infty \times H_\infty \to H_d$. In [AFP21, Section 5] we have shown how to recover the asymptotic behaviour as $\epsilon \to 0$ of u^ϵ solution of (1.3) from that of X^ϵ solution of (1.4).
As for the latter, next we introduce the limiting equation for $\bar{X} = \lim_{\epsilon\to 0} X^\epsilon$ (in some suitable sense, to be specified later). Let $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \dots\}$ be orthonormal bases of H_d, H_∞ respectively, and assume Q is diagonal with eigenvalues $q_m = \langle Q\mathbf{f}_m, \mathbf{f}_m\rangle_{H_\infty}$. First, define the so-called *Stratonovich correction* term $C : [0,T] \times H_d \to H_d$ by

$$C^i = \langle C, \mathbf{e}_i\rangle_{H_d} = \frac{1}{2} \sum_{m\in\mathbb{N}} q_m \sum_{j=1}^{d} D_j \sigma^{i,m} \sigma^{j,m}, \quad i = 1, \dots, d,$$

where $\sigma^{i,m} = \langle \sigma \mathbf{f}_m, \mathbf{e}_i\rangle_{H_d}$ for $i = 1, \dots, d$ and $m \in \mathbb{N}$ is matrix notation for the linear map $\sigma \in \mathcal{L}(H_\infty, H_d)$ with respect to our chosen basis vectors; second, let

$$b^i_{\ell,m} = \sqrt{\frac{q_\ell q_m}{2}}\, \langle \beta(\mathbf{f}_\ell, \mathbf{f}_m), \mathbf{e}_i\rangle_{H_d}, \quad i = 1, \dots, d, \quad \ell, m \in \mathbb{N}.$$

Then, our limiting equation reads

$$\dot{\bar{X}} = F(t, \bar{X}) + C(t, \bar{X}) + \sigma(t, \bar{X})\dot{W} + \sum_{\ell,m\in\mathbb{N}} b_{\ell,m} \dot{\bar{W}}^{\ell,m}, \tag{1.5}$$

where W is the same Wiener process used to define Y^ϵ, while $\{\bar{W}^{\ell,m}\}_{\ell,m\in\mathbb{N}}$ is a family of independent one-dimensional standard Wiener processes, which are also independent of W.
In this setting, we have proved:

Theorem 1.1. *Assume the coefficients of* (1.4) *satisfy:*

- $F \in C([0,T] \times H_d, H_d)$, *and* $F(t, \cdot) \in Lip_{loc}(H_d, H_d)$, *uniformly in* $t \in [0,T]$;
- $\sigma \in C^{1,\gamma}([0,T] \times H_d, \mathcal{L}(H_\infty, H_d))$ *for some* $\gamma \in (0,1)$ *and its space-differential* $D\sigma(t, \cdot) \in Lip_{loc}(H_d, \mathcal{L}(H_d, \mathcal{L}(H_\infty, H_d)))$, *uniformly in* $t \in [0,T]$;
- $\beta : H_\infty \times H_\infty \to H_d$ *is a continuous bilinear map;*
- $\sum_{\ell\in\mathbb{N}} \langle \beta(\mathbf{f}_\ell, \mathbf{f}_\ell), \mathbf{e}_i\rangle_{H_d} q_\ell = 0$, *for all* $i = 1, \dots, d$;
- *both equations* (1.4) *and* (1.5) *admit global solutions on* $[0,T]$.

Then X^ϵ *converges to* $\bar{X}$ *in law as* $\epsilon \to 0$. *Moreover, if* $\beta = 0$ *then the following stronger convergence holds true:*

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{ \sup_{t\le T} \|X_t^\epsilon - \bar{X}_t\|_{H_d} > \delta \right\} = 0, \quad \forall \delta > 0.$$

The proof of Theorem 1.1 is carried out in multiple steps. Let us discuss the strong convergence first. By a localization argument, we can restrict ourselves to $|X_t^\epsilon|, |\bar{X}_t| \le R$, for some large R, leading to Lipschitz continuity of the coefficients of (1.4) and (1.5);

second, it is possible to discretize the problem, thus reducing the desired convergence to its discrete version:

$$\lim_{\epsilon\to 0}\mathbb{P}\left\{\sup_k \|X^\epsilon_{t_k} - \bar{X}_{t_k}\|_{H_d} > \delta\right\} = 0, \quad \forall \delta > 0,$$

for only finitely many $t_k \in [0,T]$. Finally, the simplified version is proved following the lines of [IW14]. The weak convergence then comes from this preliminary result and a careful study of the quadratic term $\beta(Y^\epsilon, Y^\epsilon)$ in the limit $\epsilon \to 0$.

Notice that the limit equation (1.5) contains both additive noise $b_{\ell,m}\dot{\bar{W}}^{\ell,m}$ and multiplicative noise $\sigma(t,\bar{X})\dot{W}$, coming respectively from terms quadratic and linear with respect to Y^ϵ in the equation for X^ϵ. In particular, the multiplicative noise (in the sense of Stratonovich, because of the corrector C) has theoretical consequences on the justification of transport noise in fluid dynamics, that is the next pivotal topic of this thesis.

1.2.2 Justification of transport noise in fluid dynamics

By the theoretical point of view, model reduction has always played a primary role in geophysics and, more generally, in fluid mechanics; here model reduction is meant in the broad sense, as the operation of reducing the complexity of a model in order to conveniently describe certain phenomena. For example, if one is interested in the evolution of a certain geophysical flow on a relatively small portion of Earth's surface, then the spherical geometry of the problem is usually not so important and the use of spherical coordinates is an unnecessary complication: it is way more convenient to study the problem in Cartesian coordinates. The dynamical effects of Earth's rotation are therefore captured with the so-called f-plane approximation [Val06] (and more generally with the β-plane approximation), which constitutes a nice simplification of the problem yet capable of describing very interesting phenomena, like the motion of cyclonic flows at geostrophic balance and the Taylor–Proudman effect.

That being said, in the series of works [FP20, FP21, FP22] Franco Flandoli and I have proposed a splitting of (1.1) alternative to (1.3), in the sense that it does not come with a decomposition $H = H_d \oplus H_\infty$ but rather we impose *a priori* the evolution for u and v separately as the following system of PDEs:

$$\begin{cases} \dot{u} = Au + B(u,u) + B(v,u), \\ \dot{v} = Av + B(u,v) + B(v,v) + F. \end{cases} \tag{1.6}$$

This of course is a modelling choice; as described in details in [FP20], it is consistent with the heuristic idea that the two components of the system model the dynamics of large and small structures separately. As far as this is concerned, the splitting above is substantially equivalent to what done in the research trend called *location uncertainty* [Mé14].

Nevertheless, the variables u, v need not to represent respectively the resolved and unresolved velocity of the system under investigation. For instance, considering u, v as variables representing the *vorticity* of the fluid we recover something equivalent to the so-called *stochastic advection by Lie transport* scheme [Hol15, CGH17, FL19, GBH18].

After suitable stochastic modelling and rescaling of (1.6) (as for the stochastic model reduction paradigm, but keeping quadratic self-interaction in the second equation) we end up with the system

$$\begin{cases} \dot{u}^\epsilon = Au^\epsilon + B(u^\epsilon, u^\epsilon) + B(v^\epsilon, u^\epsilon), \\ \dot{v}^\epsilon = Av^\epsilon + B(u^\epsilon, v^\epsilon) + B(v^\epsilon, v^\epsilon) - \epsilon^{-1} v^\epsilon + \epsilon^{-1} Q^{1/2} \dot{W}. \end{cases} \tag{1.7}$$

It is worth of mention that, even if the abstract system (1.6) is completely general, the validity of this modelling assumption requires proper justification that depends on the particular system described by (1.6). For instance, in [FP20, Section 2] we showed this is the case when u describes continental-scale velocity structures and v describes human-scale fluctuations (and the units of measure in (1.7) are macroscopic); for equations in vorticity form, a justification based on different time-scales is provided in [FP21, Section 2].

This motivates the study *per se* of abstract systems of the form (1.7) and variations thereof, thanks to the plethora of potential applications. Moving to our specific contributions to the topic, I decided to present in this thesis the works [FP22] and [DP22] (the latter in collaboration with Arnaud Debussche), which are detailed in Chapter 3 and Chapter 4 respectively. Here I just give a summary of the results and ideas contained therein.

In [FP22] we have studied the following system on the two-dimensional torus $\mathbb{T}^2$, describing the coupling between large-scale Navier-Stokes ($\nu > 0$) or Euler ($\nu = 0$) equations and small-scale stochastic Euler equations:

$$\begin{cases} \partial_t \Xi^\epsilon + (u^\epsilon + v^\epsilon) \cdot \nabla \Xi^\epsilon = \nu \Delta \Xi^\epsilon + q^\epsilon, \\ \partial_t \xi^\epsilon + (u^\epsilon + v^\epsilon) \cdot \nabla \xi^\epsilon = -\epsilon^{-1} \xi^\epsilon + \epsilon^{-1} \sum_{k \in \mathbb{N}} \varsigma_k \dot{W}^k, \\ u^\epsilon = -\nabla^\perp (-\Delta)^{-1} \Xi^\epsilon, \\ v^\epsilon = -\nabla^\perp (-\Delta)^{-1} \xi^\epsilon. \end{cases}$$

The unknowns are the vorticities Ξ^ϵ, ξ^ϵ (as in the stochastic advection by Lie transport scheme) and the velocities u^ϵ, v^ϵ are reconstructed from the vorticities using the Biot-Savart law. The quantity q^ϵ is a zero-average source term, and $\sum_k \varsigma_k \dot{W}^k$ is just a more explicit expression for the additive noise $Q^{1/2} \partial_t W$ ($\varsigma_k : \mathbb{T}^2 \to \mathbb{R}^2$ is divergence-free and zero-average for all $k \in \mathbb{N}$, and the family $\{W^k\}_{k \in \mathbb{N}}$ is made of i.i.d. standard Brownian motions), introduced in order to have convenient assumptions on the noise detailed below.

In this setting, we proved that Ξ^ϵ converges towards the solution of the limiting Navier-Stokes or Euler equations with Stratonovich transport noise:

$$\begin{cases} \partial_t \Xi + u \cdot \nabla \Xi + \sum_{k \in \mathbb{N}} \sigma_k \cdot \nabla \Xi \circ \dot{W}^k = \nu \Delta \Xi + q, \\ u = -\nabla^\perp (-\Delta)^{-1} \Xi, \end{cases}$$

where $\sigma_k = -\nabla^\perp (-\Delta)^{-1} \varsigma_k$, as stated in the following:

Theorem 1.2. *Fix $T > 0$, and assume:*

- *$\xi^\epsilon_0 = 0$ and $\Xi^\epsilon_0 = \Xi_0 \in L^\infty(\mathbb{T}^2)$ is deterministic and zero-average;*
- *there exists $\ell \geq 1$ such that $\varsigma_k \in W^{\ell,\infty}(\mathbb{T}^2)$ with zero-mean for every $k \in \mathbb{N}$, and moreover $\sum_{k \in \mathbb{N}} \|\varsigma_k\|_{W^{\ell,\infty}(\mathbb{T})} < \infty$;*
- *for every $x \in \mathbb{T}^2$ it holds $\sum_{k \in \mathbb{N}} (\sigma_k \cdot \nabla \varsigma_k)(x) = 0$;*

- *there exists a constant C such that for every $\epsilon > 0$ it holds $q^\epsilon, q \in L^1([0,T], L^\infty(\mathbb{T}^2))$ and $\int_0^T \|q^\epsilon_s\|_{L^\infty(\mathbb{T}^2)} ds \leq C$, $\int_0^T \|q_s\|_{L^\infty(\mathbb{T}^2)} ds \leq C$;*
- *$q^\epsilon - q$ converges to zero in $L^1([0,T], L^\infty(\mathbb{T}^2))$.*

Then Ξ^ϵ converges towards Ξ in the following sense: for every $f \in L^1(\mathbb{T}^2)$

$$\mathbb{E}\left[\left|\int_{\mathbb{T}^2} \Xi^\epsilon_t(x) f(x) dx - \int_{\mathbb{T}^2} \Xi_t(x) f(x) dx\right|\right] \to 0 \qquad \text{as } \epsilon \to 0,$$

for every fixed $t \in [0,T]$ and in $L^p([0,T])$ for every finite p. Moreover, if $q \in L^1([0,T], Lip(\mathbb{T}^2))$ then the previous convergence holds uniformly in $t \in [0,T]$ and $f \in Lip(\mathbb{T}^2)$ with Lipschitz constant $[f]_{Lip(\mathbb{T}^2)} \leq 1$ and $\|f\|_{L^\infty(\mathbb{T}^2)} \leq 1$.

The previous result generalizes those in [FP21] by the same authors, where it is proved a similar convergence for the inviscid ($\nu = 0$), unforced ($q^\epsilon = 0$) system with no quadratic self-interaction at small-scales, that is without the term $v^\epsilon \cdot \nabla \xi^\epsilon$ in the second equation. The strategy of the proof consists in proving in the first place the convergence $\phi^\epsilon \to \phi$ for the Lagrangian particle trajectories, or *characteristics*:

$$\begin{aligned} \phi^\epsilon_t(x) &= x + \int_0^t u^\epsilon_s(\phi^\epsilon_s(x)) ds + \int_0^t v^\epsilon_s(\phi^\epsilon_s(x)) ds + \sqrt{2\nu} w_t, \\ \phi_t(x) &= x + \int_0^t u_s(\phi_s(x)) ds + \sum_{k \in \mathbb{N}} \int_0^t \sigma_k(\phi_s(x)) \circ dW^k_s + \sqrt{2\nu} w_t, \end{aligned}$$

where w is an auxiliary $\mathbb{R}^2$ valued Brownian motion. This can be done with techniques similar to those of [AFP21], although there are some differences. The main difficulties in the proof consist in the equation of characteristics containing the velocity field itself as drift (that requires a careful analysis of the Biot-Savart kernel) and the equation for ξ^ϵ having quadratic self-interaction (which is the reason we introduced the third assumption of the theorem, corresponding to a sort of isotropy of the noise).
Then, relying on the measure-preserving property of characteristics and representation formulae

$$\begin{aligned} \Xi^\epsilon_t &= \tilde{\mathbb{E}}\left[\Xi_0 \circ (\phi^\epsilon_t)^{-1} + \int_0^t q^\epsilon_s \circ \phi^\epsilon_s \circ (\phi^\epsilon_t)^{-1} ds\right], \\ \Xi_t &= \tilde{\mathbb{E}}\left[\Xi_0 \circ (\phi_t)^{-1} + \int_0^t q_s \circ \phi_s \circ (\phi_t)^{-1} ds\right], \end{aligned}$$

($\tilde{\mathbb{E}}$ denotes the expectation with respect to w) we are able to prove convergence of the vorticity fields $\Xi^\epsilon \to \Xi$ in the sense of previous theorem. In particular, when testing against test function $f \in L^1(\mathbb{T}^2)$, it holds

$$\begin{aligned} &\int_{\mathbb{T}^2} \Xi^\epsilon_t(x) f(x) dx - \int_{\mathbb{T}^2} \Xi_t(x) f(x) dx \\ &\quad = \tilde{\mathbb{E}}\left[\int_{\mathbb{T}^2} \Xi_0(y) f(\phi^\epsilon_t(y)) dy - \int_{\mathbb{T}^2} \Xi_0(y) f(\phi_t(y)) dy\right] \\ &\qquad + \tilde{\mathbb{E}}\left[\int_0^t \int_{\mathbb{T}^2} q^\epsilon_s(\phi^\epsilon_s(y)) f(\phi^\epsilon_t(y)) dy ds - \int_0^t \int_{\mathbb{T}^2} q_s(\phi_s(y)) f(\phi_t(y) dy ds\right], \end{aligned}$$

and the convergence in expectation to zero of the right-hand-side of the equation above can be shown invoking some measure theoretic arguments, see Section 3.4 for details.
Being the method in [FP22] strongly reliant on the Lagrangian formulation of the equations (that is the main reason the previous approach was limited to two space dimensions), together with Arnaud Debusche I have developed in [DP22] a completely new strategy that allows us to overcome this issue and consider more general systems, in particular three-dimensional systems in velocity form that are notoriously difficult to study in the Lagrangian formulation.
Let us illustrate the main arguments of [DP22] in the particular case of three-dimensional Navier-Stokes equations (generalizations to Surface Quasi-Geostrophic and Primitive equations are contained in [DP22, Chapter 7]). A thorough discussion is postponed to Chapter 4. Denote $H \coloneqq \{u \in [L^2(\mathbb{T}^3)]^3, \operatorname{div} u = 0\}$ the space of periodic, zero-mean, square integrable velocity fields u with null divergence in the sense of distributions and $\Pi : [L^2(\mathbb{T}^3)]^3 \to H$ the Helmotz projector. We have studied weak solutions (u^ϵ, v^ϵ) with suitable bounds on the energy (cfr. Definition 4.1 for a precise definition) of the system

$$\begin{cases} \partial_t u^\epsilon = \nu \Delta u^\epsilon - \Pi(u^\epsilon \cdot \nabla) u^\epsilon - \Pi(v^\epsilon \cdot \nabla) u^\epsilon, \\ \partial_t v^\epsilon = \nu \Delta v^\epsilon - \Pi(u^\epsilon \cdot \nabla) v^\epsilon - \Pi(v^\epsilon \cdot \nabla) v^\epsilon + \epsilon^{-1} C v^\epsilon + \epsilon^{-1} Q^{1/2} \partial_t W, \end{cases}$$

where C is a general hypodissipative term and $\nu > 0$, and we have proved the following theorem:

Theorem 1.3. *Let $T < \infty$ be fixed. Suppose $u_0, y_0 \in H$ be given and deterministic, and assume:*

- *the operator $C : D(C) \subset H \to H$ is self-adjoint and negative definite, with principal eigenvalue $-\lambda_0 < 0$;*
- *there exist $\Gamma \geq \gamma > 1/4$ such that $\|x\|^2_{H^{s+\beta\gamma}} \lesssim \|(-C)^{\beta/2} x\|^2_{H^s} \lesssim \|x\|^2_{H^{s+\beta\Gamma}}$ for every $s \in \mathbb{R}$, $\beta > 0$;*
- *Q is symmetric, positive semidefinite and commutes with C, and the operators $e^{Ct} Q e^{Ct}$ and $Q_\infty = (-C)^{-1} Q$ on H are trace-class for every $t \geq 0$;*
- *denoting $\mathcal{N}(0, Q_\infty)$ the Gaussian measure on H with covariance Q_∞ and $s_0 = \max\{5/2 + \delta, 2\Gamma\}$, $\delta > 0$ arbitrary, it holds $\int_H \|w\|^2_{H^{s_0}} d\mathcal{N}(0, Q_\infty)(w) < \infty$.*

Let μ denote the unique invariant measure on H of the linearized equation $\partial_t v = Cv + Q^{1/2} \partial_t W$, that exists by our assumptions on C and Q. Then for every $\beta > 0$ the laws of the processes $\{u^\epsilon\}_{\epsilon \in (0,1)}$ are tight as probability measures on the space $L^2([0,T], H) \cap C([0,T], H^{-\beta})$, and every weak accumulation point $(u, Q^{1/2} W)$ of $(u^\epsilon, Q^{1/2} W^\epsilon)$[1], $\epsilon \to 0$, is an analytically weak solution of the equation with transport noise and Itō-Stokes drift velocity $r = \int_H (-C)^{-1} (w \cdot \nabla) w \, d\mu(w)$:

$$\partial_t u = \nu \Delta u + \Pi(u \cdot \nabla) u + \Pi((-C)^{-1} Q^{1/2} \circ \partial_t W \cdot \nabla) u + \Pi(r \cdot \nabla) u.$$

The Itō-Stokes drift velocity r, usually defined as the difference between Lagrangian and Eulerian average flows, has important consequences in wave-induced sediment transport

[1]The Wiener process W^ϵ may depend on ϵ since we are dealing with probabilistically weak solutions.

and sandbar migration in the coastal zone, and well as transport of heat, salt and other natural or man-made tracers in the upper ocean layer [vdBB18].
Our result justifies and motivates the interest in transport noise (and Itō-Stokes drift) in fluid dynamics, and, to the best of our knowledge, this is the first rigorous derivation of a stochastic fluid model of this kind.
In fact, the transport noise in the limit equation comes from a diffusion-approximation argument, which in our case can be seen as a Wong-Zakai type result; on the other hand, the Itō-Stokes drift is due to an averaging phenomenon. As already mentioned above, the approach is very general and amenable to generalizations to different systems (in [DP22] we applied the same method also to the Surface Quasi-Geostrophic equations and to the Primitive equations, but other applications are surely possible).
Roughly speaking, the strategy of the proof consists in studying the generator $\mathscr{L}^\epsilon$ of the renormalized process (u^ϵ, y^ϵ), $y^\epsilon = \epsilon^{1/2} v^\epsilon$, and finding for every suitable test function $\varphi : H \to \mathbb{R}$ correctors $\varphi_1^\epsilon, \varphi_2^\epsilon : H \times H \to \mathbb{R}$ such that

$$\mathscr{L}^\epsilon \varphi(u^\epsilon) + \epsilon^{1/2} \mathscr{L}^\epsilon \varphi_1^\epsilon(u^\epsilon, y^\epsilon) + \epsilon \mathscr{L}^\epsilon \varphi_2^\epsilon(u^\epsilon, y^\epsilon) = \mathscr{L}^0 \varphi(u^\epsilon) + o(1), \tag{1.8}$$

for some *effective* generator $\mathscr{L}^0$ (correctors are needed because $\mathscr{L}^\epsilon \varphi(u^\epsilon)$ contains diverging-in-ϵ terms). In order to do so, a careful analysis of the Poisson equations $\langle Cy, D_y \phi \rangle + Tr(QD_y^2 \phi) = \psi$ and $\langle (C - \nu\epsilon\Delta) y, D_y \phi \rangle + Tr(QD_y^2 \phi) = \psi$ (in the unknown ϕ) is required, as well a Sobolev estimates on (bilinear functions of) u^ϵ, y^ϵ, as detailed in Chapter 4. Among the main novelties of the paper, there is the fact we are able to replace the small-scale process y^ϵ in (1.8) with its linearized version

$$dY_t^\epsilon = \epsilon^{-1} C_\epsilon Y_t^\epsilon dt + \epsilon^{-1/2} Q^{1/2} dW_t, \quad Y_0^\epsilon = 0,$$

which enjoys better space regularity than y^ϵ because of the absence of the non-linear term b. This facilitates the construction of correctors φ_1^ϵ and φ_2^ϵ, since some expression defining the correctors are just formal expression, rigorously defined in sufficiently smooth regimes only.

1.2.3 Mixing and dissipation of Ornstein-Uhlenbeck flows

The last work discussed in the main body of present thesis (more specifically in Chapter 5) is [Pap22b], where I have studied mixing and dissipation enhancement of the stationary Ornstein-Uhlenbeck flow

$$v^\epsilon = \sum_{j \in J} v_j \eta^{\epsilon,j}, \quad \dot{\eta}^{\epsilon,j} = -\epsilon^{-1} \eta^{\epsilon,j} + \epsilon^{-1} \dot{W}^j,$$

where J is a finite set and v_j is smooth, time-independent and divergence-free vector field for every $j \in J$, advecting a passive scalar ρ with molecular diffusivity $\kappa \geq 0$ via

$$\partial_t \rho + v^\epsilon \cdot \nabla \rho = \kappa \Delta \rho.$$

Introducing the *eddy diffusion* operator $\mathcal{L}$ given by

$$(\mathcal{L} f)(x) = \frac{1}{2} \sum_{j \in J} v_j(x) \cdot \nabla (v_j \cdot \nabla f)(x)$$

and the limiting equation

$$\partial_t \bar{\rho} = A\bar{\rho},$$

$A = \kappa\Delta + \mathcal{L}$, we are able to prove mixing on finite time intervals, as stated in the next

Theorem 1.4. *Let $\rho_0 \in L^2(\mathbb{T}^d)$ with zero mean, $d \geq 3$, and $T < \infty$ be fixed. Then, for every $\gamma \in (0, (d-2)/6)$ and $s > 0$ there exist coefficients $\theta, \varkappa, \varsigma > 0$ such that for every ϵ sufficiently small*

$$\mathbb{E}\left[\|\rho - \bar{\rho}\|_{C^\theta([0,T],H^{-s}(\mathbb{T}^d))}\right] \leq C\|\rho_0\|_{L^2(\mathbb{T}^d)} \left(\alpha + \epsilon^\varkappa \mu^{2+\gamma}\right)^\varsigma,$$

where α, μ are finite quantities depending on $\{v_j\}_{j\in J}$ (in particular, α can be made arbitrarily small choosing the family $\{v_j\}_{j\in J}$ properly) and $C \in (0,\infty)$ is an unimportant constant.

In the statement of the theorem, by ϵ sufficiently small we mean more precisely: there exist positive numbers $p_1, p_2 > 0$, depending only on the parameters γ, θ and κ, such that the thesis holds for every ϵ satisfying $\epsilon^{p_1} \log^{p_2}(1+\epsilon^{-1}) < 1$. The coefficients p_1, p_2 can be computed explicitly, cfr. the proof of Proposition 5.4 in Chapter 5.
It is well-known [MK99] that dissipation enhancement for advection-diffusion equations occurs for the average field $\mathbb{E}[\rho]$, when the fluid is sufficiently turbulent, and therefore $\mathbb{E}[\rho]$ is indeed expected to solve the same equation as $\bar{\rho}$ – which can be rigorously proved when u^ϵ is delta-correlated in time, interpreting $\mathcal{L}$ as a Stratonovich-to-Itō corrector. On the other hand, it is important to quantify the error made when approximating the ideal model (that with transport noise, as also emerges in the infinite scale separation limit by previous discussions) with an actual non-delta-correlated model. Thus, the interest in this result is totally justified, and moreover it would be important to understand whether similar quantitative results can be established for the general convergence results discussed above for non-simplified model, where small scales are modelled with solutions of actual fluid dynamics equations as Navier-Stokes and Euler's.
The mixing property stated in the previous theorem has many consequences, among which transfer on energy to high wavenumbers stands out. This mechanism is the main responsible for the enhanced dissipation of the $L^2(\mathbb{T}^d)$ norm of ρ. The second main result of [Pap22b] permits to estimate the rate of decay of the $L^2(\mathbb{T}^d)$ norm of ρ, when the molecular diffusivity κ is strictly positive.

Theorem 1.5. *In the same setting as above, assume in addition $\kappa > 0$. Let $c = C^{1/2}\left(\alpha + \epsilon^\varkappa \mu^{2+\gamma}\right)^{\varsigma/2} > 0$ and denote $\lambda > 0$ the principal eigenvalue of the operator $-A$. Then the following inequality holds with probability at least $1 - c$ for every $t \in [0,T]$:*

$$\|\rho_t\|_{L^2(\mathbb{T}^d)} \leq \frac{\|\rho_0\|_{L^2(\mathbb{T}^d)}}{\left(1 + \frac{\kappa}{2\lambda c^2}\log\left(\frac{c^2 e^{2\lambda t}+1}{c^2+1}\right)\right)^{1/2}}.$$

In particular, for every $t \in [0,T]$ it holds

$$\mathbb{E}\left[\|\rho_t\|_{L^2(\mathbb{T}^d)}\right] \leq c\|\rho_0\|_{L^2(\mathbb{T}^d)} + \frac{\|\rho_0\|_{L^2(\mathbb{T}^d)}}{\left(1 + \frac{\kappa}{2\lambda c^2}\log\left(\frac{c^2 e^{2\lambda t}+1}{c^2+1}\right)\right)^{1/2}}.$$

Recall that the only estimate available a priori for the $L^2(\mathbb{T}^d)$ norm of ρ is given by

$$\|\rho_t\|_{L^2(\mathbb{T}^d)} \leq e^{-\kappa t}\|\rho_0\|_{L^2(\mathbb{T}^d)},$$

and the previous inequality is in fact an equality in the inviscid case $\kappa = 0$, when dissipation does not occur. The content of our previous theorem can thus be read as follows: for every fixed $t > 0$, if $\kappa > 0$ and $\{v_j\}_{j\in J}$ is such that $\lambda \gg 1$ and $c \ll 1$ then $\mathbb{E}\left[\|\rho_t\|_{L^2(\mathbb{T}^d)}\right] \ll e^{-\kappa t}\|\rho_0\|_{L^2(\mathbb{T}^d)}$, namely dissipation of the $L^2(\mathbb{T}^d)$ norm is enhanced. Also, taking formally $c \to 0$ we obtain an augmented decay rate for the $L^2(\mathbb{T}^d)$ norm, that is

$$\mathbb{E}\left[\|\rho_t\|_{L^2(\mathbb{T}^d)}\right] \leq \frac{\lambda^{1/2}}{\kappa^{1/2}} e^{-\lambda t}\|\rho_0\|_{L^2(\mathbb{T}^d)}, \qquad \forall t \in [0,T]. \tag{1.9}$$

In order to make $\lambda \gg 1$ and $c \ll 1$ simultaneously, one can choose first the family $\{v_j\}_{j\in J}$ so that $\lambda \gg 1$ and $\alpha \ll 1$ at the same time, and then take ϵ sufficiently small so that $c \ll 1$. The problem of finding a family $\{v_j\}_{j\in J}$ that renders simultaneously λ large and α small has been previously treated (see [Gal20] and subsequent works), and it will not be discussed further here.
The strategy of the proof is as follows. First, we need a suitable bound on the quantity

$$\sup_{\substack{n,m=1,\dots,1/\delta-1 \\ n>m}} (n\delta - m\delta)^{-\theta} \left| \langle \phi, \rho_{n\delta}\rangle - \langle \phi, \rho_{m\delta}\rangle - \delta \sum_{k=m}^{n-1} \langle A\phi, \rho_{k\delta}\rangle \right|,$$

where δ is a small parameter, suitably chosen in depending on ϵ. Then, to prove Theorem 1.4, the key idea consists in introducing the random distribution f by

$$\langle \phi, f_t\rangle = \langle \phi, \rho_t\rangle - \langle \phi, \rho_0\rangle - \int_0^t \langle A\phi, \rho_s\rangle ds, \quad \forall \phi \in H^2(\mathbb{T}^d),$$

and show that $\rho - \bar\rho$ depends *path-by-path* continuously on f, thus producing an estimate on $\rho - \bar\rho$ from an estimate on f.
As for Theorem 1.5, its proof relies on the following energy inequality

$$\frac{d}{dt}\|\rho_t\|^2_{L^2(\mathbb{T}^d)} \leq -2\kappa \frac{\|\rho_t\|^4_{L^2(\mathbb{T}^d)}}{\|\rho_t\|^2_{H^{-1}(\mathbb{T}^d)}}$$

and a bound on $\|\rho_t\|^2_{H^{-1}(\mathbb{T}^d)}$ obtained applying Theorem 1.4 with $s = 1$. More details are given in the corresponding Chapter 5.

1.3 What is not included here

A part from the principal line of research outlined above, during my PhD studies I have had the possibility of doing research on different topics. As a consequence, I have authored several works that do not group together in a coherent, systematic way: some of them are a little more than exercises, which I wrote as an excuse to master new techniques (a practice that, I hope, will be forgiven to a novice PhD student); others are the starting point of lines of research never properly explored, admittedly due to my laziness; all of them are ultimately motivated, directly or indirectly, by my interest in fluid dynamics.

1.3.1 Well-posedness theory for some geophysical models in 2D

Fluid dynamics is more than Navier-Stokes and Euler equations. For geophysical applications, it often makes sense to study other equations that take into account peculiarities of the system Earth.
Together with Francesco Grotto and borrowing techniques from [AC90] and [Fla18], I have established in [GP21] existence of stationary solutions (q, V), preserving a physically relevant Gibbsian measure, of the two-dimensional Barotropic Quasi-Geostrophic equations in a channel $R = \mathbb{S}^1 \times [0, \pi]$ (see [MW06]):

$$\begin{cases} \partial_t q + \nabla^\perp \psi \cdot \nabla q = 0, \\ q = \Delta\tilde{\psi} + h + \beta y, \\ \psi = -Vy + \tilde{\psi}, \\ \dot{V} = -\frac{1}{|R|}\int_R \partial_x h(z)\tilde{\psi}(z)dz, \end{cases}$$

where q denotes the *potential vorticity*, ψ is the *stream function*, V is a function of time only describing a large scale mean flow, h is the topography and $\beta \in \mathbb{R}$ approximates the Coriolis force.
With the same techniques, jointly with Franco Flandoli and Milo Viviani [FPV22] I have shown a similar result for the Zeitlin's discretization of two-dimensional Euler equations on the sphere (cfr. [Zei04, MV20]).
In [GP22b] (co-authored by Francesco Grotto), we identified a *generalized enstrophy* formally preserved by two-dimensional Primitive equations with dissipation and additive noise:

$$\partial_t \omega + \nabla^\perp A(\omega) \cdot \nabla\omega = -(-\Delta)^\theta \omega + \sqrt{2}(-\Delta)^{\theta/2}\partial_t W,$$

where ω is a *generalized vorticity* (defined as the derivative in the vertical direction of the horizontal velocity) and $A(\omega)$ satisfies $-\partial_z A(\omega) = \omega$ (plus suitable boundary conditions). Using the formalism of [GJ13] (cfr. also [GT19]), for $\theta > 2$ we can prove existence of stationary solutions to Primitive equations preserving the generalized enstrophy; for $\theta > 3$, we can also prove pathwise uniqueness of solutions.

1.3.2 Bursts of Euler and Surface Quasi-Geostrophic vortices

Francesco Grotto and I have authored also [GP22a], where we gave a rigorous construction of solutions to the Euler point vortices system on the plane $\mathbb{R}^2 \approx \mathbb{C}$:

$$\dot{\bar{z}}_j = \frac{1}{2\pi i}\sum_{k\neq j}\frac{\xi_k}{z_j - z_k},$$

where $z_1, \dots, z_N : (0, \infty) \to \mathbb{C}$ are the positions of N point vortices, $N \geq 3$, each one with intensity $\xi_1, \dots, \xi_N \in \mathbb{R} \setminus \{0\}$, in which three vortices burst out of a single one. More precisely, we proved that given *any* configuration of $N - 2$ distinct vortices on the plane at time $t = 0$, in a small time interval $(0, T)$ there exists a solution of Euler point vortices system with N vortices, three of which burst out of a single one from the initial configuration, with their intensities summing up to the split vortex one. By time inversion, this implies existence of arbitrarily large configurations in which three vortices collide in finite time.

Systems of three vortices are integrable, and self-similar bursts and collapses of three vortices have been explicitly known since a while. On the other hand, the case $N > 3$ has been a long-standing open problem. Our strategy consists in showing first existence of bursts of three vortices under the influence of a suitable external vector field, by expressing the system in a convenient coordinate system describing closeness to the self-similar free solution, and reformulating the problem as a fixed point problem. This is an interesting result *per se*: for instance we will deduce from it the existence of bursts of three vortices in periodic or bounded domains. Existence of a burst of three vortices out of one in a system of many vortices then follows from this preliminary result, dividing the system into three bursting vortices under the external influence of the other ones, and the rest of the configuration, involving only vortices that do not collapse or burst.

1.3.3 Non-autonomous attractors of Random Dynamical Systems

In the work [FPT22], joint with Franco Flandoli and Elisa Tonello, we tried to identify a mathematical framework adequate to formal definition of concepts like weather, climate, and connection between them. The work is very speculative and this line of research is still in an embryonic state; yet the main ideas can be described here.
We start from the assumption that weather is statistically described by a family of random, time-dependent measures $\mu_\omega(t)$ on the Borel subsets of a state space X and its dynamics is encoded into a non-autonomous Random Dynamical System $U_\omega(s,t) : X \to X$, $s,t \in \mathbb{R}$, satisfying

$$U_\omega(s,t)_* \mu_\omega(s) = \mu_\omega(t).$$

The typical time-scale at which we appreciate fluctuations of μ_ω is $\epsilon \ll 1$ (usually corresponding to hours or days), assuming s,t denote macroscopic time variables proper of climate (with typical time-scale of years or decades).
How to define climate, then? Heuristically, we want climate to capture at once daily fluctuations of weather and slow-varying, long-term climate trends. In the ideal infinite time-scale separation limit, we exhibit sufficient conditions based on the random attractor of U_ω that guarantee the existence of a limit (up to subsequences) of $\mu_\omega = \mu_\omega^\epsilon$ as $\epsilon \to 0$, see [FPT22, Theorem 1.1]. The limiting object is a Young measure on the product space $\mathbb{R} \times X$ and encodes both daily fluctuations of the weather ($\mu(t,\cdot)$ is a measure on X for every fixed $t \in \mathbb{R}$) and long-term climate trends ($t \mapsto \mu(t,\cdot)$ is in general non-constant).

1.3.4 LDP for SDEs in Hilbert spaces with non-Lipschitz drift

Finally, in [Pap22a] I have studied Large Deviations, as $\epsilon \to 0$, for the family of stochastic differential equations in a infinite dimensional separable Hilbert space H

$$\dot X^\epsilon = AX^\epsilon + B(X^\epsilon) + \epsilon \dot W,$$

where $A : D(A) \subset H \to H$ is a negative-definite self-adjoint operator such that $(-A)^{1+\delta}$ is trace-class for some $\delta \in (0,1)$, $B : H \to H$ is continuous with at most linear growth and W is a cylindrical Wiener process on H. The initial condition is a deterministic $X_0^\epsilon = x_0 \in D((-A)^{\delta/2})$.

In this setting, I have proved the analogous of Freidlin-Wentzell Theorem [FW12] for X^ϵ: for every $\alpha \in (0, \delta/2)$ there exists $T > 0$ such that a Large Deviation Principle on $C([0,T], D((-A)^\alpha)$ holds for X^ϵ, with rate ϵ^2 and action functional given by

$$S(\varphi) = \frac{1}{2}\int_0^T \|\dot{\varphi}_t - A\varphi_t - B(\varphi_t)\|_H^2 dt,$$

if $\varphi \in W^{1,2}([\tau, T], H) \cap L^2([\tau, T], D(A))$ for every $\tau \in (0, T)$, $\varphi(0) = 0$, and $S(\varphi) = +\infty$ otherwise. Moreover, if B is bounded the thesis holds for every choice of $T < \infty$.
The main novelty of [Pap22a] consists in the fact that the non-linear term B need not to be (locally) Lipschitz continuous; in particular, the unperturbed equation $\epsilon = 0$ may lack uniqueness (whereas well-posedness for $\epsilon > 0$ has been proved in [DPFPR13]), so that the classical weak convergence approach by Budhiraja, Dupuis and Maroulas [BDM08] becomes unfeasible. The method of [Pap22a], inspired by [Her01], consists in the approximation of the non-linear drift B with a sequence of Lipschitz and bounded drifts. The approximation is non-trivial and relies on the Kirszbraun extension Theorem. Once such approximation is given, Large Deviation estimates for X^ϵ are recovered using an auxiliary equation, with a more regular non-linearity, for which Large Deviation estimates are classical.

1.4 Frequently used notation

Given two Banach spaces U, V, let $\mathcal{L}(U, V)$ denote the Banach space of continuous linear operators mapping U to V, endowed with the operator norm. Hilbert spaces will be usually denoted with the letter H, and the symbol $\langle \cdot, \cdot \rangle_H$ will indicate the scalar product in H, and $\|\cdot\|_H$ the norm. When no confusion may arise, we set $\langle \cdot, \cdot \rangle \coloneqq \langle \cdot, \cdot \rangle_H$ and $\|\cdot\| = \|\cdot\|_H$. Hilbert spaces will be always separable.
Given a (possibly unbounded) self-adjoint negative-definite operator $A_0 : D(A_0) \subset H \to H$, we define the H-based Sobolev space H^s, $s \in \mathbb{R}$ by the relation $H^s \coloneqq D((-A_0)^{s/2})$, with scalar product $\langle f, g \rangle_{H^s} \coloneqq \langle (-A_0)^{s/2} f, (-A_0)^{s/2} g \rangle_H$. Sobolev spaces form a Hilbert scale in the sense of Krein-Petunin [KP66]; in particular, the map $(-A_0)^{s/2} : H^r \to H^{r+s}$ is an isomorphism for every $s, r \in \mathbb{R}$ and the following interpolation inequality holds between H^{s_1} and H^{s_2}, for $s_1, s_2 \in \mathbb{R}$ and $\lambda \in (0,1)$:

$$\|f\|_{H^{s_\lambda}} \leq \|f\|_{H^{s_1}}^\lambda \|f\|_{H^{s_2}}^{1-\lambda}, \quad s_\lambda = \lambda s_1 + (1-\lambda) s_2.$$

In the following, Sobelev spaces will be often defined over $H \coloneqq \{f \in L^2(\mathbb{T}^d), \int_{\mathbb{T}^d} f = 0\}$ or $H \coloneqq \{u \in [L^2(\mathbb{T}^d)]^d, \int_{\mathbb{T}^d} u = 0, \operatorname{div} u = 0\}$, with A_0 equal to some multiple of the Laplace operator.
A finite or countable collection of standard i.i.d. Brownian motions is denoted by $\{W^k\}_{k \in I}$. We shall always assume the existence of an underlying filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ supporting the family $\{W^k\}_{k \in I}$. Filtration will always be right-continous and complete. A cylindrical Wiener process on H shall be denoted by W. We call the tuple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W)$ a *stochastic basis.*
Itō differential equations as, for instance, $dY = -Y dt + Q^{1/2} dW$ will sometimes be abbreviated in $\dot{Y} = -Y + Q^{1/2} \dot{W}$. Accordingly, stochastic integration in the sense of Stratonovich will be denoted by either $\circ dW$ or $\circ \dot{W}$. For (stochastic) partial differential equations we use the symbol ∂_t for derivatives with respect to time when we use the

concrete form of the equations, as for instance in $\partial_t u = \Delta u + \partial_t W$, whereas we use the convention $du = Au\,dt + dW$ or $\dot{u} = Au + \dot{W}$ if we interpret the equation as an evolution in some abstract Hilbert space.
We use the symbol $\lesssim$ to indicate inequality up to an unimportant constant $C \in (0, \infty)$. Whenever we need to stress that C depends of parameters $p_1, \ldots, p_n$ we use the symbol $\lesssim_{p_1,\ldots,p_n}$ instead.

Chapter 2

Stochastic model reduction

Here we recall the equations satisfied by the processes (X^ϵ, Y^ϵ)

$$\begin{cases} \dot{X}^\epsilon_t = F(t, X^\epsilon_t) + \sigma(t, X^\epsilon_t) Y^\epsilon_t + \epsilon^{1/2} \beta(Y^\epsilon_t, Y^\epsilon_t), \\ \dot{Y}^\epsilon_t = -\epsilon^{-1} Y^\epsilon_t + \epsilon^{-1} Q^{1/2} \dot{W}_t, \end{cases}$$

and $\bar{X}$

$$\dot{\bar{X}}_t = F(t, \bar{X}_t) + C(t, \bar{X}_t) + \sigma(t, \bar{X}_t) \dot{W}_t + \sum_{\ell, m \in \mathbb{N}} b_{\ell,m} \dot{\tilde{W}}^{\ell,m}_t.$$

The large-scale processes X^ϵ and $\bar{X}$ are set to start at time $t = 0$ from a deterministic initial condition $x_0 \in H_d$. As for the small-scale process Y^ϵ, it is worth mentioning that its precise initial condition is somewhat less important in applications, because of the typical decorellation time of Y^ϵ is of order $\epsilon^{1/2}$; therefore, for simplicity we impose Y^ϵ to be stationary. In particular, we set

$$Y^\epsilon_t := \int_{-\infty}^t \epsilon^{-1} e^{-\varepsilon^{-1}(t-s)} Q^{1/2} dW_s, \quad t \geq 0,$$

where W is a cylindrical Wiener process in H_∞ with real-valued time parameter.

Remark 2.1. A Wiener process with real-valued time parameter can be obtained in the following way: given two independent Wiener processes $(W^+_t)_{t\geq 0}$ and $(W^-_t)_{t\geq 0}$ defined on filtered probability spaces $(\Omega^+, (\mathcal{F}^+_t), \mathbb{P}^+)$ and $(\Omega^-, (\mathcal{F}^-_t), \mathbb{P}^-)$, respectively, set $W_t = W^+_t$, for $t \geq 0$, and $W_t = W^-_{-t}$, for $t < 0$. Using such a representation of W, we can also write

$$Y^\epsilon_t = -\int_0^\infty \epsilon^{-1} e^{-\epsilon^{-1}(t+s)} Q^{1/2} dW^-_s + \int_0^t \epsilon^{-1} e^{-\epsilon^{-1}(t-s)} Q^{1/2} dW^+_s, \quad t \geq 0,$$

which clearly is a stationary Ornstein-Uhlenbeck process on $(\Omega, \mathcal{F}^-_\infty \otimes \mathcal{F}^+_\infty, \mathbb{P})$ with initial value $Y^\epsilon_0 = -\int_0^\infty \epsilon^{-1} e^{-\epsilon^{-1}(t+s)} Q^{1/2} dW^-_s$, where $\Omega = \Omega^- \times \Omega^+$ and $\mathbb{P} = \mathbb{P}^- \otimes \mathbb{P}^+$, see [DPZ14]. Furthermore, setting up the stochastic basis for our processes (X^ϵ, Y^ϵ), let $(\Omega, \mathcal{F}, \mathbb{P})$ be the completion of $(\Omega, \mathcal{F}^-_\infty \otimes \mathcal{F}^+_\infty, \mathbb{P})$, and $(\mathcal{F}_t)_{t\geq 0}$ be the augmentation of the filtration $(\mathcal{F}^-_\infty \otimes \mathcal{F}^+_t)_{t\geq 0}$. Note that this filtration would satisfy the usual conditions.

Stochastic model reduction of finite-dimensional systems similar to (1.4) were extensively discussed in [MTVE01]. However, for the weak convergence the authors rely on a perturbation method based on a theorem by Kurtz [Kur73], and then they briefly describe a so-called *direct averaging method* for strong convergence, based on limits of solutions to

stochastic differential equations. Their approach is not immediately applicable to infinite dimensional systems like ours (not to mention some lack of rigour in their proofs).
There have been earlier attempts of proving similar abstract results of Wong-Zakai type [WZ65] in infinite dimensions, see for instance [BCF88, Twa93, TZ06]. However, we would like to emphasise that these earlier attempts dealt with piecewise linear approximations of noise rather than an infinite dimensional Ornstein-Uhlenbeck process. To see why the process Y^ϵ is an approximation of a white noise, take the time integral

$$\begin{aligned}\int_0^t Y_s^\epsilon ds &= \int_0^t Y_0^\epsilon e^{-\epsilon^{-1}s} ds + \int_0^t \left(\int_0^s \epsilon^{-1} e^{-\epsilon^{-1}(s-r)} Q^{1/2} dW_r\right) ds \\ &= \int_0^t Y_0^\epsilon e^{-\epsilon^{-1}s} ds + \int_0^t \left(\int_r^t \epsilon^{-1} e^{-\epsilon^{-1}(s-r)} ds\right) Q^{1/2} dW_r \\ &= \int_0^t Y_0^\epsilon e^{-\epsilon^{-1}s} ds + \int_0^t \left(1 - e^{-\epsilon^{-1}(s-r)}\right) Q^{1/2} dW_r \\ &= Q^{1/2} W_t + O(\epsilon^{1/2}).\end{aligned}$$

Note that it is typical for Wong-Zakai results that stochastic integral terms of limiting equations are interpreted in the sense of Stratonovich.
Finally, it is worth comparing our results with those in the literature concerning averaging principles, see for instance [FW12, Section 7.9], [PV01, PV03] and references therein. Roughly speaking, in those results the unresolved variables satisfy the equation $\dot{Y}_t^\epsilon = -\epsilon^{-1} Y_t^\epsilon + \epsilon^{-1/2} Q^{1/2} \dot{W}_t$, with a weaker noise intensity compared to ours, and therefore the resolved variables only undergo a change of drift in the limit $\epsilon \to 0$. On the contrary, in our setting a diffusion term also appears in the limit.

Remark 2.2. For notational simplicity, in what follows we shall write W instead of $Q^{1/2}W$ for a Q-Wiener process on H_∞. In this way $\int_0^t Y_s^\epsilon ds$ formally converges towards W_t as $\epsilon \to 0$ (and not towards $Q^{1/2}W_t$). This convention applies only to the present chapter.

Finally, let us describe how this chapter is organized.
In section 2.1, we give the proof of the strong convergence stated in Theorem 1.1 when $\beta = 0$. The proof relies on preliminary localization and discretization arguments which allow to consider instead its discrete version

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{\sup_k \|X_{t_k}^\epsilon - \bar{X}_{t_k}\|_{H_d} > \delta\right\} = 0, \quad \forall \delta > 0,$$

for only finitely many $t_k \in [0, T]$.
In section 2.2, we give the proof of the weak convergence of Theorem 1.1 which, at the beginning, requires a careful analysis of the quadratic term $\beta(Y_t^\epsilon, Y_t^\epsilon)$, but otherwise is an adaptation of the proof given in the previous section.
In section 2.3, we eventually use the results of section 2.1 and section 2.2 to prove Theorem 2.15 under quite natural conditions, thus making the connection to our main applications in climate modelling.

2.1 Strong convergence

In this section we give the proof of the strong convergence stated in Theorem 1.1, under the additional assumption $\beta = 0$. The proof is divided into several steps, here summarized.

First, by localization, we argue that we can restrict ourselves to $|X_t^\epsilon|$, $|\bar{X}_t| \leq R$, for some large R, which is effectively leading to Lipschitz continuity of the coefficients of (1.4) and (1.5).
Second, we discretize the problem, which allows us to reduce the desired convergence to its discrete version:

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{\sup_k |X_{t_k}^\epsilon - \bar{X}_{t_k}| > \delta\right\} = 0, \quad \forall \delta > 0,$$

for only finitely many $t_k \in [0, T]$. Here, we choose $t_k = k\Delta$, where $\Delta = \Delta_\epsilon$ is a positive parameter whose ϵ-dependence has to be carefully chosen in the proof—see Remark 2.3.
Third, we prove the above discretized version.

2.1.1 Localization

Fix $\epsilon > 0$, $\delta \in (0, 1)$, and define

$$\tau_R^\epsilon = \inf\{t \geq 0 : |X_t^\epsilon| \geq R + 1\} \wedge \inf\{t \geq 0 : |\bar{X}_t| \geq R\}, \quad \text{for } R > 0,$$

so that

$$\begin{aligned}
\mathbb{P}\left\{\sup_{t\leq T} |X_t^\epsilon - \bar{X}_t| > \delta\right\} &= \mathbb{P}\left\{\sup_{t\leq T} |X_t^\epsilon - \bar{X}_t| > \delta, \sup_{t\leq T} |\bar{X}_t| \geq R\right\} \\
&\quad + \mathbb{P}\left\{\sup_{t\leq T} |X_t^\epsilon - \bar{X}_t| > \delta, \sup_{t\leq T} |\bar{X}_t| < R\right\} \\
&= \mathbb{P}\left\{\sup_{t\leq T} |X_t^\epsilon - \bar{X}_t| > \delta, \sup_{t\leq T} |\bar{X}_t| \geq R\right\} \\
&\quad + \mathbb{P}\left\{\sup_{t\leq T\wedge\tau_R^\epsilon} |X_t^\epsilon - \bar{X}_t| > \delta, \sup_{t\leq T} |\bar{X}_t| < R\right\} \\
&\leq \mathbb{P}\left\{\sup_{t\leq T} |\bar{X}_t| \geq R\right\} + \mathbb{P}\left\{\sup_{t\leq T\wedge\tau_R^\epsilon} |X_t^\epsilon - \bar{X}_t| > \delta\right\}. \qquad (2.1)
\end{aligned}$$

Therefore, since global existence for $\bar{X}$ implies

$$\mathbb{P}\left\{\sup_{t\leq T} |\bar{X}_t| \geq R\right\} \to 0, \text{ as } Rto\infty,$$

to prove the desired convergence it is sufficient to show the convergence of the second summand on the right-hand side of (2.1), when $\epsilon \to 0$, for fixed $\delta \in (0, 1)$, $R > 0$. Furthermore, by Markov inequality,

$$\mathbb{P}\left\{\sup_{t\leq T\wedge\tau_R^\epsilon} |X_t^\epsilon - \bar{X}_t| > \delta\right\} \leq \delta^{-p}\,\mathbb{E}\left[\sup_{t\leq T\wedge\tau_R^\epsilon} |X_t^\epsilon - \bar{X}_t|^p\right], \qquad (2.2)$$

for every $p > 0$, $\delta \in (0, 1)$, and hence showing convergence of the above right-hand side, only, is enough. To keep notation light, we are going to use τ^ϵ instead of τ_R^ϵ, as $R > 0$ will be fixed in what follows.

2.1.2 Discretization

Fix $\epsilon > 0$. We show that the expectation on the right-hand side of (2.2) can be replaced by an expectation of the same quantity, but with the supremum taken over a finite number (diverging to ∞, as $\epsilon \to 0$) of times t_k, see Corollary 2.6 below.
To start with, we have the following useful a priori estimate.

Lemma 2.1. *For any $p > 1$, the Ornstein-Uhlenbeck process Y^ϵ satisfies*

$$\mathbb{E}\left[\sup_{t\le T}|Y^\epsilon_t|^p\right] \lesssim \epsilon^{-p/2}\log^{p/2}(1+\epsilon^{-1}).$$

Proof. First, using the decomposition $Y^\epsilon_t = Y^\epsilon_0 + (Y^\epsilon_t - Y^\epsilon_0)$, Gaussian estimates on Y^ϵ_0 and [JZ20, Theorem 2.2], the result is true in one dimension. In the infinite dimensional case, by Hölder's inequality we can suppose $p > 2$ without any loss of generality. Therefore, since Q is trace class with eigenvalues satisfying $\sum_{m\in\mathbb{N}} q_m < \infty$, when $\alpha = (p-2)/p$, we obtain that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\le T}|Y^\epsilon_t|^p\right] &= \mathbb{E}\left[\sup_{t\le T}\left(\sum_{m\in\mathbb{N},q_m>0} q_m^\alpha q_m^{-\alpha}|Y^{\epsilon,m}_t|^2\right)^{p/2}\right]\\ &\lesssim \left(\sum_{m\in\mathbb{N},q_m>0} q_m^{-\alpha p/2}\mathbb{E}\left[\sup_{t\le T}|Y^{\epsilon,m}_t|^p\right]\right)\left(\sum_{m\in\mathbb{N}} q_m^{\alpha p/(p-2)}\right)^{(p-2)/2}\\ &\lesssim \epsilon^{-p/2}\log^{p/2}(1+\epsilon^{-1}),\end{aligned}$$

having used the one-dimensional result for the coordinates $Y^{\epsilon,m}_t = \langle Y^\epsilon_t, \mathbf{f}_m\rangle$, for any $m \in \mathbb{N}$. □

Now, we introduce the discretization of the time interval $[0,T]$. Let $\Delta > 0$, and let $[T/\Delta]$ be the largest integer less or equal than T/Δ. In what follows, Δ will also depend on ϵ, in a way to be determined later. Also, to make it easier to bound terms by powers of ϵ or Δ, without loss of generality, we will always assume that both ϵ, Δ are less than one. The next two lemmas control the excursion of X^ϵ between adjacent nodes.

Lemma 2.2. *For any $p > 1$, and any deterministic time $\tau > 0$,*

$$\mathbb{E}\left[\sup_{\substack{k=0,1,\dots,[T/\Delta]\\ t\le\tau,\, t+k\Delta\le T\wedge\tau^\epsilon}}|X^\epsilon_{t+k\Delta} - X^\epsilon_{k\Delta}|^p\right] \lesssim \frac{\tau^p}{\epsilon^{p/2}}\log^{p/2}(1+\epsilon^{-1}).$$

Proof. Since $\beta = 0$ the increment $X^\epsilon_{t+k\Delta} - X^\epsilon_{k\Delta}$ can be written as

$$X^\epsilon_{t+k\Delta} - X^\epsilon_{k\Delta} = \int_{k\Delta}^{t+k\Delta} F(s, X^\epsilon_s)ds + \int_{k\Delta}^{t+k\Delta}\sigma(s, X^\epsilon_s)dW^\epsilon_s,$$

for $t + k\Delta \le T\wedge\tau^\epsilon$, where $W^\epsilon_t := \int_0^t Y^\epsilon_s ds$. Therefore, using our assumptions on the coefficients F, σ, boundedness of X^ϵ on $[0, \tau^\epsilon]$, and Lemma 2.1, we obtain that

$$\begin{aligned}\mathbb{E}\left[\sup_{\substack{k=0,1,\dots,[T/\Delta]\\ t\le\tau,\, t+k\Delta\le T\wedge\tau^\epsilon}}|X^\epsilon_{t+k\Delta} - X^\epsilon_{k\Delta}|^p\right] &\lesssim \tau^p\left(1+\mathbb{E}\left[\sup_{t\le T\wedge\tau^\epsilon}|Y^\epsilon_t|^p\right]\right)\\ &\lesssim \frac{\tau^p}{\epsilon^{p/2}}\log^{p/2}(1+\epsilon^{-1}).\end{aligned}$$

□

Lemma 2.3. *For any $p > 1$, and any fixed $k \in \{0, 1, \ldots, [T/\Delta]\}$ such that $k\Delta \leq T$,*

$$\mathbb{E}\left[|X^\epsilon_{(k+1)\Delta\wedge\tau^\epsilon} - X^\epsilon_{k\Delta\wedge\tau^\epsilon}|^p\right] \lesssim \Delta^{p/2} + \epsilon^{p/2}\log^{p/2}(1+\epsilon^{-1}) + \frac{\Delta^{2p}}{\epsilon^p}\log^p(1+\epsilon^{-1}).$$

Proof. It suffices to bound every single term on the right-hand side of the equation

$$\begin{aligned} X^\epsilon_{(k+1)\Delta\wedge\tau^\epsilon} - X^\epsilon_{k\Delta\wedge\tau^\epsilon} = &\int_{k\Delta\wedge\tau^\epsilon}^{(k+1)\Delta\wedge\tau^\epsilon} F(s, X^\epsilon_s)ds \\ &+ \int_{k\Delta\wedge\tau^\epsilon}^{(k+1)\Delta\wedge\tau^\epsilon} \left(\sigma(s, X^\epsilon_s) - \sigma(k\Delta\wedge\tau^\epsilon, X^\epsilon_{k\Delta\wedge\tau^\epsilon})\right) dW^\epsilon_s \\ &+ \int_{k\Delta\wedge\tau^\epsilon}^{(k+1)\Delta\wedge\tau^\epsilon} \sigma(k\Delta\wedge\tau^\epsilon, X^\epsilon_{k\Delta\wedge\tau^\epsilon})dW^\epsilon_s. \end{aligned}$$

In what follows, we shall use implicitly the regularity assumptions on F, σ without further mention. First, by boundedness of X^ϵ on $[0, \tau^\epsilon]$, we have that

$$\mathbb{E}\left[\left|\int_{k\Delta\wedge\tau^\epsilon}^{(k+1)\Delta\wedge\tau^\epsilon} F(s, X^\epsilon_s)ds\right|^p\right] \lesssim \Delta^p.$$

Second, using Hölder's inequality with $q' > 1/p$ and Lemma 2.1,

$$\begin{aligned} &\mathbb{E}\left[\left|\int_{k\Delta\wedge\tau^\epsilon}^{(k+1)\Delta\wedge\tau^\epsilon} \left(\sigma(s, X^\epsilon_s) - \sigma(k\Delta\wedge\tau^\epsilon, X^\epsilon_{k\Delta\wedge\tau^\epsilon})\right) dW^\epsilon_s\right|^p\right] \\ &\lesssim \mathbb{E}\left[\sup_{t\leq T}|Y^\epsilon_t|^p \left|\int_{k\Delta\wedge\tau^\epsilon}^{(k+1)\Delta\wedge\tau^\epsilon} \left|\sigma(s, X^\epsilon_s) - \sigma(k\Delta\wedge\tau^\epsilon, X^\epsilon_{k\Delta\wedge\tau^\epsilon})\right| ds\right|^p\right] \\ &\lesssim \epsilon^{-p/2}\log^{p/2}(1+\epsilon^{-1})\,\mathbb{E}\left[\left|\int_{k\Delta\wedge\tau^\epsilon}^{(k+1)\Delta\wedge\tau^\epsilon} \left|\sigma(s, X^\epsilon_s) - \sigma(k\Delta\wedge\tau^\epsilon, X^\epsilon_{k\Delta\wedge\tau^\epsilon})\right| ds\right|^{pq'}\right]^{1/q'}. \end{aligned}$$

Since $pq' > 1$ by assumption, we can estimate the integral above using Hölder's inequality with exponents pq' and $pq'/(pq'-1)$. Then Lemma 2.2 gives

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_{k\Delta\wedge\tau^\epsilon}^{(k+1)\Delta\wedge\tau^\epsilon} |\sigma(s,X_s^\epsilon)-\sigma(k\Delta\wedge\tau^\epsilon, X^\epsilon_{k\Delta\wedge\tau^\epsilon})|\,ds\right|^{pq'}\right]^{1/q'} \\
&\lesssim \mathbb{E}\left[\left|\int_{k\Delta\wedge\tau^\epsilon}^{(k+1)\Delta\wedge\tau^\epsilon} ds\right|^{pq'-1}\int_{k\Delta\wedge\tau^\epsilon}^{(k+1)\Delta\wedge\tau^\epsilon} |\sigma(s,X_s^\epsilon)-\sigma(k\Delta\wedge\tau^\epsilon, X^\epsilon_{k\Delta\wedge\tau^\epsilon})|^{pq'}\,ds\right]^{1/q'} \\
&\lesssim \Delta^{p-1/q'}\,\mathbb{E}\left[\int_{k\Delta\wedge\tau^\epsilon}^{(k+1)\Delta\wedge\tau^\epsilon}\left(|X_s^\epsilon - X^\epsilon_{k\Delta\wedge\tau^\epsilon}|^{pq'}+(s-k\Delta)^{pq'}\right)ds\right]^{1/q'} \\
&\lesssim \Delta^{p-1/q'}\left(\int_{k\Delta\wedge\tau^\epsilon}^{(k+1)\Delta\wedge\tau^\epsilon}\mathbb{E}\left[|X_s^\epsilon - X^\epsilon_{k\Delta\wedge\tau^\epsilon}|^{pq'}+(s-k\Delta)^{pq'}\right]ds\right)^{1/q'} \\
&\lesssim \Delta^{p-1/q'}\left(\int_{k\Delta\wedge\tau^\epsilon}^{(k+1)\Delta\wedge\tau^\epsilon}(s-k\Delta)^{pq'}\left(\epsilon^{-pq'}\log^{pq'/2}(1+\epsilon^{-2})+1\right)ds\right)^{1/q'} \\
&\lesssim \epsilon^{-p/2}\log^{p/2}(1+\epsilon^{-1})\Delta^{p-1/q'}\left(\int_{k\Delta\wedge\tau^\epsilon}^{(k+1)\Delta\wedge\tau^\epsilon}(s-k\Delta)^{pq'}ds\right)^{1/q'} \\
&\lesssim \epsilon^{-p/2}\log^{p/2}(1+\epsilon^{-1})\Delta^{2p}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathbb{E}\left[\left|\int_{k\Delta\wedge\tau^\epsilon}^{(k+1)\Delta\wedge\tau^\epsilon}\sigma(k\Delta\wedge\tau^\epsilon, X^\epsilon_{k\Delta\wedge\tau^\epsilon})dW_s^\epsilon\right|^p\right] &\lesssim \mathbb{E}\left[\left|W^\epsilon_{(k+1)\Delta\wedge\tau^\epsilon}-W^\epsilon_{k\Delta\wedge\tau^\epsilon}\right|^p\right] \\
&\lesssim \Delta^{p/2}+\epsilon^{p/2}\log^{p/2}(1+\epsilon^{-1}),
\end{aligned}
$$

because, for every $t_2 > t_1 \geq 0$,

$$
\begin{aligned}
W^\epsilon_{t_2}-W^\epsilon_{t_1} &= \int_{t_1}^{t_2}\left(\int_{-\infty}^{s}\epsilon^{-1}e^{-\epsilon^{-1}(s-r)}dW_r\right)ds \\
&= W_{t_2}-W_{t_1}-\int_{-\infty}^{t_2}e^{-\epsilon^{-1}(t_2-r)}dW_r+\int_{-\infty}^{t_1}e^{-\epsilon^{-1}(t_1-r)}dW_r.
\end{aligned}
\tag{2.3}
$$

□

The next lemma controls the excursion of the limiting process $\bar{X}$ between adjacent nodes.

Lemma 2.4. *For any $p > 1$, any deterministic time $\tau \in (0,1)$, and any fixed $k \in \{0,1,\ldots,[T/\Delta]\}$,*

$$
\mathbb{E}\left[\sup_{t\leq\tau,\, t+k\Delta\leq T\wedge\tau^\epsilon}|\bar{X}_{t+k\Delta}-\bar{X}_{k\Delta}|^p\right]\lesssim \tau^{\frac{p}{2}}.
$$

Proof. Since $\beta = 0$ the increment $\bar{X}_{t+k\Delta}-\bar{X}_{k\Delta}$ can be written as

$$
\bar{X}_{t+k\Delta}-\bar{X}_{k\Delta} = \int_{k\Delta}^{t+k\Delta}\left(F(s,\bar{X}_s)+C(s,\bar{X}_s)\right)ds + \int_{k\Delta}^{t+k\Delta}\sigma(s,\bar{X}_s)dW_s,
$$

at least for $t+k\Delta \le T\wedge\tau^\epsilon$. Therefore, using boundedness of X^ϵ on $[0,\tau^\epsilon]$ and Burkholder-Davis-Gundy's inequality, we obtain that

$$\mathbb{E}\left[\sup_{t\le\tau,\, t+k\Delta\le T\wedge\tau^\epsilon} |\bar{X}_{t+k\Delta}-\bar{X}_{k\Delta}|^p\right] \lesssim \tau^p + \mathbb{E}\left[\sup_{t\le\tau,\, t+k\Delta\le T\wedge\tau^\epsilon} \left|\int_{k\Delta}^{t+k\Delta} \sigma(s,\bar{X}_s)dW_s\right|^p\right]$$
$$\lesssim \tau^p + \tau^{\frac{p}{2}},$$

which proves the lemma since $\tau < 1$. □

Corollary 2.5. *For any $p > 1$,*

$$\mathbb{E}\left[\sup_{\substack{k=0,1,\dots,[T/\Delta]\\ t\le\Delta,\, t+k\Delta\le T\wedge\tau^\epsilon}} |\bar{X}_{t+k\Delta}-\bar{X}_{k\Delta}|^p\right] \lesssim \Delta^{\frac{p}{2}-1}.$$

Proof. The claim easily follows from Lemma 2.4 with $\tau = \Delta$, and the inequality

$$\mathbb{E}\left[\sup_{\substack{k=0,1,\dots,[T/\Delta]\\ t\le\Delta,\, t+k\Delta\le T\wedge\tau^\epsilon}} |\bar{X}_{t+k\Delta}-\bar{X}_{k\Delta}|^p\right] \lesssim \sum_{k=0}^{[T/\Delta]} \mathbb{E}\left[\sup_{t\le\Delta,\, t+k\Delta\le T\wedge\tau^\epsilon} |\bar{X}_{t+k\Delta}-\bar{X}_{k\Delta}|^p\right]$$
$$\lesssim \sum_{k=0}^{[T/\Delta]} \Delta^{p/2} = \Delta^{\frac{p}{2}-1}.$$

□

Corollary 2.6. *Let $\Delta = \Delta_\epsilon > 0$ depend on ϵ such that $\Delta/\epsilon^{1/2} \to 0$ as $\epsilon \to 0$. Then,*

$$\mathbb{E}\left[\sup_{t\le T\wedge\tau^\epsilon} |X_t^\epsilon - \bar{X}_t|^2\right] \lesssim \mathbb{E}\left[\sup_{\substack{k=0,1,\dots,[T/\Delta]\\ k\Delta\le\tau^\epsilon}} |X_{k\Delta}^\epsilon - \bar{X}_{k\Delta}|^2\right] + o(1).$$

Proof. First, by Hölder's inequality with $q > 1$ and Corollary 2.5, we have that

$$\mathbb{E}\left[\sup_{\substack{k=0,1,\dots,[T/\Delta]\\ t\le\Delta,\, t+k\Delta\le T\wedge\tau^\epsilon}} |\bar{X}_{t+k\Delta}-\bar{X}_{k\Delta}|^2\right] \lesssim \mathbb{E}\left[\sup_{\substack{k=0,1,\dots,[T/\Delta]\\ t\le\Delta,\, t+k\Delta\le T\wedge\tau^\epsilon}} |\bar{X}_{t+k\Delta}-\bar{X}_{k\Delta}|^{2q}\right]^{1/q}$$
$$\lesssim \Delta^{1-1/q} \to 0 \quad \text{as } \epsilon \to 0,$$

since we have taken $q > 1$. Thus, the proof can easily be completed by combining the above and Lemma 2.2, while taking into account

$$|X_t^\epsilon - \bar{X}_t|^2 \lesssim |X_t^\epsilon - X_{[t/\Delta]\Delta}^\epsilon|^2 + |X_{[t/\Delta]\Delta}^\epsilon - \bar{X}_{[t/\Delta]\Delta}|^2 + |\bar{X}_{[t/\Delta]\Delta} - \bar{X}_t|^2,$$

where $[t/\Delta]$ is again our notation for the floor of t/Δ. □

2.1.3 Proof of the discretized version

We now discuss our strategy to prove the discretized version of the strong convergence in Theorem 1.1. Recall that we want

$$\mathbb{P}\left\{\sup_{t\leq T}|X_t^\epsilon - \bar{X}_t| > \delta\right\} \to 0,$$

for every fixed $\delta > 0$, as $\epsilon \to 0$. As we have seen, by (2.1), (2.2) and Corollary 2.6, it suffices to prove

$$\mathbb{E}\left[\sup_{\substack{k=0,\dots,[T/\Delta]\\ k\Delta\leq\tau^\epsilon}} \left|X_{k\Delta}^\epsilon - \bar{X}_{k\Delta}\right|^2\right] \to 0, \quad \epsilon \to 0, \tag{2.4}$$

for some $\Delta = \Delta_\epsilon = o(\epsilon)$. The proof is inspired by [IW14, Section VI.7].
Hereafter, $\partial\sigma$ denotes the derivative of σ with respect its first variable, and $D\sigma$ denotes the derivative of σ with respect its second variable. To start with, by (2.3), we have that

$$\begin{aligned}
X_{(k+1)\Delta}^\epsilon &= X_{k\Delta}^\epsilon + \int_{k\Delta}^{(k+1)\Delta} F(s, X_s^\epsilon)ds + \int_{k\Delta}^{(k+1)\Delta} \sigma(s, X_s^\epsilon)dW_s^\epsilon \\
&= X_{k\Delta}^\epsilon + \int_{k\Delta}^{(k+1)\Delta} \left(F(s, X_s^\epsilon) - F(k\Delta, X_{k\Delta}^\epsilon)\right) ds \\
&\quad + \int_{k\Delta}^{(k+1)\Delta} F(k\Delta, X_{k\Delta}^\epsilon)ds \\
&\quad + \int_{k\Delta}^{(k+1)\Delta} \left(\sigma(s, X_s^\epsilon) - \sigma(k\Delta, X_{k\Delta}^\epsilon)\right) dW_s^\epsilon + \int_{k\Delta}^{(k+1)\Delta} \sigma(k\Delta, X_{k\Delta}^\epsilon)dW_s^\epsilon \\
&= X_{k\Delta}^\epsilon + \int_{k\Delta}^{(k+1)\Delta} \left(F(s, X_s^\epsilon) - F(k\Delta, X_{k\Delta}^\epsilon)\right) ds \\
&\quad + \int_{k\Delta}^{(k+1)\Delta} F(k\Delta, X_{k\Delta}^\epsilon)ds \\
&\quad + \int_{k\Delta}^{(k+1)\Delta} \left(\int_{k\Delta}^{s} \left(\partial\sigma(r, X_r^\epsilon) + D\sigma(r, X_r^\epsilon)F(r, X_r^\epsilon)\right) dr\right) dW_s^\epsilon \\
&\quad + \int_{k\Delta}^{(k+1)\Delta} \left(\int_{k\Delta}^{s} \left(D\sigma(r, X_r^\epsilon)\sigma(r, X_r^\epsilon) - D\sigma(k\Delta, X_{k\Delta}^\epsilon)\sigma(k\Delta, X_{k\Delta}^\epsilon)\right) dW_r^\epsilon\right) dW_s^\epsilon \\
&\quad + \int_{k\Delta}^{(k+1)\Delta} \left(\int_{k\Delta}^{s} \left(D\sigma(k\Delta, X_{k\Delta}^\epsilon)\sigma(k\Delta, X_{k\Delta}^\epsilon) - D\sigma(k\Delta, \bar{X}_{k\Delta})\sigma(k\Delta, \bar{X}_{k\Delta})\right) dW_r^\epsilon\right) dW_s^\epsilon \\
&\quad + \int_{k\Delta}^{(k+1)\Delta} \left(\int_{k\Delta}^{s} D\sigma(k\Delta, \bar{X}_{k\Delta})\sigma(k\Delta, \bar{X}_{k\Delta})dW_r^\epsilon\right) dW_s^\epsilon \\
&\quad + \int_{k\Delta}^{(k+1)\Delta} \sigma(k\Delta, X_{k\Delta}^\epsilon)dW_s \\
&\quad + \sigma(k\Delta, X_{k\Delta}^\epsilon)\epsilon\left(Y_{k\Delta}^\epsilon - Y_{(k+1)\Delta}^\epsilon\right) \\
&= X_{k\Delta}^\epsilon + I_1^k + I_2^k + I_3^k + I_4^k + I_5^k + I_6^k + I_7^k + I_8^k,
\end{aligned} \tag{2.5}$$

for any $k = 0, \dots, [T/\Delta]$ such that $(k+1)\Delta \leq T$.

Similarly, the process $\bar{X}$ satisfies

$$\begin{aligned}\bar{X}_{(k+1)\Delta} =& \bar{X}_{k\Delta} + \int_{k\Delta}^{(k+1)\Delta} \left(F(s,\bar{X}_s) - F(k\Delta,\bar{X}_{k\Delta})\right) ds \\ &+ \int_{k\Delta}^{(k+1)\Delta} F(k\Delta,\bar{X}_{k\Delta}) ds \\ &+ \int_{k\Delta}^{(k+1)\Delta} \left(C(s,\bar{X}_s) - C(k\Delta,\bar{X}_{k\Delta})\right) ds \\ &+ \int_{k\Delta}^{(k+1)\Delta} C(k\Delta,\bar{X}_{k\Delta}) ds \\ &+ \int_{k\Delta}^{(k+1)\Delta} \left(\sigma(s,\bar{X}_s) - \sigma(k\Delta,\bar{X}_{k\Delta})\right) dW_s + \int_{k\Delta}^{(k+1)\Delta} \sigma(k\Delta,\bar{X}_{k\Delta}) dW_s \\ =& \bar{X}_{k\Delta} + J_1^k + J_2^k + J_3^k + J_4^k + J_5^k + J_6^k.\end{aligned} \tag{2.6}$$

Having in mind to apply Gronwall's lemma, it turns out to be useful to summarise the contributions of the right-hand sides of (2.5), (2.6) as follows:

$$\begin{aligned}X^\epsilon_{h\Delta} - \bar{X}_{h\Delta} =& \sum_{k=0}^{h-1} \left(I_2^k - J_2^k\right) + \sum_{k=0}^{h-1} \left(I_6^k - J_4^k\right) + \sum_{k=0}^{h-1} \left(I_7^k - J_6^k\right) + \sum_{k=0}^{h-1} I_5^k \\ &+ \sum_{k=0}^{h-1} \left(I_1^k + I_3^k + I_4^k + I_8^k - J_1^k - J_3^k - J_5^k\right),\end{aligned} \tag{2.7}$$

for any $h = 1, \dots, [T/\Delta]$, which splits the difference $X^\epsilon_{h\Delta} - \bar{X}_{h\Delta}$ into five sums.
We at first prove that the second and the fifth sum can be neglected when proving (2.4). The summands of the fifth sum are discussed in Lemma 2.7 below. The contribution of the second sum though is more delicate and requires a martingale argument similar to that of [IW14, Theorem VI.7.1].
The remaining sums will be controlled in terms of the difference $X^\epsilon - \bar{X}$ itself, which allows them to be estimated via Gronwall's lemma.
Of course, the function F is uniformly continuous when restricted to $[0,T] \times B_R(0)$, where $B_R(0)$ is the closed ball of radius R in H_d. In what follows, we will denote by $\omega_F : [0,T] \to [0,\infty)$ the (local) modulus of continuity of $F(\cdot, x)$:

$$|F(t,x) - F(s,x)| \le \omega_F(|t-s|), \qquad \text{for every } t,s \in [0,T], \text{ and } x \in B_R(0).$$

Obviously, the function ω_F vanishes at zero, and without loss of generality, it can be chosen to be both non-decreasing and continuous. Denote by ω_σ the corresponding modulus of continuity of the derivative $D\sigma(\cdot, x)$, and let $\omega_{F,\sigma} = \omega_F + \omega_\sigma$. Recall that, under our assumption on the coefficient σ, one can take $\omega_\sigma(t) = Ct^\gamma$ for some positive constant C and $\gamma \in (0,1)$.

Lemma 2.7. *For any $p > 1$:*

$$\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau^\epsilon}}\left|\sum_{k=0}^{h-1} I_1^k\right|^p + \left|\sum_{k=0}^{h-1} I_3^k\right|^p\right] \lesssim \left(\frac{\Delta}{\epsilon^{1/2}}\right)^p \log^{p/2}(1+\epsilon^{-1}) + \omega_F(\Delta)^p;$$

$$\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau^\epsilon}}\left|\sum_{k=0}^{h-1} I_4^k\right|^p\right] \lesssim \left(\frac{\Delta^2}{\epsilon^{3/2}}\right)^p \log^{3p/2}(1+\epsilon^{-1}) + \left(\frac{\Delta}{\epsilon}\right)^p \log^p(1+\epsilon^{-1})\,\omega_\sigma(\Delta)^p;$$

$$\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau^\epsilon}}\left|\sum_{k=0}^{h-1} I_8^k\right|^p\right] \lesssim \left(\frac{\epsilon}{\Delta}\right)^{p/2} \log^{p/2}(1+\epsilon^{-1}) + \left(\frac{\epsilon}{\Delta}\right)^p \log^p(1+\epsilon^{-1}) + \left(\frac{\Delta}{\epsilon^{1/2}}\right)^p \log^p(1+\epsilon^{-1});$$

$$\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau^\epsilon}}\left|\sum_{k=0}^{h-1} J_1^k\right|^p + \left|\sum_{k=0}^{h-1} J_3^k\right|^p + \left|\sum_{k=0}^{h-1} J_5^k\right|^p\right] \lesssim \Delta^{p/2} + \omega_{F,\sigma}(\Delta)^p.$$

Proof. For $\sum I_1^k$, by Hölder's inequality and Lemma 2.2,

$$\begin{aligned}\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau^\epsilon}}\left|\sum_{k=0}^{h-1} I_1^k\right|^p\right] &\lesssim \mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau^\epsilon}}\left|\sum_{k=0}^{h-1}\int_{k\Delta}^{(k+1)\Delta} \left(|X_s^\epsilon - X_{k\Delta}^\epsilon| + \omega_F(s-k\Delta)\right) ds\right|^p\right]\\ &\lesssim \sum_{k=0}^{[T/\Delta]-1}\int_{k\Delta}^{(k+1)\Delta} \mathbb{E}\left[|X^\epsilon_{s\wedge\tau^\epsilon} - X^\epsilon_{k\Delta\wedge\tau^\epsilon}|^p + \omega_F(\Delta)^p\right] ds\\ &\lesssim \left(\frac{\Delta}{\epsilon^{1/2}}\right)^p \log^{p/2}(1+\epsilon^{-1}) + \omega_F(\Delta)^p.\end{aligned}$$

For $\sum I_3^k$, by Hölder's inequality and Lemma 2.1,

$$\begin{aligned}\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau^\epsilon}}\left|\sum_{k=0}^{h-1} I_3^k\right|^p\right] &\lesssim \mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau^\epsilon}}\left|\sup_{t\le T}|Y_t^\epsilon|\sum_{k=0}^{h-1}\int_{k\Delta}^{(k+1)\Delta}(s-k\Delta)\,ds\right|^p\right]\\ &\lesssim \mathbb{E}\left[\sup_{t\le T}|Y_t^\epsilon|^p\sum_{k=0}^{[T/\Delta]-1}\int_{k\Delta}^{(k+1)\Delta}|s-k\Delta|^p\,ds\right]\\ &\lesssim \left(\frac{\Delta}{\epsilon^{1/2}}\right)^p \log^{p/2}(1+\epsilon^{-1}).\end{aligned}$$

For $\sum I_4^k$, by Hölder's inequality, Lemma 2.1 and Lemma 2.2,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau^\epsilon}}\left|\sum_{k=0}^{h-1} I_4^k\right|^p\right]\\
&\lesssim \mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau^\epsilon}}\left|\sup_{t\le T}|Y_t^\epsilon|^2\sum_{k=0}^{h-1}\int_{k\Delta}^{(k+1)\Delta}\left(\int_{k\Delta}^{s}\left(|X_r^\epsilon - X_{k\Delta}^\epsilon| + \omega_\sigma(r-k\Delta)\right)dr\right)ds\right|^p\right]\\
&\lesssim \mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau^\epsilon}}\sup_{t\le T}|Y_t^\epsilon|^{2p}\sum_{k=0}^{h-1}\int_{k\Delta}^{(k+1)\Delta}\left|\int_{k\Delta}^{s}\left(|X_r^\epsilon - X_{k\Delta}^\epsilon| + \omega_\sigma(r-k\Delta)\right)dr\right|^p ds\right]\\
&\lesssim \epsilon^{-p}\log^p(1+\epsilon^{-1})\\
&\quad\times\left(\sum_{k=0}^{[T/\Delta]-1}\int_{k\Delta}^{(k+1)\Delta}(s-k\Delta)^{pq'-1}\int_{k\Delta}^{s}\left(\mathbb{E}\left[|X_{r\wedge\tau^\epsilon}^\epsilon - X_{k\Delta\wedge\tau^\epsilon}^\epsilon|^{pq'} + \omega_\sigma(\Delta)^{pq'}\right]dr\right)ds\right)^{1/q'}\\
&\lesssim \epsilon^{-3p/2}\log^{3p/2}(1+\epsilon^{-1})\left(\sum_{k=0}^{[T/\Delta]-1}\int_{k\Delta}^{(k+1)\Delta}(s-k\Delta)^{2pq'}ds\right)^{1/q'}\\
&\quad+\left(\frac{\Delta}{\epsilon}\right)^p\log^p(1+\epsilon^{-1})\,\omega_\sigma(\Delta)^p\\
&\lesssim \left(\frac{\Delta^2}{\epsilon^{3/2}}\right)^p\log^{3p/2}(1+\epsilon^{-1}) + \left(\frac{\Delta}{\epsilon}\right)^p\log^p(1+\epsilon^{-1})\,\omega_\sigma(\Delta)^p.
\end{aligned}
$$

We now consider $\sum I_8^k$. Here, the idea is to convert Y^ϵ-increments into X^ϵ-increments via integration by parts since X^ϵ-increments are easier to control. This way, applying

Lemma 2.1 and Lemma 2.3,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\leq\tau^\epsilon}}\left|\sum_{k=0}^{h-1} I_8^k\right|^p\right]\\
&\lesssim \mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\leq\tau^\epsilon}}\left|\sum_{k=0}^{h-1}\sigma(k\Delta, X_{k\Delta}^\epsilon)\epsilon\left(Y_{k\Delta}^\epsilon - Y_{(k+1)\Delta}^\epsilon\right)\right|^p\right]\\
&\lesssim \mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\leq\tau^\epsilon}}\left|\sum_{k=1}^{h}\left(\sigma(k\Delta, X_{k\Delta}^\epsilon) - \sigma((k-1)\Delta, X_{(k-1)\Delta}^\epsilon)\right)\epsilon Y_{k\Delta}^\epsilon\right|^p\right]\\
&\lesssim \mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\leq\tau^\epsilon}}\sup_{t\leq T}|\epsilon Y_t^\epsilon|^p\left|\sum_{k=1}^{h}\left(\left|X_{k\Delta}^\epsilon - X_{(k-1)\Delta}^\epsilon\right| + \Delta\right)\right|^p\right]\\
&\lesssim \mathbb{E}\left[\sup_{t\leq T}|\epsilon Y_t^\epsilon|^{pq}\right]^{1/q}\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\leq\tau^\epsilon}}\left|\sum_{k=1}^{h}\left(\left|X_{k\Delta}^\epsilon - X_{(k-1)\Delta}^\epsilon\right| + \Delta\right)\right|^{pq'}\right]^{1/q'}\\
&\lesssim \epsilon^{p/2}\log^{p/2}(1+\epsilon^{-1})\Delta^{1/q'-p}\\
&\quad\times\left(\sum_{k=1}^{[T/\Delta]}\mathbb{E}\left[\left|X_{k\Delta\wedge\tau^\epsilon}^\epsilon - X_{(k-1)\Delta\wedge\tau^\epsilon}^\epsilon\right|^{pq'} + \Delta^{pq'}\right]\right)^{1/q'}\\
&\lesssim \epsilon^{p/2}\log^{p/2}(1+\epsilon^{-1})\Delta^{-p}\\
&\quad\times\left(\Delta^{pq'/2} + \epsilon^{pq'/2}\log^{pq'/2}(1+\epsilon^{-1}) + \left(\frac{\Delta}{\epsilon^{1/2}}\right)^{2pq'}\log^{pq'}(1+\epsilon^{-1})\right)^{1/q'}\\
&\lesssim \left(\frac{\epsilon}{\Delta}\right)^{p/2}\log^{p/2}(1+\epsilon^{-1}) + \left(\frac{\epsilon}{\Delta}\right)^{p}\log^{p}(1+\epsilon^{-1}) + \left(\frac{\Delta}{\epsilon^{1/2}}\right)^{p}\log^{p}(1+\epsilon^{-1}).
\end{aligned}
$$

In a similar way, for $\sum J_1^k$ and $\sum J_3^k$, now applying Lemma 2.4,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\leq\tau^\epsilon}}\left|\sum_{k=0}^{h-1} J_1^k\right|^p + \left|\sum_{k=0}^{h-1} J_3^k\right|^p\right]\\
&\lesssim \mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\leq\tau^\epsilon}}\left|\sum_{k=0}^{h-1}\int_{k\Delta}^{(k+1)\Delta}\left(\left|\bar{X}_s - \bar{X}_{k\Delta}\right| + \omega_{F,\sigma}(s-k\Delta)\right)ds\right|^p\right]\\
&\lesssim \sum_{k=0}^{[T/\Delta]-1}\int_{k\Delta}^{(k+1)\Delta}\mathbb{E}\left[\left|\bar{X}_{s\wedge\tau^\epsilon} - \bar{X}_{k\Delta\wedge\tau^\epsilon}\right|^p + \omega_{F,\sigma}(\Delta)^p\right]ds\\
&\lesssim \Delta^{p/2} + \omega_{F,\sigma}(\Delta)^p.
\end{aligned}
$$

For the last sum $\sum J_5^k$, by Burkholder-Davis-Gundy's inequality and Lemma 2.4,

$$\begin{aligned}
\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau^\epsilon}}\left|\sum_{k=0}^{h-1} J_5^k\right|^p\right] &\lesssim \mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau^\epsilon}}\left|\sum_{k=0}^{h-1}\int_{k\Delta}^{(k+1)\Delta}\left(\sigma(s,\bar X_s)-\sigma(k\Delta,\bar X_{k\Delta})\right)dW_s\right|^p\right]\\
&\lesssim \mathbb{E}\left[\left|\sum_{k=0}^{[T/\Delta]-1}\int_{k\Delta\wedge\tau^\epsilon}^{(k+1)\Delta\wedge\tau^\epsilon}\left|\sigma(s,\bar X_s)-\sigma(k\Delta,\bar X_{k\Delta})\right|^2 ds\right|^{p/2}\right]\\
&\lesssim \mathbb{E}\left[\left|\sum_{k=0}^{[T/\Delta]-1}\int_{k\Delta\wedge\tau^\epsilon}^{(k+1)\Delta\wedge\tau^\epsilon}\left|\sigma(s,\bar X_s)-\sigma(k\Delta,\bar X_{k\Delta})\right|^2 ds\right|^{p}\right]^{1/2}\\
&\lesssim \left(\sum_{k=0}^{[T/\Delta]-1}\int_{k\Delta}^{(k+1)\Delta}\mathbb{E}\left[\left|\bar X_{s\wedge\tau^\epsilon}-\bar X_{k\Delta\wedge\tau^\epsilon}\right|^{2p}+(s-k\Delta)^{2p}\right]ds\right)^{1/2}\\
&\lesssim \Delta^{p/2}.
\end{aligned}$$

□

Remark 2.3. The estimates given in Lemma 2.7 motivate the following choice of how $\Delta=\Delta_\epsilon$ should behave when ϵ goes to zero:

$$\frac{\Delta^2}{\epsilon^{3/2}}\log^{3/2}(1+\epsilon^{-1})\to 0,\quad \frac{\Delta}{\epsilon}\log(1+\epsilon^{-1})\,\omega_\sigma(\Delta)\to 0,\quad \frac{\epsilon}{\Delta}\log^{1/2}(1+\epsilon^{-1})\to 0.$$

Such a choice is always possible. Indeed, one can take $\omega_\sigma(t)=Ct^\gamma$ for some positive constant C and $\gamma\in(0,2/3)$, and therefore the choice $\Delta_\epsilon=\epsilon^{\frac{1}{1+\gamma/2}}$ satisfies all the requirements above. We will maintain this choice of Δ in the remainder of the chapter.

We now discuss the second sum on the right-hand side of (2.7), that is

$$\sum_{k=0}^{h-1}\left(\int_{k\Delta}^{(k+1)\Delta}\left(\int_{k\Delta}^{s} D\sigma(k\Delta,\bar X_{k\Delta})\sigma(k\Delta,\bar X_{k\Delta})dW_r^\epsilon\right)dW_s^\epsilon-\int_{k\Delta}^{(k+1)\Delta}C(k\Delta,\bar X_{k\Delta})ds\right),$$

the i-th component of which, when plugging in the expression for the Stratonovich corrector C, reads

$$\sum_{k=0}^{h-1}\sum_{\ell,m\in\mathbb{N}}\sum_{j=1,\dots,d} D_j\sigma^{i,m}(k\Delta,\bar X_{k\Delta})\sigma^{j,\ell}(k\Delta,\bar X_{k\Delta})\left(c_{\ell,m}^k(\Delta,\epsilon)-\delta_{\ell,m}\frac{q_m}{2}\Delta\right),$$

where $c_{\ell,m}^k(\Delta,\epsilon)$ is given by

$$c_{\ell,m}^k(\Delta,\epsilon)=\int_{k\Delta}^{(k+1)\Delta}\left(\int_{k\Delta}^{s} dW_r^{\epsilon,\ell}\right)dW_s^{\epsilon,m}.$$

Taking the conditional expectation of $c^k_{\ell,m}(\Delta,\epsilon)$ with respect to $\mathcal{F}_{k\Delta}$ yields

$$\begin{aligned}\mathbb{E}\left[c^k_{\ell,m}(\Delta,\epsilon)\mid\mathcal{F}_{k\Delta}\right]=&\int_{k\Delta}^{(k+1)\Delta}\left(\int_{k\Delta}^{s}\mathbb{E}\left[Y^{\epsilon,\ell}_rY^{\epsilon,m}_s\mid\mathcal{F}_{k\Delta}\right]dr\right)ds\\ =&Y^{\epsilon,\ell}_{k\Delta}Y^{\epsilon,m}_{k\Delta}\int_{k\Delta}^{(k+1)\Delta}\left(\int_{k\Delta}^{s}e^{-\epsilon^{-1}(r+s-2k\Delta)}dr\right)ds\\ &+\delta_{\ell,m}\int_{k\Delta}^{(k+1)\Delta}\left(\int_{k\Delta}^{s}q_\ell\frac{\epsilon^{-1}}{2}\left(e^{-\epsilon^{-1}(s-r)}-e^{-\epsilon^{-1}(r+s-2k\Delta)}\right)dr\right)ds,\end{aligned}$$

where the following representation of Y^ϵ,

$$Y^{\epsilon,m}_s=Y^{\epsilon,m}_{k\Delta}e^{-\epsilon^{-1}(s-k\Delta)}+\int_{k\Delta}^{s}e^{-\epsilon^{-1}(s-r)}\epsilon^{-1}dW^m_r,$$

has been used, and this conditional expectation can easily be calculated as

$$\begin{aligned}\mathbb{E}\left[c^k_{\ell,m}(\Delta,\epsilon)\mid\mathcal{F}_{k\Delta}\right]=&\frac{\epsilon^2}{2}Y^{\epsilon,\ell}_{k\Delta}Y^{\epsilon,m}_{k\Delta}\left(e^{-\epsilon^{-1}\Delta}-1\right)^2\\ &+\delta_{\ell,m}\frac{q_m}{2}\left(\Delta+\epsilon\left(-\frac{3}{2}+2e^{-\epsilon^{-1}\Delta}-\frac{1}{2}e^{-2\epsilon^{-1}\Delta}\right)\right).\end{aligned}\tag{2.8}$$

Now, since $\sum_{j=1,\dots,d}D_j\sigma^{i,m}(k\Delta,\bar{X}_{\tau^\epsilon\wedge(k\Delta)})\sigma^{j,\ell}(k\Delta,\bar{X}_{\tau^\epsilon\wedge(k\Delta)})$ is $\mathcal{F}_{k\Delta}$ measurable, for every $\ell,m\in\mathbb{N}$, $i=1,\dots,d$, each process M^i_h, $h=1,\dots,[T/\Delta]$, given by

$$\begin{aligned}M^i_h=&\sum_{k=0}^{h-1}\sum_{\ell,m\in\mathbb{N}}\sum_{j=1,\dots,d}D_j\sigma^{i,m}(k\Delta,\bar{X}_{\tau^\epsilon\wedge(k\Delta)})\sigma^{j,\ell}(k\Delta,\bar{X}_{\tau^\epsilon\wedge(k\Delta)})\\ &\times\left(c^k_{\ell,m}(\Delta,\epsilon)-\mathbb{E}\left[c^k_{\ell,m}(\Delta,\epsilon)\mid\mathcal{F}_{k\Delta}\right]\right),\end{aligned}$$

is a discrete martingale with respect to the filtration $(\mathcal{F}_{h\Delta})_{h=1}^{[T/\Delta]}$.

Lemma 2.8. *For each $i=1,\dots,d$,*

$$\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau^\epsilon}}\left|M^i_h\right|^2\right]\lesssim\frac{\Delta^2}{\epsilon}\log(1+\epsilon^{-1})+\Delta\log^2(1+\epsilon^{-1}).$$

Proof. Combining Doob's maximal inequality and martingale property gives

$$\begin{aligned}\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau^\epsilon}}\left|M^i_h\right|^2\right]&\lesssim\mathbb{E}\left[\left|M^i_{[T/\Delta]}\right|^2\right]\\ &\lesssim\sum_{k=0}^{[T/\Delta]-1}\mathbb{E}\left[\left|\sum_{\ell,m\in\mathbb{N}}c^k_{\ell,m}(\Delta,\epsilon)-\mathbb{E}\left[c_{\ell,m}(\Delta,\epsilon)\mid\mathcal{F}_{k\Delta}\right]\right|^2\right],\end{aligned}$$

where

$$\mathbb{E}\left[\left|\sum_{\ell,m\in\mathbb{N}}c^k_{\ell,m}(\Delta,\epsilon)-\mathbb{E}\left[c^k_{\ell,m}(\Delta,\epsilon)\mid\mathcal{F}_{k\Delta}\right]\right|^2\right]\lesssim\mathbb{E}\left[\left|\sum_{\ell,m\in\mathbb{N}}c^k_{\ell,m}(\Delta,\epsilon)\right|^2\right],$$

for each $k = 0, \dots, [T/\Delta] - 1$, because the conditional expectation is an L^2-projection. Thus, by independence of $Y^{\epsilon,\ell}$ and $Y^{\epsilon,m}$, for every $\ell \neq m$, we can estimate

$$\begin{aligned}
\mathbb{E}\left[\sup_{\substack{h=1,\dots,T/\Delta \\ h\Delta\le\tau^\epsilon}} \left|M_h^i\right|^2\right] &\lesssim \sum_{k=0}^{T/\Delta-1} \sum_{\ell,m\in\mathbb{N}} \mathbb{E}\left[\left|\int_{k\Delta}^{(k+1)\Delta} \left(W_s^{\epsilon,\ell} - W_{k\Delta}^{\epsilon,\ell}\right) dW_s^{\epsilon,m}\right|^2\right] \\
&\lesssim \sum_{k=0}^{T/\Delta-1} \sum_{\ell,m\in\mathbb{N}} \Delta \int_{k\Delta}^{(k+1)\Delta} \mathbb{E}\left[\left|\left(W_s^{\epsilon,\ell} - W_{k\Delta}^{\epsilon,\ell}\right) Y_s^{\epsilon,m}\right|^2\right] ds \\
&\lesssim \sum_{k=0}^{T/\Delta-1} \sum_{\ell,m\in\mathbb{N}} \Delta \int_{k\Delta}^{(k+1)\Delta} \mathbb{E}\left[\left|W_s^{\epsilon,\ell} - W_{k\Delta}^{\epsilon,\ell}\right|^{2q}\right]^{1/q} \mathbb{E}\left[|Y_s^{\epsilon,m}|^{2q'}\right]^{1/q'} ds \\
&\lesssim \sum_{k=0}^{T/\Delta-1} \sum_{\ell,m\in\mathbb{N}} q_\ell q_m \Delta \epsilon^{-1} \log(1+\epsilon^{-1}) \int_{k\Delta}^{(k+1)\Delta} \left(\Delta + \epsilon \log(1+\epsilon^{-1})\right) ds \\
&\lesssim \frac{\Delta^2}{\epsilon} \log(1+\epsilon^{-1}) + \Delta \log^2(1+\epsilon^{-1}).
\end{aligned}$$

□

To eventually cover the remainder of the second sum on the right-hand side of (2.7), after subtracting the martingale term M_h, we introduce

$$\begin{aligned}
N_h^i = \sum_{k=0}^{h-1} \sum_{\ell,m\in\mathbb{N}} \sum_{j=1,\dots,d} & D_j \sigma^{i,m}(k\Delta, \bar{X}_{k\Delta}) \sigma^{j,\ell}(k\Delta, \bar{X}_{k\Delta}) \\
& \times \left(\mathbb{E}\left[c_{\ell,m}^k(\Delta,\epsilon) \mid \mathcal{F}_{k\Delta}\right] - \delta_{\ell,m} \frac{q_m}{2} \Delta\right).
\end{aligned}$$

Lemma 2.9. *For each $i = 1, \dots, d$,*

$$\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta] \\ h\Delta\le\tau^\epsilon}} \left|N_h^i\right|^2\right] \lesssim \left(\frac{\epsilon}{\Delta}\right)^2 \log^2(1+\epsilon^{-1}).$$

Proof. The proof is an easy consequence of (2.8). Indeed,

$$\begin{aligned}
\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta] \\ h\Delta\le\tau^\epsilon}} \left|N_h^i\right|^2\right] &\lesssim \mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta] \\ h\Delta\le\tau^\epsilon}} \left|\sum_{k=0}^{h-1} \sum_{\ell,m\in\mathbb{N}} \left|\mathbb{E}\left[c_{\ell,m}^k(\Delta,\epsilon) \mid \mathcal{F}_{k\Delta}\right] - \delta_{\ell,m}\frac{q_m}{2}\Delta\right|\right|^2\right] \\
&\lesssim \epsilon^2 \log^2(1+\epsilon^{-1}) \Delta^{-1} \sum_{k=0}^{[T/\Delta]-1} \sum_{\ell,m\in\mathbb{N}} q_\ell q_m \lesssim \left(\frac{\epsilon}{\Delta}\right)^2 \log^2(1+\epsilon^{-1}).
\end{aligned}$$

□

All in all, Lemma 2.8 and Lemma 2.9 together imply

$$\begin{aligned}
\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta] \\ h\Delta\le\tau^\epsilon}} \left|\sum_{k=0}^{h-1} \left(I_6^k - J_4^k\right)\right|^2\right] &= \mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta] \\ h\Delta\le\tau^\epsilon}} |(M_h + N_h)|^2\right] \\
&\lesssim \frac{\Delta^2}{\epsilon} \log(1+\epsilon^{-1}) + \Delta \log^2(1+\epsilon^{-1}) + \left(\frac{\epsilon}{\Delta}\right)^2 \log^2(1+\epsilon^{-1}),
\end{aligned}$$

showing that the second sum on the right-hand side of (2.7) can be neglected, like the fifth one, when $\epsilon \to 0$, and $\Delta = \Delta_\epsilon$ behaves as described in Remark 2.3.
Recall that we wanted to control the remaining sums in terms of the difference $X^\epsilon - \bar{X}$ itself, which is obvious for the first and third sum on the right-hand side of (2.7). However, in case of the fourth sum, applying almost the same martingale argument used in case of the second sum, each term I_5^k can be formally replaced by

$$\int_{k\Delta}^{(k+1)\Delta} \big(C(k\Delta, X^\epsilon_{k\Delta}) - C(k\Delta, \bar{X}_{k\Delta})\big)\, ds,$$

subject to a sufficiently small ϵ-correction, eventually leading to the wanted contraction argument in this case, too.
On the whole, we have justified that, if $\Delta = \Delta_\epsilon$ behaves as described in Remark 2.3, then for any $h = 1, \ldots, [T/\Delta]$:

$$\mathbb{E}\left[\sup_{\substack{k'=0,\ldots,h \\ k'\Delta \le \tau^\epsilon}} \left|X^\epsilon_{k'\Delta} - \bar{X}_{k'\Delta}\right|^2\right] \lesssim r(\Delta,\epsilon) + \sum_{k=0}^{h-1} \Delta \mathbb{E}\left[\sup_{\substack{k'=0,\ldots,k \\ k'\Delta \le \tau^\epsilon}} \left|X^\epsilon_{k'\Delta} - \bar{X}_{k'\Delta}\right|^2\right],$$

where $r(\Delta, \epsilon) \to 0$, $\epsilon \to 0$, finally proving (2.4), by Gronwall's lemma.
The proof of the strong convergence in Theorem 1.1 is thus complete.

2.2 Weak convergence

In this section we prove the first part of Theorem 1.1 on weak convergence: $X^\epsilon \to \bar{X}$ in law. The idea of the proof is similar to the one outlined in the previous section for strong convergence, except that now $\beta \neq 0$ is possible: it is the presence of this bilinear term which prevents us from proving convergence in probability—we only succeed in showing convergence in law, see also Remark 2.5. First, we prove weak convergence of the bilinear term alone; second, we prove convergence in law of X^ϵ, $\epsilon \to 0$.

2.2.1 Weak convergence of the bilinear term

For any $\epsilon > 0$, define the process U^ϵ by

$$U^\epsilon_t = \int_0^t \epsilon^{1/2} \beta(Y^\epsilon_s, Y^\epsilon_s) ds, \quad t \in [0,T], \tag{2.9}$$

where Y^ϵ is the stationary Ornstein-Uhlenbeck process introduced in Remark 2.1. Since by assumption $\beta : H_\infty \times H_\infty \to H_d$ is a continuous bilinear map and $\sum_{\ell\in\mathbb{N}} \langle \beta(\mathbf{f}_\ell, \mathbf{f}_\ell), \mathbf{e}_i\rangle_{H_d} q_\ell =$ 0, for all $i = 1, \ldots, d$, the process U^ϵ has zero-mean and its second moments,

$$\mathbb{E}\left[\epsilon \int_0^t \underbrace{\langle \beta(Y^\epsilon_s, Y^\epsilon_s), \mathbf{e}_i\rangle}_{\beta^i(Y^\epsilon_s, Y^\epsilon_s)} ds \int_0^t \underbrace{\langle \beta(Y^\epsilon_r, Y^\epsilon_r), \mathbf{e}_j\rangle}_{\beta^j(Y^\epsilon_r, Y^\epsilon_r)} dr\right],$$

can be calculated to be

$$\frac{1}{2} \sum_{\ell,m\in\mathbb{N}} \underbrace{\langle \beta(\mathbf{f}_\ell, \mathbf{f}_m), \mathbf{e}_i\rangle}_{\beta^i_{\ell,m}} \underbrace{\langle \beta(\mathbf{f}_\ell, \mathbf{f}_m), \mathbf{e}_j\rangle}_{\beta^j_{\ell,m}} q_\ell q_m \left(t + \frac{\epsilon}{2}\left(e^{-2\epsilon^{-1}t} - 1\right)\right),$$

for $i, j = 1, \dots, d$, and $\ell, m \in \mathbb{N}$.
Next, since $dY_t^{\epsilon,\ell} = -\epsilon^{-1} Y_t^{\epsilon,\ell} dt + \epsilon^{-1} d\langle W_t, \mathbf{f}_\ell \rangle$, Itô's formula implies

$$Y_t^{\epsilon,\ell} Y_t^{\epsilon,m} = Y_0^{\epsilon,\ell} Y_0^{\epsilon,m} - 2\epsilon^{-1} \int_0^t Y_s^{\epsilon,\ell} Y_s^{\epsilon,m} ds + \epsilon^{-1} \int_0^t Y_s^{\epsilon,\ell} d\langle W_s, \mathbf{f}_m \rangle + \epsilon^{-1} \int_0^t Y_s^{\epsilon,m} d\langle W_s, \mathbf{f}_\ell \rangle + \frac{t\epsilon^{-2}}{2} q_\ell \delta_{\ell,m},$$

for any $\ell, m \in \mathbb{N}$, and hence

$$\begin{aligned} U_t^{\epsilon,i} &= \epsilon^{1/2} \int_0^t \sum_{\ell,m\in\mathbb{N}} \beta^i_{\ell,m} Y_s^{\epsilon,\ell} Y_s^{\epsilon,m} ds \\ &= \epsilon^{1/2} \int_0^t \sum_{\ell,m\in\mathbb{N}} \beta^i_{\ell,m} Y_s^{\epsilon,\ell} d\langle W_s, \mathbf{f}_m \rangle \\ &\quad - \frac{\epsilon^{3/2}}{2} \sum_{\ell,m\in\mathbb{N}} \beta^i_{\ell,m} \left(Y_t^{\epsilon,\ell} Y_t^{\epsilon,m} - Y_0^{\epsilon,\ell} Y_0^{\epsilon,m} \right) + \frac{\epsilon^{-1/2}}{4} t \sum_{\ell\in\mathbb{N}} \beta^i_{\ell,\ell} q_\ell \\ &= M_t^{\epsilon,i} - \frac{1}{2} V_t^{\epsilon,i} + \frac{\epsilon^{-1/2}}{4} t \sum_{\ell\in\mathbb{N}} \beta^i_{\ell,\ell} q_\ell, \end{aligned}$$

where M^ϵ is a d-dimensional continuous local martingale, while the process V^ϵ satisfies

$$\begin{aligned} \mathbb{E}\left[\sup_{t\le T} |V_t^\epsilon|^p \right] &= \mathbb{E}\left[\sup_{t\le T} \left| \epsilon^{3/2} \left(\beta\left(Y_t^\epsilon, Y_t^\epsilon\right) - \beta\left(Y_0^\epsilon, Y_0^\epsilon\right) \right) \right|^p \right] \\ &\lesssim \epsilon^{p/2} \log^p(1+\epsilon^{-1}), \quad \forall\, p > 1, \end{aligned} \tag{2.10}$$

by combining bilinearity of β and Lemma 2.1.

Remark 2.4. Using $\sum_{\ell,m\in\mathbb{N}} \beta^i_{\ell,m} q_\ell q_m < \infty$ for every $i = 1, \dots, d$, it is possible to prove that M^ϵ is a square integrable martingale for every $\epsilon > 0$. However, we will not need this in the following.

The above representation of U^ϵ, though very simple, has been used in a variety of cases in a fruitful way, see for instance [Oll94] or [IPP08]. Observe that, by assumption, the Itô-correction actually cancels out, being otherwise a contribution of order $\epsilon^{-1/2}$. The process U^ϵ, nevertheless, has got an interesting limit in law:

Proposition 2.10. *The couple of processes (U^ϵ, W) converges in law, $\epsilon \to 0$, to a pair of processes (η, ω), where η is a d-dimensional Wiener process with covariance $(\sum_{\ell,m\in\mathbb{N}} b^i_{\ell,m} b^j_{\ell,m})_{i,j=1}^d$, and ω is a Q-Wiener process, like W. Furthermore, η and ω are independent.*

Proof. First, by (2.10), it is sufficient to prove the proposition for (M^ϵ, W) instead of (U^ϵ, W). Since all components of the processes M^ϵ, $\epsilon > 0$, and of W are continuous local martingales, the distributional properties of the limit (η, ω) would follow from [EK86, Chapter VII, Theorem 1.4] (readily adapted to our infinite dimensional case, thanks to the trace-class assumption on Q), if we can prove

$$\mathbb{E}\left[\left(\left[M^{\epsilon,i}, M^{\epsilon,j} \right]_t - t \sum_{\ell,m\in\mathbb{N}} b^i_{\ell,m} b^j_{\ell,m} \right)^2 \right] \to 0, \quad \epsilon \to 0, \tag{2.11}$$

for each $t \in [0,T]$, and $i,j = 1,\dots,d$, as well as

$$\mathbb{E}\left[\left(\left[M^{\epsilon,i}, \langle W, \mathbf{f}_m\rangle\right]_t\right)^2\right] \to 0, \quad \epsilon \to 0,$$

for each $t \in [0,T]$, $i = 1,\dots,d$, and $m \in \mathbb{N}$. So, let us focus on these two convergences. First, fix $t \in [0,T]$, as well as $i,j = 1,\dots,d$. Then, the quadratic covariation $[M^{\epsilon,i}, M^{\epsilon,j}]_t$ is given by

$$\left[M^{\epsilon,i}, M^{\epsilon,j}\right]_t = \epsilon \int_0^t \sum_{m\in\mathbb{N}} \sum_{\ell,\ell'\in\mathbb{N}} \beta^i_{\ell,m}\beta^j_{\ell',m} q_m Y^{\epsilon,\ell}_s Y^{\epsilon,\ell'}_s \, ds,$$

so that

$$\begin{aligned}
&\mathbb{E}\left[\left(\left[M^{\epsilon,i}, M^{\epsilon,j}\right]_t - t\sum_{\ell,m\in\mathbb{N}} b^i_{\ell,m} b^j_{\ell,m}\right)^2\right] \\
&\quad = \epsilon^2 \iint_0^t \sum_{m,\underline{m}\in\mathbb{N}} \sum_{\substack{\ell,\ell'\in\mathbb{N}\\ \underline{\ell},\underline{\ell}'\in\mathbb{N}}} \beta^i_{\ell,m}\beta^j_{\ell',m}\beta^i_{\underline{\ell},\underline{m}}\beta^j_{\underline{\ell}',\underline{m}} q_m q_{\underline{m}} \mathbb{E}\left[Y^{\epsilon,\ell}_s Y^{\epsilon,\ell'}_s Y^{\epsilon,\underline{\ell}}_r Y^{\epsilon,\underline{\ell}'}_r\right] ds dr \\
&\quad\quad - 2\epsilon \int_0^t \sum_{m\in\mathbb{N}} \sum_{\ell,\ell'\in\mathbb{N}} \beta^i_{\ell,m}\beta^j_{\ell',m} q_m \mathbb{E}\left[Y^{\epsilon,\ell}_s Y^{\epsilon,\ell'}_s\right] ds \left(t\sum_{\ell,m\in\mathbb{N}} b^i_{\ell,m} b^j_{\ell,m}\right) + \left(t\sum_{\ell,m\in\mathbb{N}} b^i_{\ell,m} b^j_{\ell,m}\right)^2.
\end{aligned}$$

Now, using the fact that one can easily calculate $\mathbb{E}\left[Y^{\epsilon,\ell}_s Y^{\epsilon,\ell'}_s\right] = \frac{\epsilon^{-1}}{2} q_\ell \delta_{\ell,\ell'}$, from Isserlis-Wick's theorem (see for instance [Jan97, Theorem 1.28]) it follows that

$$\mathbb{E}\left[Y^{\epsilon,\ell}_s Y^{\epsilon,\ell'}_s Y^{\epsilon,\underline{\ell}}_r Y^{\epsilon,\underline{\ell}'}_r\right] = \frac{\epsilon^{-2}}{4}\left(q_\ell q_{\underline{\ell}} \delta_{\ell,\ell'}\delta_{\underline{\ell},\underline{\ell}'} + q_\ell q_{\ell'} e^{-2\epsilon^{-1}|s-r|}\left(\delta_{\ell,\underline{\ell}}\delta_{\ell',\underline{\ell}'} + \delta_{\ell,\underline{\ell}'}\delta_{\ell',\underline{\ell}}\right)\right),$$

which yields

$$\mathbb{E}\left[\left(\left[M^{\epsilon,i}, M^{\epsilon,j}\right]_t - t\sum_{\ell,m\in\mathbb{N}} b^i_{\ell,m} b^j_{\ell,m}\right)^2\right] = \left(t\sum_{\ell,m\in\mathbb{N}} b^i_{\ell,m} b^j_{\ell,m} - t\sum_{\ell,m\in\mathbb{N}} b^i_{\ell,m} b^j_{\ell,m}\right)^2 + O(\epsilon) \lesssim \epsilon,$$

proving (2.11). Moving to the second desired convergence, fix $t \in [0,T]$, as well as $i = 1,\dots,d$, $m \in \mathbb{N}$. Then,

$$\left[M^{\epsilon,i}, \langle W, \mathbf{f}_m\rangle\right]_t = \int_0^t \beta^i(\epsilon^{1/2} Y^\epsilon_s, Q\mathbf{f}_m)\, ds,$$

where, using Lemma 2.1,

$$\begin{aligned}
\mathbb{E}\left[\left|\int_0^t \beta^i(\epsilon^{1/2} Y^\epsilon_s, Q\mathbf{f}_m)\, ds\right|^2\right] &= \mathbb{E}\left[\left|\beta^i(\epsilon^{1/2}\int_0^t Y^\epsilon_s ds\, , Q\mathbf{f}_m)\right|^2\right] \\
&\lesssim \mathbb{E}\left[\,\Big|\, \overbrace{\epsilon^{1/2}\int_0^t Y^\epsilon_s ds}^{\epsilon^{1/2}W_t - \epsilon^{3/2}(Y^\epsilon_t - Y^\epsilon_0)} \,\Big|^2 q_m^2\,\right] \overset{\epsilon\to 0}{\longrightarrow} 0,
\end{aligned}$$

finishing the proof of the proposition. □

Remark 2.5. *i*) Of course, a d-dimensional Wiener process with covariance $(\sum_{\ell,m\in\mathbb{N}} b^i_{\ell,m} b^j_{\ell,m})^d_{i,j=1}$ can always be represented by $\sum_{\ell,m\in\mathbb{N}} b_{\ell,m}\bar{W}^{\ell,m}$, where $\{\bar{W}^{\ell,m}\}_{\ell,m\in\mathbb{N}}$ is a family of independent one-dimensional standard Wiener processes.
ii) We would like to stress that we do not expect a much stronger convergence of U^ϵ, when $\epsilon\to 0$, as the one stated in the above proposition. Indeed, it turns out to be that the sequence $\{M^\epsilon\}_{\epsilon>0}$ is not even a Cauchy sequence in $L^2(\Omega;\mathbb{R}^d)$. To see this, for fixed $0<\epsilon<\underline{\epsilon}$, and some $1\le i\le d$, consider

$$\mathbb{E}\left[\sup_{t\le T}\left|M^{\epsilon,i}_t - M^{\underline{\epsilon},i}_t\right|^2\right] = \mathbb{E}\left[\sup_{t\le T}\left|\int_0^t \sum_{\ell,m\in\mathbb{N}} \beta^i_{\ell,m}\left(\epsilon^{1/2}Y^{\epsilon,\ell}_s - \underline{\epsilon}^{1/2}Y^{\underline{\epsilon},\ell}_s\right) d\langle W_s, \mathbf{f}_m\rangle\right|^2\right].$$

But, by Burkholder-Davis-Gundy's inequality, the above expectation can be bound from below by

$$\mathbb{E}\left[\int_0^T \sum_{m\in\mathbb{N}}\left(\sum_{\ell\in\mathbb{N}}\beta^i_{\ell,m}\left(\epsilon^{1/2}Y^{\epsilon,\ell}_s - \underline{\epsilon}^{1/2}Y^{\underline{\epsilon},\ell}_s\right)\right)^2 q_m ds\right] = T\sum_{\ell,m\in\mathbb{N}}(\beta^i_{\ell,m})^2 q_\ell q_m\left(1-\frac{2\epsilon^{-1/2}\underline{\epsilon}^{-1/2}}{\epsilon^{-1}+\underline{\epsilon}^{-1}}\right),$$

where

$$\lim_{\epsilon\to 0}\left(1-\frac{2\epsilon^{-1/2}\underline{\epsilon}^{-1/2}}{\epsilon^{-1}+\underline{\epsilon}^{-1}}\right) = 1, \quad \text{for every fixed } \underline{\epsilon}>0,$$

so that $\{M^{\epsilon,i}\}_{\epsilon>0}$ cannot be Cauchy in $L^2(\Omega)$.

2.2.2 Weak convergence of solutions

We now prove $X^\epsilon\to\bar{X}$, in law, when $\epsilon\to 0$. First, for each $\epsilon>0$, let $\hat{X}^\epsilon$ be the solution of

$$\hat{X}^\epsilon_t = x_0 + \int_0^t\left(F(s,\hat{X}^\epsilon_s)+C(s,\hat{X}^\epsilon_s)\right)ds + \int_0^t \sigma(s,\hat{X}^\epsilon_s)dW_s + U^\epsilon_t, \quad t\in[0,T], \tag{2.12}$$

where U^ϵ is given by (2.9), and let $\tau^\epsilon_R = \inf\{t\ge 0: |X^\epsilon_t|\ge R\}\wedge\inf\{t\ge 0: |\hat{X}^\epsilon_t|\ge R\}$. Notice that, if both equations (1.4) and (1.5) admit global solutions on $[0,T]$, then the coefficients F, C, σ, β must have properties such that the above equation admits global solutions on $[0,T]$, too.
Next, taking into account $\mathbb{E}\left[\left|\sup_{s\in[0,T]}\epsilon^{1/2}\beta(Y^\epsilon_s, Y^\epsilon_s)\right|^p\right]\lesssim \epsilon^{-p/2}\log^p(1+\epsilon^{-1})$ we can estimate increments of U^ϵ with

$$\mathbb{E}\left[\sup_{\substack{k=0,1,\dots,[T/\Delta]\\ t\le\tau,\, t+k\Delta\le T\wedge\tau^\epsilon_R}}|U^\epsilon_{t+k\Delta}-U^\epsilon_{k\Delta}|^p\right]\lesssim\left(\frac{\tau}{\epsilon^{1/2}}\right)^p\log^p(1+\epsilon^{-1}). \tag{2.13}$$

As a consequence, it can easily be verified that the analogous of Lemma 2.2 and Lemma 2.3 are still valid for the process X^ϵ, despite $\beta\neq 0$, on the one hand, and that the following versions Lemma 2.4 and Corollary 2.5:

$$\mathbb{E}\left[\sup_{t\le\tau,\, t+k\Delta\le T\wedge\tau^\epsilon_R}|\hat{X}^\epsilon_{t+k\Delta}-\hat{X}^\epsilon_{k\Delta}|^p\right]\lesssim\tau^{\frac{p}{2}}+\left(\frac{\tau}{\epsilon^{1/2}}\right)^p\log^p(1+\epsilon^{-1}),$$

where $p>1$, $\tau\in(0,1)$, $k\in\{0,1,\dots,[T/\Delta]\}$; and

$$\mathbb{E}\left[\sup_{\substack{k=0,1,\dots,[T/\Delta]\\ t\le\Delta,\, t+k\Delta\le T\wedge\tau_R^\epsilon}} |\hat{X}^\epsilon_{t+k\Delta}-\hat{X}^\epsilon_{k\Delta}|^p\right] \lesssim \Delta^{\frac{p}{2}-1}+\frac{\Delta^{p-1}}{\epsilon^{p/2}}\log^p(1+\epsilon^{-1}),$$

for $p>1$; would hold true when replacing $\bar{X}$ by $\hat{X}^\epsilon$, on the other. We point out that the proof of this claim differs from those in Section 5.3 only for the term U^ϵ, which however can be controlled by (2.13).
Therefore, when expanding X^ϵ and $\hat{X}^\epsilon$ as in (2.5) and (2.6), but including the β-term, and then arguing as in the proof of strong convergence in the previous section, it would immediately follow that $X^\epsilon_{\cdot\wedge\tau_R^\epsilon}-\hat{X}^\epsilon_{\cdot\wedge\tau_R^\epsilon}\to 0$, in probability, $\epsilon\to 0$, for any $R>0$, once the following lemma is also available.

Lemma 2.11. *Assume that $\Delta=\Delta_\epsilon$ behaves as described in Remark 2.3. Then,*

$$\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau_R^\epsilon}}\left|\sum_{k=0}^{h-1}\int_{k\Delta}^{(k+1)\Delta}\left(\int_{k\Delta}^{s} D\sigma(r,X^\epsilon_r)\epsilon^{1/2}\beta(Y^\epsilon_r,Y^\epsilon_r)dr\right)dW^\epsilon_s\right|^2\right]\to 0,\qquad \epsilon\to 0.$$

Proof. To start with, write

$$\begin{aligned}
&\int_{k\Delta}^{(k+1)\Delta}\left(\int_{k\Delta}^{s} D\sigma(r,X^\epsilon_r)\epsilon^{1/2}\beta(Y^\epsilon_r,Y^\epsilon_r)dr\right)dW^\epsilon_s\\
&\quad=\int_{k\Delta}^{(k+1)\Delta}\left(\int_{k\Delta}^{s}\left(D\sigma(r,X^\epsilon_r)\epsilon^{1/2}\beta(Y^\epsilon_r,Y^\epsilon_r)-D\sigma(k\Delta,X^\epsilon_{k\Delta})\epsilon^{1/2}\beta(Y^\epsilon_r,Y^\epsilon_r)\right)dr\right)dW^\epsilon_s\\
&\qquad+\int_{k\Delta}^{(k+1)\Delta}\left(\int_{k\Delta}^{s} D\sigma(k\Delta,X^\epsilon_{k\Delta})\epsilon^{1/2}\beta(Y^\epsilon_r,Y^\epsilon_r)dr\right)dW^\epsilon_s,
\end{aligned}$$

which creates two summands, for any fixed $0\le k\le[T/\Delta]-1$. We estimate the impact of each summand separately.
First, using $|D\sigma(r,X^\epsilon_r)-D\sigma(k\Delta,X^\epsilon_{k\Delta})|\lesssim|X^\epsilon_r-X^\epsilon_{k\Delta}|+\omega_\sigma(\Delta)$, we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau_R^\epsilon}}\left|\sum_{k=0}^{h-1}\int_{k\Delta}^{(k+1)\Delta}\left(\int_{k\Delta}^{s}\left(D\sigma(r,X^\epsilon_r)\epsilon^{1/2}\beta(Y^\epsilon_r,Y^\epsilon_r)-D\sigma(k\Delta,X^\epsilon_{k\Delta})\epsilon^{1/2}\beta(Y^\epsilon_r,Y^\epsilon_r)\right)dr\right)dW^\epsilon_s\right|^2\right]\\
&\lesssim\epsilon^{-2}\log^2(1+\epsilon^{-1})\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau_R^\epsilon}}\left|\sum_{k=0}^{h-1}\int_{k\Delta}^{(k+1)\Delta}\left(\int_{k\Delta}^{s}\left(|X^\epsilon_r-X^\epsilon_{k\Delta}|+\omega_\sigma(\Delta)\right)dr\right)ds\right|^2\right]\\
&\lesssim\epsilon^{-2}\log^2(1+\epsilon^{-1})\mathbb{E}\left[\sum_{k=0}^{\lceil T\wedge\tau_R^\epsilon\rceil/\Delta-1}\int_{k\Delta}^{(k+1)\Delta}\left|\int_{k\Delta}^{s}\left(|X^\epsilon_r-X^\epsilon_{k\Delta}|+\omega_\sigma(\Delta)\right)dr\right|^2ds\right]\\
&\lesssim\epsilon^{-2}\log^2(1+\epsilon^{-1})\sum_{k=0}^{[T/\Delta]-1}\int_{k\Delta}^{(k+1)\Delta}(s-k\Delta)\int_{k\Delta}^{s}\left(\mathbb{E}\left[|X^\epsilon_{r\wedge\tau_R^\epsilon}-X^\epsilon_{k\Delta\wedge\tau_R^\epsilon}|^2\right]+\omega_\sigma(\Delta)^2\right)drds\\
&\lesssim\frac{\Delta^4}{\epsilon^3}\log^3(1+\epsilon^{-1})+\left(\frac{\Delta}{\epsilon}\right)^2\log^2(1+\epsilon^{-1})\,\omega_\sigma(\Delta)^2.
\end{aligned}$$

Second, we approach

$$\sum_{k=0}^{h-1}\int_{k\Delta}^{(k+1)\Delta}\left(\int_{k\Delta}^{s} D\sigma(k\Delta, X_{k\Delta}^{\epsilon})\epsilon^{1/2}\beta(Y_r^{\epsilon}, Y_r^{\epsilon})dr\right)dW_s^{\epsilon} \tag{2.14}$$

following the method used when discussing the second sum on the right-hand side of (2.7) in the proof of strong convergence, but now for triple moments of Y^{ϵ}. Indeed, define

$$c_{\ell,m,n}^{k}(\Delta,\epsilon) = \int_{k\Delta}^{(k+1)\Delta}\left(\int_{k\Delta}^{s} Y_r^{\epsilon,\ell}Y_r^{\epsilon,m}dr\right)Y_s^{\epsilon,n}ds,$$

and take the conditional expectation with respect to $\mathcal{F}_{k\Delta}$, that is

$$\mathbb{E}\left[c_{\ell,m,n}^{k}(\Delta,\epsilon)\mid\mathcal{F}_{k\Delta}\right] = \int_{k\Delta}^{(k+1)\Delta}\left(\int_{k\Delta}^{s}\mathbb{E}\left[Y_r^{\epsilon,\ell}Y_r^{\epsilon,m}Y_s^{\epsilon,n}\mid\mathcal{F}_{k\Delta}\right]dr\right)ds.$$

Since

$$\begin{aligned}\mathbb{E}\left[Y_r^{\epsilon,\ell}Y_r^{\epsilon,m}Y_s^{\epsilon,n}\mid\mathcal{F}_{k\Delta}\right] =& Y_{k\Delta}^{\epsilon,\ell}Y_{k\Delta}^{\epsilon,m}Y_{k\Delta}^{\epsilon,n}e^{-\epsilon^{-1}(s+2r-3k\Delta)}\\ &+\left(Y_{k\Delta}^{\epsilon,\ell}\delta_{m,n}q_n + Y_{k\Delta}^{\epsilon,m}\delta_{\ell,n}q_n + Y_{k\Delta}^{\epsilon,n}\delta_{\ell,m}q_\ell\right)\frac{\epsilon^{-1}}{2}\\ &\times\left(e^{-\epsilon^{-1}(s-k\Delta)} - e^{-\epsilon^{-1}(s+2r-3k\Delta)}\right),\end{aligned}$$

we have that

$$\begin{aligned}\mathbb{E}\left[c_{\ell,m,n}^{k}(\Delta,\epsilon)\mid\mathcal{F}_{k\Delta}\right] =& Y_{k\Delta}^{\epsilon,\ell}Y_{k\Delta}^{\epsilon,m}Y_{k\Delta}^{\epsilon,n}\frac{\epsilon^2}{2}\left(1-e^{-\epsilon^{-1}\Delta}-\frac{1}{3}+\frac{1}{3}e^{-3\epsilon^{-1}\Delta}\right)\\ &+\left(Y_{k\Delta}^{\epsilon,\ell}\delta_{m,n}q_n + Y_{k\Delta}^{\epsilon,m}\delta_{\ell,n}q_n + Y_{k\Delta}^{\epsilon,n}\delta_{\ell,m}q_\ell\right)\\ &\times\frac{\epsilon}{2}\left(\frac{\Delta}{\epsilon}e^{-\epsilon^{-1}\Delta}+\frac{1}{2}-\frac{1}{2}e^{-\epsilon^{-1}\Delta}+\frac{1}{6}-\frac{1}{6}e^{-3\epsilon^{-1}\Delta}\right).\end{aligned}$$

Next, for each $i=1,\dots,d$, the process M_h^i, $h=1,\dots,[T/\Delta]$, given by

$$M_h^i = \sum_{k=0}^{h-1}\sum_{\ell,m,n\in\mathbb{N}}\sum_{j=1,\dots,d} D_j\sigma^{i,n}(k\Delta, X_{\tau_R^{\epsilon}\wedge k\Delta}^{\epsilon})\epsilon^{1/2}\beta_{\ell,m}^{j}\left(c_{\ell,m,n}^{k}(\Delta,\epsilon)-\mathbb{E}\left[c_{\ell,m,n}^{k}(\Delta,\epsilon)\mid\mathcal{F}_{k\Delta}\right]\right),$$

is a martingale with respect to the filtration $(\mathcal{F}_{h\Delta})_{h=1}^{[T/\Delta]}$, and arguing as in the proof of Lemma 2.8 yields

$$\mathbb{E}\left[\sup_{\substack{h=1,\dots,[T/\Delta]\\ h\Delta\le\tau_R^{\epsilon}}}\left|M_h^i\right|^2\right]\lesssim\frac{\Delta^3}{\epsilon^2}\log^3(1+\epsilon^{-1}),\quad i=1,\dots,d.$$

So, it remains to prove that the remainder, after subtracting the martingale term M_h from (2.14), also vanishes, when $\epsilon\to 0$. For $i=1,\dots,d$, the ith coordinate of this remainder reads

$$N_h^i = \sum_{k=0}^{h-1}\sum_{\ell,m,n\in\mathbb{N}}\sum_{j=1,\dots,d} D_j\sigma^{i,n}(k\Delta, X_{k\Delta}^{\epsilon})\epsilon^{1/2}B_{\ell,m}^{j}\mathbb{E}\left[c_{\ell,m,n}^{k}(\Delta,\epsilon)\mid\mathcal{F}_{k\Delta}\right],$$

and we can easily calculate the below bound,

$$\mathbb{E}\left[\sup_{\substack{h=1,\dots,T/\Delta\\ h\Delta\leq\tau_R^\epsilon}}\left|N_h^i\right|^2\right]\lesssim\Delta^{-1}\sum_{k=0}^{T/\Delta-1}\mathbb{E}\left[\left|\epsilon\mathbb{E}\left[c_{\ell,m,n}^k(\Delta,\epsilon)\mid\mathcal{F}_{k\Delta}\right]\right|^2\right]\lesssim\left(\frac{\epsilon}{\Delta}\right)^2\log^3(1+\epsilon^{-1}),$$

finishing the proof of the lemma. □

Corollary 2.12. *For any $R>0$, if $\Delta=\Delta_\epsilon$ behaves as described in Remark 2.3,*

$$\mathbb{E}\left[\sup_{t\leq T\wedge\tau_R^\epsilon}|X_t^\epsilon-\hat{X}_t^\epsilon|^2\right]\to 0,\quad \epsilon\to 0,$$

and hence $X^\epsilon_{\cdot\wedge\tau_R^\epsilon}-\hat{X}^\epsilon_{\cdot\wedge\tau_R^\epsilon}\to 0$, in probability, $\epsilon\to 0$, in particular.

The above corollary suggests that it would be sufficient to show that $\hat{X}^\epsilon_{\cdot\wedge\tau_R^\epsilon}\to\bar{X}_{\cdot\wedge\tau_R^\epsilon}$, in law, when $\epsilon\to 0$, subject to some procedure allowing to let R go to infinity, afterwards. So, we at first prove the weak convergence for fixed R, and then discuss the limit-procedure for $R\to\infty$.
Modify the coefficients F,σ outside the set $\{(t,x):|x|<R\}$ in such a way that the new coefficients F_R, σ_R, but also $D\sigma_R$, are globally bounded, and that both functions $F_R(t,\cdot)$ and $D\sigma_R(t,\cdot)$ are globally Lipschitz, uniformly in $t\in[0,T]$. Of course, $\hat{X}^\epsilon_{\cdot\wedge\tau_R^\epsilon}$ coincides with $\hat{X}^{\epsilon,R}_{\cdot\wedge\tau_R^\epsilon}$, where $\hat{X}^{\epsilon,R}$ denotes the solution to the equation obtained when replacing the coefficients of (2.12) by F_R,σ_R, and the Stratonovich correction C_R associated with σ_R. Also, let $\bar{X}^R$ denote the solution to the equation obtained when replacing the coefficients F,σ,C by F_R,σ_R,C_R.

Proposition 2.13. *Fix $R>0$. Then, $\hat{X}^{\epsilon,R}$ converges to $\bar{X}^R$, in law, when $\epsilon\to 0$.*

Proof. Since

$$\hat{X}_t^{\epsilon,R}-U_t^\epsilon=x_0+\int_0^t\left(F_R(s,\hat{X}_s^{\epsilon,R})+C_R(s,\hat{X}_s^{\epsilon,R})\right)ds+\int_0^t\sigma_R(s,\hat{X}_s^{\epsilon,R})dW_s,$$

by boundedness of the coefficients on the above right-hand side, we obtain that

$$\mathbb{E}\left[\sup_{t\leq T}|\hat{X}_t^{\epsilon,R}-U_t^\epsilon|\right]\lesssim|x_0|+T+\mathbb{E}\left[\sup_{t\leq T}\left|\int_0^t\sigma_R(s,\hat{X}_s^{\epsilon,R})dW_s\right|\right],$$

where Burkholder-Davis-Gundy's inequality gives $\mathbb{E}\left[\sup_{t\leq T}\left|\int_0^t\sigma_R(s,\hat{X}_s^{\epsilon,R})dW_s\right|\right]\lesssim T^{1/2}$. Similarly, $\mathbb{E}\left[|(\hat{X}_{t_2}^{\epsilon,R}-U_{t_2}^\epsilon)-(\hat{X}_{t_1}^{\epsilon,R}-U_{t_1}^\epsilon)|^p\right]\lesssim|t_2-t_1|^{p/2}$, for any $|t_2-t_1|<1$, and any $p>1$. Thus, by Kolmogorov continuity theorem, for every $\alpha\in(0,1)$, one can find $\Delta\in(0,1)$ such that

$$\mathbb{P}\left\{\sup_{t_1,t_2\in[0,T],\,|t_2-t_1|\leq\Delta}\frac{|(\hat{X}_{t_2}^{\epsilon,R}-U_{t_2}^\epsilon)-(\hat{X}_{t_1}^{\epsilon,R}-U_{t_1}^\epsilon)|}{|t_2-t_1|^\gamma}\leq K\right\}\geq 1-\alpha,\quad\forall\epsilon>0,$$

where K depends on γ, but not on ϵ, and $\gamma\in(0,1/2)$ can be freely chosen.

We therefore have equi-boundedness and equi-continuity of $\{\hat{X}^{\epsilon,R}-U^\epsilon\}_{\epsilon>0}$ with arbitrarily high probability, and hence the family $\{\hat{X}^{\epsilon,R}-U^\epsilon\}_{\epsilon>0}$ is tight with respect to the uniform topology in $C([0,T],\mathbb{R}^d)$, first applying Arzelà-Ascoli, followed by Prokhorov's theorem. Moreover, $\{U^\epsilon\}_{\epsilon>0}$ is trivially tight by Proposition 2.10, so that adding $\hat{X}^{\epsilon,R}-U^\epsilon$ and U^ϵ would make $\{\hat{X}^{\epsilon,R}\}_{\epsilon>0}$ tight, too. All in all, the family of triples $\{(\ \hat{X}^{\epsilon,R},U^\epsilon,W\ )\}_{\epsilon>0}$ is tight.
Next, for $\epsilon>0$, let $\mathbb{P}^{R,\epsilon}$ be the pushforward measure $\mathbb{P}\circ(\hat{X}^{\epsilon,R},U^\epsilon,W)^{-1}$ on the space

$$\tilde{\Omega}=C([0,T],H_d)\times C([0,T],H_d)\times C([0,T],H_\infty)$$

equipped with the Borel-σ-algebra $\mathcal{B}$, and let (ξ,η,ω) denote the coordinate process on $\tilde{\Omega}$. By tightness of $\{(\hat{X}^{\epsilon,R},U^\epsilon,W)\}_{\epsilon>0}$, there exists a subsequence $(\epsilon_n)_{n\in\mathbb{N}}$ such that $\mathbb{P}^{R,\epsilon_n}$ weakly converges to a probability measure $\mathbb{P}^R$ on $(\tilde{\Omega},\mathcal{B})$, when n
$to\infty$.
Let $\tilde{\mathcal{F}}$ be the $\mathbb{P}^R$- completion of $\mathcal{B}$, and let $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$ be the smallest filtration the process (ξ,η,ω) is adapted to, on the one hand, and which satisfies the usual conditions with respect to $\mathbb{P}^R$, on the other. Also, introduce $\tilde{\mathcal{F}}^n$, $(\tilde{\mathcal{F}}^n_t)_{t\in[0,T]}$ in a similar way with respect to $\mathbb{P}^{R,\epsilon_n}$, $n\in\mathbb{N}$.
Now, it easily follows from Proposition 2.10 that, on $(\tilde{\Omega},\tilde{\mathcal{F}},\mathbb{P}^R)$, the following distributional properties must hold for the pair of processes (η,ω): η is a d-dimensional Wiener process with covariance $(\sum_{\ell,m\in\mathbb{N}} b^i_{\ell,m}b^j_{\ell,m})^d_{i,j=1}$, ω is a Q-Wiener process, η and ω are independent.
Introduce

$$M^R_t=\xi_t-x_0-\int_0^t(F_R(s,\xi_s)+C_R(s,\xi_s))\,ds-\eta_t,\quad t\in[0,T], \tag{2.15}$$

and observe that each component of both processes M^R and ω, but also

$$M^{R,i}_tM^{R,j}_t-\int_0^t\sum_{m\in\mathbb{N}}\sigma^{i,m}_R(s,\xi_s)\sigma^{j,m}_R(s,\xi_s)q_m ds,\quad t\in[0,T],\quad i,j=1,\dots,d,$$
$$M^{R,i}_t\omega^m_t-\int_0^t\sigma^{i,m}_R(s,\xi_s)q_m ds,\quad t\in[0,T],\quad i=1,\dots,d,\quad m\in\mathbb{N},$$
$$\omega^\ell_t\omega^m_t-t\delta_{\ell,m}q_m,\quad t\in[0,T],\quad \ell,m\in\mathbb{N},$$

are continuous local martingales with respect to $(\tilde{\mathcal{F}}^n_t)_{t\in[0,T]}$ on $(\tilde{\Omega},\tilde{\mathcal{F}}^n,\mathbb{P}^{R,\epsilon_n})$, for any $n\in\mathbb{N}$, and hence they are continuous local martingales with respect to $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$ on $(\tilde{\Omega},\tilde{\mathcal{F}},\mathbb{P}^R)$, too, by [JS03, IX. Cor.1.19].
Therefore, applying [DPZ14, Theorem 8.2] to the pair of process (M^R,ω) yields

$$M^R_t=\int_0^t\sigma_R(s,\xi_s)dW^R_s,\quad \omega_t=\int_0^t 1\,dW^R_s=W^R_t,\quad t\in[0,T],$$

on $(\tilde{\Omega},\tilde{\mathcal{F}},\mathbb{P}^R)$, or an enlargement of this space we still denote by $(\tilde{\Omega},\tilde{\mathcal{F}},\mathbb{P}^R)$, where W^R is another Q-Wiener process, which, by the above representation, even $\mathbb{P}^R$- almost surely coincides with ω, so that

$$M^R_t=\int_0^t\sigma_R(s,\xi_s)d\omega_s,\quad t\in[0,T],\quad \mathbb{P}^R\text{- a.s.}$$

Thus, equation (2.15) can be written as

$$\xi_t = x_0 + \int_0^t (F_R(s,\xi_s) + C_R(s,\xi_s))\, ds + \int_0^t \sigma_R(s,\xi_s) d\omega_s + \eta_t, \quad t \in [0,T], \quad \mathbb{P}^R\text{- a.s.},$$

where ω is a Q-Wiener process, while η is a d-dimensional Wiener process, independent of ω, and with covariance $(\sum_{\ell,m\in\mathbb{N}} b^i_{\ell,m} b^j_{\ell,m})^d_{i,j=1}$. Observe that the process $\bar{X}^R$ satisfies the same type of equation, as $\sum_{\ell,m\in\mathbb{N}} b_{\ell,m} \bar{W}^{\ell,m}$ from (1.5) is a d-dimensional Wiener process with covariance $(\sum_{\ell,m\in\mathbb{N}} b^i_{\ell,m} b^j_{\ell,m})^d_{i,j=1}$, too. But, since this type of equation admits a unique strong solution, the laws of ξ and $\bar{X}^R$ must be the same, proving $\hat{X}^{\epsilon_n,R} \to \bar{X}^R$, in law, when n
$to\infty$. However, the same argument applies to any converging subsequence, and the limit will always be the same, finally proving $\hat{X}^{\epsilon,R} \to \bar{X}^R$, in law, when $\epsilon \to 0$. □

It remains to discuss how R can be taken to infinity. Recall that $\bar{X}$ is the solution of (1.5), and it is not difficult to see that $\bar{X}^R$ converges to $\bar{X}$, in law, as $R \to \infty$. Now take a function $\varphi_R \in C(C([0,T],\mathbb{R}^d),[0,1])$, such that $\varphi_R(u) = 0$, if $\sup_{t\in[0,T]} |u_t| \le R-1$, and $\varphi_R(u) = 1$, if $\sup_{t\in[0,T]} |u_t| > R$. Then,

$$\mathbb{P}\{\tau^\epsilon_R < T\} \le \mathbb{P}\left\{ \sup_{t\in[0,T]} |\hat{X}^{\epsilon,R}_t| \ge R \right\} \le \mathbb{E}\left[\varphi_R(\hat{X}^{\epsilon,R}) \right],$$

and because $\hat{X}^{\epsilon,R} \to \bar{X}^R$, in law, when $\epsilon \to 0$, we deduce that

$$\limsup_{\epsilon\to 0} \mathbb{P}\{\tau^\epsilon_R < T\} \le \mathbb{E}\left[\varphi_R(\bar{X}^R)\right] \le \mathbb{P}\left\{ \sup_{t\in[0,T]} |\bar{X}^R_t| \ge R-1 \right\} = \mathbb{P}\left\{ \sup_{t\in[0,T]} |\bar{X}_t| \ge R-1 \right\},$$

where the last probability converges to zero, when $R \to \infty$, because $\bar{X}$ is a global solution. As a consequence, for any $\psi \in C_b(C([0,T],\mathbb{R}^d),\mathbb{R})$,

$$\begin{aligned} \left|\mathbb{E}\left[\psi(X^\epsilon)\right] - \mathbb{E}\left[\psi(\bar{X})\right]\right| \le & \left|\mathbb{E}\left[\psi(X^\epsilon)\right] - \mathbb{E}\left[\psi(X^\epsilon_{\cdot\wedge\tau^\epsilon_R})\right]\right| + \left|\mathbb{E}\left[\psi(X^\epsilon_{\cdot\wedge\tau^\epsilon_R})\right] - \mathbb{E}\left[\psi(\hat{X}^{\epsilon,R}_{\cdot\wedge\tau^\epsilon_R})\right]\right| \\ & + \left|\mathbb{E}\left[\psi(\hat{X}^{\epsilon,R}_{\cdot\wedge\tau^\epsilon_R})\right] - \mathbb{E}\left[\psi(\hat{X}^{\epsilon,R})\right]\right| + \left|\mathbb{E}\left[\psi(\hat{X}^{\epsilon,R})\right] - \mathbb{E}\left[\psi(\bar{X}^R)\right]\right| \\ & + \left|\mathbb{E}\left[\psi(\bar{X}^R)\right] - \mathbb{E}\left[\psi(\bar{X})\right]\right|. \end{aligned}$$

Here, when taking R large enough, we can make all the summands on the right-hand side, except for the second and fourth, arbitrarily small, uniformly in ϵ, and, for fixed R, the remaining terms go to zero, when $\epsilon \to 0$. Thus, by a diagonal argument, the convergence in law of $X^\epsilon \to \bar{X}$, $\epsilon \to 0$, follows, completing the proof of the theorem.

2.3 Application to Climate Models

We now apply Theorem 1.1 to perform stochastic model reduction for a subclass of the stochastic climate models given by (1.4) in the Introduction: we restrict ourselves to a simpler version of the fast dynamics, omitting fast forcing $\epsilon^{-1/2}A^2_2 Y^\epsilon_t$ and $\epsilon^{-1}f^2$ and also neglecting the interactions $B^2_{12}(X^\epsilon_t, Y^\epsilon_t)$ and $B^2_{21}(Y^\epsilon_t, X^\epsilon_t)$.

For each $\epsilon > 0$, let (X^ϵ, Y^ϵ) be a pair of processes satisfying

$$\frac{dX_t^\epsilon}{dt} = F_t^1 + A_1^1 X_t^\epsilon + A_2^1 Y_t^\epsilon + B_{11}^1(X_t^\epsilon, X_t^\epsilon) + B_{12}^1(X_t^\epsilon, Y_t^\epsilon) + \epsilon^{1/2} B_{22}^1(Y_t^\epsilon, Y_t^\epsilon), \tag{2.16}$$

$$\frac{dY_t^\epsilon}{dt} = \epsilon^{-1} A_1^2 X_t^\epsilon + \epsilon^{-1} B_{11}^2(X_t^\epsilon, X_t^\epsilon) - \epsilon^{-1} Y_t^\epsilon + \epsilon^{-1}\dot{W}_t, \tag{2.17}$$

where $A_1^1 : H_d \to H_d$, $A_2^1 : H_\infty \to H_d$, $A_1^2 : H_d \to H_\infty$ are bounded linear operators, $B_{11}^1 : H_d \times H_d \to H_d$, $B_{12}^1 : H_d \times H_\infty \to H_d$, $B_{22}^1 : H_\infty \times H_\infty \to H_d$, $B_{11}^2 : H_d \times H_d \to H_\infty$ are continuous bilinear maps, and $F^1 : [0,T] \to H_d$ is a deterministic continuous external force. Notice that we have grouped together terms B_{12}^1 and B_{21}^1 without any loss of generality. Stochastic basis and Wiener process W are taken to be the same as in Remark 2.1.

In what follows, the above equations will always have initial conditions (x_0, y_0), where $x_0 \in H_d$ can be chosen arbitrarily, while $y_0 = \int_{-\infty}^0 \epsilon^{-1} e^{\epsilon^{-1}s} dW_s$ will be fixed to ensure pseudo stationarity of the scaled unresolved variables. Note that fixing $y_0 \in H_\infty$ this way would not restrict the initial data of the reduced equations.

In fluid dynamics settings it is customary to assume that A is self-adjoint, and that the full nonlinearity is skew-symmetric: $\langle B(z', z), z\rangle_H = 0$, $z, z' \in H$, see [MW06]. We therefore make the following assumptions on the projected coefficients:

(C1) $A_1^2 = (A_2^1)^*$;

(C2) $\langle B_{11}^1(x', x), x\rangle_{H_d} = 0$, for all $x, x' \in H_d$;

(C3) $\langle B_{12}^1(x', y), x\rangle_{H_d} = -\langle B_{11}^2(x', x), y\rangle_{H_\infty}$, for all $x, x' \in H_d$, $y \in H_\infty$.

Also, without loss of generality, we can assume that B_{22}^1 is symmetric in the sense of $\langle B_{22}^1(\mathbf{f}_\ell, \mathbf{f}_m), \mathbf{e}_i\rangle_{H_d} = \langle B_{22}^1(\mathbf{f}_m, \mathbf{f}_\ell), \mathbf{e}_i\rangle_{H_d}$, for all i, ℓ, m; and finally we will need the analogue assumption on β, that is

(C4) $\sum_{\ell\in\mathbb{N}} \langle B_{22}^1(\mathbf{f}_\ell, \mathbf{f}_\ell), \mathbf{e}_i\rangle_{H_d} q_\ell = 0$, for all $i = 1, \dots, d$.

Note that the latter condition is indeed satisfied for many fluid-dynamics models—it usually holds independently of the structure of the noise because $\langle B_{22}^1(\mathbf{f}_\ell, \mathbf{f}_m), \mathbf{e}_i\rangle_{H_d}$ would be zero on the diagonal, when $\ell = m$, for all i.

Next, we bring equations (2.16),(2.17) into a form which makes them comparable to (1.4). Using the definition of y_0, we have the following mild formulation of (2.17),

$$Y_t^\epsilon = \tilde{Y}_t^\epsilon + \int_0^t \epsilon^{-1} e^{-\epsilon^{-1}(t-s)} \left(A_1^2 X_s^\epsilon + B_{11}^2(X_s^\epsilon, X_s^\epsilon)\right) ds, \quad t \in [0,T], \tag{2.18}$$

where

$$\tilde{Y}_t^\epsilon = \int_{-\infty}^t \epsilon^{-1} e^{-\epsilon^{-1}(t-s)} dW_s, \quad t \in \mathbb{R},$$

is a stationary Ornstein-Uhlenbeck process. Plugging (2.18) into (2.16), X^ϵ alternatively satisfies for every $t \in [0,T]$

$$\begin{aligned} X_t^\epsilon = x_0 &+ \int_0^t \left(F_s^1 + A_1^1 X_s^\epsilon + B_{11}^1(X_s^\epsilon, X_s^\epsilon)\right) ds + \int_0^t A_2^1 Z_s^\epsilon ds + \int_0^t B_{12}^1(X_s^\epsilon, Z_s^\epsilon)\, ds \\ &+ \int_0^t A_2^1 \tilde{Y}_s^\epsilon ds + \int_0^t B_{12}^1(X_s^\epsilon, \tilde{Y}_s^\epsilon) ds \\ &+ \int_0^t \epsilon^{1/2} B_{22}^1(\tilde{Y}_s^\epsilon, \tilde{Y}_s^\epsilon) ds + 2\int_0^t \epsilon^{1/2} B_{22}^1(\tilde{Y}_s^\epsilon, Z_s^\epsilon) ds + \int_0^t \epsilon^{1/2} B_{22}^1(Z_s^\epsilon, Z_s^\epsilon)\, ds, \end{aligned} \tag{2.19}$$

when using the abbreviation

$$Z_s^\epsilon = \int_0^s \epsilon^{-1} e^{-\epsilon^{-1}(s-r)} \left(A_1^2 X_r^\epsilon + B_{11}^2(X_r^\epsilon, X_r^\epsilon)\right) dr.$$

Since Z_s^ϵ is close to $A_1^2 X_s^\epsilon + B_{11}^2(X_s^\epsilon, X_s^\epsilon)$, for small ϵ, and since both terms $B_{22}^1(\tilde{Y}_s^\epsilon, Z_s^\epsilon)$ and $B_{22}^1(Z_s^\epsilon, Z_s^\epsilon)$ will be shown to vanish with ϵ, too, the process X^ϵ should be close to $\tilde{X}^\epsilon$ satisfying

$$\begin{aligned}\tilde{X}_t^\epsilon = x_0 &+ \int_0^t \left(F_s^1 + A_1^1 \tilde{X}_s^\epsilon + B_{11}^1(\tilde{X}_s^\epsilon, \tilde{X}_s^\epsilon)\right) ds + \int_0^t A_2^1 \left(A_1^2 \tilde{X}_s^\epsilon + B_{11}^2(\tilde{X}_s^\epsilon, \tilde{X}_s^\epsilon)\right) ds \\ &+ \int_0^t B_{12}^1 \left(\tilde{X}_s^\epsilon, \left(A_1^2 \tilde{X}_s^\epsilon + B_{11}^2(\tilde{X}_s^\epsilon, \tilde{X}_s^\epsilon)\right)\right) ds \\ &+ \int_0^t A_2^1 \tilde{Y}_s^\epsilon ds + \int_0^t B_{12}^1(\tilde{X}_s^\epsilon, \tilde{Y}_s^\epsilon)\, ds + \int_0^t \epsilon^{1/2} B_{22}^1(\tilde{Y}_s^\epsilon, \tilde{Y}_s^\epsilon)\, ds, \end{aligned} \tag{2.20}$$

which is an equation of type (1.4) with

$$\begin{aligned} F(t,x) &= F_t^1 + A_1^1 x + B_{11}^1(x,x) + A_2^1\left(A_1^2 x + B_{11}^2(x,x)\right) + B_{12}^1\left(x, \left(A_1^2 x + B_{11}^2(x,x)\right)\right), \\ \sigma(t,x) &= A_2^1 + B_{12}^1(x,\cdot)\,, \\ \beta &= B_{22}^1\,. \end{aligned}$$

Thus, in this setting, the analogue of (1.5) would read

$$\begin{aligned}\bar{X}_t = x_0 &+ \int_0^t \left(F_s^1 + A_1^1 \bar{X}_s + B_{11}^1(\bar{X}_s, \bar{X}_s)\right) ds + \int_0^t A_2^1 \left(A_1^2 \bar{X}_s + B_{11}^2(\bar{X}_s, \bar{X}_s)\right) ds \\ &+ \int_0^t B_{12}^1 \left(\bar{X}_s, \left(A_1^2 \bar{X}_s + B_{11}^2(\bar{X}_s, \bar{X}_s)\right)\right) ds + \int_0^t C(\bar{X}_s)\, ds \\ &+ A_2^1 W_t + \int_0^t B_{12}^1(\bar{X}_s, dW_s) + \sum_{\ell,m\in\mathbb{N}} b_{\ell,m} \bar{W}_t^{\ell,m}, \end{aligned} \tag{2.21}$$

where the Stratonovich correction term $C : H_d \to H_d$ simplifies to

$$\langle C(x), \mathbf{e}_i \rangle_{H_d} = \frac{1}{2} \sum_{m\in\mathbb{N}} q_m \sum_{j=1}^d \langle B_{12}^1(\mathbf{e}_j, \mathbf{f}_m), \mathbf{e}_i \rangle_{H_d} \langle B_{12}^1(x, \mathbf{f}_m), \mathbf{e}_j \rangle_{H_d}, \quad i = 1, \ldots, d,$$

and

$$b_{\ell,m}^i = \langle B_{22}^1(\mathbf{f}_\ell, \mathbf{f}_m), \mathbf{e}_i \rangle_{H_d} \sqrt{\frac{q_\ell q_m}{2}}, \quad i = 1, \ldots, d,\ \ell, m \in \mathbb{N}.$$

Proposition 2.14. *When assuming (C1)-(C3), equation* (2.21) *admits a unique global strong solution on* $[0,T]$.

Proof. First, regularity of coefficients guarantees the existence of a unique local strong solution. Second, by Itô's formula,

$$\begin{aligned}\frac{1}{2}|\bar{X}_{t\wedge\tau}|^2 &= \frac{1}{2}|x_0|^2 + \int_0^{t\wedge\tau} \langle F_s^1 + A_1^1\bar{X}_s + B_{11}^1(\bar{X}_s,\bar{X}_s), \bar{X}_s\rangle\, ds \\ &+ \int_0^{t\wedge\tau} \langle A_2^1\left(A_1^2\bar{X}_s + B_{11}^2(\bar{X}_s,\bar{X}_s)\right), \bar{X}_s\rangle\, ds \\ &+ \int_0^{t\wedge\tau} \langle B_{12}^1\left(\bar{X}_s, \left(A_1^2\bar{X}_s + B_{11}^2(\bar{X}_s,\bar{X}_s)\right)\right), \bar{X}_s\rangle\, ds + \int_0^{t\wedge\tau}\langle C(\bar{X}_s),\bar{X}_s\rangle\, ds \\ &+ \int_0^{t\wedge\tau}\langle A_2^1 dW_s, \bar{X}_s\rangle + \int_0^{t\wedge\tau}\langle B_{12}^1(\bar{X}_s, dW_s),\bar{X}_s\rangle + \sum_{\ell,m\in\mathbb{N}}\int_0^{t\wedge\tau}\langle b_{\ell,m},\bar{X}_s\rangle\, d\bar{W}_s^{\ell,m} \\ &+ \frac{1}{2}\sum_{m\in\mathbb{N}}|A_2^1\mathbf{f}_m|^2 q_m(t\wedge\tau) + \frac{1}{2}\sum_{m\in\mathbb{N}}\int_0^{t\wedge\tau}|B_{12}^1(\bar{X}_s,\mathbf{f}_m)|^2 q_m\, ds + \frac{1}{2}\sum_{\ell,m\in\mathbb{N}}|b_{\ell,m}|^2(t\wedge\tau),\end{aligned}$$

for any fixed $t \in [0,T]$, and any stopping time τ smaller than a possible explosion time. Applying (C1)-(C3), we have the identities

$$\begin{aligned}\langle B_{11}^1(\bar{X}_s,\bar{X}_s),\bar{X}_s\rangle_{H_d} &= 0,\\ \langle A_2^1 B_{11}^2(\bar{X}_s,\bar{X}_s),\bar{X}_s\rangle_{H_d} &= \langle B_{11}^2(\bar{X}_s,\bar{X}_s), A_1^2\bar{X}_s\rangle_{H_\infty},\\ \langle B_{12}^1(\bar{X}_s, A_1^2\bar{X}_s),\bar{X}_s\rangle_{H_d} &= -\langle B_{11}^2(\bar{X}_s,\bar{X}_s), A_1^2\bar{X}_s\rangle_{H_\infty},\\ \langle B_{12}^1(\bar{X}_s, B_{11}^2(\bar{X}_s,\bar{X}_s)),\bar{X}_s\rangle_{H_d} &= -\|B_{11}^2(\bar{X}_s,\bar{X}_s)\|_{H_\infty}^2,\end{aligned}$$

leading to

$$\mathbb{E}\left[\sup_{t'\le t}|\bar{X}_{t'\wedge\tau}|^2\right] \lesssim \left(1 + \int_0^t \mathbb{E}\left[\sup_{s'\le s}|\bar{X}_{s'\wedge\tau}|^2\right] ds\right),$$

again using the regularity of the coefficients combined with Burkholder-Davis-Gundy's inequality. Thus, by Gronwall, the local solution $\bar{X}$ has to be global on $[0,T]$. □

Remark 2.6. In a very similar way, it can be shown that both equations (2.19) and (2.20) admit unique global strong solutions on $[0,T]$, too, and hence those proofs are omitted. As a consequence, simply substituting the solution of (2.19) into (2.18), for each $\epsilon > 0$, there is a unique pair of processes (X^ϵ, Y^ϵ) satisfying (2.16), (2.17) on $[0,T]$.

Theorem 2.15. *Assume (C1)-(C3), fix $\epsilon > 0$, and let (X^ϵ, Y^ϵ) be the unique pair of processes satisfying* (2.16),(2.17) *on a given climate time interval $[0,T]$. Then:*
i) If (C4), then X^ϵ converges in law, $\epsilon \to 0$, to the unique process $\bar{X}$ satisfying (2.21).
ii) However, if (C4) comes via $B_{22}^1 = 0$, then the strong convergence in probability holds true.

Proof. Recall the process $\tilde{X}^\epsilon$ satisfying (2.20), which is an equation of type (1.4) with coefficients F, σ, β satisfying the assumptions of Theorem 1.1. Furthermore, by Proposition 2.14 and Remark 2.6, global existence of solutions is satisfied, too, while assumptions on β descend from (C4).
All in all, Theorem 1.1 implies that both parts $i)$ and $ii)$ of Theorem 2.15 hold true when replacing X^ϵ by $\tilde{X}^\epsilon$. Thus, it is sufficient to prove convergence in probability of $X^\epsilon - \tilde{X}^\epsilon$ to zero, $\epsilon \to 0$, uniformly on compact subsets of a localising stochastic interval, which

can easily be shown following the lines of proof of Theorem 1.1. Indeed, by localization and discretization arguments, one would first derive for any $h = 1, \dots, [T/\Delta]$

$$\mathbb{E}\left[\sup_{\substack{k'=0,\dots,h \\ k'\Delta \le \tau_R^\epsilon}} \left| X_{k'\Delta}^\epsilon - \tilde{X}_{k'\Delta}^\epsilon \right|^2\right] \lesssim r(\Delta, \epsilon) + \sum_{k=0}^{h-1} \Delta \mathbb{E}\left[\sup_{\substack{k'=0,\dots,k \\ k'\Delta \le \tau_R^\epsilon}} \left| X_{k'\Delta}^\epsilon - \tilde{X}_{k'\Delta}^\epsilon \right|^2\right],$$

where $\tau_R^\epsilon = \inf\{t \ge 0 : |X_t^\epsilon| \ge R\} \wedge \inf\{t \ge 0 : |\tilde{X}_t^\epsilon| \ge R\}$, and $r(\Delta, \epsilon) \to 0$, $\epsilon \to 0$, for a suitable choice of $\Delta = \Delta_\epsilon$. Then, combining Gronwall's lemma and Markov's inequality, one would obtain

$$\lim_{\epsilon \to 0} \mathbb{P}\left\{\sup_{t \le T \wedge \tau_R^\epsilon} \|X_t^\epsilon - \tilde{X}_t^\epsilon\|_{H_d} > \delta\right\} = 0, \quad \forall\, \delta > 0,$$

which yields the convergences stated in parts $i)$ and $ii)$ of Theorem 2.15 up to time τ_R^ϵ. Since $\bar{X}$ is globally defined, both types of convergence can be extended to the whole interval $[0, T]$, using similar arguments given in the proof of the corresponding parts of Theorem 1.1. □

Chapter 3

From additive to transport noise in 2D fluids

In this chapter we intend to prove Theorem 1.2 on the convergence of the large-scale component of the system

$$\begin{cases} d\Xi^\epsilon_t + (u^\epsilon_t + v^\epsilon_t)\cdot\nabla\Xi^\epsilon_t dt = \nu\Delta\Xi^\epsilon_t dt + q^\epsilon_t dt,\\ d\xi^\epsilon_t + (u^\epsilon_t + v^\epsilon_t)\cdot\nabla\xi^\epsilon_t dt = -\epsilon^{-1}\xi^\epsilon_t dt + \epsilon^{-1}\sum_{k\in\mathbb{N}}\varsigma_k dW^k_t,\\ u^\epsilon_t = -\nabla^\perp(-\Delta)^{-1}\Xi^\epsilon_t,\\ v^\epsilon_t = -\nabla^\perp(-\Delta)^{-1}\xi^\epsilon_t. \end{cases} \tag{3.1}$$

towards the solution of

$$\begin{cases} d\Xi_t + u_t\cdot\nabla\Xi_t dt + \sum_{k\in\mathbb{N}}\sigma_k\cdot\nabla\Xi_t\circ dW^k_t = \nu\Delta\Xi_t dt + q_t dt,\\ u_t = -\nabla^\perp(-\Delta)^{-1}\Xi_t, \end{cases} \tag{3.2}$$

where $\sigma_k = -\nabla^\perp(-\Delta)^{-1}\varsigma_k$.
Let $(\tilde\Omega, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}})$ be an auxiliary probability space and let w be a standard $\mathbb{R}^2$-valued Wiener process defined on $(\tilde\Omega, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}})$. The strategy of the proof of Theorem 1.2 is based on the study of the stochastic characteristics

$$\phi^\epsilon_t(x) = x + \int_0^t u^\epsilon_s(\phi^\epsilon_s(x))ds + \int_0^t v^\epsilon_s(\phi^\epsilon_s(x))ds + \sqrt{2\nu}w_t, \tag{3.3}$$

$$\phi_t(x) = x + \int_0^t u_s(\phi_s(x))ds + \sum_{k\in\mathbb{N}}\int_0^t \sigma_k(\phi_s(x))\circ dW^k_s + \sqrt{2\nu}w_t, \tag{3.4}$$

for $t\in[0,T]$, $x\in\mathbb{T}^2$, and the representation formulae

$$\Xi^\epsilon_t = \tilde{\mathbb{E}}\left[\Xi_0\circ(\phi^\epsilon_t)^{-1} + \int_0^t q^\epsilon_s\circ\phi^\epsilon_s\circ(\phi^\epsilon_t)^{-1}ds\right], \tag{3.5}$$

$$\Xi_t = \tilde{\mathbb{E}}\left[\Xi_0\circ(\phi_t)^{-1} + \int_0^t q_s\circ\phi_s\circ(\phi_t)^{-1}ds\right], \tag{3.6}$$

where $\tilde{\mathbb{E}}$ is the expectation on $\tilde\Omega$ with respect to $\tilde{\mathbb{P}}$.
We adopt an abstract point of view on the problem, dropping the assumption that the large-scale velocity u^ϵ is generated by Ξ^ϵ via the Biot-Savart law. We highlight very

minimal conditions on u, u^ϵ and other quantities involved in (3.1),(3.2) allowing us to prove the desired convergence, see assumptions (A1)-(A7) below. In this way we are able to include at once Navier-Stokes, Euler and passive scalar equations at large scales into the scope of applicability of our Theorem 1.2.
With respect to other works on Wong-Zakai approximation results, [FP22] here discussed is the first work where the velocity field approximating the white noise one is the solution of a nonlinear fluid mechanics equation.
The present chapter is organized as follows.
In Section 3.1 we introduce some notation and recall classical results, among others: main properties of the Biot-Savart kernel on the torus; a useful Gronwall-type lemma for ODEs with log-Lipschitz drift; notions of solution and well-posedness results for stochastic Euler equations (unresolved component of (3.1)), equations of characteristics (3.3) and (3.4), and large-scale dynamics in (3.1) and (3.2). Also, here we introduce our main working assumptions (A1)-(A7), and in the last part of this section we state our main result on convergence of characteristics Theorem 3.1.1.
In the first part of Section 3.2, we define a linearized version of (the second component of) (3.1), where we neglect the nonlinear term. This approach is similar to that of [FP21], and the key idea is that, although the solution θ^ϵ of linearized equation is not close to the actual solution ξ^ϵ of (3.1), the characteristics generated by θ^ϵ are close to the characteristics generated by ξ^ϵ, in particular they have the same limit as $\epsilon \to 0$.
In the same section we present two main technical results, needed in the proof of Theorem 3.1.1. The first of those results is Proposition 3.6, which ensures that the linear part θ^ϵ of the small-scale dynamics behaves as a Stratonovich white-in-time noise as $\epsilon \to 0$, at least in a distributional sense. The second result Proposition 3.7, instead, aims to rigorously prove the closeness of the characteristics generated by θ^ϵ and ξ^ϵ, and it is one of the main novelties of [FP22] with respect to [FP21].
The proof of Theorem 3.1.1 is contained in Section 3.3, and it is based on a Gronwall-type lemma and Itō Formula applied to a smooth approximation $g_\delta(x)$ of the absolute value $|x|$, $x \in \mathbb{R}^2$. The proof of Theorem 1.2 can be found in Section 3.4, and it relies on representation formulae (3.5) and (3.6) and a measure-theoretic argument.
Finally, in Section 3.5 we discuss how our main motivational examples fit our abstract setting. In particular, the non-trivial one is the coupled system given by deterministic Navier-Stokes equations at large scales plus stochastic Euler equations at small scales; we identify an additional but very natural condition (A8) on the limit external source q that allows to verify assumptions (A1)-(A7) for the system under consideration.

3.1 Notation and preliminaries

3.1.1 Properties of the Biot-Savart kernel

Here we briefly recall some useful properties of the Biot-Savart kernel K. We refer to [MP94, BFM16] for details and proofs.
First of all, the Biot-Savart kernel K is defined as $K := -\nabla^\perp G = (\partial_2 G, -\partial_1 G)$, where G is the Green function of the Laplace operator on the torus $\mathbb{T}^2$ with zero mean.
For $p \in (1, \infty)$ and $\xi \in L^p(\mathbb{T}^2)$ with zero-mean, the convolution with K represents the Biot-Savart operator:

$$K * \xi = -\nabla^\perp(-\Delta)^{-1}\xi,$$

that to every zero-mean $\xi \in L^p(\mathbb{T}^2)$ associates the unique zero-mean, divergence-free velocity vector field $u \in W^{1,p}(\mathbb{T}^2, \mathbb{R}^2)$ such that $\operatorname{curl} u = \xi$. Moreover, for every $p \in (1, \infty)$ there exist constants c, C such that for every zero-mean $\xi \in L^p(\mathbb{T}^2)$

$$c\|\xi\|_{L^p(\mathbb{T}^2)} \leq \|K * \xi\|_{W^{1,p}(\mathbb{T}^2,\mathbb{R}^2)} \leq C\|\xi\|_{L^p(\mathbb{T}^2)}.$$

Also, recall that since $K \in L^1(\mathbb{T}^2, \mathbb{R}^2)$ the convolution $K * \xi$ is well-defined for every $\xi \in L^p(\mathbb{T}^2)$, $p \in [1, \infty]$ and the following estimate holds:

$$\|K * \xi\|_{L^p(\mathbb{T}^2,\mathbb{R}^2)} \leq \|K\|_{L^1(\mathbb{T}^2,\mathbb{R}^2)}\|\xi\|_{L^p(\mathbb{T}^2)}. \tag{3.7}$$

Let $r \geq 0$. Denote $\gamma : [0, \infty) \to \mathbb{R}$ the concave function:

$$\gamma(r) = r(1 - \log r)\mathbf{1}_{\{0<r<1/e\}} + (r + 1/e)\mathbf{1}_{\{r\geq 1/e\}}.$$

The following two lemmas are proved in [MP94] and [BFM16].

Lemma 3.1. *There exists a constant C such that:*

$$\int_{\mathbb{T}^2} |K(x - y) - K(x' - y)|\, dy \leq C\gamma(|x - x'|)$$

for every $x, x' \in \mathbb{T}^2$.

Lemma 3.2. *Let $T > 0$, $\lambda > 0$, $a_0 \in [0, \exp(1 - 2e^{\lambda T})]$ be constants. Let $a : [0, T] \to \mathbb{R}$ be such that for every $t \in [0, T]$:*

$$a_t \leq a_0 + \lambda \int_0^t \gamma(a_s)ds.$$

Then for every $t \in [0, T]$ the following estimate holds:

$$a_t \leq e a_0^{\exp(-\lambda t)}.$$

3.1.2 Stochastic flows of measure-preserving homeomorphisms

As a convention, in the following we say that $\mathcal{N} \subset \Omega$ (respectively $\tilde{\mathcal{N}} \subset \tilde{\Omega}$) is *negligible* if it is measurable and $\mathbb{P}(\mathcal{N}) = 0$ (respectively $\tilde{\mathbb{P}}(\tilde{\mathcal{N}}) = 0$), without explicit mention of the reference probability measure. Unless otherwise specified, we will always denote with $\mathcal{N}$ negligible sets in Ω, and with $\tilde{\mathcal{N}}$ negligible sets in $\tilde{\Omega}$.
Let us begin this paragraph with the following fundamental definition.

Definition 3.1. A measurable map $\phi : \Omega \times \tilde{\Omega} \times [0, T] \times \mathbb{T}^2 \to \mathbb{T}^2$ is a *stochastic flow of measure-preserving homeomorphisms* provided there exist negligible sets $\mathcal{N} \subset \Omega$ and $\tilde{\mathcal{N}} \subset \tilde{\Omega}$ such that:

- for every $\omega \in \mathcal{N}^c$, $\tilde{\omega} \in \tilde{\mathcal{N}}^c$ and $t \in [0, T]$, the map $\phi(\omega, \tilde{\omega}, t, \cdot) : \mathbb{T}^2 \to \mathbb{T}^2$ is a homeomorphism of the torus and

$$\int_{\mathbb{T}^2} f(x)dx = \int_{\mathbb{T}^2} f(\phi(\omega, \tilde{\omega}, t, y))dy$$

  for every $f \in L^1(\mathbb{T}^2)$;

- for every $\tilde{\omega} \in \tilde{\mathcal{N}}^c$ and $x \in \mathbb{T}^2$, the stochastic process $\phi(\cdot, \tilde{\omega}, \cdot, x) : \Omega \times [0,T] \to \mathbb{T}^2$ is progressively measurable with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$.

In some circumstances it can be useful to have the following:

Definition 3.2. A stochastic flow of measure-preserving homeomorphisms ϕ is called *inviscid* if there exist negligible sets $\mathcal{N} \subset \Omega$ and $\tilde{\mathcal{N}} \subset \tilde{\Omega}$, and a measurable map $\psi : \Omega \times [0,T] \times \mathbb{T}^2 \to \mathbb{T}^2$ such that for every $\omega \in \mathcal{N}^c$, $\tilde{\omega} \in \tilde{\mathcal{N}}^c$, $t \in [0,T]$ and $x \in \mathbb{T}^2$

$$\phi(\omega, \tilde{\omega}, t, x) = \psi(\omega, t, x).$$

With a little abuse of notation, hereafter we identify an inviscid stochastic flow of measure-preserving homeomorphisms ϕ with its $\tilde{\omega}$-independent representative ψ.
Let us now clarify the meaning of (3.3), (3.4).
A measurable map $\phi^\epsilon : \Omega \times \tilde{\Omega} \times [0,T] \times \mathbb{T}^2 \to \mathbb{T}^2$ is a solution of (3.3) if there exist negligible sets $\mathcal{N} \subset \Omega$ and $\tilde{\mathcal{N}} \subset \tilde{\Omega}$ such that for every $\omega \in \mathcal{N}^c$, $\tilde{\omega} \in \tilde{\mathcal{N}}^c$, $t \in [0,T]$ and $x \in \mathbb{T}^2$:

$$\begin{aligned}\phi^\epsilon(\omega, \tilde{\omega}, t, x) = x &+ \int_0^t u^\epsilon(\omega, s, \phi^\epsilon(\omega, \tilde{\omega}, s, x)) ds \\ &+ \int_0^t v^\epsilon(\omega, s, \phi^\epsilon(\omega, \tilde{\omega}, s, x)) ds + \sqrt{2\nu} w(\tilde{\omega}, t),\end{aligned}$$

where the previous identity can be interpreted as an equation on $\mathbb{T}^2$ since one can check $\phi^\epsilon(\omega, \tilde{\omega}, t, x + 2\pi\mathbf{e}) = \phi^\epsilon(\omega, \tilde{\omega}, t, x) + 2\pi\mathbf{e}$ for $\mathbf{e} = (1,0)$ and $\mathbf{e} = (0,1)$.
Similarly, a measurable map $\phi : \Omega \times \tilde{\Omega} \times [0,T] \times \mathbb{T}^2 \to \mathbb{T}^2$ is a solution of (3.4) if there exist negligible sets $\mathcal{N} \subset \Omega$ and $\tilde{\mathcal{N}} \subset \tilde{\Omega}$ such that for every $\tilde{\omega} \in \tilde{\mathcal{N}}^c$ and $x \in \mathbb{T}^2$, the stochastic process $\phi(\cdot, \tilde{\omega}, \cdot, x) : \Omega \times [0,T] \to \mathbb{T}^2$ is progressively measurable with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, and for every $\omega \in \mathcal{N}^c$, $\tilde{\omega} \in \tilde{\mathcal{N}}^c$, $t \in [0,T]$ and $x \in \mathbb{T}^2$:

$$\begin{aligned}\phi(\omega, \tilde{\omega}, t, x) = x &+ \int_0^t u(\omega, s, \phi(\omega, \tilde{\omega}, s, x)) ds \\ &+ \sum_{k\in\mathbb{N}} \left(\int_0^t \sigma_k(\phi(\cdot, \tilde{\omega}, s, x)) \circ dW^k_s \right)(\omega) + \sqrt{2\nu} w(\tilde{\omega}, t).\end{aligned}$$

Notice that progressive measurability of the process $\phi(\cdot, \tilde{\omega}, \cdot, x) : \Omega \times [0,T] \to \mathbb{T}^2$ is necessary to make sense of the Stratonovich stochastic integral appearing in the equation above.

3.1.3 Notions of solution and some well-posedness results

The aim of the present subsection is twofold. On the one hand, we provide a suitable notion of solution for (3.1), in some sense highlighting the minimal requirements on the solutions to prove our results. On the other hand, we show the existence of solutions in the general case, as well as uniqueness in the case of the large-scale process being a passive scalar. In the following, we say that a field u^ϵ is *compatible* with the large-scale process Ξ^ϵ if: either Ξ^ϵ is a passive scalar, or: Ξ^ϵ is an active scalar and u^ϵ is reconstructed from the latter by the Biot–Savart law. We adopt a similar terminology for the limiting quantities u, Ξ. In this subsection we make assumptions directly on the fields u^ϵ, u. we shall see in Section 3.5 that, even for active scalars, fields compatible with large-scale processes satisfy our assumptions.

Well-posedness of small-scale dynamics and characteristics

First we make the following assumptions on the external fields:

(A1) $u^\epsilon, u : \Omega \times [0,T] \times \mathbb{T}^2 \to \mathbb{R}^2$ and for every $t \in [0,T]$ the maps $u^\epsilon, u|_{\Omega\times[0,t]} : \Omega \times [0,t] \times \mathbb{T}^2 \to \mathbb{R}^2$ are $\mathcal{F}_t \otimes \mathcal{B}_{[0,t]} \otimes \mathcal{B}_{\mathbb{T}^2}$ measurable, where $\mathcal{B}$ denotes the Borel sigma-field;

(A2) there exist a constant C and a negligible set $\mathcal{N} \subset \Omega$ such that, for every $\omega \in \mathcal{N}^c$, $\epsilon > 0$ and $t \in [0,T]$: $\operatorname{div} u^\epsilon(\omega,t,\cdot) = \operatorname{div} u(\omega,t,\cdot) = 0$, and

$$|u^\epsilon(\omega,t,x)| \le C, \quad |u^\epsilon(\omega,t,x) - u^\epsilon(\omega,t,y)| \le C\gamma(|x-y|),$$
$$|u(\omega,t,x)| \le C, \quad |u(\omega,t,x) - u(\omega,t,y)| \le C\gamma(|x-y|),$$

for every $x, y \in \mathbb{T}^2$.

Also, we make the following assumption on the coefficients $(\varsigma_k)_{k\in\mathbb{N}}$:

(A3) there exists $\ell \ge 1$ such that $\varsigma_k \in W^{\ell,\infty}(\mathbb{T}^2)$ with zero-mean for every $k \in \mathbb{N}$, and moreover

$$\sum_{k\in\mathbb{N}} \|\varsigma_k\|_{W^{\ell,\infty}(\mathbb{T}^2)} < \infty.$$

Given a stochastic flow of measure-preserving homeomorphisms ϕ we will use $\phi_t(x)$ as a notational shortcut for $\phi(\omega,\tilde\omega,t,x)$, thus making implicit the dependence of the randomness variables $\omega,\tilde\omega$. The same convention may be used for the fields u, v, *et cetera*.
The next result can be proved repeating the arguments contained in [BFM16] and [FP21].

Proposition 3.3. *Assume (A1)-(A3). Then:*

- *for every $\epsilon > 0$ there exist a unique Lagrangian solution ξ^ϵ of (3.1), namely there exists a unique stochastic process $\xi^\epsilon : \Omega \times [0,T] \to L^\infty(\mathbb{T}^2)$ weakly progressively measurable with respect to $(\mathcal{F}_t)_{t\in[0,T]}$ such that the equation*

$$\psi^\epsilon_t(x) = x + \int_0^t u^\epsilon_s(\psi^\epsilon_s(x))ds + \int_0^t v^\epsilon_s(\psi^\epsilon_s(x))ds,$$

  with $v^\epsilon = K * \xi^\epsilon$, admits a unique inviscid stochastic flow of measure-preserving homeomorphisms ψ^ϵ as a solution, and moreover

$$\xi^\epsilon_t(\psi^\epsilon_t(x)) = \epsilon^{-1} \sum_{k\in\mathbb{N}} \int_0^t e^{-\epsilon^{-1}(t-s)} \varsigma_k(\psi^\epsilon_s(x)) dW^k_s; \tag{3.8}$$

- *for every $\epsilon > 0$ there exists a unique stochastic flow of measure-preserving homeomorphisms ϕ^ϵ solution of (3.3), with $v^\epsilon = K * \xi^\epsilon$;*

- *there exists a unique stochastic flow of measure-preserving homeomorphisms ϕ solution of (3.4).*

Remark 3.1. If $\nu = 0$, then both ϕ^ϵ and ϕ are inviscid stochastic flows of measure-preserving homeomorphisms, and actually $\phi^\epsilon = \psi^\epsilon$. The terminology is thus justified, since $\nu = 0$ corresponds to null diffusivity/viscosity in the equations for the large-scale dynamics in (3.1) and (3.2).

Remark 3.2. Formula (3.8) above corresponds to the solution of (3.1) with initial condition $\xi_0^\epsilon = 0$, that we assume throughout this paper for the sake of simplicity. More general initial conditions, as those considered in [FP21], can be taken into account by simply modifying (3.8) into

$$\xi_t^\epsilon(\psi_t^\epsilon(x)) = e^{-\epsilon^{-1}t}\xi_0^\epsilon(x) + \epsilon^{-1}\sum_{k\in\mathbb{N}}\int_0^t e^{-\epsilon^{-1}(t-s)}\varsigma_k(\psi_s^\epsilon(x))dW_s^k.$$

Notion of solution to the large-scale dynamics

By previous Proposition 3.3, under assumption (A1)-(A3) we can use the Euler flow to represent the large-scale solutions of (3.1) and (3.2). To be more precise, our notion of solution is given exactly by those processes Ξ^ϵ, Ξ for which (3.5) and (3.6) hold true, and it is inspired by the notion of generalized solution in [BF95, Definition 2.2].

Definition 3.3. Assume (A1)-(A3), $q^\epsilon, q \in L^1([0,T], L^\infty(\mathbb{T}^2))$ for every $\epsilon > 0$ and $\Xi_0 \in L^\infty(\mathbb{T}^2)$ with zero mean. Then:

- for every $\epsilon > 0$, a measurable map $\Xi^\epsilon : \Omega \times [0,T] \times \mathbb{T}^2 \to \mathbb{R}$ is called *generalized solution* to (the first component of) (3.1) if it is compatible with u^ϵ and for every $t \in [0,T]$ it holds

$$\Xi_t^\epsilon = \tilde{\mathbb{E}}\left[\Xi_0 \circ (\phi_t^\epsilon)^{-1} + \int_0^t q_s^\epsilon \circ \phi_s^\epsilon \circ (\phi_t^\epsilon)^{-1} ds\right],$$

  as an equality in $L^\infty(\Omega \times \mathbb{T}^2)$, where ϕ^ϵ is the unique stochastic flow of measure-preserving homeomorphisms solution of (3.3);

- a measurable map $\Xi : \Omega \times [0,T] \times \mathbb{T}^2 \to \mathbb{R}$ is called *generalized solution* to (3.2) if it is compatible with u and for every $t \in [0,T]$ it holds

$$\Xi_t = \tilde{\mathbb{E}}\left[\Xi_0 \circ (\phi_t)^{-1} + \int_0^t q_s \circ \phi_s \circ (\phi_t)^{-1} ds\right],$$

  as an equality in $L^\infty(\Omega \times \mathbb{T}^2)$, where ϕ is the unique stochastic flow of measure-preserving homeomorphisms solution of (3.4).

Notice that this notion of solution immediately implies existence and uniqueness in the case of passive large-scale dynamics: we can state that in the following

Proposition 3.4. *Under the same assumptions as above, suppose Ξ^ϵ (resp. Ξ) are passive scalars. Then the exists a unique generalized solution to (3.1) (resp. (3.2)).*

Proof. Indeed, for passive scalars the compatibility condition is void, and Ξ^ϵ (resp. Ξ) depends only on the initial datum Ξ_0, the external sources q^ϵ (resp. q), and the characteristics ϕ^ϵ (resp. ϕ), the latter existing and being unique by Proposition 3.3. □

For active dynamics the previous picture is not correct, since the compatibility condition between the external field and the large-scale variable is not encoded in the representation formula itself. However, we will not investigate in this paper well-posedness for this notion of solution in full generality. For active scalars, we limit ourselves to show existence of generalized solutions, see Proposition 3.5 below.

Also, it is worth of mention that every sufficiently smooth generalized solution of the first component of (3.1) or (3.2) is also a classical solution, as can be proved following the lines of [CI08, Theorem 2.2 and Proposition 2.7]. On the other hand, our notion of generalized solution is weaker than the notion of L^∞-weak solution contained in [BFM16], that we recall now:

Definition 3.4. Assume (A1)-(A3), $q^\epsilon, q \in L^1([0,T], L^\infty(\mathbb{T}^2))$ for every $\epsilon > 0$ and $\Xi_0 \in L^\infty(\mathbb{T}^2)$ with zero mean. For $f, g : \mathbb{T}^2 \to \mathbb{R}$, denote $\langle f, g\rangle := \int_{\mathbb{T}^2} f(x)g(x)dx$. Then:

- for every $\epsilon > 0$, a stochastic process $\Xi^\epsilon : \Omega \times [0,T] \to L^\infty(\mathbb{T}^2)$ is called a L^∞-*weak solution* of (3.1) if it is weakly progressively measurable with respect to $(\mathcal{F}_t)_{t\in[0,T]}$, it is compatible with u^ϵ and for every smooth test function $f \in C^\infty(\mathbb{T}^2)$ it holds $\mathbb{P}$-a.s. for every $t \in [0,T]$:

$$\begin{aligned}\langle \Xi^\epsilon_t, f\rangle - \langle \Xi^\epsilon_0, f\rangle &= \int_0^t \langle \Xi^\epsilon_s, (u^\epsilon_s + v^\epsilon_s)\cdot\nabla f\rangle ds\\ &\quad + \int_0^t \langle \Xi^\epsilon_s, \nu\Delta f\rangle ds + \int_0^t \langle q^\epsilon_s, f\rangle ds;\end{aligned}$$

- a stochastic process $\Xi : \Omega \times [0,T] \to L^\infty(\mathbb{T}^2)$ is called a L^∞-*weak solution* of (3.2) if it is weakly progressively measurable with respect to $(\mathcal{F}_t)_{t\in[0,T]}$, it is compatible with u and for every smooth test function $f \in C^\infty(\mathbb{T}^2)$ it holds $\mathbb{P}$-a.s. for every $t \in [0,T]$:

$$\begin{aligned}\langle \Xi_t, f\rangle - \langle \Xi_0, f\rangle &= \int_0^t \langle \Xi_s, u_s\cdot\nabla f\rangle ds + \sum_{k\in\mathbb{N}} \int_0^t \langle \Xi_s, \sigma_k\cdot\nabla f\rangle \circ dW^k_s\\ &\quad + \int_0^t \langle \Xi_s, \nu\Delta f\rangle ds + \int_0^t \langle q_s, f\rangle ds.\end{aligned}$$

In [BFM16] well-posedness of L^∞-weak solution to stochastic Euler Equations is shown. With minor modifications in the argument one can prove existence of L^∞-weak solutions to (3.1) and (3.2) in the general case. For active scalars, those provide generalized solutions in the sense of Definition 3.3, that is the content of the following:

Proposition 3.5. *Assume (A1)-(A3), $q^\epsilon, q \in L^1([0,T], L^\infty(\mathbb{T}^2))$ for every $\epsilon > 0$ and $\Xi_0 \in L^\infty(\mathbb{T}^2)$. Then every L^∞-weak solution to the second component of (3.1) is also a generalized solution, and every L^∞-weak solution to (3.2) is also a generalized solution.*

Proof. The strategy of the proof is similar to [BFM16, Proposition 5.3] and [FGP10, Theorem 20], and consists in taking the convolution of a L^∞-weak solution with a smooth mollifier $\vartheta_\delta = \delta^{-2}\vartheta(\delta\cdot)$, $\delta > 0$, and then taking the limit for $\delta \to 0$.
Let Ξ^ϵ be a L^∞-weak solution of (3.1) and Ξ be a L^∞-weak solution of (3.2), in the sense of the previous definition. Using $f = \vartheta_\delta(y - \cdot)$ as a test function, $y \in \mathbb{T}^2$, and denoting $\Xi^\epsilon_\delta := \vartheta_\delta * \Xi^\epsilon$, $\Xi_\delta := \vartheta_\delta * \Xi$ we get (omitting the parameter ω)

$$\begin{aligned}\Xi^\epsilon_\delta(t,y) - \Xi^\epsilon_\delta(0,y) &= \int_0^t \int_{\mathbb{T}^2} \Xi^\epsilon(s,x)(u^\epsilon(s,x) + v^\epsilon(s,x))\cdot\nabla_x\vartheta_\delta(y-x)dxds\\ &\quad + \nu\int_0^t \int_{\mathbb{T}^2} \Xi^\epsilon(s,x)\Delta_x\vartheta_\delta(y-x)dxds\\ &\quad + \int_0^t \int_{\mathbb{T}^2} q^\epsilon(s,x)\vartheta_\delta(y-x)dxds,\end{aligned}$$

and

$$
\begin{aligned}
\Xi_\delta(t,y)-\Xi_\delta(0,y) &= \int_0^t\int_{\mathbb{T}^2}\Xi(s,x)u(s,x)\cdot\nabla_x\vartheta_\delta(y-x)dxds \\
&+\sum_{k\in\mathbb{N}}\int_0^t\int_{\mathbb{T}^2}\Xi(s,x)\sigma_k(x)\cdot\nabla_x\vartheta_\delta(y-x)dx\circ dW^k_s \\
&+\nu\int_0^t\int_{\mathbb{T}^2}\Xi(s,x)\Delta_x\vartheta_\delta(y-x)dxds \\
&+\int_0^t\int_{\mathbb{T}^2}q(s,x)\vartheta_\delta(y-x)dxds.
\end{aligned}
$$

Since Ξ^ϵ_δ, Ξ_δ are smooth functions in the variable y, we can write the equivalent expressions in differential notation

$$
\begin{aligned}
&d\Xi^\epsilon_\delta(t,y)+\nabla\Xi^\epsilon_\delta(t,y)\cdot(u^\epsilon(t,y)+v^\epsilon(t,y))dt \\
&\quad=\int_{\mathbb{T}^2}\Xi^\epsilon(t,x)(u^\epsilon(t,x)+v^\epsilon(t,x))\cdot\nabla_x\vartheta_\delta(y-x)dxdt \\
&\qquad+\nu\int_{\mathbb{T}^2}\Xi^\epsilon(t,x)\Delta_x\vartheta_\delta(y-x)dxdt+\int_{\mathbb{T}^2}q^\epsilon(t,x)\vartheta_\delta(y-x)dxdt \\
&\qquad+\nabla\Xi^\epsilon_\delta(t,y)\cdot(u^\epsilon(t,y)+v^\epsilon(t,y))dt,
\end{aligned}
$$

and

$$
\begin{aligned}
&d\Xi_\delta(t,y)+\nabla\Xi_\delta(t,y)\cdot u(t,y)dt+\sum_{k\in\mathbb{N}}\nabla\Xi_\delta(t,y)\cdot\sigma_k(y)\circ dW^k_t \\
&\quad=\int_{\mathbb{T}^2}\Xi(t,x)u(t,x)\cdot\nabla_x\vartheta_\delta(y-x)dxdt \\
&\qquad+\sum_{k\in\mathbb{N}}\int_{\mathbb{T}^2}\Xi(t,x)\sigma_k(x)\cdot\nabla_x\vartheta_\delta(y-x)dx\circ dW^k_t \\
&\qquad+\nu\int_{\mathbb{T}^2}\Xi(t,x)\Delta_x\vartheta_\delta(y-x)dxdt+\int_{\mathbb{T}^2}q(t,x)\vartheta_\delta(y-x)dxdt \\
&\qquad+\nabla\Xi_\delta(t,y)\cdot u(t,y)dt+\sum_{k\in\mathbb{N}}\nabla\Xi_\delta(t,y)\cdot\sigma_k(y)\circ dW^k_t.
\end{aligned}
$$

Notice that the following formulas for the gradient of the convolution hold true: $\nabla\Xi^\epsilon_\delta(t,y)=-\int_{\mathbb{T}^2}\Xi^\epsilon(t,x)\nabla_x\vartheta_\delta(y-x)$, and $\nabla\Xi_\delta(t,y)=-\int_{\mathbb{T}^2}\Xi(t,x)\nabla_x\vartheta_\delta(y-x)$; also, $\Delta_x\vartheta_\delta(y-x)=\Delta_y\vartheta_\delta(y-x)$. Substituting into the previous expressions, we get

$$
\begin{aligned}
&d\Xi^\epsilon_\delta(t,y)+\nabla\Xi^\epsilon_\delta(t,y)\cdot(u^\epsilon(t,y)+v^\epsilon(t,y))dt \\
&\quad=\left[-\vartheta_\delta*(\nabla\Xi^\epsilon_t\cdot(u^\epsilon_t+v^\epsilon_t))+(u^\epsilon_t+v^\epsilon_t)\cdot(\vartheta_\delta*\nabla\Xi^\epsilon_t)\right](y)dt \\
&\qquad+\nu\Delta\Xi^\epsilon_\delta(t,y)dt+q^\epsilon_\delta(t,y)dt \\
&\quad=R_\delta\left[u^\epsilon_t+v^\epsilon_t,\Xi^\epsilon_t\right](y)dt+\nu\Delta\Xi^\epsilon_\delta(t,y)dt+q^\epsilon_\delta(t,y)dt,
\end{aligned}
$$

and

$$\begin{aligned}
d\Xi_\delta(t,y) &+ \nabla\Xi_\delta(t,y)\cdot u(t,y)dt + \sum_{k\in\mathbb{N}} \nabla\Xi_\delta(t,y)\cdot\sigma_k(y)\circ dW_t^k \\
&= \left[-\vartheta_\delta * (\nabla\Xi_t\cdot u_t) + u_t\cdot(\vartheta_\delta * \nabla\Xi_t)\right](y)dt \\
&\quad + \sum_{k\in\mathbb{N}} \left[-\vartheta_\delta * (\nabla\Xi_t\cdot\sigma_k) + \sigma_k\cdot(\vartheta_\delta * \nabla\Xi_t)\right](y)\circ dW_t^k \\
&\quad + \nu\Delta\Xi_\delta(t,y)dt + q_\delta(t,y)dt \\
&= R_\delta\left[u_t,\Xi_t\right](y)dt + \sum_{k\in\mathbb{N}} R_\delta\left[\sigma_k,\Xi_t\right](y)\circ dW_t^k \\
&\quad + \nu\Delta\Xi_\delta(t,y)dt + q_\delta(t,y)dt,
\end{aligned}$$

where we have defined $q_\delta^\epsilon \coloneqq \vartheta_\delta * q^\epsilon$, $q_\delta \coloneqq \vartheta_\delta * q$ and the commutator

$$R_\delta\left[v,\Xi\right] \coloneqq -\vartheta_\delta * (\nabla\Xi\cdot v) + v\cdot(\vartheta_\delta * \nabla\Xi).$$

We have obtained differential equations for the spatially smooth processes Ξ_δ^ϵ and Ξ_δ. Applying the backwards Itō Formula to the processes $s\mapsto\Xi_\delta^\epsilon(s,\phi_s^\epsilon((\phi_t^\epsilon)^{-1}(y)))$ and $s\mapsto\Xi_\delta(s,\phi_s((\phi_t)^{-1}(y)))$, for fixed $t\in[0,T]$, and taking the expectation with respect to $\tilde{\mathbb{P}}$, we obtain that the process Ξ_δ^ϵ is given by

$$\begin{aligned}
\Xi_\delta^\epsilon(t,y) &= \tilde{\mathbb{E}}\left[\Xi_\delta^\epsilon(0,(\phi_t^\epsilon)^{-1}(y)) + \int_0^t q_\delta^\epsilon(s,\phi_s^\epsilon((\phi_t^\epsilon)^{-1}(y)))ds\right] \\
&\quad + \tilde{\mathbb{E}}\left[\int_0^t R_\delta\left[u_s^\epsilon + v_s^\epsilon,\Xi_s^\epsilon\right](\phi_s^\epsilon((\phi_t^\epsilon)^{-1}(y)))ds\right],
\end{aligned} \tag{3.9}$$

whereas the process Ξ_δ is given by

$$\begin{aligned}
\Xi_\delta(t,y) &= \tilde{\mathbb{E}}\left[\Xi_\delta(0,(\phi_t)^{-1}(y)) + \int_0^t q_\delta(s,\phi_s((\phi_t)^{-1}(y)))ds\right] \\
&\quad + \tilde{\mathbb{E}}\left[\int_0^t R_\delta\left[u_s,\Xi_s\right](\phi_s((\phi_t)^{-1}(y)))ds\right] \\
&\quad + \sum_{k\in\mathbb{N}}\tilde{\mathbb{E}}\left[\int_0^t R_\delta\left[\sigma_k,\Xi_s\right](\phi_s((\phi_t)^{-1}(y)))\circ dW_s^k\right]
\end{aligned} \tag{3.10}$$

Let us focus on (3.9). By well-known properties of mollifiers, for every fixed $\omega\in\Omega$ and $t\in[0,T]$, the right-hand side $\Xi_\delta^\epsilon(\omega,t,\cdot)\to\Xi^\epsilon(\omega,t,\cdot)$ in $L^1(\mathbb{T}^2)$ as $\delta\to 0$. Concerning the left-hand side, a commutator lemma [FGP10, Lemma 17] yields for every fixed $\epsilon>0$

$$\lim_{\delta\to 0}\int_{\mathbb{T}^2}\left|\tilde{\mathbb{E}}\left[\int_0^t R_\delta\left[u_s^\epsilon+v_s^\epsilon,\Xi_s^\epsilon\right](\phi_s^\epsilon((\phi_t^\epsilon)^{-1}(y)))ds\right]\right|dy = 0,$$

and by well-known properties of mollifiers and Lebesgue dominated convergence Theorem we can prove the convergence

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\Xi_\delta^\epsilon(0,(\phi_t^\epsilon)^{-1}) + \int_0^t q_\delta^\epsilon(s,\phi_s^\epsilon((\phi_t^\epsilon)^{-1}))ds\right] \\
&\quad + \tilde{\mathbb{E}}\left[\int_0^t R_\delta\left[u_s^\epsilon+v_s^\epsilon,\Xi_s^\epsilon\right](\phi_s^\epsilon((\phi_t^\epsilon)^{-1}))ds\right] \\
&\to \tilde{\mathbb{E}}\left[\Xi^\epsilon(0,(\phi_t^\epsilon)^{-1}) + \int_0^t q^\epsilon(s,\phi_s^\epsilon((\phi_t^\epsilon)^{-1}))ds\right]
\end{aligned}$$

in $L^1(\mathbb{T}^2)$ as $\delta \to 0$, for almost every $\omega \in \Omega$ and $t \in [0,T]$. Therefore, by (3.10) we have and the uniqueness of the $L^1(\mathbb{T}^2)$ limit, for almost every $\omega \in \Omega$, $t \in [0,T]$ and $y \in \mathbb{T}^2$:

$$\Xi^\epsilon(t,y) = \tilde{\mathbb{E}}\left[\Xi^\epsilon(0,(\phi^\epsilon_t)^{-1}(y)) + \int_0^t q^\epsilon(s,\phi^\epsilon_s((\phi^\epsilon_t)^{-1}(y)))ds\right],$$

that is exactly the desired representation formula (3.5). The argument for (3.10) is similar, with only a little complication due to the stochastic integral, and we leave it to the reader. □

As a final remark, since we have seen that the notion of generalized solution is weaker than the notion of L^∞-weak solution, our results are indeed very general: they can be applied at least to every L^∞-weak solution.

3.1.4 Convergence of characteristics

We remind the reader that, for externally given u^ϵ and u satisfying (A1)-(A3), there exist unique solutions of the characteristic equations and the large-scale dynamics, assuming the latter is passive (cfr. Proposition 3.4). Strictly speaking, the results in this section are formulated for passive scalars; however, we shall see in Section 3.5 that, a posteriori, even in the active case, fields generated by large-scale processes satisfy all the needed assumptions. Therefore the following Theorem 3.1.1 holds true in the more general case, simply looking at an active large-scale process as a passive scalar compatible with the external fields it generates.
Denote $|x-y|$ the geodesic distance on the flat two dimensional torus between points $x, y \in \mathbb{T}^2$. To keep the notation simple, we define the following quantity associated with a measurable map $\phi : \mathbb{T}^2 \to \mathbb{T}^2$:

$$\|\phi\|_{L^1(\mathbb{T}^2,\mathbb{T}^2)} \coloneqq \int_{\mathbb{T}^2} |\phi(x)|dx.$$

Notice that $\|\cdot\|_{L^1(\mathbb{T}^2,\mathbb{T}^2)}$ is not a norm on the space of measurable maps $\phi : \mathbb{T}^2 \to \mathbb{T}^2$, in particular it is not positively homogeneous. However, $\|\cdot\|_{L^1(\mathbb{T}^2,\mathbb{T}^2)}$ induces a distance on the space $C(\mathbb{T}^2,\mathbb{T}^2)$ of continuous maps $\phi : \mathbb{T}^2 \to \mathbb{T}^2$. Similarly, we define $\|\cdot\|_{L^\infty(\mathbb{T}^2,\mathbb{T}^2)}$ as

$$\|\phi\|_{L^\infty(\mathbb{T}^2,\mathbb{T}^2)} \coloneqq \operatorname*{ess\,sup}_{x\in\mathbb{T}^2} |\phi(x)|.$$

In order to prove convergence of characteristics $\phi^\epsilon \to \phi$, it is clear that one needs some sort of control for the difference $u^\epsilon - u$. Therefore, we assume:

(A4) there exist a constant C and a negligible set $\mathcal{N} \subset \Omega$ such that for every $\omega \in \mathcal{N}^c$, $\epsilon > 0$ and $t \in [0,T]$:

$$\|u^\epsilon(\omega,t,\cdot) - u(\omega,t,\cdot)\|_{L^1(\mathbb{T}^2,\mathbb{R}^2)} \leq C\gamma\left(\tilde{\mathbb{E}}\left[\|\phi^\epsilon_t - \phi_t\|_{L^1(\mathbb{T}^2,\mathbb{T}^2)}\right]\right) + C\int_0^t \gamma\left(\tilde{\mathbb{E}}\left[\|\phi^\epsilon_s - \phi_s\|_{L^1(\mathbb{T}^2,\mathbb{T}^2)}\right]\right)ds + c_\epsilon,$$

where $c_\epsilon \in \mathbb{R}$ is infinitesimal as $\epsilon \to 0$, $\phi^\epsilon_t = \phi^\epsilon(\omega,\tilde{\omega},t,\cdot)$ is the unique solution of (3.3), and $\phi_t = \phi(\omega,\tilde{\omega},t,\cdot)$ is the unique solution of (3.4).

A little less clear, at this point, is our next assumption on the coefficients $(\varsigma_k)_{k\in\mathbb{N}}$:

(A5) for every $x \in \mathbb{T}^2$ it holds $\sum_{k\in\mathbb{N}}((K * \varsigma_k)\cdot\nabla\varsigma_k)(x) = 0$.

The motivations for assuming (A5) will become evident during the proof of Proposition 3.7 in Section 3.2.
We are ready to state our main result on the convergence of characteristics:

Theorem 3.1.1. *Assume (A1)-(A5). Let $\hat{\mathbb{E}}[\cdot] := \mathbb{E}\tilde{\mathbb{E}}[\cdot]$ denote the expectation on $\hat{\Omega} := \Omega\times\tilde{\Omega}$ with respect to the probability measure $\hat{\mathbb{P}} := \mathbb{P}\otimes\tilde{\mathbb{P}}$. Then*

$$\sup_{t\in[0,T]} \hat{\mathbb{E}}\left[\|\phi^\epsilon_t - \phi_t\|_{L^1(\mathbb{T}^2,\mathbb{T}^2)}\right] \to 0 \qquad \text{as } \epsilon\to 0.$$

3.2 Technical results

3.2.1 Linearized dynamics

For $\epsilon > 0$, denote θ^ϵ the solution of the linear problem

$$d\theta^\epsilon_t = -\epsilon^{-1}\theta^\epsilon_t dt + \epsilon^{-1}\sum_{k\in\mathbb{N}}\varsigma_k dW^k_t,$$

with initial condition $\theta^\epsilon|_{t=0} = 0$. The process θ^ϵ is explicitly given by the formula $\theta^\epsilon_t = \sum_{k\in\mathbb{N}}\varsigma_k\eta^{\epsilon,k}_t$, where

$$\eta^{\epsilon,k}_t := \epsilon^{-1}\int_0^t e^{-\epsilon^{-1}(t-s)}dW^k_s, \quad k\in\mathbb{N},$$

is the so called Ornstein-Uhlenbeck process with null initial condition. By [JZ20, Theorem 2.2], for every fixed $p\geq 1$ it holds uniformly in $k\in\mathbb{N}$

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\eta^{\epsilon,k}_t|^p\right] \lesssim \epsilon^{-p/2}\log^{p/2}(1+\epsilon^{-1}), \tag{3.11}$$

and therefore by assumption (A3)

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\theta^\epsilon_t\|^p_{W^{1,\infty}(\mathbb{T}^2)}\right] \lesssim \epsilon^{-p/2}\log^{p/2}(1+\epsilon^{-1}). \tag{3.12}$$

The difference $\zeta^\epsilon := \xi^\epsilon - \theta^\epsilon$ between the small-scale vorticity ξ^ϵ and θ^ϵ solves the equation

$$d\zeta^\epsilon_t + (u^\epsilon_t + v^\epsilon_t)\cdot\nabla\zeta^\epsilon_t dt = -\epsilon^{-1}\zeta^\epsilon_t dt - (u^\epsilon_t + v^\epsilon_t)\cdot\nabla\theta^\epsilon_t dt$$

with initial condition $\zeta^\epsilon_0 = 0$, whose solution satisfies

$$\zeta^\epsilon_t(\psi^\epsilon_t(x)) = -\int_0^t e^{-\epsilon^{-1}(t-s)}((u^\epsilon_s + v^\epsilon_s)\cdot\nabla\theta^\epsilon_s)(\psi^\epsilon_s(x))ds. \tag{3.13}$$

In the following, for $t\in[0,T]$ and $x\in\mathbb{T}^2$ we denote $z^\epsilon_t(x) = (K * \zeta^\epsilon_t)(x)$.

3.2.2 Main technical results

We are going to prove two main technical results, needed for the proof of Theorem 3.1.1. Since our strategy consists in replicating the proof of [FP21, Proposition 4.1], the first result we need is the following:

Proposition 3.6. *Assume (A1)-(A3). Then the following inequality holds:*

$$\hat{\mathbb{E}}\left[\sup_{t\in[0,T]}\left\|\sum_{k\in\mathbb{N}}\int_0^t \sigma_k(\phi_s^\epsilon(\cdot))\eta_s^{\epsilon,k}ds-\sum_{k\in\mathbb{N}}\int_0^t \sigma_k(\phi_s^\epsilon(\cdot))\circ dW_s^k\right\|_{L^1(\mathbb{T}^2,\mathbb{R}^2)}\right] \lesssim \epsilon^{1/42}\log^{47/42}(1+\epsilon^{-1}).$$

In [FP21, Section 4] a similar estimate was proven along the way, using a considerable amount of auxiliary lemmas and computations. In view of this, here we refrain from going again into full detail, and the proof of Proposition 3.6 will only be sketched.
On the other hand, the nonlinear term in (3.1) produces a new term in the equation of characteristcs, that was absent in [FP21]. Although the final results is not affected by this new term, it is not trivial to actually prove so. We need the following:

Proposition 3.7. *Assume (A1)-(A5). Then:*

$$\hat{\mathbb{E}}\left[\sup_{t\in[0,T]}\left\|\int_0^t z_s^\epsilon(\phi_s^\epsilon(\cdot))ds\right\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^2)}\right]\lesssim \epsilon^{1/12}\log^{11/12}(1+\epsilon^{-1}).$$

The proof of Proposition 3.7 relies strongly on assumption (A5) and the following Itō Formulas, yielding for every fixed $t\in[0,T]$ and $k,h\in\mathbb{N}$:

$$\begin{aligned}
\eta_t^{\epsilon,k}\eta_t^{\epsilon,h} &= -\epsilon^{-1}\int_0^t e^{-\epsilon^{-1}(t-s)}\eta_s^{\epsilon,k}\eta_s^{\epsilon,h}ds \\
&\quad+\epsilon^{-1}\int_0^t e^{-\epsilon^{-1}(t-s)}\eta_s^{\epsilon,k}dW_s^h+\epsilon^{-1}\int_0^t e^{-\epsilon^{-1}(t-s)}\eta_s^{\epsilon,h}dW_s^k \\
&\quad+\delta_{k,h}\frac{\epsilon^{-2}}{2}\int_0^t e^{-\epsilon^{-1}(t-s)}ds, \\
\eta_t^{\epsilon,k}\eta_t^{\epsilon,h} &= -2\epsilon^{-1}\int_0^t \eta_s^{\epsilon,k}\eta_s^{\epsilon,h}ds \\
&\quad+\epsilon^{-1}\int_0^t \eta_s^{\epsilon,k}dW_s^h+\epsilon^{-1}\int_0^t \eta_s^{\epsilon,h}dW_s^k+\delta_{k,h}\frac{\epsilon^{-2}t}{2},
\end{aligned}$$

with $\delta_{k,h}$ being the Kronecker delta function, allowing to control the time integral of quadratics $\eta_s^{\epsilon,k}\eta_s^{\epsilon,h}$. In the formula above we have used $\eta_0^{\epsilon,k}=\eta_0^{\epsilon,h}=0$, although the computations could be performed also for more general initial conditions.

3.2.3 Proof of Proposition 3.6

In this paragraph we recall the argument contained in [FP21]. Roughly speaking, Proposition 3.6 is a sort of Wong-Zakai result for the Ornstein-Uhlenbeck process $\eta^{\epsilon,k}$ converging to a white-in-time noise, that is the formal time derivative of the Wiener process W^k.

We need to exploit a discretization of (3.3) to show the closeness, in a certain sense to be specified, between the Stratonovich-to-Itō corrector $c:\mathbb{T}^2\to\mathbb{R}^2$, given by:

$$c(x)=\frac{1}{2}\sum_{k\in\mathbb{N}}\nabla\sigma_k(x)\cdot\sigma_k(x),\quad x\in\mathbb{T}^2,$$

coming from the stochastic integral, and the iterated time integral of the Ornstein-Uhlenbeck process.
In order to discretize the problem, for every $\epsilon>0$ take a mesh $\delta>0$ such that T/δ is an integer. For any $n=0,\dots,T/\delta-1$ and fixed $x\in\mathbb{T}^2$, consider the following decomposition:

$$\begin{aligned}\sum_{k\in\mathbb{N}}\int_{n\delta}^{(n+1)\delta}\sigma_k(\phi_s^\epsilon(x))\eta_s^{\epsilon,k}ds&=\sum_{k\in\mathbb{N}}\int_{n\delta}^{(n+1)\delta}\left(\int_{n\delta}^{s}\nabla\sigma_k(\phi_r^\epsilon(x))\cdot u_r^\epsilon(\phi_r^\epsilon(x))dr\right)\eta_s^{\epsilon,k}ds\\&+\sum_{k\in\mathbb{N}}\int_{n\delta}^{(n+1)\delta}\left(\int_{n\delta}^{s}\nabla\sigma_k(\phi_r^\epsilon(x))\cdot z_r^\epsilon(\phi_r^\epsilon(x))dr\right)\eta_s^{\epsilon,k}ds\\&+\sum_{k,h\in\mathbb{N}}\int_{n\delta}^{(n+1)\delta}\left(\int_{n\delta}^{s}\nabla\sigma_k(\phi_r^\epsilon(x))\cdot\sigma_h(\phi_r^\epsilon(x))\eta_r^{\epsilon,h}dr\right)\eta_s^{\epsilon,k}ds\\&+\sum_{k\in\mathbb{N}}\int_{n\delta}^{(n+1)\delta}\left(\int_{n\delta}^{s}\nabla\sigma_k(\phi_r^\epsilon(x))\cdot\sqrt{2\nu}dw_r\right)\eta_s^{\epsilon,k}ds\\&+\sum_{k\in\mathbb{N}}\int_{n\delta}^{(n+1)\delta}\sigma_k(\phi_{n\delta}^\epsilon(x))dW_s^k\\&-\sum_{k\in\mathbb{N}}\int_{n\delta}^{(n+1)\delta}\sigma_k(\phi_{n\delta}^\epsilon(x))\epsilon d\eta_s^{\epsilon,k}\\&=:I_1^\epsilon(n)+I_2^\epsilon(n)+I_3^\epsilon(n)+I_4^\epsilon(n)+I_5^\epsilon(n)+I_6^\epsilon(n),\end{aligned}$$

where the terms $I_2^\epsilon(n)$ and $I_3^\epsilon(n)$ come from the identity

$$v_r^\epsilon(\phi_r^\epsilon(x))=z_r^\epsilon(\phi_r^\epsilon(x))+\sum_{h\in\mathbb{N}}\sigma_h(\phi_r^\epsilon(x))\eta_r^{\epsilon,h},$$

which can be obtained applying the Biot-Savart law to the identity $\zeta^\epsilon=\xi^\epsilon-\theta^\epsilon$ defining ζ^ϵ. Regarding the Stratonovich integral, we can rewrite:

$$\begin{aligned}\sum_{k\in\mathbb{N}}\int_{n\delta}^{(n+1)\delta}\sigma_k(\phi_s^\epsilon(x))\circ dW_s^k&=\sum_{k\in\mathbb{N}}\int_{n\delta}^{(n+1)\delta}\left(\sigma_k(\phi_s^\epsilon(x))-\sigma_k(\phi_{n\delta}^\epsilon(x))\right)dW_s^k\\&+\sum_{k\in\mathbb{N}}\int_{n\delta}^{(n+1)\delta}\sigma_k(\phi_{n\delta}^\epsilon(x))dW_s^k\\&+\int_{n\delta}^{(n+1)\delta}\left(c(\phi_s^\epsilon(x))-c(\phi_{n\delta}^\epsilon(x))\right)ds\\&+\int_{n\delta}^{(n+1)\delta}c(\phi_{n\delta}^\epsilon(x))ds\\&=:J_1^\epsilon(n)+J_2^\epsilon(n)+J_3^\epsilon(n)+J_4^\epsilon(n).\end{aligned}$$

The ingredients for the proof of Proposition 3.6 are:

- a good estimate on $\hat{\mathbb{E}}\left[\sup_{t\in[0,T]}|z_t^\epsilon(\phi_t^\epsilon(x))|\right]$ (cfr. Lemma 3.8), needed to control $I_2^\epsilon(n)$;
- a good estimate on $\hat{\mathbb{E}}\left[\sup_{\tau\le\delta}|\phi_{\tau+n\delta}^\epsilon(x)-\phi_{n\delta}^\epsilon(x)|\right]$ (cfr. Lemma 3.9), needed to approximate $I_3^\epsilon(n)$ with

$$\sum_{k,h\in\mathbb{N}}\nabla\sigma_k(\phi_{n\delta}(x))\cdot\sigma_h(\phi_{n\delta}(x))\int_{n\delta}^{(n+1)\delta}\left(\int_{n\delta}^s\eta_r^{\epsilon,h}dr\right)\eta_s^{\epsilon,k}ds; \tag{3.14}$$

- a better estimate on $\hat{\mathbb{E}}\left[|\phi_{(n+1)\delta}^\epsilon(x)-\phi_{n\delta}^\epsilon(x)|\right]$ (cfr. Lemma 3.10), needed to control $I_6^\epsilon(n)$ with a discrete integration by parts.

Notice that $I_5^\epsilon(n)=J_2^\epsilon(n)$. Also, the expression in (3.14) (which approximates $I_3^\epsilon(n)$) must be compensated by subtracting $J_4^\epsilon(n)$.

Lemma 3.8. *Assume (A1)-(A3). Then for every fixed $p\ge1$ it holds*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\zeta_t^\epsilon\|_{L^\infty(\mathbb{T}^2)}^p\right]\lesssim\log^p(1+\epsilon^{-1}).$$

In particular, since $z_t^\epsilon=K*\zeta_t^\epsilon$ we alse have

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|z_t^\epsilon\|_{L^\infty(\mathbb{T}^2)}^p\right]\lesssim\log^p(1+\epsilon^{-1}).$$

Proof. We prove in the first place the weaker estimate:

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\zeta_t^\epsilon\|_{L^\infty(\mathbb{T}^2)}^p\right]\lesssim\epsilon^{-p}. \tag{3.15}$$

Since θ^ϵ satisfies the bound above by (3.12), it suffices to prove it for ξ^ϵ. Denote $M_t^\epsilon(x)=\sum_{k\in\mathbb{N}}\int_0^t\varsigma_k(\psi_s^\epsilon(x))dW_s^k$. Since for every $s,t\in[0,T]$

$$\mathbb{E}\left[\|M_t^\epsilon-M_s^\epsilon\|_{L^\infty(\mathbb{T}^2)}^4\right]\lesssim\left(\sum_{k\in\mathbb{N}}\|\varsigma_k\|_{L^\infty(\mathbb{T}^2)}^2\right)^2(t-s)^2,$$

by (A3) and Kolmogorov continuity Theorem the process $M^\epsilon:\Omega\times[0,T]\to L^\infty(\mathbb{T}^2)$ has a modification $\tilde{M}^\epsilon$ that is α-Hölder continuous for every $\alpha<1/4$, with α-Hölder constant $K_{\epsilon,\alpha}$ bounded in $L^p(\Omega)$ for every $p<\infty$ uniformly in ϵ. Since M^ϵ has continuous trajectories, $M_t^\epsilon=\tilde{M}_t^\epsilon$ a.s. as random variables in $L^\infty(\mathbb{T}^2)$ and

$$\begin{aligned}\xi_t^\epsilon(\psi_t^\epsilon(x))&=\epsilon^{-1}\int_0^t e^{-\epsilon^{-1}(t-s)}dM_s^\epsilon(x)\\&=\epsilon^{-1}\int_0^t e^{-\epsilon^{-1}(t-s)}d(M_s^\epsilon(x)-M_t^\epsilon(x))\\&=\epsilon^{-1}\left[e^{-\epsilon^{-1}(t-s)}(M_s^\epsilon(x)-M_t^\epsilon(x))\right]_{s=0}^{s=t}\\&\quad-\epsilon^{-2}\int_0^t e^{-\epsilon^{-1}(t-s)}(M_s^\epsilon(x)-M_t^\epsilon(x))ds.\end{aligned}$$

Clearly $\|\xi^\epsilon_t\|_{L^\infty(\mathbb{T}^2)} = \|\xi^\epsilon_t \circ \psi^\epsilon_t\|_{L^\infty(\mathbb{T}^2)}$, and therefore

$$\|\xi^\epsilon_t\|_{L^\infty(\mathbb{T}^2)} \leq \epsilon^{-1} e^{-\epsilon^{-1}t} \|M^\epsilon_t\|_{L^\infty(\mathbb{T}^2)} + \epsilon^{-1} K_{\epsilon,\alpha},$$

and (3.15) follows.
Recalling (3.13), the following inequality holds

$$\|\zeta^\epsilon_t\|_{L^\infty(\mathbb{T}^2)} \leq \int_0^t e^{-\epsilon^{-1}(t-s)} \|(u^\epsilon_s + v^\epsilon_s)\cdot \nabla\theta^\epsilon_s\|_{L^\infty(\mathbb{T}^2)} ds. \tag{3.16}$$

Using assumption (A2) and $v^\epsilon_s = K * \zeta^\epsilon_s + K * \theta^\epsilon_s$ we get

$$\|(u^\epsilon_s + v^\epsilon_s)\cdot \nabla\theta^\epsilon_s\|_{L^\infty(\mathbb{T}^2)} \lesssim \left(1 + \|\zeta^\epsilon_s\|_{L^\infty(\mathbb{T}^2)} + \|\theta^\epsilon_s\|_{L^\infty(\mathbb{T}^2)}\right) \|\nabla\theta^\epsilon_s\|_{L^\infty(\mathbb{T}^2)},$$

that can be plugged back into (3.16) to produce the recursive estimate

$$\begin{aligned}
\|\zeta^\epsilon_t\|_{L^\infty(\mathbb{T}^2)} &\lesssim \int_0^t e^{-\epsilon^{-1}(t-s)} \left(1 + \|\theta^\epsilon_s\|_{L^\infty(\mathbb{T}^2)}\right) \|\nabla\theta^\epsilon_s\|_{L^\infty(\mathbb{T}^2)} ds \\
&\quad + \int_0^t e^{-\epsilon^{-1}(t-s)} \|\zeta^\epsilon_s\|_{L^\infty(\mathbb{T}^2)} \|\nabla\theta^\epsilon_s\|_{L^\infty(\mathbb{T}^2)} ds \\
&\lesssim \epsilon \left(\sup_{s\in[0,T]} \|\theta^\epsilon_s\|_{L^\infty(\mathbb{T}^2)} + \sup_{s\in[0,T]} \|\zeta^\epsilon_s\|_{L^\infty(\mathbb{T}^2)} \right) \sup_{s\in[0,T]} \|\nabla\theta^\epsilon_s\|_{L^\infty(\mathbb{T}^2)}.
\end{aligned}$$

By Hölder inequality and (3.15) we deduce from the previous inequality

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\zeta^\epsilon_t\|^p_{L^\infty(\mathbb{T}^2)} \right] \lesssim \log^p(1+\epsilon^{-1}) + \epsilon^{-p/2} \log^{p/2}(1+\epsilon^{-1}),$$

improving the bound (3.15) itself. Iterating the same argument one more time we obtain the desired estimate. □

Lemma 3.9. *Assume (A1)-(A3). Then for every fixed $p \geq 1$ and $\alpha \in (0, 1/2)$*

$$\hat{\mathbb{E}}\left[\sup_{\substack{t+\tau\leq T \\ \tau\leq\delta}} \|\phi^\epsilon_{t+\tau} - \phi^\epsilon_t\|^p_{L^\infty(\mathbb{T}^2,\mathbb{T}^2)} \right] \lesssim \delta^p \epsilon^{-p/2} \log^{p/2}(1+\epsilon^{-1}) + \delta^{p\alpha}.$$

Proof. The increment $\phi^\epsilon_{t+\tau}(x) - \phi^\epsilon_t(x)$ can be written as

$$\begin{aligned}
\phi^\epsilon_{t+\tau}(x) - \phi^\epsilon_t(x) &= \int_t^{t+\tau} u^\epsilon_s(\phi^\epsilon_s(x)) ds + \sum_{k\in\mathbb{N}} \int_t^{t+\tau} \sigma_k(\phi^\epsilon_s(x)) \eta^{\epsilon,k}_s ds \\
&\quad + \int_t^{t+\tau} z_s(\phi^\epsilon_s(x)) ds + \sqrt{2\nu}(w_{t+\tau} - w_t),
\end{aligned}$$

therefore, by assumption (A2) we have

$$\begin{aligned}
\sup_{t+\tau\leq T} \|\phi^\epsilon_{t+\tau} - \phi^\epsilon_t\|_{L^\infty(\mathbb{T}^2,\mathbb{T}^2)} &\lesssim \tau + \tau \sum_{k\in\mathbb{N}} \|\sigma_k\|_{L^\infty(\mathbb{T}^2)} \sup_{s\in[0,T]} |\eta^{\epsilon,k}_s| \\
&\quad + \tau \sup_{s\in[0,T]} \|\zeta^\epsilon_s\|_{L^\infty(\mathbb{T}^2)} + K_\alpha \tau^\alpha,
\end{aligned}$$

where K_α denotes the α-Hölder constant of w. The thesis follows easily by (A3), (3.11) and Lemma 3.8. □

Lemma 3.10. *Assume (A1)-(A3). Then for every fixed $p \geq 1$ we have, uniformly in $n = 0, \dots, T/\delta - 1$:*

$$\hat{\mathbb{E}}\left[\|\phi^\epsilon_{(n+1)\delta} - \phi^\epsilon_{n\delta}\|^p_{L^\infty(\mathbb{T}^2,\mathbb{T}^2)}\right] \lesssim \delta^{2p}\epsilon^{-p}\log^p(1+\epsilon^{-1}) \\ + \delta^{p(1+\alpha)}\epsilon^{-p/2}\log^{p/2}(1+\epsilon^{-1}) \\ + \delta^{p/2} + \epsilon^{p/2}\log^{p/2}(1+\epsilon^{-1}).$$

Proof. The increment $\phi^\epsilon_{(n+1)\delta}(x) - \phi^\epsilon_{n\delta}(x)$ can be written as

$$\phi^\epsilon_{(n+1)\delta}(x) - \phi^\epsilon_{n\delta}(x) = \int_{n\delta}^{(n+1)\delta} u^\epsilon_s(\phi^\epsilon_s(x))ds \\ + \sum_{k\in\mathbb{N}}\int_{n\delta}^{(n+1)\delta} \left(\sigma_k(\phi^\epsilon_s(x)) - \sigma_k(\phi^\epsilon_{n\delta}(x))\right)\eta^{\epsilon,k}_s ds \\ + \sum_{k\in\mathbb{N}}\int_{n\delta}^{(n+1)\delta} \sigma_k(\phi^\epsilon_{n\delta}(x))\eta^{\epsilon,k}_s ds \\ + \int_{n\delta}^{(n+1)\delta} z^\epsilon_s(\phi^\epsilon_s(x))ds + \sqrt{2\nu}(w_{(n+1)\delta} - w_{n\delta}).$$

The first, fourth and fifth term are easy. The second one is bounded in $L^\infty(\mathbb{T}^2,\mathbb{T}^2)$ uniformly in n by

$$\int_0^\delta \sum_{k\in\mathbb{N}} \|\nabla\sigma_k\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^4)} \sup_{t+s\leq T}\|\phi^\epsilon_{t+s} - \phi^\epsilon_t\|_{L^\infty(\mathbb{T}^2,\mathbb{T}^2)} \sup_{s\in[0,T]}|\eta^{\epsilon,k}_s|ds,$$

and by (A3) and Hölder inequality with exponent $q > 1$

$$\hat{\mathbb{E}}\left[\left(\int_0^\delta \sum_{k\in\mathbb{N}} \|\nabla\sigma_k\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^4)} \sup_{t+s\leq T}\|\phi^\epsilon_{t+s} - \phi^\epsilon_t\|_{L^\infty(\mathbb{T}^2,\mathbb{T}^2)} \sup_{s\in[0,T]}|\eta^{\epsilon,k}_s|ds\right)^p\right] \\ \leq \delta^{p-1}\left(\sum_{k\in\mathbb{N}}\|\nabla\sigma_k\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^4)}\right)^{p-1}\int_0^\delta \sum_{k\in\mathbb{N}}\|\nabla\sigma_k\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^4)} \\ \times \hat{\mathbb{E}}\left[\sup_{t+s\leq T}\|\phi^\epsilon_{t+s} - \phi^\epsilon_t\|^{pq}_{L^\infty(\mathbb{T}^2,\mathbb{T}^2)}\right]^{1/q} \hat{\mathbb{E}}\left[\sup_{s\in[0,T]}|\eta^{\epsilon,k}_s|^{pq'}\right]^{1/q'} ds \\ \lesssim \delta^{p-1}\int_0^\delta \left(s^p\epsilon^{-p}\log^p(1+\epsilon^{-1})ds + s^{p\alpha}\epsilon^{-p/2}\log^{p/2}(1+\epsilon^{-1})\right)ds \\ \lesssim \delta^{2p}\epsilon^{-p}\log^p(1+\epsilon^{-1}) + \delta^{p(1+\alpha)}\epsilon^{-p/2}\log^{p/2}(1+\epsilon^{-1}).$$

The third term is bounded in $L^\infty(\mathbb{T}^2,\mathbb{R}^2)$ by

$$\sum_{k\in\mathbb{N}}\|\sigma_k\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^2)}\left|\int_{n\delta}^{(n+1)\delta}\eta^{\epsilon,k}_s ds\right| = \sum_{k\in\mathbb{N}}\|\sigma_k\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^2)}\left|W^k_{(n+1)\delta} - W^k_{n\delta}\right| \\ + \sum_{k\in\mathbb{N}}\|\sigma_k\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^2)}\epsilon\left|\eta^{\epsilon,k}_{(n+1)\delta} - \eta^{\epsilon,k}_{n\delta}\right|,$$

from which we deduce as usual

$$\hat{\mathbb{E}}\left[\left(\sum_{k\in\mathbb{N}}\|\sigma_k\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^2)}\left|\int_{n\delta}^{(n+1)\delta}\eta_s^{\epsilon,k}ds\right|\right)^p\right]\lesssim \delta^{p/2}+\epsilon^{p/2}\log^{p/2}(1+\epsilon^{-1}).$$

Putting all together, the thesis follows. □

Proof of Proposition 3.6. For any given $t\in[0,T]$, let $\lfloor t\rfloor =: m\delta$ be the largest multiple of δ strictly smaller than t. We can therefore decompose

$$\begin{aligned}\sum_{k\in\mathbb{N}}\int_0^t\sigma_k(\phi_s^\epsilon(x))\eta_s^{\epsilon,k}ds &= \sum_{k\in\mathbb{N}}\int_0^{m\delta}\sigma_k(\phi_s^\epsilon(x))\eta_s^{\epsilon,k}ds+\sum_{k\in\mathbb{N}}\int_{m\delta}^t\sigma_k(\phi_s^\epsilon(x))\eta_s^{\epsilon,k}ds\\ &= \sum_{j=1}^6\sum_{n=0}^{m-1}I_j^\epsilon(n)+\sum_{k\in\mathbb{N}}\int_{m\delta}^t\sigma_k(\phi_s^\epsilon(x))\eta_s^{\epsilon,k}ds,\end{aligned}$$

and in a similar fashion

$$\begin{aligned}\sum_{k\in\mathbb{N}}\int_0^t\sigma_k(\phi_s^\epsilon(x))\circ dW_s^k &= \sum_{k\in\mathbb{N}}\int_0^{m\delta}\sigma_k(\phi_s^\epsilon(x))\circ dW_s^k+\sum_{k\in\mathbb{N}}\int_{m\delta}^t\sigma_k(\phi_s^\epsilon(x))\circ dW_s^k\\ &= \sum_{j=1}^4\sum_{n=0}^{m-1}J_j^\epsilon(n)+\sum_{k\in\mathbb{N}}\int_{m\delta}^t\sigma_k(\phi_s^\epsilon(x))\circ dW_s^k.\end{aligned}$$

By (3.11), the following estimate holds true

$$\hat{\mathbb{E}}\left[\sup_{\substack{m=0,\dots,T/\delta-1\\ t\le\delta}}\left\|\sum_{k\in\mathbb{N}}\int_{m\delta}^t\sigma_k(\phi_s^\epsilon(\cdot))\eta_s^{\epsilon,k}ds\right\|_{L^1(\mathbb{T}^2,\mathbb{R}^2)}\right]\lesssim\delta\epsilon^{-1/2}\log^{1/2}(1+\epsilon^{-1}).$$

Also, by (A3) and Kolmogorov continuity Theorem, for every fixed $\alpha\in(0,1/2)$ we have

$$\hat{\mathbb{E}}\left[\sup_{\substack{m=0,\dots,T/\delta-1\\ t\le\delta}}\left\|\sum_{k\in\mathbb{N}}\int_{m\delta}^t\sigma_k(\phi_s^\epsilon(\cdot))\circ dW_s^k\right\|_{L^1(\mathbb{T}^2,\mathbb{R}^2)}\right]\lesssim\delta^\alpha.$$

Finally, by calculations similar to those performed in Lemma 4.6 and Lemma 4.7 of [FP21], for every fixed $\alpha\in(0,1/2)$

$$\begin{aligned}&\hat{\mathbb{E}}\left[\sup_{m=0,\dots,T/\delta-1}\left\|\sum_{j=1}^6\sum_{n=0}^{m-1}I_j^\epsilon(n)-\sum_{j=1}^4\sum_{n=0}^{m-1}J_j^\epsilon(n)\right\|_{L^1(\mathbb{T}^2,\mathbb{R}^2)}\right]\\ &\lesssim\delta\epsilon^{-1/2}\log^{3/2}(1+\epsilon^{-1})+\delta^{\alpha-1}\epsilon^{1/2}\log(1+\epsilon^{-1})\\ &+\delta^2\epsilon^{-3/2}\log^{3/2}(1+\epsilon^{-1})+\delta^{1+\alpha}\epsilon^{-1}\log(1+\epsilon^{-1})+\delta^\alpha.\end{aligned}$$

We conclude the proof fixing α close to $1/2$ so that $(1+\alpha)^{-1}<3/4<(2-2\alpha)^{-1}$, for instance $\alpha=3/8$, and optimizing over δ: for $\delta=\epsilon^{16/21}\log^{-4/21}(1+\epsilon^{-1})$, it follows the desired inequality

$$\begin{aligned}&\hat{\mathbb{E}}\left[\sup_{t\in[0,T]}\left\|\sum_{k\in\mathbb{N}}\int_0^t\sigma_k(\phi_s^\epsilon(\cdot))\eta_s^{\epsilon,k}ds-\sum_{k\in\mathbb{N}}\int_0^t\sigma_k(\phi_s^\epsilon(\cdot))\circ dW_s^k\right\|_{L^1(\mathbb{T}^2,\mathbb{R}^2)}\right]\\ &\lesssim\epsilon^{1/42}\log^{47/42}(1+\epsilon^{-1}).\end{aligned}$$

□

3.2.4 Proof of Proposition 3.7

Recall the content of Proposition 3.7: we need to prove, under assumptions (A1)-(A5)

$$\hat{\mathbb{E}}\left[\sup_{t\in[0,T]}\left\|\int_0^t z_s^\epsilon(\phi_s^\epsilon(\cdot))ds\right\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^2)}\right] \lesssim \epsilon^{1/12}\log^{11/12}(1+\epsilon^{-1}).$$

Comparing the desired inequality with Lemma 3.8, one realizes that time integration of the process $z_s^\epsilon(\phi_s^\epsilon(x))$ allows a better control due to cancellation of opposite-sign oscillations, even if the latter may become of large magnitude for ϵ going to zero.
Concerning the strategy of the proof, in the first place we prove the following:

Lemma 3.11. *For every fixed $t\in[0,T]$ it holds*

$$\hat{\mathbb{E}}\left[\left\|\int_0^t z_s^\epsilon(\phi_s^\epsilon(\cdot))ds\right\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^2)}\right] \lesssim \epsilon^{1/6}\log^{5/6}(1+\epsilon^{-1}).$$

Having at hands the previous result, the proof of Proposition 3.7 goes as follows: for some parameter $\delta = T/m > 0$, $m\in\mathbb{N}$ to be chosen, write

$$\begin{aligned}
\sup_{t\in[0,T]}\left\|\int_0^t z_s^\epsilon(\phi_s^\epsilon(\cdot))ds\right\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^2)} &\le \sup_{n=0,\dots,m-1}\left\|\int_0^{n\delta} z_s^\epsilon(\phi_s^\epsilon(\cdot))ds\right\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^2)}\\
&\quad+\sup_{\substack{n=0,\dots,m-1\\ t\le\delta}}\left\|\int_{n\delta}^{n\delta+t} z_s^\epsilon(\phi_s^\epsilon(\cdot))ds\right\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^2)}\\
&\le \sum_{n=0}^{m-1}\left\|\int_0^{n\delta} z_s^\epsilon(\phi_s^\epsilon(\cdot))ds\right\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^2)}\\
&\quad+\delta\sup_{s\in[0,T]}\|z_s^\epsilon(\phi_s^\epsilon(\cdot))\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^2)}.
\end{aligned}$$

Hence, by Lemma 3.8 and Lemma 3.11

$$\begin{aligned}
\hat{\mathbb{E}}\left[\sup_{t\in[0,T]}\left\|\int_0^t z_s^\epsilon(\phi_s^\epsilon(\cdot))ds\right\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^2)}\right] &\le \sum_{n=0}^{m-1}\hat{\mathbb{E}}\left[\left\|\int_0^{n\delta} z_s^\epsilon(\phi_s^\epsilon(\cdot))ds\right\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^2)}\right]\\
&\quad+\delta\hat{\mathbb{E}}\left[\sup_{s\in[0,T]}\|z_s^\epsilon(\phi_s^\epsilon(\cdot))\|_{L^\infty(\mathbb{T}^2,\mathbb{R}^2)}\right]\\
&\lesssim \delta^{-1}\epsilon^{1/6}\log^{5/6}(1+\epsilon^{-1})+\delta\log(1+\epsilon^{-1}),
\end{aligned}$$

and the thesis follows by optimizing the choice of δ.

Proof of Lemma 3.11. We will work with fixed $x\in\mathbb{T}^2$. The reader can easily check that all the inequalities present in the proof hold uniformly in x. Recall $z_t^\epsilon = K*\zeta_t^\epsilon$, and for $\psi_{t,s}^\epsilon(x) := \psi_s^\epsilon((\psi_t^\epsilon)^{-1}(x))$ the formula

$$\begin{aligned}
\zeta_t^\epsilon(x) = &-\int_0^t e^{-\epsilon^{-1}(t-s)}((u_s^\epsilon + K*\zeta_s^\epsilon)\cdot\nabla\theta_s^\epsilon)(\psi_{t,s}^\epsilon(x))ds\\
&-\int_0^t e^{-\epsilon^{-1}(t-s)}((K*\theta_s^\epsilon)\cdot\nabla\theta_s^\epsilon)(\psi_{t,s}^\epsilon(x))ds.
\end{aligned}$$

For notational simplicity let $\Theta^\epsilon_s := (K * \theta^\epsilon_s) \cdot \nabla \theta^\epsilon_s$, and rewrite

$$
\begin{aligned}
\zeta^\epsilon_t(x) = &- \int_0^t e^{-\epsilon^{-1}(t-s)} ((u^\epsilon_s + K * \zeta^\epsilon_s) \cdot \nabla \theta^\epsilon_s)(\psi^\epsilon_{t,s}(x)) ds \\
&- \int_0^t e^{-\epsilon^{-1}(t-s)} \left(\Theta^\epsilon_s(\psi^\epsilon_{t,s}(x)) - \Theta^\epsilon_s(x) \right) ds \\
&- \int_0^t e^{-\epsilon^{-1}(t-s)} \Theta^\epsilon_s(x) ds \\
&=: \zeta^{\epsilon,1}_t(x) + \zeta^{\epsilon,2}_t(x) + \zeta^{\epsilon,3}_t(x).
\end{aligned}
$$

Let us focus on the terms $\zeta^{\epsilon,j}$, $j = 1, 2, 3$ separately. Concerning $\zeta^{\epsilon,1}$,

$$
\begin{aligned}
\|\zeta^{\epsilon,1}_t\|_{L^\infty(\mathbb{T}^2)} \lesssim &\int_0^t e^{-\epsilon^{-1}(t-s)} ds \left(1 + \sup_{s \in [0,T]} \|\zeta^\epsilon_s\|_{L^\infty(\mathbb{T}^2)} \right) \\
&\times \sup_{s \in [0,T]} \|\nabla \theta^\epsilon_s\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)},
\end{aligned}
$$

and thus the following holds by assumption (A2) and Lemma 3.8

$$
\sup_{t \in [0,T]} \hat{\mathbb{E}} \left[\|\zeta^{\epsilon,1}_t\|_{L^\infty(\mathbb{T}^2)} \right] \lesssim \epsilon^{1/2} \log^{3/2}(1 + \epsilon^{-1}). \tag{3.17}
$$

Moving to $\zeta^{\epsilon,2}$, notice that $|\psi^\epsilon_{t,s}(x) - x| = |\psi^\epsilon_{t,s}(x) - \psi^\epsilon_{t,t}(x)|$, and letting $y = (\psi^\epsilon_t)^{-1}(x)$ we have

$$
\begin{aligned}
|\psi^\epsilon_{t,s}(x) - \psi^\epsilon_{t,t}(x)| &= |\psi^\epsilon_s(y) - \psi^\epsilon_t(y)| \\
&\leq \int_s^t |u^\epsilon_r(\psi^\epsilon_r(y))| dr + \int_s^t |v^\epsilon_r(\psi^\epsilon_r(y))| dr \\
&\lesssim |t - s| \left(1 + \sup_{r \in [0,T]} \|\zeta^\epsilon_r\|_{L^\infty(\mathbb{T}^2)} + \sup_{r \in [0,T]} \|\theta^\epsilon_r\|_{L^\infty(\mathbb{T}^2)} \right),
\end{aligned}
$$

therefore

$$
\begin{aligned}
\|\zeta^{\epsilon,2}_t\|_{L^\infty(\mathbb{T}^2)} \lesssim &\int_0^t e^{-\epsilon^{-1}(t-s)} |t - s| ds \sup_{s \in [0,T]} \|\nabla \Theta^\epsilon_s\|_{L^\infty(\mathbb{T}^2, \mathbb{R}^2)} \\
&\times \left(1 + \sup_{r \in [0,T]} \|\zeta^\epsilon_r\|_{L^\infty(\mathbb{T}^2)} + \sup_{r \in [0,T]} \|\theta^\epsilon_r\|_{L^\infty(\mathbb{T}^2)} \right),
\end{aligned}
$$

that implies

$$
\sup_{t \in [0,T]} \hat{\mathbb{E}} \left[\|\zeta^{\epsilon,2}_t\|_{L^\infty(\mathbb{T}^2)} \right] \lesssim \epsilon^{1/2} \log^{3/2}(1 + \epsilon^{-1}). \tag{3.18}
$$

Finally, let us consider the term $\zeta^{\epsilon,3}$, which requires a preliminary manipulation. Since $\theta^\epsilon_s(x) = \sum_{k \in \mathbb{N}} \sigma_k(x) \eta^{\epsilon,k}_s$, we can rewrite for every $x \in \mathbb{T}^2$

$$
\Theta^\epsilon_s(x) = \sum_{k,h \in \mathbb{N}} (\sigma_k \cdot \nabla \varsigma_h)(x) \eta^{\epsilon,k}_s \eta^{\epsilon,h}_s =: \sum_{k,h \in \mathbb{N}} \Theta_{k,h}(x) \eta^{\epsilon,k}_s \eta^{\epsilon,h}_s,
$$

where we have used $\sigma_k = K * \varsigma_k$ and $\Theta_{k,h} \coloneqq \sigma_k \cdot \nabla \varsigma_h$. Also, rewrite:

$$\begin{aligned} \zeta_t^{\epsilon,3}(x) &= -\int_0^t e^{-\epsilon^{-1}(t-s)} \Theta_s^\epsilon(x) ds \\ &= -\sum_{k,h\in\mathbb{N}} \Theta_{k,h}(x) \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,k} \eta_s^{\epsilon,h} ds. \end{aligned}$$

By Itō Formula, for every fixed t and $k, h \in \mathbb{N}$ it holds

$$\begin{aligned} \eta_t^{\epsilon,k}\eta_t^{\epsilon,h} &= -\epsilon^{-1} \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,k} \eta_s^{\epsilon,h} ds \\ &\quad + \epsilon^{-1} \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,k} dW_s^h + \epsilon^{-1} \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,h} dW_s^k \\ &\quad + \frac{\epsilon^{-2}}{2} \delta_{k,h} \int_0^t e^{-\epsilon^{-1}(t-s)} ds, \end{aligned}$$

with $\delta_{k,h}$ being the Kronecker delta function: $\delta_{k,h} = 1$ if $k = h$ and $\delta_{k,h} = 0$ if $k \neq h$. Otherwise said:

$$\begin{aligned} \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,k} \eta_s^{\epsilon,h} ds &= -\epsilon \eta_t^{\epsilon,k} \eta_t^{\epsilon,h} \\ &\quad + \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,k} dW_s^h + \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,h} dW_s^k \\ &\quad + \frac{1 - e^{\epsilon^{-1}t}}{2} \delta_{k,h}. \end{aligned} \tag{3.19}$$

By (3.19) and assumption (A5), for every $x \in \mathbb{T}^2$ we have

$$\begin{aligned} \zeta_t^{\epsilon,3}(x) &= \sum_{k,h\in\mathbb{N}} \Theta_{k,h}^\epsilon(x) \epsilon \eta_t^{\epsilon,k} \eta_t^{\epsilon,h} \\ &\quad - \sum_{k,h\in\mathbb{N}} \Theta_{k,h}^\epsilon(x) \left(\int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,k} dW_s^h + \int_0^t e^{-\epsilon^{-1}(t-s)} \eta_s^{\epsilon,h} dW_s^k \right), \end{aligned}$$

and therefore we can rewrite

$$\begin{aligned} \int_0^t (K * \zeta_s^{\epsilon,3})(\phi_s^\epsilon(x)) ds &= \sum_{k,h\in\mathbb{N}} \int_0^t (K * \Theta_{k,h}^\epsilon)(\phi_s^\epsilon(x)) \epsilon \eta_s^{\epsilon,k} \eta_s^{\epsilon,h} ds \\ - \sum_{k,h\in\mathbb{N}} \int_0^t (K * \Theta_{k,h})(\phi_s^\epsilon(x)) &\left(\int_0^s e^{-\epsilon^{-1}(s-r)} \eta_r^{\epsilon,k} dW_r^h + \int_0^s e^{-\epsilon^{-1}(s-r)} \eta_r^{\epsilon,h} dW_r^k \right) ds. \end{aligned}$$

Thus,

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\left|\int_0^t (K*\Theta^\epsilon_{k,h})(\phi^\epsilon_s(x))\int_0^s e^{-\epsilon^{-1}(s-r)}\eta^{\epsilon,k}_r dW^h_r ds\right|\right]\\
&\quad= \hat{\mathbb{E}}\left[\left|\int_0^t\left(\int_r^t (K*\Theta^\epsilon_{k,h})(\phi^\epsilon_s(x))e^{-\epsilon^{-1}(s-r)}ds\right)\eta^{\epsilon,k}_r dW^h_r\right|\right]\\
&\quad\lesssim \hat{\mathbb{E}}\left[\left|\int_0^t\left(\int_r^t (K*\Theta^\epsilon_{k,h})(\phi^\epsilon_s(x))e^{-\epsilon^{-1}(s-r)}ds\right)\eta^{\epsilon,k}_r dW^h_r\right|^2\right]^{1/2}\\
&\quad\lesssim \hat{\mathbb{E}}\left[\int_0^t\left(\int_r^t (K*\Theta^\epsilon_{k,h})(\phi^\epsilon_s(x))e^{-\epsilon^{-1}(s-r)}ds\right)^2|\eta^{\epsilon,k}_r|^2 dr\right]^{1/2}\\
&\quad\lesssim \epsilon^{1/2}\log^{1/2}(1+\epsilon^{-1}).
\end{aligned}$$

The last non-trivial term is manipulated as follows. Let $\delta = t/m > 0$, $m\in\mathbb{N}$ to be suitably chosen. We have

$$\begin{aligned}
&\sum_{k,h\in\mathbb{N}}\int_0^t (K*\Theta_{k,h})(\phi^\epsilon_s(x))\epsilon\eta^{\epsilon,k}_s\eta^{\epsilon,h}_s ds \qquad (3.20)\\
&\quad=\sum_{k,h\in\mathbb{N}}\sum_{n=0}^{m-1}\int_{n\delta}^{(n+1)\delta}\left((K*\Theta_{k,h})(\phi^\epsilon_s(x))-(K*\Theta_{k,h})(\phi^\epsilon_{n\delta}(x))\right)\epsilon\eta^{\epsilon,k}_s\eta^{\epsilon,h}_s ds\\
&\qquad+\sum_{k,h\in\mathbb{N}}\sum_{n=0}^{m-1}(K*\Theta_{k,h})(\phi^\epsilon_{n\delta}(x))\int_{n\delta}^{(n+1)\delta}\epsilon\eta^{\epsilon,k}_s\eta^{\epsilon,h}_s ds.
\end{aligned}$$

Recalling (3.3), for every $\alpha\in(0,1/2)$ it holds

$$\begin{aligned}
|\phi^\epsilon_t(x)-\phi^\epsilon_s(x)| &\leq \int_s^t |u^\epsilon_r(\phi^\epsilon_r(x))|dr + \int_s^t |v^\epsilon_r(\phi^\epsilon_r(x))|dr + \sqrt{2\nu}(w_t-w_s)\\
&\lesssim |t-s|\left(1+\sup_{r\in[0,T]}\|\zeta^\epsilon_r\|_{L^\infty(\mathbb{T}^2)}+\sup_{r\in[0,T]}\|\theta^\epsilon_r\|_{L^\infty(\mathbb{T}^2)}\right)+|t-s|^\alpha,
\end{aligned}$$

which implies

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\left|\sum_{k,h\in\mathbb{N}}\sum_{n=0}^{m-1}\int_{n\delta}^{(n+1)\delta}\left((K*\Theta_{k,h})(\phi^\epsilon_s(x))-(K*\Theta_{k,h})(\phi^\epsilon_{n\delta}(x))\right)\epsilon\eta^{\epsilon,k}_s\eta^{\epsilon,h}_s ds\right|\right] \qquad (3.21)\\
&\quad\lesssim \delta\epsilon^{-1/2}\log^{3/2}(1+\epsilon^{-1})+\delta^\alpha\log(1+\epsilon^{-1}).
\end{aligned}$$

Also, we can apply Itō Formula again to find an alternative representation for the time integral of the quadratics $\eta^{\epsilon,k}_s\eta^{\epsilon,h}_s$, similar to (3.19). Indeed,

$$\begin{aligned}
\eta^{\epsilon,k}_{(n+1)\delta}\eta^{\epsilon,h}_{(n+1)\delta}-\eta^{\epsilon,k}_{n\delta}\eta^{\epsilon,h}_{n\delta} &= -2\epsilon^{-1}\int_{n\delta}^{(n+1)\delta}\eta^{\epsilon,k}_t\eta^{\epsilon,h}_t dt\\
&\quad+\epsilon^{-1}\int_{n\delta}^{(n+1)\delta}\eta^{\epsilon,k}_t dW^h_t+\epsilon^{-1}\int_{n\delta}^{(n+1)\delta}\eta^{\epsilon,h}_t dW^k_t\\
&\quad+\frac{\epsilon^{-2}\delta}{2}\delta_{k,h},
\end{aligned}$$

and rearranging the terms we obtain

$$\int_{n\delta}^{(n+1)\delta} \epsilon\eta_t^{\epsilon,k}\eta_t^{\epsilon,h}dt = \frac{\epsilon^2}{2}\left(\eta_{n\delta}^{\epsilon,k}\eta_{n\delta}^{\epsilon,h} - \eta_{(n+1)\delta}^{\epsilon,k}\eta_{(n+1)\delta}^{\epsilon,h}\right) + \frac{\epsilon}{2}\int_{n\delta}^{(n+1)\delta} \eta_t^{\epsilon,k}dW_t^h + \frac{\epsilon}{2}\int_{n\delta}^{(n+1)\delta} \eta_t^{\epsilon,h}dW_t^k + \frac{\delta}{4}\delta_{k,h}. \tag{3.22}$$

Finally, making use of (3.22) above and assumption (A5) we can rewrite

$$\begin{aligned}
&\sum_{k,h\in\mathbb{N}}\sum_{n=0}^{m-1}(K*\Theta_{k,h})(\phi_{n\delta}^\epsilon(x))\int_{n\delta}^{(n+1)\delta} \epsilon\eta_s^{\epsilon,k}\eta_s^{\epsilon,h}ds \\
&\quad= \sum_{k,h\in\mathbb{N}}\sum_{n=0}^{m-1}(K*\Theta_{k,h})(\phi_{n\delta}^\epsilon(x))\frac{\epsilon^2}{2}\left(\eta_{n\delta}^{\epsilon,k}\eta_{n\delta}^{\epsilon,h} - \eta_{(n+1)\delta}^{\epsilon,k}\eta_{(n+1)\delta}^{\epsilon,h}\right) \\
&\quad+ \sum_{k,h\in\mathbb{N}}\sum_{n=0}^{m-1}(K*\Theta_{k,h})(\phi_{n\delta}^\epsilon(x))\left(\frac{\epsilon}{2}\int_{n\delta}^{(n+1)\delta} \eta_t^{\epsilon,k}dW_t^h + \frac{\epsilon}{2}\int_{n\delta}^{(n+1)\delta} \eta_t^{\epsilon,h}dW_t^k\right).
\end{aligned}$$

We have

$$\hat{\mathbb{E}}\left[\left|\sum_{k,h\in\mathbb{N}}\sum_{n=0}^{m-1}(K*\Theta_{k,h})(\phi_{n\delta}^\epsilon(x))\frac{\epsilon^2}{2}\left(\eta_{n\delta}^{\epsilon,k}\eta_{n\delta}^{\epsilon,h} - \eta_{(n+1)\delta}^{\epsilon,k}\eta_{(n+1)\delta}^{\epsilon,h}\right)\right|\right] \lesssim \delta^{-1}\epsilon\log(1+\epsilon^{-1}), \tag{3.23}$$

and

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\left|\sum_{k,h\in\mathbb{N}}\sum_{n=0}^{m-1}(K*\Theta_{k,h})(\phi_{n\delta}^\epsilon(x))\frac{\epsilon}{2}\int_{n\delta}^{(n+1)\delta} \eta_t^{\epsilon,k}dW_t^h\right|\right] \qquad (3.24)\\
&\lesssim \sum_{n=0}^{m-1}\epsilon\hat{\mathbb{E}}\left[\left|\int_{n\delta}^{(n+1)\delta} \eta_t^{\epsilon,k}dW_t^h\right|^2\right]^{1/2} \lesssim \delta^{-1/2}\epsilon^{1/2}\log^{1/2}(1+\epsilon^{-1}).
\end{aligned}$$

It only remains to choose δ in a suitable way, so that all the terms (3.21), (3.23) and (3.23) are infinitesimal in the limit $\epsilon \to 0$. Taking for instance $\alpha = 1/3$ and optimizing over δ gives

$$\hat{\mathbb{E}}\left[\left|\int_0^t (K*\zeta_s^{\epsilon,3})(\phi_s^\epsilon(x))ds\right|\right] \lesssim \epsilon^{1/6}\log^{5/6}(1+\epsilon^{-1}). \tag{3.25}$$

Considering (3.17), (3.18) and (3.25), we finally get the desired estimate: the proof is complete. □

3.3 Convergence of characteristics

In this section we prove our major result on convergence of characteristics Theorem 3.1.1. The proof is based on Itō Formula for a smooth approximation $g_\delta(x)$ of the absolute value $|x|$.

Proof of Theorem 3.1.1. The difference $\phi^\epsilon - \phi$ solves $\hat{\mathbb{P}}$-a.s. for every $t \in [0,T]$ and $x \in \mathbb{T}^2$:

$$\begin{aligned}\phi^\epsilon_t(x) - \phi_t(x) &= \int_0^t u^\epsilon_s(\phi^\epsilon_s(x))ds - \int_0^t u_s(\phi^\epsilon_s(x))ds \\ &+ \int_0^t u_s(\phi^\epsilon_s(x))ds - \int_0^t u_s(\phi_s(x))ds \\ &+ \sum_{k\in\mathbb{N}} \int_0^t \sigma_k(\phi^\epsilon_s(x))\eta^{\epsilon,k}_s ds - \sum_{k\in\mathbb{N}} \int_0^t \sigma_k(\phi^\epsilon_s(x)) \circ dW^k_s \\ &+ \sum_{k\in\mathbb{N}} \int_0^t \sigma_k(\phi^\epsilon_s(x)) \circ dW^k_s - \sum_{k\in\mathbb{N}} \int_0^t \sigma_k(\phi_s(x)) \circ dW^k_s \\ &+ \int_0^t z^\epsilon_s(\phi^\epsilon_s(x))ds.\end{aligned}$$

For $\delta > 0$, introduce the smooth function $g_\delta : \mathbb{R}^2 \to \mathbb{R}$ defined by $g_\delta(x) := (|x|^2 + \delta)^{1/2}$. It holds $\partial_{x_j} g_\delta(x) = x_j g_\delta(x)^{-1}$ and $\partial_{x_j}\partial_{x_i} g_\delta(x) = g_\delta(x)^{-1}(\delta_{i,j} - x_i x_j g_\delta(x)^{-2})$ for every $x \in \mathbb{R}^2$ and $j = 1,2$, and moreover $|x| \le g_\delta(x) \le |x| + \delta^{1/2}$.
Denote

$$R^\epsilon_t(x) := \int_0^t z^\epsilon_s(\phi^\epsilon_s(x))ds + \sum_{k\in\mathbb{N}} \int_0^t \sigma_k(\phi^\epsilon_s(x))\eta^{\epsilon,k}_s ds - \sum_{k\in\mathbb{N}} \int_0^t \sigma_k(\phi^\epsilon_s(x)) \circ dW^k_s,$$

and

$$Z^\epsilon_t(x) := \phi^\epsilon_t(x) - \phi_t(x) - R^\epsilon_t(x),$$

both seen as functions on the whole plane $\mathbb{R}^2$. Applying Itō Formula to $g_\delta(Z^\epsilon_t(x))$ yields:

$$\begin{aligned}dg_\delta(Z^\epsilon_t(x)) &= Z^\epsilon_t(x) g_\delta(Z^\epsilon_t(x))^{-1} \cdot (u^\epsilon_t(\phi^\epsilon_t(x)) - u_t(\phi^\epsilon_t(x)))\,dt \\ &+ Z^\epsilon_t(x) g_\delta(Z^\epsilon_t(x))^{-1} \cdot (u_t(\phi^\epsilon_t(x)) - u_t(\phi_t(x)))\,dt \\ &+ \sum_{k\in\mathbb{N}} Z^\epsilon_t(x) g_\delta(Z^\epsilon_t(x))^{-1} \cdot (\sigma_k(\phi^\epsilon_t(x)) - \sigma_k(\phi_t(x)))\,dW^k_t \\ &+ Z^\epsilon_t(x) g_\delta(Z^\epsilon_t(x))^{-1} \cdot (c(\phi^\epsilon_t(x)) - c(\phi_t(x)))\,dt \\ &+ \sum_{k\in\mathbb{N}} \sum_{i,j=1}^2 g_\delta(Z^\epsilon_t(x))^{-1}(\delta_{i,j} - (Z^\epsilon_t(x))^i (Z^\epsilon_t(x))^j g_\delta(Z^\epsilon_t(x))^{-2}) \\ &\quad \times (\sigma_k(\phi^\epsilon_t(x)) - \sigma_k(\phi_t(x)))^i\,(\sigma_k(\phi^\epsilon_t(x)) - \sigma_k(\phi_t(x)))^j\,dt,\end{aligned}$$

and therefore

$$\begin{aligned}\hat{\mathbb{E}}\left[|\phi^\epsilon_t(x) - \phi^\epsilon_t(x)|\right] &\le \hat{\mathbb{E}}\left[|Z^\epsilon_t(x)|\right] + \hat{\mathbb{E}}\left[|R^\epsilon_t(x)|\right] \le \hat{\mathbb{E}}\left[g_\delta(Z^\epsilon_t(x))\right] + \hat{\mathbb{E}}\left[|R^\epsilon_t(x)|\right] \\ &\lesssim \delta^{1/2} + \hat{\mathbb{E}}\left[|R^\epsilon_t(x)|\right] + \hat{\mathbb{E}}\left[\int_0^t |u^\epsilon_s(\phi^\epsilon_s(x)) - u_s(\phi^\epsilon_s(x))|\,ds\right] \\ &+ \hat{\mathbb{E}}\left[\int_0^t |u_s(\phi^\epsilon_s(x)) - u_s(\phi_s(x))|\,ds\right] \\ &+ \hat{\mathbb{E}}\left[\int_0^t |\phi^\epsilon_s(x) - \phi_s(x)|\,ds\right] + \delta^{-1/2}\hat{\mathbb{E}}\left[\sup_{t\in[0,T]} |R^\epsilon_t(x)|\right],\end{aligned}$$

where in the last line we have used $g_\delta(Z^\epsilon_s(x))^{-1} \le \delta^{-1/2}$ and $|\phi^\epsilon_s(x) - \phi_s(x)| \lesssim |Z^\epsilon_s(x)| + |R^\epsilon_s(x)|$.

Taking the integral over $x \in \mathbb{T}^2$ and using assumptions (A2), (A4), concavity of the function γ, Jensen inequality, Proposition 3.6 and Proposition 3.7 we get

$$\hat{\mathbb{E}}\left[\|\phi^\epsilon_t - \phi_t\|_{L^1(\mathbb{T}^2,\mathbb{T}^2)}\right] \lesssim \delta^{1/2} + \delta^{-1/2}\epsilon^{1/42}\log^{47/42}(1+\epsilon^{-1}) + c_\epsilon + \int_0^t \gamma\left(\hat{\mathbb{E}}\left[\|\phi^\epsilon_s - \phi_s\|_{L^1(\mathbb{T}^2,\mathbb{T}^2)}\right]\right) ds$$

uniformly in $t \in [0,T]$ and $\delta > 0$. Taking $\delta = \epsilon^{1/42}\log^{47/42}(1+\epsilon^{-1})$ we deduce the desired result by Lemma 3.2. □

3.4 Convergence of large-scale dynamics

In order to prove the convergence of the large-scale processes $\Xi^\epsilon \to \Xi$, we need assumptions on the source terms. Hence, let $q^\epsilon, q : [0,T] \times \mathbb{T}^2 \to \mathbb{R}$ be such that:

(A6) there exists a constant C such that for every $\epsilon > 0$ it holds $q^\epsilon, q \in L^1([0,T], L^\infty(\mathbb{T}^2))$ and

$$\int_0^T \|q^\epsilon_s\|_{L^\infty(\mathbb{T}^2)} ds \le C, \qquad \int_0^T \|q_s\|_{L^\infty(\mathbb{T}^2)} ds \le C;$$

(A7) $q^\epsilon - q$ converges to zero in $L^1([0,T], L^\infty(\mathbb{T}^2))$.

Recall the representation formulas for the solutions of (3.1) and (3.2)

$$\Xi^\epsilon_t = \tilde{\mathbb{E}}\left[\Xi_0 \circ (\phi^\epsilon_t)^{-1} + \int_0^t q^\epsilon_s \circ \phi^\epsilon_s \circ (\phi^\epsilon_t)^{-1} ds\right],$$
$$\Xi_t = \tilde{\mathbb{E}}\left[\Xi_0 \circ (\phi_t)^{-1} + \int_0^t q_s \circ \phi_s \circ (\phi_t)^{-1} ds\right],$$

with ϕ^ϵ and ϕ solving respectively (3.3) and (3.4).

As made clear by the following proof, these representation formulas are the key ingredient needed to show Theorem 1.2, thus justifying our Definition 3.3 in terms of these identities.

Proof of Theorem 1.2. Let $f \in L^1(\mathbb{T}^2)$ and $t \in [0,T]$. We have

$$
\begin{aligned}
&\left|\int_{\mathbb{T}^2} \Xi^\epsilon_t(x) f(x)dx - \int_{\mathbb{T}^2} \Xi_t(x) f(x)dx\right| \\
&\leq \left|\int_{\mathbb{T}^2} \tilde{\mathbb{E}}\left[\Xi_0((\phi^\epsilon_t)^{-1}(x))\right] f(x)dx - \int_{\mathbb{T}^2} \tilde{\mathbb{E}}\left[\Xi_0((\phi_t)^{-1}(x))\right] f(x)dx\right| \\
&\quad + \left|\int_{\mathbb{T}^2} \tilde{\mathbb{E}}\left[\int_0^t q^\epsilon_s(\phi^\epsilon_s((\phi^\epsilon_t)^{-1}(x)))ds\right] f(x)dx - \int_{\mathbb{T}^2} \tilde{\mathbb{E}}\left[\int_0^t q_s(\phi_s((\phi_t)^{-1}(x)))ds\right] f(x)dx\right| \\
&= \left|\tilde{\mathbb{E}}\left[\int_{\mathbb{T}^2} \Xi_0((\phi^\epsilon_t)^{-1}(x)) f(x)dx - \int_{\mathbb{T}^2} \Xi_0((\phi_t)^{-1}(x)) f(x)dx\right]\right| \\
&\quad + \left|\tilde{\mathbb{E}}\left[\int_{\mathbb{T}^2}\int_0^t q^\epsilon_s(\phi^\epsilon_s((\phi^\epsilon_t)^{-1}(x)))ds f(x)dx - \int_{\mathbb{T}^2}\int_0^t q_s(\phi_s((\phi_t)^{-1}(x)))ds f(x)dx\right]\right| \\
&= \left|\tilde{\mathbb{E}}\left[\int_{\mathbb{T}^2} \Xi_0(y) f(\phi^\epsilon_t(y))dy - \int_{\mathbb{T}^2} \Xi_0(y) f(\phi_t(y))dy\right]\right| \\
&\quad + \left|\tilde{\mathbb{E}}\left[\int_0^t\int_{\mathbb{T}^2} q^\epsilon_s(\phi^\epsilon_s(y)) f(\phi^\epsilon_t(y))dyds - \int_0^t\int_{\mathbb{T}^2} q_s(\phi_s(y)) f(\phi_t(y)dyds\right]\right|.
\end{aligned}
$$

Taking expectation with respect to $\mathbb{P}$, the first summand is bounded by

$$
\begin{aligned}
&\mathbb{E}\left|\tilde{\mathbb{E}}\left[\int_{\mathbb{T}^2} \Xi_0(y) f(\phi^\epsilon_t(y))dy - \int_{\mathbb{T}^2} \Xi_0(y) f(\phi_t(y))dy\right]\right| \\
&\quad \leq \|\Xi_0\|_{L^\infty(\mathbb{T}^2)} \hat{\mathbb{E}}\left[\int_{\mathbb{T}^2} \left|f(\phi^\epsilon_t(y)) - f(\phi_t(y))\right| dy\right].
\end{aligned} \tag{3.26}
$$

As for the second term, we can rewrite

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{T}^2} q^\epsilon_s(\phi^\epsilon_s(y)) f(\phi^\epsilon_t(y))dyds - \int_0^t\int_{\mathbb{T}^2} q_s(\phi_s(y)) f(\phi_t(y))dyds \\
&\quad = \int_0^t\int_{\mathbb{T}^2} q^\epsilon_s(\phi^\epsilon_s(y)) f(\phi^\epsilon_t(y))dyds - \int_0^t\int_{\mathbb{T}^2} q^\epsilon_s(\phi^\epsilon_s(y)) f(\phi_t(y))dyds \\
&\quad\quad + \int_0^t\int_{\mathbb{T}^2} q^\epsilon_s(\phi^\epsilon_s(y)) f(\phi_t(y))dyds - \int_0^t\int_{\mathbb{T}^2} q_s(\phi^\epsilon_s(y)) f(\phi_t(y))dyds \\
&\quad\quad + \int_0^t\int_{\mathbb{T}^2} q_s(\phi^\epsilon_s(y)) f(\phi_t(y))dyds - \int_0^t\int_{\mathbb{T}^2} q_s(\phi_s(y)) f(\phi_t(y))dyds,
\end{aligned}
$$

with estimates

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\left|\int_0^t\int_{\mathbb{T}^2} q^\epsilon_s(\phi^\epsilon_s(y)) f(\phi^\epsilon_t(y))dyds - \int_0^t\int_{\mathbb{T}^2} q^\epsilon_s(\phi^\epsilon_s(y)) f(\phi_t(y))dyds\right|\right] \\
&\quad \leq \int_0^t \|q^\epsilon_s\|_{L^\infty(\mathbb{T}^2)} ds \hat{\mathbb{E}}\left[\int_{\mathbb{T}^2} |f(\phi^\epsilon_t(y)) - f(\phi_t(y))| dy\right];
\end{aligned} \tag{3.27}
$$

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\left|\int_0^t\int_{\mathbb{T}^2} q^\epsilon_s(\phi^\epsilon_s(y)) f(\phi_t(y))dyds - \int_0^t\int_{\mathbb{T}^2} q_s(\phi^\epsilon_s(y)) f(\phi_t(y))dyds\right|\right] \\
&\quad \leq \int_0^t \|q^\epsilon_s - q_s\|_{L^\infty(\mathbb{T}^2)} ds \|f\|_{L^1(\mathbb{T}^2)};
\end{aligned} \tag{3.28}
$$

and

$$\begin{aligned}\hat{\mathbb{E}}\left[\left|\int_0^t\int_{\mathbb{T}^2} q_s(\phi_s^\epsilon(y))f(\phi_t(y))dyds - \int_0^t\int_{\mathbb{T}^2} q_s(\phi_s(y))f(\phi_t(y))dyds\right|\right] \\ \leq \hat{\mathbb{E}}\left[\int_0^t\int_{\mathbb{T}^2} |q_s(\phi_s^\epsilon(y)) - q_s(\phi_s(y))||f(\phi_t(y))|dyds\right] \\ =: \hat{\mathbb{E}}\left[\int_0^t\int_{\mathbb{T}^2} |q_s(\phi_s^\epsilon(y)) - q_s(\phi_s(y))|d\mu(y)ds\right], \end{aligned} \tag{3.29}$$

where $d\mu(y) := |f(\phi_t(y))|dy$ is a random Radon measure on $\mathbb{T}^2$.
By assumptions (A6) and (A7), the terms (3.26), (3.27) and (3.28) go to zero as $\epsilon \to 0$, using the same reasoning of [FP21, Theorem 5.1]. Therefore, here we restrict ourselves to only consider the remaining term (3.29).
Let us argue *per absurdum*. Suppose by contradiction that there exists a subsequence $\epsilon_k \to 0$ such that

$$\hat{\mathbb{E}}\left[\int_0^t\int_{\mathbb{T}^2} |q_s(\phi_s^{\epsilon_k}(y)) - q_s(\phi_s(y))|d\mu(y)ds\right] \geq C \tag{3.30}$$

for some $C > 0$ and for every ϵ_k.
Let $\mathcal{N}$ and $\tilde{\mathcal{N}}$ be negligible sets such that ϕ_t is measure preserving for every $\omega \in \mathcal{N}^c$ and $\tilde{\omega} \in \tilde{\mathcal{N}}^c$.
Take $\delta > 0$. By Lusin Theorem [Rud70, Theorem 2.23] there exists a measurable set $C_\delta \subset [0,t]\times\mathbb{T}^2$ with $\mathscr{L}_{[0,t]}\otimes\mathscr{L}_{\mathbb{T}^2}([0,t]\times\mathbb{T}^2\setminus C_\delta) < \delta$ and a continuous function $Q_\delta \in C([0,t]\times\mathbb{T}^2)$ that coincides with q on C_δ. Therefore

$$\begin{aligned}\int_0^t\int_{\mathbb{T}^2} |q_s(\phi_s^{\epsilon_k}(y)) - q_s(\phi_s(y))|d\mu(y)ds &= \int_{C_\delta} |q_s(\phi_s^{\epsilon_k}(y)) - q_s(\phi_s(y))|d\mu(y)ds \\ &\quad + \int_{[0,t]\times\mathbb{T}^2\setminus C_\delta} |q_s(\phi_s^{\epsilon_k}(y)) - q_s(\phi_s(y))|d\mu(y)ds \\ &\leq \int_{[0,t]\times\mathbb{T}^2} |Q_\delta(s,\phi_s^{\epsilon_k}(y)) - Q_\delta(s,\phi_s(y))|d\mu(y)ds \\ &\quad + 2\int_{[0,t]\times\mathbb{T}^2\setminus C_\delta} \|q_s\|_{L^\infty(\mathbb{T}^2)}d\mu(y)ds.\end{aligned}$$

Let us consider the second term first. Recalling $d\mu(y) = |f(\phi_t(y))|dy$, we have

$$\begin{aligned}\int_{[0,t]\times\mathbb{T}^2\setminus C_\delta} \|q_s\|_{L^\infty(\mathbb{T}^2)}d\mu(y)ds &= \int_{[0,t]\times\mathbb{T}^2\setminus C_\delta} \|q_s\|_{L^\infty(\mathbb{T}^2)}|f(\phi_t(y))|dyds \\ &= \int_{\phi_t^{-1}(C_\delta^c)} \|q_s\|_{L^\infty(\mathbb{T}^2)}|f(y)|dyds,\end{aligned}$$

with $\phi_t^{-1}(C_\delta^c) := \{(s,y) : (s,\phi_t(y)) \in C_\delta^c\}$. Since ϕ_t is measure preserving for every $\omega \in \mathcal{N}^c$ and $\tilde{\omega} \in \tilde{\mathcal{N}}^c$, it is easy to check

$$\mathscr{L}_{[0,t]}\otimes\mathscr{L}_{\mathbb{T}^2}(\phi_t^{-1}(C_\delta^c)) = \mathscr{L}_{[0,t]}\otimes\mathscr{L}_{\mathbb{T}^2}(C_\delta^c) < \delta$$

$\hat{\mathbb{P}}$-almost surely, and since $\|q\|_{L^\infty(\mathbb{T}^2)}|f| \in L^1([0,t]\times\mathbb{T}^2)$, absolute continuity of Lebesgue integral gives the existence of $\delta > 0$ such that for every $\omega \in \mathcal{N}^c$ and $\tilde{\omega} \in \tilde{\mathcal{N}}^c$

$$\int_{[0,t]\times\mathbb{T}^2\setminus C_\delta} \|q_s\|_{L^\infty(\mathbb{T}^2)} d\mu(y) ds < C/3.$$

We fix such a δ hereafter. For the first term we argue as follows: since we have proved

$$\sup_{t\in[0,T]} \hat{\mathbb{E}}\left[\|\phi_t^{\epsilon_k} - \phi_t\|_{L^1(\mathbb{T}^2,\mathbb{T}^2)}\right] \to 0$$

as $\epsilon_k \to 0$, then for every fixed $s \in [0,T]$ there exists a subsequence (that we still denote ϵ_k) such that the maps

$$\Phi_s^{\epsilon_k} : \hat{\Omega}\times\mathbb{T}^2 \to [0,T]\times\mathbb{T}^2,$$
$$\Phi_s^{\epsilon_k}(\hat{\omega}, y) = (s, \phi^\epsilon(\hat{\omega}, s, y))$$

converge $\hat{\mathbb{P}}\otimes\mathscr{L}_{\mathbb{T}^2}$-almost everywhere to Φ_s given by $\Phi_s(\hat{\omega}, y) = (s, \phi(\hat{\omega}, s, y))$. By almost sure continuity in time of $\Phi_s^{\epsilon_k}$ and Φ_s, it is possible to extract a common subsequence $\epsilon_k \to 0$ such that $\Phi_s^{\epsilon_k}$ converges $\hat{\mathbb{P}}\otimes\mathscr{L}_{\mathbb{T}^2}$-almost everywhere to Φ_s simultaneously for all $s \in [0,T]$.
Therefore, since Q_δ is continuous on $[0,t]\times\mathbb{T}^2$, also $Q_\delta(\Phi^{\epsilon_k})$ converges $\hat{\mathbb{P}}\otimes\mathscr{L}_{[0,t]}\otimes\mathscr{L}_{\mathbb{T}^2}$-almost everywhere to $Q_\delta(\Phi)$, and since μ is absolutely continuous with respect to $\mathscr{L}_{\mathbb{T}^2}$ for almost every $\hat{\omega}\in\hat{\Omega}$, the convergence is actually $\hat{\mathbb{P}}\otimes\mathscr{L}_{[0,t]}\otimes\mu_{\hat{\omega}}$-almost everywhere; moreover, $Q_\delta(\Phi^{\epsilon_k})$ is dominated by the constant $\sup_{s\in[0,t],y\in\mathbb{T}^2}|Q_\delta(s,y)|$, and Lebesgue dominated convergence yields convergence in $L^1(\hat{\Omega}\times[0,T]\times\mathbb{T}^2, \hat{\mathbb{P}}\otimes\mathscr{L}_{[0,t]}\otimes\mu_{\hat{\omega}})$, that is

$$\hat{\mathbb{E}}\left[\int_{[0,t]\times\mathbb{T}^2} |Q_\delta(s,\phi_s^{\epsilon_k}(y)) - Q_\delta(s,\phi_s(y))| d\mu(y) ds\right] \to 0,$$

as $\epsilon_k \to 0$. This contradicts (3.30), and therefore we have proved: for every $f \in L^1(\mathbb{T}^2)$

$$\mathbb{E}\left|\int_{\mathbb{T}^2} \Xi_t^\epsilon(x) f(x) dx - \int_{\mathbb{T}^2} \Xi_t(x) f(x) dx\right| \to 0 \quad \text{as } \epsilon \to 0,$$

for every fixed $t \in [0,T]$. Since $\|\Xi_t^\epsilon\|_{L^\infty(\mathbb{T}^2)}$ is bounded uniformly in $\epsilon > 0$ and $t \in [0,T]$, pointwise converges implies convergence in $L^p([0,T])$ for every finite p by Lebesgue dominated convergence Theorem.
Finally, if $q \in L^1([0,T], Lip(\mathbb{T}^2))$ and $f \in Lip(\mathbb{T}^2)$ with $[f]_{Lip(\mathbb{T}^2)} \le 1$, we have

$$\begin{aligned}\hat{\mathbb{E}}\left[\int_{\mathbb{T}^2} |f(\phi_t^\epsilon(y)) - f(\phi_t(y))|\, dy\right] &\le \hat{\mathbb{E}}\left[\int_{\mathbb{T}^2} |\phi_t^\epsilon(y) - \phi_t(y)|\, dy\right] \\ &\le \sup_{t\in[0,T]} \hat{\mathbb{E}}\left[\|\phi_t^\epsilon - \phi_t\|_{L^1(\mathbb{T}^2,\mathbb{T}^2)}\right],\end{aligned}$$

controlling (3.26) and (3.27) uniformly in f; also, since $\|f\|_{L^\infty(\mathbb{T}^2)} \le 1$ it holds

$$\begin{aligned}&\hat{\mathbb{E}}\left[\int_0^t \int_{\mathbb{T}^2} |q_s(\phi_s^\epsilon(y)) - q_s(\phi_s(y))||f(\phi_t(y))| dy ds\right] \\ &\le \hat{\mathbb{E}}\left[\int_0^t \int_{\mathbb{T}^2} \|q_s\|_{Lip(\mathbb{T}^2)} |\phi_s^\epsilon(y) - \phi_s(y)| dy ds\right] \\ &\le \int_0^t \|q_s\|_{Lip(\mathbb{T}^2)} ds \sup_{s\in[0,T]} \hat{\mathbb{E}}\left[\|\phi_s^\epsilon - \phi_s\|_{L^1(\mathbb{T}^2,\mathbb{T}^2)}\right],\end{aligned}$$

allowing to bound (3.29) in a simpler way. Putting all together, we have proved the desired convergence uniformly in $t \in [0,T]$ and $f \in Lip(\mathbb{T}^2)$ with $[f]_{Lip(\mathbb{T}^2)} \leq 1$, $\|f\|_{L^\infty(\mathbb{T}^2)} \leq 1$. The proof is complete. □

3.5 Examples

Let us finally discuss how assumptions (A1)-(A7) are fulfilled by our main motivational examples, namely advection-diffusion or Navier-Stokes equations at large scales coupled with stochastic Euler equations at small scales.
First of all, notice that in the case of passive scalars, like in the advection-diffusion equations, there is nothing to actually prove since all the subjects of assumptions (A1)-(A7) are given *a priori*. On the other hand, in the Navier-Stokes system the fields u^ϵ, u are given by $u^\epsilon = K * \Xi^\epsilon$, $u = K * \Xi$, and therefore (A1), (A2) and (A4) need to be checked. The verification of (A4) needs an additional requirement on the external source q: assume

(A8) there exists a constant C such that for almost every $t \in [0,T]$ and almost every $x, y \in \mathbb{T}^2$
$$|q(t,x) - q(t,y)| \leq C\gamma(|x-y|).$$

Proposition 3.12. *Let $\nu \geq 0$, $\Xi_0 \in L^\infty(\mathbb{T}^2)$ with zero spatial average and consider the Navier-Stokes ($\nu > 0$) or Euler ($\nu = 0$) system*
$$\begin{cases} d\Xi^\epsilon_t + (u^\epsilon_t + v^\epsilon_t)\cdot \nabla \Xi^\epsilon_t dt = \nu\Delta\Xi^\epsilon_t dt + q^\epsilon_t dt, \\ d\xi^\epsilon_t + (u^\epsilon_t + v^\epsilon_t)\cdot\nabla\xi^\epsilon_t dt = -\epsilon^{-1}\xi^\epsilon_t dt + \epsilon^{-1}\sum_{k\in\mathbb{N}} \varsigma_k dW^k_t, \\ u^\epsilon_t = -\nabla^\perp(-\Delta)^{-1}\Xi^\epsilon_t, \\ v^\epsilon_t = -\nabla^\perp(-\Delta)^{-1}\xi^\epsilon_t, \end{cases}$$
and the limiting large-scale dynamics
$$\begin{cases} d\Xi_t + u_t\cdot\nabla\Xi_t dt + \sum_{k\in\mathbb{N}}\sigma_k\cdot\nabla\Xi_t \circ dW^k_t = \nu\Delta\Xi_t dt + q_t dt, \\ u_t = -\nabla^\perp(-\Delta)^{-1}\Xi_t. \end{cases}$$
Assume (A3), (A5)-(A8) and take q^ϵ_t, q_t with zero spatial average for almost every $t \in [0,T]$. Then the velocity fields u^ϵ, u satisfy (A1), (A2) and (A4).

Proof. Concerning (A1), measurability can be deduced by $u^\epsilon = K * \Xi^\epsilon$, $u = K * \Xi$, representation formulas (3.5) and (3.6), and the fact that ϕ^ϵ, ϕ are stochastic flows of measure-preserving homeomorphisms. Assumption (A2) is given by $u^\epsilon = K * \Xi^\epsilon$, $u = K * \Xi$, (3.7) and Lemma 3.1.
Finally, let us then verify (A4). Recall
$$\begin{aligned} u^\epsilon_t(x) &= \int_{\mathbb{T}^2} K(x-y)\Xi^\epsilon_t(y)dy \\ &= \int_{\mathbb{T}^2} K(x-y)\tilde{\mathbb{E}}\left[\Xi_0((\phi^\epsilon_t)^{-1}(y)) + \int_0^t q^\epsilon_s(\phi^\epsilon_s((\phi^\epsilon_t)^{-1}(y)))ds\right]dy \\ &= \tilde{\mathbb{E}}\left[\int_{\mathbb{T}^2} K(x-\phi^\epsilon_t(y))\Xi_0(y)dy\right] + \tilde{\mathbb{E}}\left[\int_{\mathbb{T}^2} K(x-\phi^\epsilon_t(y))\int_0^t q^\epsilon_s(\phi^\epsilon_s(y))dsdy\right], \end{aligned}$$

and

$$\begin{aligned} u_t(x) &= \int_{\mathbb{T}^2} K(x-y)\Xi_t(y)dy \\ &= \int_{\mathbb{T}^2} K(x-y)\tilde{\mathbb{E}}\left[\Xi_0((\phi_t)^{-1}(y)) + \int_0^t q_s(\phi_s((\phi_t)^{-1}(y)))ds\right]dy \\ &= \tilde{\mathbb{E}}\left[\int_{\mathbb{T}^2} K(x-\phi_t(y))\Xi_0(y)dy\right] + \tilde{\mathbb{E}}\left[\int_{\mathbb{T}^2} K(x-\phi_t(y))\int_0^t q_s(\phi_s(y))dsdy\right]. \end{aligned}$$

We have

$$\begin{aligned} &\int_{\mathbb{T}^2} |u_t^\epsilon(x) - u_t(x)|dx \\ &\quad \leq \tilde{\mathbb{E}}\left[\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} |K(x-\phi_t^\epsilon(y)) - K(x-\phi_t(y))|\,|\Xi_0(y)|dydx\right] \\ &\qquad + \tilde{\mathbb{E}}\left[\int_{\mathbb{T}^2}\left|\int_{\mathbb{T}^2} K(x-\phi_t^\epsilon(y))\int_0^t q_s^\epsilon(\phi_s^\epsilon(y))dsdy - \int_{\mathbb{T}^2} K(x-\phi_t(y))\int_0^t q_s(\phi_s(y))dsdy\right| dx\right] \\ &\quad \leq \tilde{\mathbb{E}}\left[\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} |K(x-\phi_t^\epsilon(y)) - K(x-\phi_t(y))|\,|\Xi_0(y)|dydx\right] \\ &\qquad + \tilde{\mathbb{E}}\left[\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} |K(x-\phi_t^\epsilon(y)) - K(x-\phi_t(y))|\left|\int_0^t q_s^\epsilon(\phi_s^\epsilon(y))ds\right| dydx\right] \\ &\qquad + \tilde{\mathbb{E}}\left[\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} |K(x-\phi_t(y))|\int_0^t |q_s^\epsilon(\phi_s^\epsilon(y)) - q_s(\phi_s^\epsilon(y))ds|\,dydx\right] \\ &\qquad + \tilde{\mathbb{E}}\left[\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} |K(x-\phi_t(y))|\int_0^t |q_s(\phi_s^\epsilon(y)) - q_s(\phi_s(y))ds|\,dydx\right] \\ &\quad \lesssim \gamma\left(\tilde{\mathbb{E}}\left[\|\phi_t^\epsilon - \phi_t\|_{L^1(\mathbb{T}^2,\mathbb{T}^2)}\right]\right) + \int_0^t \|q_s^\epsilon - q_s\|_{L^\infty(\mathbb{T}^2)}ds \\ &\qquad + \int_0^t \gamma\left(\tilde{\mathbb{E}}\left[\|\phi_s^\epsilon - \phi_s\|_{L^1(\mathbb{T}^2,\mathbb{T}^2)}\right]\right)ds, \end{aligned}$$

that is the desired estimate, since $\int_0^t \|q_s^\epsilon - q_s\|_{L^\infty(\mathbb{T}^2)}ds \to 0$ as $\epsilon \to 0$ by assumption (A7). □

Chapter 4

From additive to transport noise in 3D fluids

In this chapter we will prove Theorem 1.3 on the convergence of the slow component of the coupled fast-slow Navier-Stokes system. For the sake of generality here we look at

$$\begin{cases} du^\epsilon_t = Au^\epsilon_t dt + b(u^\epsilon_t, u^\epsilon_t)dt + b(v^\epsilon_t, u^\epsilon_t)dt, \\ dv^\epsilon_t = \epsilon^{-1} C v^\epsilon_t dt + Av^\epsilon dt + b(u^\epsilon_t, v^\epsilon_t)dt + b(v^\epsilon_t, v^\epsilon_t)dt + \epsilon^{-1} Q^{1/2} dW_t, \end{cases} \tag{4.1}$$

where A and C are (possibly unbounded) negative definite linear operators on a separable Hilbert space H, and the map $b : H \times H \to H$ is bilinear and enjoys suitable properties detailed below in assumptions (B1)-(B4). We shall always assume solutions (u^ϵ, v^ϵ) satisfy $\operatorname{div} u^\epsilon = \operatorname{div} v^\epsilon = 0$ (in distributional sense) with deterministic initial condition $(u_0, y_0) \in H \times H$.
Under suitable conditions, we are able to prove the convergence of u^ϵ towards a solution of

$$du_t = Au_t dt + b(u_t, u_t)dt + b((-C)^{-1} Q^{1/2} \circ dW_t, u_t) + b(r, u_t)dt, \tag{4.2}$$

in the sense of Theorem 1.3; of course the latter theorem on the Navier-Stokes system descends from the general case by specialization.
Let us describe the strategy of the proof. The method here presented is originally due to Papanicolaou, Stroock and Varadhan [PSV88]. New developments and a presentation may be found in the book [FGPS10]. It has been recently extended to infinite dimension and partial differential equations, see for instance [dBG12, DdMV16, DV12, DV21].
Consider the normalized small-scale process $y^\epsilon := \epsilon^{1/2} v^\epsilon$. The evolution of the Markov process (u^ϵ, y^ϵ) is described by its infinitesimal generator $\mathscr{L}^\epsilon$, which takes the following form when applied to a suitable test function φ:

$$\begin{aligned} \mathscr{L}^\epsilon \varphi(u, y) = \langle Au + b(u, u), D_u\varphi \rangle + \epsilon^{-1/2} \langle b(y, u), D_u \varphi \rangle \\ + \langle Ay + b(u, y), D_y \varphi \rangle + \epsilon^{-1/2} \langle b(y, y), D_y \varphi \rangle \\ + \epsilon^{-1} \langle Cy, D_y \varphi \rangle + \frac{\epsilon^{-1}}{2} Tr(Q D^2_y \varphi). \end{aligned}$$

Since we are interested in the limiting behaviour of u^ϵ as the parameter ϵ goes to zero, we add correctors to φ in order to cancel out singular terms in the expression of $\mathscr{L}^\epsilon \varphi$, on the one hand, and simultaneously eliminate the dependence on y in the terms of order one, on the other. Thus, consider the *perturbed test function*

$$\varphi^\epsilon(u,y) = \varphi(u) + \epsilon^{1/2}\varphi_1^\epsilon(u,y) + \epsilon\varphi_2^\epsilon(u,y),$$

where φ_1^ϵ and φ_2^ϵ are suitable correctors.
It is immediately clear that terms of order ϵ^{-1} in the expression of $\mathscr{L}^\epsilon\varphi^\epsilon$ vanish, since φ does not depend on y.
Denoting $\mathscr{L}_y^\epsilon$ the operator

$$\mathscr{L}_y^\epsilon = \langle C_\epsilon y, D_y \cdot\rangle + \frac{1}{2}Tr(QD_y^2\cdot), \quad C_\epsilon \coloneqq C + \epsilon A,$$

and wanting to cancel out terms of order $\epsilon^{-1/2}$, we impose φ_1^ϵ to be a solution to the Poisson equation:

$$\mathscr{L}_y^\epsilon\varphi_1^\epsilon(u,y) = -\langle b(y,u), D_u\varphi\rangle. \tag{4.3}$$

As for the terms of order one in the expression of $\mathscr{L}^\epsilon\varphi^\epsilon$, they equal:

$$\langle Au + b(u,u), D_u\varphi\rangle + \langle b(y,u), D_u\varphi_1^\epsilon\rangle + \langle b(y,y), D_y\varphi_1^\epsilon\rangle + \mathscr{L}_y^\epsilon\varphi_2^\epsilon.$$

Now we make the following key observation: replacing the process y^ϵ by its linear counterpart Y^ϵ, satisfying:

$$dY_t^\epsilon = \epsilon^{-1}C_\epsilon Y_t^\epsilon dt + \epsilon^{-1/2}Q^{1/2}dW_t, \quad Y_0^\epsilon = 0,$$

one can rewrite (we shall see that any φ_1^ϵ satisfying (4.3) is linear in both u and y)

$$\begin{aligned}
\langle b(y,u), D_u\varphi_1^\epsilon(y)\rangle &= \langle b(Y,u), D_u\varphi_1^\epsilon(Y)\rangle + \langle b(y-Y,u), D_u\varphi_1^\epsilon(Y)\rangle \\
&\quad + \langle b(y,u), D_u\varphi_1^\epsilon(y-Y)\rangle, \\
\langle b(y,y), D_y\varphi_1^\epsilon(u)\rangle &= \langle b(Y,Y), D_y\varphi_1^\epsilon(u)\rangle + \langle b(y-Y,Y), D_y\varphi_1^\epsilon(u)\rangle \\
&\quad + \langle b(y,y-Y), D_y\varphi_1^\epsilon(u)\rangle,
\end{aligned}$$

and prove that the terms involving the difference $y - Y$ are infinitesimal as $\epsilon \to 0$ when evaluated at $y = y_t^\epsilon$, $Y = Y_t^\epsilon$, and integrated with respect to time. Therefore, the actual terms of order one in the expression of $\mathscr{L}^\epsilon\varphi^\epsilon$ are given by

$$\langle Au + b(u,u), D_u\varphi\rangle + \langle b(Y,u), D_u\varphi_1^\epsilon(Y)\rangle + \langle b(Y,Y), D_y\varphi_1^\epsilon(u)\rangle + \mathscr{L}_y^\epsilon\varphi_2^\epsilon(u,Y), \tag{4.4}$$

It is not necessary to require the previous quantity to be zero, but it is sufficient to just seek for φ_2^ϵ such that it does not depend on Y, namely

$$\begin{aligned}
\mathscr{L}^0\varphi(u) = \langle Au + b(u,u), D_u\varphi\rangle &+ \langle b(Y,u), D_u\varphi_1^\epsilon(Y)\rangle \\
&+ \langle b(Y,Y), D_y\varphi_1^\epsilon(u)\rangle + \mathscr{L}_y^\epsilon\varphi_2^\epsilon(u,Y)
\end{aligned}$$

for some *effective* generator $\mathscr{L}^0$. In this way, one would formally get $\mathscr{L}^\epsilon\varphi^\epsilon(u^\epsilon,y^\epsilon) = \mathscr{L}^0\varphi(u^\epsilon)$ and identify the limit behaviour of $\mathscr{L}^\epsilon\varphi(u^\epsilon)$, up to an infinitesimal correction as $\epsilon \to 0$.
As already mentioned our results are very general and we are able to study many different systems, some of them notably difficult to study in the Lagrangian formulation, highlighting the fundamental nature of transport noise in fluids. More precisely, in [DP22, Section

7] we show that our method also applies (with minor modifications) also to the Surface Quasi-Geostrophic equations and to the Primitive equations.
Let us briefly describe how the present chapter is structured.
In Section 4.1 we introduce the necessary notation and preliminaries for our analysis. In particular, we introduce the abstract spaces and operators governing our system, and we identify their key properties. Also, here we present the notion of bounded-energy family $\{(u^\epsilon, y^\epsilon)\}_{\epsilon\in(0,1)}$ of weak martingale solutions to our system, that is a family of solutions enjoying some uniform-in-ϵ bound on the energy.
In Section 4.2 we introduce a class of test functions ψ for which it is possible to solve implicitly the Poisson equation $\mathscr{L}_y^\epsilon \phi = -\psi$. The class consists in quadratic functions on H that are continuous maps from some Sobolev space H^θ to $\mathbb{R}$. We also show that, depending on the regularity of C, solutions of the Poisson equations so constructed are more regular than the datum ψ, and recover bounds on the regularity of ϕ and its derivative in terms of the regularity of ψ and C.
In Section 4.3 we apply abstract results on the Poisson equation to carry on the program presented in the Introduction; we identify suitable correctors φ_1^ϵ and φ_2^ϵ to cancel out divergent terms in the expression of $\mathscr{L}^\epsilon\varphi$, and recover the limiting behaviour of the slow variable u^ϵ alone.
In Section 4.4, we prove our main Theorem 1.3 dividing the proof into three different steps: at first, we prove that the family (of the laws of) $\{u^\epsilon\}_{\epsilon\in(0,1)}$ is tight in a suitable space of functions; then, checking that the contribution due to correctors φ_1^ϵ and φ_2^ϵ is actually negligible as $\epsilon \to 0$, we prove that every weak accumulation point u is a solution of a limit closed equation; finally, we recognize the different terms in the equation solved by u as the sum of the original slow dynamics, a Stratonovich transport noise and an Itō-Stokes drift.

4.1 Preliminaries and assumptions

4.1.1 Abstract spaces and operators

The linear operator A and Sobolev spaces

In what follows, the operator $A : D(A) \subset H \to H$ is unbounded, self-adjoint and negative definite. For $s \in \mathbb{R}$, Sobolev space H^s is defined by the relation $H^s := D((-A)^{s/2})$.
For $\alpha \in (0,1)$, $p \geq 1$ and $s \in \mathbb{R}$, we define $W^{\alpha,p}([0,T], H^s)$ as the Sobolev-Slobodeckij space of all $u \in L^p([0,T], H^s)$ such that

$$\int_0^T \int_0^T \frac{\|u_t - u_s\|_{H^s}^p}{|t-s|^{1+\alpha p}} dtds < \infty,$$

endowed with the norm

$$\|u\|_{W^{\alpha,p}([0,T],H^s)}^p := \int_0^T \|u_t\|_{H^s}^p dt + \int_0^T \int_0^T \frac{\|u_t - u_s\|_{H^s}^p}{|t-s|^{1+\alpha p}} dtds.$$

We recall the following compactness criterium from [Sim86].

Lemma 4.1. *For $\sigma > 0$, $\alpha > 1/p$ and $\beta \in (0,\sigma)$ we have the compact embeddings:*

$$L^2([0,T], H^1) \cap W^{\alpha,p}([0,T], H^{-\sigma}) \subset L^2([0,T], H);$$
$$L^\infty([0,T], H) \cap W^{\alpha,p}([0,T], H^{-\sigma}) \subset C([0,T], H^{-\beta}).$$

Denote $\mathscr{S} := \cap_{s\in\mathbb{R}} H^s$ the class of smooth elements $h \in H$, and define

$$F := \{\varphi : H \to \mathbb{R}, \quad \exists h \in \mathscr{S} \text{ such that } \varphi(u) = \langle u, h\rangle\}.$$

Distributions on H are elements of the space $\mathscr{S}' := \cup_{s\in\mathbb{R}} H^s$. Every $\varphi \in F$ is continuous from $\mathscr{S}'$ to $\mathbb{R}$.

The linear operator C

We assume

(C1) The operator $C : D(C) \subset H \to H$ is self-adjoint and negative definite, with principal eigenvalue $-\lambda_0 < 0$;

(C2) There exist $\Gamma \geq \gamma > 1/4$ such that $\|x\|^2_{H^{s+\beta\gamma}} \lesssim \|(-C)^{\beta/2}x\|^2_{H^s} \lesssim \|x\|^2_{H^{s+\beta\Gamma}}$ for every $s \in \mathbb{R}$, $\beta > 0$.

The previous assumptions imply that the operators C and $C_\epsilon := C + \epsilon A$ generate C_0-semigroups on H, that we denote respectively e^{Ct} and $e^{C_\epsilon t}$, $t > 0$. Moreover, for every $s \in \mathbb{R}$ and $\beta_1 > 0$ it holds uniformly in $t > 0$ and $\epsilon \in (0, 1)$:

$$\|(-C_\epsilon)^{\beta_1} e^{C_\epsilon t}\|_{H^s\to H^s} \lesssim \frac{e^{-\lambda_0 t/2}}{t^{\beta_1}};$$

by interpolation, since the operators $(-C_\epsilon)^{-1}C$ and $(-C_\epsilon)^{-1}\epsilon A$ are bounded, we also have for every $\theta \in [\gamma, 1]$, $\lambda = \frac{\lambda_0(1-\theta)}{2(1-\gamma)}$:

$$\|e^{C_\epsilon t}\|_{H^s\to H^{s+2\theta\beta_1}} \lesssim \|(-C)^{\beta_1} e^{C_\epsilon t}\|^{\frac{1-\theta}{1-\gamma}}_{H^s\to H^s} \|(-A)^{\beta_1} e^{C_\epsilon t}\|^{\frac{\theta-\gamma}{1-\gamma}}_{H^s\to H^s} \lesssim \epsilon^{-\beta_1 \frac{\theta-\gamma}{1-\gamma}} \frac{e^{-\lambda t}}{t^{\beta_1}}.$$

In addition, for every $s \in \mathbb{R}$ and $\beta_2 \in [0, 1]$ they hold:

$$\|(-C_\epsilon)^{-\beta_2}(e^{C_\epsilon t} - 1)\|_{H^s\to H^s} \lesssim t^{\beta_2}, \quad \|e^{C_\epsilon t} - 1\|_{H^s\to H^{s-2\Gamma\beta_2}} \lesssim t^{\beta_2}$$

uniformly in $t > 0$ and $\epsilon \in (0, 1)$, and moreover the difference of the semigroups $e^{C_\epsilon t} - e^{Ct}$ satisfies[1] $\|e^{C_\epsilon t} - e^{Ct}\|_{H^{\theta+2\beta_2}\to H^\theta} \lesssim \epsilon^{\beta_2}$ uniformly in $t > 0$.
Finally, for every $\beta_2 \in [0, 1]$ the operator $G_\epsilon := (-C_\epsilon)^{-1} - (-C)^{-1} = \epsilon(-C)^{-1}A(-C_\epsilon)^{-1}$ satisfies $\|G_\epsilon\|_{H^s\to H^{s+2\gamma(1+\beta_2)-2\beta_2}} \lesssim \epsilon^{\beta_2}$.

The bilinear operator b

Concerning the nonlinearity b, we suppose the validity of the following properties:

(B1) $b : H^s \times H^{\theta_0} \to H^s$ is bilinear and continuous for every $s \in \mathbb{R}$, $s < 3/2$, $\theta_0 > 5/2$;

(B2) $b : H^s \times H^{\theta_1} \to H^s$ is bilinear and continuous for every $s \in \mathbb{R}$, $s \geq 3/2$ and $\theta_1 > 1 + s$;

(B3) $b : H^s \times H^r \to H^{s+r-5/2}$ is bilinear and continuous if $s, r-1 \in (-3/2, 3/2)$, $s+r > 1$.

[1] To see this, one can define $y_t := e^{C_\epsilon t}x - e^{Ct}x$, $x \in H$ and notice that $y_t = \int_0^t Cy_s ds + \epsilon\int_0^t Ae^{C_\epsilon s}x ds$; since $y_0 = 0$, Duhamel's Formula gives $y_t = \epsilon\int_0^t e^{C(t-s)}Ae^{C_\epsilon s}x ds$, and using $\|\epsilon^{1-\beta_2}(-A)^{1-\beta_2}e^{C_\epsilon t}\|_{H^s\to H^s} \lesssim e^{-\lambda_0 t/2}t^{\beta_2-1}$ produces the desired inequality.

(B4) $\langle b(x_1,x_2),x_3\rangle = -\langle b(x_1,x_3),x_2\rangle$ for every $x_i \in \mathscr{S}'$, $i=1,2,3$ such that either one of the scalar products is well-defined.

We point out that properties (B1)-(B4) hold true for the Navier-Stokes system in velocity form, with spatial domain equal to the three-dimensional torus equipped with periodic boundary conditions. In the following, we will denote without explicit mention $\theta_0, \theta_1 = \theta_1(s)$ any constants such that (B1) and (B2) hold.

The covariance operator Q

We assume that the covariance operator $Q : H \to H$ satisfies the following properties:

(Q1) Q is symmetric, positive semidefinite and commutes with C. The following operators on H are trace-class for every $t \geq 0$:

$$e^{Ct}Qe^{Ct}, \quad Q_\infty := \int_0^\infty e^{Ct}Qe^{Ct}dt = \frac{1}{2}(-C)^{-1}Q;$$

(Q2) Denoting $\mathcal{N}(0,Q_\infty)$ the Gaussian measure on H with covariance Q_∞ and $s_0 = \max\{\theta_0, 2\Gamma\}$, it holds $\int_H \|w\|^2_{H^{s_0}} d\mathcal{N}(0,Q_\infty)(w) < \infty$.

In (Q2) above, θ_0 can be any real number such that (B1) holds true and Γ is as in (C2). It is easy to see that under (Q1)-(Q2) to following hold true: $e^{C_\epsilon t}Qe^{C_\epsilon t}$, $Q^\epsilon_\infty := \int_0^\infty e^{C_\epsilon t}Qe^{C_\epsilon t}dt$ are trace-class (although in general $Q^\epsilon_\infty \neq \frac{1}{2}(-C_\epsilon)^{-1}Q$ since we do not assume A and Q commuting) and $\int_H \|w\|^2_{H^{\theta_0}} d\mathcal{N}(0,Q^\epsilon_\infty)(w) \leq 1 + \int_H \|w\|^2_{H^{\theta_0}} d\mathcal{N}(0,Q_\infty)(w) < \infty$ for every $\epsilon \in (0,1)$.

4.1.2 Ornstein-Uhlenbeck semigroup

Assume (Q1)-(Q2). For every $\epsilon \in (0,1)$ and $y \in H$ there exists a unique solution $Y^y = Y^y(\epsilon)$ of the Ornstein-Uhlenbeck equation

$$dY^y_t = C_\epsilon Y^y_t dt + Q^{1/2}dW_t, \quad Y^y_0 = y,$$

that is explicitly given by the formula

$$Y^y_t = e^{C_\epsilon t}y + W^{C_\epsilon,Q}_t, \quad W^{C_\epsilon,Q}_t = \int_0^t e^{C_\epsilon(t-s)}Q^{1/2}dW_s.$$

The Ornstein-Uhlenbeck semigroup $P^\epsilon_t : C_b(H) \to C_b(H)$ is defined by

$$P^\epsilon_t\psi(y) := \mathbb{E}\left[\psi(Y^y_t)\right], \quad \psi \in C_b(H), \; y \in H,$$

and it is a semigroup by Markov property. It can be extended uniquely to a strongly continuous semigroup of 1-Lipschitz maps on $L^2(H,\mu^\epsilon)$, $\mu^\epsilon := \mathcal{N}(0,Q^\epsilon_\infty)$, see [DPZ02, Theorem 10.1.5]. The Gaussian measure μ^ϵ is concentrated on $H^{\theta_0} \subset H$, and μ^ϵ is invariant for P^ϵ_t, i.e.

$$\int_H P^\epsilon_t\psi(y)d\mu^\epsilon(y) = \int_H \psi(y)d\mu^\epsilon(y), \quad \forall \psi \in L^2(H,\mu^\epsilon).$$

The domain $D(\mathscr{L}^\epsilon_y)$ of the generator $\mathscr{L}^\epsilon_y : D(\mathscr{L}^\epsilon_y) \to L^2(H,\mu^\epsilon)$ is defined as the set

$$D(\mathscr{L}^\epsilon_y) := \left\{ \psi \in L^2(H,\mu^\epsilon) : \exists \lim_{t\to 0^+} \frac{P^\epsilon_t\psi - \psi}{t} \in L^2(H,\mu^\epsilon) \right\},$$

and $\mathscr{L}^\epsilon_y$ acts on $\psi \in D(\mathscr{L}^\epsilon_y)$ as $\mathscr{L}^\epsilon_y\psi := \lim_{t\to 0^+} \frac{P^\epsilon_t\psi - \psi}{t}$. The generator $\mathscr{L}^\epsilon_y$ is a closed operator on $L^2(H,\mu^\epsilon)$.

4.1.3 Notion of solution and energy estimates

Let us consider again system (4.1), and denote $y^\epsilon_t = \epsilon^{1/2} v^\epsilon_t$:

$$\begin{cases} du^\epsilon_t = Au^\epsilon_t dt + b(u^\epsilon_t, u^\epsilon_t)dt + \epsilon^{-1/2} b(y^\epsilon_t, u^\epsilon_t)dt, \\ dy^\epsilon_t = \epsilon^{-1} C y^\epsilon_t dt + A y^\epsilon_t dt + b(u^\epsilon_t, y^\epsilon_t)dt + \epsilon^{-1/2} b(y^\epsilon_t, y^\epsilon_t)dt + \epsilon^{-1/2} Q^{1/2} dW_t. \end{cases} \tag{4.5}$$

In the following, H_w denotes the space H endowed with the weak topology.

Definition 4.1. We say that the family $\{(u^\epsilon, y^\epsilon)\}_{\epsilon\in(0,1)}$ is a *bounded-energy family of weak martingale solutions* to (4.5) if for every $\epsilon \in (0,1)$ there exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W)$ such that the following hold :

(S1) $(u^\epsilon, y^\epsilon) : \Omega \times [0,T] \to H \times H$ is $\{\mathcal{F}_t\}$-progressively measurable, with paths $u^\epsilon, y^\epsilon \in C([0,T], H_w) \cap L^2([0,T], H^1)$, $\mathbb{P}$-almost surely;

(S2) for every $h \in \mathscr{S}$, the following equalities hold $\mathbb{P}$-almost surely for every $t \in [0,T]$:

$$\langle u^\epsilon_t, h\rangle = \langle u_0, h\rangle + \int_0^t \langle u^\epsilon_s, Ah\rangle + \int_0^t \langle b(u^\epsilon_s, u^\epsilon_s), h\rangle ds + \epsilon^{-1/2} \int_0^t \langle b(y^\epsilon_s, u^\epsilon_s), h\rangle ds,$$

$$\langle y^\epsilon_t, h\rangle = \langle y_0, h\rangle + \epsilon^{-1} \int_0^t \langle y^\epsilon_s, Ch\rangle ds + \int_0^t \langle y^\epsilon_s, Ah\rangle ds + \int_0^t \langle b(u^\epsilon_s, y^\epsilon_s), h\rangle ds$$
$$+ \epsilon^{-1/2} \int_0^t \langle b(y^\epsilon_s, y^\epsilon_s), h\rangle ds + \epsilon^{-1/2} \langle Q^{1/2} W_t, h\rangle;$$

(S3) the family $\{u^\epsilon\}_{\epsilon\in(0,1)}$ is uniformly bounded in

$$\mathcal{U} := L^\infty(\Omega, C([0,T], H_w)) \cap L^\infty(\Omega, L^2([0,T], H^1));$$

(S4) for every fixed $p < \infty$, the family $\{y^\epsilon\}_{\epsilon\in(0,1)}$ is uniformly bounded in

$$\mathcal{Y} := L^p(\Omega, C([0,T], H_w)) \cap L^2(\Omega, L^2([0,T], H^1)).$$

It is worth to comment on the previous definition.
First of all, since we are working on the intersection of two fields and to avoid any confusion with the terminology, let us specify that here we are working with *analytically weak, probabilistically martingale* solutions. Solutions are analytically weak since they solve (4.5) only when tested again smooth test functions $h \in \mathscr{S}$. They are martingale solutions (sometimes referred to also as probabilistically weak solutions) since the stochastic basis in not given a priori (that would be called pathwise or probabilistically strong solutions). To avoid any misunderstanding we point out that hereafter the stochastic basis $\{(\Omega^\epsilon, \mathcal{F}^\epsilon, \{\mathcal{F}^\epsilon_t\}_{t\geq 0}, \mathbb{P}^\epsilon, W^\epsilon)\}_{\epsilon\in(0,1)}$ will be always dependent on ϵ, but we shall drop the indices for notational simplicity.
Second, our solutions form a *bounded-energy family* since in (S3)-(S4) we require suitable energy bounds to hold uniformly in $\epsilon \in (0,1)$ (recall that the range of every elements in $C([0,T], H_w)$ is bounded in H by Banach-Steinhaus Theorem). In classical theory of deterministic Navier-Stokes equations subject to external forcing $f \in L^1([0,T], H)$:

$$\begin{cases} du_t + (u_t \cdot \nabla) u_t dt = \nu \Delta u_t dt + \nabla p_t dt + f_t dt, \\ \operatorname{div} u_t = 0, \end{cases}$$

a very fundamental concept is that of *Leray-Hopf* weak solutions, namely (analytically) weak solutions u enjoying the energy inequality

$$\frac{1}{2}\|u_t\|_H^2 + \int_0^t \|u_s\|_{H^1}^2 ds \leq \frac{1}{2}\|u_0\|_H^2 + \int_0^t \langle u_s, f_s \rangle ds.$$

In the stochastic setting the picture is more complicated since, when the external forcing $f = Q^{1/2}W$ is a stochastic process: i) sensible bounds can only be obtained in expected value; and ii) formally applying Itō Formula to $\|u_t\|_H^2$ introduces and additional term $Tr(Q)dt$ on the right hand side of the estimate.
In [FR08], the authors propose a notion of solution which encodes the energy inequality in the requirement that the process

$$\begin{aligned} E_t^p := \frac{1}{2}\|u_t\|_H^{2p} + p\int_0^t \|u_s\|_H^{2p-2}\|u_s\|_{H^1}^2 ds - \frac{1}{2}\|u_0\|_H^{2p} \\ - \frac{p(2p-1)}{2} Tr(Q) \int_0^t \|u_s\|_H^{2p-2} ds \end{aligned}$$

be an *almost sure super martingale* for every positive integer p, namely $\mathbb{E}\left[E_t^p\right] < \infty$ for all $t \in [0,T]$ and there exists a Lebesgue measurable set $\mathcal{T} \subset (0,T]$, with null Lebesgue measure, such that $\mathbb{E}\left[E_t^p \mathbf{1}_A\right] \leq \mathbb{E}\left[E_s^p \mathbf{1}_A\right]$ for every $s \in \mathcal{T}$, every $t \geq s$ and every $A \in \mathcal{F}_s$.
However, for our purposes there are some limitations in considering solutions satisfying some kind of energy inequality, since: i) it does not seem immediate to recover uniform bounds in $\epsilon \in (0,1)$, and ii) we do not need energy inequality but just energy bounds, and recents developments in convex integration suggest that the class of weak solutions to Navier-Stokes equations with bounded energy may be stricly larger than the class of Leray-Hopf weak solutions, see [BV19] for a deterministic result and [HZZ21] for a stochastic one (even though the solutions constructed there are not known to satisfy H^1 bounds in the space variable).
In order to construct a bounded-energy family of weak martingale solutions to (4.5), one can make use of classical compactness arguments involving the Galerkin approximation scheme:

$$\begin{cases} du_t^{\epsilon,n} = Au_t^{\epsilon,n}dt + \Pi_n b(u_t^{\epsilon,n}, u_t^{\epsilon,n})dt + \epsilon^{-1/2}\Pi_n b(y_t^{\epsilon,n}, u_t^{\epsilon,n})dt, \\ dy_t^{\epsilon,n} = \epsilon^{-1}Cy_t^{\epsilon,n}dt + Ay_t^{\epsilon,n}dt + \Pi_n b(u_t^{\epsilon,n}, y_t^{\epsilon,n})dt + \epsilon^{-1/2}\Pi_n b(y_t^{\epsilon,n}, y_t^{\epsilon,n})dt \\ \qquad\qquad + \epsilon^{-1/2}\Pi_n Q^{1/2} dW_t, \end{cases} \tag{4.6}$$

where $\{\Pi_n\}_{n\in\mathbb{N}}$ is a family of Galerkin projectors and the initial condition is $(u_0^{\epsilon,n}, y_0^{\epsilon,n}) = (\Pi_n u_0, \Pi_n y_0)$. Indeed, since solutions of (4.6) above are smooth in space, uniform energy estimates (S3)-(S4) can be rigorously proved for $(u^{\epsilon,n}, y^{\epsilon,n})$ making use of Itō Formula; then, for every fixed $\epsilon \in (0,1)$, it is possible to prove via Ascoli-Arzelà Theorem that there exist u^ϵ, y^ϵ such that $u^{\epsilon,n} \to u^\epsilon$ and $y^{\epsilon,n} \to y^\epsilon$ with respect to a topology that permits to take the limit in the energy estimates (S3)-(S4), on the one hand, and in the weak formulation of the equation (S2), on the other (up to a possible change in the underlying stochastic basis, in order to gain adaptedness of the processes u^ϵ, y^ϵ).

Proposition 4.2. *There exists at least one bounded-energy family* $\{(u^\epsilon, y^\epsilon)\}_{\epsilon\in(0,1)}$ *of weak martingale solutions to* (4.5).

Existence of a weak martingale solution for fixed $\epsilon \in (0,1)$ is known since [FG95]. The only difference here is uniform in ϵ energy bounds, which require suitable estimates at the level of Galerkin truncations (cfr. Lemma 4.3 and Lemma 4.4 below) and compactness arguments well-suited for the passage to the limit $n \to \infty$.

Remark 4.1. Notice that if (4.5) admits pathwise uniqueness then one obtain the existence of probabilistically strong solution, namely the stochastic basis can be taken independent of ϵ.

Before the proof of Proposition 4.2, we show the needed energy bounds in the following lemmas.

Lemma 4.3. *For every positive integer p it holds*

$$\sup_{\substack{\epsilon\in(0,1),\\ n\in\mathbb{N}}} \int_0^T \mathbb{E}\left[\|y_s^{\epsilon,n}\|_H^{2p-2}\|y_s^{\epsilon,n}\|_{H^\gamma}^2\right] ds \lesssim 1.$$

Proof. Let $\epsilon \in (0,1)$ and $n \in \mathbb{N}$ be fixed, and take an arbitrary $t \in [0,T]$. Applying Itō Formula to $\frac{1}{2}\|y_t^{\epsilon,n}\|_H^{2p}$ we get

$$\begin{aligned}
\frac{1}{2}\|y_t^{\epsilon,n}\|_H^{2p} &+ \epsilon^{-1} p \int_0^t \|y_s^{\epsilon,n}\|_H^{2p-2}\|(-C)^{1/2}y_s^{\epsilon,n}\|_H^2 ds + p\int_0^t \|y_s^{\epsilon,n}\|_H^{2p-2}\|y_s^{\epsilon,n}\|_{H^1}^2 ds \\
&= \frac{1}{2}\|\Pi_n y_0\|_H^{2p} + \epsilon^{-1/2} p \int_0^t \|\Pi_n y_s^{\epsilon}\|_H^{2p-2}\langle y_s^{\epsilon,n}, \Pi_n Q^{1/2} dW_s\rangle \\
&\quad + \epsilon^{-1}\frac{p(2p-1)}{2} Tr(\Pi_n Q \Pi_n)\int_0^t \|y_s^{\epsilon,n}\|_H^{2p-2} ds.
\end{aligned}$$

Taking expectations in the expression above with $p = 1$ we obtain

$$\epsilon^{-1}\int_0^t \mathbb{E}\left[\|y_s^{\epsilon,n}\|_{H^\gamma}^2\right] ds \le \frac{1}{2M}\|y_0\|^2 + \epsilon^{-1}\frac{Tr(Q)}{2M} t,$$

where we have used $\|(-C)^{1/2}y_s^{\epsilon,n}\|_H^2 \ge M\|y_s^{\epsilon,n}\|_{H^\gamma}^2$ for some unimportant constant M; thus we deduce

$$\sup_{\substack{\epsilon\in(0,1),\\ n\in\mathbb{N}}} \int_0^T \mathbb{E}\left[\|y_s^{\epsilon,n}\|_{H^\gamma}^2\right] ds \le \frac{\epsilon}{2M}\|y_0\|_H^2 + \frac{Tr(Q)T}{2M} \lesssim 1.$$

For $p > 1$, we argue as follows: first, recalling $\|y\|_{H^\gamma}^2 \ge \nu_0^\gamma \|y\|_H^2$ for some $\nu_0 > 0$ (the principal eigenvalue of the operator $-A$), for every $t \in [0,T]$ we have

$$\begin{aligned}
&\int_0^t \mathbb{E}\left[\|y_s^{\epsilon,n}\|_H^{2p-2}\|y_s^{\epsilon,n}\|_{H^\gamma}^2\right] ds \\
&\quad\le \frac{\epsilon}{2pM}\|y_0\|_H^{2p} + \frac{2p-1}{2M} Tr(Q)\int_0^t \mathbb{E}\left[\|y_s^{\epsilon,n}\|_H^{2p-2}\right] ds \\
&\quad\le \frac{\epsilon}{2pM}\|y_0\|_H^{2p} + \frac{2p-1}{2M}\nu_0^\gamma Tr(Q)\int_0^t \mathbb{E}\left[\|y_s^{\epsilon,n}\|_H^{2p-4}\|y_s^{\epsilon,n}\|_{H^\gamma}^2\right] ds;
\end{aligned}$$

then, since $p-1$ is a positive integer, by induction we have the desired inequality uniformly in $\epsilon \in (0,1)$ and $n \in \mathbb{N}$. ☐

Lemma 4.4. *For every $p \geq 2$ it holds*

$$\sup_{\substack{\epsilon\in(0,1),\\ n\in\mathbb{N}}} \sup_{t\in[0,T]} \left(\mathbb{E}\left[\|y_t^{\epsilon,n}\|_H^p\right] + \int_0^t \mathbb{E}\left[\|y_s^{\epsilon,n}\|_H^{p-2}\|y_s^{\epsilon,n}\|_{H^1}^2\right] ds \right) \lesssim 1.$$

Proof. As in the previous Lemma 4.3, it is sufficient to prove the result for every positive even integer p. Let us introduce the auxiliary process $Y_t^{\epsilon,n}$ solution of

$$dY_t^{\epsilon,n} = \epsilon^{-1} C_\epsilon Y_t^{\epsilon,n} dt + \epsilon^{-1/2}\Pi_n Q^{1/2} dW_s, \qquad Y_0^{\epsilon,n} = 0,$$

so that, by Itō Formula, the difference process $\zeta_t^{\epsilon,n} := y_t^{\epsilon,n} - Y_t^{\epsilon,n}$ satisfies, for every $t \in [0,T]$ and $p \geq 2$

$$\begin{aligned}
\|\zeta_t^{\epsilon,n}\|_H^p &+ \epsilon^{-1} pM \int_0^t \|\zeta_s^{\epsilon,n}\|_H^{p-2}\|\zeta_s^{\epsilon,n}\|_{H^\gamma}^2 ds + p\int_0^t \|\zeta_s^{\epsilon,n}\|_H^{p-2}\|\zeta_s^{\epsilon,n}\|_{H^1}^2 ds \qquad (4.7)\\
&\leq \|y_0\|_H^p + p\int_0^t \|\zeta_s^{\epsilon,n}\|_H^{p-2}\langle b(u_s^{\epsilon,n}, Y_s^{\epsilon,n}), \zeta_s^{\epsilon,n}\rangle ds\\
&\quad + \epsilon^{-1/2} p \int_0^t \|\zeta_s^{\epsilon,n}\|_H^{p-2}\langle b(y_s^{\epsilon,n}, Y_s^{\epsilon,n}), \zeta_s^{\epsilon,n}\rangle ds\\
&\leq \|y_0\|^p + M_1 \int_0^t \|\zeta_s^{\epsilon,n}\|_H^{p-1}\|u_s^{\epsilon,n}\|_H\|Y_s^{\epsilon,n}\|_{H^{\theta_0}} ds\\
&\quad + \epsilon^{-1/2} M_1 \int_0^t \|\zeta_s^{\epsilon,n}\|_H^{p-1}\|y_s^{\epsilon,n}\|_H\|Y_s^{\epsilon,n}\|_{H^{\theta_0}} ds,
\end{aligned}$$

where M_1 is another unimportant constant. By Young inequality

$$\|Y_s^{\epsilon,n}\|_{H^{\theta_0}}\|\zeta_s^{\epsilon,n}\|_H^{p-1} \leq \frac{\|Y_s^{\epsilon,n}\|_{H^{\theta_0}}^p}{p} + \frac{p-1}{p}\|\zeta_s^{\epsilon,n}\|_H^p,$$

and, for every positive constant c:

$$\begin{aligned}
\epsilon^{-1/2}\|y_s^{\epsilon,n}\|_H\|Y_s^{\epsilon,n}\|_{H^{\theta_0}}\|\zeta_s^{\epsilon,n}\|_H^{p-1} &\leq \frac{c^{-p}}{2p}\|y_s^{\epsilon,n}\|_H^{2p} + \frac{c^{-p}}{2p}\|Y_s^{\epsilon,n}\|_{H^{\theta_0}}^{2p}\\
&\quad + \epsilon^{-\frac{p}{2(p-1)}} \frac{(p-1)c^{\frac{p}{p-1}}}{p}\|\zeta_s^{\epsilon,n}\|_H^p.
\end{aligned}$$

Choosing $c = \left(\frac{p^2 M}{2(p-1)\nu_0^\gamma M_1}\right)^{\frac{p-1}{p}}$, the previous inequalities can be plugged into (4.7) to get

$$\begin{aligned}
\|\zeta_t^{\epsilon,n}\|_H^p + &+ \epsilon^{-1}\int_0^t \|\zeta_s^{\epsilon,n}\|_H^{p-2}\|\zeta_s^{\epsilon,n}\|_{H^\gamma}^2 ds + \int_0^t \|\zeta_s^{\epsilon,n}\|_H^{p-2}\|\zeta_s^{\epsilon,n}\|_{H^1}^2 ds\\
&\lesssim \|y_0\|_H^p + \int_0^t \|Y_s^{\epsilon,n}\|_{H^{\theta_0}}^p ds + \int_0^t \|y_s^{\epsilon,n}\|_H^{2p} ds + \int_0^t \|Y_s^{\epsilon,n}\|_{H^{\theta_0}}^{2p} ds.
\end{aligned}$$

Since $\mathbb{E}\left[\|Y_s^{\epsilon,n}\|_{H^{\theta_0}}^{2p}\right]$ is bounded uniformly in ϵ, n and s by assumption (Q2), and invoking previous Lemma 4.3, the previous inequality produces the bounds for $\zeta^{\epsilon,n}$:

$$\sup_{\substack{\epsilon\in(0,1),\\ n\in\mathbb{N}}} \sup_{t\in[0,T]} \left(\mathbb{E}\left[\|\zeta_t^{\epsilon,n}\|_H^p\right] + \int_0^t \mathbb{E}\left[\|\zeta_s^{\epsilon,n}\|_H^{p-2}\|\zeta_s^{\epsilon,n}\|_{H^1}^2\right] ds \right) \lesssim 1, \qquad (4.8)$$

$$\sup_{\substack{\epsilon\in(0,1),\\ n\in\mathbb{N}}} \epsilon^{-1}\int_0^T \mathbb{E}\left[\|\zeta_s^{\epsilon,n}\|_H^{p-2}\|\zeta_s^{\epsilon,n}\|_{H^\gamma}^2\right] ds \lesssim 1. \qquad (4.9)$$

Since $y^{\epsilon,n} = Y^{\epsilon,n} + \zeta^{\epsilon,n}$, from (4.8) we deduce the thesis. □

We also recall the following result, which is an immediate corollary of Ascoli-Arzelà Theorem.

Lemma 4.5. *Let E be a separable Banach space and let $F \subset E$ be a dense subset. Let $\{f^n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions such that $f^n : [0,T] \to E^*$. Assume that for every $t \in [0,T]$ the sequence $\{f^n_t\}_{n\in\mathbb{N}}$ is equibounded in E^*, and for any fixed $h \in F$ the sequence of real-valued functions $\{t \mapsto \langle f^n_t, h\rangle\}_{n\in\mathbb{N}}$ is equicontinuous. Then, $f^n \in C([0,T];(E^*)_w)$ for every $n \in \mathbb{N}$, and there exists $f \in C([0,T];(E^*)_w)$ such that, up to a subsequence,*

$$f^n \to f \text{ strongly in } C([0,T];(E^*)_w).$$

We are ready to prove our existence result.

Proof of Proposition 4.2. Fix a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W)$. Since the Galerkin system (4.6) is finite-dimensional for every $\epsilon \in (0,1)$ and $n \in \mathbb{N}$, it is classical to show that for every $\epsilon \in (0,1)$ and $n \in \mathbb{N}$ a strong solution to (4.6) exists on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W)$. Hereafter, we fix $\epsilon \in (0,1)$ and we focus on the sequences $\{u^{\epsilon,n}\}_{n\in\mathbb{N}}$ and $\{y^{\epsilon,n}\}_{n\in\mathbb{N}}$. Our aim is to prove that the aforementioned sequences are relatively compact with respect to a suitable topology.
Applying Itō Formula to the function $\|u^{\epsilon,n}_t\|^2_H$, $t \in [0,T]$, we get

$$\|u^{\epsilon,n}_t\|^2_H + 2\int_0^t \|u^{\epsilon,n}_s\|^2_{H^1} ds = \|\Pi_n u_0\|^2_H.$$

Moreover, recalling Lemma 4.4:

$$\sup_{\substack{\epsilon\in(0,1),\\ n\in\mathbb{N}}} \sup_{t\in[0,T]} \left(\mathbb{E}\left[\|y^{\epsilon,n}_t\|^2_H\right] + \int_0^t \mathbb{E}\left[\|y^{\epsilon,n}_s\|^2_{H^1}\right] ds \right) \lesssim 1.$$

Therefore, for any $p < \infty$ the sequences $\{u^{\epsilon,n}\}_{n\in\mathbb{N}}$ and $\{y^{\epsilon,n}\}_{n\in\mathbb{N}}$ are uniformly bounded respectively in the space $L^p(\Omega, L^2([0,T],H^1))$ and $L^2(\Omega, L^2([0,T],H^1))$ and there exist $u^\epsilon \in L^p(\Omega, L^2([0,T],H^1))$ and $y^\epsilon \in L^2(\Omega, L^2([0,T],H^1))$ such that, up to a subsequence that we still denote n:

$$\begin{aligned} u^{\epsilon,n} &\rightharpoonup u^\epsilon, \quad \text{weakly in } L^p(\Omega, L^2([0,T],H^1)),\\ y^{\epsilon,n} &\rightharpoonup y^\epsilon, \quad \text{weakly in } L^2(\Omega, L^2([0,T],H^1)). \end{aligned}$$

Also, again by Lemma 4.4, for every $p < \infty$ the functions $u^{\epsilon,n}$, $y^{\epsilon,n}$ are measurable maps from $[0,T]$ with values in $L^p(\Omega,H) = (L^q(\Omega,H))^*$, $1/p + 1/q = 1$, that are equibounded in $L^p(\Omega,H)$ for every fixed $t \in [0,T]$ (actually uniformly in $t \in [0,T]$). Moreover, for every fixed $h \in L^\infty(\Omega, \mathscr{S})$ and $s,t \in [0,T]$, $s < t$ we have

$$\begin{aligned} |\mathbb{E}\left[\langle u^{\epsilon,n}_t - u^{\epsilon,n}_s, h\rangle\right]| &\leq \int_s^t \mathbb{E}\left[|\langle u^{\epsilon,n}_r, Ah\rangle|\right] dr + \int_s^t \mathbb{E}\left[|\langle b(u^{\epsilon,n}_r, u^{\epsilon,n}_r), h\rangle|\right] dr \\ &\quad \epsilon^{-1/2} \int_s^t \mathbb{E}\left[|\langle b(u^{\epsilon,n}_r, u^{\epsilon,n}_r), h\rangle|\right] dr \\ &\lesssim |t-s| \ \left(1 + \epsilon^{-1/2}\right) \|h\|_{L^\infty(\Omega, H^{\theta_0})}, \end{aligned}$$

and

$$\begin{aligned}|\mathbb{E}\left[\langle y_t^{\epsilon,n} - y_s^{\epsilon,n}, h\rangle\right]| &\leq \epsilon^{-1} \int_s^t \mathbb{E}\left[|\langle y_r^{\epsilon,n}, C_\epsilon h\rangle|\right] dr + \int_s^t \mathbb{E}\left[|\langle b(u_r^{\epsilon,n}, y_r^{\epsilon,n}), h\rangle|\right] dr \\ &\quad \epsilon^{-1/2} \int_s^t \mathbb{E}\left[|\langle b(y_r^{\epsilon,n}, y_r^{\epsilon,n}), h\rangle|\right] dr \\ &\quad + \epsilon^{-1/2} \mathbb{E}\left[\langle \Pi_n Q^{1/2}(W_t - W_s), h\rangle\right] \\ &\lesssim |t-s| \left(1 + \epsilon^{-1}\right) \|h\|_{L^\infty(\Omega, H^{\theta_0})} + |t-s|^{1/2} \epsilon^{-1/2} \|h\|_{L^\infty(\Omega, H)},\end{aligned}$$

meaning that the sequences of real-valued functions $\{t \mapsto \langle u_t^{\epsilon,n}, h\rangle\}_{n\in\mathbb{N}}$ and $\{t \mapsto \langle y_t^{\epsilon,n}, h\rangle\}_{n\in\mathbb{N}}$ are equicontinuous for every fixed $h \in L^\infty(\Omega, \mathscr{S})$. Since $L^\infty(\Omega, \mathscr{S})$ is dense in $L^q(\Omega, H)$, by previous Lemma 4.5 we have, up to a subsequence that we still denote n:

$$u^{\epsilon,n} \to u^\epsilon,\ y^{\epsilon,n} \to y^\epsilon, \qquad \text{strongly in } C([0,T], (L^p(\Omega, H))_w).$$

Therefore, for every $p < \infty$ and $t \in [0,T]$ the limiting process u^ϵ satisfies

$$\begin{aligned}&\|u_t^\epsilon\|^2_{L^p(\Omega,H)} + 2\|u^\epsilon\|^2_{L^p(\Omega, L^2([0,T],H^1))} \\ &\quad \leq \liminf_{n\to\infty} \|u_t^{\epsilon,n}\|^2_{L^p(\Omega,H)} + 2\|u^{\epsilon,n}\|^2_{L^p(\Omega,L^2([0,T],H^1))} \lesssim 1,\end{aligned}$$

which implies that the same bound holds $\mathbb{P}$-almost surely being it uniform in $p < \infty$, and condition (S3) of Definition 4.1 follows; similarly, the process y^ϵ satisfies (S4) for every fixed $p < \infty$.
In order to finish the proof, we are left to check (S1) and (S2). Since $(u^{\epsilon,n}, y^{\epsilon,n})$ is a strong solution of (4.6) for every $\epsilon \in (0,1)$ and $n \in \mathbb{N}$, for every fixed $t \in [0,T]$ and $h \in L^\infty(\Omega, \mathscr{S})$ we have

$$\begin{aligned}\mathbb{E}\left[\langle u_t^{\epsilon,n}, h\rangle\right] &= \langle u_0, h\rangle + \int_0^t \mathbb{E}\left[\langle u_s^{\epsilon,n}, Ah\rangle\right] + \int_0^t \mathbb{E}\left[\langle b(u_s^{\epsilon,n}, u_s^{\epsilon,n}), \Pi_n h\rangle\right] ds \\ &\quad + \epsilon^{-1/2} \int_0^t \mathbb{E}\left[\langle b(y_s^{\epsilon,n}, u_s^{\epsilon,n}), \Pi_n h\rangle\right] ds, \\ \mathbb{E}\left[\langle y_t^{\epsilon,n}, h\rangle\right] &= \langle y_0, h\rangle + \epsilon^{-1} \int_0^t \mathbb{E}\left[\langle y_s^{\epsilon,n}, C_\epsilon h\rangle\right] ds + \int_0^t \mathbb{E}\left[\langle b(u_s^{\epsilon,n}, y_s^{\epsilon,n}), \Pi_n h\rangle\right] ds \\ &\quad + \epsilon^{-1/2} \int_0^t \mathbb{E}\left[\langle b(y_s^{\epsilon,n}, y_s^{\epsilon,n}), \Pi_n h\rangle\right] ds + \epsilon^{-1/2} \mathbb{E}\left[\langle Q^{1/2} W_t, \Pi_n h\rangle\right].\end{aligned}$$

Let us restrict to $h \in L^\infty(\Omega, \mathscr{S})$ of the form $h = \phi g$, for some $\phi \in L^\infty(\Omega)$ and $g \in \mathscr{S}$. Since $u^{\epsilon,n} \to u^\epsilon$ and $y^{\epsilon,n} \to y^\epsilon$ strongly in $C([0,T], (L^p(\Omega, H))_w)$, and $\Pi_n g \to g$ strongly in H, letting $n \to \infty$ in the expression above yields

$$\begin{aligned}\mathbb{E}\left[\langle u_t^\epsilon, g\rangle \phi\right] &= \mathbb{E}\left[\langle u_0, g\rangle \phi\right] + \int_0^t \mathbb{E}\left[\langle u_s^\epsilon, Ag\rangle \phi\right] + \int_0^t \mathbb{E}\left[\langle b(u_s^\epsilon, u_s^\epsilon), g\rangle \phi\right] ds \\ &\quad + \epsilon^{-1/2} \int_0^t \mathbb{E}\left[\langle b(y_s^\epsilon, u_s^\epsilon), g\rangle \phi\right] ds, \\ \mathbb{E}\left[\langle y_t^\epsilon, g\rangle \phi\right] &= \mathbb{E}\left[\langle y_0, g\rangle \phi\right] + \epsilon^{-1} \int_0^t \mathbb{E}\left[\langle y_s^\epsilon, C_\epsilon g\rangle \phi\right] ds + \int_0^t \mathbb{E}\left[\langle b(u_s^\epsilon, y_s^\epsilon), g\rangle \phi\right] ds \\ &\quad + \epsilon^{-1/2} \int_0^t \mathbb{E}\left[\langle b(y_s^\epsilon, y_s^\epsilon), g\rangle \phi\right] ds + \epsilon^{-1/2} \mathbb{E}\left[\langle Q^{1/2} W_t, g\rangle \phi\right].\end{aligned}$$

Being $\phi \in L^\infty(\Omega)$ arbitrary, we deduce the desired $\mathbb{P}$-almost sure identities (S2). Finally, if pathwise uniqueness holds for (4.5) then solutions are necessarily adapted, that is (S1); otherwise, arguing as in the proof of Proposition 4.21 and using martingale representation [DPZ14, Theorem 8.2], we also get adaptedness up to a possible change in the underlying stochastic basis.

□

The previous Lemma 4.4 gives the uniform-in-ϵ bounds necessary for the proof of Proposition 4.2. As a by-product of the previous proof we have also obtained (4.9), that permits to control the difference between the Galerkin approximation of the small scale process $y^{\epsilon,n}$ and its linearised counterpart $Y^{\epsilon,n}$ in the Sobolev space H^γ. Recall that we have assumed $\gamma > 1/4$, and we can assume $\gamma \in (1/4, 1]$ without loss of generality. A close inspection of the proof of Proposition 4.2 shows that the bound (4.9) is stable under passage to the limit $n \to \infty$; therefore, we can deduce the following

Proposition 4.6. *Let $\{(u^\epsilon, y^\epsilon)\}_{\epsilon\in(0,1)}$ be a bounded-energy family of weak martingale solutions to (4.5). For every $\epsilon \in (0,1)$ let Y^ϵ be the unique strong solution of*

$$dY^\epsilon_t = \epsilon^{-1} C_\epsilon Y^\epsilon_t dt + \epsilon^{-1/2} Q^{1/2} dW_s, \qquad Y^\epsilon_0 = 0. \tag{4.10}$$

Then

$$\sup_{\epsilon\in(0,1)} \epsilon^{-1} \int_0^T \mathbb{E}\left[\|y^\epsilon_s - Y^\epsilon_s\|^2_{H^\gamma}\right] ds \lesssim 1.$$

The previous result will be fundamental in performing the linearisation trick presented at the beginning of this chapter; cfr. also Proposition 4.14.

4.2 Quadratic functions and solution to the Poisson equation

Recall that we shall define correctors φ^ϵ_1, φ^ϵ_2 as solutions to certain Poisson equations $\mathscr{L}^\epsilon_y \phi = -\psi$. In this section we develop the technology needed to solve the Poisson equation for a class of functions ψ that is large enough for our purposes, namely the class of quadratic functions on Sobolev spaces H^θ. Moreover, we also provide partial regularity estimates for the so obtained solution ϕ in terms of analogous bounds on ψ, showing improved regularity (see Corollary 4.11). To avoid any confusion for the reader, we point out that all the estimates in the present section are uniform in $\epsilon \in (0,1)$.

4.2.1 Quadratic functions

Denote $\mathcal{E}_\theta \subset L^2(H, \mu^\epsilon)$, $\theta \in (-\infty, \theta_0]$ the space of quadratic functions $\psi : H^\theta \to \mathbb{R}$, namely $\psi \in \mathcal{E}$ if there exist $a_0 \in \mathbb{R}$, $a_1 : H^\theta \to \mathbb{R}$ linear and bounded, and $a_2 : H^\theta \times H^\theta \to \mathbb{R}$ bilinear, symmetric and bounded such that $\psi(y) = a_0 + a_1(y) + a_2(y,y)$ for every $y \in H^\theta$. The inclusion in $L^2(H, \mu^\epsilon)$ holds true by (Q2). Notice that every $\psi \in \mathcal{E}_\theta$ admits a unique rewriting as $\psi = a_0 + a_1 + a_2$: indeed $\psi(ry) = a_0 + r a_1(y) + r^2 a_2(y,y)$, and therefore

$$a_0 = \psi(0), \quad a_1(y) = \frac{d}{dr}\psi(ry)\bigg|_{r=0},$$

and by taking the difference $a_2(y,y) = \psi(y) - a_1(y) - a_0$, defining uniquely the quadratic form $a_2(y,y)$ and also its associated symmetric bilinear map, via polarization formula. For future purposes define

$$\begin{aligned}\|\psi\|_{\mathcal{E}_\theta} &:= |a_0| + \|a_1\|_{H^\theta\to\mathbb{R}} + \|a_2\|_{H^\theta\times H^\theta\to\mathbb{R}} \\ &= |a_0| + \sup_{\substack{y\in H^\theta,\\ \|y\|_{H^\theta}=1}} |a_1(y)| + \sup_{\substack{y,y'\in H^\theta,\\ \|y\|_{H^\theta}=\|y'\|_{H^\theta}=1}} |a_2(y,y')|.\end{aligned}$$

The space $\mathcal{E}_\theta$ is Banach when endowed with the norm $\|\cdot\|_{\mathcal{E}_\theta}$, and $\mathcal{E}_\theta \subset \mathcal{E}_{\theta'}$ with continuous embedding if $\theta \le \theta'$. As a notational convention, denote $\mathcal{E} := \mathcal{E}_0$.

Lemma 4.7. *Let $\psi \in \mathcal{E}$, then for every $t \ge 0$ it holds $P_t^\epsilon\psi \in D(\mathscr{L}_y^\epsilon)$ and $\mathscr{L}_y^\epsilon P_t^\epsilon\psi = P_t^\epsilon\mathscr{L}_y^\epsilon\psi$.*

Proof. By Markov property $P_s^\epsilon P_t^\epsilon\psi = P_{t+s}^\epsilon\psi$. Recalling that Y^y is a strong solution of $dY_t^y = C_\epsilon Y_t^y dt + Q^{1/2}dW_t$, $Y_0^y = y$, by Itō Formula we have

$$\begin{aligned}P_s^\epsilon P_t^\epsilon\psi - P_t^\epsilon\psi &= \mathbb{E}\left[\psi(Y_{t+s}^\cdot)\right] - \mathbb{E}\left[\psi(Y_t^\cdot)\right] \\ &= \int_t^{t+s} \mathbb{E}\left[\langle C_\epsilon Y_r^\cdot, D_y\psi(Y_r^\cdot)\rangle + \frac{1}{2}Tr(QD_y^2\psi(Y_r^\cdot))\right] dr.\end{aligned}$$

Let $\psi(y) = a_0 + a_1(y) + a_2(y,y)$ be the canonical decomposition of $\psi \in \mathcal{E}$. Since $\int_H \|C_\epsilon y\|_H\|y\|_H d\mu^\epsilon(y) < \infty$ uniformly in ϵ by our assumptions on C and Q, we have

$$\begin{aligned}\bar{\psi} &:= \langle C_\epsilon y, D_y\psi(y)\rangle + \frac{1}{2}Tr(QD_y^2\psi) \\ &= a_1(C_\epsilon y) + 2a_2(C_\epsilon y, y) + \frac{1}{2}Tr(QD_y^2 a_2) \in L^2(H,\mu^\epsilon).\end{aligned}$$

In addition, the semigroup $P_\cdot\bar{\psi}$ is right continuous at time t, with respect to the $L^2(H,\mu^\epsilon)$ topology, and therefore we have in the limit $s \to 0^+$

$$\begin{aligned}\left\|\frac{P_s^\epsilon P_t^\epsilon\psi - P_t^\epsilon\psi}{s} - P_t^\epsilon\bar{\psi}\right\|_{L^2(H,\mu^\epsilon)} &= \left\|\frac{1}{s}\int_t^{t+s} P_r^\epsilon\bar{\psi}dr - P_t^\epsilon\bar{\psi}\right\|_{L^2(H,\mu^\epsilon)} \\ &\le \frac{1}{s}\int_t^{t+s} \left\|P_r^\epsilon\bar{\psi} - P_t^\epsilon\bar{\psi}\right\|_{L^2(H,\mu^\epsilon)} dr \to 0.\end{aligned}$$

In particular, this means $P_t^\epsilon\psi \in D(\mathscr{L}_y^\epsilon)$ and $\mathscr{L}_y^\epsilon P_t^\epsilon\psi = P_t^\epsilon\bar{\psi}$. Finally, taking $t=0$ in the previous formula we deduce $\mathscr{L}_y^\epsilon\psi = \bar{\psi}$, that yields $\mathscr{L}_y^\epsilon P_t^\epsilon\psi = P_t^\epsilon\bar{\psi} = P_t^\epsilon\mathscr{L}_y^\epsilon\psi$. □

Lemma 4.8. *The semigroup P_t^ϵ is exponentially mixing at zero when restricted to $\mathcal{E}_\theta$, $\theta \in (-\infty, \theta_0]$, namely for every $\psi \in \mathcal{E}_\theta$ and $t \ge 0$*

$$\left|P_t^\epsilon\psi(0) - \int_H \psi(w)d\mu^\epsilon(w)\right| \lesssim \|\psi\|_{\mathcal{E}_\theta} e^{-\lambda_0 t}.$$

Proof. Let $\psi \in \mathcal{E}_\theta$ be given by $\psi(y) = a_0 + a_1(y) + a_2(y,y)$, $y \in H^\theta$. Recall

$$\begin{aligned}P_t^\epsilon\psi(y) &= a_0 + \mathbb{E}\left[a_2(W_t^{C_\epsilon,Q}, W_t^{C_\epsilon,Q})\right] + a_1(e^{C_\epsilon t}y) + a_2(e^{C_\epsilon t}y, e^{C_\epsilon t}y) \\ &= P_t^\epsilon\psi(0) + a_1(e^{C_\epsilon t}y) + a_2(e^{C_\epsilon t}y, e^{C_\epsilon t}y),\end{aligned}$$

and therefore

$$\begin{aligned}|P^\epsilon_t\psi(0)-P^\epsilon_t\psi(y)| &\leq \left|a_1(e^{C_\epsilon t}y)\right|+\left|a_2(e^{C_\epsilon t}y,e^{C_\epsilon t}y)\right| \\ &\lesssim \|\psi\|_{\mathcal{E}_\theta}e^{-\lambda_0 t}\|y\|_{H^\theta}+\|\psi\|_{\mathcal{E}_\theta}e^{-2\lambda_0 t}\|y\|^2_{H^\theta}.\end{aligned}$$

Since μ^ϵ is invariant for P^ϵ_t,

$$\begin{aligned}\left|P^\epsilon_t\psi(0)-\int_H\psi(w)d\mu^\epsilon(w)\right| &= \left|P^\epsilon_t\psi(0)-\int_H P^\epsilon_t\psi(w)d\mu^\epsilon(w)\right| \\ &\leq \int_H |P^\epsilon_t\psi(0)-P^\epsilon_t\psi(w)|\,d\mu^\epsilon(w) \\ &\lesssim \|\psi\|_{\mathcal{E}_\theta}e^{-\lambda_0 t}\int_H\|w\|_{H^\theta}d\mu^\epsilon(w) \\ &\quad+\|\psi\|_{\mathcal{E}_\theta}e^{-2\lambda_0 t}\int_H\|w\|^2_{H^\theta}d\mu^\epsilon(w) \\ &\lesssim \|\psi\|_{\mathcal{E}_\theta}e^{-\lambda_0 t}.\end{aligned}$$

□

4.2.2 Solution to the Poisson equation

Recall that the operator $\mathscr{L}^\epsilon_y$ is closed under our assumptions, and therefore the space $D(\mathscr{L}^\epsilon_y)$ is complete when endowed with the graph norm

$$\|\psi\|_{D(\mathscr{L}^\epsilon_y)}\coloneqq\|\psi\|_{L^2(H,\mu^\epsilon)}+\|\mathscr{L}^\epsilon_y\psi\|_{L^2(H,\mu^\epsilon)}.$$

Also, by Lemma 4.7 we have $\mathcal{E}\subset D(\mathscr{L}^\epsilon_y)$ with continuous embedding, in virtue of the following inequality:

$$\begin{aligned}\|\psi\|^2_{D(\mathscr{L}^\epsilon_y)} &\lesssim \int_H|\psi(w)|^2d\mu^\epsilon(w)+\int_H\left|\langle C_\epsilon w,D_y\psi(w)\rangle\right|^2d\mu^\epsilon(w)+Tr(QD^2_y\psi)^2 \\ &\lesssim \|\psi\|^2_{\mathcal{E}}\left(\int_H\|w\|^2_H d\mu^\epsilon(w)+\int_H\|w\|_{H^{2\Gamma}}(1+\|w\|_H)d\mu^\epsilon(w)+Tr(Q)^2\right) \\ &\lesssim \|\psi\|^2_{\mathcal{E}}.\end{aligned}$$

Lemma 4.9. *Let $\psi\in\mathcal{E}_\theta$, $\theta\in[0,\theta_0)$, be given by $\psi(y)=a_1(y)$. Then for every $T>0$ we have $\psi(e^{C_\epsilon T}\cdot)\in\mathcal{E}$, $\int_{1/T}^T\psi(e^{C_\epsilon t}\cdot)dt\in\mathcal{E}$, and*

$$\lim_{T\to\infty}\int_{1/T}^T\psi(e^{C_\epsilon t}\cdot)dt=\psi((-C_\epsilon)^{-1}\cdot),$$

the limit being understood with respect to the $D(\mathscr{L}^\epsilon_y)$ topology.

Proof. First of all, there exists a vector $\mathbf{a}_1\in H^{-\theta}$ such that $a_1(y)=\langle y,\mathbf{a}_1\rangle$ for every $y\in H^\theta$. Hence, for every $T>0$ we have $\psi(e^{C_\epsilon T}\cdot)\in\mathcal{E}$ since $|\psi(e^{C_\epsilon T}y)|=|\langle e^{C_\epsilon T}y,\mathbf{a}_1\rangle|\leq\|e^{C_\epsilon T}y\|_{H^\theta}\|\mathbf{a}_1\|_{H^{-\theta}}\lesssim e^{-\lambda_0 T/2\gamma}T^{-\theta/2}\|y\|_H\|\mathbf{a}_1\|_{H^{-\theta}}$. As a consequence, $\int_{1/T}^T\psi(e^{C_\epsilon t}\cdot)dt\in\mathcal{E}\subset D(\mathscr{L}^\epsilon_y)$ as well, since

$$\left\|\int_{1/T}^T\psi(e^{C_\epsilon t}\cdot)dt\right\|_{\mathcal{E}}\leq\int_{1/T}^T\|\psi(e^{C_\epsilon t}\cdot)\|_{\mathcal{E}}dt<\infty.$$

Let us finally check $\int_{1/T}^T \psi(e^{C_\epsilon t}\cdot)dt \to \psi((-C_\epsilon)^{-1}\cdot)$ in $D(\mathscr{L}_y^\epsilon)$ as $T \to \infty$. For every $T > 0$ and $y \in H$ we have

$$\begin{aligned}\int_{1/T}^T \psi(e^{C_\epsilon t}y)dt &= \int_{1/T}^T \langle e^{C_\epsilon t}y, \mathbf{a}_1\rangle dt = \langle \left(\int_{1/T}^T e^{C_\epsilon t}dt\right) y, \mathbf{a}_1\rangle \\ &= \langle (e^{C_\epsilon/T} - e^{C_\epsilon T})(-C_\epsilon)^{-1}y, \mathbf{a}_1\rangle,\end{aligned}$$

and therefore we only have to check $\langle (e^{C_\epsilon/T} - e^{C_\epsilon T} - 1)(-C_\epsilon)^{-1}\cdot, \mathbf{a}_1\rangle \to 0$ and $\langle (e^{C_\epsilon/T} - e^{C_\epsilon T} - 1)\cdot, \mathbf{a}_1\rangle \to 0$ in $L^2(H, \mu^\epsilon)$ (recall that $D_y^2\psi = 0$ since ψ is linear). We only prove the second convergence, the former being easier.

$$\begin{aligned}\int_H \left|\langle (e^{C_\epsilon/T} - e^{C_\epsilon T} - 1)w, \mathbf{a}_1\rangle\right|^2 d\mu^\epsilon(w) &\lesssim \int_H \left|\langle (e^{C_\epsilon/T} - 1)w, \mathbf{a}_1\rangle\right|^2 d\mu^\epsilon(w) \\ &\quad + \int_H \left|\langle e^{C_\epsilon T}w, \mathbf{a}_1\rangle\right|^2 d\mu^\epsilon(w) \\ &\le \int_H \|(e^{C_\epsilon/T} - 1)w\|_{H^\theta}^2 \|\mathbf{a}_1\|_{H^{-\theta}}^2 d\mu^\epsilon(w) \\ &\quad + \int_H \|e^{C_\epsilon T}w\|_{H^\theta}^2 \|\mathbf{a}_1\|_{H^{-\theta}}^2 d\mu^\epsilon(w) \\ &\lesssim T^{(\theta-\theta_0)/\Gamma} \|\mathbf{a}_1\|_{H^{-\theta}}^2 \int_H \|w\|_{H^{\theta_0}}^2 d\mu^\epsilon(w) \\ &\quad + e^{-\lambda_0 T}\|\mathbf{a}_1\|_{H^{-\theta}}^2 \int_H \|w\|_{H^\theta}^2 d\mu^\epsilon(w) \to 0.\end{aligned}$$

□

Lemma 4.10. *Let $\psi \in \mathcal{E}_\theta$, $\theta \in [0, \theta_0)$, be given by $\psi(y) = a_2(y, y)$. Then for every $T > 0$ we have $\psi(e^{C_\epsilon T}\cdot) \in \mathcal{E}$, $\int_{1/T}^T \psi(e^{C_\epsilon t}\cdot)dt \in \mathcal{E}$, and there exists $\phi \in \mathcal{E}_{\theta-\delta} \cap D(\mathscr{L}_y^\epsilon)$ for every $\delta \in (0, \gamma)$ satisfying $\|\phi\|_{\mathcal{E}_{\theta-\delta}} \lesssim \|\psi\|_{\mathcal{E}_\theta}$, such that*

$$\lim_{T\to\infty} \int_{1/T}^T \psi(e^{C_\epsilon t}\cdot)dt = \phi,$$

the limit being understood with respect to the $D(\mathscr{L}_y^\epsilon)$ topology.

Proof. First of all, there exists a linear bounded operator $A_2 : H^\theta \to H^{-\theta}$ such that $a_2(y, v) = \langle y, A_2 v\rangle = \langle A_2 y, v\rangle$ for every $y, v \in H^\theta$, and $\|\psi\|_{H^\theta} = \|A_2\|_{H^\theta \to H^{-\theta}}$. Then, for every $T > 0$ we have $\psi(e^{C_\epsilon T}\cdot) \in \mathcal{E}$ since $|\psi(e^{C_\epsilon T}y)| \le \|\psi\|_{H^\theta}\|e^{C_\epsilon T}y\|_{H^\theta}^2 \lesssim \|\psi\|_{H^\theta} e^{-\lambda_0 T}T^{-\theta/\gamma}\|y\|_H^2$, and similarly $\int_{1/T}^T \psi(e^{C_\epsilon t}\cdot)dt \in \mathcal{E}$. In addition, for every $T > 0$ and $y \in H^\theta$

$$\int_{1/T}^T \psi(e^{C_\epsilon t}y)dt = \int_{1/T}^T \langle e^{C_\epsilon t}y, A_2 e^{C_\epsilon t}y\rangle dt = \langle y, \left(\int_{1/T}^T e^{C_\epsilon t}A_2 e^{C_\epsilon t}dt\right) y\rangle,$$

and since for every $\delta_1, \delta_2 \ge 0$ satisfying $\delta_1 + \delta_2 = \delta < \gamma$ it holds

$$\int_0^\infty \left\|e^{C_\epsilon t}A_2 e^{C_\epsilon t}\right\|_{H^{\theta-2\delta_1} \to H^{2\delta_2-\theta}} dt \lesssim \|A_2\|_{H^\theta \to H^{-\theta}} \int_0^\infty \frac{e^{-\lambda_0 t}}{t^{\delta/\gamma}}dt < \infty,$$

there exists a linear bounded operator $A_2^\infty : H^{\theta-2\delta_1} \to H^{2\delta_2-\theta}$ such that

$$\int_{1/T}^T e^{C_\epsilon t} A_2 e^{C_\epsilon t} dt \to A_2^\infty$$

strongly as $T \to \infty$ for every $\delta_1 + \delta_2 = \delta < \gamma$, and $\langle y, A_2^\infty v\rangle = \langle A_2^\infty y, v\rangle$ for every $y, v \in H^{\theta-\delta}$. In particular, using $\delta_1 = \delta_2 = \delta/2$, we can define $\phi \in \mathcal{E}_{\theta-\delta}$ given by

$$\phi(y) = \langle y, A_2^\infty y\rangle, \quad y \in H^{\theta-\delta},$$

which of course satisfies $\|\phi\|_{\mathcal{E}_{\theta-\delta}} = \|A_2^\infty\|_{H^{\theta-\delta}\to H^{\delta-\theta}} \lesssim \|A_2\|_{H^\theta \to H^{-\theta}} = \|\psi\|_{\mathcal{E}_\theta}$. Let us now check $\phi \in D(\mathscr{L}_y^\epsilon)$: we have for every $y \in H^\theta$

$$\langle C_\epsilon y, D_y \phi(y)\rangle = 2\langle C_\epsilon y, A_2^\infty y\rangle = -\langle y, A_2 y\rangle,$$

where the last equality comes from an integration by parts. Also, given a complete orthonormal system $\{e_k\}_{k\in\mathbb{N}}$ of H and choosing $\delta \in (0,\gamma)$ such that $\theta - \delta \leq \theta_0 - \gamma$, by (Q2) it holds

$$\begin{aligned}\frac{1}{2} Tr(Q D_y^2 \phi) &= \sum_{k\in\mathbb{N}} \langle Q^{1/2} e_k, A_2^\infty Q^{1/2} e_k\rangle \\ &\leq \|A_2^\infty\|_{H^{\theta-\delta}\to H^{\delta-\theta}} \sum_{k\in\mathbb{N}} \|Q^{1/2} e_k\|^2_{H^{\theta-\delta}} < \infty.\end{aligned}$$

Putting all together,

$$\begin{aligned}\|\phi\|^2_{D(\mathscr{L}_y)} &\lesssim \int_H |\phi(w)|^2 d\mu^\epsilon(w) + \int_H |\langle y, A_2 y\rangle|^2 d\mu^\epsilon(w) + Tr(Q D_y^2\phi)^2 \\ &\lesssim \|A_2^\infty\|_{H^{\theta-\delta}\to H^{\delta-\theta}} \int_H \|w\|^2_{H^{\theta-\delta}} d\mu^\epsilon(w) \\ &\quad + \|A_2\|_{H^\theta\to H^{-\theta}} \int_H \|w\|^2_{H^\theta} d\mu^\epsilon(w) \\ &\quad + Tr(Q D_y^2 \phi)^2 < \infty.\end{aligned}$$

Let us finally prove $\lim_{T\to\infty} \int_{1/T}^T \psi(e^{C_\epsilon t}\cdot) dt = \phi$ in the $D(\mathscr{L}_y^\epsilon)$ topology. To ease the notation, denote $A_2^T = \int_{1/T}^T e^{C_\epsilon t} A_2 e^{C_\epsilon t} dt$, so that $\int_{1/T}^T \psi(e^{C_\epsilon t} y) dt = \langle y, A_2^T y\rangle$ for every $y \in H$. First, we have

$$\begin{aligned}\left\| \int_{1/T}^T \psi(e^{C_\epsilon t}\cdot) dt - \phi \right\|^2_{L^2(H,\mu^\epsilon)} &= \int_H |\langle w, (A_2^T - A_2^\infty) w\rangle|^2 d\mu^\epsilon(w) \\ &\leq \|A_2^T - A_2^\infty\|_{H^{\theta-\delta}\to H^{\delta-\theta}} \int_H \|w\|^2_{H^{\theta-\delta}} d\mu^\epsilon(w) \to 0,\end{aligned}$$

as $T \to \infty$, since $A_2^T \to A_2^\infty$ strongly. Second, the following identities hold true:

$$\begin{aligned}\mathscr{L}_y^\epsilon \left(\int_{1/T}^T \psi(e^{C_\epsilon t}\cdot) dt \right)(y) &= 2\langle C_\epsilon y, A_2^T y\rangle + Tr(Q A_2^T) \\ &= \langle y, (e^{C_\epsilon T} A_2 e^{C_\epsilon T} - e^{C_\epsilon/T} A_2 e^{C_\epsilon/T}) y\rangle + Tr(Q A_2^T); \\ \mathscr{L}_y^\epsilon \phi(y) &= -\langle y, A_2 y\rangle + Tr(Q A_2^\infty),\end{aligned}$$

from which we get

$$
\begin{aligned}
&\left\|\mathscr{L}^\epsilon_y\left(\int_{1/T}^T \psi(e^{C_\epsilon t}\cdot)dt\right) - \mathscr{L}^\epsilon_y \phi\right\|^2_{L^2(H,\mu^\epsilon)} \\
&\quad \lesssim \int_H |\langle w, (e^{C_\epsilon T} A_2 e^{C_\epsilon T} + A_2 - e^{C_\epsilon/T} A_2 e^{C_\epsilon/T})w\rangle|^2 d\mu^\epsilon(w) \\
&\qquad + |Tr(QA_2^T) - Tr(QA_2^\infty)|^2 \\
&\quad \lesssim \int_H |\langle w, e^{C_\epsilon T} A_2 e^{C_\epsilon T} w\rangle|^2 d\mu^\epsilon(w) + \int_H |\langle w, A_2(1 - e^{C_\epsilon/T})w\rangle|^2 d\mu^\epsilon(w) \\
&\qquad + \int_H |\langle w, (1 - e^{C_\epsilon/T}) A_2 e^{C_\epsilon/T} w\rangle|^2 d\mu^\epsilon(w) \\
&\qquad + \left|\sum_{k\in\mathbb{N}} \langle Q^{1/2} e_k, (A_2^T - A_2^\infty) Q^{1/2} e_k\rangle\right|^2 \\
&\lesssim e^{-\lambda_0 T} \|A_2\|_{H^\theta \to H^{-\theta}} \int_H \|w\|^2_{H^\theta} d\mu^\epsilon(w) \\
&\quad + T^{(\theta-\theta_0)/2\Gamma} \|A_2\|_{H^\theta \to H^{-\theta}} \left(1 + e^{-\lambda_0 T}\right) \int_H \|w\|_{H^\theta} \|w\|_{H^{\theta_0}} d\mu^\epsilon(w) \\
&\quad + \|A_2^T - A_2^\infty\|^2_{H^{\theta-\delta} \to H^{\delta-\theta}} \left|\sum_{k\in\mathbb{N}} \|Q^{1/2} e_k\|^2_{H^{\theta-\delta}}\right|^2 \to 0.
\end{aligned}
$$

□

Corollary 4.11. *Under the hypotheses Lemma 4.10, let* $\phi = \lim_{T\to\infty} \int_{1/T}^T \psi(e^{C_\epsilon t}\cdot)dt$. *Then for every* $\delta_1, \delta_2 \geq 0$, $\delta_1 + \delta_2 < \gamma$ *it holds* $\langle D_y\phi(\cdot), v\rangle \in \mathcal{E}_{\theta-2\delta_1}$ *for every* $v \in H^{\theta-2\delta_2}$, *with* $\|\langle D_y\phi(\cdot), v\rangle\|_{\mathcal{E}_{\theta-2\delta_1}} \lesssim \|\psi\|_{\mathcal{E}_\theta} \|v\|_{H^{\theta-2\delta_2}}$.

Proof. It is sufficient to recall the expression $\phi(y) = \langle y, A_2^\infty y\rangle$, valid for $y \in \mathscr{S}$, and notice that for every $v \in H^{\theta-2\delta_2}$ it holds $\langle v, D_y\phi(y)\rangle = 2\langle v, A_2^\infty y\rangle$. To conclude, notice that $|\langle v, D_y\phi(y)\rangle| \lesssim \|y\|_{H^{\theta-2\delta_1}} \|\psi\|_{\mathcal{E}_\theta} \|v\|_{H^{\theta-2\delta_2}}$ because $A_2^\infty : H^{\theta-2\delta_1} \to H^{2\delta_2-\theta}$ is continuous with $\|A_2^\infty\|_{H^{\theta-2\delta_1} \to H^{2\delta_2-\theta}} \lesssim \|\psi\|_{\mathcal{E}_\theta}$, and therefore the identity $\langle v, D_y\phi(y)\rangle = 2\langle v, A_2^\infty y\rangle$ extends to every $y \in H^{\theta-2\delta_1}$. □

The next proposition permits to solve the Poisson equation $\mathscr{L}^\epsilon_y \phi = -\psi$ in the unknown ϕ, under suitable assumptions on the datum ψ. We need, in particular, ψ to be a quadratic function on some Sobolev space H^θ, $\theta \in [0, \theta_0)$, with zero average with respect to the invariant measure μ^ϵ, namely $\int_H \psi(w) d\mu^\epsilon(w) = 0$. This latter condition being necessary is clear from invariance of μ^ϵ under the Ornstein-Uhlenbeck semigroup P^ϵ_t and

$$
\left|\int_H \mathscr{L}^\epsilon_y \phi(w) d\mu^\epsilon(w)\right| \leq \left|\int_H \left(\mathscr{L}^\epsilon_y \phi(w) - \frac{1}{t}\left(P^\epsilon_t \phi - \phi\right)(w)\right) d\mu^\epsilon(w)\right| \to 0
$$

as $t \to 0^+$; by the proposition, the zero-average condition on ψ is also sufficient, at least when we restrict ourselves to $\psi \in \mathcal{E}_\theta$. Finally, notice that the solution of the Poisson equation is more regular than the datum, namely: if $\psi \in \mathcal{E}_\theta$, then $\phi \in \mathcal{E}_{\theta-\delta}$ for every $\delta \in (0, \gamma)$, and $\langle D_y\phi(\cdot), v\rangle \in \mathcal{E}_{\theta-2\delta_1}$ for every $v \in H^{\theta-2\delta_2}$, $\delta_1, \delta_2 \geq 0$, $\delta_1 + \delta_2 < \gamma$.

Proposition 4.12. *Let $\psi \in \mathcal{E}_\theta$, $\theta \in [0, \theta_0)$ be such that $\int_H \psi(w) d\mu^\epsilon(w) = 0$. Then there exists $\phi \in \mathcal{E}_{\theta-\delta} \cap D(\mathscr{L}_y^\epsilon)$ for every $\delta \in (0, \gamma)$ satisfying $\|\phi\|_{\mathcal{E}_{\theta-\delta}} \lesssim \|\psi\|_{\mathcal{E}_\theta}$, such that*

$$\phi = \lim_{T\to\infty} \int_{1/T}^T P_t^\epsilon \psi dt$$

with respect to topology of $D(\mathscr{L}_y^\epsilon)$, and $\mathscr{L}_y^\epsilon \phi = -\psi$. Moreover, $\langle D_y\phi(\cdot), v\rangle \in \mathcal{E}_{\theta-2\delta_1}$ for every $v \in H^{\theta-2\delta_2}$, $\delta_1, \delta_2 \geq 0$, $\delta_1 + \delta_2 < \gamma$, with $\|\langle D_y\phi(\cdot), v\rangle\|_{\mathcal{E}_{\theta-2\delta_1}} \lesssim \|\psi\|_{\mathcal{E}_\theta} \|v\|_{H^{\theta-2\delta_2}}$.

Proof. First we prove that the limit exists. Let $\psi(y) = a_0 + a_1(y) + a_2(y, y)$. We have

$$P_t\psi(y) = P_t\psi(0) + a_1(e^{C_\epsilon t}y) + a_2(e^{C_\epsilon t}y, e^{C_\epsilon t}y),$$

and by Lemma 4.9 and Lemma 4.10, the quantity $\int_{1/T}^T \left(a_1(e^{C_\epsilon t}y) + a_2(e^{C_\epsilon t}y, e^{C_\epsilon t}y)\right) dt$ converges with respect to the $D(\mathscr{L}_y^\epsilon)$ topology to some $\phi_\star \in \mathcal{E}_{\theta-\delta} \cap D(\mathscr{L}_y^\epsilon)$ for every $\delta \in (0, \gamma)$. Moreover, by Lemma 4.8

$$|P_t^\epsilon \psi(0)| = |P_t^\epsilon \psi(0) - \int_H \psi(w) d\mu^\epsilon(w)| \lesssim \|\psi\|_{\mathcal{E}_\theta} e^{-\lambda_0 t}$$

is integrable with respect to time, and so it converges with respect to the $D(\mathscr{L}_y^\epsilon)$ topology to a constant ϕ_0. Putting all together,

$$\lim_{T\to\infty} \int_{1/T}^T P_t^\epsilon \psi dt = \phi_0 + \phi_\star =: \phi \in \mathcal{E}_{\theta-\delta} \cap D(\mathscr{L}_y^\epsilon).$$

Let us show that ϕ is indeed a solution of the Poisson equation $\mathscr{L}_y^\epsilon \phi = -\psi$. Notice that $\mathscr{L}_y^\epsilon : D(\mathscr{L}_y^\epsilon) \to L^2(H, \mu^\epsilon)$ is bounded, and therefore by continuity we have

$$\mathscr{L}_y^\epsilon \phi = \mathscr{L}_y^\epsilon \left(\lim_{T\to\infty} \int_{1/T}^T P_t^\epsilon \psi dt\right) = \lim_{T\to\infty} \mathscr{L}_y^\epsilon \left(\int_{1/T}^T P_t^\epsilon \psi dt\right),$$

where the first limit is understood with respect to the $D(\mathscr{L}_y^\epsilon)$ topology, and the second one with respect to the $L^2(H, \mu^\epsilon)$ topology. Since $\int_{1/T}^T \|P_t^\epsilon \psi\|_{\mathcal{E}} dt < \infty$ for every $T > 0$ we have $\int_{1/T}^T P_t^\epsilon \psi dt \in \mathcal{E} \subset D(\mathscr{L}_y^\epsilon)$ for every $T > 0$, and by Lemma 4.7 we have

$$\mathscr{L}_y^\epsilon \left(\int_{1/T}^T P_t^\epsilon \psi dt\right) = \int_{1/T}^T \mathscr{L}_y^\epsilon P_t^\epsilon \psi dt = \int_{1/T}^T P_t^\epsilon \mathscr{L}_y \psi dt = P_T^\epsilon \psi - P_{1/T}^\epsilon \psi.$$

In particular,

$$\mathscr{L}_y^\epsilon \phi = \lim_{T\to\infty} \left(P_T^\epsilon \psi - P_{1/T}^\epsilon \psi\right).$$

Since we already know that $P_{1/T}^\epsilon \psi \to \psi \in L^2(H, \mu^\epsilon)$ as $T \to \infty$ by continuity of the semigroup, we are left to check $\lim_{T\to\infty} P_T^\epsilon \psi = 0 \in L^2(H, \mu^\epsilon)$. We have:

$$\begin{aligned}
|P_T^\epsilon \psi(y)| &\leq |P_T^\epsilon \psi(0)| + |a_1(e^{C_\epsilon T}y)| + |a_2(e^{C_\epsilon T}y, e^{C_\epsilon T}y)| \\
&\leq |P_T^\epsilon \psi(0)| + \|\psi\|_{\mathcal{E}_\theta}(\|e^{C_\epsilon T}y\|_{H^\theta} + \|e^{C_\epsilon T}y\|_{H^\theta}^2) \\
&\lesssim \|\psi\|_{\mathcal{E}_\theta} e^{-\lambda_0 T} + \|\psi\|_{\mathcal{E}_\theta}(e^{-\lambda_0 T}\|y\|_{H^\theta} + e^{-2\lambda_0 T}\|y\|_{H^\theta}^2) \to 0
\end{aligned}$$

as $T \to \infty$ in $L^2(H, \mu^\epsilon)$. Finally, the assertion about the derivative $D_y\phi$ follows by the explicit construction of Lemma 4.9 and Corollary 4.11. □

4.3 Perturbed test function method

Let us move back to the problem of identifying φ_1^ϵ, φ_2^ϵ in the expression of the test function φ^ϵ. Recall we are looking for a perturbation of $\varphi = \varphi(u)$ of the following form

$$\varphi^\epsilon(u,y) = \varphi(u) + \epsilon^{1/2}\varphi_1^\epsilon(u,y) + \epsilon\varphi_2^\epsilon(u,Y).$$

For our purposes it is sufficient to consider $\varphi \in F$, namely $\varphi(u) = \langle u, h\rangle$ for some given smooth test function $h \in \mathscr{S}$. With this choice of φ, we have in particular

$$D_u\varphi = h, \quad D_u^2\varphi = 0.$$

4.3.1 Finding φ_1

Recalling (4.3), the first corrector φ_1^ϵ needs to solve the Poisson equation

$$\mathscr{L}_y^\epsilon\varphi_1^\epsilon(u,y) = -\langle b(y,u), h\rangle.$$

For every fixed $u \in H$, we can apply Proposition 4.12 to the function $\psi_u = \langle b(\cdot,u), h\rangle \in \mathcal{E}$. Indeed $\int_H \psi_u(w)d\mu(w) = 0$, therefore there exists $\phi_u^\epsilon \in \mathcal{E}$ such that $\mathscr{L}_y^\epsilon\phi_u^\epsilon = -\psi_u$. Moreover, since ψ_u is linear in y, following the construction of Lemma 4.9 it is easy to check

$$\phi_u^\epsilon = \langle b((-C_\epsilon)^{-1}\cdot, u), h\rangle.$$

Finally, we define:

$$\varphi_1^\epsilon(u,y) = \phi_u^\epsilon(y) = \langle b((-C_\epsilon)^{-1}y, u), h\rangle. \tag{4.11}$$

Notice that for every $v \in \mathscr{S}$:

$$\begin{aligned}\langle D_y\varphi_1^\epsilon(u), v\rangle &= \langle b((-C_\epsilon)^{-1}v, u), h\rangle\\ \langle D_u\varphi_1^\epsilon(y), v\rangle &= \langle b((-C_\epsilon)^{-1}y, v), h\rangle.\end{aligned}$$

Proposition 4.13. *For every $u, y \in H^s$, $s \in \mathbb{R}$, we have $D_y\varphi_1^\epsilon(u), D_u\varphi_1^\epsilon(y) \in H^{2\theta+s}$ for every $\theta \in [\gamma, 1]$, with*

$$\|D_y\varphi_1^\epsilon(u)\|_{H^{2\theta+s}} \lesssim \epsilon^{-\frac{\theta-\gamma}{1-\gamma}}\|h\|_{H^{\theta_1}}\|u\|_{H^s}, \quad \|D_u\varphi_1^\epsilon(y)\|_{H^{2\theta+s}} \lesssim \epsilon^{-\frac{\theta-\gamma}{1-\gamma}}\|h\|_{H^{\theta_1}}\|y\|_{H^s},$$

for some $\theta_1 = \theta_1(s)$ sufficiently large.

Proof. Take $\theta_1 = \theta_1(s)$ such that (B2) holds true. For every $v \in H^{2\gamma+s}$ we have

$$\begin{aligned}|\langle D_y\varphi_1^\epsilon(u), (-A)^{\gamma+s/2}v\rangle| &= |\langle b((-C_\epsilon)^{-1}(-A)^{\gamma+s/2}v, u), h\rangle|\\ &\lesssim \|(-C_\epsilon)^{-1}(-A)^{\gamma+s/2}v\|_{H^{-s}}\|h\|_{H^{\theta_1}}\|u\|_{H^s}\\ &\lesssim \|(-C)^{-1}(-A)^{\gamma+s/2}v\|_{H^{-s}}\|h\|_{H^{\theta_1}}\|u\|_{H^s}\\ &\lesssim \|v\|_H\|h\|_{H^{\theta_1}}\|u\|_{H^s},\end{aligned}$$

and similarly for every $v \in H^{2+s}$

$$
\begin{aligned}
|\langle D_y\varphi_1^\epsilon(u), (-A)^{1+s/2}v\rangle| &= |\langle b((-C_\epsilon)^{-1}(-A)^{1+s/2}v, u), h\rangle| \\
&\lesssim \|(-C_\epsilon)^{-1}(-A)^{1+s/2}v\|_{H^{-s}}\|h\|_{H^{\theta_1}}\|u\|_{H^s} \\
&\lesssim \|(-\epsilon A)^{-1}(-A)^{1+s/2}v\|_{H^{-s}}\|h\|_{H^{\theta_1}}\|u\|_{H^s} \\
&\lesssim \epsilon^{-1}\|v\|_H\|h\|_{H^{\theta_1}}\|u\|_{H^s}.
\end{aligned}
$$

Since v is arbitrary, by interpolation we deduce

$$
\|D_y\varphi_1^\epsilon(u)\|_{H^{2\theta+s}} \lesssim \|D_y\varphi_1^\epsilon(u)\|_{H^{2\gamma+s}}^{\frac{1-\theta}{1-\gamma}}\|D_y\varphi_1^\epsilon(u)\|_{H^{2+s}}^{\frac{\theta-\gamma}{1-\gamma}} \lesssim \epsilon^{-\frac{\theta-\gamma}{1-\gamma}}\|h\|_{H^{\theta_1}}\|u\|_{H^s}.
$$

The argument is similar for the term involving $D_u\varphi_1^\epsilon$: first, for every $v \in H^{2\gamma+s}$

$$
\begin{aligned}
|\langle D_u\varphi_1^\epsilon(y), (-A)^{\gamma+s/2}v\rangle| &\lesssim |\langle b((-C_\epsilon)^{-1}y, (-A)^{\gamma+s/2}v), h\rangle| \\
&\lesssim \|(-C_\epsilon)^{-1}y\|_{H^{2\gamma+s}}\|h\|_{H^{\theta_1}}\|(-A)^{\gamma+s/2}v\|_{H^{-2\gamma-s}} \\
&\lesssim \|(-C)^{-1}y\|_{H^{2\gamma+s}}\|h\|_{H^{\theta_1}}\|(-A)^{\gamma+s/2}v\|_{H^{-2\gamma-s}} \\
&\lesssim \|y\|_{H^s}\|h\|_{H^{\theta_1}}\|v\|_H,
\end{aligned}
$$

whereas for every $v \in H^{2+s}$

$$
\begin{aligned}
|\langle D_u\varphi_1^\epsilon(y), (-A)^{1+s/2}v\rangle| &\lesssim |\langle b((-C_\epsilon)^{-1}y, (-A)^{1+s/2}v), h\rangle| \\
&\lesssim \|(-C_\epsilon)^{-1}y\|_{H^{2+s}}\|h\|_{H^{\theta_1}}\|(-A)^{1+s/2}v\|_{H^{-2-s}} \\
&\lesssim \|(-\epsilon A)^{-1}y\|_{H^{2+s}}\|h\|_{H^{\theta_1}}\|(-A)^{1+s/2}v\|_{H^{-2-s}} \\
&\lesssim \epsilon^{-1}\|y\|_{H^s}\|h\|_{H^{\theta_1}}\|v\|_H.
\end{aligned}
$$

The thesis follows by interpolation. □

4.3.2 Finding φ_2

Let us move to equation (4.4) for the second corrector:

$$
\langle Au + b(u,u), h\rangle + \langle b(Y,u), D_u\varphi_1^\epsilon(Y)\rangle + \langle b(Y,Y), D_y\varphi_1^\epsilon(u)\rangle + \mathscr{L}_y^\epsilon\varphi_2^\epsilon(u,Y).
$$

As already discussed, the previous expression is obtained by manipulating the analogous expression with y replacing Y.
In order to motivate this substitution, let $\zeta \in H$ indicate the difference $\zeta = y - Y$. We can prove the following:

Proposition 4.14. *For every $\delta \in (0, \gamma - 1/4)$, $u \in H^{1-\delta}$, $y \in H^\gamma$ and $Y \in H^{\theta_0-\gamma}$ it holds:*

$$
\begin{aligned}
|\langle b(y,u), D_u\varphi_1^\epsilon(y)\rangle - \langle b(Y,u), D_u\varphi_1^\epsilon(Y)\rangle| &\lesssim \|\zeta\|_H\|h\|_{H^{\theta_1}}\|Y\|_{H^{\theta_0-\gamma}}\|u\|_H \\
&\quad + \epsilon^{-\frac{1-\gamma-\delta}{2(1-\gamma)}}\|\zeta\|_{H^\gamma}\|h\|_{H^{\theta_1}}\|y\|_H\|u\|_{H^{1-\delta}}, \\
|\langle b(y,y), D_y\varphi_1^\epsilon(u)\rangle - \langle b(Y,Y), D_y\varphi_1^\epsilon(u)\rangle| &\lesssim \|\zeta\|_H\|h\|_{H^{\theta_1}}\|Y\|_{H^{\theta_0-\gamma}}\|u\|_H \\
&\quad + \epsilon^{-\frac{1-\gamma-\delta}{2(1-\gamma)}}\|\zeta\|_{H^\gamma}\|h\|_{H^{\theta_1}}\|y\|_H\|u\|_{H^{1-\delta}}.
\end{aligned}
$$

Proof. Recall

$$\langle b(y,u), D_u\varphi_1^\epsilon(y)\rangle - \langle b(Y,u), D_u\varphi_1^\epsilon(Y)\rangle = \langle b(\zeta,u), D_u\varphi_1^\epsilon(Y)\rangle + \langle b(y,u), D_u\varphi_1^\epsilon(\zeta)\rangle,$$
$$\langle b(y,y), D_y\varphi_1^\epsilon(u)\rangle - \langle b(Y,Y), D_y\varphi_1^\epsilon(u)\rangle = \langle b(\zeta,Y), D_y\varphi_1^\epsilon(u)\rangle + \langle b(y,\zeta), D_y\varphi_1^\epsilon(u)\rangle,$$

and by (B3) and Proposition 4.13 with $\theta = \frac{1+\gamma-\delta}{2}$ the following estimates hold:

$$|\langle b(\zeta,u), D_u\varphi_1^\epsilon(Y)\rangle| \lesssim \|\zeta\|_H\|h\|_{H^{\theta_1}}\|Y\|_{H^{\theta_0-\gamma}}\|u\|_H,$$
$$|\langle b(y,u), D_u\varphi_1^\epsilon(\zeta)\rangle| \lesssim \epsilon^{-\frac{1-\gamma-\delta}{2(1-\gamma)}}\|\zeta\|_{H^\gamma}\|h\|_{H^{\theta_1}}\|y\|_H\|u\|_{H^{1-\delta}},$$
$$|\langle b(\zeta,Y), D_y\varphi_1^\epsilon(u)\rangle| \lesssim \|\zeta\|_H\|h\|_{H^{\theta_1}}\|Y\|_{H^{\theta_0-\gamma}}\|u\|_H,$$
$$|\langle b(y,\zeta), D_y\varphi_1^\epsilon(u)\rangle| \lesssim \epsilon^{-\frac{1-\gamma-\delta}{2(1-\gamma)}}\|\zeta\|_{H^\gamma}\|h\|_{H^{\theta_1}}\|y\|_H\|u\|_{H^{1-\delta}},$$

where we have used $b : H \times H^{2\gamma+1-\delta} \to H^{\delta-1}$ and $b : H \times H^{2+\gamma-2\delta} \to H^{-\gamma}$ continuous by (B3) and our choice of δ. □

Remark 4.2. The previous proposition will be used in Section 4.4 to check rigorously that we can actually replace the small-scale process y^ϵ with Y^ϵ, up to a correction which is infinitesimal as ϵ. Indeed, since $\frac{1-\gamma-\delta}{2(1-\gamma)} < \frac{1}{2}$ we can compensate diverging factors in ϵ in the previous proposition with a factor $\epsilon^{1/2}$ coming from Proposition 4.6, having taken expectation and time integral.

Thus, our goal is to find $\varphi_2^\epsilon = \varphi_2^\epsilon(u,Y)$ such that (4.4) is independent of Y. Let then $u \in H$ be fixed. The idea is again to apply Proposition 4.12 to

$$\psi_u^\epsilon = \langle b(\cdot,u), D_u\varphi_1^\epsilon(\cdot)\rangle + \langle b(\cdot,\cdot), D_y\varphi_1^\epsilon(u)\rangle.$$

For every $\theta > \frac{5}{4} - \gamma$ it holds $\psi_u^\epsilon \in \mathcal{E}_\theta$, with $\|\psi_u^\epsilon\|_{\mathcal{E}_\theta} \lesssim \|h\|_{H^{\theta_1}}\|u\|_H$. However, ψ_u^ϵ does not satisfy the hypotheses of that proposition: indeed, it has not necessarily zero average with respect to the invariant measure μ^ϵ. To deal with this issue, let us consider instead

$$\Psi_u^\epsilon = \psi_u^\epsilon - \int_H \psi_u^\epsilon(w)d\mu^\epsilon(w).$$

With this choice of Ψ_u^ϵ we have $\Psi_u^\epsilon \in \mathcal{E}_\theta$ and $\int_H \Psi_u^\epsilon(w)d\mu^\epsilon(w) = 0$, thus Proposition 4.12 applies. Given $u \in H$ and $\Phi_u^\epsilon \in \mathcal{E}_{\theta'} \cap D(\mathscr{L}_y^\epsilon)$, $\theta' > \frac{5}{4} - 2\gamma$ such that $\mathscr{L}_y^\epsilon\Phi_u^\epsilon = -\Psi_u^\epsilon$, we finally define

$$\varphi_2^\epsilon(u,Y) = \Phi_u^\epsilon(Y), \qquad \|\varphi_2^\epsilon(u,\cdot)\|_{\mathcal{E}_{\theta'}} \lesssim \|\Psi_u^\epsilon\|_{\mathcal{E}_\theta} \lesssim \|h\|_{H^{\theta_1}}\|u\|_H. \tag{4.12}$$

With this choice of φ_2^ϵ, (4.4) can be rewritten as

$$\langle Au + b(u,u), h\rangle + \int_H \psi_u^\epsilon(w)d\mu^\epsilon(w) =: \mathscr{L}^{0,\epsilon}\varphi(u), \tag{4.13}$$

which is indeed a function of the sole variable u.

In the following, specifically when computing $\mathscr{L}^\epsilon\varphi^\epsilon = \mathscr{L}^\epsilon\left(\varphi + \epsilon^{1/2}\varphi_1^\epsilon + \epsilon\varphi_2^\epsilon\right)$, we will need control over the derivatives $D_u\varphi_2^\epsilon$, $D_Y\varphi_2^\epsilon$ to check that the corrections we impose on the test function φ do not change the underlying dynamics in the limit $\epsilon \to 0$, i.e. $\mathscr{L}^\epsilon\varphi^\epsilon$ is close to $\mathscr{L}^{0,\epsilon}\varphi$ in a suitable sense. Control over $D_Y\varphi_2^\epsilon$ is already encoded in the statement of Proposition 4.12, as a straightforward consequence of Corollary 4.11: indeed, for every

$\theta > \frac{5}{4} - \gamma$, $\delta_1, \delta_2 \geq 0$, $\delta_1 + \delta_2 < \gamma$, and $v \in H^{\theta-2\delta_2}$ it holds $\langle D_Y \varphi_2^\epsilon(u, \cdot), v\rangle \in \mathcal{E}_{\theta-2\delta_1}$, with uniform-in-$\epsilon$ bound:

$$\|\langle D_Y \varphi_2^\epsilon(u,\cdot), v\rangle\|_{\mathcal{E}_{\theta-2\delta_1}} \lesssim \|\Psi_u^\epsilon\|_{\mathcal{E}_\theta} \|v\|_{H^{\theta-2\delta_2}} \lesssim \|h\|_{H^{\theta_1}} \|u\|_H \|v\|_{H^{\theta-2\delta_2}}. \tag{4.14}$$

On the other hand, to control $D_u \varphi_2^\epsilon$ we need the following preliminary lemma:

Lemma 4.15. *For every $\theta < \theta_0 + 2\gamma - 1$ there exists θ_1 sufficiently large such that $\langle v, D_u \Psi_u^\epsilon(\cdot)\rangle \in \mathcal{E}_{\theta_0}$ for every $v \in H^{-\theta}$ with*

$$\|\langle v, D_u \Psi_u^\epsilon(\cdot)\rangle\|_{\mathcal{E}_{\theta_0}} \lesssim \|h\|_{H^{\theta_1}} \|v\|_{H^{-\theta}}.$$

Proof. By (B4) and (B2), for every $\theta < \theta_0 + 2\gamma - 1$ we have $b : H^{\theta_0} \times H^{\theta_0+2\gamma} \to H^\theta$ and $b : H^{\theta_0} \times H^{\theta_0} \to H^{\theta-2\gamma}$ continuous, hence by Proposition 4.13 it holds for every $v \in H^{-\theta}$

$$|\langle b(Y, v), D_u \varphi_1^\epsilon(Y)\rangle| \lesssim \|b(Y, D_u \varphi_1^\epsilon(Y))\|_{H^\theta} \|v\|_{H^{-\theta}} \lesssim \|Y\|^2_{H^{\theta_0}} \|h\|_{H^{\theta_1}} \|v\|_{H^{-\theta}},$$

and,

$$\begin{aligned} |\langle v, \langle b(Y,Y), D_u D_y \varphi_1^\epsilon(u)\rangle\rangle| &= |\langle b((-C_\epsilon)^{-1} b(Y,Y), v), h\rangle| \\ &\lesssim \|b(Y,Y)\|_{H^{\theta-2\gamma}} \|v\|_{H^{-\theta}} \|h\|_{\theta_1} \\ &\lesssim \|h\|_{\theta_1} \|Y\|^2_{H^{\theta_0}} \|v\|_{H^{-\theta}}. \end{aligned}$$

Thus, the desired result is true for if we replace Ψ_u^ϵ by ψ_u^ϵ. To conclude the proof, just notice that by the same computation as above,

$$\int_H |\langle v, D_u \Psi_u^\epsilon(w)| d\mu^\epsilon(w) \lesssim \|h\|_{\theta_1} \|v\|_{H^{-\theta}} \int_H \|w\|^2_{H^{\theta_0}} d\mu^\epsilon(w) \lesssim \|h\|_{\theta_1} \|v\|_{H^{-\theta}},$$

since the integral is finite by (Q2). □

Proposition 4.16. *For every $\theta < \theta_0 + 2\gamma - 1$ and $v \in H^{-\theta}$ it holds $\langle v, D_u \varphi_2^\epsilon(\cdot)\rangle \in \mathcal{E}_{\theta_0 - \delta}$ for every $\delta \in (0, \gamma)$ with*

$$\|\langle v, D_u \varphi_2^\epsilon(\cdot)\rangle\|_{\mathcal{E}_{\theta_0-\delta}} \lesssim \|h\|_{H^{\theta_1}} \|v\|_{H^{-\theta}}.$$

Proof. Recalling

$$\begin{aligned} \psi_u^\epsilon(Y) &= \langle b(Y,u), D_u \varphi_1^\epsilon(Y)\rangle + \langle b(Y,Y), D_y \varphi_1^\epsilon(u)\rangle, \\ \Psi_u^\epsilon(Y) &= \psi_u^\epsilon(Y) - \int_H \psi_u^\epsilon(w) d\mu^\epsilon(w), \end{aligned}$$

we have for every $v \in \mathscr{S}$

$$\begin{aligned} \langle D_u \psi_u^\epsilon(Y), v\rangle &= \langle b(Y,v), D_u \varphi_1^\epsilon(Y)\rangle + \langle b(Y,Y), \langle D_u D_y \varphi_1^\epsilon, v\rangle\rangle, \\ \langle D_u \Psi_u^\epsilon(Y), v\rangle &= \langle D_u \psi_u^\epsilon(Y), v\rangle - \int_H \langle D_u \psi_u^\epsilon(w), v\rangle d\mu(w), \end{aligned}$$

and the previous quantity is independent of $u \in H$. Recall that we have defined $\varphi_2^\epsilon(u,\cdot) = \Phi_u^\epsilon = \lim_{T\to\infty} \int_{1/T}^T P_t^\epsilon \Psi_u^\epsilon dt$, where the limit is taken in $D(\mathscr{L}_y^\epsilon)$; we prove now that for every $v \in H$ we have

$$\langle D_u \varphi_2^\epsilon(\cdot), v\rangle = \lim_{T\to\infty} \int_{1/T}^T P_t^\epsilon \langle D_u \Psi_u^\epsilon, v\rangle dt.$$

Denote $\varphi_2^{\epsilon,T}(u,\cdot) = \int_{1/T}^T P_t^\epsilon \Psi_u^\epsilon dt \in \mathcal{E}$, and consider for $r \in \mathbb{R}$

$$\begin{aligned}\varphi_2^{\epsilon,T}(u+rv,\cdot) - \varphi_2^{\epsilon,T}(u,\cdot) &= \int_{1/T}^T P_t^\epsilon \Psi_{u+rv}^\epsilon dt - \int_{1/T}^T P_t^\epsilon \Psi_u^\epsilon dt \\ &= \int_{1/T}^T P_t^\epsilon \left(\Psi_{u+rv}^\epsilon - \Psi_u^\epsilon\right) dt \\ &= \int_{1/T}^T P_t^\epsilon \left(r\langle D_u \Psi_u^\epsilon, v\rangle\right) dt \\ &= r \int_{1/T}^T P_t^\epsilon \langle D_u \Psi_u^\epsilon, v\rangle dt,\end{aligned}$$

where we have used linearity of P_t^ϵ and the fact that Ψ_u^ϵ is linear in u. The map $y \mapsto \langle D_u \Psi_u^\epsilon(y), v\rangle$ satisfies the assumptions of Proposition 4.12, and therefore we can take the limit in $D(\mathscr{L}_y^\epsilon)$ of the previous expression, as $T \to \infty$, to obtain

$$\varphi_2^\epsilon(u+rv,\cdot) - \varphi_2^\epsilon(u,\cdot) = r \lim_{T\to\infty} \int_{1/T}^T P_t^\epsilon \langle D_u \Psi_u^\epsilon, v\rangle dt.$$

Finally, rearranging and taking the limit as $r \to 0$ we get

$$\langle D_u \varphi_2^\epsilon(\cdot), v\rangle = \lim_{r\to 0} \frac{1}{r}\left(\varphi_2^\epsilon(u+rv,\cdot) - \varphi_2^\epsilon(u,\cdot)\right) = \lim_{T\to\infty} \int_{1/T}^T P_t^\epsilon \langle D_u \Psi_u^\epsilon, v\rangle dt.$$

Since it holds $\langle D_u \Psi_u^\epsilon(\cdot), v\rangle \in \mathcal{E}_{\theta_0}$ with $\|\langle v, D_u \Psi_u^\epsilon(\cdot)\rangle\|_{\mathcal{E}_{\theta_0}} \lesssim \|h\|_{H^{\theta_1}} \|v\|_{H^{-\theta}}$ for every $v \in H^{-\theta}$ and $\theta < \theta_0 + 2\gamma - 1$, by Proposition 4.12 we have $\langle D_u \varphi_2^\epsilon(\cdot), v\rangle \in \mathcal{E}_{\theta_0 - \delta}$ for every $\delta \in (0,\gamma)$ with

$$\|\langle D_u \varphi_2^\epsilon(\cdot), v\rangle\|_{\mathcal{E}_{\theta_0-\delta}} \lesssim \|\langle v, D_u \Psi_u^\epsilon(\cdot)\rangle\|_{\mathcal{E}_{\theta_0}} \lesssim \|h\|_{H^{\theta_1}} \|v\|_{H^{-\theta}}.$$

Finally, the previous bound extend to all $v \in H^{-\theta}$ by continuity. The proof is complete. □

4.4 Convergence to transport noise

In this section we state and prove convergence of u^ϵ stated in Theorem 1.3. The proof is split in three parts. In the first place, invoking Simon compactness criterium (Lemma 4.1), we prove that the laws of the processes $\{u^\epsilon\}_{\epsilon\in(0,1)}$ are tight as probability measures on the space $L^2([0,T],H) \cap C([0,T],H^{-\beta})$ for every $\beta > 0$ (see Proposition 4.20 below); next, in subsequent Proposition 4.21 we show that every weak accumulation point $(u, Q^{1/2}W)$ is an analytically weak solution of the equation with effective generator $\mathscr{L}^0$ and Itō transport noise (cfr. (4.23)); finally, we check that the generator $\mathscr{L}^0$ can be split into the sum of the Itō-to-Stratonovich corrector (which together with the Itō integral gives the Stratonovich transport noise) and the Itō-Stokes drift (Equations (4.26), (4.27), and (4.28)).

Remark 4.3. When the limit equation does not admit uniqueness, we do not know whether or not different subsequences can converge towards different solutions of the limit equation. It might well be that, notwithstanding the fact that the limit equation admits multiple solutions, this approximating procedure only selects some special solution with enjoys additional properties. However, we are not able to answer this question at the moment: we can only provide a partial selection criterium based on the fact that every selected solution u must satisfy the same energy bounds (S3) of the approximating sequence $\{u^\epsilon\}_{\epsilon\in(0,1)}$ (this latter property can be deduced first on Galerkin projections $\{\Pi_m u\}_{m\in\mathbb{N}}$, and then checked to be uniform in $m \in \mathbb{N}$). This is of particular interest if we start with solutions satisfying the energy inequality as in [FR08].

As a preliminary step towards the proof of Theorem 1.3, we need a version of the celebrated Itō Formula suited for our solution processes (u^ϵ, y^ϵ). Indeed, since (4.5) only holds in analytically weak sense (S2), the classical Itō Formula [DPZ02, Theorem 4.32] is not a priori applicable to the process $\Phi(u^\epsilon_t, y^\epsilon_t)$ unless Φ only depends on a finite number of projections $\langle u^\epsilon_t, h_i\rangle$, $\langle y^\epsilon_t, k_i\rangle$, for some $h_i, k_i \in \mathscr{S}$. Thus, our approach consists in applying first the classical Itō Formula to Galerkin projections $\Pi_n u^\epsilon$, $\Pi_n y^\epsilon$, and then pass to the limit as $n \to \infty$, under suitable controls over $D_u\Phi, D_y\Phi$.

Lemma 4.17 (Itō Formula)**.** *Let $\Phi : H \times H \to \mathbb{R}$ be such that, for every fixed $u, y \in H$, it holds $\Phi(u,\cdot), \Phi(\cdot, y) \in \mathcal{E}$ and*

$$\|D_u\Phi(u,y)\|_{H^1} \lesssim 1 + \|u\|_H + \|y\|_H, \quad \|D_y\Phi(u,y)\|_{H^1} \lesssim 1 + \|u\|_H + \|y\|_H. \tag{4.15}$$

Let (u^ϵ, y^ϵ) be a solution to (4.5) in the sense of Definition 4.1. Then for every fixed $\epsilon \in (0,1)$ the following Itō Formula holds $\mathbb{P}$-a.s. for every $t \in [0,T]$

$$\Phi(u^\epsilon_t, y^\epsilon_t) = \Phi(u_0, y_0) + \int_0^t \mathscr{L}^\epsilon \Phi(u^\epsilon_s, y^\epsilon_s)ds + \int_0^t \langle D_y\Phi(u^\epsilon_s, y^\epsilon_s), Q^{1/2}dW_s\rangle.$$

Proof. Let $\{\Pi_n\}_{n\in\mathbb{N}}$ be a family of Galerkin projectors and let $h \in H$ be fixed. Since $\Pi_n h \in \mathscr{S}$ for every $n \in \mathbb{N}$, by (S2) we have $\mathbb{P}$-a.s. for every $t \in [0,T]$:

$$\begin{aligned}
\langle u^\epsilon_t, \Pi_n h\rangle &= \langle u_0, \Pi_n h\rangle + \int_0^t \langle u^\epsilon_s, A\Pi_n h\rangle + \int_0^t \langle b(u^\epsilon_s, u^\epsilon_s), \Pi_n h\rangle ds \\
&\quad + \epsilon^{-1/2}\int_0^t \langle b(y^\epsilon_s, u^\epsilon_s), \Pi_n h\rangle ds, \\
\langle y^\epsilon_t, \Pi_n h\rangle &= \langle y_0, \Pi_n h\rangle + \epsilon^{-1}\int_0^t \langle y^\epsilon_s, C_\epsilon \Pi_n h\rangle ds + \int_0^t \langle b(u^\epsilon_s, y^\epsilon_s), \Pi_n h\rangle ds \\
&\quad + \epsilon^{-1/2}\int_0^t \langle b(y^\epsilon_s, y^\epsilon_s), \Pi_n h\rangle ds + \epsilon^{-1/2}\langle Q^{1/2}W_t, \Pi_n h\rangle.
\end{aligned}$$

Letting h freely vary in H in the previous expression, we deduce that the process $(\Pi_n u^\epsilon, \Pi_n y^\epsilon)$ is an Itō process satisfying

$$\begin{aligned}
\Pi_n u^\epsilon_t &= \Pi_n u_0 + \int_0^t \Pi_n A u^\epsilon_s ds + \int_0^t \Pi_n b(u^\epsilon_s, u^\epsilon_s)ds + \epsilon^{-1/2}\int_0^t \Pi_n b(y^\epsilon_s, u^\epsilon_s)ds, \\
\Pi_n y^\epsilon_t &= \Pi_n y_0 + \epsilon^{-1}\int_0^t \Pi_n C_\epsilon y^\epsilon_s ds + \int_0^t \Pi_n b(u^\epsilon_s, y^\epsilon_s)ds \\
&\quad + \epsilon^{-1/2}\int_0^t \Pi_n b(y^\epsilon_s, y^\epsilon_s)ds + \epsilon^{-1/2}\Pi_n Q^{1/2}W_t
\end{aligned}$$

in strong analytical sense. In particular, by (S1) and classical Itō Formula, the following a.s. identity holds for every $t \in [0,T]$:

$$\Phi(\Pi_n u_t^\epsilon, \Pi_n y_t^\epsilon) = \Phi(\Pi_n u_0, \Pi_n y_0) + \int_0^t \mathscr{L}^{\epsilon,n}_{u_s^\epsilon, y_s^\epsilon} \Phi(\Pi_n u_s^\epsilon, \Pi_n y_s^\epsilon) ds + \int_0^t \langle D_y \Phi(\Pi_n u_s^\epsilon, \Pi_n y_s^\epsilon), \Pi_n Q^{1/2} dW_s \rangle,$$

where $\mathscr{L}^{\epsilon,n}_{u_s^\epsilon, y_s^\epsilon} \Phi(\Pi_n u_s^\epsilon, \Pi_n y_s^\epsilon)$ is given by

$$\begin{aligned}
\mathscr{L}^{\epsilon,n}_{u_s^\epsilon, y_s^\epsilon} \Phi(\Pi_n u_s^\epsilon, \Pi_n y_s^\epsilon) = & \langle \Pi_n A u_s^\epsilon + \Pi_n b(u_s^\epsilon, u_s^\epsilon), D_u \Phi(\Pi_n u_s^\epsilon, \Pi_n y_s^\epsilon) \rangle \\
& + \epsilon^{-1/2} \langle \Pi_n b(y_s^\epsilon, u_s^\epsilon), D_u \Phi(\Pi_n u_s^\epsilon, \Pi_n y_s^\epsilon) \rangle \\
& + \langle \Pi_n b(u_s^\epsilon, y_s^\epsilon), D_y \Phi(\Pi_n u_s^\epsilon, \Pi_n y_s^\epsilon) \rangle \\
& + \epsilon^{-1/2} \langle \Pi_n b(y_s^\epsilon, y_s^\epsilon), D_y \Phi(\Pi_n u_s^\epsilon, \Pi_n y_s^\epsilon) \rangle \\
& + \epsilon^{-1} \langle \Pi_n C_\epsilon y_s^\epsilon, D_y \Phi(\Pi_n u_s^\epsilon, \Pi_n y_s^\epsilon) \rangle \\
& + \frac{\epsilon^{-1}}{2} Tr(\Pi_n Q \Pi_n D_y^2 \Phi(\Pi_n u_s^\epsilon)).
\end{aligned}$$

Because $\Phi \in \mathcal{E}$ whenever any of its two argument is fixed, it is easy to check that $\mathbb{P}$-a.s. for every $t \in [0,T]$ the convergences $\Phi(\Pi_n u_t^\epsilon, \Pi_n y_t^\epsilon) \to \Phi(u_t^\epsilon, y_t^\epsilon)$ and $\Phi(\Pi_n u_0, \Pi_n y_0) \to \Phi(u_0, y_0)$ hold true as $n \to \infty$. By (S3), (4.15) and Lebesgue dominated convergence, we have, up to subsequences, $\mathbb{P}$-a.s. for every $t \in [0,T]$

$$\int_0^t \mathscr{L}^{\epsilon,n} \Phi(\Pi_n u_s^\epsilon, \Pi_n y_s^\epsilon) ds \to \int_0^t \mathscr{L}^\epsilon \Phi(u_s^\epsilon, y_s^\epsilon) ds.$$

Similarly, since $\int_0^t \|\Pi_n D_y \Phi(\Pi_n u_s^\epsilon, \Pi_n y_s^\epsilon)\|^2 ds \to \int_0^t \|D_y \Phi(u_s^\epsilon, y_s^\epsilon)\|^2 ds$ a.s. as $n \to \infty$, the following convergence in probability holds true

$$\int_0^t \langle D_y \Phi(\Pi_n u_s^\epsilon, \Pi_n y_s^\epsilon), \Pi_n Q^{1/2} dW_s \rangle \to \int_0^t \langle D_y \Phi(u_s^\epsilon, y_s^\epsilon), Q^{1/2} dW_s \rangle,$$

and the convergence is almost sure up to extracting a subsequence, concluding the proof. □

Remark 4.4. *i*) As a consequence of Lemma 4.17 and the definition of correctors φ_1^ϵ, φ_2^ϵ from the previous section, we immediately deduce that φ_1^ϵ and φ_2^ϵ belong to the domain of the generator $\mathscr{L}^\epsilon$ and the Itō Formula holds for the processes $\varphi_1^\epsilon(u^\epsilon, y^\epsilon)$ and $\varphi_2^\epsilon(u^\epsilon, y^\epsilon)$. However, strictly speaking we do not actually need such a strong result. For instance, it would have been sufficient to show the existence of *generalized correctors* $\tilde{\varphi}_1^\epsilon$, $\tilde{\varphi}_2^\epsilon$ and adapted processes $H^{\epsilon,i}$, $i = 1, 2$ such that $\mathbb{P}$-a.s. for every $t \in [0,T]$:

$$\tilde{\varphi}_i^\epsilon(u_t^\epsilon, y_t^\epsilon) = \tilde{\varphi}_i^\epsilon(u_0, y_0) + \int_0^t H_s^{\epsilon,i} ds + \int_0^t \langle D_y \tilde{\varphi}_i^\epsilon(u_s^\epsilon, y_s^\epsilon), Q^{1/2} dW_s \rangle, \quad i = 1, 2,$$

with the additional requirement that $H^\epsilon + \epsilon^{\alpha_1} H^{\epsilon,1} + \epsilon^{\alpha_2} H^{\epsilon,2}$ converges to some explicit process H^0 in a suitable sense, for some $\alpha_i > 0$ (here H^ϵ denotes an adapted process such that $\varphi(u_t^\epsilon) = \varphi(u_0) + \int_0^t H_s^\epsilon ds$, that exists by (S2)). In particular, it is not necessary that generalized correctors $\tilde{\varphi}_1^\epsilon$ and $\tilde{\varphi}_2^\epsilon$ are in the domain of $\mathscr{L}^\epsilon$.

On the other hand, whenever the arguments of previous sections work and produce correctors φ_1^ϵ, φ_2^ϵ within the domain of $\mathscr{L}^\epsilon$, it is natural to choose them to apply the perturbed function method. Moreover, we can not avoid proving the validity of *some* Itō Formula for the processes $\varphi_1^\epsilon(u^\epsilon, y^\epsilon)$ and $\varphi_2^\epsilon(u^\epsilon, y^\epsilon)$, since it does not descend directly from our notion of solution (whereas an Itō Formula for $\varphi(u^\epsilon)$ does); thus, proving previous Lemma 4.17 is a fully justified effort.
ii) Since Y^ϵ is regular, it is possible to consider functions $\Phi_1(u, y, Y) = \Phi_1(\Phi(u, y), Y)$ and prove an analogous Itō Formula for the process $\Phi_1(u_t^\epsilon, y_t^\epsilon, Y_t^\epsilon)$.

4.4.1 Tightness

In this paragraph we prove that the laws of the processes $\{u^\epsilon\}_{\epsilon\in(0,1)}$ are tight as probability measures on the space $L^2([0,T],H)\cap C([0,T],H^{-\beta})$ for every $\beta > 0$. The idea is to apply Simon compactness criterium Lemma 4.1. In order to do so, we need estimates on the increments $\mathbb{E}\left[\|u_t^\epsilon - u_s^\epsilon\|_{H^{-\sigma}}^p\right]$, $s, t \in [0,T]$, where $p > 2$ and $\sigma > 0$ are suitable parameters. Making use of the formula

$$\|u_t^\epsilon - u_s^\epsilon\|_{H^{-\sigma}}^2 = \sum_{k\in\mathbb{N}} \frac{\left(\varphi^k(u_t^\epsilon) - \varphi^k(u_s^\epsilon)\right)^2}{\lambda_k^{2\sigma}}, \qquad \varphi^k(u) = \langle u, e_k\rangle, \tag{4.16}$$

for $\{e_k\}_{k\in\mathbb{N}}$ a complete orthonormal system in H, we reduce the problem to providing suitable estimates for the quantity $\mathbb{E}\left[(\varphi^k(u_t^\epsilon) - \varphi^k(u_s^\epsilon))^p\right]$, which can be obtained applying Itō Formula Lemma 4.17 to the test function $\varphi^{k,\epsilon}(u^\epsilon, y^\epsilon) = \varphi^k(u^\epsilon) + \epsilon^{1/2}\varphi_1^{k,\epsilon}(u^\epsilon, y^\epsilon)$, with $\varphi_1^{k,\epsilon}$ being given by (4.11).
We prove first the following auxiliary lemma, consisting of an estimate on some negative Sobolev norm of the time increments $u_t^\epsilon - u_s^\epsilon$ and $y_t^\epsilon - y_s^\epsilon$.

Lemma 4.18. *Let $\{(u^\epsilon, y^\epsilon)\}_{\epsilon\in(0,1)}$ be a bounded-energy family of weak martingale solutions to (4.5), and for every $\epsilon \in (0,1)$ let Y^ϵ be the unique strong solution of (4.10). Then for every $p \geq 1$ and $\theta = \max\{\theta_0, \Gamma\}$ the following estimates hold:*

$$\mathbb{E}\left[\|u_t^\epsilon - u_s^\epsilon\|_{H^{-\theta_0}}^p\right] \lesssim \epsilon^{-p/2}|t-s|^p;$$
$$\mathbb{E}\left[\|Y_t^\epsilon - Y_s^\epsilon\|_{H^{-\theta_0}}^p\right] \lesssim \epsilon^{-p/2}|t-s|^{p/2};$$
$$\mathbb{E}\left[\|y_t^\epsilon - y_s^\epsilon - (Y_t^\epsilon - Y_s^\epsilon)\|_{H^{-\theta}}^p\right] \lesssim \epsilon^{-p}|t-s|^p.$$

Proof. Let us start from the estimate on u^ϵ. We have for every $h \in H^{\theta_0}$

$$\langle u_t^\epsilon - u_s^\epsilon, h\rangle = \int_s^t \langle u_r^\epsilon, Ah\rangle dr + \int_s^t \langle b(u_r^\epsilon, u_r^\epsilon), h\rangle dr + \epsilon^{-1/2}\int_s^t \langle b(y_r^\epsilon, u_r^\epsilon), h\rangle dr,$$

hence, using (B1)

$$|\langle u_t^\epsilon - u_s^\epsilon, h\rangle| \lesssim \int_s^t \|u_r^\epsilon\|_H \|h\|_{H^2} dr + \int_s^t \|u_r^\epsilon\|_H^2 \|h\|_{H^{\theta_0}} dr + \epsilon^{-1/2}\int_s^t \|y_r^\epsilon\|_H \|u_r^\epsilon\|_H \|h\|_{H^{\theta_0}} dr.$$

Therefore, taking the supremum over $h \in H^{\theta_0}$ with $\|h\|_{H^{\theta_0}} = 1$, elevating to the p-th power and taking expectations:

$$\mathbb{E}\left[\|u_t^\epsilon - u_s^\epsilon\|_{H^{-\theta_0}}^p\right] \lesssim \epsilon^{-p/2}|t-s|^p.$$

In order to get the estimate on Y^ϵ, we preliminarily rewrite the increment $Y^\epsilon_t - Y^\epsilon_s$ using the mild formulation of (4.10):

$$Y^\epsilon_t - Y^\epsilon_s = \left(e^{\epsilon^{-1}C_\epsilon(t-s)} - 1\right) Y^\epsilon_s + \epsilon^{-1/2}\int_s^t e^{\epsilon^{-1}C_\epsilon(t-r)} Q^{1/2} dW_r,$$

from which we are able to deduce, on the one hand

$$\mathbb{E}\left[\left\|\left(e^{\epsilon^{-1}C_\epsilon(t-s)} - 1\right) Y^\epsilon_s\right\|^p_{H^{-\theta_0}}\right] \lesssim \epsilon^{-p/2}|t-s|^{p/2}\mathbb{E}\left[\|Y^\epsilon_s\|^p_{H^{\Gamma-\theta_0}}\right] \lesssim \epsilon^{-p/2}|t-s|^{p/2},$$

and, applying Itō Isometry:

$$\mathbb{E}\left[\left\|\epsilon^{-1/2}\int_s^t e^{\epsilon^{-1}C_\epsilon(t-r)} Q^{1/2} dW_r\right\|^p_{H^{-\theta_0}}\right] \lesssim |t-s|^{p/2},$$

on the other.
Let us move to the estimate on y^ϵ. First, since Y^ϵ is a strong solution of (4.10), for every fixed $h \in H^{\theta_0}$ and $s,t \in [0,T]$, $s < t$, we have the following weak reformulation of (4.10):

$$\langle Y^\epsilon_t - Y^\epsilon_s, h\rangle = \epsilon^{-1}\int_s^t \langle Y^\epsilon_r, C_\epsilon h\rangle dr + \epsilon^{-1/2}\langle Q^{1/2}(W_t - W_s), h\rangle,$$

so that putting the previous expression together with (S2) we get

$$\begin{aligned}\langle y^\epsilon_t - y^\epsilon_s - (Y^\epsilon_t - Y^\epsilon_s), h\rangle = \epsilon^{-1}\int_s^t \langle y^\epsilon_r - Y^\epsilon_r, C_\epsilon h\rangle dr + \int_s^t \langle b(u^\epsilon_r, y^\epsilon_r), h\rangle dr \\ + \epsilon^{-1/2}\int_s^t \langle b(y^\epsilon_r, y^\epsilon_r), h\rangle dr.\end{aligned}$$

Hence, arguing as with $u^\epsilon_t - u^\epsilon_s$ we obtain

$$\mathbb{E}\left[\|y^\epsilon_t - y^\epsilon_s - (Y^\epsilon_t - Y^\epsilon_s)\|^p_{H^{-\theta}}\right] \lesssim \epsilon^{-p}|t-s|^p.$$

□

We move now to the main computation of this subsection. We have:

Lemma 4.19. *There exists $\alpha > 0$ depending only on γ, Γ and θ_0 such that the following holds. For every $p > 2$ there exists $\sigma > 0$ such that, for every $s,t \in [0,T]$:*

$$\mathbb{E}\left[\|u^\epsilon_t - u^\epsilon_s\|^p_{H^{-\sigma}}\right] \lesssim |t-s|^{\alpha p}.$$

Proof. Let us consider as test function $\varphi^{k,\epsilon} = \varphi^k + \epsilon^{1/2}\varphi^{k,\epsilon}_1$ as above, namely $\varphi^k(u) = \langle u, e_k\rangle$ and $\{e_k\}_{k\in\mathbb{N}}$ a complete orthonormal system in H, and $\varphi^{k,\epsilon}_1$ given by (4.11). With this choice of φ^k we have $D_u\varphi^k = e_k$ and

$$\begin{aligned}\varphi^{k,\epsilon}_1(u^\epsilon, y^\epsilon) &= \langle b((-C_\epsilon)^{-1}y^\epsilon, u^\epsilon), e_k\rangle,\\ D_u\varphi^{k,\epsilon}_1(y^\epsilon) &= -b((-C_\epsilon)^{-1}y^\epsilon, e_k),\\ D_y\varphi^{k,\epsilon}_1(u^\epsilon) &= \langle b((-C_\epsilon)^{-1}\cdot, u^\epsilon), e_k\rangle.\end{aligned}$$

Applying Itō Formula to $\varphi^{k,\epsilon}(u^\epsilon, y^\epsilon)$ we get almost surely for any given $s, t \in [0,T]$, $s < t$:

$$\varphi^k(u_t^\epsilon) - \varphi^k(u_s^\epsilon) = \epsilon^{1/2}(\varphi_1^{k,\epsilon}(u_s^\epsilon, y_s^\epsilon) - \varphi_1^{k,\epsilon}(u_t^\epsilon, y_t^\epsilon)) \\ + \int_s^t \mathscr{L}^\epsilon \varphi^{k,\epsilon}(u_r^\epsilon, y_r^\epsilon) dr + \int_s^t \langle D_y \varphi_1^{k,\epsilon}(u_r^\epsilon), Q^{1/2} dW_r \rangle. \tag{4.17}$$

Therefore, using (4.16) and Hölder inequality, for every $\sigma > 0$ satisfying $\sum_{k\in\mathbb{N}} \lambda_k^{-2\sigma} < \infty$ we get the following inequality

$$\mathbb{E}\left[\|u_t^\epsilon - u_s^\epsilon\|_{H^{-\sigma}}^p\right] = \mathbb{E}\left[\left(\sum_{k\in\mathbb{N}} \frac{\left(\varphi^k(u_t^\epsilon) - \varphi^k(u_s^\epsilon)\right)^2}{\lambda_k^{2\sigma}}\right)^{p/2}\right] \\ \leq \left(\sum_{k\in\mathbb{N}} \frac{1}{\lambda_k^{2\sigma}}\right)^{\frac{p-2}{p}} \mathbb{E}\left[\sum_{k\in\mathbb{N}} \frac{\left(\varphi^k(u_t^\epsilon) - \varphi^k(u_s^\epsilon)\right)^p}{\lambda_k^{2\sigma}}\right]. \tag{4.18}$$

Let us estimate the summands on the right-hand-side of (4.17) to complete the proof of the proposition. We start from the terms involving the time increment $\varphi_1^{k,\epsilon}(u_s^\epsilon, y_s^\epsilon) - \varphi_1^{k,\epsilon}(u_t^\epsilon, y_t^\epsilon)$: for every $s, t \in [0,T]$, $s < t$ it holds

$$|\varphi_1^{k,\epsilon}(u_s^\epsilon, y_s^\epsilon) - \varphi_1^{k,\epsilon}(u_t^\epsilon, y_t^\epsilon)| \leq |\varphi_1^{k,\epsilon}(u_s^\epsilon - u_t^\epsilon, y_s^\epsilon)| + |\varphi_1^{k,\epsilon}(u_t^\epsilon, y_s^\epsilon - y_t^\epsilon)| \\ = |\langle b((-C_\epsilon)^{-1} y_s^\epsilon, e_k), u_s^\epsilon - u_t^\epsilon\rangle| + |\langle b((-C_\epsilon)^{-1}(y_s^\epsilon - y_t^\epsilon), e_k), u_t^\epsilon\rangle| \\ \lesssim \|u_s^\epsilon - u_t^\epsilon\|_{H^{-2\gamma}} \|y_s^\epsilon\|_H \|e_k\|_{H^{\theta_1}} + \|y_s^\epsilon - y_t^\epsilon\|_{H^{-2\gamma}} \|u_t^\epsilon\|_H \|e_k\|_{H^{\theta_0}}.$$

We can invoke Lemma 4.18 and interpolation inequality to estimate the $H^{-2\gamma}$ norm of the time increments $u_s^\epsilon - u_t^\epsilon$ and $y_s^\epsilon - y_t^\epsilon$, and get (without loss of generality we assume $\gamma \leq \theta_0/4$)

$$\epsilon^{p/2}\, \mathbb{E}\left[\left(\varphi_1^{k,\epsilon}(u_s^\epsilon, y_s^\epsilon) - \varphi_1^{k,\epsilon}(u_t^\epsilon, y_t^\epsilon)\right)^p\right] \\ \lesssim \epsilon^{p/2} \|e_k\|_{H^{\theta_1}}^p\, \mathbb{E}\left[\|u_t^\epsilon - u_s^\epsilon\|_{H^{-\theta_0}}^{2\gamma p/\theta_0} \|u_t^\epsilon - u_s^\epsilon\|_H^{(1-2\gamma/\theta_0)p} \|y_t^\epsilon\|_H^p\right] \\ + \epsilon^{p/2} \|e_k\|_{H^{\theta_0}}^p\, \mathbb{E}\left[\|y_t^\epsilon - y_s^\epsilon\|_{H^{-\theta}}^{2\gamma p/\theta} \|y_t^\epsilon - y_s^\epsilon\|_H^{(1-2\gamma/\theta)p} \|u_s^\epsilon\|_H^p\right] \\ \lesssim \|e_k\|_{H^{\theta_1}}^p \epsilon^{p/2} \epsilon^{-2\gamma p/\theta_0} |t-s|^{2\gamma p/\theta_0} + \|e_k\|_{H^{\theta_0}}^p \epsilon^{p/2} \epsilon^{-2\gamma p/\theta} |t-s|^{\gamma p/\theta} \\ \lesssim \|e_k\|_{H^{\theta_1}}^p |t-s|^{2\gamma p/\theta_0} + \|e_k\|_{H^{\theta_0}}^p |t-s|^{\gamma p/\theta},$$

where we recall $\theta = \max\{\theta_0, \Gamma\}$. Let us move now the term with the time integral of $\mathscr{L}^\epsilon \varphi^{k,\epsilon}(u_r^\epsilon, y_r^\epsilon)$. We conveniently rewrite this term as $\mathscr{L}^\epsilon \varphi^{k,\epsilon} = \Phi^{k,\epsilon} + \epsilon^{1/2} \Phi_1^{k,\epsilon}$, where for $r \in [0,T]$, $\Phi^{k,\epsilon}, \Phi_1^{k,\epsilon}$ are implicitly given by

$$\mathscr{L}^\epsilon \varphi^{k,\epsilon}(u_r^\epsilon, y_r^\epsilon) = \langle A u_r^\epsilon + b(u_r^\epsilon, u_r^\epsilon), D_u \varphi^k\rangle + \langle b(y_r^\epsilon, u_r^\epsilon), D_u \varphi_1^{k,\epsilon}(y_r^\epsilon)\rangle + \langle b(y_r^\epsilon, y_r^\epsilon), D_y \varphi_1^{k,\epsilon}(u_r^\epsilon)\rangle \\ + \epsilon^{1/2}\left(\langle A u_r^\epsilon + b(u_r^\epsilon, u_r^\epsilon), D_u \varphi_1^{k,\epsilon}(y_r^\epsilon)\rangle + \langle b(u_r^\epsilon, y_r^\epsilon), D_y \varphi_1^{k,\epsilon}(u_r^\epsilon)\rangle\right) \\ =: \Phi^{k,\epsilon}(u_r^\epsilon, y_r^\epsilon) + \epsilon^{1/2} \Phi_1^{k,\epsilon}(u_r^\epsilon, y_r^\epsilon).$$

We have the inequalities

$$|\langle A u_r^\epsilon + b(u_r^\epsilon, u_r^\epsilon), D_u \varphi^k\rangle| = |\langle A u_r^\epsilon + b(u_r^\epsilon, u_r^\epsilon), e_k\rangle| \\ \lesssim \|u_r^\epsilon\|_H \|e_k\|_{H^2} + \|u_r^\epsilon\|_H^2 \|e_k\|_{H^{\theta_0}}, \\ |\langle b(y_r^\epsilon, u_r^\epsilon), D_u \varphi_1^{k,\epsilon}(y_r^\epsilon)\rangle| \lesssim \|e_k\|_{H^{\theta_1}} \|y_r^\epsilon\|_H \|y_r^\epsilon\|_{H^{3/2-2\gamma}} \|u_r^\epsilon\|_{H^1}, \\ |\langle b(y_r^\epsilon, y_r^\epsilon), D_y \varphi_1^{k,\epsilon}(u_r^\epsilon)\rangle| \lesssim \|e_k\|_{H^{\theta_1}} \|y_r^\epsilon\|_{H^1} \|y_r^\epsilon\|_{H^{3/2-2\gamma}} \|u_r^\epsilon\|_H,$$

so that the time integral of $\Phi^{k,\epsilon}(u_r^\epsilon, y_r^\epsilon)$ satisfies:

$$\mathbb{E}\left[\left(\int_s^t \Phi^{k,\epsilon}(u_r^\epsilon, y_r^\epsilon)dr\right)^p\right] \lesssim \|e_k\|_{H^{\theta_1}}^p |t-s|^{(\gamma-1/4)p}$$

uniformly in ϵ. Similarly,

$$\begin{aligned}|\langle Au_r^\epsilon, D_u\varphi_1^{k,\epsilon}(y_r^\epsilon)\rangle| &\lesssim \|e_k\|_{H^{\theta_1}}\|u_r^\epsilon\|_{H^1}\|y_r^\epsilon\|_{H^{1-2\gamma}},\\ |\langle b(u_r^\epsilon, u_r^\epsilon), D_u\varphi_1^{k,\epsilon}(y_r^\epsilon)\rangle| &\lesssim \|e_k\|_{H^{\theta_1}}\|u_r^\epsilon\|_{H^1}\|u_r^\epsilon\|_{H^{3/2-2\gamma}}\|y_r^\epsilon\|_{H},\\ |\langle b(u_r^\epsilon, y_r^\epsilon), D_y\varphi_1^{k,\epsilon}(u_r^\epsilon)\rangle| &\lesssim \|e_k\|_{H^{\theta_1}}\|y_r^\epsilon\|_{H^1}\|u_r^\epsilon\|_{H^{3/2-2\gamma}}\|u_r^\epsilon\|_{H},\end{aligned}$$

and we can bound the time integral of $\Phi_1^{k,\epsilon}(u_r^\epsilon, y_r^\epsilon)$ with

$$\epsilon^{p/2}\mathbb{E}\left[\left(\int_s^t \Phi_1^{k,\epsilon}(u_r^\epsilon, y_r^\epsilon)dr\right)^p\right] \lesssim \epsilon^{p/2}\|e_k\|_{H^{\theta_1}}^p |t-s|^{(\gamma-1/4)p}.$$

The last term remaining is the stochastic integral; we have by Itō Isometry

$$\begin{aligned}&\mathbb{E}\left[\left(\int_s^t \langle D_y\varphi_1^{k,\epsilon}(u_r^\epsilon), Q^{1/2}dW_r\rangle\right)^p\right]\\ &= \mathbb{E}\left[\left(\int_s^t \langle b((-C_\epsilon)^{-1}Q^{1/2}dW_r, u_r^\epsilon), e_k\rangle\right)^p\right] \lesssim \|e_k\|_{H^{\theta_0}}^p |t-s|^{p/2}.\end{aligned}$$

Putting all together, we finally arrive to the following bound, uniform in ϵ and valid for every $k \in \mathbb{N}$, $s,t \in [0,T]$, $s<t$ and for every $p>2$:

$$\mathbb{E}\left[\left(\varphi^k(u_t^\epsilon) - \varphi^k(u_s^\epsilon)\right)^p\right] \lesssim \|e_k\|_{H^{\theta_1}}^p |t-s|^{\alpha p}, \quad \alpha = \min\{2\gamma/\theta_0, \gamma/\theta, \gamma - 1/4\}. \tag{4.19}$$

Recall that in order to estimate the $H^{-\sigma}$ norm of $u_t^\epsilon - u_s^\epsilon$ we have to sum (4.19) above over all $k \in \mathbb{N}$; for this reason, we further require that σ is such that

$$\sum_{k\in\mathbb{N}} \frac{\|e_k\|_{H^{\theta_1}}^p}{\lambda_k^{2\sigma}} < \infty,$$

so that by Fubini-Tonelli Theorem, (4.18) and (4.19)

$$\begin{aligned}\mathbb{E}\left[\|u_t^\epsilon - u_s^\epsilon\|_{H^{-\sigma}}^p\right] &= \mathbb{E}\left[\left(\sum_{k\in\mathbb{N}} \frac{\left(\varphi^k(u_t^\epsilon) - \varphi^k(u_s^\epsilon)\right)^2}{\lambda_k^{2\sigma}}\right)^{p/2}\right]\\ &\le \left(\sum_{k\in\mathbb{N}} \frac{1}{\lambda_k^{2\sigma}}\right)^{\frac{p-2}{p}} \mathbb{E}\left[\sum_{k\in\mathbb{N}} \frac{\left(\varphi^k(u_t^\epsilon) - \varphi^k(u_s^\epsilon)\right)^p}{\lambda_k^{2\sigma}}\right] \lesssim |t-s|^{\alpha p}.\end{aligned}$$

□

Thus, we are ready to prove the first part of Theorem 1.3, that is:

Proposition 4.20. *For every $\beta > 0$, the laws of the processes $\{u^\epsilon\}_{\epsilon\in(0,1)}$ are tight as probability measures on the space $L^2([0,T],H) \cap C([0,T],H^{-\beta})$.*

Proof. Let α_0 be given by previous Lemma 4.19, and take $\alpha \in (0,\alpha_0)$, $p > 1/\alpha$ and $\sigma > 0$ such that the lemma holds. By the aforementioned lemma, $\mathbb{E}\left[\|u^\epsilon\|_{W^{\alpha,p}([0,T],H^{-\sigma})}\right]$ is bounded uniformly in ϵ; since in addition $\mathbb{E}\left[\|u^\epsilon\|_{L^\infty([0,T],H)}\right]$ and $\mathbb{E}\left[\|u^\epsilon\|_{L^2([0,T],H^1)}\right]$ are also bounded uniformly in ϵ by assumption (S3), Simon compactness criterium Lemma 4.1 yields tightness of the sequence of laws of the processes $\{u^\epsilon\}_{\epsilon\in(0,1)}$ in the space $L^2([0,T],H)$ $\cap$ $C([0,T],H^{-\beta})$ for every $\beta > 0$. □

4.4.2 Identification of the limit

Let $\varphi = \langle \cdot, h\rangle \in F$ be a test function, and denote $\varphi^\epsilon(u,y,Y) = \varphi(u) + \epsilon^{1/2}\varphi_1^\epsilon(u,y) + \epsilon\varphi_2^\epsilon(u,Y)$, where φ_1^ϵ and φ_2^ϵ are given by (4.11) and (4.12) respectively. Let us also introduce the homogeneous corrector φ_1 via the formula

$$\varphi_1(u,y) \coloneqq \langle b((-C)^{-1}y,u),h\rangle, \tag{4.20}$$

and the limiting effective generator $\mathscr{L}^0$ by

$$\mathscr{L}^0\varphi(u) = \langle Au + b(u,u),h\rangle + \int_H \psi_u(w)d\mu(w), \tag{4.21}$$

where $\psi_u(w) = \langle b(w,u), D_u\varphi_1(w)\rangle + \langle b(w,w), D_y\varphi_1(u)\rangle$ and $\mu = \mathcal{N}(0,Q_\infty)$.
Since (u^ϵ, y^ϵ) is a weak solution of system (4.5) and Y^ϵ is a strong solution to (4.10), by Itō Formula Lemma 4.17 we have almost surely for every $t \in [0,T]$:

$$\begin{aligned}\varphi^\epsilon(u_t^\epsilon, y_t^\epsilon, Y_t^\epsilon) = \varphi^\epsilon(u_0,y_0,0) &+ \int_0^t \mathscr{L}^\epsilon\varphi^\epsilon(u_s^\epsilon,y_s^\epsilon,Y_s^\epsilon)ds \\ &+ \epsilon^{-1/2}\int_0^t \langle D_y\varphi^\epsilon(u_s^\epsilon,y_s^\epsilon,Y_s^\epsilon), Q^{1/2}dW_s\rangle,\end{aligned}$$

or equivalently,

$$\begin{aligned}\varphi(u_t^\epsilon) = \varphi(u_0) &+ \int_0^t \mathscr{L}^0\varphi(u_s^\epsilon)ds + \int_0^t \langle b((-C)^{-1}Q^{1/2}dW_s,u_s^\epsilon),h\rangle \\ &+ \int_0^t \left(\mathscr{L}^{0,\epsilon}\varphi(u_s^\epsilon) - \mathscr{L}^0\varphi(u_s^\epsilon)\right)ds + \int_0^t \langle b(\left((-C_\epsilon)^{-1} - (-C)^{-1}\right)Q^{1/2}dW_s,u_s^\epsilon),h\rangle \\ &+ \epsilon^{1/2}\left(\varphi_1^\epsilon(u_0,y_0) - \varphi_1^\epsilon(u_t^\epsilon,y_t^\epsilon)\right) + \epsilon\left(\varphi_2^\epsilon(u_0,0) - \varphi_2^\epsilon(u_t^\epsilon,Y_t^\epsilon)\right) \\ &+ \int_0^t \Phi_0^\epsilon(u_s^\epsilon,y_s^\epsilon,Y_s^\epsilon)ds + \epsilon^{1/2}\int_0^t \Phi_1^\epsilon(u_s^\epsilon,y_s^\epsilon,Y_s^\epsilon)ds + \epsilon\int_0^t \Phi_2^\epsilon(u_s^\epsilon,Y_s^\epsilon)ds \\ &+ \epsilon^{1/2}\int_0^t \langle D_Y\varphi_2^\epsilon(u_s^\epsilon,Y_s^\epsilon), Q^{1/2}dW_s\rangle,\end{aligned} \tag{4.22}$$

where $\mathscr{L}^0$ is the limiting effective generator defined by (4.21), $\mathscr{L}^{0,\epsilon}$ is the effective generator defined by (4.13) and we have denoted for notational simplicity

$$\begin{aligned}\Phi_0^\epsilon(u,y,Y) &= \langle b(y,u), D_u\varphi_1^\epsilon(y)\rangle + \langle b(y,y), D_y\varphi_1^\epsilon(u)\rangle \\ &\quad - \langle b(Y,u), D_u\varphi_1^\epsilon(Y)\rangle + \langle b(Y,Y), D_y\varphi_1^\epsilon(u)\rangle, \\ \Phi_1^\epsilon(u,y,Y) &= \langle Au + b(u,u), D_u\varphi_1^\epsilon(y)\rangle + \langle b(u,y), D_y\varphi_1^\epsilon(u)\rangle \\ &\quad + \langle b(y,u), D_u\varphi_2^\epsilon(Y)\rangle, \\ \Phi_2^\epsilon(u,Y) &= \langle Au + b(u,u), D_u\varphi_2^\epsilon(Y)\rangle.\end{aligned}$$

Equation (4.22) clearly indicates the candidate limit dynamics - first line of the equation - and the remainder terms - lines second to fifth. Our aim consists in proving, on the one hand the convergence of the first line to the same quantity evaluated at $u^\epsilon = u$ (for a possibly different Wiener process W; recall that at this stage the stochastic basis is still dependent on ϵ), and on the other hand the convergence of all remainders to zero.

In order to conveniently pass to the limit $\epsilon \to 0$, we invoke a standard combination of Prokhorov Theorem and Skorokhod Theorem. Indeed, since the family of laws of the processes $\{u^\epsilon\}_{\epsilon\in(0,1)}$ is tight on the space $L^2([0,T],H) \cap C([0,T],H^{-\beta})$ for every $\beta > 0$, and $\{Q^{1/2}W = Q^{1/2}W^\epsilon\}_{\epsilon\in(0,1)}$ is a family of identically distributed $C([0,T],H)$-valued random variables, by Prokhorov Theorem there exists a subsequence $\epsilon_n \to 0$ such that $(u^{\epsilon_n}, Q^{1/2}W^{\epsilon_n})$ converges in distribution as $n \to \infty$ towards a process $(u, Q^{1/2}W)$[2] taking values in the space:

$$\mathcal{X} := \big(L^2([0,T],H) \cap C([0,T],H^{-\beta})\big) \times C([0,T],H).$$

Then, given *any* subsequence such that $(u^{\epsilon_n}, Q^{1/2}W^{\epsilon_n}) \to (u, Q^{1/2}W)$ in distribution (not necessarily that one provided by Prokhorov Theorem), in virtue of Skorokhod Theorem there exists a new probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ supporting $\mathcal{X}$-valued random variables $(\tilde u, Q^{1/2}\tilde W) \sim (u, Q^{1/2}W)$ and $(\tilde u^n, Q^{1/2}\tilde W^n) \sim (u^{\epsilon_n}, Q^{1/2}W^{\epsilon_n})$ for every $n \in \mathbb{N}$ such that $(\tilde u^n, Q^{1/2}\tilde W^n) \to (\tilde u, Q^{1/2}\tilde W)$ $\tilde{\mathbb{P}}$-almost surely as random variables in $\mathcal{X}$. Of course, as usually done in these situations we drop the tildes in what follows.

Proposition 4.21. *Let $(u^n, Q^{1/2}W^n) \to (u, Q^{1/2}W)$ as above. Then for every $\varphi \in F$ we have the almost sure identity*

$$\varphi(u_t) = \varphi(u_0) + \int_0^t \mathscr{L}^0\varphi(u_s)ds + \int_0^t \langle b((-C)^{-1}Q^{1/2}dW_s, u_s), h\rangle, \quad \forall t \in [0,T]. \tag{4.23}$$

Proof. We divide the proof in three steps. First, we show that the remainder terms are infinitesimal in mean square as $n \to \infty$; second, we prove that the deterministic effective dynamics is a continuous function of the path $\xi \in C([0,T],H^{-\beta}) \cap L^2([0,T],H)$; finally, we invoke a martingale representation theorem to identify the limit behaviour of the martingale term in (4.23).

Step 1. Let us focus on the remainder terms in the right-hand-side of (4.22). They are of several kinds: *i*) terms involving the differences

$$\int_0^t \mathscr{L}^{0,\epsilon_n}\varphi(u_s^{\epsilon_n})ds - \int_0^t \mathscr{L}^0\varphi(u_s^{\epsilon_n})ds, \qquad \int_0^t \langle b(G_{\epsilon_n}Q^{1/2}dW_s, u_s^{\epsilon_n}), h\rangle,$$

where the operator $G_{\epsilon_n} := (-C_{\epsilon_n})^{-1} - (-C)^{-1} = \epsilon_n(-C)^{-1}A(-C_\epsilon)^{-1}$, which are controlled using the bounds $\|G_{\epsilon_n}\|_{H^s\to H^{s+2\gamma(1+\beta)-2\beta}} \lesssim {\epsilon_n}^\beta$ and $\|e^{C_{\epsilon_n}t} - e^{Ct}\|_{H^{\theta+2\beta}\to H^\theta} \lesssim {\epsilon_n}^\beta$ for every $\beta \in [0,1]$, uniformly in $t \in [0,\infty)$, and go to zero in mean square as $n \to \infty$ (and $\epsilon_n \to 0$); *ii*) terms of the form $\epsilon_n^{1/2}\varphi_1^{\epsilon_n}(u_t^n, y_t^n)$ or $\epsilon_n\varphi_2^{\epsilon_n}(u_t^n, Y_t^n)$, $t \in [0,T]$, $Y^n := Y^{\epsilon_n}$, which can be easily shown to converge to zero in mean square as $n \to \infty$ as a consequence of energy bounds for (u^ϵ, y^ϵ), the bound $\|\varphi_2^{\epsilon_n}(u_t^n, \cdot)\|_{\mathcal{E}^{\theta'}} \lesssim \|h\|_{H^{\theta_1}}\|u_t^n\|_H$ for some $\theta' > 5/4 - 2\gamma$ and

$$\begin{aligned}|\varphi_1^{\epsilon_n}(u_t^n, y_t^n)| &\lesssim \|y_t^n\|_H\|u_t^n\|_H\|h\|_{H^{\theta_0}},\\ |\varphi_2^{\epsilon_n}(u_t^n, Y_t^n)| &\lesssim (1 + \|Y_t^n\|^2_{H^{\theta'}})\|h\|_{H^{\theta_1}}\|u_t^n\|_H;\end{aligned}$$

iii) the term $\int_0^t \Phi_0^{\epsilon_n}(u_s^n, y_s^n, Y_s^n)ds$, which is infinitesimal in mean square by Proposition 4.14 and Proposition 4.6; *iv*) the terms involving the time integrals of $\Phi_1^{\epsilon_n}(u_s^n, y_s^n, Y_s^n)$

[2] Recall that any Wiener process with covariance operator Q can be written as $Q^{1/2}W$ for some cylindrical Wiener process W on H.

and $\Phi_2^{\epsilon_n}(u_s^n, Y_s^n)$, which are controlled by Proposition 4.13, Proposition 4.16 and the estimates:

$$\begin{aligned}
|\Phi_1^{\epsilon_n}(u_s^n, y_s^n, Y_s^n)| &\lesssim |\langle Au_s^n + b(u_s^n, u_s^n), D_u\varphi_1^{\epsilon_n}(y_s^n)\rangle| + |\langle b(u_s^n, y_s^n), D_y\varphi_1^{\epsilon_n}(u_s^n)\rangle| \\
&\quad + |\langle b(y_s^n, u_s^n), D_u\varphi_2^{\epsilon_n}(Y_s^n)\rangle| \\
&\lesssim \|u_s^n\|_H(1 + \|u_s^n\|_{H^1})\|y_s^n\|_H\|h\|_{H^{\theta_1}} \\
&\quad + \|u_s^n\|_H\|y_s^n\|_H\|Y_s^n\|_{H^{\theta_0}}^2\|h\|_{H^{\theta_1}}, \\
|\Phi_2^{\epsilon_n}(u_s^n, Y_s^n)| &\lesssim \|u_s^n\|_H(1 + \|u_s^n\|_H)\|Y_s^n\|_{H^{\theta_0}}^2\|h\|_{H^{\theta_1}};
\end{aligned}$$

and finally, v) the stochastic integral $\epsilon_n^{1/2}\int_0^t\langle D_Y\varphi_2^{\epsilon_n}(u_s^n, Y_s^n), Q^{1/2}dW_s^n\rangle$, which by Itō Isometry satifies

$$\epsilon_n\mathbb{E}\left[\left(\int_0^t\langle D_Y\varphi_2^{\epsilon_n}(u_s^n, Y_s^n), Q^{1/2}dW_s^n\rangle\right)^2\right] = \epsilon_n\mathbb{E}\left[\int_0^t\|Q^{1/2}D_Y\varphi_2^{\epsilon_n}(u_s^n, Y_s^n)\|_H^2ds\right] \to 0.$$

Thus all the remainders converge to zero in mean square.
Step 2. Let us consider on the path space $\mathcal{X}$ equipped with its Borel sigma field $\mathcal{B}$ the pushforward probability measures

$$\mathbb{Q}^n := \mathbb{P}\circ(u^n, Q^{1/2}W^n)^{-1}, \quad \mathbb{Q} := \mathbb{P}\circ(u, Q^{1/2}W)^{-1}.$$

Of course $\mathbb{Q}^n$ weakly converges towards $\mathbb{Q}$ as $n\to\infty$. Let $\mathcal{A}$ be the $\mathbb{Q}$-completion of $\mathcal{B}$, and let $\{\mathcal{A}_t\}_{t\in[0,T]}$ be the smallest filtration of $\mathcal{A}$ that satisfies the usual conditions with respect to $\mathbb{Q}$ and such that the coordinate process (ξ,ω) on $\mathcal{X}$ is adapted. Introduce $\mathcal{A}^n$ and $\{\mathcal{A}_t^n\}_{t\in[0,T]}$ similarly. Define the process

$$\rho_t := \varphi(\xi_t) - \varphi(\xi_0) - \int_0^t \mathscr{L}^0\varphi(\xi_s)ds, \quad t\in[0,T]. \tag{4.24}$$

Let us show that ρ_t is a continuous function of ξ.
First of all, notice that every $\varphi\in F$, $\varphi(\xi) = \langle\xi, h\rangle$ for some $h\in\mathscr{S}$, is a continuous function from $H^{-\beta}$ to $\mathbb{R}$. Therefore if $\xi^n\to\xi$ in $C([0,T], H^{-\beta})$ we have $\varphi(\xi^n)\to\varphi(\xi)$ in $C([0,T])$ as well. Let us now consider the term involving the effective generator $\mathscr{L}^0$. Recall

$$\mathscr{L}^0\varphi(\xi) = \langle A\xi + b(\xi,\xi), h\rangle + \int_H\psi_\xi(w)d\mu(w).$$

Let us show that the map $L^2([0,T], H)\ni\xi\mapsto\int_0^\cdot\mathscr{L}^0\varphi(\xi_s)ds\in C([0,T])$ is sequentially continuous, or equivalently

$$\int_0^t\mathscr{L}^0\varphi(\xi_s^n)ds\to\int_0^t\mathscr{L}^0\varphi(\xi_s)ds \tag{4.25}$$

in $C([0,T])$. For every $s\in[0,t]$, rewrite

$$\begin{aligned}
\mathscr{L}^0\varphi(\xi_s^n) - \mathscr{L}^0\varphi(\xi_s) &= \langle A\xi_s^n + b(\xi_s^n,\xi_s^n), h\rangle - \langle A\xi_s + b(\xi_s,\xi_s), h\rangle \\
&\quad + \int_H\psi_{\xi_s^n}(w)d\mu(w) - \int_H\psi_{\xi_s}(w)d\mu(w).
\end{aligned}$$

Let us bound the terms in the right-hand-side of the previous expression separately. Making use of the usual estimates on b, we have

$$\begin{aligned}|\langle A(\xi_s^n-\xi_s),h\rangle| &\leq \|\xi_s^n-\xi_s\|_H\|h\|_{H^2};\\ |\langle b(\xi_s^n,\xi_s^n)-b(\xi_s,\xi_s),h\rangle| &\leq |\langle b(\xi_s^n,\xi_s^n-\xi_s),h\rangle|+|\langle b(\xi_s^n-\xi_s,\xi_s),h\rangle|\\ &\lesssim (\|\xi_s^n\|_H+\|\xi_s\|_H)\|\xi_s^n-\xi_s\|_H\|h\|_{H^{\theta_0}};\\ \left|\int_H \psi_{\xi_s^n}(w)d\mu(w)-\int_H \psi_{\xi_s}(w)d\mu(w)\right| &\lesssim \|h\|_{H^{\theta_1}}\|\xi_s^n-\xi_s\|_H\int_H \|w\|_{H^{\theta_0}}^2 d\mu(w)\\ &\lesssim \|h\|_{H^{\theta_1}}\|\xi_s^n-\xi_s\|_H.\end{aligned}$$

Putting all together, we finally obtain the following bound

$$|\mathscr{L}^0\varphi(\xi_s^n)-\mathscr{L}^0\varphi(\xi_s)| \lesssim (1+\|\xi_s^n\|_H+\|\xi_s\|_H)\|\xi_s^n-\xi_s\|_H.$$

In particular, recalling that $\xi^n\to\xi$ in $L^2([0,T],H)$ we have:

$$\begin{aligned}\sup_{t\in[0,T]}\left|\int_0^t \mathscr{L}^0\varphi(\xi_s^n)ds-\int_0^t \mathscr{L}^0\varphi(\xi_s)ds\right| &\leq \int_0^T |\mathscr{L}^0\varphi(\xi_s^n)-\mathscr{L}^0\varphi(\xi_s)|ds\\ \lesssim \left(\int_0^T (1+\|\xi_s^n\|_H+\|\xi_s\|_H)^2 ds\right)^{1/2} &\left(\int_0^T \|\xi_s^n-\xi_s\|_H^2 ds\right)^{1/2}\to 0.\end{aligned}$$

Step 3. By weak convergence $\mathbb{Q}^n\to\mathbb{Q}$ and previous steps it is easy to show (cfr. for instance [FG95, Theorem 3.1] or [DPZ14, Chapter 8.4]) that the couple (ρ,ω) is a continuous square-integrable martingale on $(\mathcal{X},\mathcal{A},\{\mathcal{A}_t\}_{t\in[0,T]},\mathbb{Q})$ with quadratic covariations ($\{e_k\}_{k\in\mathbb{N}}$ is a complete orthonormal system of H):

$$\begin{aligned}[\rho,\rho]_t &= \int_0^t \|Q^{1/2}D_y\varphi_1(\xi_s)\|_H^2 ds,\\ [\rho,\langle e_k,\omega\rangle]_t &= \int_0^t \langle Q^{1/2}D_y\varphi_1(\xi_s),Q^{1/2}e_k\rangle ds,\\ [\langle e_k,\omega\rangle,\langle e_k,\omega\rangle]_t &= \langle Q^{1/2}e_k,Q^{1/2}e_k\rangle t.\end{aligned}$$

This is basically due to the fact that ρ can be written as the sum of a martingale on $(\mathcal{X},\mathcal{A},\{\mathcal{A}_t^n\}_{t\in[0,T]},\mathbb{Q}^n)$ plus remainder terms which are infinitesimal in mean square as $n\to\infty$. By [DPZ14, Theorem 8.2], up to a possible enlargement of the underlying probability space, there exists a cylindrical Wiener process $\tilde\omega$ on $(\mathcal{X},\mathcal{A},\{\mathcal{A}_t\}_{t\in[0,T]},\mathbb{Q})$ such that the following martingale representation formulae hold $\mathbb{Q}$-almost surely for every $t\in[0,T]$:

$$\begin{aligned}\omega_t &= \int_0^t Q^{1/2}d\tilde\omega_s = Q^{1/2}\tilde\omega_t,\\ \rho_t &= \int_0^t \langle D_y\varphi_1(\xi_s),Q^{1/2}d\tilde\omega_s\rangle = \int_0^t \langle D_y\varphi_1(\xi_s),d\omega_s\rangle.\end{aligned}$$

In particular, since the auxiliary Wiener process $\tilde\omega$ satisfies $\omega_t=Q^{1/2}\tilde\omega_t$, the equation for ρ_t above holds true also in the original probability space, without necessarily taking an enlargement thereof. Thus, recalling (4.24) we have the following $\mathbb{Q}$-almost sure identity on the path space $\mathcal{X}$:

$$\varphi(\xi_t) = \varphi(\xi_0) + \int_0^t \mathscr{L}^0 \varphi(\xi_s) ds + \int_0^t \langle D_y \varphi_1(\xi_s), d\omega_s \rangle, \quad \forall t \in [0,T],$$

that by $\mathbb{Q} = \mathbb{P} \circ (u, Q^{1/2} W)^{-1}$ and the explicit expression of φ_1 (4.20) is equivalent to our thesis.

□

4.4.3 Itō-Stokes drift and Stratonovich corrector

In this subsection, we provide an interpretation of the limiting equation in terms of different contribution to the dynamics. Recall that every weak accumulation point u of the family $\{u^\epsilon_t\}_{\epsilon \in (0,1)}$ satisfies, for every $\varphi \in F$, $\varphi(u) = \langle u, h \rangle$ for some $h \in \mathscr{S}$, $t \in [0,T]$ the almost sure identity (4.23):

$$\varphi(u_t) = \varphi(u_0) + \int_0^t \mathscr{L}^0 \varphi(u_s) ds + \int_0^t \langle b((-C)^{-1} Q^{1/2} dW_s, u_s), h \rangle,$$

where the limiting effective generator $\mathscr{L}^0$ is given by (4.21). For the reader's convenience, here we rewrite $\mathscr{L}^0 \varphi$ more explicitly as

$$\begin{aligned}
\mathscr{L}^0 \varphi(u) &= \langle Au + b(u,u), h \rangle + \int_H \langle b(w,u), D_u \varphi_1(w) \rangle d\mu(w) + \int_H \langle b(w,w), D_y \varphi_1(u) \rangle d\mu(w) \\
&= \langle Au + b(u,u), h \rangle + \int_H \langle b((-C)^{-1} w, b(w,u)), h \rangle d\mu(w) \\
&\quad + \int_H \langle b((-C)^{-1} b(w,w), u), h \rangle d\mu(w).
\end{aligned}$$

Let us compare (4.23) with the dynamics of u^ϵ, $\epsilon \in (0,1)$. Of course, the term $\langle Au_s + b(u_s,u_s), h \rangle$ reflects the deterministic dynamics $\langle Au^\epsilon_s + b(u^\epsilon_s, u^\epsilon_s), h \rangle$ in the evolution of u^ϵ. On the other hand, the fast-oscillating term $\epsilon^{-1/2} \langle b(y^\epsilon_s, u^\epsilon_s), h \rangle$ in the equation for u^ϵ is responsible for the additional terms in the limit. We can distinguish three different contributions:

- The Itō integral:

$$\int_0^t \langle b((-C)^{-1} Q^{1/2} dW_s, u_s), h \rangle; \tag{4.26}$$

- The Stratonovich corrector:

$$\int_0^t \int_H \langle b((-C)^{-1} w, b(w, u_s)), h \rangle d\mu(w) ds; \tag{4.27}$$

- The Itō-Stokes drift:

$$\int_0^t \int_H \langle b((-C)^{-1} b(w,w), u_s), h \rangle d\mu(w) ds. \tag{4.28}$$

Of course the term denoted "Itō-Stokes drift" equals $\int_0^t \langle b(r,u_s),h\rangle ds$ by the very definition of the Itō-Stokes drift velocity $r = \int_H (-C)^{-1} b(w,w) d\mu(w)$. Moreover, we shall see that the term called "Stratonovich corrector" (4.27) above is indeed the Stratonovich corrector of the term called "Ito integral" (4.26), namely:

$$\begin{aligned}\int_0^t \langle D_y\varphi_1(u_s), Q^{1/2}\circ dW_s\rangle &= \int_0^t \langle D_y\varphi_1(u_s), Q^{1/2} dW_s\rangle \\ &\quad + \int_0^t \int_H \langle b(w,u_s), D_u\varphi_1(w)\rangle d\mu(w) ds.\end{aligned} \tag{4.29}$$

Therefore, (4.23) is exactly the weak formulation of the equation in Theorem 1.3, which is then proved.

We are left to check the validity of (4.29). Let $h \in \mathscr{S}$ be fixed, and let $W = \sum_k e_k W^k$, where $\{e_k\}_{k\in\mathbb{N}}$ is a complete orthonormal system in H and $\{W^k\}_{k\in\mathbb{N}}$ is a family of one-dimensional i.i.d. Wiener processes. As a matter of fact, one can rewrite the Itō integral (4.26) as

$$\int_0^t \langle b((-C)^{-1}Q^{1/2}dW_s, u_s), h\rangle = -\sum_{k\in\mathbb{N}} \int_0^t \langle b((-C)^{-1}Q^{1/2}e_k, h), u_s\rangle dW_s^k;$$

since for $h \in \mathscr{S}$ it holds $b((-C)^{-1}Q^{1/2}e_k, h) \in \mathscr{S}$ as well, the quadratic variation between the processes $\langle b((-C)^{-1}Q^{1/2}e_k,h),u\rangle$ and W^k is given by

$$\left[\langle b((-C)^{-1}Q^{1/2}e_k,h),u\rangle, W^k\right]_t = -\int_0^t \langle b((-C)^{-1}Q^{1/2}e_k,u_s), b((-C)^{-1}Q^{1/2}e_k,h)\rangle ds. \tag{4.30}$$

On the other hand, (4.27) equals

$$\begin{aligned}\int_0^t \int_H \langle b((-C)^{-1}w, b(w,u_s)), h\rangle d\mu(w) ds &= \sum_{k\in\mathbb{N}} \int_0^t \langle b((-C)^{-1}Q_\infty^{1/2}e_k, b(Q_\infty^{1/2}e_k,u_s)), h\rangle ds \\ &= -\sum_{k\in\mathbb{N}} \int_0^t \langle b((-C)^{-1}Q_\infty^{1/2}e_k, h), b(Q_\infty^{1/2}e_k,u_s)\rangle ds.\end{aligned} \tag{4.31}$$

Recall that, since C and Q commute, the covariance operator Q_∞ of the invariant measure $\mu = \mathcal{N}(0,Q_\infty)$ can be written as $Q_\infty = \frac{1}{2}(-C)^{-1}Q$. In particular, there exists a complete orthonormal system $\{e_k\}_{k\in\mathbb{N}}$ of H that diagonalizes C, Q and Q_∞ simultaneously, namely

$$Ce_k = -\lambda_k e_k, \quad Qe_k = q_k e_k, \quad Q_\infty e_k = \frac{q_k}{2\lambda_k} e_k,$$

and therefore by (4.30) and (4.31) we finally get

$$\frac{1}{2}\left[\langle b((-C)^{-1}Q^{1/2}e_k,h),u\rangle, W^k\right]_t = \int_0^t \int_H \langle b((-C)^{-1}w, b(w,u_s)), h\rangle d\mu(w) ds.$$

Remark 4.5. *i*) The attentive reader would have noticed that this is the first time we are actually using that C and Q commute. In particular, the convergence result described in this chapter still holds true in the case $CQ \neq QC$. In this case, we believe that

$$\sum_{k\in\mathbb{N}} \int_0^t \langle b((-C)^{-1}Q^{1/2}e_k, u_s), b((-C)^{-1}Q^{1/2}e_k, h)\rangle ds$$
$$\neq \sum_{k\in\mathbb{N}} \int_0^t \langle b((-C)^{-1}Q_\infty^{1/2}e_k, h), b(Q_\infty^{1/2}e_k, u_s)\rangle ds$$

and the limit equation cannot be interpreted as an equation with Strotonovich transport noise.

ii) Assume that Q is isotropic, i.e. every basis of eigenvectors of A also diagonalizes Q. Under this circumstance, the Itō-Stokes drift equals zero for the Navier-Stokes system, since for the particular choice

$$e_{\mathbf{k},i} = a_{\mathbf{k},i}\cos(2\pi\mathbf{k}\cdot x), \quad \text{or } e_{\mathbf{k},i} = a_{\mathbf{k},i}\sin(2\pi\mathbf{k}\cdot x),$$

where $\mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, $i = 1, \dots, d-1$, and $a_{\mathbf{k},i} \in \mathbf{k}^\perp$ for every $\mathbf{k}, i$, it holds

$$b(e_{\mathbf{k},i}, e_{\mathbf{k},i}) = -\Pi\left((e_{\mathbf{k},i}\cdot\nabla)e_{\mathbf{k},i}\right) = \pm\Pi\left(\underbrace{(a_{\mathbf{k},i}\cdot 2\pi\mathbf{k})}_{=0}\cos(2\pi\mathbf{k}\cdot x)\sin(2\pi\mathbf{k}\cdot x)\right) = 0.$$

Chapter 5

Quantitative mixing and enhanced dissipation of Ornstein-Uhlenbeck flow

We recall the equation satisfied by ρ:

$$\partial_t \rho + v^\epsilon \cdot \nabla \rho = \kappa \Delta \rho \quad \text{in } [0,1] \times \mathbb{T}^d, \tag{5.1}$$

with initial value $\rho|_{t=0} = \rho_0 \in L^2(\mathbb{T}^d)$ with zero mean, and by $\bar{\rho}$, with the same initial condition:

$$\partial_t \bar{\rho} = (\kappa \Delta + \mathcal{L}) \bar{\rho} \quad \text{in } [0,1] \times \mathbb{T}^d. \tag{5.2}$$

The vector field v^ϵ has been defined as

$$v^\epsilon = \sum_{j \in J} v_j \eta^{\epsilon, j},$$

where J is a finite set, $\{v_j\}_{j \in J}$ is a family of smooth, divergence-free vector fields and $\eta^{\epsilon,j}$ are stationary one-dimensional i.i.d. Ornstein-Uhlenbeck processes. Details are given below.

In this chapter we prove mixing (Theorem 1.4) and dissipation enhancement (Theorem 1.5) for the solution ρ of (5.1). The key point is that our result is truly quantitative. This is important because, as we have already seen, transport noise in Stratonovich form is a good model to take into account small-scales unresolved variables in several systems only in the infinite scale-separation limit, that is obviously a crude idealization; in reality, scale-separation is always finite (or there is no scale-separation at all) and thus it is useful to quantify what error we make when using the ideal white-in-time noise.

The chapter is structured as follows.

In Section 5.1 we recall some classical result concerning well-posedness of (5.1) and (5.2). A precise description of our model is carried on in Section 5.1.2.

In Section 5.2 we collect some auxiliary results concerning the time increments of the process ρ solution of (5.1). There we prove that ρ has a.s. regular trajectories as a process taking value in a space of distributions, and we obtain good bounds for the expectation of its increments – see Lemma 5.2 and Lemma 5.3. In addition, we state the key result Proposition 5.4, needed for the proof of Theorem 1.4, that consists in a bound on the quantity

$$\sup_{\substack{n,m=1,\dots,1/\delta-1 \\ n>m}} (n\delta - m\delta)^{-\theta} \left| \langle \phi, \rho_{n\delta} \rangle - \langle \phi, \rho_{m\delta} \rangle - \delta \sum_{k=m}^{n-1} \langle A\phi, \rho_{k\delta} \rangle \right|,$$

where δ is a small parameter, suitably chosen in depending on ϵ.
In Section 5.3, we give the proof of Theorem 1.4 and Theorem 1.5.
Concerning the first result, the key idea consists in introducing the random distribution f defined via the formula

$$\langle \phi, f_t \rangle = \langle \phi, \rho_t \rangle - \langle \phi, \rho_0 \rangle - \int_0^t \langle A\phi, \rho_s \rangle ds, \quad \forall \phi \in H^2(\mathbb{T}^d), \quad \forall t \in [0,1] :$$

of course if one could prove $f = 0$, then one would have $\rho_t = \bar{\rho}_t$ for every $t \in [0,1]$; however, it turns out that the difference $\rho - \bar{\rho}$ depends *path-by-path* continuously on f, (Lemma 5.6) and therefore we are able to deduce an estimate on $\rho - \bar{\rho}$ from an estimate on f – cfr. Proposition 5.5.
As for Theorem 1.5, its proof relies on the following energy inequality

$$\frac{d}{dt} \|\rho_t\|^2_{L^2(\mathbb{T}^d)} \leq -2\kappa \frac{\|\rho_t\|^4_{L^2(\mathbb{T}^d)}}{\|\rho_t\|^2_{H^{-1}(\mathbb{T}^d)}}$$

and a bound on $\|\rho_t\|^2_{H^{-1}(\mathbb{T}^d)}$ obtained applying Theorem 1.4 with $s = 1$. Since our result on mixing is truly quantitative, we are able to say that $\|\rho_t\|^2_{H^{-1}(\mathbb{T}^d)}$ is smaller than a certain threshold (depending on t) with high probability, and using this information in the previous inequality we deduce an explicit rate of decay for the $L^2(\mathbb{T}^d)$ norm of ρ.
Finally, in Section 5.4 we prove Proposition 5.4. Its proof is based on a discretization procedure very common in the literature about averaging and Wong-Zakai approximations theorems for stochastic differential equations, and are inspired by our previous works [AFP21, FP21]. The advection-diffusion equation (5.1) is interpreted in a pathwise sense, although the velocity field v is random – it is usually referred to as random PDE rather than SPDE; the key Lemma 5.6 is analytic as well; the results contained in Section 5.4 are more probabilistic in the spirit, and rely on explicit computability of the Ornstein-Uhlenbeck process, Doob maximal inequality, Burkholder-Davis-Gundy inequality and Kolmogorov continuity criterion.

5.1 Notation and preliminaries

5.1.1 Functional analytic setting

Let $e_k(x) = (2\pi)^{-d/2} e^{ik\cdot x}$, $k \in \mathbb{Z}^d$. The set $\{e_k\}_{k\in\mathbb{Z}^d}$ is a complete orthonormal system of $L^2(\mathbb{T}^d, \mathbb{C})$ made of eigenfunctions of the Laplace operator: $\Delta e_k = -|k|^2 e_k$ for every $k \in \mathbb{Z}^d$. A generic function $f \in L^2(\mathbb{T}^d, \mathbb{C})$ can be then represented as a Fourier series:

$$f(x) = \sum_{k\in\mathbb{Z}^d} \hat{f}_k e_k(x), \quad x \in \mathbb{T}^d,$$

for a (unique) square summable sequence $\{\hat{f}_k\}_{k\in\mathbb{Z}^d}$ of Fourier coefficients. The Fourier map $\mathfrak{F} : L^2(\mathbb{T}^d, \mathbb{C}) \to \ell^2(\mathbb{Z}^d, \mathbb{C})$, that associates to every f the sequence of its Fourier coefficients, is an isomorphism of Hilbert spaces; it can be extended to an isomorphism between the space of (complex-valued) tempered distributions $\mathcal{S}'(\mathbb{T}^d, \mathbb{C})$ and space of sequences of Fourier coefficients $\{\hat{f}_k\}_{k\in\mathbb{Z}^d}$ such that $\sum_{k\in\mathbb{Z}^d} |k|^{2s}|\hat{f}_k|^2 < \infty$ for some $s \in \mathbb{R}$. Unless otherwise stated, we will only work in the following with the space $\mathcal{S}'(\mathbb{T}^d) \subset \mathcal{S}'(\mathbb{T}^d, \mathbb{C})$ of real-valued distributions with zero mean, that corresponds to sequences $\{\hat{f}_k\}_{k\in\mathbb{Z}^d}$ such that $\hat{f}_0 = 0$ and $\hat{f}_k = \overline{\hat{f}_{-k}}$ for every $k \in \mathbb{Z}^d$. We denote $\mathbb{Z}^d_0 = \mathbb{Z}^d \setminus \{\mathbf{0}\}$.
For $s \in \mathbb{R}$, define the Sobolev space

$$H^s(\mathbb{T}^d) = \left\{ f \in \mathcal{S}'(\mathbb{T}^d) : \|f\|_{H^s} \coloneqq \sum_{k\in\mathbb{Z}^d_0} |k|^{2s}|\hat{f}_k|^2 < \infty \right\},$$

which is a Hilbert space when equipped with the scalar product

$$\langle f, g\rangle_{H^s(\mathbb{T}^d)} = \sum_{k\in\mathbb{Z}^d_0} |k|^{2s} \hat{f}_k \hat{g}_{-k}, \quad f, g \in H^s(\mathbb{T}^d).$$

In the special case $s = 0$, the Sobolev space corresponds to the space $L^2(\mathbb{T}^d)$ of real-valued, square integrable functions on the torus with zero mean. The scalar product $\langle\cdot,\cdot\rangle \coloneqq \langle\cdot,\cdot\rangle_{H^0(\mathbb{T}^d)}$ is also a duality map between $H^s(\mathbb{T}^d)$ and $H^{-s}(\mathbb{T}^d)$ for every $s \in \mathbb{R}$:

$$\langle f, g\rangle = \sum_{k\in\mathbb{Z}^d_0} \hat{f}_k \hat{g}_{-k}, \quad f \in H^s(\mathbb{T}^d),\ g \in H^{-s}(\mathbb{T}^d).$$

Sobolev spaces form a Hilbert scale with respect to the operator $(-\Delta)^{1/2}$; in particular, the following interpolation inequality holds between $H^{s_1}(\mathbb{T}^d)$ and $H^{s_2}(\mathbb{T}^d)$, for $s_1, s_2 \in \mathbb{R}$, $s_1 < s_2$ and $\theta \in (0, 1)$:

$$\|f\|_{H^{s_\theta}(\mathbb{T}^d)} \le \|f\|^{\theta}_{H^{s_1}(\mathbb{T}^d)} \|f\|^{1-\theta}_{H^{s_2}(\mathbb{T}^d)}, \quad s_\theta = \theta s_1 + (1-\theta)s_2. \tag{5.3}$$

Finally, we have the following lemma, cfr. [BCD11, Corollary 2.55] for a proof in the full space:

Lemma 5.1. *Let $s_1, s_2 \in \mathbb{R}$ such that $s_1, s_2 < d/2$ and $s_1 + s_2 > 0$. Then for every $f \in H^{s_1}(\mathbb{T}^d)$ and $g \in H^{s_2}(\mathbb{T}^d)$ the product $fg \in H^{s_1+s_2-d/2}(\mathbb{T}^d)$ and*

$$\|fg\|_{H^{s_1+s_2-d/2}(\mathbb{T}^d)} \lesssim_{s_1,s_2} \|f\|_{H^{s_1}(\mathbb{T}^d)} \|g\|_{H^{s_2}(\mathbb{T}^d)}.$$

Remark 5.1. Condition $d \ge 3$ stated in the introduction is a technical limitation of our method, due to application of Lemma 5.1, but there is no physical reason for this constraint. Also, the case $d = 2$ in Theorem 1.4 and Theorem 1.5 is readily implied by our results in dimension $d = 3$ and the following observation: for every $s \in \mathbb{R}$ and $f \in H^s(\mathbb{T}^2)$, the function $g : \mathbb{T}^3 \to \mathbb{R}$ defined by $g(x_1, x_2, x_3) \coloneqq f(x_1, x_2)$ satisfies for every $k = (k_1, k_2, k_3) \in \mathbb{Z}^3_0$

$$\hat{g}_{(k_1,k_2,k_3)} = \begin{cases} \hat{f}_{(k_1,k_2)} & \text{if } k_3 = 0, \\ 0 & \text{if } k_3 \neq 0, \end{cases}$$

and thus $g \in H^s(\mathbb{T}^3)$ with $\|g\|_{H^s(\mathbb{T}^3)} = \|f\|_{H^s(\mathbb{T}^2)}$.

5.1.2 The model

Stationary Ornstein-Uhlenbeck processes

Let J be a finite index set of cardinality $|J|$, and let $(\Omega^+, \{\mathcal{F}_t^+\}_{t\geq 0}, \mathbb{P}^+)$ and $(\Omega^-, \{\mathcal{F}_t^-\}_{t\geq 0}, \mathbb{P}^-)$ be two filtered probability spaces, satisfying the usual conditions, which support two families of i.i.d. Brownian motions $\{W^{+,j}\}_{j\in J}$ and $\{W^{-,j}\}_{j\in J}$. Set $W_t^j = W_t^{+,j}$, for $t \geq 0$, and $W_t^j = W_{-t}^{-,j}$, for $t < 0$.
For every $\epsilon < 1$, the processes

$$\eta^{\epsilon,j}(t) := \int_{-\infty}^t \epsilon^{-1} e^{-\epsilon^{-1}(t-s)} dW_s^j, \quad t \in [0,1], \quad j \in J, \tag{5.4}$$

constitute a family of i.i.d. stationary Ornstein-Uhlenbeck processes solutions of

$$d\eta^{\epsilon,j} = -\epsilon^{-1}\eta^{\epsilon,j} dt + \epsilon^{-1} dW_t^j, \quad t \in [0,1], \quad j \in J,$$

on the filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\in[0,1]}, \mathbb{P})$ defined by $\Omega := \Omega^- \times \Omega^+$, $\mathbb{P} := \mathbb{P}^- \otimes \mathbb{P}^+$, and where $\{\mathcal{F}_t\}_{t\geq 0}$ is the augmentation of the filtration $\{\mathcal{F}_\infty^- \otimes \mathcal{F}_t^+\}_{t\in[0,1]}$. Notice that this filtration satisfies the usual conditions. Moreover, by (5.4) it holds for every $t \in [0,1]$,

$$\begin{aligned} \eta^{\epsilon,j}(t) &= -\int_0^\infty \epsilon^{-1} e^{-\epsilon^{-1}(t+s)} dW_s^{-,j} + \int_0^t \epsilon^{-1} e^{-\epsilon^{-1}(t-s)} dW_s^{+,j} \\ &= e^{-\epsilon^{-1}t}\eta^{\epsilon,j}(0) + \epsilon^{-1}\int_0^t e^{-\epsilon^{-1}(t-s)} dW_s^j, \end{aligned} \tag{5.5}$$

where in the second line we have used $W_s^{+,j} = W_s^j$ for every $s \geq 0$ and

$$\eta^{\epsilon,j}(0) = -\int_0^\infty \epsilon^{-1} e^{-\epsilon^{-1}s} dW_s^{-,j}.$$

5.1.3 Notion of solution to (5.1) and (5.2)

We assume $\{v_j\}_{j\in J}$ to be a family of smooth vector fields $v_j : \mathbb{T}^d \to \mathbb{R}^d$ such that $\operatorname{div} v_j = 0$ for every $j \in J$.
We give now the notion of solution for the random PDE (5.1)

$$\partial_t \rho + v^\epsilon \cdot \nabla \rho = \kappa \Delta \rho \quad \text{in } [0,1] \times \mathbb{T}^d,$$

with (deterministic) initial value $\rho|_{t=0} = \rho_0 \in L^2(\mathbb{T}^d)$. We shall use this notion of solution throughout the chapter. A general time interval $[0,T]$ can be handled with no difficulties In what follows, $\mathcal{D}(\mathbb{T}^d)$ stands for the space of real-valued, zero-mean, smooth test functions.

Definition 5.1. Assume $\kappa > 0$. A stochastic process $\rho : \Omega \times [0,1] \to L^2(\mathbb{T}^d)$, adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,1]}$, is a (analytically weak, probabilistically strong) solution of (5.1) if there exists a full-measure set $\tilde\Omega \subset \Omega$ such that for every $\omega \in \tilde\Omega$ it holds: $\rho(\omega,\cdot) \in L^\infty([0,1], L^2(\mathbb{T}^d)) \cap L^2([0,1], H^1(\mathbb{T}^d))$ and

$$\langle \phi, \rho_t \rangle = \langle \phi, \rho_s \rangle + \int_s^t \langle v^\epsilon(r) \cdot \nabla\phi, \rho_r \rangle dr + \kappa \int_s^t \langle \Delta\phi, \rho_r \rangle dr,$$

for every $s < t$ and $\phi \in \mathcal{D}(\mathbb{T}^d)$.

For every initial datum $\rho_0 \in L^2(\mathbb{T}^d)$ and $\kappa > 0$, (5.1) is well-posed in the sense of the previous definition. Indeed, the advection velocity v^ϵ is almost surely smooth in space and Hölder continuous in time, thus by [Paz83, Corollary 2.2] the only solution is given by Duhamel formula

$$\rho_t = e^{\kappa\Delta t}\rho_0 + \int_0^t e^{\kappa\Delta(t-s)}\left(v^\epsilon(s)\cdot\nabla\rho_s\right)ds.$$

Existence of (probabilistically strong) solution can be proved by standard approximation schemes and Yamada-Watanabe theorem. Also, by linearity of the equation and mollification we can also check that the map $t \mapsto \|\rho_t\|^2_{L^2(\mathbb{T}^d)}$ has a.s. absolutely continuous trajectories and the following energy inequality (as variables in $L^1([0,1])$) holds with probability one:

$$\frac{d}{dt}\|\rho_t\|^2_{L^2(\mathbb{T}^d)} = -2\kappa\|\rho_t\|^2_{H^1(\mathbb{T}^d)}, \tag{5.6}$$

In particular, from (5.6) we deduce the following almost sure energy estimate, which we will use extensively in the following:

$$\sup_{t\in[0,1]}\left(\|\rho_t\|^2_{L^2(\mathbb{T}^d)} + 2\kappa\int_0^t\|\rho_s\|^2_{H^1(\mathbb{T}^d)}ds\right) \leq \|\rho_0\|^2_{L^2(\mathbb{T}^d)}. \tag{5.7}$$

By (5.6) and using the inequality $\|\rho_t\|^2_{L^2(\mathbb{T}^d)} \leq \|\rho_t\|^2_{H^1(\mathbb{T}^d)}$, one can deduce the following almost sure decay of $L^2(\mathbb{T}^d)$ norm for the solution of (5.1):

$$\|\rho_t\|_{L^2(\mathbb{T}^d)} \leq e^{-\kappa t}\|\rho_0\|_{L^2(\mathbb{T}^d)}. \tag{5.8}$$

In the inviscid case $\kappa = 0$, we must introduce the flow $\mathbf{V}^\epsilon$ associated to v^ϵ to exhibit the (Lagrangian) solution $\rho_t = \rho_0 \circ (\mathbf{V}^\epsilon)^{-1}$, which however need not to have trajectories in $L^2([0,T],H^1)$. Energy inequalities (5.7) and (5.8) in this case read as $\|\rho_t\|_{L^2(\mathbb{T}^d)} \leq \|\rho_0\|_{L^2(\mathbb{T}^d)}$.

Concerning equation (5.2), it is well known [Paz83, Theorem 5.2] that the operator A generates an analytic semigroup of negative type $e^{A\cdot}$ on $\mathcal{S}'(\mathbb{T}^d)$, and the unique solution of (5.2) is given by the Duhamel formula

$$\bar\rho_t = e^{At}\rho_0, \quad t\in[0,1].$$

Decay of $L^2(\mathbb{T}^d)$ norm of $\bar\rho$ can be estimated as follows:

$$\|\bar\rho_t\|_{L^2(\mathbb{T}^d)} \leq e^{-\lambda t}\|\bar\rho_0\|_{L^2(\mathbb{T}^d)},$$

where λ is the principal eigenvalue of the operator $-A$. Notice that $\lambda \geq \kappa$ since $\mathcal{L}$ is a negative-semidefinite operator.

Remark 5.2. The inequality $\|\rho_t\|^2_{L^2(\mathbb{T}^d)} \leq \|\rho_t\|^2_{H^1(\mathbb{T}^d)}$, used to deduce (5.8) above, may be very loose if the energy of ρ is distributed at high wavenumbers, viz. the Fourier coefficients $\{\hat\rho_k\}_{k\in\mathbb{Z}^d_0}$ do not decrease sufficiently fast as $|k|\to\infty$. This is in fact the case: indeed, by Theorem 1.4, for every fixed $t > 0$

$$\begin{aligned}\mathbb{E}\left[\|\pi_N\rho_t\|_{L^2(\mathbb{T}^d)}\right] &\leq N\mathbb{E}\left[\|\rho_t\|_{H^{-1}(\mathbb{T}^d)}\right]\\ &\leq N\left(\|\bar\rho_t\|_{H^{-1}(\mathbb{T}^d)} + \mathbb{E}\left[\|\rho_t-\bar\rho_t\|_{H^{-1}(\mathbb{T}^d)}\right]\right)\\ &\lesssim N\|\rho_0\|_{L^2(\mathbb{T}^d)}\left(e^{-\lambda t} + \left(\alpha+\epsilon^{\varkappa}\mu^{2+\gamma}\right)^{\varsigma}\right),\end{aligned}$$

where $\pi_N : \mathcal{S}'(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$ denotes the projector onto Fourier modes $|k| \leq N$, $N \in \mathbb{N}$. The previous inequality suggests that energy is actually transferred to high wavenumbers if $t > 0$, $\alpha, \epsilon \ll 1$ and $\lambda \gg 1$, and hence we do not expect inequality (5.8) to be sharp – that is indeed the content of Theorem 1.5.

5.2 Useful estimates

In this section, we prove some auxiliary results concerning the time increments of the process ρ solution of (5.1). Proofs are mostly inspired by our previous works [AFP21, FP21].
Recall that ρ has trajectories taking values in $L^\infty([0,1], L^2(\mathbb{T}^d))$ almost surely. Forthcoming Lemma 5.2 and (5.3) allow to estimate the increments $\rho_{t+\delta} - \rho_t$ with respect to Sobolev norms $H^s(\mathbb{T}^d)$, $s \in [-1, 0]$. It turns out that ρ is actually a.s. Hölder continuous as a variable taking values in $H^s(\mathbb{T}^d)$ for $s \in (-1, 0]$, and it is a.s. Lipschitz continuous as variable taking values in $H^{-1}(\mathbb{T}^d)$; however, its Hölder and Lipschitz constants diverge to infinity for $\epsilon \to 0$, and therefore we need Lemma 5.3 to better control them in expected value.
The subsequent Proposition 5.4 aims to control the error between the actual solution ρ of (5.1) and a discretized version of (5.2). We will make an essential use of the latter in the proof of Proposition 5.5 in Section 5.3. Its proof, however, is quite long: for the sake of a clear and effective presentation we postpone it to Section 5.4.
To start with, we recall that for every $p \geq 1$ and $j \in J$ the supremum of the Ornstein-Uhlenbeck process $\eta^{\epsilon,j}$ can be estimated in expected value with

$$\mathbb{E}\left[\sup_{s\in[0,1]} |\eta^{\epsilon,j}(s)|^p\right] \lesssim_p \epsilon^{-p/2} \log(1+\epsilon^{-1})^{p/2}. \tag{5.9}$$

From (5.9) and the definition of μ, one can deduce the following inequalities:

$$\mathbb{E}\left[\sup_{s\in[0,1]} \|v^\epsilon(s)\|^p_{H^{d/2-\gamma}(\mathbb{T}^d)}\right] \lesssim_p \mu^p \epsilon^{-p/2} \log^{p/2}(1+\epsilon^{-1}); \tag{5.10}$$

$$\mathbb{E}\left[\sup_{s\in[0,1]} \|v^\epsilon(s)\|^p_{L^\infty(\mathbb{T}^d)}\right] \lesssim_p \mu^p \epsilon^{-p/2} \log^{p/2}(1+\epsilon^{-1}). \tag{5.11}$$

We prove now the following result, which allows to control the time increments of the process ρ in a Sobolev space of negative order.

Lemma 5.2. *Let $\delta_\star > 0$ such that $\delta_\star \epsilon^{-1} \log(1+\epsilon^{-1}) > 1$. Then for every $p \geq 1$ the following inequality holds:*

$$\mathbb{E}\left[\sup_{\substack{t+\delta\leq 1\\ \delta\leq\delta_\star}} \|\rho_{t+\delta} - \rho_t\|^p_{H^{-1}(\mathbb{T}^d)}\right] \lesssim_p \|\rho_0\|^p_{L^2(\mathbb{T}^d)} \mu^p \delta_\star^p \epsilon^{-p/2} \log^{p/2}(1+\epsilon^{-1}).$$

Proof. By the very definition of weak solution of (5.1) and (5.7), for every test function $\phi \in \mathcal{D}(\mathbb{T}^d)$ one has the following almost sure inequality

$$\begin{aligned}|\langle \phi, \rho_{t+\delta} - \rho_t \rangle| &\leq \int_t^{t+\delta} |\langle v^\epsilon(s) \cdot \nabla \phi, \rho_s \rangle| ds + \kappa \int_t^{t+\delta} |\langle \Delta \phi, \rho_s \rangle| ds \\ &\leq \|\phi\|_{H^1(\mathbb{T}^d)} \int_t^{t+\delta} \|v^\epsilon(s)\|_{L^\infty(\mathbb{T}^d)} \|\rho_s\|_{L^2(\mathbb{T}^d)} ds \\ &\quad + \kappa \|\phi\|_{H^1(\mathbb{T}^d)} \int_t^{t+\delta} \|\rho_s\|_{H^1(\mathbb{T}^d)} ds \\ &\leq \|\phi\|_{H^1(\mathbb{T}^d)} \|\rho_0\|_{L^2(\mathbb{T}^d)} \delta \sup_{s \in [0,1]} \|v^\epsilon(s)\|_{L^\infty(\mathbb{T}^d)} \\ &\quad + \|\phi\|_{H^1(\mathbb{T}^d)} \|\rho_0\|_{L^2(\mathbb{T}^d)} \kappa^{1/2} \delta^{1/2}.\end{aligned}$$

Since ϕ is arbitrary, we deduce

$$\|\rho_{t+\delta} - \rho_t\|_{H^{-1}(\mathbb{T}^d)} \leq \|\rho_0\|_{L^2(\mathbb{T}^d)} \left(\delta \sup_{s \in [0,1]} \|v^\epsilon(s)\|_{L^\infty(\mathbb{T}^d)} + \kappa^{1/2} \delta^{1/2} \right).$$

Taking the supremum (raised to power p) over δ, and then expectation, (5.11) yields

$$\mathbb{E} \left[\sup_{\substack{t+\delta \leq 1 \\ \delta \leq \delta_\star}} \|\rho_{t+\delta} - \rho_t\|^p_{H^{-1}(\mathbb{T}^d)} \right] \lesssim_p \|\rho_0\|^p_{L^2(\mathbb{T}^d)} \left(\delta_\star^p \mu^p \epsilon^{-p/2} \log^{p/2}(1+\epsilon^{-1}) + \kappa^{p/2} \delta_\star^{p/2} \right).$$

The thesis follows recalling that $\kappa^{p/2}\delta_\star^{p/2} < \delta_\star^p \mu^p \epsilon^{-p/2} \log^{p/2}(1+\epsilon^{-1})$, due to our choice of parameters. □

The previous lemma can be used to deduce the following result. In view of interpolation inequality (5.3), the next lemma can be used together with Lemma 5.2 in order to obtain better estimates – needed in the following – on the increments of ρ in Sobolev spaces $H^s(\mathbb{T}^d)$, with $s \in (-2, -1)$.

Lemma 5.3. *Let $d \geq 3$, $\delta \in (0,1)$ such that $\delta\epsilon^{-1}\log(1+\epsilon^{-1}) > 1$ and $\mu\delta^{4/3}\epsilon^{-1}\log^{1/3}(1+\epsilon^{-1}) < 1$. Then for every $\gamma \in (0,(d-2)/2)$ and $p \geq 1$ the following inequality holds for every fixed $t \in [0, 1-\delta]$:*

$$\mathbb{E}\left[\|\rho_{t+\delta} - \rho_t\|^p_{H^{-2-\gamma}(\mathbb{T}^d)} \right] \lesssim_{\gamma,p} \|\rho_0\|^p_{L^2(\mathbb{T}^d)} \mu^p \left(\delta^{p/2} + \epsilon^{p/2} \log^{p/2}(1+\epsilon^{-1}) \right).$$

Proof. As in the proof of Lemma 5.2, we have the following almost sure inequality for every given test function $\phi \in \mathcal{D}(\mathbb{T}^d)$

$$\begin{aligned}|\langle \phi, \rho_{t+\delta} - \rho_t \rangle| &\leq \int_t^{t+\delta} |\langle v^\epsilon(s) \cdot \nabla \phi, \rho_s - \rho_t \rangle| ds \\ &\quad + \left| \int_t^{t+\delta} \langle v^\epsilon(s) \cdot \nabla \phi, \rho_t \rangle ds \right| + \kappa \int_t^{t+\delta} |\langle \Delta \phi, \rho_s \rangle| ds.\end{aligned}$$

Let us deal with each term separately. Using Lemma 5.1 with $s_1 = d/2 - \gamma$, $s_2 = 1 + \gamma$ and Lemma 5.2 we get

$$\begin{aligned}\int_t^{t+\delta} |\langle v^\epsilon(s) \cdot \nabla\phi, \rho_s - \rho_t \rangle| ds &\le \int_t^{t+\delta} \|v^\epsilon(s) \cdot \nabla\phi\|_{H^1(\mathbb{T}^d)} \|\rho_s - \rho_t\|_{H^{-1}(\mathbb{T}^d)} ds \\ &\lesssim_\gamma \|\phi\|_{H^{2+\gamma}(\mathbb{T}^d)} \int_t^{t+\delta} \|v^\epsilon(s)\|_{H^{d/2-\gamma}(\mathbb{T}^d)} \|\rho_s - \rho_t\|_{H^{-1}(\mathbb{T}^d)} ds,\end{aligned}$$

where we use $d/2 - \gamma < d/2$, $1 + \gamma < d/2$ to apply Lemma 5.1, which correspond to the conditions $d \ge 3$, $\gamma \in (0, (d-2)/2)$.
Moving to the next term, recall that $v^\epsilon(s)ds = \sum_{j\in J} v_j \eta^{\epsilon,j}(s)ds = \sum_{j\in J} v_j dW^j_s - \epsilon \sum_{j\in J} v_j d\eta^{\epsilon,j}(s)$, and thus

$$\begin{aligned}\left| \int_t^{t+\delta} \langle v^\epsilon(s) \cdot \nabla\phi, \rho_t \rangle ds \right| &= \left| \left\langle \left(\int_t^{t+\delta} v^\epsilon(s) ds \right) \cdot \nabla\phi, \rho_t \right\rangle \right| \\ &\le \|\phi\|_{H^1(\mathbb{T}^d)} \|\rho_0\|_{L^2(\mathbb{T}^d)} \sum_{j\in J} \|v_j\|_{L^\infty(\mathbb{T}^d)} \left| W^j_{t+\delta} - W^j_t \right| \\ &\quad + \|\phi\|_{H^1(\mathbb{T}^d)} \|\rho_0\|_{L^2(\mathbb{T}^d)} \epsilon \sup_{s\in[0,1]} \|v^\epsilon(s)\|_{L^\infty(\mathbb{T}^d)}.\end{aligned}$$

Finally,

$$\int_t^{t+\delta} |\langle \Delta\phi, \rho_s \rangle| ds \le \|\phi\|_{H^2(\mathbb{T}^d)} \|\rho_0\|_{L^2(\mathbb{T}^d)} \delta.$$

Therefore, since ϕ is arbitrary and $\|\phi\|_{H^1(\mathbb{T}^d)}, \|\phi\|_{H^2(\mathbb{T}^d)} \le \|\phi\|_{H^{2+\gamma}(\mathbb{T}^d)}$ for every $\gamma > 0$:

$$\begin{aligned}\|\rho_{t+\delta} - \rho_t\|_{H^{-2-\gamma}(\mathbb{T}^d)} \lesssim_\gamma &\sup_{s\in[0,1]} \|v^\epsilon(s)\|_{H^{d/2-\gamma}(\mathbb{T}^d)} \int_t^{t+\delta} \|\rho_s - \rho_t\|_{H^{-1}(\mathbb{T}^d)} ds \\ &+ \|\rho_0\|_{L^2(\mathbb{T}^d)} \sum_{j\in J} \|v_j\|_{L^\infty(\mathbb{T}^d)} \left| W^j_{t+\delta} - W^j_t \right| \\ &+ \|\rho_0\|_{L^2(\mathbb{T}^d)} \epsilon \sup_{s\in[0,1]} \|v^\epsilon(s)\|_{L^\infty(\mathbb{T}^d)} + \|\rho_0\|_{L^2(\mathbb{T}^d)} \delta.\end{aligned}$$

Using (5.10), (5.11), Lemma 5.2, and the inequality $\mathbb{E}\left[\left| W^j_{t+\delta} - W^j_t \right|^p \right] \lesssim_p \delta^{p/2}$ valid for every $p \ge 1$ we get (recall that $\delta < 1$ and therefore $\delta < \delta^{1/2}$)

$$\begin{aligned}\mathbb{E}\left[\|\rho_{t+\delta} - \rho_t\|^p_{H^{-2-\gamma}(\mathbb{T}^d)} \right] \lesssim_{\gamma,p} &\|\rho_0\|^p_{L^2(\mathbb{T}^d)} \mu^{2p} \delta^{2p} \epsilon^{-p} \log^p(1 + \epsilon^{-1}) \\ &+ \|\rho_0\|^p_{L^2(\mathbb{T}^d)} \mu^p \delta^{p/2} \\ &+ \|\rho_0\|^p_{L^2(\mathbb{T}^d)} \mu^p \epsilon^{p/2} \log^{p/2}(1 + \epsilon^{-1}).\end{aligned}$$

The thesis now follows recalling our choice of parameters. □

For the next proposition we need some preparation. Divide the interval $[0, 1]$ into subintervals of the form $[n\delta, (n+1)\delta]$, for some $\delta \in (0, 1)$. In the following δ will be taken small, and it may depend on the parameter ϵ. The idea behind this subdivision is to control the following quantity:

$$\sup_{\substack{n,m=1,\dots,1/\delta-1 \\ n>m}} (n\delta - m\delta)^{-\theta} \left| \langle \phi, \rho_{n\delta} \rangle - \langle \phi, \rho_{m\delta} \rangle - \delta \sum_{k=m}^{n-1} \langle A\phi, \rho_{k\delta} \rangle \right|, \tag{5.12}$$

for any given test function $\phi \in \mathcal{D}(\mathbb{T}^d)$ and some $\theta > 0$, where we have used the symbol A as an abbreviation for the operator $\kappa\Delta + \mathcal{L}$. To be precise, we will prove the following:

Proposition 5.4. *Let $d \geq 3$, $\beta > d/2 + 1$ and $\gamma \in (0, (d-2)/6)$ be fixed. Then there exist $\theta > 0$, $\varkappa > 0$ and $\delta > 0$ such that $1/\delta$ is an integer and for every ϵ sufficiently small*

$$\mathbb{E}\left[\sup_{\substack{n,m=1,\dots,1/\delta-1 \\ n>m}} (n\delta - m\delta)^{-\theta} \left| \langle \phi, \rho_{n\delta} \rangle - \langle \phi, \rho_{m\delta} \rangle - \delta \sum_{k=m}^{n-1} \langle A\phi, \rho_{k\delta} \rangle \right| \right]$$
$$\lesssim_\gamma \|\phi\|_{H^\beta(\mathbb{T}^d)} \|\rho_0\|_{L^2(\mathbb{T}^d)} \left(\alpha + \epsilon^{\varkappa} \mu^{2+\gamma} \right).$$

The proof of the previous result is quite long and technical, and we postpone it to Section 5.4. The particular choice of the parameters $\theta, \varkappa$ and δ is specified therein. In the statement of the proposition, ϵ sufficiently small means more precisely: $\mu\epsilon^{p_1} \log^{p_2}(1 + \epsilon^{-1}) < 1$, for some $p_1, p_2 > 0$ depending only on the parameters γ, θ and $\varkappa$. More details are given in the proof of the proposition.
Moreover, since we always work at fixed γ, hereafter we omit the dependence of γ in the symbol $\lesssim_\gamma$.

5.3 Quantitative mixing and dissipation enhancement

In the first part of this section we prove our main result Theorem 1.4. The idea is very simple, but effective.
Define the random distribution $f : \Omega \times [0, 1] \to H^{-2}(\mathbb{T}^d)$ as follows: for every test function $\phi \in H^2(\mathbb{T}^d)$,

$$\langle \phi, f_t \rangle \coloneqq \langle \phi, \rho_t \rangle - \langle \phi, \rho_0 \rangle - \int_0^t \langle A\phi, \rho_s \rangle ds, \quad t \in [0, 1].$$

If one could prove $f = 0$, then by [Bal77] we would have $\rho_t - e^{At}\rho_0 = \rho_t - \bar{\rho}_t = 0$ for every $t \in [0, 1]$. Of course this is not the case, since $f \neq 0$; however, owing to [GLT06, Theorem 1] we can prove that the difference $\rho - \bar{\rho}$ is a continuous function of f (with respect to suitable topologies), and therefore $\rho - \bar{\rho}$ is small if also f is.
The "right" topology in which to prove smallness of f turns out to be that of Hölder continuous functions $C^\theta([0, 1], H^{-\beta}(\mathbb{T}^d))$, for some small $\theta \in (0, 1)$ and $\beta > d/2 + 1$ (cfr. Lemma 5.6). We would like to stress that *any* $\theta \in (0, 1)$ sufficiently small works: in particualr, it can be taken *arbitrarily* small. This comes with no surprise: the reader familiar with SPDEs would recognize f as a sort of additive noise perturbing the linear equation (5.2).
In the forthcoming Proposition 5.5 we provide suitable estimates for the Hölder norm of f. As just discussed, we will use this result in the subsequent Section 5.3.2 to prove Theorem 1.4.

Finally, we will prove Theorem 1.5 in Section 5.3.3. We will deduce this theorem from Theorem 1.4 and energy equality (5.6), using as an intermediate step Markov inequality to prove that the quantity $\sup_{t\in[0,1]} \|\rho_t - \bar{\rho}_t\|_{H^{-1}(\mathbb{T}^d)}$ is small with high probability.

5.3.1 Estimate on f

We begin with the following remark: since $f_0 = 0$ and the time interval is compact, the Hölder norm of f is equivalent to the Hölder seminorm

$$\|f\|_{C^\theta([0,1],H^{-\beta}(\mathbb{T}^d))} \sim \sup_{0<s<t<1} \frac{\|f_t - f_s\|_{H^{-\beta}(\mathbb{T}^d)}}{|t-s|^\theta}.$$

Proposition 5.5. *Let $d \geq 3$, $\beta > d/2 + 1$ and $\gamma \in (0, (d-2)/6)$. Then there exists θ sufficiently small such that, for every ϵ sufficiently small:*

$$\mathbb{E}\left[\|f\|_{C^\theta([0,1],H^{-\beta}(\mathbb{T}^d))}\right] \lesssim \|\rho_0\|_{L^2(\mathbb{T}^d)} \left(\alpha + \epsilon^{\varkappa}\mu^{2+\gamma}\right)$$

Proof. Let $s, t \in [0,1]$, $s < t$ and let $\phi \in \mathcal{D}(\mathbb{T}^d)$ be a test function. Given δ as in Proposition 5.4, we distinguish two cases:

- if $|t-s| \leq \delta$, then arguing as in the proof of Lemma 5.2 one can prove the following a.s. inequality:

$$\begin{aligned}
|\langle \phi, (\rho_t - \rho_s)\rangle| &\leq \|\phi\|_{H^2(\mathbb{T}^d)} \|\rho_t - \rho_s\|_{H^{-2}(\mathbb{T}^d)} \\
&\leq \|\phi\|_{H^2(\mathbb{T}^d)} \|\rho_0\|_{L^2(\mathbb{T}^d)} |t-s| \left(\sup_{s\in[0,1]} \|v^\epsilon\|_{L^\infty(\mathbb{T}^d)} + \kappa \right) \\
&\leq \|\phi\|_{H^2(\mathbb{T}^d)} \|\rho_0\|_{L^2(\mathbb{T}^d)} |t-s|^\theta \delta^{1-\theta} \left(\sup_{s\in[0,1]} \|v^\epsilon\|_{L^\infty(\mathbb{T}^d)} + \kappa \right).
\end{aligned}$$

  The previous inequality, together with

$$\left| \int_s^t \langle A\phi, \rho_r \rangle dr \right| \lesssim \|\phi\|_{H^2(\mathbb{T}^d)} \|\rho_0\|_{L^2(\mathbb{T}^d)} \mu^2 |t-s|,$$

  gives

$$\sup_{\substack{0<s<t<1,\\ |t-s|\leq\delta}} \frac{\|f_t - f_s\|_{H^{-\beta}(\mathbb{T}^d)}}{|t-s|^\theta} \lesssim \|\rho_0\|_{L^2(\mathbb{T}^d)} \delta^{1-\theta} \left(\sup_{s\in[0,1]} \|v^\epsilon\|_{L^\infty(\mathbb{T}^d)} + \mu^2 \right).$$

- if $|t-s| > \delta$, then there exist $n, m \in \mathbb{N}$, $n \geq m$, such that $t \in (n\delta, (n+1)\delta]$ and $s \in [(m-1)\delta, m\delta)$. Of course with this choice we have $|t - n\delta|, |n\delta - m\delta|, |m\delta - s| \leq |t-s|$, and therefore

$$\begin{aligned}
\frac{\|f_t - f_s\|_{H^{-\beta}(\mathbb{T}^d)}}{|t-s|^\theta} \leq \frac{\|f_t - f_{n\delta}\|_{H^{-\beta}(\mathbb{T}^d)}}{|t-n\delta|^\theta} &+ \frac{\|f_{n\delta} - f_{m\delta}\|_{H^{-\beta}(\mathbb{T}^d)}}{|n\delta - m\delta|^\theta} \\
&+ \frac{\|f_{m\delta} - f_s\|_{H^{-\beta}(\mathbb{T}^d)}}{|m\delta - s|^\theta},
\end{aligned}$$

whenever $n > m$, and

$$\frac{\|f_t - f_s\|_{H^{-\beta}(\mathbb{T}^d)}}{|t-s|^\theta} \le \frac{\|f_t - f_{n\delta}\|_{H^{-\beta}(\mathbb{T}^d)}}{|t-n\delta|^\theta} + \frac{\|f_{n\delta} - f_s\|_{H^{-\beta}(\mathbb{T}^d)}}{|n\delta - s|^\theta},$$

in the special case $n = m$. Since $|t - n\delta|, |m\delta - s| \le \delta$, we can estimate

$$\sup_{0<s<t<1} \frac{\|f_t - f_s\|_{H^{-\beta}(\mathbb{T}^d)}}{|t-s|^\theta} \lesssim \sup_{\substack{0<s<t<1,\\ |t-s|\le\delta}} \frac{\|f_t - f_s\|_{H^{-\beta}(\mathbb{T}^d))}}{|t-s|^\theta} + \sup_{\substack{n,m=1,\dots,1/\delta-1\\ n>m}} \frac{\|f_{n\delta} - f_{m\delta}\|_{H^{-\beta}(\mathbb{T}^d)}}{|n\delta - m\delta|^\theta}.$$

Putting all together, we finally get

$$\|f\|_{C^\theta([0,1],H^{-\beta}(\mathbb{T}^d))} \lesssim \|\rho_0\|_{L^2(\mathbb{T}^d)} \delta^{1-\theta} \left(\sup_{s\in[0,1]} \|v^\epsilon\|_{L^\infty(\mathbb{T}^d)} + \mu^2 \right) + \sup_{\substack{n,m=1,\dots,1/\delta-1\\ n>m}} \frac{\|f_{n\delta} - f_{m\delta}\|_{H^{-\beta}(\mathbb{T}^d)}}{|n\delta - m\delta|^\theta},$$

and therefore the thesis follows by (5.11), Proposition 5.4 and the choice of δ, whenever θ is sufficiently small. □

5.3.2 Proof of Theorem 1.4

We are ready to prove our main result Theorem 1.4. In the first place, we recall [GLT06, Theorem 1], that in our setting reads as follows:

Lemma 5.6. *Let $\theta \in (0,1)$. Then for all $\vartheta < \theta$ there exists a linear map*

$$\mathfrak{S} : C^\theta([0,1], H^{-\beta}(\mathbb{T}^d)) \to C^\vartheta([0,1], H^{-\beta}(\mathbb{T}^d))$$

that associates to every $X \in C^\theta([0,1], H^{-\beta}(\mathbb{T}^d))$ the unique weak solution u of the evolution equation

$$u_t - \int_0^t A u_s ds = X_t, \qquad u_0 = 0.$$

Moreover,

$$\|\mathfrak{S}(X)\|_{C^\vartheta([0,1],H^{-\beta}(\mathbb{T}^d))} \lesssim \|X\|_{C^\theta([0,1],H^{-\beta}(\mathbb{T}^d))}.$$

Proof of Theorem 1.4. By definition of f, the process ρ_t is almost surely the weak solution of

$$\rho_t - \rho_0 - \int_0^t A\rho_s ds = f_t,$$

whereas $\bar\rho$ solves

$$\bar\rho_t - \rho_0 - \int_0^t A\bar\rho_s ds = 0.$$

Applying Lemma 5.6 with $u = \rho - \bar{\rho}$ and $X = f$ we get the almost sure inequality

$$\|\rho - \bar{\rho}\|_{C^{\vartheta}([0,1],H^{-\beta}(\mathbb{T}^d))} \lesssim \|f\|_{C^{\theta}([0,1],H^{-\beta}(\mathbb{T}^d))}. \tag{5.13}$$

Recall that, by Proposition 5.5, there exists θ sufficiently small such that, for every ϵ sufficiently small

$$\mathbb{E}\left[\|f\|_{C^{\theta}([0,1],H^{-\beta}(\mathbb{T}^d))}\right] \lesssim \|\rho_0\|_{L^2(\mathbb{T}^d)} \left(\alpha + \epsilon^{\varkappa}\mu^{2+\gamma}\right);$$

hence, the thesis of the theorem follows for every $s \geq \beta$, while for $s \in (0, \beta)$, (5.7), (5.13) and interpolation inequality (5.3) yield

$$\begin{aligned}
\mathbb{E}\left[\|\rho - \bar{\rho}\|_{C^{\vartheta}([0,1],H^{-s}(\mathbb{T}^d))}\right] &\leq \mathbb{E}\left[\|\rho_0\|_{L^2(\mathbb{T}^d)}^{1-s/\beta}\|\rho - \bar{\rho}\|_{C^{\vartheta}([0,1],H^{-\beta}(\mathbb{T}^d))}^{s/\beta}\right] \\
&\lesssim \mathbb{E}\left[\|\rho_0\|_{L^2(\mathbb{T}^d)}^{1-s/\beta}\|f\|_{C^{\theta}([0,1],H^{-\beta}(\mathbb{T}^d))}^{s/\beta}\right] \\
&\leq \|\rho_0\|_{L^2(\mathbb{T}^d)}^{1-s/\beta}\mathbb{E}\left[\|f\|_{C^{\theta}([0,1],H^{-\beta}(\mathbb{T}^d))}\right]^{s/\beta} \\
&\lesssim \|\rho_0\|_{L^2(\mathbb{T}^d)} \left(\alpha + \epsilon^{\varkappa}\mu^{2+\gamma}\right)^{s/\beta}.
\end{aligned}$$

The proof is complete. □

5.3.3 Proof of Theorem 1.5

In this paragraph we are concerned with the proof of Theorem 1.5, that quantifies dissipation enhancement of $L^2(\mathbb{T}^d)$ for the solution of (5.1) when $\kappa > 0$. We insist once again that our result states that a suitable velocity field v^{ϵ} can dissipate energy almost *instantaneously*, i.e. for every fixed $t > 0$, without the necessity of taking t large enough to have $\mathbb{E}\left[\|\rho_t\|_{L^2(\mathbb{T}^d)}\right]$ below a certain threshold.

Proof of Theorem 1.5. The following proof is mostly inspired by the proof of [BBPS20, Theorem 1.4]. By energy equality (5.6) and interpolation inequality (5.3), we have

$$\frac{d}{dt}\|\rho_t\|_{L^2(\mathbb{T}^d)}^2 = -2\kappa\|\rho_t\|_{H^1(\mathbb{T}^d)}^2 \leq -2\kappa\frac{\|\rho_t\|_{L^2(\mathbb{T}^d)}^4}{\|\rho_t\|_{H^{-1}(\mathbb{T}^d)}^2}. \tag{5.14}$$

From the previous inequality it is clear that, in order to control $\|\rho_t\|_{L^2(\mathbb{T}^d)}$, it is sufficient to have a good bound from above on the quantity $\|\rho_t\|_{H^{-1}(\mathbb{T}^d)}$. Notice that the trivial bound $\|\rho_t\|_{H^{-1}(\mathbb{T}^d)} \leq \|\rho_t\|_{L^2(\mathbb{T}^d)}$ produces the equally trivial estimate (5.8). To have a better control on $\|\rho_t\|_{H^{-1}(\mathbb{T}^d)}$ we will exploit Theorem 1.4 as follows.
Denote $c = C^{1/2}\left(\alpha + \epsilon^{\varkappa}\mu^{2+\gamma}\right)^{\varsigma/2} > 0$, as in the statement of the theorem. By Theorem 1.4 and Markov inequality we have

$$\mathbb{P}\left\{\sup_{t\in[0,1]}\|\rho_t - \bar{\rho}_t\|_{H^{-1}(\mathbb{T}^d)} > c\|\rho_0\|_{L^2(\mathbb{T}^d)}\right\} \leq c, \tag{5.15}$$

hence with probability at least $1 - c$ it holds

$$\|\rho_t\|_{H^{-1}(\mathbb{T}^d)}^2 \leq 2\|\bar{\rho}_t\|_{H^{-1}(\mathbb{T}^d)}^2 + 2\|\rho_t - \bar{\rho}_t\|_{H^{-1}(\mathbb{T}^d)}^2 \leq 2\|\rho_0\|_{L^2(\mathbb{T}^d)}^2\left(e^{-2\lambda t} + c^2\right).$$

In view of this, (5.14) implies on a set of probability at least $1-c$

$$\|\rho_t\|^2_{L^2(\mathbb{T}^d)} \leq \frac{\|\rho_0\|^2_{L^2(\mathbb{T}^d)}}{1+\kappa\int_0^t \frac{ds}{e^{-2\lambda s}+c^2}} = \frac{\|\rho_0\|^2_{L^2(\mathbb{T}^d)}}{1+\frac{\kappa}{2\lambda c^2}\log\left(\frac{c^2e^{2\lambda t}+1}{c^2+1}\right)},$$

where in the last equality we have used $c>0$. This completes the proof of the first part of the theorem. As for the second part, it immediately follows from the a.s. inequality (5.8), (5.15) and

$$\begin{aligned}\mathbb{E}\left[\|\rho_t\|_{L^2(\mathbb{T}^d)}\right] &= \mathbb{E}\left[\|\rho_t\|_{L^2(\mathbb{T}^d)}\mathbf{1}_{\left\{\sup_{t\in[0,1]}\|\rho_t-\bar{\rho}_t\|_{H^{-1}(\mathbb{T}^d)}>c\|\rho_0\|_{L^2(\mathbb{T}^d)}\right\}}\right] \\ &\quad+\mathbb{E}\left[\|\rho_t\|_{L^2(\mathbb{T}^d)}\mathbf{1}_{\left\{\sup_{t\in[0,1]}\|\rho_t-\bar{\rho}_t\|_{H^{-1}(\mathbb{T}^d)}\leq c\|\rho_0\|_{L^2(\mathbb{T}^d)}\right\}}\right].\end{aligned}$$

□

Remark 5.3. Looking back at (5.14) one realizes that an alternative approach could be that of producing a lower bound for $\|\rho_t\|_{H^1(\mathbb{T}^d)}$ instead of an upper bound for $\|\rho_t\|_{H^{-1}(\mathbb{T}^d)}$. In order to do this, we present an heuristic argument. Write

$$\begin{aligned}\|\rho_t\|^2_{H^1(\mathbb{T}^d)} &= \|\pi_N\rho_t\|^2_{H^1(\mathbb{T}^d)} + \|(I-\pi_N)\rho_t\|^2_{H^1(\mathbb{T}^d)} \\ &\geq \|\pi_N\rho_t\|^2_{L^2(\mathbb{T}^d)} + N^2\|(I-\pi_N)\rho_t\|^2_{L^2(\mathbb{T}^d)} \\ &= (1-N^2)\|\pi_N\rho_t\|^2_{L^2(\mathbb{T}^d)} + N^2\|\rho_t\|^2_{L^2(\mathbb{T}^d)},\end{aligned}$$

where π_N denotes the Fourier projector onto modes $|k|\leq N$, $N\in\mathbb{N}$. Plugging into (5.14), and assuming $N^2\gg\kappa^{-1}$, we have formally

$$\begin{aligned}\|\rho_t\|^2_{L^2(\mathbb{T}^d)} &\leq e^{-2\kappa N^2 t}\|\rho_0\|^2_{L^2(\mathbb{T}^d)} + \int_0^t e^{-2\kappa N^2(t-s)}2\kappa(N^2-1)\|\pi_N\rho_s\|^2_{L^2(\mathbb{T}^d)}ds \\ &\sim e^{-2\kappa N^2 t}\|\rho_0\|^2_{L^2(\mathbb{T}^d)} + \|\pi_N\rho_t\|^2_{L^2(\mathbb{T}^d)}.\end{aligned} \tag{5.16}$$

Using the inequality above with $N^2=\lambda/\kappa$, $\lambda\gg 1$ and recalling Remark 5.2

$$\mathbb{E}\left[\|\pi_N\rho_t\|_{L^2(\mathbb{T}^d)}\right] \lesssim N\|\rho_0\|_{L^2(\mathbb{T}^d)}\left(e^{-\lambda t}+\left(\alpha+\epsilon^{\varkappa}\mu^{2+\gamma}\right)^{\varsigma}\right),$$

gives

$$\mathbb{E}\left[\|\rho_t\|_{L^2(\mathbb{T}^d)}\right] \lesssim \frac{\lambda^{1/2}}{\kappa^{1/2}}\left(e^{-\lambda t}+\left(\alpha+\epsilon^{\varkappa}\mu^{2+\gamma}\right)^{\varsigma}\right)\|\rho_0\|_{L^2(\mathbb{T}^d)}.$$

Comparing with (1.9), the previous estimate has in addition the term $(\alpha+\epsilon^{\varkappa}\mu^{2+\gamma})^{\varsigma}$ on the right hand side, and the implicit constant in the inequality. Moreover, it has been recovered assuming $\lambda\gg 1$ in order to approximate the time integral in (5.16) with $\|\pi_N\rho_t\|^2_{L^2(\mathbb{T}^d)}$. On the other hand, the statement of Theorem 1.5 is valid for every value of λ.

5.4 Proof of Proposition 5.4

In this section we give the proof of Proposition 5.4. Recall that we are interested in the expression (5.12), given by

$$\sup_{\substack{n,m=1,\ldots,1/\delta-1\\ n>m}} (n\delta - m\delta)^{-\theta} \left| \langle \phi, \rho_{n\delta}\rangle - \langle \phi, \rho_{m\delta}\rangle - \delta \sum_{k=m}^{n-1} \langle A\phi, \rho_{k\delta}\rangle \right|,$$

where $\phi \in \mathcal{D}(\mathbb{T}^d)$ is a smooth test function, $\theta > 0$ and $A = \kappa\Delta + \mathcal{L}$.
The choice of the parameter δ in the expression above is crucial. We will see that, in order to have the result of Proposition 5.4, the parameters δ and ϵ must satisfy very particular relations.

5.4.1 A convenient decomposition

In order to control (5.12), we first consider the quantity

$$\langle \phi, \rho_{n\delta}\rangle - \langle \phi, \rho_{m\delta}\rangle - \delta \sum_{k=m}^{n-1} \langle A\phi, \rho_{k\delta}\rangle, \qquad n > m,$$

or equivalently

$$\langle \phi, \rho_{(n+1)\delta}\rangle - \langle \phi, \rho_{m\delta}\rangle - \delta \sum_{k=m}^{n} \langle A\phi, \rho_{k\delta}\rangle, \qquad n \geq m.$$

Let us preliminarily rewrite the previous expression in a more convenient way. For every $k = 0, \ldots, 1/\delta - 1$ it holds

$$\begin{aligned}\langle \phi, \rho_{(k+1)\delta}\rangle - \langle \phi, \rho_{k\delta}\rangle &= \int_{k\delta}^{(k+1)\delta} \langle v^\epsilon(s)\cdot\nabla\phi, \rho_s\rangle ds + \kappa \int_{k\delta}^{(k+1)\delta} \langle \Delta\phi, \rho_s\rangle ds \\ &= I_1(k) + I_2(k).\end{aligned} \tag{5.17}$$

Let us further decompose

$$\begin{aligned} I_1(k) &= \int_{k\delta}^{(k+1)\delta} \langle v^\epsilon(s)\cdot\nabla\phi, \rho_s\rangle ds \\ &= \int_{k\delta}^{(k+1)\delta} \langle v^\epsilon(s)\cdot\nabla\phi, (\rho_s - \rho_{k\delta})\rangle ds + \int_{k\delta}^{(k+1)\delta} \langle v^\epsilon(s)\cdot\nabla\phi, \rho_{k\delta}\rangle ds \\ &= \int_{k\delta}^{(k+1)\delta} \int_{k\delta}^{s} \langle v^\epsilon(r)\cdot\nabla(v^\epsilon(s)\cdot\nabla\phi), (\rho_r - \rho_{k\delta})\rangle dr ds \\ &\quad + \int_{k\delta}^{(k+1)\delta} \int_{k\delta}^{s} \langle v^\epsilon(r)\cdot\nabla(v^\epsilon(s)\cdot\nabla\phi), \rho_{k\delta}\rangle dr ds \\ &\quad + \kappa \int_{k\delta}^{(k+1)\delta} \int_{k\delta}^{s} \langle \Delta(v^\epsilon(s)\cdot\nabla\phi), \rho_r\rangle dr ds \\ &\quad + \int_{k\delta}^{(k+1)\delta} \langle v^\epsilon(s)\cdot\nabla\phi, \rho_{k\delta}\rangle ds \\ &= I_{11}(k) + I_{12}(k) + I_{13}(k) + I_{14}(k), \end{aligned}$$

where we have used $v^\epsilon(s)\cdot\nabla\phi \in \mathcal{D}(\mathbb{T}^d)$ as a test function in order to pass from the second to the third line. Indeed, since $\operatorname{div} v^\epsilon(s) = 0$ for every $s \in [0,1]$ we have

$$\langle 1, v^\epsilon(s)\cdot\nabla\phi\rangle = \langle 1, \operatorname{div}(v^\epsilon(s)\phi)\rangle = -\langle \nabla 1, v^\epsilon(s)\phi\rangle = 0.$$

The term $I_{12}(k)$ can be rewritten as follows:

$$\begin{aligned}
I_{12}(k) &= \int_{k\delta}^{(k+1)\delta}\int_{k\delta}^{s}\langle v^\epsilon(r)\cdot\nabla(v^\epsilon(s)\cdot\nabla\phi), \rho_{k\delta}\rangle dr ds \\
&= \sum_{j,j'\in J}\int_{k\delta}^{(k+1)\delta}\int_{k\delta}^{s}\langle v_{j'}\eta^{\epsilon,j'}(r)\cdot\nabla(v_j\eta^{\epsilon,j}(s)\cdot\nabla\phi), \rho_{k\delta}\rangle dr ds \\
&= \sum_{j,j'\in J}\langle v_{j'}\cdot\nabla(v_j\cdot\nabla\phi), \rho_{k\delta}\rangle \int_{k\delta}^{(k+1)\delta}\int_{k\delta}^{s}\eta^{\epsilon,j}(s)\eta^{\epsilon,j'}(r) dr ds \\
&= \sum_{j,j'\in J}\langle v_{j'}\cdot\nabla(v_j\cdot\nabla\phi), \rho_{k\delta}\rangle \left(\int_{k\delta}^{(k+1)\delta}\int_{k\delta}^{s}\eta^{\epsilon,j}(s)\eta^{\epsilon,j'}(r) dr ds - \delta_{j,j'}\frac{\delta}{2}\right) \\
&\quad + \delta\langle \mathcal{L}\phi, \rho_{k\delta}\rangle \\
&= I_{121}(k) + I_{122}(k).
\end{aligned}$$

As for the term $I_2(k)$ in (5.17), we have

$$\begin{aligned}
I_2(k) &= \kappa\int_{k\delta}^{(k+1)\delta}\langle \Delta\phi, \rho_s\rangle ds \\
&= \kappa\int_{k\delta}^{(k+1)\delta}\langle \Delta\phi, (\rho_s - \rho_{k\delta})\rangle ds + \kappa\int_{k\delta}^{(k+1)\delta}\langle \Delta\phi, \rho_{k\delta}\rangle ds \\
&= I_{21}(k) + I_{22}(k).
\end{aligned}$$

Taking the sum of (5.17) over $k = m, \dots, n$ we get

$$\begin{aligned}
&\langle \phi, \rho_{(n+1)\delta}\rangle - \langle \phi, \rho_m\rangle - \delta\sum_{k=m}^{n}\langle A\phi, \rho_{k\delta}\rangle \\
&\qquad = \sum_{k=m}^{n}\left(I_{11}(k) + I_{121}(k) + I_{13}(k) + I_{14}(k) + I_{21}(k)\right).
\end{aligned} \tag{5.18}$$

5.4.2 Controlling the terms $I_{11}(k)$, $I_{13}(k)$ and $I_{21}(k)$

Lemma 5.7. *Let $d \geq 3$ and $\gamma \in (0,(d-2)/6)$. Let $\delta > 0$ be such that $\delta\epsilon^{-1}\log(1+\epsilon^{-1}) > 1$ and $\mu\delta^{4/3}\epsilon^{-1}\log^{1/3}(1+\epsilon^{-1}) < 1$. Then the following estimates hold:*

$$\mathbb{E}\left[\sup_{\substack{n,m=1,\dots,1/\delta-1\\ n\geq m}}\left|\sum_{k=m}^{n} I_{11}(k)\right|\right] \lesssim \|\phi\|_{H^{2+3\gamma}(\mathbb{T}^d)}\|\rho_0\|_{L^2(\mathbb{T}^d)}\,\mu^{2+\gamma}\delta^{1+\gamma}\epsilon^{-1-\gamma/2}\log^{1+\gamma/2}(1+\epsilon^{-1});$$

$$\mathbb{E}\left[\sup_{\substack{n,m=1,\dots,1/\delta-1\\ n\geq m}}\left|\sum_{k=m}^{n} I_{13}(k)\right|\right] \lesssim \|\phi\|_{H^{2+2\gamma}(\mathbb{T}^d)}\|\rho_0\|_{L^2(\mathbb{T}^d)}\mu\delta^{(1+\gamma)/2}\epsilon^{-1/2}\log^{1/2}(1+\epsilon^{-1});$$

$$\mathbb{E}\left[\sup_{\substack{n,m=1,\dots,1/\delta-1\\ n\geq m}}\left|\sum_{k=m}^{n} I_{21}(k)\right|\right] \lesssim \|\phi\|_{H^{2+\gamma}(\mathbb{T}^d)}\|\rho_0\|_{L^2(\mathbb{T}^d)}\mu^{\gamma}\delta^{\gamma}\epsilon^{-\gamma/2}\log^{\gamma/2}(1+\epsilon^{-1}).$$

Proof. Throughout the proof, we will use without explicit mention the following key inequality:

$$\mathbb{E}\left[\sup_{\substack{n,m=1,\dots,1/\delta-1\\ n\geq m}}\left|\sum_{k=m}^{n} I(k)\right|\right] \leq \sum_{k=1}^{1/\delta-1}\mathbb{E}\left[|I(k)|\right], \quad I = I_{11}, I_{13}, I_{21}.$$

Let us start from $I_{11}(k)$:

$$I_{11}(k) = \int_{k\delta}^{(k+1)\delta}\int_{k\delta}^{s}\langle v^\epsilon(r)\cdot\nabla(v^\epsilon(s)\cdot\nabla\phi),(\rho_r-\rho_{k\delta})\rangle drds.$$

Using Lemma 5.1 and (5.3), for every $\gamma \in (0,(d-2)/6)$

$$\begin{aligned}|I_{11}(k)| &\leq \int_{k\delta}^{(k+1)\delta}\int_{k\delta}^{s}\|v^\epsilon(s)\cdot\nabla\phi\|_{H^{1+2\gamma}(\mathbb{T}^d)}\|v^\epsilon(r)(\rho_r-\rho_{k\delta})\|_{H^{-2\gamma}(\mathbb{T}^d)}drds\\ &\lesssim \int_{k\delta}^{(k+1)\delta}\int_{k\delta}^{s}\|v^\epsilon(s)\|_{H^{d/2-\gamma}(\mathbb{T}^d)}\|\phi\|_{H^{2+3\gamma}(\mathbb{T}^d)}\|v^\epsilon(r)\|_{H^{d/2-\gamma}(\mathbb{T}^d)}\|(\rho_r-\rho_{k\delta})\|_{H^{-\gamma}(\mathbb{T}^d)}drds\\ &\leq \|\phi\|_{H^{2+3\gamma}(\mathbb{T}^d)}\sup_{s\in[0,1]}\|v^\epsilon(s)\|^2_{H^{d/2-\gamma}(\mathbb{T}^d)}\|\rho_0\|^{1-\gamma}_{L^2(\mathbb{T}^d)}\int_{k\delta}^{(k+1)\delta}\int_{k\delta}^{s}\|(\rho_r-\rho_{k\delta})\|^{\gamma}_{H^{-1}(\mathbb{T}^d)}drds.\end{aligned}$$

Hence Lemma 5.2 and (5.10) yield

$$\begin{aligned}\mathbb{E}\left[\sup_{\substack{n,m=1,\dots,1/\delta-1\\ n\geq m}}\left|\sum_{k=m}^{n} I_{11}(k)\right|\right] &\leq \sum_{k=1}^{1/\delta-1}\mathbb{E}\left[|I_{11}(k)|\right]\\ &\lesssim \|\phi\|_{H^{2+3\gamma}(\mathbb{T}^d)}\|\rho_0\|_{L^2(\mathbb{T}^d)}\,\mu^{2+\gamma}\delta^{1+\gamma}\epsilon^{-1-\gamma/2}\log^{1+\gamma/2}(1+\epsilon^{-1}).\end{aligned}$$

As for the term $I_{13}(k)$ we have

$$I_{13}(k) = \kappa\int_{k\delta}^{(k+1)\delta}\int_{k\delta}^{s}\langle\Delta(v^\epsilon(s)\cdot\nabla\phi),\rho_r\rangle drds.$$

By Lemma 5.1, Hölder inequality and (5.7), for every $\gamma \in (0,(d-2)/4)$

$$\begin{aligned}
|I_{13}(k)| &\leq \kappa \int_{k\delta}^{(k+1)\delta} \int_{k\delta}^{s} \|v^\epsilon(s)\cdot\nabla\phi\|_{H^{1+\gamma}(\mathbb{T}^d)} \|\rho_r\|_{H^{1-\gamma}(\mathbb{T}^d)} dr ds \\
&\lesssim \kappa \int_{k\delta}^{(k+1)\delta} \int_{k\delta}^{s} \|v^\epsilon(s)\|_{H^{d/2-\gamma}(\mathbb{T}^d)} \|\phi\|_{H^{2+2\gamma}(\mathbb{T}^d)} \|\rho_r\|_{H^{1-\gamma}(\mathbb{T}^d)} dr ds \\
&\leq \kappa \|\phi\|_{H^{2+2\gamma}(\mathbb{T}^d)} \sup_{s\in[0,1]} \|v^\epsilon(s)\|_{H^{d/2-\gamma}(\mathbb{T}^d)} \|\rho_0\|^{\gamma}_{L^2(\mathbb{T}^d)} \int_{k\delta}^{(k+1)\delta} \int_{k\delta}^{s} \|\rho_r\|^{1-\gamma}_{H^1(\mathbb{T}^d)} dr ds \\
&\leq \kappa^{(1+\gamma)/2} \|\phi\|_{H^{2+2\gamma}(\mathbb{T}^d)} \sup_{s\in[0,1]} \|v^\epsilon(s)\|_{H^{d/2-\gamma}(\mathbb{T}^d)} \|\rho_0\|_{L^2(\mathbb{T}^d)} \delta^{(3+\gamma)/2}.
\end{aligned}$$

Hence using (5.10)

$$\begin{aligned}
\mathbb{E}\left[\sup_{\substack{n,m=1,\dots,1/\delta-1\\ n\geq m}} \left|\sum_{k=m}^{n} I_{13}(k)\right|\right] &\leq \sum_{k=1}^{1/\delta-1} \mathbb{E}\left[|I_{13}(k)|\right] \\
&\lesssim \|\phi\|_{H^{2+2\gamma}(\mathbb{T}^d)} \|\rho_0\|_{L^2(\mathbb{T}^d)} \mu \delta^{(1+\gamma)/2} \epsilon^{-1/2} \log^{1/2}(1+\epsilon^{-1}).
\end{aligned}$$

Let us move finally to the term $I_{21}(k)$:

$$I_{21}(k) = \kappa \int_{k\delta}^{(k+1)\delta} \langle \Delta\phi, (\rho_s - \rho_{k\delta})\rangle ds.$$

By (5.3) we have

$$\begin{aligned}
|I_{21}(k)| &\leq \kappa \int_{k\delta}^{(k+1)\delta} \|\phi\|_{H^{2+\gamma}(\mathbb{T}^d)} \|\rho_s - \rho_{k\delta}\|_{H^{-\gamma}(\mathbb{T}^d)} ds \\
&\leq \kappa \|\rho_0\|^{1-\gamma}_{L^2(\mathbb{T}^d)} \int_{k\delta}^{(k+1)\delta} \|\phi\|_{H^{2+\gamma}(\mathbb{T}^d)} \|\rho_s - \rho_{k\delta}\|^{\gamma}_{H^{-1}(\mathbb{T}^d)} ds.
\end{aligned}$$

Hence using (5.11) and Lemma 5.2 we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{\substack{n,m=1,\dots,1/\delta-1\\ n\geq m}} \left|\sum_{k=m}^{n} I_{21}(k)\right|\right] &\leq \sum_{k=1}^{1/\delta-1} \mathbb{E}\left[|I_{21}(k)|\right] \\
&\lesssim \|\phi\|_{H^{2+\gamma}(\mathbb{T}^d)} \|\rho_0\|_{L^2(\mathbb{T}^d)} \mu^{\gamma} \delta^{\gamma} \epsilon^{-\gamma/2} \log^{\gamma/2}(1+\epsilon^{-1}).
\end{aligned}$$

□

5.4.3 Controlling the term $I_{121}(k)$

The term $I_{121}(k)$ requires more care. We will deduce estimates for this term using a martingale argument due to Nakao, that can be found for instance in [IW81]. Recall

$$I_{121}(k) = \sum_{j,j'\in J} \langle v_{j'}\cdot\nabla(v_j\cdot\nabla\phi), \rho_{k\delta}\rangle \left(\int_{k\delta}^{(k+1)\delta} \int_{k\delta}^{s} \eta^{\epsilon,j}(s)\eta^{\epsilon,j'}(r) dr ds - \delta_{j,j'}\frac{\delta}{2}\right).$$

Define the following quantity

$$c_{j,j'}(k) = \int_{k\delta}^{(k+1)\delta} \int_{k\delta}^{s} \eta^{\epsilon,j}(s)\eta^{\epsilon,j'}(r) dr ds.$$

By the explicit expression of the Ornstein-Uhlenbeck process (5.5), the conditional expectation of $c_{j,j'}(k)$ with respect to $\mathcal{F}_{k\delta}$ gives

$$\begin{aligned}
\mathbb{E}\left[c_{j,j'}(k) \mid \mathcal{F}_{k\delta}\right] &= \int_{k\delta}^{(k+1)\delta} \int_{k\delta}^{s} \mathbb{E}\left[\eta^{\epsilon,j}(s)\eta^{\epsilon,j'}(r) \mid \mathcal{F}_{k\delta}\right] dr ds \\
&= \eta^{\epsilon,j}(k\delta)\eta^{\epsilon,j'}(k\delta) \int_{k\delta}^{(k+1)\delta} \int_{k\delta}^{s} e^{-\epsilon^{-1}(s-k\delta)} e^{-\epsilon^{-1}(r-k\delta)} dr ds \\
&\quad + \delta_{j,j'} \int_{k\delta}^{(k+1)\delta} \int_{k\delta}^{s} \mathbb{E}\left[\epsilon^{-2} \int_{k\delta}^{s} e^{-\epsilon^{-1}(s-s')} dW^j_{s'} \int_{k\delta}^{r} e^{-\epsilon^{-1}(r-r')} dW^j_{r'}\right] dr ds \\
&= \eta^{\epsilon,j}(k\delta)\eta^{\epsilon,j'}(k\delta) \int_{k\delta}^{(k+1)\delta} \int_{k\delta}^{s} e^{-\epsilon^{-1}(s-k\delta)} e^{-\epsilon^{-1}(r-k\delta)} dr ds \\
&\quad + \delta_{j,j'} \epsilon^{-2} \int_{k\delta}^{(k+1)\delta} \int_{k\delta}^{s} \int_{k\delta}^{r} e^{-\epsilon^{-1}(s-r')} e^{-\epsilon^{-1}(r-r')} dr' dr ds \\
&= \eta^{\epsilon,j}(k\delta)\eta^{\epsilon,j'}(k\delta) \frac{\epsilon^2}{2} \left(1 - e^{-\epsilon^{-1}\delta}\right)^2 \\
&\quad + \delta_{j,j'} \left(\frac{\delta}{2} + \epsilon \left(e^{-\epsilon^{-1}\delta} - 1 + \frac{1}{4}\left(1 - e^{-2\epsilon^{-1}\delta}\right)\right)\right).
\end{aligned}$$

We introduce now the following auxiliary processes:

$$\begin{aligned}
M_n &= \sum_{k=1}^{n-1} \sum_{j,j' \in J} \langle v_{j'} \cdot \nabla(v_j \cdot \nabla\phi), \rho_{k\delta}\rangle \left(c_{j,j'}(k) - \mathbb{E}\left[c_{j,j'}(k) \mid \mathcal{F}_{k\delta}\right]\right). \\
R_n &= \sum_{k=1}^{n-1} \sum_{j,j' \in J} \langle v_{j'} \cdot \nabla(v_j \cdot \nabla\phi), \rho_{k\delta}\rangle \left(\mathbb{E}\left[c_{j,j'}(k) \mid \mathcal{F}_{k\delta}\right] - \delta_{j,j'}\frac{\delta}{2}\right).
\end{aligned}$$

Since $\rho_{k\delta}$ is $\mathcal{F}_{k\delta}$-measurable, the process $\{M_n\}_{n=1,\dots,1/\delta}$ is a discrete martingale with respect to the filtration $\mathcal{G}_n := \mathcal{F}_{(n-1)\delta}$ with initial condition $M_1 = 0$. By Doob maximal inequality and the martingale property we have the following

$$\begin{aligned}
\mathbb{E}\left[\sup_{n=1,\dots,1/\delta} M_n^2\right] &\lesssim \mathbb{E}\left[M_{1/\delta}^2\right] = \sum_{k=1}^{1/\delta-1} \mathbb{E}\left[|M_{k+1} - M_k|^2\right] \\
&= \sum_{k=1}^{1/\delta-1} \mathbb{E}\left[\left|\sum_{j,j' \in J} \langle v_{j'} \cdot \nabla(v_j \cdot \nabla\phi), \rho_{k\delta}\rangle \left(c_{j,j'}(k) - \mathbb{E}\left[c_{j,j'}(k) \mid \mathcal{F}_{k\delta}\right]\right)\right|^2\right].
\end{aligned}$$

Using the inequality, valid for $\gamma \in (0, (d-2)/2)$,

$$\begin{aligned}
|\langle v_{j'} \cdot \nabla(v_j \cdot \nabla\phi), \rho_{k\delta}\rangle| &\leq \|v_j \cdot \nabla\phi\|_{H^1(\mathbb{T}^d)} \|v_{j'}\rho_{k\delta}\|_{L^2(\mathbb{T}^d)} \\
&\lesssim \|v_j\|_{H^{d/2-\gamma}(\mathbb{T}^d)} \|\phi\|_{H^{2+\gamma}(\mathbb{T}^d)} \|v_{j'}\|_{L^\infty(\mathbb{T}^d)} \|\rho_0\|_{L^2(\mathbb{T}^d)},
\end{aligned}$$

we arrive at the following estimate

$$
\begin{aligned}
&\left|\sum_{j,j'\in J}\langle v_{j'}\cdot\nabla(v_j\cdot\nabla\phi),\rho_{k\delta}\rangle\left(c_{j,j'}(k)-\mathbb{E}c_{j,j'}(k)\mid\mathcal{F}_{k\delta}\right)\right|^2\\
&\lesssim\sum_{j,j'\in J}\|\phi\|^2_{H^{2+\gamma}(\mathbb{T}^d)}\|\rho_0\|^2_{L^2(\mathbb{T}^d)}\|v_j\|^2_{H^{d/2-\gamma}(\mathbb{T}^d)}\|v_{j'}\|^2_{L^\infty(\mathbb{T}^d)}\\
&\qquad\times\sum_{j,j'\in J}\left(c_{j,j'}(k)-\mathbb{E}\left[c_{j,j'}(k)\mid\mathcal{F}_{k\delta}\right]\right)^2\\
&=\|\phi\|_{H^{2+\gamma}(\mathbb{T}^d)}\|\rho_0\|_{L^2(\mathbb{T}^d)}\sum_{j\in J}\|v_j\|^2_{H^{d/2-\gamma}(\mathbb{T}^d)}\sum_{j'\in J}\|v_{j'}\|^2_{L^\infty(\mathbb{T}^d)}\\
&\qquad\times\sum_{j,j'\in J}\left(c_{j,j'}(k)-\mathbb{E}\left[c_{j,j'}(k)\mid\mathcal{F}_{k\delta}\right]\right)^2.
\end{aligned}
$$

Since the conditional expectation is a $L^2(\Omega)$-projection,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{n=1,\dots,1/\delta}|M_n|\right]&\le\mathbb{E}\left[\sup_{n=1,\dots,1/\delta}M_n^2\right]^{1/2}\\
&\lesssim\|\phi\|_{H^{2+\gamma}(\mathbb{T}^d)}\|\rho_0\|_{L^2(\mathbb{T}^d)}\left(\sum_{j\in J}\|v_j\|^2_{H^{d/2-\gamma}(\mathbb{T}^d)}\sum_{j'\in J}\|v_{j'}\|^2_{L^\infty(\mathbb{T}^d)}\right)^{1/2}\\
&\qquad\times\left(\sum_{k=1}^{1/\delta-1}\sum_{j,j'\in J}\mathbb{E}\left[\left(c_{j,j'}(k)-\mathbb{E}\left[c_{j,j'}(k)\mid\mathcal{F}_{k\delta}\right]\right)^2\right]\right)^{1/2}\\
&\le\|\phi\|_{H^{2+\gamma}(\mathbb{T}^d)}\|\rho_0\|_{L^2(\mathbb{T}^d)}\left(\sum_{j\in J}\|v_j\|^2_{H^{d/2-\gamma}(\mathbb{T}^d)}\sum_{j'\in J}\|v_{j'}\|^2_{L^\infty(\mathbb{T}^d)}\right)^{1/2}\\
&\qquad\times\left(\sum_{k=1}^{1/\delta-1}\sum_{j,j'\in J}\mathbb{E}\left[c_{j,j'}(k)^2\right]\right)^{1/2}\\
&\lesssim\|\phi\|_{H^{2+\gamma}(\mathbb{T}^d)}\|\rho_0\|_{L^2(\mathbb{T}^d)}\mu^2\delta\epsilon^{-1/2}\log^{1/2}(1+\epsilon^{-1}),
\end{aligned}
$$

where the last inequality follows from

$$
\begin{aligned}
\mathbb{E}\left[c_{j,j'}(k)^2\right]&=\mathbb{E}\left[\left(\int_{k\delta}^{(k+1)\delta}\int_{k\delta}^{s}\eta^{\epsilon,j}(s)\eta^{\epsilon,j'}(r)drds\right)^2\right]\\
&=\mathbb{E}\left[\left(\int_{k\delta}^{(k+1)\delta}\eta^{\epsilon,j}(s)\left(W^{j'}_s-W^{j'}_{k\delta}-\epsilon\left(\eta^{\epsilon,j'}(s)-\eta^{\epsilon,j'}(k\delta)\right)\right)ds\right)^2\right]\\
&\lesssim\delta\int_{k\delta}^{(k+1)\delta}\mathbb{E}\left[\sup_{s\in[0,1]}|\eta^{\epsilon,j'}(s)|^2\left(|W^{j'}_s-W^{j'}_{k\delta}|^2-\epsilon^2\sup_{s\in[0,1]}|\eta^{\epsilon,j'}(s)|^2\right)\right]ds\\
&\lesssim\delta^3\epsilon^{-1}\log(1+\epsilon^{-1})+\delta^2\log(1+\epsilon^{-1}).
\end{aligned}
$$

As for the remaining term,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{n=1,\dots,1/\delta}|R_n|\right] \\
&\lesssim \|\phi\|_{H^{2+\gamma}(\mathbb{T}^d)}\|\rho_0\|_{L^2(\mathbb{T}^d)}\sum_{j\in J}\|v_j\|_{H^{d/2-\gamma}(\mathbb{T}^d)}\sum_{j'\in J}\|v_{j'}\|_{L^\infty(\mathbb{T}^d)} \\
&\quad\times\sum_{k=1}^{1/\delta-1}\sum_{j,j'\in J}\mathbb{E}\left[\left|\mathbb{E}\left[c_{j,j'}(k)\mid\mathcal{F}_{k\delta}\right]-\delta_{j,j'}\frac{\delta}{2}\right|\right] \\
&\lesssim \|\phi\|_{H^{2+\gamma}(\mathbb{T}^d)}\|\rho_0\|_{L^2(\mathbb{T}^d)}\mu^2\delta^{-1}\epsilon\log(1+\epsilon^{-1}),
\end{aligned}$$

where we have used

$$\mathbb{E}\left[\left|\mathbb{E}\left[c_{j,j'}(k)\mid\mathcal{F}_{k\delta}\right]-\delta_{j,j'}\frac{\delta}{2}\right|\right]\lesssim\epsilon\log(1+\epsilon^{-1}).$$

Putting all together, and recalling $\sum_{k=m}^{n}I_{121}(k)=M_{n+1}+R_{n+1}-M_m-R_m$, we deduce the following:

Lemma 5.8. *Let d, γ, δ and ϵ^{-1} as in Lemma 5.7. Then*

$$\mathbb{E}\left[\sup_{\substack{n,m=1,\dots,1/\delta-1\\ n\geq m}}\left|\sum_{k=m}^{n}I_{121}(k)\right|\right]\lesssim\|\phi\|_{H^{2+\gamma}(\mathbb{T}^d)}\|\rho_0\|_{L^2(\mathbb{T}^d)}\mu^2\delta^{-1}\epsilon\log(1+\epsilon^{-1}).$$

5.4.4 Controlling the remaining terms

Consider now the term $I_{14}(k)$ in (5.18):

$$I_{14}(k)=\int_{k\delta}^{(k+1)\delta}\langle v^\epsilon(s)\cdot\nabla\phi,\rho_{k\delta}\rangle ds=\left\langle\left(\int_{k\delta}^{(k+1)\delta}v^\epsilon(s)ds\right)\cdot\nabla\phi,\rho_{k\delta}\right\rangle.$$

Recall $v^\epsilon(s)ds = \sum_{j\in J} v_j \eta^{\epsilon,j}(s)ds = \sum_{j\in J} v_j dW_s^j - \epsilon \sum_{j\in J} v_j d\eta^{\epsilon,j}(s)$, and thus

$$\begin{aligned}
I_{14}(k) &= \sum_{j\in J} \langle v_j \cdot \nabla\phi, \rho_{k\delta}\rangle \left(W^j_{(k+1)\delta} - W^j_{k\delta}\right) \\
&\quad - \sum_{j\in J} \langle v_j \cdot \nabla\phi, \rho_{k\delta}\rangle \epsilon \left(\eta^{\epsilon,j}((k+1)\delta) - \eta^{\epsilon,j}(k\delta)\right) \\
&= \sum_{j\in J} \langle v_j \cdot \nabla\phi, \rho_{k\delta}\rangle \left(W^j_{(k+1)\delta} - W^j_{k\delta}\right) - \sum_{j\in J} \int_{k\delta}^{(k+1)\delta} \langle v_j \cdot \nabla\phi, \rho_s\rangle dW_s^j \\
&\quad + \sum_{j\in J} \int_{k\delta}^{(k+1)\delta} \langle v_j \cdot \nabla\phi, \rho_s\rangle dW_s^j \\
&\quad - \sum_{j\in J} \langle v_j \cdot \nabla\phi, \rho_{k\delta}\rangle \epsilon \left(\eta^{\epsilon,j}((k+1)\delta) - \eta^{\epsilon,j}(k\delta)\right) \\
&= \sum_{j\in J} \int_{k\delta}^{(k+1)\delta} \langle v_j \cdot \nabla\phi, \rho_{k\delta} - \rho_s\rangle dW_s^j \\
&\quad + \sum_{j\in J} \int_{k\delta}^{(k+1)\delta} \langle v_j \cdot \nabla\phi, \rho_s\rangle dW_s^j \\
&\quad - \sum_{j\in J} \langle v_j \cdot \nabla\phi, \rho_{k\delta}\rangle \epsilon \left(\eta^{\epsilon,j}((k+1)\delta) - \eta^{\epsilon,j}(k\delta)\right) \\
&= I_{141}(k) + I_{142}(k) + I_{143}(k).
\end{aligned}$$

We have the forthcoming:

Lemma 5.9. *Let d, γ, δ and ϵ be as in Lemma 5.7 and denote $\theta = \frac{1+\gamma}{1+2\gamma}$. Then*

$$\mathbb{E}\left[\sup_{\substack{n,m=1,\dots,1/\delta-1\\ n\geq m}} \left|\sum_{k=m}^n I_{141}(k)\right|\right] \lesssim \|\phi\|_{H^{2+\gamma}(\mathbb{T}^d)} \|\rho_0\|_{L^2(\mathbb{T}^d)} \mu^2 \delta \epsilon^{-1/2} \log^{1/2}(1+\epsilon^{-1});$$

$$\begin{aligned}
\mathbb{E}\left[\sup_{\substack{n,m=1,\dots,1/\delta-1\\ n\geq m}} \left|\sum_{k=m}^n I_{143}(k)\right|\right] &\lesssim \|\phi\|_{H^{2+2\gamma}(\mathbb{T}^d)} \|\rho_0\|_{L^2(\mathbb{T}^d)} \mu^2 \\
&\quad \times \left(\delta^{(\theta-1)/2} \epsilon^{(1-\theta)/2} \log^{(1+\theta)/2}(1+\epsilon^{-1}) + \delta^{\theta-1}\epsilon^{1-\theta} \log(1+\epsilon^{-1})\right).
\end{aligned}$$

Proof. Concerning the term $I_{141}(k)$, we have

$$\sum_{k=m}^n I_{141}(k) = \sum_{j\in J} \int_{m\delta}^{(n+1)\delta} \langle v_j \cdot \nabla\phi, \rho_{[s]} - \rho_s\rangle dW_s^j,$$

where we denote by $[s]$ the largest multiple of δ smaller than s. Therefore by Burkholder-

Davis-Gundy inequality and Lemma 5.2,

$$\mathbb{E}\left[\sup_{\substack{n,m=1,\dots,1/\delta-1\\ n\geq m}}\left|\sum_{k=m}^{n} I_{141}(k)\right|\right] \lesssim \mathbb{E}\left[\sup_{t\in[0,1]}\left|\sum_{j\in J}\int_0^t \langle v_j\cdot\nabla\phi, \rho_{[s]}-\rho_s\rangle dW^j_s\right|\right]$$

$$\lesssim \mathbb{E}\left[\left|\sum_{j\in J}\int_0^1 \langle v_j\cdot\nabla\phi, \rho_{[s]}-\rho_s\rangle^2 ds\right|^{1/2}\right]$$

$$\lesssim \mathbb{E}\left[\left|\int_0^1 \sum_{j\in J}\|v_j\|^2_{H^{d/2-\gamma}(\mathbb{T}^d)}\|\phi\|^2_{H^{2+\gamma}(\mathbb{T}^d)}\|\rho_{[s]}-\rho_s\|^2_{H^{-1}(\mathbb{T}^d)}ds\right|^{1/2}\right]$$

$$\lesssim \|\phi\|_{H^{2+\gamma}(\mathbb{T}^d)}\|\rho_0\|_{L^2(\mathbb{T}^d)}\mu^2\delta\epsilon^{-1/2}\log^{1/2}(1+\epsilon^{-1}).$$

Let us move now to $I_{143}(k)$. Since the increments of $\eta^{\epsilon,j}$ are difficult to control, we perform a discrete integration by parts to get

$$\sum_{k=m}^{n} I_{143}(k) = -\sum_{k=m}^{n}\sum_{j\in J}\langle v_j\cdot\nabla\phi, \rho_{k\delta}\rangle\epsilon\left(\eta^{\epsilon,j}((k+1)\delta)-\eta^{\epsilon,j}(k\delta)\right)$$

$$= -\epsilon\sum_{k=m}^{n}\langle (v^\epsilon((k+1)\delta)-v^\epsilon(k\delta))\cdot\nabla\phi, \rho_{k\delta}\rangle$$

$$= \epsilon\sum_{k=m+1}^{n}\langle v^\epsilon(k\delta)\cdot\nabla\phi, (\rho_{k\delta}-\rho_{(k-1)\delta})\rangle$$

$$-\epsilon\sum_{j\in J}\langle v^\epsilon(m\delta)\cdot\nabla\phi, \rho_{m\delta}\rangle$$

$$-\epsilon\langle v^\epsilon((n+1)\delta)\cdot\nabla\phi, \rho_{n\delta}\rangle.$$

By the usual estimates, taking $\gamma\in(0,(d-2)/4)$ we have

$$\left|\sum_{k=m}^{n} I_{143}(k)\right| \lesssim \epsilon\|\phi\|_{H^{2+2\gamma}(\mathbb{T}^d)}\sup_{s\in[0,1]}\|v^\epsilon(s)\|_{H^{d/2-\gamma}(\mathbb{T}^d)}$$

$$\times\left(\|\rho_0\|_{L^2(\mathbb{T}^d)}+\sum_{k=m+1}^{n}\|\rho_{k\delta}-\rho_{(k-1)\delta}\|_{H^{-1-\gamma}(\mathbb{T}^d)}\right)$$

$$=: \sum_{k=m}^{n} I'_{143}(k).$$

Interpolation inequality (5.3) with $\theta=\frac{1+\gamma}{1+2\gamma}$ gives:

$$\|\rho_{k\delta}-\rho_{(k-1)\delta}\|_{H^{-1-\gamma}(\mathbb{T}^d)} \leq \|\rho_{k\delta}-\rho_{(k-1)\delta}\|^{\theta}_{H^{-1}(\mathbb{T}^d)}\|\rho_{k\delta}-\rho_{(k-1)\delta}\|^{1-\theta}_{H^{-2-2\gamma}(\mathbb{T}^d)},$$

and therefore by Lemma 5.2 and Lemma 5.3:

$$\mathbb{E}\left[\sup_{\substack{n,m=1,\dots,1/\delta-1\\ n\geq m}}\left|\sum_{k=m}^{n} I_{143}(k)\right|\right] \leq \mathbb{E}\left[\sum_{k=1}^{1/\delta-1} I'_{143}(k)\right]$$

$$\lesssim \|\phi\|_{H^{2+2\gamma}(\mathbb{T}^d)}\|\rho_0\|_{L^2(\mathbb{T}^d)}\mu^2$$

$$\times\left(\delta^{(\theta-1)/2}\epsilon^{(1-\theta)/2}\log^{(1+\theta)/2}(1+\epsilon^{-1})+\delta^{\theta-1}\epsilon^{1-\theta}\log(1+\epsilon^{-1})\right).$$

□

5.4.5 Proof of Proposition 5.4

In this paragraph we are going to prove Proposition 5.4. We want to show

$$\mathbb{E}\left[\sup_{\substack{n,m=1,\dots,1/\delta-1\\ n>m}} (|n-m|\delta)^{-\theta}\left|\langle\phi,\rho_{k\delta}\rangle-\langle\phi,\rho_{m\delta}\rangle-\delta\sum_{k=m}^{n-1}\langle A\phi,\rho_{k\delta}\rangle\right|\right]$$
$$\lesssim \|\phi\|_{H^\beta(\mathbb{T}^d)}\|\rho_0\|_{L^2(\mathbb{T}^d)}\left(\alpha+\epsilon^{\varkappa}\mu^{2+\gamma}\right),$$

for some $\theta,\varkappa$ sufficiently small and $\epsilon,\beta,\gamma,\delta$ as in the statement of the proposition. Let us preliminarily discuss the condition on δ. First, in order to apply Lemma 5.3, the parameter δ must be chosen depending on ϵ (and μ) so that

$$\delta\epsilon^{-1}\log(1+\epsilon^{-1})>1,\quad \mu\delta^{4/3}\epsilon^{-1}\log^{1/3}(1+\epsilon^{-1})<1. \tag{5.19}$$

Moreover, recall the following decomposition (5.18)

$$\langle\phi,\rho_{n\delta}\rangle-\langle\phi,\rho_{m\delta}\rangle-\delta\sum_{k=m}^{n-1}\langle A\phi,\rho_{k\delta}\rangle$$
$$=\sum_{k=m}^{n-1}\left(I_{11}(k)+I_{121}(k)+I_{13}(k)+I_{141}(k)+I_{142}(k)+I_{143}(k)+I_{21}(k)\right).$$

We are assuming $1/\delta$ to be an integer, so that the previous decomposition is well-calibrated – the interval $[0,1]$ is split exactly into $1/\delta$ subintervals of length δ.
To simplify the notation, write $I(k)$ as an abbreviation for $|I_{11}(k)|+|I_{12}(k)|+|I_{13}(k)|+|I_{141}(k)|+|I_{143}(k)|+|I_{21}(k)|$. The only term remaining is that involving $|I_{142}(k)|$, that will be treated separately. We shall prove next that for every θ sufficiently small there exists $\varkappa>0$ such that

$$\mathbb{E}\left[\sup_{\substack{n,m=1,\dots,1/\delta-1\\ n>m}} (|n-m|\delta)^{-\theta}\sum_{k=m}^{n-1}I(k)\right]\lesssim \|\phi\|_{H^{2+\gamma}(\mathbb{T}^d)}\|\rho_0\|_{L^2(\mathbb{T}^d)}\epsilon^{\varkappa}\mu^{2+\gamma}. \tag{5.20}$$

Invoking Lemma 5.7, and in particular the estimate for the term involving I_{11}, one immediately realizes that for the previous estimate to be true it is necessary that

$$\delta^{1+\gamma-\theta}\epsilon^{-(1+\gamma/2+\varkappa)}\log^{1+\gamma/2}(1+\epsilon^{-1})<1. \tag{5.21}$$

Then, once (5.19) and (5.21) are both satisfied, from Lemma 5.7, Lemma 5.8 and Lemma 5.9 we deduce (5.20) (possibly taking smaller θ and $\varkappa$ if needed). Let θ, $\varkappa$ such that

$$1<\frac{1+\gamma-\theta}{1+\gamma/2+\varkappa},$$

which is always possible taking θ and $\varkappa$ sufficiently small. Then δ is chosen by

$$\delta=c_1\epsilon^{c_2},\quad \max\left\{\frac{4}{5},\frac{1+\gamma/2+\varkappa}{1+\gamma-\theta}\right\}<c_2<1,$$

and c_1 is an auxiliary constant such that $1/\delta$ is an integer. With such a choice of δ, conditions (5.19) and (5.21) hold true for small ϵ, since logarithmic factors become negligible when compared with powers of ϵ.
Let us move next to the term $\sum_{k=m}^{n-1} I_{142}(k)$, given by

$$\sum_{k=m}^{n-1} I_{142}(k) = \sum_{j\in J} \int_{m\delta}^{n\delta} \langle v_j \cdot \nabla\phi, \rho_r\rangle dW_r^j.$$

Since by Sobolev embedding Theorem $\|\nabla\phi\|_{L^\infty(\mathbb{T}^d)} \lesssim \|\phi\|_{H^\beta(\mathbb{T}^d)}$ for every $\beta > d/2+1$, Burkholder-Davis-Gundy inequality gives

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{j\in J}\int_s^t \langle v_j \cdot \nabla\phi, \rho_r\rangle dW_r^j\right|^3\right] &\lesssim \mathbb{E}\left[\left|\sum_{j\in J}\int_s^t |\langle v_j \cdot \nabla\phi, \rho_r\rangle|^2 dr\right|^{3/2}\right] \\ &\lesssim |t-s|^{3/2}\alpha^3\|\phi\|^3_{H^\beta(\mathbb{T}^d)}\|\rho_0\|^3_{L^2(D)}.\end{aligned}$$

Therefore, by Kolmogorov continuity criterion the stochastic integral in the expression above is θ-Hölder continuous for every $\theta < 1/6$, and its Hölder constant K_θ satisfies

$$\mathbb{E}\left[\sup_{0<s<t<1} \frac{\left|\sum_{j\in J}\int_s^t \langle v_j \cdot \nabla\phi, \rho_r\rangle dW_r^j\right|}{|t-s|^\theta}\right] = \mathbb{E}\left[K_\theta\right] \lesssim \alpha\|\phi\|_{H^\beta(\mathbb{T}^d)}\|\rho_0\|_{L^2(\mathbb{T}^d)}.$$

The proof is complete.

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