Scuola Normale Superiore

Classe di Scienze Matematiche, Fisiche e Naturali

Geometric aspects of PDEs on sub-Riemannan manifolds

Tesi di Perfezionamento in Matematica

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Introduction

A CR structure on a $2n+1$-dimensional manifold M is a n-dimensional subbundle $T^{(1,0)}M$ of $TM \otimes \mathbf{C}$ such that $T^{(1,0)}M \cap \overline{T^{(1,0)}M} = 0$ and $[T^{(1,0)}M, T^{(1,0)}M] \subseteq T^{(1,0)}M$. This definition is inspired by complex analysis: in fact every real hypersurface of $\mathbf{C}^{n+1}$ has a natural CR structure, and this definition permits to study in an abstract way the geometry of these hypersurfaces.

A CR structure is called nondegenerate if $H(M) = \mathfrak{Re}(T^{(1,0)}M \oplus T^{(0,1)}M)$ is a contact distribution, and in such case a contact form on it is called a pseudo-Hermitian structure.

The choice of a contact form determines a rich geometric structure. For example it determines a pseudo-Riemannian metric g_θ, and the CR manifold is said pseudoconvex if it is Riemannian. Furthermore θ determines a connection called Tanaka-Webster connection, and a curvature tensor.

Since a pseudo-Hermitian structure on a nondegenerate CR structure is defined up to the multiplication by a smooth function, their study leads naturally to study problems of conformal geometry.

In particular, since g_θ can be used to contract the curvature tensor, the choice of θ determines a scalar curvature invariant R called Webster curvature. Thus it is natural to study the Webster curvature prescription problem, and as a particular case the CR Yamabe problem. If $\widetilde{\theta} = u^{\frac{4}{Q-2}}\theta$ then the Webster curvature $\widetilde{R}$ relative to $\widetilde{\theta}$ is

$$\widetilde{R} = u^{\frac{Q+2}{Q-2}}(-b_n \Delta_b + R)u, \tag{1}$$

where Δ_b is a second order differential operator called sublaplacian, and $Q = 2n+2$ is the homogeneous dimension. The similarity between this formula and the formula for the conformal change of scalar curvature shows the strong relationship between CR geometry and conformal Riemannian geometry.

The most important CR manifold is the Heisenberg group $\mathbf{H}^n$. It plays a role in CR geometry analogous to the one of $\mathbf{R}^n$ in Riemannian geometry. In particular it is flat, and every pseudoconvex CR manifold has local coordinates with domain in $\mathbf{H}^n$ which preserve the structure at first order, analogously to the normal coordinates of Riemannian geometry.

For these reasons it is interesting to study contact forms on $\mathbf{H}^n$ for which the curvature satisfies special properties. A natural choice is to look for constant positive Webster curvature. By formula (1) the problem is equivalent to find positive solutions to the equation

$$-\Delta_b u = u^{\frac{Q+2}{Q-2}}. \tag{2}$$

In $\mathbf{R}^n$ positive solutions of the analogous equation

$$-\Delta u = u^{\frac{n+2}{n-2}}, \tag{3}$$

which correspond to metrics $u^{\frac{4}{n-2}} g_{\mathbf{R}^n}$ with positive constant scalar curvature (up to an inessential constant) are completely classified (see [CGS]), and are

$$u_{\lambda,x_0}(x) = \left(\frac{\sqrt{n(n-2)}\lambda}{\lambda^2 + |x-x_0|^2} \right)^{\frac{n-2}{2}}.$$

Geometrically the metrics corresponding to this family of solutions are the round metric of the sphere pushed forward to $\mathbf{R}^n$ through the stereographic projection.

The problem of classifying solution of (3) on $\mathbf{R}^n \setminus \{0\}$ has also been studied and completely solved. In fact has been proven (see [CGS]) that all solutions (besides the restrictions of regular solutions) are radial, and this allows to study a one dimensional problem, which turns out to be a Hamiltonian dynamic system. It turns out that there are two kinds of solutions: a homogeneous solution, $\frac{c_n}{|x|^{(n-2)/2}}$, corresponding to the standard metric on the cylinder, and a family of solutions called Fowler solutions verifying the homogeneity property $u(\lambda x) = \lambda^{-(n-2)/2} u$ for some λ, which geometrically correspond to periodic metrics on the cylinder, known as Delaunay metrics (see [MP] and the references cited therein). These solutions have been used to study general singular solutions, in blow-up theory, and to build singular metrics.

These classification results were proved using the method of moving planes, which cannot be extended to the Heisenberg group because it does not have enough symmetries.

Positive solutions of Equation (1) on $\mathbf{H}^n$ were classified by Jerison and Lee (see [JL2]) under the integrability hypothesis $u \in L^{\frac{Q+2}{Q-2}}$. Such solutions are

$$\omega(z,t) = \frac{c_0}{\left(t^2 + (1+|z|^2)^2\right)^{(Q-2)/4}}$$

with dilations and translates thereof, where (z,t) are the standard coordinates of $\mathbf{H}^n$ defined in Chapter 2. Geometrically these solutions correspond to the standard contact form on the sphere S^{2n+1} transported on $\mathbf{H}^n$ through the Cayley transform, and thus there is a strong analogy with the Riemannian case.

In this thesis we investigate singular solutions of Equation (1) on $\mathbf{H}^n \setminus \{0\}$, and in particular we prove the existence of solutions analogous to the homogeneous one and to the Fowler solutions.

The appropriate generalization of the notion of homogeneity is obtained by the natural dilations of $\mathbf{H}^n$ defined as $\delta_\lambda(z,t) = (\lambda z, \lambda^2 t)$ (see Chapter 2).

In Chapter 3 we prove the following theorem about the existence of solutions analogous to the Fowler's ones on $\mathbf{R}^n$ (see [A1]).

Theorem 1. *There exists T_0 such that for $T \geq T_0$ there exists a positive solution of the equation*

$$-\Delta_b u = u^{\frac{Q+2}{Q-2}}$$

on $\mathbf{H}^n \setminus \{0\}$ such that $u \circ \delta_T = T^{-\frac{Q-2}{2}} u$, and T is the smallest period.

In order to prove the above Theorem, we will formulate the problem as the Euler-Lagrange equation for the functional, defined on a suitable functional space X_T,

$$\mathscr{I}_T(u) = \int_{\Omega_T} \left(|\nabla_{\mathbf{H}^n} u|^2 - \frac{1}{2^*} |u|^{2^*} \right),$$

where $\Omega_T = \{1 \leq |x| \leq T\}$ and $|X|$ is the subriemannian norm. Then we will define a family $\mathscr{Z}_T$ of approximate solutions $\Psi_{\lambda,T}$, depending on a parameter $\lambda \in \mathbf{R}$, by adding infinitely many dilates of ω. We will prove various estimates that show that $\Psi_{\lambda,T}$ is in fact an approximate critical point of $\mathscr{I}_T$, and that $d^2 \mathscr{I}_T$ is nondegenerate on $\mathscr{Z}_T$ on the orthogonal of the tangent. This will allow us to use the Lyapunov-Schmidt method to find an exact solution as a perturbation of the approximate one.

In Chapter 4 we will prove the existence of a homogeneous solution (see [A2]).

Theorem 2. *There exists a non zero solution Ψ of the equation*

$$-\Delta \Psi = \Psi^{\frac{Q+2}{Q-2}},$$

defined on $\mathbf{H}^n \setminus \{0\}$, such that $\Psi \circ \delta_\lambda = \lambda^{\frac{Q-2}{2}} \Psi$ for all $\lambda > 0$ and $\Psi(z,t) = \Psi(|z|, t)$.

The strategy that we adopt to prove the above theorem is to perform a conformal change with the form $\widetilde{\theta} = \frac{1}{|x|^2} \theta$, which is invariant by Heisenberg dilations, and then to impose symmetries to get a one-dimensional variational problem. In order to perform the conformal change on the annulus $\{1 \leq |x| \leq T\}$ in the Riemannian case one can use the fact the problem of prescribing scalar curvature on the interior of a manifold and mean curvature on the boundary (known in the literature as Escobar problem) has a variational formulation through a conformally covariant functional. In CR geometry such formulation did not exist, so we provided it by defining an appropriate notion of curvature of a hypersurface of a CR manifold which is conformally covariant (see [CHMY, CCWYY] for related problems in dimension three).

In $\mathbf{R}^n$ the Fowler solutions bifurcate from the homogeneous one, as can be proved by ODE analysis. In Chapter 4 we prove the following bifurcation theorem for the homogeneous solution Ψ found by us.

Theorem 3. *There exists arbitrarily large values of T for which $d^2 \mathscr{I}_T(\Psi)$ is singular, and every such value is a bifurcation value of non-homogeneous solutions.*

In Chapter 5 we move to another important matter in CR geometry, that is the Einstein-Hilbert functional.

In Riemannian geometry the Einstein-Hilbert functional is defined as

$$\mathscr{R}(g) = \int_M R_g dV_g,$$

and the renormalized, scaling-invariant, version thereof is

$$\widetilde{\mathscr{R}}(g) = \mathrm{Vol}(M)^{\frac{2-n}{n}} \mathscr{R}(g).$$

The Einstein-Hilbert functional plays a fundamental role in the study of Riemannian manifolds. Critical metrics for $\mathscr{R}$ are Ricci-flat metrics, and critical metrics for $\widetilde{\mathscr{R}}$ are Einstein metrics. There is also a significant relation to the Yamabe problem, ince if the infimum of $\widetilde{\mathscr{R}}$ in a given conformal is smaller than that on the standard sphere, this infimum is attained by a Yamabe metric.

In particular S^n with the standard metric is a critical point for $\widetilde{\mathscr{R}}(g)$. It can be proved that it is a saddle point: $d^2\widetilde{\mathscr{R}}(g)$ is zero on the tangent of the space of variations that arise by pulling back the metric by diffeomorphisms, positive definite on the tangent to the space of conformal variations, and negative definite of the orthogonal of the sum of these two subspaces.

Given a compact, non degenerate, $2n+1$-dimensional pseudo-Hermitian manifold, we define the Einstein-Hilbert functional as

$$\mathscr{W} = \int_M W\, \theta \wedge (d\theta)^n$$

and the renormalized version as

$$\widetilde{\mathscr{W}} = \mathrm{Vol}_\theta(M)^{-\frac{Q-2}{Q}} \mathscr{W}.$$

In Chapter 5 first we characterize the critical CR structures for $\widetilde{\mathscr{W}}$.

Theorem 4. *A pseudo-Hermitian structure is stationary for $\widetilde{\mathscr{W}}$ if and only if it has constant Webster curvature and zero torsion.*

If $c_1(T^{(1,0)}M) = 0$, this is equivalent to the pseudo-Hermitian structure being pseudo-Einstein.

Then we study the second variation of $\widetilde{\mathscr{W}}$ on S^{2n+1} with its standard CR structure and contact form, which, thanks to the former theorem, is a critical point. Thanks to a theorem of Gray, we can suppose that a variation has the same contact distribution, and so we have to consider variations of the couple (J, θ). As for the variations of θ, they are conformal variations, and so $d^2\widetilde{\mathscr{W}}$ is positive semidefinite on these direction by the solution of the CR Yamabe problem.

As for the variations of J, let us denote $\dot{J} = 2E$. S^1 acts naturally on $S^{2n+1} \subset \mathbf{C}^{n+1}$ by

$$\rho(e^{i\theta})(z_1, \ldots, z_{n+1}) = (e^{i\theta} z_1, \ldots, e^{i\theta} z_{n+1}),$$

and accordingly on all tensor spaces on S^{2n+1}. Let us denote by $E^{(m)}$ the m-th Fourier component corresponding of E with respect to this action. Then we will prove the following formula.

Theorem 5. *With respect to the Fourier decomposition defined above, the second variation of $\widetilde{\mathscr{W}}$ with respect to J is given by*

$$d_J^2\widetilde{\mathscr{W}}(\theta_0, J_0)[E,E] = n\sum_{m\in\mathbf{Z}}(m+4)\int_{S^3}\left|E^{(m)}\right|^2\theta_0\wedge(d\theta_0)^n.$$

This formula can be interpreted thanks to results of Bland and Duchamp characterizing the Fourier components $E^{(m)}$. When $n \geq 2$, it turns out that $d^2\widetilde{\mathscr{W}}$ is negative definite, as in the Riemannian case. When $n = 1$ the second variation $d^2\widetilde{\mathscr{W}}$ is negative definite on the subspace of embeddable perturbations, while it is negative definite on the orthogonal thereof.

We recall that every pseudoconvex CR manifold of dimension greater or equal than 5 is embeddable by a result of Boutet de Monvel, while in dimension 3 CR manifolds in general are not CR embeddable. CR embeddability is strictly connected to the analytic and geometric properties of CR manifolds, and in various questions embeddable CR manifold behave like the Riemannian case, in opposition to non embeddable CR manifolds. For example this is the case for the positivity of pseudo-Hermitian mass and for the behavior of the CR Yamabe functional (see [CMY1, CMY2]). So our result on the variation of the CR Einstein-Hilbert functional is a further result that corroborates this pattern.

The plan of the thesis is as follows.

In Chapter 1 we will give an introduction to CR geometry, trying to give as best as possible the motivations from complex analysis, and introducing the funfamental concepts that will be used in the following chapters such as the Tanaka-Webster connection and the associated curvature.

In Chapter 2 we will introduce the Heisenberg group $\mathbf{H}^n$, which is most important example of CR manifold and the main subject of this thesis. We will motivate its definition by symmetry arguments, then we will describe the main tools we will need like the fundamental solution of the sublaplacian and the Jerison-Lee solution of Equation 2.

In Chapter 3 we will prove the existence of the Fowler-type solutions of Equation 2 on $\mathbf{H}^n \setminus \{0\}$.

In Chapter 4 we will prove the existence of a homogeneous solution of Equation 2 on $\mathbf{H}^n \setminus \{0\}$ and the bifurcation result described above.

In Chapter 5 we will study the first variation of the CR Einstein-Hilbert functional on pseudo-Hermitian manifolds and the second variation on the spheres.

Chapter 1

CR geometry

1.1 CR manifolds

The study of analysis in several complex variables shows that real hypersurfaces of $\mathbf{C}^n$ or of complex manifolds are not holomorphically equivalent and possess geometric properties that distinguish them, and, in the case of boundaries, hugely influence the complex analytic properties of domains. CR structures are a way to study the geometry of these hypersurfaces abstractly and in an intrinsic way that does not depend on a specific immersion in $\mathbf{C}^n$.

To find a way to achieve this goal, one cannot simply act through restriction from the definition of complex manifold through charts, so a more suitable equivalent definition is used.

We recall that given complex coordinates $z^1, \ldots, z^n$, the usual coframe used is the local frame of the complexified cotangent bundle given by the differentials of the coordinates and their conjugates, $dz^1, \ldots, dz^n, d\overline{z}^1, \ldots, d\overline{z}^n$, and the local frame of vector fields used is the dual frame of the latter,

$$\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \overline{z}^1}, \ldots, \frac{\partial}{\partial \overline{z}^n},$$

given explicitly by

$$\frac{\partial}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - i\frac{\partial}{\partial y^\alpha}\right).$$

First of all, let us recall that a function $\mathbf{C}^n \to \mathbf{C}^k$ is holomorphic if and only if it is real differentiable, and the differential is complex linear. Furthermore a real linear operator between complex vector spaces is complex linear if and only if it commutes with the multiplication by i, which, by the real point of view, is a linear operator. When considering this on the tangent spaces, the operator of "multiplication by i" on $T_p\mathbf{C}^n$ which characterizes the differentials of holomorphic functions in this way is evidently the one that maps $\frac{\partial}{\partial x^\alpha}\big|_p$ to $\frac{\partial}{\partial y^\alpha}\Big|_p$ and $\frac{\partial}{\partial y^\alpha}\Big|_p$ to $-\frac{\partial}{\partial x^\alpha}\big|_p$. We cannot indicate it with the symbol i because it already denotes the multiplication by i in the complexification, so we denote it by J_p. On the complex frame, $J\frac{\partial}{\partial z^\alpha} = i\frac{\partial}{\partial z^\alpha}$ and $\frac{\partial}{\partial \bar{z}^\alpha} = -i\frac{\partial}{\partial \bar{z}^\alpha}$.

It turns out that the tensor J is invariant by biholomorphism, and so it can be defined on complex manifolds. This can be verified through a simple computation, or alternatively it can be noted that J_0 is the differential of the multiplication by i, and apply the elementary formula $d(\varphi(iz)) = i(d\varphi)(iz)$ in $z = 0$.

Because of the latter considerations, a function $\varphi : M \to N$ is holomorphic if and only if $d\varphi \circ J_M = J_N \circ d\varphi$. Furthermore, applying this to the identity, we see that J determines uniquely the structure of a complex manifold.

We recall that operator J on a real vector space V is the multiplication by i with respect to a complex vector structure extending the real one if and only if $J^2 = -I$, and that such an operator is called a complex structure on V. In light of the preceding considerations, the following definition arises naturally.

Definition 1.1 (Almost complex structure)**.** An almost complex structure on a manifold M is a tensor $J \in \mathrm{End}(TM)$ which is a complex structure on the tangent space of every point $p \in M$.

J is diagonalizable and has i and $-i$ as eigenvalues, and so is determined by its eigenspaces, which we call $T^{(1,0)}M$ and $T^{(0,1)}M$. Since J is a linear operator, $\overline{T^{(0,1)}M} = T^{(1,0)}M$, and so J is determined by $T^{(1,0)}M$ alone. Hence an equivalent definition of almost complex structure on a $2n$-dimensional manifold is a subbundle $T^{(1,0)}M \subset TM \otimes \mathbf{C}$ such that $\dim T^{(1,0)}M = n$ and $T^{(1,0)}M \cap \overline{T^{(1,0)}M} = \{0\}$.

In the case of a complex manifold, with respect to local coordinates

$$T^{(1,0)}M = \mathrm{span}\left\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}\right\},$$

and so $T^{(1,0)}M$ respects the formal integrability condition

$$[\Gamma(T^{(1,0)}M), \Gamma(T^{(1,0)}M)] \subseteq \Gamma(T^{(1,0)}M).$$

A deep theorem establishes that the converse is true.

Theorem 1.2 (Newlander–Nirenberg)**.** *An almost complex manifold is a complex manifold if and only if* $[\Gamma(T^{(1,0)}M), \Gamma(T^{(1,0)}M)] \subseteq \Gamma(T^{(1,0)}M)$. [1]

Therefore a $2n$-dimensional complex manifold can be equivalently characterized as a subbundle $T^{(1,0)}M \subset TM \otimes \mathbf{C}$ such that $\dim T^{(1,0)}M = n$, $T^{(1,0)}M \cap \overline{T^{(1,0)}M} = \{0\}$ and $[\Gamma(T^{(1,0)}M), \Gamma(T^{(1,0)}M)] \subseteq \Gamma(T^{(1,0)}M)$. This definition is suitable to restriction, and hence the following definition is motivated.

Definition 1.3 (CR manifold)**.** A CR structure on a $2n+1$-dimensional manifold M is a n-dimensional subbundle $T^{(1,0)}M$ of $TM \otimes \mathbf{C}$ such that:

[1] For a proof see [CS, Theorem 5.4.4] and the references cited therein at the end of the chapter

- $T^{(1,0)}M \cap T^{(0,1)}M = 0$, where $T^{(0,1)}M = \overline{T^{(1,0)}M}$,
- $[\Gamma(T^{(1,0)}M), \Gamma(T^{(1,0)}M)] \subseteq \Gamma(T^{(1,0)}M)$.

As anticipated, real hypersurfaces of a complex manifold carry a natural CR structure.

Proposition 1.4. *If M is a real hypersurface of a n-dimensional complex manifold X, then $T^{(1,0)}X \cap (TM \otimes \mathbf{C})$ is a CR structure on M.*

If M is a $2k+1$-dimensional real submanifold of X, with $k < n$, then $T^{(1,0)}X \cap (TM \otimes \mathbf{C})$ is not in general a CR structure on M because it can have dimension less than k. If $\dim(T^{(1,0)}X \cap (TM \otimes \mathbf{C})) = k$ then it is a CR structure.

We define
$$H(M) = \mathfrak{Re}(T^{(1,0)}M \oplus T^{(0,1)}M).$$
Inspired by complex manifolds, we define a complex structure J on $H(M)$ such that the i-eigenspace on the complexification $H(M) \otimes \mathbf{C} = T^{(1,0)}M \oplus T^{(0,1)}M$ is $T^{(1,0)}M$; that is, by the explicit formula,
$$J(Z + \overline{Z}) = i(Z - \overline{Z}).$$
$H(M)$ and J determine the CR structure. This can be used to give an alternative definition of CR structure as a subbundle of TM and a complex structure J on it verifying certain integrability conditions (see [DT, formulas 1.8 and 1.9]).

1.2 Pseudo-Hermitian structures

Let E be a subbundle of TM of codimension one. We want to define a notion of "maximal non-integrability" for E. Let us consider the antisymmetric map
$$B : \Gamma(E) \times \Gamma(E) \to \Gamma(TM/E)$$
defined as $B(X,Y) = [X,Y] \mod E$. It is well defined because
$$B(fX, gY) = -gY(f)X + fX(g)Y + fg[X,Y] \mod E = fg[X,Y] \mod E.$$
E is integrable if and only if B is zero. Then we say that E is a contact distribution if B is nondegenerate at every point. [2] Since B in antisymmetric, this implies that $\dim E$ is even, and so $\dim M$ is odd.

Definition 1.5. A CR structure is said nondegenerate if $H(M)$ is a contact distribution.

Definition 1.6. Given a contact distribution E, a differential form θ such that $E = \ker\theta$ is called a contact form.

[2] Most authors use the equivalent characterization of Proposition 1.16 as a definition

Definition 1.7. Given a CR structure, a pseudo-Hermitian structure is a differential form such that $H(M) = \ker\theta$.

Given a pseudo-Hermitian structure θ, every other pseudo-Hermitian structure is of the form $\varphi\theta$ with φ nowhere zero.

Since such a form descends to a nowhere zero section of $(TM/H(M))^* \simeq TM/H(M)$, it exists only if $TM/H(M)$ has nowhere zero sections, or equivalently, given that $\dim(TM/H(M)) = 1$, if and only if $TM/H(M)$ is orientable. Since $H(M)$ is oriented by the complex structure J, this happens if and only if M is orientable.

As we will show, a pseudo-Hermitian structure on a CR manifold allows to build a rich geometric structure on it, especially in the nondegenerate case.

Lemma 1.8. *θ is a contact form if and only if it is nowhere zero and the restriction of $d\theta$ to $\ker\theta$ is nondegenerate at every point.*

Proof. Since θ descends to a nowhere zero section of $(TM/\ker\theta)^*$, it is contact if and only if the map $(X, Y) \mapsto \theta([X, Y])$ is nondegenerate on $\ker\theta$. Since, for $X, Y \in \ker\theta$, $\theta([X, Y]) = -d\theta(X, Y)$, the thesis follows. □

Definition 1.9. The Levi form of a pseudo-Hermitian manifold is the section of $(T^{(1,0)}M)^* \otimes (T^{(0,1)}M)^*$ defined by

$$L_\theta(Z, \overline{W}) = i\theta([Z, \overline{W}]) = -id\theta(Z, \overline{W}).$$

The Levi form is Hermitian:

$$\overline{L_\theta(W, \overline{Z})} = -i\theta([\overline{W}, Z]) = i\theta([Z, \overline{W}]) = L_\theta(Z, \overline{W}).$$

If $\widetilde{\theta} = \varphi\theta$ with $\varphi > 0$ smooth, then, since

$$d\widetilde{\theta} = d(\varphi\theta) = d\varphi \wedge \theta + \varphi d\theta$$

and θ (identified with its complexified) vanishes on $H(M) \otimes \mathbf{C} = T^{(1,0)}M \oplus T^{(0,1)}M$, the Levi form transforms as

$$\widetilde{\theta} = \varphi L_\theta. \tag{1.1}$$

Proposition 1.10. *A pseudo-Hermitian manifold is nondegenerate if and only if its Levi form is nondegenerate at every point.*

Proof. The nondegeneracy of a pseudo-Hermitian structure is equivalent to that of $d\theta$ restricted to $H(M)$ by Lemma 1.8, and since degeneracy is left invariant by complexification, $d\theta(\cdot, \cdot)$ is zero on $(T^{(1,0)}M \otimes T^{(1,0)}M) \oplus (T^{(0,1)}M \otimes T^{(0,1)}M)$, and $\overline{T^{(1,0)}M \otimes T^{(0,1)}M} = T^{(0,1)}M \otimes T^{(1,0)}M$, this condition is equivalent to $d\theta$ being nondegenerate on $T^{(1,0)}M \otimes T^{(0,1)}M$, and so to the nondegeneracy of the Levi form. □

Definition 1.11. A pseudo-Hermitian structure is said strictly pseudoconvex if L_θ is positive definite everywhere. A CR structure is called strictly pseudoconvex if the Levi form of one (and hence all, by formula 1.1) pseudo-Hermitian structure on it is definite at every point (and hence positive definite up to changing the sign of the contact form).

Given a vector space V, if $V \otimes \mathbf{C} = W \oplus \overline{W}$ then the complexification $B_{\mathbf{C}}$ of every bilinear symmetric form B on V can be restricted to $W \otimes \overline{W}$ to generate a Hermitian form $H = \Phi(B)$. Every such a Hermitian form arises in this way, but not from a unique B. If we impose that its complexified $B_{\mathbf{C}}$ is zero restricted to $W \otimes W$, then a unique form $\Psi(H)$ is determined. Ψ is a right inverse of Φ which is a natural way to associate a symmetric form on V to any Hermitian form on W. If J is the complex structure on V whose i-eigenspace is W, $\Psi(H)$ can be alternatively determined by the condition that $\Psi(H)(Jv, Jw) = \Psi(H)(v, w)$.

Proposition 1.12. *The signature of $\Psi(H)$ is the double of the signature of H.*

Proof. Let $W = W_+ \oplus W_- \oplus W_0$ where H is positive definite on W_+, negative definite on W_-, null on W_0, and W_+, W_-, W_0 are orthogonal with respect to H. Then the signature of H is $(\dim W_+, \dim W_-, \dim W_0)$. If $V_+ = \mathfrak{Re}(W_+ \oplus \overline{W_+})$, $V_- = \mathfrak{Re}(W_- \oplus \overline{W_-})$ and $V_0 = \mathfrak{Re}(W_0 \oplus \overline{W_0})$ then it can be easily verified that V_+, V_- and V_0 are orthogonal and that $\Psi(H)|_{V_+ \times V_+} = \Psi\left(H|_{W_+ \times W_+}\right)$. Since Φ preserves definite or null products, $\Psi(H)$ is positive definite on V_+, negative definite on V_- and null on V_0, and so the thesis follows. □

Applying the latter considerations to the Levi form, we get that there exists a canonical symmetric form G_θ on $H(M)$, which is nondegenerate, or positive definite, if and only if the Levi form is, and such that

$$G_\theta(JX, JY) = G_\theta(X, Y).$$

An explicit formula for G_θ is

$$G_\theta(X, Y) = d\theta(X, JY)$$

(the only non trivial thing to verify to prove this is that $d\theta(\cdot, J\cdot)$ is symmetric; see [DT, Section 1.1.2] for a proof).

When $(M, T^{(1,0)}M, \theta)$ is pseudoconvex, $(M, H(M), G_\theta)$ ia a sub-Riemannian manifold.

Proposition 1.13. *Given a contact form θ there exists a unique vector field T such that*

$$\theta(T) = 1 \quad \textit{and} \quad i_T d\theta = 0.$$

Proof. Since $d\theta$ is antisymmetric and its restriction to $\ker\theta$ is nondegenerate, its radical has dimension one and must be transverse to $\ker\theta$, and from this the thesis easily follows. □

We call the vector field T of the preceding theorem the Reeb vector field.

We can use T to extend G_θ to a pseudo-Riemannian metric on M.

Definition 1.14. The Webster metric g_θ is the pseudo-Riemannian metric on M that coincides with G_θ on $H(M)$ and such that T is orthogonal to $H(M)$ and $g_\theta(T,T) = 1$.

Furthermore, we extend J to the whole TM by imposing $JT = 0$.

Definition 1.15. The horizontal gradient of a function $u \in \mathscr{C}^1(M)$ is the vector field $\nabla^\theta u \in \Gamma(H(M))$ such that

$$G_\theta(\nabla^\theta u, X) = du(X)$$

for every $X \in \Gamma(H(M))$. It coincides with the projection onto $H(M)$ of the gradient with respect to g_θ.

On pseudo-Hermitian manifolds there is a canonical volume form.

Proposition 1.16. *θ is a contact form if and only if $\theta \wedge (d\theta)^n$ is a volume form.*

Thanks to this property the divergence operator for vector fields can be defined by

$$\mathscr{L}_X(\theta \wedge (d\theta)^n) = \mathrm{div}_\theta(X)\theta \wedge (d\theta)^n.$$

Finally we define the sublaplacian Δ_b as the divergence of the horizontal gradient:

$$\Delta_b u = \mathrm{div}_\theta(\nabla^\theta u).$$

1.2.1 Hypersurfaces

Let M be an orientable hypersurface on the complex manifold X, and let $j : M \to X$ be the inclusion. We want to define a pseudo-Hermitian structure on M with its natural CR structure as a restriction of a form on X.

We recall that the decomposition $TX \otimes \mathbf{C} = T^{(1,0)}X \oplus T^{(0,1)}X$ gives rise to a dual decomposition on 1-forms $\Omega^1(X) \otimes \mathbf{C} = \Omega^{(1,0)}(X) \oplus \Omega^{(0,1)}(X)$. In local coordinates $\Omega^{(1,0)}(X)$ has $dz^1, \dots, dz^n$ as a frame, $\Omega^{(0,1)}(X)$ has $d\bar{z}^1, \dots, d\bar{z}^n$.

Composing the exterior differential with the projections we get two operators $\partial : \mathscr{C}^\infty(X) \to \Omega^{(1,0)}(X)$ and $\overline{\partial} : \mathscr{C}^\infty(X) \to \Omega^{(0,1)}(X)$ such that $d = \partial + \overline{\partial}$.

Since M is orientable, there exist a defining function $\varphi : X \to \mathbf{R}$ such that $M = \varphi^{-1}(0)$ and $d\varphi \neq 0$ on M. Since φ is real, this implies that $\overline{\partial}\varphi \neq 0$ on M, and so $\ker(\overline{\partial}\varphi) = T^{(1,0)}X$, and therefore

$$\ker(j^*\overline{\partial}\varphi) = T^{(1,0)}M. \tag{1.2}$$

The real and imaginary part of $j^*\overline{\partial}\varphi$ are real forms which are zero on $H(M)$. The former is always zero because

$$2\mathfrak{Re}(j^*\overline{\partial}\varphi) = j^*(\overline{\partial}\varphi + \partial\varphi) = j^*(d\varphi) = d(\varphi \circ j) = 0$$

and hence the latter is nowhere zero, because $j^*\overline{\partial}\varphi$ is nowhere zero due to formula (1.2). Therefore $\theta_\varphi = -2\mathfrak{Im}(j^*\overline{\partial}\varphi) = j^*(i(\partial - \overline{\partial})\varphi)$ is a pseudo-Hermitian structure on M.

If u is a nowhere zero function then $u\varphi$ is another defining function for M, and

$$\theta_{u\varphi} = j^*(i(\partial - \overline{\partial})(u\varphi)) = j^*(i\varphi(\partial - \overline{\partial})u + iu(\partial - \overline{\partial})\varphi) = uj^*(i(\partial - \overline{\partial})\varphi) = u\theta_\varphi$$

and so every pseudo-Hermitian structure on M coincides with θ_φ for some defining function φ.

1.3 The Tanaka-Webster connection

To study the geometry of pseudo-Hermitian manifolds it would be useful to have an affine connection on them. In the nondegenerate case, since there is the canonical Webster metric, one could think that the Levi-Civita connection could be used, but $T^{(1,0)}M$ is not parallel with respect to it, and so it is not a good connection to study pseudo-Hermitian manifolds.

Actually $T^{(1,0)}M$ is not parallel with respect to any torsion-free connection: in fact for any connection ∇ for which $T^{(1,0)}M$ is parallel

$$\theta(T_\nabla(Z,\overline{W})) = \theta(\nabla_Z\overline{W} - \nabla_{\overline{W}}Z - [Z,\overline{W}]) = -\theta([Z,\overline{W}]) = iL_\theta(Z,\overline{W}). \quad (1.3)$$

Let us suppose that ∇ is a connection such that

$$H(M) \text{ is parallel,} \quad (1.4)$$

$$\nabla g_\theta = 0, \quad (1.5)$$

$$\nabla J = 0. \quad (1.6)$$

It is not hard to prove that this implies that also $T^{(1,0)}M$, θ and T are parallel. Since the obstruction that we found to ∇ having zero torsion is (1.3), which fixes the T-component of T_∇ restricted to $T^{(1,0)}M \times T^{(0,1)}M$, let us impose that it is the only component, that is that

$$T_\nabla(Z,\overline{W}) = iL_\theta(Z,\overline{W})T. \quad (1.7)$$

By a similar computation it can be proved that $\theta(T_\nabla(Z,W)) = 0$, so let us impose further that

$$T_\nabla(Z,W) = 0. \quad (1.8)$$

These conditions are compatible, but are not enough to determine a unique connection, so let us look for conditions to impose on the remaining parts of the torsion, which are determined by the tensor $\tau : TM \to TM$ given by $\tau(X) = T_\nabla(T,X)$.

Given a connection ∇, it is well known that all other connections on M can be expressed as $\nabla_X Y + A(X)Y$ where $A \in \Omega^1(\operatorname{End} TM)$, and it holds that

$$T_{\nabla+A}(X,Y) = T_\nabla(X,Y) + A(X)Y - A(Y)X.$$

If ∇ is a connection which verifies conditions (1.4)-(1.8), in order that $\nabla + A$ also verifies them, the following conditions (of standard verification) on A are required:

1. $H(M)$ is parallel with respect to $\nabla + A$ if and only if it is $A(X)$-invariant for every X;

2. g_θ is parallel with respect to $\nabla + A$ if and only if $A(X)$ is anti-self-adjoint with respect to g_θ for every X;

3. J is parallel with respect to $\nabla + A$ if and only if $A(X)J = JA(X)$ for every X;

4. $\nabla + A$ verifies conditions 1.7 and 1.8 if and only if $A(X)Y = A(Y)X$ for every $X, Y \in \Gamma(H(M))$

(the proof of these is a standard computation).

Condition 3 implies that $A(X)T = 0$ for every X, and so

$$\tau_{\nabla+A} = \tau_\nabla + A(T) \tag{1.9}$$

and condition 1 implies that $A(T)$ is an endomorphism of $H(M)$.

Condition 2 and formula (1.9) imply that formulas (1.4)-(1.8) determine the self-adjoint part of τ, which in general is not zero, and suggest that we could try to make zero the anti-self-adjoint part of τ; so a possible condition to impose on τ is self-adjointness.

To analyse condition 4, let us define $\Lambda : \Gamma(\mathrm{End}(H(M))) \to \Gamma(\mathrm{End}(H(M)))$ as $\Lambda A = J \circ A \circ J$. Since it is an involution, $\Gamma(\mathrm{End}(H(M))) = P_+ \oplus P_-$, where

$$P_+ = \{A \mid \Lambda A = A\} = \{A \mid A \circ J + J \circ A = 0\}$$

and

$$P_- = \{A \mid \Lambda A = -A\} = \{A \mid A \circ J - J \circ A = 0\}.$$

So condition 4 and formula (1.9) imply that formulas (1.4)-(1.8) determine the projection on P_+ of τ, and suggest that we could try to impose that $\tau \in P_+$.

Condition 4 does not involve $A(T)$ directly, so it is of no help.

It turns out that the two conditions on τ that we conjectured, that is self-adjointness and the condition $\tau \circ J + J \circ \tau$, can be both be achieved. Even more, the second alone implies the first, and with all the other conditions determines a unique connection.

Theorem 1.17. *On a nondegenerate pseudo-Hermitian manifold there exists a unique connection ∇ such that:*

- *$H(M)$ is parallel with respect to ∇,*
- *$\nabla J = 0$, $\nabla g_\theta = 0$,*
- *$T_\nabla(Z, W) = 0$ for every $Z, W \in \Gamma(T^{(1,0)}M)$,*
- *$T_\nabla(Z, W) = iL_\theta(Z, \overline{W})T$ for every $Z, W \in \Gamma(T^{(1,0)}M)$,*
- *if $\tau(X) = T_\nabla(T, X)$ then $\tau \circ J + J \circ \tau = 0$.*

For a proof, see [DT, Theorem 1.3] or [T, Proposition 3.1].

The connection of Theorem 1.17 is called the Tanaka-Webster connection of the pseudo-Hermitian structure.

The restriction of the torsion to $T^{(1,0)}M \oplus T^{(0,1)}M$ can be expressed in terms of L_θ and T, and so it is trivial in a certain sense. Since the remaining part of the torsion is determined by the tensor τ, this gets called pseudo-Hermitian torsion.

Proposition 1.18. *The Tanaka-Webster connection also satisfies the following properties:*

(I) $T^{(1,0)}M$ *is parallel;*

(II) $\nabla T = 0$, $\nabla\theta = 0$, $\nabla d\theta = 0$*;*

(III) $\tau(T^{(1,0)}M) \subseteq T^{(0,1)}M$*;*

(IV) τ *is self-adjoint with respect to* g_θ*;*

(V) $\operatorname{tr}\tau = 0$.

Proof. (I) It is an easy verification.

(II) The first is [DT, formula 1.41], the other two follow easily.

(III) See [DT, Lemma 1.2].

(IV) See [DT, Lemma 1.4].

(V) It is sufficient to notice that

$$\operatorname{tr}\tau = -\operatorname{tr}(J^{-1}\tau J) = -\operatorname{tr}\tau.$$

□

1.4 Local computations

Let $(M, T^{(1,0)}M, \theta)$ be a nondegenerate pseudo-Hermitian manifold and let $Z_1, \dots, Z_n$ be a local frame of $T^{(1,0)}M$. Define $Z_{\overline{\alpha}} = \overline{Z}_\alpha$ and $Z_0 = T$. Let $(\theta^1, \dots, \theta^n, \theta^{\overline{1}}, \dots, \theta^{\overline{n}}, \theta^0)$ be the dual frame of $(Z_1, \dots, Z_n, Z_{\overline{1}}, \dots, Z_{\overline{n}}, Z_0)$ (in particular we take $\theta^0 = \theta$). Given a tensor $\mathfrak{T}$ of type (k, ℓ) and indexes $A_i, B_i \in \{1, \dots, n, \overline{1}, \dots, \overline{n}, 0\}$ let us define

$$\mathfrak{T}^{A_1 \dots A_k}{}_{B_1 \dots B_\ell} = \mathfrak{T}(\theta^{A_1}, \dots, \theta^{A_k}, Z_{B_1}, \dots, Z_{B_\ell}).$$

We separate indexes relative to covariant derivatives with respect to the Tanaka-Webster connection through a comma.

Let $h_{\alpha\overline{\beta}} = L_\theta(Z_\alpha, Z_{\overline{\beta}})$ be the coefficients of the Levi form. $h_{\alpha\overline{\beta}}$ and its inverse $h^{\alpha\overline{\beta}}$ can be used to raise and lower indexes. By definition

$$d\theta = i h_{\alpha\overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}}.$$

Let ${\omega_\alpha}^\beta$ be the connection forms defined by

$$\nabla Z_\alpha = {\omega_\alpha}^\beta \otimes Z_\beta$$

which implies that $\nabla \theta^\beta = -{\omega_\alpha}^\beta \otimes \theta^\alpha$. The equation $\nabla d\theta = 0$ in coordinates becomes

$$dh_{\alpha\overline{\beta}} = h_{\alpha\overline{\gamma}}{\omega_{\overline{\beta}}}^{\overline{\gamma}} + h_{\gamma\overline{\beta}}{\omega_\alpha}^\gamma.$$

Define the functions ${A_\alpha}^{\overline{\beta}}$ by

$$\tau(Z_\alpha) = {A_\alpha}^{\overline{\beta}} Z_{\overline{\beta}}$$

and the torsion forms $\tau^\beta = {A_{\overline{\alpha}}}^\beta \theta^{\overline{\alpha}}$, so that

$$\tau = \tau^\alpha \otimes Z_\alpha + \tau^{\overline{\alpha}} \otimes Z_{\overline{\alpha}}.$$

Then

$$\begin{aligned} d\theta^\alpha(X,Y) &= X(\theta^\alpha(Y)) - Y(\theta^\alpha(X)) - \theta^\alpha([X,Y]) = \\ &= \nabla_X(\theta^\alpha(Y)) - \nabla_Y(\theta^\alpha(X)) - \theta^\alpha([X,Y]) = \\ &= -{\omega_\beta}^\alpha(X)\theta^\beta(Y) + \theta^\alpha(\nabla_X Y) + {\omega_\beta}^\alpha(Y)\theta^\beta(X) - \theta^\alpha(\nabla_Y X) - \theta^\alpha([X,Y]) = \\ &= (\theta^\beta \wedge {\omega_\beta}^\alpha)(X,Y) + \theta^\alpha(T_\nabla(X,Y)). \end{aligned}$$

Simple computations show that $\theta^\alpha(T_\nabla(X,Y)) = (\theta \wedge \tau^\alpha)(X,Y)$. Therefore

$$d\theta^\alpha = \theta^\beta \wedge {\omega_\beta}^\alpha + \theta \wedge \tau^\alpha.$$

Since $G_\theta(\tau(X),Y) = G_\theta(X,\tau(Y))$ for X, Y sections of $H(M)$,

$$d\theta(\tau(X),JY) = G_\theta(\tau(X),Y) = G_\theta(X,\tau(Y)) = d\theta(X,J\tau(Y)) = -d\theta(X,\tau(JY))$$

so, since J is invertible on $H(M)$,

$$d\theta(\tau(X),Y) + d\theta(X.\tau(Y)) = 0.$$

Applying this formula with $X = Z_\alpha$, $Y = Z_\beta$ one gets that $A_{\alpha\beta} = A_{\beta\alpha}$. In terms of the torsion forms, this is equivalent to $\tau_\alpha \wedge \theta^\alpha = 0$.

The properties proved in this section characterize the Tanaka-Webster connection.

Theorem 1.19. *On a nondegenerate pseudo-Hermitian manifold there exist unique forms ${\omega_\alpha}^\beta$, τ^β such that*

- $d\theta^\alpha = \theta^\beta \wedge {\omega_\beta}^\alpha + \theta \wedge \tau^\alpha$,
- $dh_{\alpha\overline{\beta}} = h_{\alpha\overline{\gamma}}{\omega_{\overline{\beta}}}^{\overline{\gamma}} + h_{\gamma\overline{\beta}}{\omega_\alpha}^\gamma$,
- $\tau_\alpha \wedge \theta^\alpha = 0$.

They are the connection and torsion forms of the Tanaka-Webster connection.

Proof. We proved above that the connection and torsion forms of the Tanaka-Webster connection verify these equations. Uniqueness is proved in [We]. □

Given a function u, since $du = u_{,0}\theta + u_{,\alpha}\theta^\alpha + u_{,\overline{\alpha}}\theta^{\overline{\alpha}}$ then the horizontal gradient of u is

$$\nabla^\theta u = u^{,\alpha} Z_\alpha + u^{,\overline{\alpha}} Z_{\overline{\alpha}}.$$

It is standard to prove that there always exists a frame such that $h_{\alpha\overline{\beta}} = \pm\delta_{\alpha\overline{\beta}}$ (an orthonormal frame in the pseudoconvex case).

Proposition 1.20. *If $X = X^\alpha Z_\alpha + X^{\overline{\alpha}} Z_{\overline{\alpha}}$ then*

$$\mathrm{div}_\theta(Z) = X^\alpha{}_{,\alpha} + X^{\overline{\alpha}}{}_{,\overline{\alpha}}.$$

Proof. Since the formula is tensorial it can be proved using any frame, and in particular we can assume that $h_{\alpha\overline{\beta}} = \pm\delta_{\alpha\overline{\beta}}$. In this case using

$$\theta \wedge (d\theta)^n = i^n(-1)^{n(n-1)/2}(-1)^p \theta \wedge \theta^1 \wedge \ldots \wedge \theta^n \wedge \theta^{\overline{1}} \wedge \ldots \wedge \theta^{\overline{n}}$$

(where p is the negativity ndex of $h_{\alpha\overline{\beta}}$), and differentiating the relation $L_\theta(Z_\alpha, Z_{\overline{\alpha}}) = 0$ one gets

$$\theta^\alpha(\nabla_X Z_\alpha) + \theta^{\overline{\alpha}}(\nabla_X Z_{\overline{\alpha}}) = L_\theta(\nabla_X Z_\alpha, Z_{\overline{\alpha}}) + L_\theta(Z_\alpha, \nabla_X Z_{\overline{\alpha}}) = 0$$

and so, evaluating the formula $\mathrm{div}_X(\theta \wedge (d\theta)^n) = \mathscr{L}_X(\theta \wedge (d\theta)^n)$ on $(T, Z_1 \ldots, Z_n, Z_{\overline{1}}, \ldots, Z_{\overline{n}})$ we get that

$$\mathrm{div}(X) = -\sum_A \theta^A(\mathscr{L}_X Z_A) = -\sum_A \theta^A([X, Z_A]) =$$

$$= -\sum_A \theta^A(\nabla_X Z_A - \nabla_{Z_A} X - T_\nabla(X, Z_A)) =$$

$$= -\sum_\alpha \left(\theta^\alpha(\nabla_X Z_\alpha) + \theta^{\overline{\alpha}}(\nabla_X Z_{\overline{\alpha}})\right) + \sum_\alpha \left(\theta^\alpha(\nabla_{Z_\alpha} X) + \theta^{\overline{\alpha}}(\nabla_{Z_{\overline{\alpha}}} X)\right) =$$

$$= X^\alpha{}_{,\alpha} + X^{\overline{\alpha}}{}_{,\overline{\alpha}}.$$

□

In particular we get the following formula for the sublaplacian:

$$\Delta_b u = u^{,\alpha}{}_\alpha + u^{,\overline{\alpha}}{}_{\overline{\alpha}} = u_{,\alpha\overline{\alpha}} + u_{,\overline{\alpha}\alpha}. \tag{1.10}$$

1.5 Curvature

As for every connection, the Tanaka-Webster connection has a curvature tensor

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

which is an $\mathrm{End}(TM)$-valued 2-form on M. In pseudo-Hermitian geometry, a certain restriction of the curvature tensor has particular importance.

Definition 1.21. The pseudo-Hermitian curvature tensor is the section of $(T^{(1,0)}M)^* \otimes (T^{(0,1)}M)^* \otimes (T^{(1,0)}M)^* \otimes T^{(1,0)}M$ defined as

$$R(Z,\overline{W})X = \nabla_Z \nabla_{\overline{W}} X - \nabla_{\overline{W}} \nabla_Z X - \nabla_{[Z,\overline{W}]} X$$

for $Z, W, X \in \Gamma(T^{(1,0)}M)$. It is well defined because $T^{(1,0)}M$ is parallel.

Given a frame Z_α we indicate the components of the pseudo-Hermitian curvature by

$$R_{\alpha\overline{\beta}\mu}{}^{\nu} = L_\theta(R(Z_\alpha, Z_{\overline{\beta}})Z_\mu, Z_\nu).$$

As in Riemannian geometry, from the pseudo-Hermitian curvature tensor we can derive other notions of curvature.

Definition 1.22. The pseudo-Hermitian Ricci is the tensor with components $R_{\alpha\overline{\beta}} = R_{\mu\overline{\beta}\alpha}{}^{\mu}$. The pseudo-Hermitian or Webster scalar curvature is $R = h^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}$.

We will need the the transformation law for the Webster curvature under conformal changes of metric (see [JL1, Chapter 3] or [DT, page 160]).

Proposition 1.23. *If* $\widetilde{\theta} = u^{\frac{4}{Q-2}}\theta$ *then*

$$\widetilde{R} = u^{\frac{Q+2}{Q-2}}(-b_n \Delta_b + R)u \tag{1.11}$$

where $b_n = \frac{2Q}{Q-2}$.

1.5.1 Pseudo-Hermitian sectional curvature

The pseudo-Hermitian sectional curvature of a J-invariant plane $\sigma \subset H(M)_p$ is

$$k_\theta(\sigma) = -\frac{1}{4}\frac{R(X,JX)X,JX)}{G_\theta(X,X)^2}$$

where X is a non zero element of σ (the proof of the well-posedness of this definition is standard). Equivalently, if Z is a non zero vector in $\sigma \otimes \mathbf{C}$, the pseudo-Hermitian sectional curvature of σ is

$$k_\theta(\sigma) = \frac{1}{4}\frac{R(Z,\overline{Z})Z,\overline{Z})}{L_\theta(Z,\overline{Z})^2}.$$

A classification theorem for manifolds with constant pseudo-Hermitian sectional curvature similar to the Riemannian one holds, under the assumption that the pseudo-Hermitian torsion is zero.

The natural candidates for zero and positive curvature will be defined in the next Chapter, and $\mathbf{H}^n$ and S^{2n+1} respecively. Inspired by the hyperbolic space, we define

$$Q^n = \left\{ z \in \mathbf{C}^{n+1} \middle| q(z) = 1 \right\}$$

where $q(z) = |z_{n+1}|^2 - \sum_{\alpha=1}^n |z_\alpha|^2$. Q^n has a unique (up to constant multiples) pseudo-Hermitian structure invariant by the natural CR action of $U(n,1)$, given by $\theta_{Q^n} = \frac{i}{2}(\partial - \overline{\partial})q$. Q_n is a trivial S^1-bundle on the complex hyperbolic space. Let $\widetilde{Q}^n$ be its universal covering, and $\theta_{\widetilde{Q}^n}$ the pull-back of θ_{Q^n}.

Theorem 1.24. *If a simply connected pseudo-Hermitian manifold complete with respect the Webster metric has constant pseudo-Hermitian sectional curvature K and zero pseudo-Hermitian torsion then it is isomorphic to:*

- *the Heisenberg group $(\mathbf{H}^n, \theta_{\mathbf{H}^n})$ if $K = 0$;*
- *the sphere $(S^{2n+1}, \frac{1}{K}\theta_{S^{2n+1}})$ if $K > 0$;*
- *$(\widetilde{Q}^n, \frac{1}{K}\theta_{\widetilde{Q}^n})$ if $K < 0$.*

1.5.2 Invariant decomposition

We want to better motivate the definition of the pseudo-Hermitian curvature tensor, following [M]. We do it in the pseudoconvex case for simplicity of notation, but mutatis mutandis this could be extended to the general case.

First of all, we note that a pseudoconvex pseudo-Hermitian structure is associated naturally to a $U(n)$ principal bundle. Let $\mathfrak{P}$ the $SO(2n+1)$ principal bundle of orthogonal frames with respect to the Webster metric, and let σ the standard action of $SO(2n+1)$ on $\mathbf{R}^{2n+1}$, so that $TM \simeq \mathfrak{P} \times_\sigma \mathbf{R}^{2n+1}$.

Let $\mathfrak{Q}$ be the principal bundle over $U(n)$ with fiber

$$\mathfrak{Q}_p = \Big\{ (v_1, \dots, v_n) \text{ orthonormal base of } T^{(1,0)}M \text{ with respect to } L_\theta \Big\}$$

and the natural $U(n)$ action. Then, if ρ is the natural action of $U(n)$ on $\mathbf{C}^n$,

$$T^{(1,0)}M \simeq \mathfrak{Q} \times_\rho \mathbf{C}^n. \tag{1.12}$$

Since $TM \otimes \mathbf{C} = T^{(1,0)}M \oplus T^{(0,1)}M \oplus \mathscr{C}^\infty(M)T$, we can also recover the complexified tangent bundle:

$$TM \otimes \mathbf{C} \simeq \mathfrak{Q} \times_{\rho \oplus \overline{\rho} \oplus \varepsilon} (\mathbf{C}^n \times \mathbf{C}^n \times \mathbf{C}) \tag{1.13}$$

where $\overline{\rho}$ is the conjugate representation of ρ and ε is the trivial representation of $U(n)$ on $\mathbf{C}$.

The operator $c : \mathbf{C}^n \times \mathbf{C}^n \times \mathbf{C} \to \mathbf{C}^n \times \mathbf{C}^n \times \mathbf{C}$ given by $c(z, w, \zeta) = (\overline{w}, \overline{z}, \overline{\zeta})$ intertwines the representation $\rho \oplus \overline{\rho} \oplus \varepsilon$, and thus gives rise (through the isomorphism of equation (1.13)) to an automorphism of the bundle $TM \otimes \mathbf{C}$ which is obviously the complex conjugation. So TM, which is the subset of vectors of TM invariant by conjugation, can be recovered through the restriction ρ' of $\rho \oplus \overline{\rho} \oplus \varepsilon$ to $V = \ker(c - id)$:

$$TM = \mathfrak{Q} \times_{\rho'} V.$$

Given the natural isomorphism $\varphi : V \to \mathbf{R}^{2n+1}$ given by $\varphi(z, \overline{z}, t) = (\mathfrak{Re} z, \mathfrak{Im} z, t)$, $\varphi \circ \rho'(P) \in SO(2n+1)$ for every $P \in U(n)$, there exists a homomorphism $\Phi : U(n) \to O(2n+1)$ such that $\sigma(\Phi(P)) = \varphi(\rho'(P))$ for every $P \in U(n)$, given explicitly by

$$\Phi(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathfrak{Re} P & \mathfrak{Im} P \\ 0 & -\mathfrak{Im} P & \mathfrak{Re} P \end{pmatrix}.$$

Composing ρ' with φ one gets a representation $\widetilde{\rho}$ of $U(n)$ on $\mathbf{R}^{2n+1}$ such that

$$TM = \mathfrak{Q} \times_{\widetilde{\rho}} \mathbf{R}^{2n+1}. \tag{1.14}$$

Since T is parallel and the curvature tensor is real, the curvature $R(X, Y)Z$ is completely determined by its restriction to $Z \in T^{(1,0)}M$, which has values in $T^{(1,0)}M$. Since it is antisymmetric in X and Y and its action on $T^{(1,0)}M$ is skew-Hermitian, it can be seen as a section of $\mathfrak{so}(TM) \otimes \mathfrak{u}(T^{(1,0)}M)$, and so, thanks to (1.12) and (1.14), it corresponds to a $U(n)$-equivariant map from $\mathfrak{Q}$ to $\mathfrak{so}(\mathbf{R}^{2n+1}) \otimes \mathfrak{u}(\mathbf{C}^n)$. $d\widetilde{\rho} : \mathfrak{u}(n) \to \mathfrak{so}(\mathbf{R}^{2n+1})$ is an injective morphism of representations, given explicitly by

$$d\widetilde{\rho}(A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathfrak{Re} A & \mathfrak{Im} A \\ 0 & -\mathfrak{Im} A & \mathfrak{Re} A \end{pmatrix}.$$

The orthogonal of the subrepresentation $d\widetilde{\rho}(\mathfrak{u}(n)) \subset \mathfrak{so}(\mathbf{R}^{2n+1})$ is the subrepresentation

$$\mathfrak{w} = \left\{ \begin{pmatrix} 0 & v^T & w^T \\ v & A & B \\ w & B & -A \end{pmatrix} \middle| A, B \in \mathfrak{so}(n), v, w \in \mathbf{R}^n \right\}.$$

The decomposition $\mathfrak{so}(\mathbf{R}^{2n+1}) = d\widetilde{\rho}(\mathfrak{u}(n)) \oplus \mathfrak{w}$ gives rise to a decomposition $\mathfrak{so}(\mathbf{R}^{2n+1}) \otimes \mathfrak{u}(\mathbf{C}^n) = (d\widetilde{\rho}(\mathfrak{u}(n)) \otimes \mathfrak{u}(\mathbf{C}^n)) \oplus (\mathfrak{w} \otimes \mathfrak{u}(\mathbf{C}^n))$. It can be easily shown that the pseudo-Hermitian curvature tensor corresponds to the projection on the first factor.

The second factor is zero, and so the pseudo-Hermitian curvature represents the whole curvature tensor, if and only if the pseudo-Hermitian torsion is zero. This follows from [DT, Equations 1.85 and 1.86].

Chapter 2

The Heisenberg group

2.1 Definition

For any geometric structure it is important to know the most symmetric examples of it. For example in Riemannian geometry the Euclidean space, the sphere and the hyperbolic space are the most fundamental examples of Riemannian manifold.

So we want to find very symmetric examples of CR or pseudo-Hermitian manifolds, expecially pseudoconvex ones. The easiest way that we have to define a CR manifold is to take a real hypersurface of a complex manifold, so we look for boundaries of symmetric complex manifolds.

The natural complex manifolds to be considered are $\mathbf{C}^{n+1}$, $\mathbf{P}^{n+1}(\mathbf{C})$ and $B^{n+1} = B_1(0) \subset \mathbf{C}^{n+1}$. [1]

Since $\mathbf{C}^{n+1}$ and $\mathbf{P}^{n+1}(\mathbf{C})$ do not have a boundary, we use B^{n+1}. So we find our first CR manifold with many symmetries, the sphere $\partial B^{n+1} = S^{2n+1}$.

We recall that B^{n+1} is a model of the complex hyperbolic space. If we define the Hermitian product $\langle\cdot|\cdot\rangle$ on $\mathbf{C}^{n+2}$ as

$$\langle z, w\rangle = \sum_{k=1}^{n+1} z_k \overline{w}_k - z_{n+2}\overline{w}_{n+2}$$

then the complex hyperbolic space is defined as

$$\mathscr{H}^{n+1} = \{[z] \in \mathbf{P}^{n+1}(\mathbf{C}) \mid \langle z, z\rangle < 0\}.$$

The function $\varphi : B^{n+1} \to \mathscr{H}^{n+1}$ defined as $\varphi(z) = [(z, 1)]$ is a biholomorphism. The group $PU(n+1, 1)$ acts holomorphically on $\mathscr{H}^{n+1}$, and it turns out to

[1] For example they are the only simply connected manifolds which support complete Kähler metrics with constant curvature (see [KN, Theorems 7.8 and 7.9]). Unlike the Riemannian case, they are not characterized by having a biholomorphism group of maximal dimension; in particular $\dim \mathrm{Aut}(\mathbf{P}^n(\mathbf{C})), \dim \mathrm{Aut}(B^n) < \infty$ while many complex manifolds have infinite dimensional biholomorphism group. However they are characterized by a fairly weak homogeneity property (see [GKK, Theorems 6.1.1 and 6.1.3])

be the group of biholomorphisms $\mathrm{Aut}(\mathscr{H}^{n+1})$ (this follows from [R, Theorem 2.1.1]).

$\mathscr{H}^{n+1}$ (and hence B^{n+1}) can be endowed with a natural Hermitian metric, which is the unique (up to a multiplicative constant) invariant by $\mathrm{Aut}(\mathscr{H}^{n+1})$.

Every element of $\mathrm{Aut}(B^{n+1})$ extends to a CR automorphism of S^{2n+1}; all CR automorphisms of S^{2n+1} arise in this way, [2] so $\mathrm{Aut}_{\mathrm{CR}}(S^{2n+1}) \simeq \mathrm{Aut}(B^{n+1})$.

In the study of the real hyperbolic space, in addition to the hyperboloid model and the ball model, there is a third important and useful model, the half-space model, which for example is useful to study isometries with a fixed point at the infinity. It is obtained by the ball model by a spherical inversion, and it is relevant to us because that isometry extends to the boundary of the ball, the Euclidean sphere, minus a point, and takes it to $\mathbf{R}^n$, and so we would like to do the same to get an analogue of $\mathbf{R}^n$ in CR geometry. In the complex case spherical inversion are not holomorphic, and if $n \geq 2$ the sphere is not biholomorphically equivalent to a half-space. So we try to apply the simplest biholomorphism which takes a point of ∂B^{n+1} to the infinity, that is a projectivity:

$$F(z_1, \ldots, z_{n+1}) = \left(\frac{z_1}{1+z_1}, \ldots, \frac{z_n}{1+z_n}, i\frac{1-z_{n+1}}{1+z_{n+1}} \right).$$

F takes B^{n+1} to the Siegel domain

$$\mathscr{U} = \left\{ \mathfrak{Im} z_{n+1} > \sum_{k=1}^{n} |z_k|^2 \right\}$$

and the sphere $S^{2n+1} \setminus \{p\}$, where $p = (0, \ldots, 0, -1)$, to its boundary

$$\partial\mathscr{U} = \left\{ \mathfrak{Im} z_{n+1} = \sum_{k=1}^{n} |z_k|^2 \right\}.$$

CR automorphisms of S^{2n+1} which fix p lead to CR automorphisms of $\partial\mathscr{U}$. By studying the stabilizer of a null line with respect to $PU(n+1,1)$, it can be proved that the automorphisms that arise in this way can be decomposed uniquely as $\Lambda_\lambda \circ \Xi_P \circ \tau_{(w,t)}$ where, if $(z, \zeta) \in \mathbf{C}^{n-1} \times \mathbf{C}$:

- $\Lambda_\lambda(z, \zeta) = (\lambda z, \lambda^2 \zeta)$ for $\lambda \in (0, \infty)$;
- $\Xi_P(z, \zeta) = (Pz, \zeta)$ for $P \in U(n-1)$;
- $\tau_{(w,t)}(z, \zeta) = (z + w, \zeta + t + 2iz \cdot \overline{w} + i|w|^2)$. [3]

[2] To prove this fact, one can use the results of [JL2] to prove this only for automorphisms which preserve the standard pseudo-Hermitian structure on S^{2n+1} which will be defined at the end of this section, and then apply dimensional considerations as in [We, Theorem (1.2)] or [M, Theorem 4.10]

[3] this decomposition could be seen as an Iwasawa decomposition for the group

This resembles what happens with the real hyperbolic space, when doing the same thing gives rise, on the boundary $\mathbf{R}^n$ of the half-space model, to dilations, linear isometries and translations.

As for the translations in $\mathbf{R}^n$, the set $\{\tau_{(w,t)}\}_{(z,t)\in\mathbf{C}^{n-1}\times\mathbf{R}}$ forms a subgroup of $\mathrm{Aut}(\partial\mathscr{U})$, since

$$\tau_{(z,t)}\circ\tau_{(w,s)}=\tau_{(z+w,t+s+2\mathfrak{Im}(z\cdot\overline{w}))}\text{ and }\tau^{-1}_{(z,t)}=\tau_{(-z,-t)}.$$

So we are lead to the following definition.

Definition 2.1. The Heisenberg group $\mathbf{H}^n$ is $\mathbf{C}^n\times\mathbf{R}$ with the group law

$$(z,t)\cdot(w,s)=(z+w,t+s+2\mathfrak{Im}(z\cdot\overline{w})).$$

$\tau_{(\cdot,\cdot)}$ defines a CR action of $\mathbf{H}^n$ on $\partial\mathscr{U}$. Since this action is free and transitive, we can fix the point $0\in\mathbf{C}^{n+1}$ as an origin and identify the Heisenberg group with $\partial\mathscr{U}$ by defining

$$\psi(z,t)=\tau_{(z,t)}0=(z,t+i|z|^2).$$

In this way the Heisenberg group can be endowed with a CR structure, given explicitly by

$$T^{(1,0)}\mathbf{H}^n=d(\psi^{-1})(T^{(1,0)}(\partial\mathscr{U})).$$

Let us define the left translation by $x\in\mathbf{H}^n$ as $L_xy=x\cdot y$. Then

$$(\psi\circ L_x)(y)=\tau_{xy}0=\tau_x(\tau_y(0))=(\tau_x\circ\psi)(y)$$

and so $L_x=\psi^{-1}\circ\tau_x\circ\psi$ is a composition of CR automorphisms, therefore it is a CR automorphism, or, in other words, the CR structure on $\mathbf{H}^n$ is left-invariant.

Through ψ we can transport the families of CR automorphisms Λ_λ and Ξ_P to $\mathbf{H}^n$: we define

$$\delta_\lambda=\psi^{-1}\circ\Lambda_\lambda\circ\psi$$

and

$$\rho_P=\psi^{-1}\circ\Xi_P\circ\psi$$

for $\lambda\in(0,\infty)$ and $P\in U(n)$. Explicitly

$$\delta_\lambda(z,t)=(\lambda z,\lambda^2t)\tag{2.1}$$

and

$$\rho_P(z,t)=(Pz,t).\tag{2.2}$$

It holds that $\delta_{\lambda_1}\circ\delta_{\lambda_2}=\delta_{\lambda_1\lambda_2}$ and $\rho_{P_1}\circ\rho_{P_2}=\rho_{P_1P_2}$.

Furthermore both δ_λ and ρ_P are, in addition to CR automorphisms, also group automorphisms of $\mathbf{H}^n$.

We need to compute a frame for $T^{(1,0)}\mathbf{H}^n$. We have that $T(\partial\mathscr{U})|_0\simeq\mathbf{C}^n\times\{0\}$, and so $T^{(1,0)}(\partial\mathscr{U})|_0\simeq T^{(1,0)}\mathbf{C}^n\times\{0\}$ and, calling 0 the identity of $\mathbf{H}^n$, $T^{(1,0)}\mathbf{H}^n|_0=d(\psi^{-1})(T^{(1,0)}(\partial\mathscr{U})|_0)\simeq T^{(1,0)}\mathbf{C}^n\times\{0\}$. Therefore

$$\left(\left.\frac{\partial}{\partial z^1}\right|_0,\dots,\left.\frac{\partial}{\partial z^n}\right|_0\right)$$

is a basis of $T^{(1,0)}\mathbf{H}^n|_0$. Since $T^{(1,0)}\mathbf{H}^n$ is left-invariant, the left-invariant extensions of this vectors form a frame of it, which can be computed esplicitly as

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} + i\overline{z}^\alpha \frac{\partial}{\partial t}.$$

The coframe associated with this frame is $dz^1, \ldots, dz^n$,

Now we want to define a pseudo-Hermitian structure on $\mathbf{H}^n$. Obviously there exists only one such structure that is left-invariant, up to multiplication by a constant. At the identity $dt|_0$ is an element of T_0^*M such that it annihilates $Z_1|_0, \ldots, Z_n|_0$, and extending it to a left invariant form one gets the pseudo-Hermitian structure

$$\theta = dt + i\sum_{\alpha=1}^{n} \left(z^\alpha d\overline{z}^\alpha - \overline{z}^\alpha dz^\alpha\right).$$

θ is invariant also by the operators ρ_P.

The Levi form is

$$L_\theta = -id\theta = \sum_{\alpha=1}^{n} \left(dz^\alpha \wedge d\overline{z}^\alpha - d\overline{z}^\alpha \wedge dz^\alpha\right) = 2\sum_{\alpha=1}^{n} dz^\alpha \wedge d\overline{z}^\alpha$$

and so $h_{\alpha\overline{\beta}} = 2\delta_{\alpha\overline{\beta}}$ and $\mathbf{H}^n$ is strictly pseudoconvex. [4]

The Reeb vector field with respect to θ is $T = \frac{\partial}{\partial t}$.

Applying Theorem 1.19 we deduce that ${\omega_\alpha}^\beta = 0$ and $\tau^\alpha = 0$ for every α, β. So we proved the following.

Proposition 2.2. *On the Heisenberg group $\mathbf{H}^n$ any left invariant vector field is parallel, and thus the Tanaka-Webster connection is flat; furthermore the pseudo-Hermitian torsion τ is zero.*

This suggests that $\mathbf{H}^n$ should have the same role among pseudoconvex pseudo-Hermitian manifolds that $\mathbf{R}^n$ has among Riemannian manifolds. We will show that this intuition is true.

$F^{-1}\circ\psi$ is a CR isomorphism between $\mathbf{H}^n$ ans $S^{2n+1}\setminus\{p\}$. The push-forward of θ $(F^{-1}\circ\psi_*\theta$ is a pseudo-Hermitian structure on $S^{2n+1}\setminus\{p\}$, but we will verify that it does not extend continuously to S^{2n+1}. To define a canonical contact form on S^{2n+1} the natural condition to impose is invariance by the elements of $\mathrm{Aut}_{\mathrm{CR}}(S^{2n+1})$ which preserve the usual Riemannian metric. It can be shown that this subgroup is $U(n+1)$. Since $U(n+1)$ acts transitively on S^{2n+1}, there can exist only one $U(n+1)$-invariant pseudo-Hermitian structure on S^{2n+1} up to multiplication by a constant. Thanks to the results of Subsection 1.2.1 it is quite easy find explicitly such a structure, namely $\theta_{S^{2n+1}} = \frac{i}{2}(\partial - \overline{\partial})|z|^2$.

The commutation relations are

$$[Z_\alpha, \overline{Z}_\beta] = -2i\delta_{\alpha\beta}T \text{ and } [Z_\alpha, Z_\beta] = [Z_\alpha, T] = 0$$

[4] This could have been known beforehand by who knows analysis in several complex variables by noting that $\mathbf{H}^n$ is, by construction, locally CR equivalent to S^{2n+1}, which is the boundary of a strictly convex domain

or, with respect to the real frame,

$$[X_\alpha, Y_\beta] = -4\delta_{\alpha\beta}T \text{ and } [X_\alpha, X_\beta] = [X_\alpha, T] = [Y_\alpha, T] = 0$$

and thus $\mathbf{H}^n$ is a nilpotent Lie group.

Thanks to formula (1.10), and since $h_{\alpha\overline{\beta}} = 2\delta_{\alpha\overline{\beta}}$,

$$\Delta_b = \frac{1}{2}\sum_{\alpha=1}^{n}(Z_\alpha Z_{\overline{\alpha}} + Z_{\overline{\alpha}} Z_\alpha) = \frac{1}{4}\sum_{\alpha=1}^{n}(X_\alpha^2 + Y_\alpha^2).$$

The volume form associated with the contact form is

$$\theta \wedge (d\theta)^n = 4^n n! dt \wedge dx^1 \wedge dy^1 \wedge \ldots \wedge dx^n \wedge dy^n$$

which induces a multiple of the Lebesgue measure; since θ is left invariant, the Lebesgue measure is a left Haar measure. Since the group inverse map is $(z,t) \mapsto (-z,-t)$ preserves the Lebesgue measure, it is also a right Haar measure, and so $\mathbf{H}^n$ is unimodular. [5]

2.2 Symmetry and homogeneity

The group automorphisms of $\mathbf{H}^n$ can be explicitly classified (see [F2, Theorem 1.22]).

Proposition 2.3. *Every group automorphism of* $\mathbf{H}^n$ *can be decomposed uniquely as* $\varphi_1\varphi_2\varphi_3\varphi_4$ *where:*

- $\varphi_1(x,y,t) = (A(x,y),t)$ *where* A *is a symplectic operator;*
- φ_2 *is an inner automorphism; it can be computed that*

$$(a,b,s)(x,y,t)(a,b,s)^{-1} = (x,y,t + a\cdot y - b\cdot x);$$

- $\varphi_3 = \delta_\lambda$ *for some* $\lambda \in (0,\infty)$ *(as defined in* (2.1)*);*
- φ_4 *is either the identity or the involution*

$$(x,y,t) \mapsto (y,x,-t).$$

It is also interesting to study which automorphisms preserve the other structures we put on $\mathbf{H}^n$.

Proposition 2.4. *An automorphism of* $\mathbf{H}^n$ *preserves the distribution* $H(\mathbf{H}^n)$ *if and only if, in the notation of Proposition 2.3,* $\varphi_2 = \mathrm{id}$.

An automorphism of $\mathbf{H}^n$ *preserves the distribution* $H(\mathbf{H}^n)$ *and the sub-Riemannian metric* G_θ *on it if and only if* $\varphi_2 = \varphi_3 = \mathrm{id}$ *and* $\varphi_1(z,t) = (Pz,t)$ *with* $P \in U(n)$.

[5] in fact every nilpotent group is unimodular, see [K, Corollary 8.31]

Proof. The proof consists of routine verifications, except for the fact that automorphisms φ_1 which preserve the metric must be unitary, which follows from the fact that $O(2n) \cap Sp(2n) = U(n)$ (see [F2, Proposition 4.6]; this is part of the famous "two out of three" property of the unitary group). $\square$

The family of operators δ_λ enjoys many similarities with the dilations of $\mathbf{R}^n$. We give a general definition of dilation on a Lie group.

Definition 2.5. A family of dilations on Lie algebra $\mathfrak{g}$ is a family of automorphisms of the form $e^{A \log \lambda}$, where A is a diagonalizable operator on A with positive eigenvalues. The corresponding family of operators on the simply connected group with Lie algebra G are also called dilations.

Many results of this section could be formulated in this more general contest.

The push forward of the Lebesgue measure under the dilations is

$$(\delta_\lambda)_\# \mathscr{L}^n = \lambda^{2n+2} \mathscr{L}^n. \tag{2.3}$$

Because of this, as we will see, in many matters the number $Q = 2n + 2$ plays the role of the dimension of $\mathbf{R}^n$; because of this Q is known as homogeneous dimension of $\mathbf{H}^n$.

Definition 2.6. A distribution Λ on $\mathbf{H}^n$ (or, more generally, on an open cone, that is a set closed under dilations) is said homogeneous of degree α if $\langle \Lambda, \varphi \circ \delta_\lambda \rangle = \lambda^{\alpha - Q} \langle \Lambda, \varphi \rangle$ for every $\varphi \in \mathscr{D}(\mathbf{H}^n)$. If Λ arises from a function u in L^1_{loc} this is equivalent to $u \circ \delta_\lambda = \lambda^\alpha u$.

For example the Dirac distribution at the origin δ_0 is homogeneous of degree $-Q$.

Definition 2.7. A linear operator $T : \mathscr{D}(\mathbf{H}^n) \to \mathscr{D}'(\mathbf{H}^n)$ is said homogeneous of degree α if $\langle T(\varphi \circ \delta_\lambda), \psi \rangle = \lambda^{\alpha - Q} \left\langle T\varphi, \psi \circ \delta_{1/\lambda} \right\rangle$.

The vector fields Z_α, X_α and Y_α are operators homogeneous of degree -1, while T is homogeneous of degree -2. Since the composition (on the appropriate domain) of an operator homogeneous of degree α and an operator homogeneous of degree β is homogeneous of degree $\alpha + \beta$, the sublaplacian is homogeneous of degree -2.

Let Z be the infinitesimal generator of the group of dilations, that is, the vector field defined by

$$\left.\frac{d}{d\lambda}\right|_{\lambda=1} (u \circ \delta_\lambda) = Zu$$

for $u \in \mathscr{C}^1(\mathbf{H}^n)$. An explicit expression for it is

$$Z = \sum_{k=1}^n x_k \frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial y_k} + 2t \frac{\partial}{\partial t}.$$

It is easy to verify that

$$\lambda \frac{d}{d\lambda} (u \circ \delta_\lambda) = Z(u \circ \delta_\lambda) = (Zu) \circ \delta_\lambda. \tag{2.4}$$

Using this formula, it is easy to prove the following analogue of Euler's theorem on homogeneous functions on $\mathbf{R}^n$.

Proposition 2.8. *A function u is homogeneous of degree α if and only if $Zu = \alpha u$.*

On $\mathbf{R}^n$ a known characterization of the laplacian is that every linear differential operator commuting with translations and isometries is a polynomial of the Laplacian (see [F3, Theorem 8.51]). We want to study analogous characterizations of invariant operators on $\mathbf{H}^n$.

Proposition 2.9. *A linear differential operator L on $\mathbf{H}^n$ commutes with left translations and with the operators ρ_P for $P \in U(n)$ defined in* (2.2) *if and only if $L = p(T, \Delta_b)$ for some polynomial p in two variables.*

Proof. Let G be the group generated by the left translations and the operators ρ_P, and H be the stabilizer of the origin with respect to the natural action of G on $\mathbf{H}^n$, that is the subgroup formed by the ρ_P, so that $H \simeq U(n)$. Let $\mathfrak{m} \subset \mathfrak{g}$ be the Lie algebra of the subgroup of left translations. Now we apply [H, Theorem 4.9]. [6] Thanks to that general theorem, the thesis becomes equivalent to prove that a polynomial $p(\mathfrak{Re} z, \mathfrak{Im} z, t)$ invariant by the action of $U(n)$ on z is of the form $p(\mathfrak{Re} z, \mathfrak{Im} z, t) = p(|z|^2, t)$. Writing $p(\mathfrak{Re} z, \mathfrak{Im} z, t) = \sum_\alpha t^\alpha p_\alpha(\mathfrak{Re} z, \mathfrak{Im} z)$ this becomes equivalent to prove that a polynomial $p_\alpha(\mathfrak{Re} z, \mathfrak{Im} z)$ invariant by the action of $U(n)$ is a polynomial of $|z|^2$, and this is elementary. $\square$

2.3 Analysis

$$|(z,t)| = \left(|z|^4 + t^2\right)^{1/4}.$$

It verifies the following properties:

- it is continuous and $|x| = 0 \iff x = 0$;
- $|\delta_\lambda x| = \lambda |x|$;
- $|x^{-1}| = |x|$;
- $|xy| \leq |x| + |y|$.

As in the case of $\mathbf{R}^n$, it can be proved that all norms are equivalent.

The convolution of two measurable functions f and g on $\mathbf{H}^n$ is

$$(f * g)(x) = \int_{\mathbf{H}^n} f(xy^{-1}) g(y) = \int_{\mathbf{H}^n} f(y) g(y^{-1}x)$$

[6] For convenience of the reader, we point out that in the cited reference $\mathbf{D}(G/H)$ is defined in Chapter 2, section 4.1, page 274; $\mathfrak{m}$ is any $\mathrm{Ad}_G(\mathfrak{h})$-invariant subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (see page 284); $I(\mathfrak{m})$ is defined in Corollary 4.8; $S(\mathfrak{m})$ and λ are defined in Theorem 4.3; the definition of "reductive" is given at the bottom of page 284.

whenever the integral is defined for almost every x. Convolution can be defined in general on locally compact Hausdorff groups with respect to a left Haar measure, with some care required if the group is not unimodular.

As in the case of $\mathbf{R}^n$, convolution can be extended to measures and distributions.

Convolution is associative, but unlike $\mathbf{R}^n$ it is not commutative as a consequence of the non commutativity of the group structure, as shown by the formula $\delta_x * \delta_y = \delta_{xy}$.

Proposition 2.10. *If X is a left invariant vector field, then $X(f*g) = f*(Xg)$.*

We point out that it is not true that $X(f*g) = (Xf)*g$. This can be deduced for example from the not hard to prove formula $\delta_x * Xf * \delta_{x^{-1}} = \mathrm{Ad}_x(X)f$.

We will need integral inequalities for the convolution. To state them in the most general form, we recall briefly the notion of Lorentz spaces. Given a σ-finite measure space (X, μ) and $1 \leq p < \infty$, $1 \leq q \leq \infty$, the Lorentz quasinorm is defined as

$$\|u\|_{L^{p,q}(X)} = p^{1/q} \left\| \lambda \mu\{|u| > \lambda\}^{1/p} \right\|_{L^q(dt/t)}.$$

Furthermore we define $\|u\|_{L^{\infty,\infty}(X)} = \|u\|_{L^\infty(X)}$. The Lorentz space $L^{p,q}(X)$ is the set of functions such that this quantity is finite. When $p = q$, $\|u\|_{L^{p,p}} = \|u\|_{L^p}$, while when $q = \infty$, $L^{p,\infty}$ coincides with the weak L^p space.

Theorem 2.11. *If $1 < p, p_1, p_2 < \infty$, $1 \leq q, q_1, q_2 \leq \infty$ are such that*

$$\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p} \quad \text{and} \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$$

then there exists C such that for every $f \in L^{p_1,q_1}(\mathbf{H}^n)$, $g \in L^{p_2,q_2}(\mathbf{H}^n)$ it holds

$$\|f * g\|_{L^{p,q}(\mathbf{H}^n)} \leq C \|f\|_{L^{p_1,q_1}(\mathbf{H}^n)} \|g\|_{L^{p_2,q_2}(\mathbf{H}^n)}.$$

Whenever $p = q$, $p_1 = q_1$ and $p_2 = q_2$ the inequality holds with $C = 1$:

$$\|f * g\|_{L^p(\mathbf{H}^n)} \leq \|f\|_{L^{p_1}(\mathbf{H}^n)} \|g\|_{L^{p_2}(\mathbf{H}^n)}$$

and also for $p = 1, \infty$.

Proof. The theorem follows from [ON, Theorem 2.6] (with the corrections in [Y]) and [Gr, Theorem 1.2.12, Remark 1.2.11]. □

Lemma 2.12. *If $0 < s \leq Q$ then a measurable function homogeneous of degree $-s$ and boundend on $\{|x| = 1\}$ belongs to $L^{Q/s,\infty}$.*

Proof. It follows easily from formula (2.3). □

All of this suggests that the study of fundamental solutions for left invariant differential operators on $\mathbf{H}^n$ should have the same role of fundamental solutions for constant coefficients operators (which is equivalent to translation invariant) operators on $\mathbf{R}^n$.

Regarding the sublaplacian, since it is homogeneous of degree -2, and δ_0 is homogeneous of order $-Q$, we can expect that if it has a fundamental solution, it is homogeneous of order $-Q+2$. It turns out that this is the case, and the fundamental solution, discovered by Folland, is known explicitly.

Proposition 2.13. *There exists a constant c_0 such that $-\Delta_b |x|^{-Q+2} = c_0 \delta_0$.*

For the proof, see [F1], [FS, Theorem 6.2], [CS, Theorem 10.1.1] [DT, Theorem 3.9]. The latter three references study, more generally, the homogeneous operators $-\Delta_b + \alpha T$ ($\alpha \in \mathbf{C}$), which is useful to study the Kohn Laplacian (a version of the Hodge Laplacian for pseudo-Hermitian manifolds). We define $\Phi = \frac{1}{c_0 |x|^{Q-2}}$.

Proposition 2.14. *If $u \in \mathscr{E}'(\mathbf{H}^n)$ then $-\Delta_b(u * \Phi) = (-\Delta_b u) * \Phi = u$.*

Proof. The fact that $-\Delta_b(u * \Phi) = u$ follows from Proposition 2.10. If $v \in \mathscr{D}(\mathbf{H}^n)$ then

$$\langle (-\Delta_b u) * \Phi, v \rangle = \langle -\Delta_b u, v * \Phi \rangle = \langle u, -\Delta_b (v * \Phi) \rangle = \langle u, v \rangle .$$

□

Now we introduce the analogues of the Sobolev spaces. To define them, if $I = (\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)$ is a multi-index, we use the notation

$$X_I u = X_1^{\alpha_1} \dots X_n^{\alpha_n} Y_1^{\alpha_1} \dots Y_n^{\alpha_n} u.$$

Definition 2.15. If $U \subseteq \mathbf{H}^n$ is open, we define the Folland-Stein space

$$S^{k,p}(U) = \{ u \in L^p(U) | X_I u \in L^p(U) \text{ for every multi-index } I \text{ with } |I| \leq k \}$$

with norm

$$\|u\|_{S^{k,p}(U)} = \sum_{|I| \leq k} X_I u.$$

Proposition 2.16 (Properties of Folland-Stein spaces).

- *$S^{k,p}(U)$ is a Banach space;*
- *smooth functions are dense in $S^{k,p}(U)$; smooth compactly supported functions are dense in $S^{k,p}(\mathbf{H}^n)$;*
- *If $u \in S^{k,p}(U)$, $|u| \in S^{k,p}(U)$ and $\| |u| \|_{S^{k,p}} = \|u\|_{S^{k,p}}$.*

Theorem 2.17. *If $u \in S^{1,p}(\mathbf{H}^n)$ with $1 \leq p < Q$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q}$ then*

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p} .$$

Proof. Let us denote $X_{n+\alpha} = Y_\alpha$. By Proposition 2.16 we can suppose that u is smooth and compactly supported. Then by Proposition 2.14

$$u = u * \delta = u * (-\Delta_b \Phi) = u * (-\Delta_b \delta) * \Phi = -\frac{1}{4} \sum_{\alpha=1}^{2n} u * (X_\alpha^2 \delta) * \Phi =$$

$$= -\frac{1}{4}\sum_{\alpha=1}^{2n} u * (X_\alpha\delta * X_\alpha\delta) * \Phi = -\frac{1}{4}\sum_{\alpha=1}^{2n}(X_\alpha u) * (X_\alpha\delta * \Phi).$$

$X_\alpha\delta * \Phi$ is equal to $X_\alpha^r\Phi$ where X_α^r is the right invariant vector field coinciding with X_α at the origin. In particular, if $R_\alpha = -\frac{1}{4}X_\alpha^r\Phi$, then

$$u = \sum_{\alpha=1}^{2n}(X_\alpha u) * R_\alpha \tag{2.5}$$

and R_α are functions smooth outside the origin and homogeneous of degree $-Q+1$. By Theorem 2.11 and Lemma 2.12, if $1 < p < Q$,

$$\|u\|_{L^{p^*}} \leq \sum_{\alpha=1}^{2n} \|(X_\alpha u) * R_\alpha\|_{L^{p^*}} \leq C_1 \sum_{\alpha=1}^{2n} \|X_\alpha u\|_{L^p} \leq C\|\nabla u\|_{L^p}.$$

This proof does not work for $p = 1$, because the operator $f \mapsto f * \Phi$ does not map L^1 to $L^{\frac{Q}{Q-1}}$. Nevertheless if K is a homogeneous function of degree $-Q+1$ the operator $f \mapsto f * K$ maps L^1 continuously to $L^{\frac{Q}{Q-1},\infty}$, that is, explicitly,

$$\sup_{t>0} t|\{|u * K| > t\}|^{\frac{Q-1}{Q}} \leq \|u\|_{L^1}.$$

and this is sufficient to prove the desired inequality thanks to a trick by Maz'ya.

By Proposition 2.16 we can suppose that $u \geq 0$. Let us set $A_k = \{2^k < u \leq 2^{k+1}\}$ and $u_k = \min\{0, \max\{u - 2^k, 2^k\}\}$. From formula (2.5) we deduce that

$$|u| \leq \sum_{\alpha=1}^{2n} |X_\alpha u| * |R_\alpha| \leq C|\nabla u| * K$$

where $K = \sum R_\alpha$. Then

$$|A_{k+1}| \leq |\{u_k > 2^{k-1}\}| \leq \left|\{|\nabla u_k| * K| > C^{-1}2^{k-1}\}\right| \leq$$

$$\leq C_1\left(\frac{1}{2^k}\int_{\mathbf{H}^n}|\nabla u_k|\right)^{\frac{Q}{Q-1}} \leq C_1\left(\frac{1}{2^k}\int_{A_k}|\nabla u|\right)^{\frac{Q}{Q-1}}$$

and thus

$$\int_{\mathbf{H}^n} u^{\frac{Q}{Q-1}} = \sum_{k\in\mathbf{Z}}\int_{A_k} u^{\frac{Q}{Q-1}} \leq \sum_{k\in\mathbf{Z}}(2^{k+1})^{\frac{Q}{Q-1}}|A_k| \leq C_2\sum_{k\in\mathbf{Z}}\left(\int_{A_k}|\nabla u|\right)^{\frac{Q}{Q-1}} \leq$$

$$\leq C_2\left(\sum_{k\in\mathbf{Z}}\int_{A_k}|\nabla u|\right)^{\frac{Q}{Q-1}} = C_2\left(\int_{\mathbf{H}^n}|\nabla u|\right)^{\frac{Q}{Q-1}}.$$

□

2.4 Contact form with constant Webster curvature in $\mathbf{H}^n$

Because of formula (1.11), the problem of finding pseudo-Hermitian structures with positive constant scalar curvature K is equivalent, up to some non essential multiplicative constant, to finding positive solutions to the equation

$$-\Delta_b u = \frac{Q-2}{2Q} K u^{\frac{Q+2}{Q-2}}.$$

Without loss of generality we can assume $K = \frac{2Q}{Q-2}$, so that we are lead to the equation

$$-\Delta_b u = u^{\frac{Q+2}{Q-2}}. \tag{2.6}$$

There is an obvious family of solutions. In fact we know, from Section 2.1, that there exists a CR isomorphism from the sphere minus a point $S^{2n+1} \setminus \{p\}$ to $\mathbf{H}^n$, given by the formula

$$F(z) = \left(\frac{z_1}{1+z_{n+1}}, \ldots, \frac{z_n}{1+z_{n+1}}, \mathfrak{Re} \left(\frac{1-z_{n+1}}{1+z_{n+1}} \right) \right).$$

F is known as Cayley transform.

On S^{2n+1} the form $\theta_{S^{2n+1}} = \frac{i}{2}(\partial - \overline{\partial})|z|^2$ has constant scalar curvature by symmetry, and we know that it is positive. So the form $(F^{-1})^* \theta_{S^{2n+1}}$ is a pseudo-Hermitian structure on $\mathbf{H}^n$ with constant positive scalar curvature, and computing its conformal factor with respect to the standard Heisenberg form, it can be found that a solution of equation (2.6) is

$$\omega(z,t) = \frac{c_0}{\left(t^2 + (1+|z|^2)^2 \right)^{(Q-2)/4}} \tag{2.7}$$

for some constant c_0.

From ω we can find other solutions through CR automorphisms: if $\Psi = \rho_P \circ \delta_\lambda \circ L_x \in \mathrm{Aut}_{\mathrm{CR}}(\mathbf{H}^n)$ then

$$\Psi^*(\omega^{\frac{4}{Q-2}}\theta) = (\rho_P \circ \delta_\lambda \circ L_x)^*(\omega^{\frac{4}{Q-2}}\theta) = (\omega^{\frac{4}{Q-2}} \circ \rho_P \circ \delta_\lambda \circ L_x) \cdot (\rho_P \circ \delta_\lambda \circ L_x)^*\theta =$$

$$= (\omega^{\frac{4}{Q-2}} \circ \delta_\lambda \circ L_x) \cdot \lambda^2 \theta = \left(\lambda^{\frac{Q-2}{2}} \omega \circ \delta_\lambda \circ L_x \right)^{\frac{4}{Q-2}} \theta,$$

so $\lambda^{\frac{Q-2}{2}} \omega \circ \delta_\lambda \circ L_x$ is a solution of Equation (2.6) too. Let us denote $\omega_{\lambda,x} = \lambda^{-\frac{Q-2}{2}} \omega \circ \delta_{1/\lambda} \circ L_{x^{-1}}$.

On $\mathbf{R}^n$ it is known that all metrics with constant scalar curvature conformal to the flat one are pull-backs, through the stereographic projection, of the standard metric on the sphere (see [CGS]). Jerison and Lee proved a similar result on the Heisenberg group, under some integrability hypotheses (see [JL2, Corollary 4.2]).

Theorem 2.18. *All positive solutions of Equation* (2.6) *on the Heisenberg group belonging to* $L^{\frac{2Q}{Q-2}}$ *are of the form* $\omega_{\lambda,x}$ *for some* $\lambda \in (0, \infty)$ *and* $x \in \mathbf{H}^n$.

We remark that since the volume form associated to $u^{\frac{4}{Q-2}}\theta$ is $u^{\frac{2Q}{Q-2}}\theta \wedge (d\theta)^n$, the hypothesis that $u \in L^{\frac{2Q}{Q-2}}$ is equivalent to say that it has finite volume. In particular this holds for pull-backs from metrics on S^{2n+1} through the Cayley transform. So we get the following.

Corollary 2.19.
- *Pseudo-Hermitian structures on* $\mathbf{H}^n$ *with its CR structure which have finite volume and constant scalar curvature are* $(F^{-1})^*\theta_{S^{2n+1}}$ *and its multiples and pull-backs through CR automorphisms.*
- *Pseudo-Hermitian structures on* S^{2n+1} *with its CR structure are pull-backs of* $\theta_{S^{2n+1}}$ *through CR automorphisms.*

The proof of the classification in $\mathbf{R}^n$ is through the method of moving planes. On $\mathbf{H}^n$ this method does not work because the group of pseudo-Hermitian isomorphisms is too small. The proof of Jerison and Lee was inspired by a proof of Obata of the classification of Riemannian metrics of constant scalar curvature on S^n conformal to the standard one (see [Ob, Proposition 6.1]). Obata's proof is based on the fact that if $\widetilde{g} = \varphi^{-2} g_{S^n}$ has constant scalar curvature, then the formula

$$\operatorname{div}(B^{ij}\varphi_{,i}\partial_j) = \varphi|B|^2$$

holds, where B is the traceless Ricci tensor. Integrating over S^n then leads to deduce that $B = 0$, and this permits to prove the theorem.

Jerison and Lee conjectured the existence of some formula that equates the divergence of a vector field with a sum of squares so that integration would lead to deduce the vanishing of the squares. Through a computer program, they found that in fact formulas of his kind exist, and they were able to use them to prove the classification theorem.

The hypothesis that $u \in L^{\frac{2Q}{Q-2}}$ is used, in the proof of Jerison and Lee, to prove that after the integration by parts the boundary term tends to zero at infinity. It is conjectured that, as in $\mathbf{R}^n$, this integrability hypothesis is not necessary, but this is still an open problem.

In [Wa], Jerison and Lee's formula was generalized to prove an Obata-like theorem on pseudo-Einstein manifolds.

Chapter 3

Periodic singular CR Yamabe structures

As seen in the introduction, in $\mathbf{R}^n \setminus \{0\}$ constant curvature metrics conformal to the Euclidean one are completely classified. The problem is equivalent to classify all positive solutions of the equation

$$-\Delta u = u^{\frac{2n}{n-2}}.$$

On $\mathbf{H}^n$ the analogous problem brings to the equation

$$-\Delta_b u = u^{\frac{2Q}{Q-2}}. \tag{3.1}$$

We want to find conformal metrics analogues to the Fowler solutions on $\mathbf{R}^n$, that is, such that there exist $T > 0$ such that δ_T is a pseudo-Hermitian isometry of the associated form $u^{\frac{4}{Q-2}}\theta$. This condition is equivalent to

$$u^{\frac{4}{Q-2}}\theta = (\delta_T)^*(u^{\frac{4}{Q-2}}\theta) = (u \circ \delta_T)^{\frac{4}{Q-2}} T^2 \theta$$

that is, $u \circ \delta_T = T^{-\frac{Q-2}{2}} u$.

Let $\Omega_T = \{1 \leq |x| \leq T\}$. The natural space in which study the problem is the Hilbert space

$$X_T = \left\{ u \in S^{1,2}_{\text{loc}}(\mathbf{H}^n) \middle| u \circ \delta_T = T^{-\frac{Q-2}{2}} u \right\}$$

with the product

$$\langle u, v \rangle = \int_{\Omega_T} \nabla u \cdot \nabla v.$$

Lemma 3.1.

$$\int_{\delta_\lambda(E)} |u|^{\frac{2Q}{Q-2}} = \int_E |\lambda^{\frac{Q-2}{2}} u \circ \delta_\lambda|^{\frac{2Q}{Q-2}}$$

and

$$\int_{\delta_\lambda(E)} |\nabla u|^2 = \int_E |\lambda^{\frac{Q-2}{2}} \nabla(u \circ \delta_\lambda)|^2$$

whwnever u is a function such that the terms in the formulas are defined.

If $u \in X_T$ then

$$\int_{\delta_\lambda \Omega_T} |u|^{\frac{2Q}{Q-2}} = \int_{\Omega_T} |u|^{\frac{2Q}{Q-2}}, \tag{3.2}$$

$$\int_{\delta_\lambda \Omega_T} \nabla u \cdot \nabla v = \int_{\Omega_T} \nabla u \cdot \nabla v. \tag{3.3}$$

Proof. The first two formulas follow a simple change of variables.

If $1 \leq \lambda \leq T$ then

$$\int_{\delta_\lambda \Omega_T} |u|^{\frac{2Q}{Q-2}} = \int_{\Omega_T \setminus \Omega_\lambda} |u|^{\frac{2Q}{Q-2}} + \int_{\Omega_{\lambda T} \setminus \Omega_T} |u|^{\frac{2Q}{Q-2}} =$$

$$= \int_{\Omega_T \setminus \Omega_\lambda} |u|^{\frac{2Q}{Q-2}} + \int_{\Omega_\lambda} |T^{\frac{Q-2}{2}} u \circ \delta_T|^{\frac{2Q}{Q-2}} = \int_{\Omega_T} |u|^{\frac{2Q}{Q-2}}.$$

From this by induction Formula (3.2) for $\lambda \geq 1$ is proved, and simply extended to all λ. Formula (3.3) is proved analogously. □

Lemma 3.2. *If $u.v \in X_T$ then*

$$\int_{\Omega_T} \nabla u \cdot \nabla v = -\int_{\Omega_T} \Delta_b u \cdot v.$$

Proof. Thanks to a partition of unity we can write $v = v_1 + v_2$ with $u_1, u_2 \in X_T$, $\operatorname{supp} u_1 \cap \Omega_T \subset \mathring{\Omega}_T$ and $\operatorname{supp} u_2 \cap \delta_{\sqrt{T}} \Omega_T \subset \delta_{\sqrt{T}} \mathring{\Omega}_T$.

Then, using formula (3.3),

$$\int_{\Omega_T} \nabla u \cdot \nabla v = \int_{\Omega_T} \nabla u \cdot \nabla v_1 + \int_{\Omega_T} \nabla u \cdot \nabla v_2 =$$

$$= -\int_{\Omega_T} \Delta_{\mathbf{H}^n} u \cdot v_1 + \int_{\delta_{\sqrt{T}} \Omega_T} \nabla u \cdot \nabla v_2 =$$

$$= -\int_{\Omega_T} \Delta_{\mathbf{H}^n} u \cdot v_1 - \int_{\delta_{\sqrt{T}} \Omega_T} \Delta_{\mathbf{H}^n} u \cdot v_2 =$$

$$= -\int_{\Omega_T} \Delta_{\mathbf{H}^n} u \cdot v_1 - \int_{\Omega_T} \Delta_{\mathbf{H}^n} u \cdot v_2 = -\int_{\Omega_T} \Delta_{\mathbf{H}^n} u \cdot v.$$

□

We want to give a variational formulation to Equation (3.1) in X_T. Let us define

$$\mathscr{J}_T(u) = \int_{\Omega_T} \left(|\nabla u|^2 - \frac{1}{2^*} |u|^{2^*} \right).$$

Since

$$d\mathscr{I}_T(u)[\varphi] = \int_{\Omega_T} \nabla u \cdot \nabla \varphi - u|u|^{2^*-2}\varphi$$

thanks to Lemma 3.2 critical points of $\mathscr{I}_T$ are solutions of Equation (3.1).

In the proof we will need to work in the closed subspace of X_T of the functions of the form $u(|z|,t)$, which we denote by $\widetilde{X}_T$. Functions on X_T are functions invariant by the action of $U(n)$ on $\mathbf{H}^n$, so, thanks to Palais' symmetric criticality principle (see [P]), critical points of $\mathscr{I}_T$ restricted to $\widetilde{X}_T$ are critical points of $\mathscr{I}_T$ on X_T.

The second differential of $\mathscr{I}_T$ is

$$d^2\mathscr{I}_T(u)[\varphi,\psi] = \int_{\Omega_T} \left(\nabla\varphi \cdot \nabla\psi - (2^*-1)|u|^{2^*-2}\varphi\psi\right).$$

We call $\mathscr{I}''_T$ the associated operator:

$$\langle \mathscr{I}''_T(u)[\varphi],\psi\rangle = d^2\mathscr{I}_T(u)[\varphi,\psi].$$

3.1 Estimate of the Sobolev constant on X_T

In order to carry out the estimates in the next Sections, we will need an explicit bound on the Sobolev constant on X_T.

Proposition 3.3. *If f is an L^p_{loc} function on $\mathbf{H}^n \setminus \{0\}$ such that $f \circ \delta_T = T^{-\alpha} f$ and $\alpha p = Q$ then*

$$\left(\frac{T^Q-1}{T^Q}\right)^{1/p} \|u\|_{L^{p,\infty}(\mathbf{H}^n)} \le C_2 \|u\|_{L^p(\Omega_T)} \le Q^{1/p}(\log T)^{1/p} \|u\|_{L^{p,\infty}(\mathbf{H}^n)}.$$

Proof. Let us call $f(\lambda) = \mu\{x \in \Omega_T \mid u(x) > \lambda\}$ and $g(t) = \mu\{x \in \mathbf{H}^n \mid u(x) > \lambda\}$. Then it holds that

$$g(\lambda) = \sum_{k\in\mathbf{Z}} T^{Qk} f(\lambda T^\alpha).$$

Therefore for every $\lambda > 0$, since f is decreasing,

$$\|u\|^p_{L^p(\Omega_T)} = p\int_0^\infty \xi^{p-1} f(\xi)d\xi = p\sum_{k\in\mathbf{Z}} \int_{\lambda T^{\alpha(k-1)}}^{\lambda T^{\alpha k}} \xi^{p-1} f(\xi)d\xi \ge$$

$$\ge p\sum_{k\in\mathbf{Z}} f(\lambda T^{\alpha k}) \int_{\lambda T^{\alpha(k-1)}}^{\lambda T^{\alpha k}} \xi^{p-1} d\xi = \sum_{k\in\mathbf{Z}} f(\lambda T^{\alpha k})\left((\lambda T^{\alpha k})^p - (\lambda T^{\alpha(k-1)})^p\right) =$$

$$= \frac{T^Q-1}{T^Q}\lambda^p \sum_{k\in\mathbf{Z}} T^{Qk} f(\lambda T^{\alpha k}) = \frac{T^Q-1}{T^Q}\lambda^p g(\lambda).$$

Taking the supremum with respect to λ we get the first inequality.

For the other one, let us pick an integer $N > 0$ and write

$$\|u\|^p_{L^p(\Omega_T)} = p\int_0^\infty \xi^{p-1} f(\xi)d\xi = p\sum_{k\in\mathbf{Z}}\int_{T^{\alpha k/N}}^{T^{\alpha(k+1)/N}} \xi^{p-1} f(\xi)d\xi \leq$$

$$\leq p\sum_{k\in\mathbf{Z}} f(T^{\alpha k/N})\int_{T^{\alpha k/N}}^{T^{\alpha(k+1)/N}} \xi^{p-1} d\xi =$$

$$= \sum_{m=1}^{N}\sum_{j\in\mathbf{Z}}(T^{\alpha p/N}-1)T^{\alpha pj}T^{\alpha pm/N} f(T^{\alpha j}T^{\alpha m/N}) =$$

$$= (T^{Q/N}-1)\sum_{m=1}^{N} T^{Qm/N}\sum_{j\in\mathbf{Z}} T^{Qj} f(T^{\alpha j}T^{\alpha m/N}) =$$

$$= (T^{Q/N}-1)\sum_{m=1}^{N} T^{Qm/N} g(T^{\alpha m/N}) \leq N(T^{Q/N}-1)\,\|u\|^p_{L^{p,\infty}}\,.$$

Taking the limit for $N \to \infty$ we get the second inequality. $\square$

Proposition 3.4. *If $u \in L^{2,\infty}(\mathbf{H}^n)$ is such that $\nabla u \in L^{2,\infty}(\mathbf{H}^n)$ then*

$$\|u\|_{L^{\frac{2Q}{Q-2},\infty}} \leq C\,\|\nabla u\|_{L^{2,\infty}}\,.$$

Proof. The proof is essentially the same as that one of (2.17), that is, proving that

$$u = \sum_{\alpha=1}^{2n}(X_\alpha u) * R_\alpha \tag{3.4}$$

and then applying Theorem 2.11. There is a technical complication, that is, in the proof Theorem 2.17, Formula (3.4) is proved for smooth compactly supported functions and then extended by density, but smooth compactly supported functions are not dense in weak L^p spaces.

To overcome the problem, let us define $E = u > 1$, $E^c = \mathbf{H}^n \setminus E$, $u_1 = u\chi_{E^c} + \chi_E$ and $u_2 = (u-1)\chi_E$, so that $u = u_1 + u_2$. It is standard to prove that u_1 and u_2 have weak sub-Riemannian gradient and that $\nabla u_1 = (\nabla u)\chi_{E^c}$, $\nabla u_2 = (\nabla u)\chi_E$. It is easy to prove that $u_1 \in S^{1,p}(\mathbf{H}^n)$ for $p > 2$ and that $u_2 \in S^{1,q}(\mathbf{H}^n)$ for $q < 2$. Therefore

$$u_k = \sum_{\alpha=1}^{2n}(X_\alpha u_k) * R_\alpha$$

for $k = 1, 2$, and by summing, Formula (3.4) is proved. $\square$

Combining the last two Propositions, we get a Sobolev inequality for X_T with an explicit constant.

Proposition 3.5. *There exist a constant C independent by T such that for every $u \in X_T$*

$$\|u\|_{L^{\frac{2Q}{Q-2}}(\Omega_T)} \leq C(\log T)^{\frac{Q-2}{2Q}} \left(\frac{T^Q}{T^Q-1}\right)^{1/2} \|u\|_{X_T}.$$

3.2 Construction of a family of approximate solutions

We define a family of approximate critical points of $\mathscr{J}_T$, with the intention to apply a perturbative method to find an exact critical point. The family is

$$\Psi_{\lambda,T} = \sum_{k\in\mathbf{Z}} \omega_{\lambda/T^k} = \sum_{k\in\mathbf{Z}} T^{\frac{Q-2}{2}k} \omega_\lambda \circ \delta_{T^k}$$

where $\omega_\lambda = \lambda^{-\frac{Q-2}{2}} \omega \circ \delta_{1/\lambda}$. Dependence on T will be omitted whenever not relevant.

We notice that ω verifies the elementary estimates

$$\omega(x) \leq \frac{C}{1+|x|^{Q-2}}, \quad |\nabla\omega(x)| \leq \frac{C}{1+|x|^{Q-1}}.$$

Lemma 3.6. *$\Psi_{\lambda,T}$ is well defined and belongs to X_T.*

Proof. The series defining $\Psi_{\lambda,T}$ converges uniformly on compact sets K, because, if $x \in K$

$$\Psi_{\lambda,T}(x) = \sum_{k\in\mathbf{Z}} T^{\frac{Q-2}{2}k} \omega_\lambda \circ \delta_{T^k} \leq C_{\lambda,K} \sum_{k\in\mathbf{Z}} T^{\frac{Q-2}{2}k} \frac{1}{1+T^{(Q-2)k}} \leq C'_{\lambda,K}.$$

Analogously

$$|\nabla\Psi_{\lambda,T}(x)| = \left| \sum_{k\in\mathbf{Z}} T^{\frac{Q-2}{2}k} T^k \left(\nabla\omega_\lambda \circ \delta_{T^k}\right)(x) \right| \leq$$

$$\leq C_{\lambda,K} \sum_{k\in\mathbf{Z}} T^{\frac{Q}{2}k} \frac{1}{1+T^{k(Q-1)}} \leq C'_{\lambda,K}$$

(where, we recall, $\nabla(u\circ\delta_\lambda) = \lambda(\nabla u)\circ\delta_\lambda$ because the sub-Riemannian gradient is a homogeneous operator of degree -1). So $\Psi_{\lambda,T}$ is of class $\mathscr{C}^1$. Since

$$\Psi_{\lambda,T} \circ \delta_T = \sum_{k\in\mathbf{Z}} T^{\frac{Q-2}{2}k} \omega_\lambda \circ \delta_{T^k} \circ \delta_T =$$

$$= T^{-\frac{Q-2}{2}} \sum_{k\in\mathbf{Z}} T^{\frac{Q-2}{2}(k+1)} \omega_\lambda \circ \delta_{T^{k+1}} = T^{-\frac{Q-2}{2}} \Psi_\lambda$$

then $\Psi_{\lambda,T} \in X_T$. $\square$

It holds that

$$\Psi_{T\lambda} = \sum_{k\in\mathbf{Z}} T^{\frac{Q-2}{2}k}\omega_{T\lambda}\circ\delta_{T^k} = \sum_{k\in\mathbf{Z}} T^{\frac{Q-2}{2}k}\frac{1}{T^{\frac{Q-2}{2}}}\omega_\lambda\circ\delta_{1/T}\circ\delta_{T^k} =$$

$$= \sum_{k\in\mathbf{Z}} T^{\frac{Q-2}{2}(k-1)}\omega_\lambda\circ\delta_{T^{k-1}} = \sum_{k\in\mathbf{Z}} T^{\frac{Q-2}{2}(k-1)}\omega_\lambda\circ\delta_{T^{k-1}} = \Psi_\lambda.$$

Therefore the set $\mathscr{Z}_T = \{\Psi_\lambda \mid \lambda\in(0,\infty)\}$ is a closed curve in X_T.

Moreover, using formula (2.4), it can be computed that

$$\frac{\partial\Psi_\lambda}{\partial\lambda} = \frac{\partial}{\partial\lambda}\sum_{k\in\mathbf{Z}}\omega_{\lambda/T^k} = \sum_{k\in\mathbf{Z}}\frac{\partial}{\partial\lambda}\left(\lambda^{-\frac{Q-2}{2}}\omega_{1/T^k}\circ\delta_{\lambda^{-1}}\right) =$$

$$= \sum_{k\in\mathbf{Z}}\left(-\frac{Q-2}{2}\frac{1}{\lambda}\omega_{\lambda/T^k} - \lambda^{-\frac{Q-2}{2}}\frac{1}{\lambda^2}\lambda Z(\omega_{1/T^k}\circ\delta_{\lambda^{-1}})\right) =$$

$$= \sum_{k\in\mathbf{Z}}\left(-\frac{Q-2}{2}\frac{1}{\lambda}\omega_{\lambda/T^k} - \frac{1}{\lambda}Z(\omega_{\lambda/T^k})\right) =$$

$$= -\frac{Q-2}{2}\frac{1}{\lambda}\Psi_\lambda - \frac{1}{\lambda}Z(\Psi_\lambda). \tag{3.5}$$

This implies that the curve $\mathscr{Z}_T$ is immersed for T big enough, because if $\frac{\partial\Psi_\lambda}{\partial\lambda}$ was zero then $Z(\Psi_\lambda) = -\frac{Q-2}{2}\Psi_\lambda$ would be zero, and by Proposition 2.8 Ψ_λ would be homogeneous of degree $-\frac{Q-2}{2}$; but it is clearly not by construction if T is big enough.

Now we want to prove that Φ_λ is an approximate critical point of $\mathscr{J}_T$ if T is big enough. So we compute the differential of $\mathscr{J}_T$ in Ψ_λ:

$$d\mathscr{J}_T(\Psi_\lambda)[u] = \int_{\Omega_T}\nabla\Psi_\lambda\cdot\nabla u - \Psi_\lambda^{2^*-1}u =$$

$$= \int_{\Omega_T}\sum_{k\in\mathbf{Z}}\nabla\omega_{\lambda/T^k}\cdot\nabla u - \left(\sum_{k\in\mathbf{Z}}\omega_{\lambda/T^k}\right)^{2^*-1}u =$$

$$= \sum_{k\in\mathbf{Z}}\left(\int_{\Omega_T}\nabla\omega_{\lambda/T^k}\cdot\nabla u - \omega_{\lambda/T^k}^{\frac{Q+2}{Q-2}}u\right) +$$

$$-\int_{\Omega_T}\left[\left(\sum_{k\in\mathbf{Z}}\omega_{\lambda/T^k}\right)^{\frac{Q+2}{Q-2}} - \sum_{k\in\mathbf{Z}}\omega_{\lambda/T^k}^{\frac{Q+2}{Q-2}}\right]u =$$

$$:= A + B. \tag{3.6}$$

Lemma 3.7. *In the above notation, $A = 0$.*

Proof. We have

$$A = \sum_{k\in\mathbf{Z}} \int_{\Omega_T} T^{\frac{Q-2}{2}k} (\nabla\omega_\lambda \circ \delta_{T^k}) \cdot \nabla u - (T^{\frac{Q-2}{2}k})^{\frac{Q+2}{Q-2}} (\omega_\lambda \circ \delta_{T^k})^{\frac{Q+2}{Q-2}} u =$$

$$= \sum_{k\in\mathbf{Z}} \int_{\Omega_T} T^{\frac{Q}{2}k} (\nabla\omega_\lambda) \circ \delta_{T^k} \cdot \nabla u - T^{\frac{Q+2}{2}k} (\omega_\lambda \circ \delta_{T^k})^{\frac{Q+2}{Q-2}} u =$$

$$= \sum_{k\in\mathbf{Z}} \int_{\delta_{T^k}(\Omega_T)} T^{-kQ} \left[T^{\frac{Q}{2}k} \nabla\omega_\lambda \cdot (\nabla u) \circ \delta_{T^{-k}} - T^{\frac{Q+2}{2}k} \omega_\lambda^{\frac{Q+2}{Q-2}} u \circ \delta_{T^{-k}} \right] =$$

$$= \sum_{k\in\mathbf{Z}} \int_{\delta_{T^k}(\Omega_T)} T^{-\frac{Q}{2}k} T^k \nabla\omega_\lambda \cdot \nabla(u \circ \delta_{T^{-k}}) - \omega_\lambda^{\frac{Q+2}{Q-2}} u =$$

$$= \sum_{k\in\mathbf{Z}} \int_{\delta_{T^k}(\Omega_T)} \nabla\omega_\lambda \cdot \nabla u - \omega_\lambda^{\frac{Q+2}{Q-2}} u = \int_{\mathbf{H}^n} \nabla\omega_\lambda \cdot \nabla u - \omega_\lambda^{\frac{Q+2}{Q-2}} u.$$

Let us pick a family of smooth functions $\varphi_{\varepsilon,R}$ such that $\varphi_{\varepsilon,R} \equiv 1$ on $B_R \setminus B_{2\varepsilon}$, $\varphi_{\varepsilon,R} \equiv 0$ on B_ε and $\mathbf{H}^n \setminus B_{R+1}$, $|\nabla\varphi_{\varepsilon,R}| \leq \frac{C}{\varepsilon}$ on $B_{2\varepsilon} \setminus B_\varepsilon$ and $|\nabla\varphi_{\varepsilon,R}| \leq C$ on $B_{R+1} \setminus B_R$. Then

$$A = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{\mathbf{H}^n} \left(\nabla\omega_\lambda \cdot \nabla u - \omega_\lambda^{\frac{Q+2}{Q-2}} u \right) \varphi_{\varepsilon,R} =$$

$$= \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{\mathbf{H}^n} -(\Delta_{\mathbf{H}^n}\omega_\lambda + \omega_\lambda^{\frac{Q+2}{Q-2}}) u\varphi_{\varepsilon,R} - u\nabla\omega_\lambda \cdot \nabla\varphi_{\varepsilon,R} =$$

$$= -\lim_{R\to\infty} \int_{B_{R+1}\setminus B_R} u\nabla\omega_\lambda \cdot \nabla\varphi_{\varepsilon,R} - \lim_{\varepsilon\to 0} \int_{B_{2\varepsilon}\setminus B_\varepsilon} u\nabla\omega_\lambda \cdot \nabla\varphi_{\varepsilon,R}.$$

If $x \to \infty$ then $\nabla\omega_\lambda \lesssim \frac{1}{|x|^{Q-1}}$ and $u \lesssim |x|^{-\frac{Q-2}{2}}$, and so the first limit is zero. If $x \to 0$ then $\nabla\omega_\lambda \lesssim 1$ and $u \lesssim |x|^{-\frac{Q-2}{2}}$, and so also the second limit is zero. Therefore $A = 0$. □

Lemma 3.8. *The term B in formula* (3.6) *verifies*

$$|B| \leq C \log T \left(\frac{1}{T}\right)^{\frac{Q(Q-2)}{2(Q+2)}} \|u\|_{X_T}.$$

Proof. For $x \in \Omega_T$, let $\lambda(x)$ be the number such that $\frac{|x|}{\lambda(x)} \in \left[\frac{1}{\sqrt{T}}, \sqrt{T}\right)$. Then

$$|B| \leq \int_{\Omega_T} \left[\left(\sum_{k\in\mathbf{Z}} \omega_{\lambda/T^k} \right)^{\frac{Q+2}{Q-2}} - \sum_{k\in\mathbf{Z}} \omega_{\lambda/T^k}^{\frac{Q+2}{Q-2}} \right] |u| \leq$$

$$\leq \int_{\Omega_T} \left[\left(\sum_{k\in\mathbf{Z}} \omega_{\lambda/T^k} \right)^{\frac{Q+2}{Q-2}} - \omega_{\lambda(x)}^{\frac{Q+2}{Q-2}} \right] |u| \leq$$

$$\leq \left\{ \int_{\Omega_T} \left[\left(\sum_{k \in \mathbf{Z}} \omega_{\lambda/T^k} \right)^{\frac{Q+2}{Q-2}} - \omega_{\lambda(x)}^{\frac{Q+2}{Q-2}} \right]^{\frac{2Q}{Q+2}} \right\}^{\frac{Q+2}{2Q}} \|u\|_{L^{\frac{2Q}{Q-2}}(\Omega_T)} \leq$$

$$\leq C(\log T)^{\frac{Q-2}{2Q}} \|u\|_{X_T} \left\{ \int_{\Omega_T} \left[\left(\sum_{k \in \mathbf{Z}} \omega_{\lambda/T^k} \right)^{\frac{Q+2}{Q-2}} - \omega_{\lambda(x)}^{\frac{Q+2}{Q-2}} \right]^{\frac{2Q}{Q+2}} \right\}^{\frac{Q+2}{2Q}} =$$

$$= C(\log T)^{\frac{Q-2}{2Q}} \|u\|_{X_T} \cdot$$

$$\cdot \left\{ \int_{\Omega_T} \left[\left(\sum_{k \in \mathbf{Z}} |x|^{\frac{Q-2}{2}} \omega_{\lambda/T^k} \right)^{\frac{Q+2}{Q-2}} - (|x|^{\frac{Q-2}{2}} \omega_{\lambda(x)})^{\frac{Q+2}{Q-2}} \right]^{\frac{2Q}{Q+2}} \frac{dx}{|x|^Q} \right\}^{\frac{Q+2}{2Q}}$$

by Proposition 3.5 (taking $T \geq T_0$ big, since all following estimates are valid for T big enough). Let us define $\eta_\lambda = |x|^{\frac{Q-2}{2}} \omega_\lambda$. Then

$$|B| \leq C(\log T)^{\frac{Q-2}{2Q}} \|u\|_{X_T} \left\{ \int_{\Omega_T} \left[\left(\sum_{k \in \mathbf{Z}} \eta_{\lambda/T^k} \right)^{\frac{Q+2}{Q-2}} - \eta_{\lambda(x)}^{\frac{Q+2}{Q-2}} \right]^{\frac{2Q}{Q+2}} \frac{dx}{|x|^Q} \right\}^{\frac{Q+2}{2Q}}.$$

η is bounded, and if $k \geq 0$ and T is large enough then $\eta_{\lambda(x)/T^k}$ satisfies estimates

$$|\eta_{\lambda(x)/T^k}(x)| \lesssim \left(\frac{\left(\frac{T^k}{\lambda}\right)|x|}{1 + \left(\frac{T^k}{\lambda}\right)^2 |x|^2} \right)^{\frac{Q-2}{2}} \lesssim \left(T^k \frac{|x|}{\lambda} \right)^{-\frac{Q-2}{2}} \leq \left(\frac{1}{T} \right)^{\left(k - \frac{1}{2}\right)\frac{Q-2}{2}}$$

and

$$|\eta_{\lambda(x)/T^{-k}}(x)| \lesssim \left(\frac{\left(\frac{T^{-k}}{\lambda}\right)|x|}{1 + \left(\frac{T^{-k}}{\lambda}\right)^2 |x|^2} \right)^{\frac{Q-2}{2}} \lesssim \left(\frac{1}{T^k} \frac{|x|}{\lambda} \right)^{\frac{Q-2}{2}} \leq \left(\frac{1}{T} \right)^{\left(k - \frac{1}{2}\right)\frac{Q-2}{2}}$$

uniformly in λ. It is elementary to verify that, for $\alpha, \beta \geq 1$ the function

$$\frac{[(x+y)^\alpha - x^\alpha]^\beta}{x^{(\alpha-1)\beta} y^\beta + y^{\alpha\beta}}$$

is bounded on $(0, \infty)^2$, and so there exist C such that

$$[(x+y)^\alpha - x^\alpha]^\beta \leq C(x^{(\alpha-1)\beta} y^\beta + y^{\alpha\beta})$$

for $x, y \geq 0$. Taking

$$x = \eta_\lambda, \quad y = \sum_{k \in \mathbf{Z} \setminus \{0\}} \eta_{\lambda(x)/T^k}, \quad \alpha = \frac{Q+2}{Q-2} \quad \text{and} \quad \beta = \frac{2Q}{Q+2}$$

one gets that

$$|B| \leq C(\log T)^{\frac{Q-2}{2Q}} \|u\|_{X_T} \left\{ \int_{\Omega_T} \left[\eta_{\lambda(x)}^{\frac{8Q}{(Q+2)(Q-2)}} \left(\sum_{k \in \mathbf{Z} \setminus \{0\}} \eta_{\lambda(x)/T^k} \right)^{\frac{2Q}{Q+2}} + \right.\right.$$

$$\left.\left. + \left(\sum_{k \in \mathbf{Z} \setminus \{0\}} \eta_{\lambda(x)/T^k} \right)^{\frac{2Q}{Q-2}} \right] \frac{dx}{|x|^Q} \right\}^{\frac{Q+2}{2Q}} \lesssim$$

$$\lesssim C(\log T)^{\frac{Q-2}{2Q}} \|u\|_{X_T} \left\{ \int_{\Omega_T} \left[\left(\frac{1}{T}\right)^{\frac{Q-2}{4} \cdot \frac{2Q}{Q+2}} + \left(\frac{1}{T}\right)^{\frac{Q-2}{4} \cdot \frac{2Q}{Q-2}} \right] \frac{dx}{|x|^Q} \right\}^{\frac{Q+2}{2Q}} \lesssim$$

$$\lesssim C(\log T)^{\frac{Q-2}{2Q}} \|u\|_{X_T} \left\{ \left(\frac{1}{T}\right)^{\frac{Q(Q-2)}{2(Q+2)}} \int_{\Omega_T} \frac{dx}{|x|^Q} \right\}^{\frac{Q+2}{2Q}} \lesssim$$

$$\lesssim C(\log T)^{\frac{Q-2}{2Q}} \|u\|_{X_T} \left\{ \left(\frac{1}{T}\right)^{\frac{Q(Q-2)}{2(Q+2)}} \log T \right\}^{\frac{Q+2}{2Q}} \longrightarrow 0$$

uniformly in λ. □

Putting the former lemmas together, we proved that the functions Ψ_λ are an approximate critical points of $\mathscr{J}_T$.

Proposition 3.9. *There exist T_0 and C, depending only on by n, such that if $T \geq T_0$ then $\|\nabla \mathscr{J}_T\| < C \log T \left(\frac{1}{T}\right)^{\frac{Q(Q-2)}{2(Q+2)}}$ on $\mathscr{Z}_T$.*

3.3 Non degeneracy of the second differential

Now we want to prove the nondegeneracy of $d^2 \mathscr{J}_T(\Psi_\lambda)$ on the orthogonal of the tangent to $\mathscr{Z}_T$ when restricted to $\widetilde{X}_T$ (which contains $\mathscr{Z}_T$). To prove this, we will use the characterization of the kernel of $d^2 \mathscr{J}(\omega)$, where

$$\mathscr{J}(u) = \int_{\mathbf{H}^n} \left(|\nabla u|^2 - \frac{1}{2^*} |u|^{2^*} \right),$$

operated in [MU, Lemma 5].

Theorem 3.10. *A function $u \in S^1(\mathbf{H}^n)$ is in the kernel of $\mathscr{J}''(\omega)$, or equivalently a solution of the following equation:*

$$-\Delta_{\mathbf{H}^n} u = (2^* - 1)\omega^{2^*-2} u \tag{3.7}$$

if and only if there exist coefficients $\mu, \nu_1, \ldots, \nu_{2n} \in \mathbf{R}$ such that

$$u = \mu \left.\frac{\partial \omega_\lambda}{\partial \lambda}\right|_{\lambda=1} + \sum_{i=0}^{2n} \nu_i X_i(\omega_\lambda)$$

(where $X_0 = T$ and $X_{n+k} = Y_k$). Furthermore $\mathscr{J}''(\omega)$ is an operator of the form $I + \mathscr{K}$ with $\mathscr{K}$ compact, and has Morse index one, with negative eigenspace generated by ω.

The fact that $\mathscr{J}''(\omega)$ has Morse index one is not explicitly stated in [MU], but follows from the proof of Lemma 5. The fact that ω is an eigenvector and has negative eigenvalue follows from the fact that it solves Equation (3.1).

Theorem 3.10 implies that there exists a constant C such that if $u \in S^1(\mathbf{H}^n)$ and

$$\int_{\mathbf{H}^n} \nabla u \cdot \nabla \frac{\partial \omega_\lambda}{\partial \lambda} = 0, \quad \int_{\mathbf{H}^n} \nabla u \cdot \nabla X_i(\omega_\lambda) = 0, \int_{\mathbf{H}^n} \nabla u \cdot \nabla \omega_\lambda = 0 \tag{3.8}$$

then

$$d^2 \mathscr{J}(\omega_\lambda)[u,u] \geq C \int_{\mathbf{H}^n} |\nabla u|^2. \tag{3.9}$$

Furthermore, under the same hypotheses, since $\mathscr{J}''$ is selfadjoint and ω_λ is an eigenfunction,

$$d^2 \mathscr{J}(\omega_\lambda)[\omega_\lambda, u] = 0 \tag{3.10}$$

and since the corresponding eigenvalue is negative,

$$d^2 \mathscr{J}(\omega_\lambda)[\omega_\lambda, \omega_\lambda] \leq -C \int_{\mathbf{H}^n} |\nabla \omega_\lambda|^2. \tag{3.11}$$

In the following we will need the Hardy inequalities for $S^{1,2}(\mathbf{H}^n)$ and for X_T.

Proposition 3.11. *For every $u \in S^{1,2}(\mathbf{H}^n)$*

$$\int_{\mathbf{H}^n} \frac{|u|^2}{|x|^2} \leq C \int_{\mathbf{H}^n} |\nabla u|^2.$$

For every $u \in X_T$

$$\int_{\Omega_T} \frac{|u|^2}{|x|^2} \leq C \log T \int_{\Omega_T} |\nabla u|^2$$

Proof. The first follows from the Young-O'Neil inequality 2.11, the second from Proposition 3.5 and Hölder's inequality. ☐

To carry out the estimates of this Section, it will be convenient to introduce on X_T the norm

$$\|u\|^2_{T,\mathfrak{H}} = \int_{\Omega_T} \left(|\nabla u|^2 + \left|\frac{u}{|x|}\right|^2 \right).$$

Thanks to Proposition 3.11,

$$\|u\|_{X_T} \leq \|u\|_{T,\mathfrak{H}} \leq C(\log T)^{1/2} \|u\|_{X_T}. \tag{3.12}$$

Our strategy to prove that $d^2 \mathscr{J}_T(\Psi_\lambda)$ is nondegenerate on X_T in the orthogonal of $\frac{\partial \Psi_\lambda}{\partial \lambda}$ is to use Formulas 3.9, 3.10 and 3.11 to prove that if $u \in \widetilde{X}_T$,

$$\int_{\Omega_T} \nabla u \cdot \nabla \frac{\partial \Psi_\lambda}{\partial \lambda} = 0 \tag{3.13}$$

and

$$\int_{\Omega_T} \nabla u \cdot \nabla \Psi_\lambda = 0, \tag{3.14}$$

then, given $\varepsilon > 0$, for T large

$$d^2 \mathscr{J}_T(\Psi_\lambda)[u,u] \geq C \int_{\Omega_T} |\nabla u|^2 + \left|\frac{u}{|x|}\right|^2,$$

$$d^2 \mathscr{J}_T(\Psi_\lambda)[\Psi_\lambda,\Psi_\lambda] \geq -C \int_{\Omega_T} |\nabla \Psi_\lambda|^2 + \left|\frac{\Psi_\lambda}{|x|}\right|^2$$

and

$$|d^2 \mathscr{J}_T(\Psi_\lambda)[\Psi_\lambda,u]| < \varepsilon \|\Psi_\lambda\|_{T,\mathfrak{H}} \|u\|_{T,\mathfrak{H}}.$$

The estimates

$$|\nabla \Psi_{\lambda,T}| \leq C \frac{1}{|x|^{\frac{Q}{2}}}, \quad \left|\nabla \frac{\partial \Psi_{\lambda,T}}{\partial \lambda}\right| \leq \frac{C}{\lambda} \frac{1}{|x|^{\frac{Q}{2}}}. \tag{3.15}$$

hold, the first because $\Psi_{\lambda,T} \in X_T$ and is of class $\mathscr{C}^1$, the second follows from formula (3.5).

Meanwhile by the computations in formula (3.5) follows that

$$\frac{\partial \omega_\lambda}{\partial \lambda} = -\frac{Q-2}{2} \frac{1}{\lambda} \omega_\lambda - \frac{1}{\lambda} Z(\omega_\lambda)$$

and from this it can be proved that $\frac{\partial \omega_\lambda}{\partial \lambda}$ satisfies the estimate

$$\left|\nabla \frac{\partial \omega_\lambda}{\partial \lambda}\right| \leq \frac{C}{\lambda} \frac{1}{|x|^{\frac{Q}{2}}}. \tag{3.16}$$

Now let $u \in \widetilde{X}_T$ be fixed. Let $W = (B_{2T} \setminus B_T) \cup (B_1 \setminus B_{1/2})$. The quantity

$$\lambda^Q \int_W \left(|\nabla (u \circ \delta_\lambda)|^2 + \left|\frac{u}{|x|} \circ \delta_\lambda\right|^2 \right)$$

is continuous with respect to λ and periodic with respect to $\log\lambda$, so we can suppose that it is minimal for $\lambda = 1$. Since in $\Omega_T \cup \delta_T\Omega_T$ there are $\sim \log T$ mutually disjoint sets of the form $\delta_\lambda W$, there holds the inequality

$$\int_W \left(|\nabla u|^2 + \left|\frac{u}{|x|}\right|^2\right) \leq \frac{C}{\log T}\int_{\Omega_T}\left(|\nabla u|^2 + \left|\frac{u}{|x|}\right|^2\right) = \frac{C}{\log T}\,\|u\|^2_{T,\mathfrak{H}}\,. \quad (3.17)$$

Let us take a radial function $\rho = \rho(|x|)$ such that $\rho = 1$ on Ω_T, $\rho = 0$ on $B_{1/2} \cup (\mathbf{H}^n \setminus B_{2T}$, $0 \leq \rho \leq 1$, $|\nabla\rho| \leq C$ on $B_1 \setminus B_{1/2}$, $|\nabla\rho| \leq C/T$ on $B_{2T} \setminus B_T$. In particular $|\nabla\rho(x)| \leq C/|x|$ everywhere.

Lemma 3.12. *If ρ is a cut-off function as above, there exists T_0 such that for $T \geq T_0$ if* (3.13) *and* (3.14) *hold then for some C*

$$\left|\int_{\mathbf{H}^n}\nabla(\rho u)\nabla\frac{\partial\Psi_{T,\lambda}}{\partial\lambda}\right| \leq \frac{C}{(\log T)^{1/2}}\frac{1}{\lambda}\,\|u\|_{T,\mathfrak{H}}$$

and

$$\left|\int_{\mathbf{H}^n}\nabla(\rho u)\nabla\Psi_{T,\lambda}\right| \leq \frac{C}{(\log T)^{1/2}}\,\|u\|_{T,\mathfrak{H}}\,.$$

Proof.

$$\int_{\mathbf{H}^n}\nabla(\rho u)\nabla\frac{\partial\Psi_\lambda}{\partial\lambda} = \int_{\mathbf{H}^n}\nabla(\rho u)\nabla\frac{\partial\Psi_\lambda}{\partial\lambda} - \int_{\Omega_T}\nabla u\cdot\nabla\frac{\partial\Psi_\lambda}{\partial\lambda} =$$

$$= \int_W\left[(\rho\nabla u + u\nabla\rho)\nabla\frac{\partial\Psi_\lambda}{\partial\lambda}\right]$$

so, thanks to formulas (3.15) and (3.17),

$$\left|\int_{\mathbf{H}^n}\nabla(\rho u)\nabla\frac{\partial\Psi_\lambda}{\partial\lambda}\right| \leq \frac{C}{\lambda}\left(\int_W(\rho\nabla u + u\nabla\rho)^2\right)^{1/2}\left(\int_W\frac{1}{|x|^Q}\right)^{1/2} \leq$$

$$\leq \frac{C}{\lambda}\left(\int_W\left(|\nabla u|^2 + \left|\frac{u}{|x|}\right|^2\right)\right)^{1/2} \leq \frac{C}{(\log T)^{1/2}}\frac{1}{\lambda}\,\|u\|_{T,\mathfrak{H}}\,.$$

The proof of the second estimate is identical. $\square$

Lemma 3.13. *For every ε there exists T_0 such that for $T \geq T_0$ if* (3.13) *and* (3.14) *hold then*

$$\left|\int_{\mathbf{H}^n}\nabla(\rho u)\nabla\frac{\partial\omega_\lambda}{\partial\lambda}\right| \leq \frac{C}{(\log T)^{1/2}}\frac{1}{\lambda}\,\|u\|_{T,\mathfrak{H}}$$

and

$$\left|\int_{\mathbf{H}^n}\nabla(\rho u)\nabla\omega_\lambda\right| \leq \frac{C}{(\log T)^{1/2}}\,\|u\|_{T,\mathfrak{H}}\,.$$

Proof. Applying Lemma 3.12 we can estimate

$$\left|\int_{\mathbf{H}^n} \nabla(\rho u)\nabla\frac{\partial\omega_\lambda}{\partial\lambda}\right| \leq \left|\int_{\mathbf{H}^n} \nabla(\rho u)\nabla\frac{\partial\Psi_{T,\lambda}}{\partial\lambda}\right| + \left|\int_{\mathbf{H}^n} \nabla(\rho u)\left(\nabla\frac{\partial\omega_\lambda}{\partial\lambda} - \frac{\partial\Psi_{T,\lambda}}{\partial\lambda}\right)\right| \leq$$

$$\leq \frac{C}{(\log T)^{1/2}}\frac{1}{\lambda}\|u\|_{T,\mathfrak{H}} + C\|u\|_{T,\mathfrak{H}}\left(\int_{\Omega_T\cup W}\left|\nabla\frac{\partial\Psi_\lambda}{\partial\lambda} - \nabla\frac{\partial\omega_\lambda}{\partial\lambda}\right|^2\right)^{1/2}.$$

Using Formula (3.16) the term

$$\left(\int_{\Omega_T\cup W}\left|\nabla\frac{\partial\Psi_\lambda}{\partial\lambda} - \nabla\frac{\partial\omega_\lambda}{\partial\lambda}\right|^2\right)^{1/2}$$

can be estimated in an essentially identical way as in the proof of Lemma 3.8, getting the thesis.

The proof of the second inequality estimate is identical. □

Lemma 3.14. *There exist constants T_0 and C such that for $T \geq T_0$ if* (3.13) *and* (3.14) *hold then*

$$|d^2\mathscr{J}(\omega_\lambda)[\rho u, \rho u]| \geq C\int_{\mathbf{H}^n}|\nabla(\rho u)|^2 + \left|\frac{\rho u}{|x|}\right|^2,$$

and

$$|d^2\mathscr{J}(\omega_\lambda)[\omega_\lambda, \rho u]| \leq \frac{C}{(\log T)^{1/2}}\|u\|_{T,\mathfrak{H}}.$$

Proof. Since $u \in \widetilde{X}_T$, $u\rho$ is invariant with respect to the symmetry $(x,t) \mapsto (-x,t)$, one has

$$\int_{\mathbf{H}^n}\nabla(\rho u)\cdot\nabla T_i(\omega_\lambda) = 0.$$

The claim follows by Lemma 3.13, by equations (3.9) and (3.10), and elementary Hilbert space theory. □

Lemma 3.15. *There exist constants T_0 and C such that for $T \geq T_0$ if conditions* (3.13) *and* (3.14) *hold, then*

$$|d^2\mathscr{J}_T(\Psi_\lambda)[u,u]| \geq C\|u\|^2_{T,\mathfrak{H}},$$

$$d^2\mathscr{J}_T(\Psi_\lambda)[\Psi_\lambda, \Psi_\lambda] \leq -C\|\Psi_\lambda\|^2_{T,\mathfrak{H}}$$

and

$$|d^2\mathscr{J}_T(\Psi_\lambda)[\Psi_\lambda, u]| \leq \frac{C}{(\log T)^{1/2}}\|\Psi_\lambda\|_{T,\mathfrak{H}}\|u\|_{T,\mathfrak{H}}.$$

Proof. By direct computation we find

$$
\begin{aligned}
&\left|d^2\mathscr{J}(\omega_\lambda)[\rho u,\rho u]-d^2\mathscr{J}_T(\Psi_\lambda)[u,u]\right| = \\
&=\left|\int_{\mathbf{H}^n}|\nabla(\rho u)|^2-(2^*-1)|\omega_\lambda|^{2^*-2}\rho^2u^2+\right. \\
&\left.-\int_{\Omega_T}|\nabla u|^2-(2^*-1)|\Psi_\lambda|^{2^*-2}u^2\right|\leq \\
&\leq(2^*-1)\left|\int_{\Omega_T}\left(|\Psi_\lambda|^{2^*-2}-|\omega_\lambda|^{2^*-2}\right)u^2\right|+ \\
&+(2^*-1)\left|\int_W|\omega_\lambda|^{2^*-2}\rho^2u^2\right|+2\left|\int_W(u^2|\nabla\rho|^2+\rho^2|\nabla u|^2)\right|.
\end{aligned}
$$

The first term can be estimated as in Lemma 3.8, the second in a standard way, and the third has been essentially already estimated, to prove that for every ε there exists T big enough to ensure that the whole sum is bounded by $\varepsilon\,\|u\|^2_{T,\mathfrak{H}}$.

Analogously

$$
\left|\int_{\mathbf{H}^n}\left(|\nabla(\rho u)|^2+\left|\frac{\rho u}{|x|}\right|^2\right)-\int_{\Omega_T}\left(|\nabla u|^2+\left|\frac{u}{|x|}\right|^2\right|^2\right)\leq\varepsilon\,\|u\|^2_{T,\mathfrak{H}}.
$$

This and Lemma 3.14 imply the first part of the thesis. The other statements are deduced in an analogous manner. ☐

Proposition 3.16. *There exist constants T_0 and C such that for $T\geq T_0$ the operator $\mathscr{J}''_T(\Psi_\lambda)$ is invertible on the orthogonal space of $\frac{\partial\Psi_\lambda}{\partial\lambda}$ in X_T, and $\left\|\mathscr{J}''_T(\Psi_\lambda)^{-1}\right\|_{\mathscr{L}(X_T)}\leq C\log T$.*

Proof. It follows from Lemma 3.15 and basic Hilbert space theory that $\mathscr{J}''_T(\Psi_\lambda)$ is invertible on the orthogonal space of $\frac{\partial\Psi_\lambda}{\partial\lambda}$ in X_T, with norm uniformly bounded in T with respect to operator norm associated to $\|\cdot\|_{T,\mathfrak{H}}$. This and Formula 3.3 imply the thesis. ☐

3.4 Existence of solutions

We have proved that, for T big enough, on the orthogonal in $\widetilde{X}_T$ of the tangent of the curve $\mathscr{Z}_T$ the second differential of $\mathscr{J}_T$ is non degenerate. Let us call W this orthogonal in the point $\Psi_\lambda\in\mathscr{Z}$ and π the orthogonal projection on W. We remember that our aim is to solve $\nabla\mathscr{J}_T(u)=0$. Following the standard reasoning in [AM, Section 2.2] we note that this is equivalent to solve

$$\pi\nabla\mathscr{J}_T(\Psi_\lambda+w)=0$$

(auxiliary equation) and

$$(I-\pi)\nabla\mathscr{J}_T(\Psi_\lambda+w)=0$$

(bifurcation equation) with $w\in W$.

Lemma 3.17. *There exists T_0 such that the auxiliary equation has a unique solution $w_T(\lambda)$; furthermore $\sup_\lambda \|w_T(\lambda)\| \to 0$ for $T \to \infty$.*

Proof. Write

$$\nabla \mathscr{J}_T(\Psi_\lambda + w) = \nabla \mathscr{J}_T(\Psi_\lambda) + \mathscr{J}_T''[w] + R(\Psi_\lambda, w)$$

with $R(\Psi_\lambda, w) = o(\|w\|)$ and $R(\Psi_\lambda, w) - R(\Psi_\lambda, v) = o(\|w - v\|)$, so that the auxiliary equation becomes

$$\pi\nabla \mathscr{J}_T(\Psi_\lambda) + \pi \mathscr{J}_T''(\Psi_\lambda)[w] + \pi R(\Psi_\lambda, w) = 0,$$

namely

$$w = -(\pi \mathscr{J}_T''(\Psi_\lambda))^{-1} \left[\pi\nabla \mathscr{J}_T(\Psi_\lambda) + \pi R(\Psi_\lambda, w)\right] := N_\lambda(w).$$

By Propositions 3.9 and 3.16, N is a contraction if T is big enough, and so the auxiliary equation has a unique solution $w = w_T(\lambda)$. Furthermore for every $r > 0$ there exists T big enough such that $B_r(\Psi_\lambda) \cap W$ is mapped into itself by N. So $\sup_\lambda \|w_T(\lambda)\|$ tends to zero for $T \to \infty$. □

Finally we can prove that the desired solutions exist.

Theorem 3.18. *There exists T_0 such that for $T \geq T_0$ there exists a positive solution of the equation*

$$-\Delta_b u = u^{\frac{Q+2}{Q-2}}$$

on $\mathbf{H}^n \setminus \{0\}$ such that $u \circ \delta_T = T^{-\frac{Q-2}{2}} u$, and T is the smallest period.

Proof. Let us consider the function

$$\Phi(\lambda) = \mathscr{J}_T(\Psi_\lambda + w(\lambda)).$$

Φ is continuous and periodic, so it has stationary point λ_0. Following the standard argument of Theorem 2.12 and Remark 2.14 in [AM], with the need for only formal modifications, the fact that $\Phi'(\lambda_0) = \mathscr{J}_T'(\Psi_{\lambda_0} + w(\lambda_0)) \cdot (\frac{\partial \Psi_{\lambda_0}}{\partial \lambda} + w'(\lambda_0))$ implies $u = \Psi_{\lambda_0} + w(\lambda_0)$ to solve the bifurcation equation, and so to be a stationary point of $\mathscr{J}_T$.

The smoothness of the solution can be proved with the same method of Appendix B in [S].

Also $\lambda^{(2-Q)/2} u \circ \delta_{\lambda^{-1}}$ is a critical point of $\mathscr{J}_T$, and by the unicity in the fixed point theorem it must be equal to $\Psi_{\lambda_0\lambda} + w(\lambda_0\lambda)$, and so the whole curve $\widetilde{\mathscr{Z}}_T = \{\Psi_\lambda + w(\lambda)\}$ consists of critical points of $\mathscr{J}$.

To prove the positivity, let us notice that from the proof of Proposition 3.16 follows that $\mathscr{J}(\omega_\lambda)$ has Morse index one on $\left\{\lambda \frac{\partial \omega_\lambda}{\partial \lambda}\right\}^\perp$. By continuity, the same holds for the orthogonal to the tangent space to $\widetilde{\mathscr{Z}}_T$. Since $d\mathscr{J}_T$ is zero on $\widetilde{\mathscr{Z}}_T$, the tangent of $\widetilde{\mathscr{Z}}_T$ is in the kernel of $\mathscr{J}_T''$. So the Morse index of $\mathscr{J}_T$ on $\widetilde{X}_T$ is one.

By a slight adaptation of the proof of Proposition 3.2 in [BCD] the set $\{u \neq 0\}$ has at most one connected component modulo δ_T, and so u does not change sign. By construction it is evident that it must be weakly positive (and even if it was not, it would be enough to change sign). The strict positivity follows from Bony's maximum principle (see [Bo]).

The last assertion follows by construction. $\square$

Chapter 4

Cylindrical and nearly cylindrical singular solutions

This chapter is devoted to prove the existence of a pseudo-Hermitian structure on $\mathbf{H}^n \setminus \{0\}$ conformal to the standard one and such that the dilations δ_λ are isometries. This is equivalent to find solutions of the equation

$$-\Delta_b u = u^{\frac{Q+2}{Q-2}} \tag{4.1}$$

such that $u \circ \delta_\lambda = \lambda^{\frac{Q-2}{2}} u$.

The above result is proved by posing the problem in a variational form, and then performing a conformal change that trasforms $\mathbf{H}^n \setminus \{0\}$ in a pseudohermitian cylinder, and imposing symmetries in order to reduce the problem to an ODE with variational structure.

The main difficulty is that, because of the non compactness of $\mathbf{H}^n \setminus \{0\}$, the problem has to be formulated on a closed annulus $\{1 \leq |x| \leq r\}$ (where $|\cdot|$ is the homogeneous norm), and so one has to put boundary conditions that, under a conformal change, behave in a treatable way. It is known that the mean curvature behaves in such a way, indeed the prescription of the mean curvature of the boundary is considered the most natural boundary condition in the prescribed curvature problem for manifolds with boundary (see, for example, [E]). In our case there is not such a concept, except in dimension three (see [CHMY]). So we introduce, in arbitrary dimension, the notion of canonical pseudohermitian normal curvature. In such a way we can formulate variationally the problem of the prescription of the Webster curvature with boundary conditions, with a functional that is conformally invariant.

In this section we will need the following formulas for conformal changes of pseudo-Hermitian metric from [L1].

Proposition 4.1. *Under the conformal change $\theta \mapsto \widetilde{\theta} = e^{2f}\theta$, the connection forms trasform as*

$$\widetilde{\omega}_\alpha^\beta = \omega_\alpha^\beta + 2(f_\beta \theta^\alpha - f_\alpha \theta^\beta) + \delta_\alpha^\beta (f_\gamma \theta^\gamma - f^\gamma \theta_\gamma) + F \cdot \theta$$

(where F is a function of f explicitly known, but whose expression is irrelevant for our purposes).

4.1 The canonical pseudohermitian normal curvature

Let M be a pseudo-Hermitian manifold, and let Σ be a two-sided hypersurface in M such that, calling $V = T\Sigma \cap H(M)$, $\dim V = 2n-1$ at every point. If N is a normal vector field to Σ with respect to g_θ, the normalization of his orthogonal projection on $H(M)$, ν, is normal to V. Equivalently, ν is (one of the two) unit length vector orthogonal to V. Let $\xi = -J\nu \in V$. This is a canonical direction (given an orientation on Σ). So we define the canonical pseudohermitian normal curvature of Σ as

$$\kappa = g_\theta(\nabla_\xi \xi, \nu).$$

Proposition 4.2. *Under the conformal change $\theta \mapsto \widetilde{\theta} = u^{2/n}\theta$, κ the canonical pseudohermitian normal curvature of the new pseudohermitian metric is given by the formula*

$$\kappa u - \frac{3}{n}\nu(u) = u^{1+\frac{1}{n}}\widetilde{\kappa}.$$

Proof. Since $\xi \in T^{(1,0)}M$ and is an unit vector with respect to g_θ, there exists an orthonormal frame $Z_1, \dots, Z_n$ for $T^{(1,0)}M$ such that $Z_1 + \overline{Z}_1 = \sqrt{2}\xi$. Then $\nu = \frac{i}{\sqrt{2}}(Z_1 - \overline{Z}_1)$ Because of Theorem 1.19, $\omega_{\overline{1}}^{\overline{1}} = -\omega_1^1$, and so

$$\kappa = g_\theta(\nabla_\xi \xi, \nu) = \frac{1}{2} g_\theta(\nabla_\xi Z_1 + \nabla_\xi \overline{Z}_1, i(Z_1 - \overline{Z}_1)) =$$

$$= -\frac{1}{2} d\theta(\omega_1^\alpha(\xi) Z_\alpha + \omega_{\overline{1}}^{\overline{\alpha}}(\xi) Z_{\overline{\alpha}}, Z_1 + \overline{Z}_1) =$$

$$= -\frac{i}{2}\omega_1^1(\xi) L_\theta(Z_1, \overline{Z}_1) - \frac{i}{2}\omega_1^1(\xi) L_\theta(Z_1, \overline{Z}_1) = -i\omega_1^1(\xi).$$

Applying Proposition 4.1, since $h_{1\overline{1}} = 1$ we obtain that

$$\widetilde{\omega}_1^1 = \omega_1^1 + 2(f_1\theta^1 - f_1\theta^1) + \delta_1^1(f_1\theta^1 - f^1\theta_1) + F \cdot \theta =$$

$$= \omega_1^1 + 3(Z_1 f \theta^1 - \overline{Z}_1 f \overline{\theta}^1) \qquad \mod \theta.$$

Considering that after the conformal change the Levi form is multiplied by e^{2f}, and so the canonical tangent vector becomes $\widetilde{\xi} = e^{-f}\xi$, we obtain that

$$\widetilde{\kappa} = -i\widetilde{\omega}_1^1(\widetilde{\xi}) = -ie^{-f}(\omega_1^1 + 3(Z_1 f\theta^1 - \overline{Z}_1 f\overline{\theta}^1))(\xi) =$$

$$= e^{-f}\kappa - \frac{3i}{\sqrt{2}}(Z_1 f\theta^1 - \overline{Z}_1 f\overline{\theta}^1))(Z_1 + \overline{Z}_1) = e^{-f}\kappa - \frac{3i}{\sqrt{2}}(Z_1 - \overline{Z}_1)f =$$

$$= e^{-f}\kappa - 3\nu(f).$$

□

Now let us pick a local orthonormal frame $Z_1, \dots, Z_n$ of $T^{(1,0)}M$ such that $\xi = \frac{1}{\sqrt{2}}(Z_n + \overline{Z}_n)$ and $\nu = \frac{i}{\sqrt{2}}(Z_n - \overline{Z}_n)$. Then if $e_{2\alpha-1} = \frac{1}{\sqrt{2}}(Z_\alpha + \overline{Z}_\alpha)$ and $e_{2\alpha} = \frac{i}{\sqrt{2}}(Z_\alpha - \overline{Z}_\alpha)$, $e_1, \dots, e_{2n}$ is an orthononormal frame with respect to G_θ. Let $e^1, \dots, e^{2n}$ be the associated coframe. Then $\theta^\alpha = \frac{1}{\sqrt{2}}(e^{2\alpha-1} + ie^{2\alpha})$.

So, by definition of the Levi form, we have

$$d\theta = i\sum_{\alpha=1}^{n} \theta^\alpha \wedge \theta^{\overline{\alpha}} = \frac{i}{2}\sum_{\alpha=1}^{n}(e^{2\alpha-1} + ie^{2\alpha}) \wedge (e^{2\alpha-1} - ie^{2\alpha}) = \sum_{\alpha=1}^{n} e^{2\alpha-1} \wedge e^{2\alpha},$$

therefore

$$\theta \wedge (d\theta)^n = \theta \wedge \left(\sum_{\alpha=1}^{n} e^{2\alpha-1} \wedge e^{2\alpha}\right)^n = n!\theta \wedge e^1 \wedge \dots \wedge e^n = n! \operatorname{vol}_{g_\theta}.$$

Inspired by Riemannian geometry, we want to give a variational formulation to the problem of the prescription of the Webster curvature and the prescription of the canonical pseudohermitian normal curvature on the boundary.

Proposition 4.3. *The functional*

$$Q(v) = \int_M (b_n|\nabla v|^2 + Wv^2)\theta \wedge (d\theta)^n - c_n \int_{\partial M} \kappa v^2 \sigma \wedge \theta,$$

where $c_n = n\frac{b_n}{3}n!$ *and* $\sigma = e^1 \wedge e^2 \wedge \dots \wedge e^{2n-1}$, *is invariant by the transformation*

$$\theta \mapsto \widetilde{\theta} = u^{2/n}\theta, \quad v \mapsto \widetilde{v} = vu^{-1}.$$

Proof. Under this conformal change $G_\theta \mapsto u^{2/n}G_\theta$, and so $\widetilde{\nabla} = u^{-2/n}\nabla$. Therefore

$$\int_M |\widetilde{\nabla}\widetilde{v}|^2 \widetilde{\theta} \wedge (d\widetilde{\theta})^n = \int_M u^{-2/n}|\nabla(u^{-1}v)|^2 u^{2(n+1)/n}\theta \wedge (d\theta)^n =$$

$$= u^2 \int_M |u^{-1}\nabla v - u^{-2}v\nabla u|^2 \theta \wedge (d\theta)^n =$$

$$= \int_M \left(|\nabla v|^2 + u^{-2}v^2|\nabla u|^2 - 2u^{-1}v\nabla u \cdot \nabla v\right)\theta \wedge (d\theta)^n =$$

$$= \int_M |\nabla v|^2 + \int_M \left(v^2|\nabla \log u|^2 - \nabla \log u \cdot \nabla(v^2)\right)\theta \wedge (d\theta)^n =$$

$$= \int_M |\nabla v|^2 + \int_M v^2\left(|\nabla \log u|^2 + \Delta_b \log u\right)\theta \wedge (d\theta)^n - 2^n n! \int_{\partial M} v^2 g_\theta(\nabla \log u, \xi)\mathscr{V},$$

where $\mathscr{V}$ is the volume form associated to the restriction of g_θ. It is easy to verify that $\mathscr{V} = \sigma \wedge \theta$, and that for every X in $H(M)$, $g_\theta(\xi, X) = e^{2n}(X)$. So

$$\int_M |\widetilde{\nabla}\widetilde{v}|^2 \widetilde{\theta} \wedge (d\widetilde{\theta})^n =$$

$$= \int_M |\nabla v|^2 + \int_M v^2 \left(|\nabla \log u|^2 + \Delta_b \log u \right) \theta \wedge (d\theta)^n - 2^n n! \int_{\partial M} v^2 \nu(\log u) \sigma \wedge \theta.$$

Thanks to the conformal change formula,

$$\int_M \widetilde{W}\widetilde{v}^2 \widetilde{\theta} \wedge (d\widetilde{\theta})^n = \int_M (-b_n u^{-1-2/n} \Delta_b u + W u^{-2/n}) v^2 u^{-2} u^{2+2/n} \theta \wedge (d\theta)^n =$$

$$= \int_M (-b_n u^{-1} \Delta_b u + W) v^2 \theta \wedge (d\theta)^n.$$

It holds that

$$\Delta_b \log u = \operatorname{div}(\nabla \log u) = \operatorname{div}\left(\frac{\nabla u}{u}\right) = \frac{\Delta_b u}{u} - \frac{|\nabla u|^2}{u^2} = \frac{\Delta_b u}{u} - |\nabla \log u|^2,$$

and so

$$\int_M \widetilde{W}\widetilde{v}^2 \widetilde{\theta} \wedge (d\widetilde{\theta})^n = \int_M W v^2 \theta \wedge (d\theta)^n - b_n \int_M (\Delta_b \log u + |\nabla \log u|^2) v^2 \theta \wedge (d\theta)^n.$$

Finally

$$\int_{\partial M} \widetilde{\kappa}\widetilde{v}^2 \widetilde{\sigma} \wedge \widetilde{\theta} = \int_{\partial M} \left(u^{-1/n}\kappa - \frac{3}{n} u^{-1-1/n} \nu(u) \right) v^2 u^{-2} u^{2+1/n} \sigma \wedge \theta =$$

$$= \int_{\partial M} \kappa v^2 \sigma \wedge \theta - \frac{3}{n} \int_{\partial M} \nu(\log u) v^2 \sigma \wedge \theta.$$

By summing the above identities we get the desired result. □

Proposition 4.4. *A conformal change has Webster curvature W_1 and canonical pseudohermitian normal curvature κ_1 if and only if it is a stationary point of the functional*

$$I_{W_1,\kappa_1}(v) = Q(v) - \frac{n}{n+1} \int_M W_1 v^{2+2/n} \theta \wedge (d\theta)^n + \frac{n c_n}{2n+1} \int_{\partial M} \kappa_1 v^{2+1/n} \sigma \wedge \theta,$$

that is invariant for the same transformation of Proposition 4.3.

4.2 Existence of a homogeneous solution

Now that we have a variational and conformally covariant formulation of the problem of prescribed curvature with boundary conditions, thanks to Proposition 4.4, we study this problem on suitable annuli, imposing that the boundary has zero curvature, a natural condition because of the simmetry given by the Cayley transform. So let us study the problem

$$\begin{cases} -b_n \Delta_b u = u^{1+2/n} & \text{on} \quad A_r \\ -\frac{3}{n}\nu(u) + \kappa_{A_r} u = 0 & \text{on} \quad \partial A_r \end{cases}$$

where $A_r = B_r \setminus \overline{B}_1$, and $B_r = B_r(0)$. The latter problem is equivalent to find the critical points of

$$I(v) = b_n \int_M |\nabla v|^2 \theta \wedge (d\theta)^n - c_n \int_{\partial M} \kappa v^2 \sigma \wedge \theta - \frac{n}{n+1} \int_M v^{2+2/n} \theta \wedge (d\theta)^n.$$

Let us consider the conformal change

$$\theta \mapsto \widetilde{\theta} = \rho^{-2}\theta,$$

where $\rho = |x|$.

Lemma 4.5. *The Webster curvature of* $\widetilde{\theta}$ *is*

$$\widetilde{W} = -\frac{b_n}{4} u^{-1-\frac{2}{n}} \Delta_b u = -b_n \rho^{n+2} \Delta_b(\rho^{-n}) = \frac{b_n}{4} n^2 \frac{|x|^2}{\rho^2},$$

and the mean curvature of the boundary of A_r *is zero.*

Proof. We have

$$X_\alpha(\rho^4) = \left(\frac{\partial}{\partial x^\alpha} + 2y^\alpha \frac{\partial}{\partial t}\right)(|x|^4 + t^2) =$$

$$= 4(x_\alpha^3 + (|x|^2 - x_\alpha^2)x_\alpha + y_\alpha t) = 4(|x|^2 x_\alpha + y_\alpha t);$$

$$Y_\alpha(\rho^4) = \left(\frac{\partial}{\partial y^\alpha} - 2x^\alpha \frac{\partial}{\partial t}\right)(|x|^4 + t^2) = 4(|x|^2 y_\alpha - x_\alpha t);$$

$$X_\alpha^2(\rho^4) = 4\left(\frac{\partial}{\partial x^\alpha} + 2y^\alpha \frac{\partial}{\partial t}\right)(x_\alpha^3 + |y_\alpha|^2 x_\alpha + y_\alpha t) = 4(|x|^2 + 2|x_\alpha|^2 + 2|y_\alpha|^2);$$

$$Y_\alpha^2(\rho^4) = 4\left(\frac{\partial}{\partial y^\alpha} - 2x^\alpha \frac{\partial}{\partial t}\right)(y_\alpha^3 + |x_\alpha|^2 y_\alpha - x_\alpha t) = 4(|x|^2 + 2|x_\alpha|^2 + 2|y_\alpha|^2).$$

$$X_\alpha^2(\rho^{-n}) = X_\alpha(X_\alpha((\rho^4)^{-n/4})) = -\frac{n}{4} X_\alpha(\rho^{-n-4} X_\alpha(\rho^4)) =$$

$$= \frac{n(n+4)}{16} \rho^{-n-8} |X_\alpha(\rho^4)|^2 - \frac{n}{4} \rho^{-n-4} X_\alpha^2(\rho^4) =$$

$$= \frac{n(n+4)}{16} \rho^{-n-8} 16(|x|^2 x_\alpha + y_\alpha t)^2 - \frac{n}{4} \rho^{-n-4} 4(|x|^2 + 2|x_\alpha|^2 + 2|y_\alpha|^2) =$$

$$= n(n+4)\rho^{-n-8}(|x|^2 x_\alpha + y_\alpha t)^2 - n\rho^{-n-4}(|x|^2 + 2|x_\alpha|^2 + 2|y_\alpha|^2),$$

and analogously

$$Y_\alpha^2(\rho^{-n}) = n(n+4)\rho^{-n-8}(|x|^2 y_\alpha - x_\alpha t)^2 - n\rho^{-n-4}(|x|^2 + 2|x_\alpha|^2 + 2|y_\alpha|^2),$$

so

$$\Delta_b(\rho^{-n}) = \frac{1}{4}\sum_{\alpha=1}^{n}(X_\alpha^2 + Y_\alpha^2)(\rho^{-n}) =$$

$$= \frac{1}{4}n(n+4)\rho^{-n-8}(|x|^6 + |x|^2 t^2) - 2n(n+2)\rho^{-n-4}|x|^2 = -\frac{n^2}{4}\rho^{-n-4}|x|^2.$$

Using $u = \rho^{-n}$ in Proposition (1.23) we get the desired result.

It can be readily verified that the Kelvin transform is isopseudohermitian with respect to $\widetilde{\theta}$ (that is, it preserves the pseudohermitian structure). Since transformations of $\mathbf{H}^n$ of the form $(z,t) \mapsto (Az,t)$ with A unitary are isopseudohermitian, for every point x of ∂A_r, there is a isopseudohermitian transformation that fixes x, leaves its component of ∂A_r invariant, but reverses the orientation. Since reversing the orientation changes sign to $\widetilde{\kappa}$, it follows that $\widetilde{\kappa} = 0$. $\square$

Therefore, thanks to Proposition 4.4,

$$\widetilde{I}(v) = b_n \int_{A_r} \left(|\widetilde{\nabla} v|^2_{\widetilde{\theta}} + n^2 \frac{|x|^2}{\rho^2} v^2 \right) \widetilde{\theta} \wedge (d\widetilde{\theta})^n - \frac{n}{n+1} \int_{A_r} v^{2+2/n} \widetilde{\theta} \wedge (d\widetilde{\theta})^n.$$

We want to impose that the solution is homogeneous and symmetric, in the sense that $u \circ \delta_\lambda = \lambda^{\frac{Q-2}{2}} u$ and $u(x,t) = u(|x|,t)$.

We need to express this functional in suitable coordinates.

Lemma 4.6. *If $v = v(|x|,t)$, in the coordinates $l = \frac{1}{n}\log\rho \in \mathbf{R}$, $\tau = t/\rho^2 \in [-1,1]$ and $\gamma = x/|x| \in \mathbf{S}^{2n-1}$, it holds that*

$$|\widetilde{\nabla} v|^2_{\widetilde{\theta}} = (1-\tau^2)^{3/2} \left| \frac{\partial v}{\partial \tau} \right|^2 + \frac{1}{4n^2}(1-\tau^2)^{1/2} \left| \frac{\partial v}{\partial l} \right|^2.$$

Proof. Since transformations of $\mathbf{H}^n$ of the form $(x,t) \mapsto (Ax,t)$ with A unitary are isomorphisms of the pseudohermitian structure, this kind of transformations preserves the sphere of unit radius, and the action of the unitary group is transitive between vectors of the same length, we can calculate $|\widetilde{\nabla} v|^2_{\widetilde{\theta}}$ on the points of the curve

$$(\sqrt[4]{1-t^2}, 0, \ldots, 0, t).$$

At such points

$$X_\alpha = \frac{\partial}{\partial x_\alpha}$$

for every $\alpha = 1, \ldots, n$,

$$Y_\alpha = \frac{\partial}{\partial y_\alpha}$$

for every $\alpha \neq 1$, and

$$Y_1 = \frac{\partial}{\partial y_1} - 2\sqrt[4]{1-t^2}\frac{\partial}{\partial t}.$$

Using the symmetry of v in x, we get

$$|\widetilde{\nabla} v|^2_{\widetilde{\theta}} = v^{-2/n}|\nabla v|^2_{\theta} = \frac{1}{4}|X_1 u|^2 + \frac{1}{4}|Y_1 u|^2 =$$

$$= \frac{1}{4}\left|\frac{\partial v}{\partial x_1}\right|^2 + (1-t)^{1/2}\left|\frac{\partial v}{\partial t}\right|^2.$$

Since

$$\frac{\partial \tau}{\partial x_1} = -\frac{1}{2}\tau^3\frac{\partial(\tau^{-2})}{\partial x_1} = -\frac{1}{2}\frac{\tau^3}{t^2}\frac{\partial}{\partial x_1}(x_1^4 + t^2) = -2tx_1^3;$$

$$\frac{\partial l}{\partial x_1} = \frac{1}{n}\frac{\partial \log \rho}{\partial x_1} = \frac{1}{4n}\frac{\partial \rho^4}{\partial x_1} = \frac{1}{n}x_1^3;$$

$$\frac{\partial \tau}{\partial t} = \frac{1}{2}\tau^{-1}\frac{\partial(\tau^2)}{\partial t} = \frac{1}{2}t^{-1}(2t - 2t^3) = (1-t^2);$$

$$\frac{\partial l}{\partial t} = \frac{1}{4n}\frac{\partial \rho^4}{\partial t} = \frac{1}{2n}t,$$

we obtain

$$|\widetilde{\nabla} v|^2_{\widetilde{\theta}} = \frac{1}{4}\left|\frac{\partial v}{\partial x_1}\right|^2 + (1-t^2)^{1/2}\left|\frac{\partial v}{\partial t}\right|^2 =$$

$$= \frac{1}{4}\left|-2tx_1^3\frac{\partial v}{\partial \tau} + \frac{x_1^3}{n}\frac{\partial v}{\partial l}\right|^2 + (1-t^2)^{1/2}\left||x|^4\frac{\partial v}{\partial \tau} + \frac{t}{2n}\frac{\partial v}{\partial l}\right|^2 =$$

$$= t^2(1-t^2)^{3/2}\left|\frac{\partial v}{\partial \tau}\right|^2 + \frac{1}{4n^2}(1-t^2)^{3/2}\left|\frac{\partial v}{\partial l}\right|^2 - \frac{1}{n}t(1-t^2)^{3/2}\frac{\partial v}{\partial \tau}\frac{\partial v}{\partial l} +$$

$$+(1-t^2)^{5/2}\left|\frac{\partial v}{\partial \tau}\right|^2 + \frac{1}{4n^2}t^2(1-t^2)^{1/2}\left|\frac{\partial v}{\partial l}\right|^2 + \frac{1}{n}(1-t^2)^{3/2}t\frac{\partial v}{\partial \tau}\frac{\partial v}{\partial l} =$$

$$= (1-t^2)^{3/2}\left|\frac{\partial v}{\partial \tau}\right|^2 + \frac{1}{4n^2}(1-t^2)^{1/2}\left|\frac{\partial v}{\partial l}\right|^2.$$

By the dilation invariance of $\widetilde{\theta}$ we obtain the formula in the general case. □

Now we compute the volume form.

Lemma 4.7. *In the coordinates of Lemma 4.6*

$$\widetilde{\theta} \wedge (d\widetilde{\theta})^n = n!(1-\tau^2)^{(n-2)/2} dl \wedge d\gamma \wedge d\tau.$$

Proof. The volume form becomes

$$\widetilde{\theta} \wedge (d\widetilde{\theta})^n = \rho^{-2(n+1)}\theta \wedge (d\theta)^n = \frac{n!}{\rho^{2(n+1)}} \operatorname{vol}_{g_\theta} = \frac{n!}{\rho^{2(n+1)}} |x|^{2n-1} d|x| \wedge d\gamma \wedge dt.$$

By an easy computation

$$\begin{aligned} d|x| &= \frac{1}{4|x|^3} d(\rho^4 - t^2) = \frac{1}{4|x|^3} d(e^{4ln}(1-\tau^2)) = \\ &= \frac{1}{4(1-\tau^2)^{3/4} e^{3nl}} e^{4ln}(4n(1-\tau^2)dl - 2\tau d\tau) = \\ &= e^{ln}\left(n(1-\tau^2)^{1/4}dl - \frac{\tau}{2(1-\tau^2)^{3/4}} d\tau\right); \\ dt &= d(e^{2nl}\tau) = e^{2nl}(2n\tau dl + d\tau); \\ d|x| \wedge dt &= e^{3nl} \frac{1}{(1-\tau^2)^{3/4}} dl \wedge d\tau, \end{aligned}$$

so

$$\begin{aligned} \widetilde{\theta} \wedge (d\widetilde{\theta})^n &= -\frac{n!}{\rho^{2(n+1)}}(1-\tau^2)^{(2n-1)/4} \rho^{2n-1} d|x| \wedge dt \wedge d\gamma = \\ &= -\frac{n!}{e^{3nl}}(1-\tau^2)^{(2n-1)/4} e^{3nl} \frac{1}{(1-\tau^2)^{3/4}} dl \wedge d\tau \wedge d\gamma = \\ &= n!(1-\tau^2)^{(n-2)/2} dl \wedge d\gamma \wedge d\tau, \end{aligned}$$

as desired. □

Using Lemmas 4.6 and 4.7 we have that

$$\begin{aligned} \widetilde{I}(v) = b_n \int_{A_r} \Bigg(&(1-\tau^2)^{3/2} \left|\frac{\partial v}{\partial \tau}\right|^2 + \frac{1}{4n^2}(1-\tau^2)^{1/2} \left|\frac{\partial v}{\partial l}\right|^2 v^2 + \\ &+ n^2(1-\tau^2)^{1/2}\Bigg) n!(1-\tau^2)^{(n-2)/2} dl \wedge d\gamma \wedge d\tau + \\ &- \frac{n}{n+1}\int_{A_r} v^{2+2/n} n!(1-\tau^2)^{(n-2)/2} dl \wedge d\gamma \wedge d\tau = \\ = b_n 2^n n! \int_0^{\frac{\log r}{n}} \int_{-1}^{1} \Bigg(&(1-\tau^2)^{(n+1)/2} \left|\frac{\partial v}{\partial \tau}\right|^2 + \frac{1}{4n^2}(1-\tau^2)^{(n-1)/2} \left|\frac{\partial v}{\partial l}\right|^2 v^2 + \\ + n^2(1-\tau^2)^{(n-1)/2}\Bigg) &dl \wedge d\tau - \frac{nn!}{n+1} \int_0^{\frac{\log r}{n}} \int_{-1}^{1} v^{2+2/n}(1-\tau^2)^{(n-2)/2} dl \wedge d\tau. \end{aligned}$$

If $\tau = \sin s$, then

$$\widetilde{I}(v) = b_n 2^n n! \int_0^{\frac{\log r}{n}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{(\cos s)^{n+1}}{(\cos s)^2} \left|\frac{\partial v}{\partial s}\right|^2 + \frac{1}{4n^2}(\cos s)^{n-1} \left|\frac{\partial v}{\partial l}\right|^2 + \right.$$

$$+n^2(\cos s)^{n-1}v^2\big)\,dl(\cos s)ds - \frac{nn!}{n+1}\int_0^{\frac{\log r}{n}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v^{2+2/n}(\cos s)^{n-2}dl(\cos s)ds =$$

$$= b_n 2^n n!\int_0^{\frac{\log r}{n}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos s)^n\left(\left|\frac{\partial v}{\partial s}\right|^2+\frac{1}{4n^2}\left|\frac{\partial v}{\partial l}\right|^2+n^2v^2\right)dlds+$$

$$-\frac{nn!}{n+1}\int_0^{\frac{\log r}{n}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v^{2+2/n}(\cos s)^{n-1}dlds.$$

Now let us look for homogeneous solutions. Homogeous solutions in the original setting correspond to solutions invariant by translation (in the l direction), and so let us set $\frac{\partial v}{\partial l}=0$, and $v=v(s)$. In this special case we have

$$\widetilde{I}(v)=b_n n!\frac{\log r}{n}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos s)^n\left((v')^2+n^2v^2\right)ds+$$

$$-\frac{nn!}{n+1}\frac{\log r}{n}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v^{2+2/n}(\cos s)^{n-1}ds.$$

The Euler-Lagrange equation for this functional is

$$-\frac{d}{ds}((\cos s)^n v'(s))+n^2(\cos s)^n v(s)=\frac{n}{2(n+1)}(\cos s)^{n-1}v(s)^{1+2/n},$$

or equivalently

$$-\cos s v''(s)+n\sin s v'(s)+n^2\cos s v(s)=\frac{n}{2(n+1)}v(s)^{1+2/n},$$

on the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, with Neumann boundary conditions, that is also the Euler-Lagrange equation (up to rescaling, thanks to homogeneity) of

$$J(v)=\frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos s)^n\left((v')^2+n^2v^2\right)ds}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v^{2+2/n}(\cos s)^{n-1}ds}.$$

Let us define the weighted Sobolev and Lebesgue spaces

$$X=\left\{u\in H^1_{\mathrm{loc}}\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\;\middle|\;\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos s)^n\left((v')^2+v^2\right)ds<\infty\right\},$$

$$Y=\left\{u\in L^1_{\mathrm{loc}}\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\;\middle|\;\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos s)^{n-1}v^{2+2/n}ds<\infty\right\}.$$

Proposition 4.8. *X embeds compactly in Y.*

Proof. Let Z be the subspace of $H^1(S^{n+1})$ formed by functions invariant by rotation around the last coordinate axis. Every such function is of the form $v(x) = u(\cos x^{n+2})$, and it is easy to verify thar under such an identification

$$\|v\|_Z = \|u\|_X .$$

So this is an isometric isomorphism between X and Z. By Rellich--Kondrachov's theorem, Z embeds compactly into $L^p(S^n)$ for every $p \in [1, 2\frac{n+1}{n-1})$, which by similar arguments is isometrically isomorphic to $L^p\left(((-\frac{\pi}{2}, \frac{\pi}{2}), (\cos s)^n ds\right)$. Given $\alpha > 0$, $q > 1$,

$$\int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}} (\cos s)^{n-1} v^{2+2/n} ds = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{(\cos s)^{n-1+\alpha}}{(\cos s)^{\alpha}} ds \leq$$

$$\leq \left(\int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}} v^{(2+2/n)q} (\cos s)(n-1+\alpha) q ds \right)^{1/q} \left(\int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}} (\cos s)^{-\alpha q'} ds \right)^{1/q'} .$$

If we impose that $(n-1+\alpha)q = n$, then taking α small enough, we can find that $p = \left(2 + \frac{2}{n} q\right) < 2\frac{n+1}{n-1}$ and $\alpha q' < n$, getting that

$$\int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}} (\cos s)^{n-1} v^{2+2/n} ds \leq C \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}} \left(v^p (\cos s)^n ds\right)^{1/q},$$

that is

$$\|v\|_Y \leq C \|v\|_{L^p((\cos s)^n ds)},$$

and so $L^p\left(\left(\frac{\pi}{2}, \frac{\pi}{2}\right) (\cos s)^n ds\right)$ embeds into Y. So we get the thesis. $\square$

Finally we can prove the existence of the desired homogeneous solution.

Theorem 4.9. *There exists a non zero solution Ψ of the equation*

$$-\Delta \Psi = \Psi^{\frac{Q+2}{Q-2}},$$

defined on $\mathbf{H}^n \setminus \{0\}$, such that $\Psi \circ \delta_\lambda = \lambda^{\frac{Q-2}{2}} \Psi$ and $\Psi(z,t) = \Psi(|z|, t)$.

Proof. The existence of a solution on Ω_r can be proved, in a standard way, by the direct methods of the calculus of variation. Since it is homogeneous by construction, it does not depend on r, and can be extended to $\mathbf{H}^n \setminus \{0\}$ by homogeneity. $\square$

4.3 Bifurcation of non-homogeneous solutions

We proved the existence of a homogeneous solution Ψ of degree $-\frac{Q-2}{2}$ of Equation

$$-\Delta_b u = u^{\frac{Q+2}{Q-2}} \tag{4.2}$$

on $\mathbf{H}^n \setminus \{0\}$, and in Chapter 3 we studied solution that are verify the condition $u \circ \delta_T = T^{-\frac{Q-2}{2}} u$. Obviously Ψ verifies the condition $\Psi \circ \delta_T = T^{-\frac{Q-2}{2}} \Psi$. It is natural to ask whether non homogeneous solutions verifying $u \circ \delta_T = T^{-\frac{Q-2}{2}} u$ bifurcate from the homogeneous one for certain values of T. In the Euclidean case, in which explicit computation may be performed, the answer is affermative. In this Chapter we will prove that also in $\mathbf{H}^n$ the answer is affermative.

Proposition 4.10. *The Morse index of $\mathscr{J}_T(\Psi)$ is finite and tends to infinity as $T \to \infty$.*

Proof. The Morse index is finite because the operator on X_T associated to the bilinear form

$$d^2 \mathscr{J}_T(\Psi)[u, v] = \int_{\Omega_T} \left(\nabla u \nabla v - (2^* - 1)\Psi^{2^*-2} uv \right)$$

is the sum of the identity and a compact operator (thanks to the Rellich-Kondrachov theorem for the Stein-Folland space).

The signature of a simmetric bilinear form remains invariant passing to the complexification and extending it to a hermitian form. So let us take

$$u(x) = \exp\left(i \frac{1}{M} \log|x|\right) \Psi(x),$$

with

$$\frac{\log T}{2\pi M} \in \mathbf{Z}.$$

Then

$$d^2 \mathscr{J}_T(\Psi)[u, \overline{u}] = \int_{\Omega_T} \left(\left| \nabla \left(\exp\left(i \frac{1}{M} \log|x|\right) \Psi \right) \right|^2 - (2^* - 1)\Psi^{2^*-2}\Psi^2 \right) =$$

$$= \int_{\Omega_T} \left(\frac{1}{M^2|x|^2} \Psi^2 + |\nabla \Psi|^2 + 2\frac{1}{M|x|} \Psi \nabla \Psi \cdot \nabla |x| - (2^* - 1)\Psi^{2^*} \right) =$$

$$= \int_{\Omega_T} \left(\frac{1}{M^2|x|^2} \Psi^2 + 2\frac{1}{M|x|} \Psi \nabla \Psi \cdot \nabla |x| - (2^* - 2)\Psi^{2^*} \right) =$$

$$= -(2^* - 2) \int_{\Omega_T} \Psi^{2^*} + \int_{\Omega_T} \left(\frac{1}{M^2|x|^2} \Psi^2 + 2\frac{1}{M|x|} \Psi \nabla \Psi \cdot \nabla |x| \right).$$

By homogeneity the three integrals

$$\int_{\Omega_T} \Psi^{2^*}, \quad \int_{\Omega_T} \frac{1}{|x|^2} \Psi^2, \quad \int_{\Omega_T} \frac{1}{|x|} \Psi \nabla \Psi \cdot \nabla |x|$$

are constant multiples of $\log T$, so there exists a constant C such that if $M \geq C$ then $d^2 \mathscr{J}_T(\Psi)[u, u]$ is negative. Given $k \in \mathbf{N}$, let T be big enough so that

$$\frac{2\pi k}{\log T} \leq \frac{1}{C}.$$

Then the functions

$$u_m(x) = \exp\left(i\frac{2\pi m}{\log T}\log|x|\right)\Psi(x)$$

with $m = 1, \ldots, k$, are such that $d^2\mathscr{I}_T(\Psi)[u_m, u_m] \leq -\varepsilon \log T$ is negative. If f is a homogeneous function of degree zero and $m \neq 0$ then

$$\int_{\Omega_T} \exp\left(i\frac{2\pi m}{\log T}\log|x|\right)\frac{f(x)}{|x|^Q} = \int_{S_1} d\sigma f \int_1^T dr \frac{\exp\left(i\frac{2\pi m}{\log T}\log r\right)}{r} = 0.$$

When calculating

$$d^2\mathscr{I}_T(\Psi)[u_m, u_j]$$

with $m \neq j$, the result is a sum of terms of this kind, so it is zero. So the functions u_m span a vector space of dimension k on which $d^2\mathscr{I}_T(\Psi)$ is negative definite. □

Now let us apply the conformal transformation corresponding to the homogeneous solution Ψ. Thanks to Proposition 4.4, we get the functional

$$\widetilde{\mathscr{I}_T}(u) = \int_{\Omega_T}\left(|\widetilde{\nabla}_{\mathbf{H}^n} u|^2 + \frac{1}{2}u^2 - \frac{1}{2^*}|u|^{2^*}\right),$$

defined on the space

$$Y_T = \left\{\frac{u}{\Psi} \;\middle|\; u \in X_T\right\} = \left\{u \in S^1_{\text{loc}}(\mathbf{H}^n) \;\middle|\; u \circ \delta_T = u\right\}.$$

Let Σ be the unit sphere with respect to the Euclidean metric [1]. Let φ_j be a complete set in $L^2(\Sigma)$ consisting of analytic functions. So

$$\gamma_{j,m,T}(x) = \varphi_j\left(\frac{x}{|x|_{\text{eucl}}}\right)\sin\left(i\frac{2\pi m}{\log T}\log|x|_{\text{eucl}}\right)$$

is a complete set of functions in $H^1(\Omega_T)$, analytic with respect to the couple (x, T). So it is complete also in $S^1(\Omega_T)$. With the Gram-Schmidt algorithm, we can obtain a family of Hilbert bases $\psi_{k,T}$ of $S^1(\Omega_T)$, and preserve the analyticity property. Let us define the isometry Ψ_T between Y_T and Y_2 obtained sending $\psi_{k,T}$ into $\psi_{k,2}$. Let us call

$$L_T = \Psi_T \circ \widetilde{\mathscr{I}}''_T(1) \circ \Psi_T^{-1}.$$

Then, for every l, k,

$$\langle L_T u_{2,k}, u_{2,k}\rangle_{Y_2} = \left\langle \widetilde{\mathscr{I}}''_T(1) u_{k,T}, u_{l,T}\right\rangle_{Y_T} =$$

[1] this is necessary to perform the next steps of the proofs because the sphere with respect to the Heisenberg metric is not smooth

$$= \int_{\Omega_T} \widetilde{\nabla} u_{k,T} \widetilde{\nabla} u_{l,T} + u_{k,T} u_{l,T} - (2^* - 1) u_{k,T} u_{l,T}$$

is an analytic operator-valued function, and, calling Π the projection of Y_T onto the constant functions, it holds that

$$\left\langle \widetilde{\mathscr{J}}_T''(1)u, v \right\rangle = \int_{\Omega_T} \widetilde{\nabla} u \widetilde{\nabla} v + uv - (2^* - 1)uv =$$

$$= \int_{\Omega_T} \widetilde{\nabla} u \widetilde{\nabla} v - (2^* - 2) \left(\Delta(G_T((I - \Pi)u)) + \Pi u \right) v =$$

$$= \int_{\Omega_T} \widetilde{\nabla} u \widetilde{\nabla} v + (2^* - 2) \widetilde{\nabla}(G_T((I - \Pi)u)) \widetilde{\nabla} v - (2^* - 2)(\Pi u) v,$$

where $G_T : Y_T \to Y_T$ is the Green's operator, so

$$\widetilde{\mathscr{J}}_T''(1) = I + (2^* - 2) G_T + (\widetilde{\mathscr{J}}_T''(1) - I) \circ \Pi.$$

Since L_T is, by definition, conjugated to $\widetilde{\mathscr{J}}_T''(1)$, it is of the form $I - K(T)$, where $K(T)$ is an analytic operator-valued function of compact operators.

From this we can prove the following bifurcation theorem for $\mathscr{J}_T$.

Theorem 4.11. *There exists arbitrarily large values of T for which $d^2 \mathscr{J}_T(\Psi)$ is singular, and every such value is a bifurcation value.*

Proof. It suffices to apply Theorem 8.9 in [MW]. In our case the hypotheses of that Theorem are all either trivial or standardly verifiable, with exception of hypothesis γ, that is consequence of Corollary 8.3 in the same book. $\square$

Chapter 5

Variation of the Einstein-Hilbert functional on spheres

The Eistein-Hilbert functional (or action) of a Riemannian manifold (M, g)

$$\mathscr{R}(g) = \int_M R_g dV_g$$

and the renormalized, scaling-invariant, version thereof,

$$\widetilde{\mathscr{R}}(g) = \mathrm{Vol}(M)^{\frac{2-n}{n}} \mathscr{R}(g)$$

are of central importance in Riemannian geometry.

A metric is critical for $\mathscr{R}(g)$ if and only if it is Ricci flat, and is critical for $\widetilde{\mathscr{R}}(g)$ if and only if it is Einstein.

In particular S^n with the standard metric is a critical point for $\widetilde{\mathscr{R}}(g)$. Computing the second variation shows that it is a saddle point: $d^2\widetilde{\mathscr{R}}(g)$ is zero on the tangent of the space of variations that arise by pulling back the metric by diffeomorphisms, positive definite on the tangent to the space of conformal variations, and negative definite of the orthogonal of the sum of these two subspaces.

Given a compact, non degenerate, pseudo-Hermitian manifold, the Einstein-Hilbert functional is defined, in a similar manner of Riemannian geometry, as

$$\mathscr{W} = \int_M W\,\theta \wedge (d\theta)^n$$

and the renormalized version as

$$\widetilde{\mathscr{W}} = (\mathrm{Vol}_\theta(M))^{-\frac{Q-2}{Q}}\,\mathscr{W}.$$

In this Chapter we study the variation of the functional $\widetilde{\mathscr{W}}$ on S^{2n+1} with its standard pseudo-Hermitian structure.

5.1 Formulas for variations

In order to study variations of a pseudo-Hermitian structure, we use a well known theorem of Gray, to study variations of pseudo-Hermitian structures we can suppose that they have the same underlying contact distribution. So we can study variations of the contact form and of the complex structure. It is not hard to verify that these two kinds of variations are orthogonal with respect to $d^2\mathscr{W}$, so they can be studied separately.

In the case of S^{2n+1}, $d^2\mathscr{W}$ is positive definite on variations of the contact form, so we can study variations of the complex structure. This follows from the fact that the CR Einstein-Hilbert functional resticted to a conformal class obviously coincides with the CR Yamabe functional, and from the solution of the CR Yamabe problem on spheres (see [JL1, JL2]).

Let $J_{(t)}$ be a family of complex structures. Let $(Z_{\alpha(t)})_\alpha$ a local frame orthononormal with respect to the Levi form such that $J_{(t)}Z_{\alpha(t)} = iZ_{\alpha(t)}$. Thus

$$J_{(t)} = i\theta^\alpha_{(t)} \otimes Z_{\alpha(t)} - i\theta^{\overline{\alpha}}_{(t)} \otimes Z_{\overline{\alpha}(t)}.$$

Let $\dot{J} := \frac{d}{dt}J_{(t)}$.

Lemma 5.1. *The variation $\dot{J}$ of J can be expressed as*

$$\dot{J} = 2E = 2E_\alpha{}^{\overline{\beta}}\theta^\alpha \otimes Z_{\overline{\beta}} + \text{conj.}.$$

Proof. By definition $\dot{J}(Z_\alpha) = 2E_\alpha{}^{\overline{\beta}}Z_{\overline{\beta}} + 2E_\alpha{}^\beta Z_\beta$. Differentiate the relation $J^2_{(t)} = -Id$ with respect to t (at $t = 0$) to obtain:

$$\dot{J} \circ J + J \circ \dot{J} = 0.$$

Expressing this relation with respect to the basis $(Z_\alpha)_\alpha$ and $(\overline{Z}_\alpha)_\alpha$, we obtain

$$\begin{pmatrix} E_\alpha{}^\beta & E_\alpha{}^{\overline{\beta}} \\ E_{\overline{\alpha}}{}^\beta & E_{\overline{\alpha}}{}^{\overline{\beta}} \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} E_\alpha{}^\beta & E_\alpha{}^{\overline{\beta}} \\ E_{\overline{\alpha}}{}^\beta & E_{\overline{\alpha}}{}^{\overline{\beta}} \end{pmatrix} = 0,$$

which implies $E_\alpha{}^\beta = E_{\overline{\alpha}}{}^{\overline{\beta}} = 0$, as claimed. □

We consider next the variation of the basis vector fields with respect to t.

Lemma 5.2. *Let us write the derivative of the frame $(Z_\alpha)_\alpha$ in the form*

$$\dot{Z}_\alpha = F_\alpha{}^\beta Z_\beta + G_\alpha{}^{\overline{\beta}} Z_{\overline{\beta}}.$$

Then we can assume that $F_\alpha{}^\beta \in \mathbf{R}$, and that there holds

$$G_\alpha{}^{\overline{\beta}} = -iE_\alpha{}^{\overline{\beta}}; \qquad F_\alpha{}^\beta + F_\beta{}^\alpha = 0. \tag{5.1}$$

Moreover, at $t = 0$ we can take $F^\alpha_\beta = 0$.

Proof. We have $d\theta = i\sum \theta^{\alpha}_{(t)} \wedge \theta^{\overline{\alpha}}_{(t)}$ (from $h_{\alpha\overline{\beta}} = \delta^{\beta}_{\alpha}$), so

$$-2id\theta(Z_{\alpha(t)} \wedge Z_{\overline{\beta}(t)}) = \delta^{\beta}_{\alpha}.$$

Differentiating this relation in t, we get

$$\begin{aligned} 0 &= -2id\theta(\dot{Z}_\alpha \wedge Z_{\overline{\beta}}) - 2id\theta(Z_\alpha \wedge \dot{Z}_{\overline{\beta}}) \\ &= -2id\theta((F_\alpha{}^\gamma Z_\gamma + G_\alpha{}^{\overline{\gamma}} Z_{\overline{\gamma}}) \wedge Z_{\overline{\beta}}) - 2id\theta(Z_\alpha \wedge (\overline{F_\beta{}^\gamma} Z_{\overline{\gamma}} + \overline{G_\beta{}^{\overline{\gamma}}} Z_\gamma)) \\ &= -2iF_\alpha{}^\beta - 2i\overline{F_\beta{}^\alpha}. \end{aligned} \tag{5.2}$$

On the other hand, we can always compose the frame $(Z_\alpha)_\alpha$ with an element of $SU(n)$, which infinitesimally means adding to F^{β}_{α} a matrix B^{β}_{α} such that $B^{\beta}_{\alpha} = -\overline{B}^{\alpha}_{\beta}$. We can choose for example to add $\overline{F}^{\beta}_{\alpha}$, which satisfies this property by the above relation (5.2): this means that we can choose F to be real, and implies the second relation in (5.1).

To get the first one, differentiate $J_{(t)}Z_{\alpha(t)} = iZ_{\alpha(t)}$ with respect to t, to find

$$\begin{aligned} 0 &= \dot{J}Z_\alpha + J\dot{Z}_\alpha - i\dot{Z}_\alpha \\ &= 2E_\alpha{}^{\overline{\beta}} Z_{\overline{\beta}} + F_\alpha{}^\beta iZ_\beta + G_\alpha{}^{\overline{\beta}}(-i)Z_{\overline{\beta}} - i(F_\alpha{}^\beta Z_\beta + G_\alpha{}^{\overline{\beta}} Z_{\overline{\beta}}) \\ &= 2E_\alpha{}^{\overline{\beta}} Z_{\overline{\beta}} - 2iG_\alpha{}^{\overline{\beta}} Z_{\overline{\beta}}, \end{aligned}$$

so $G_\alpha{}^{\overline{\beta}} = -iE_\alpha{}^{\overline{\beta}}$, as desired.

To prove that we can take $F = 0$ at $t = 0$, let $\mathfrak{F} = F^{\beta}_{\alpha}$ at $t = 0$ and consider the new frame

$$\tilde{Z}_{\alpha(t)} = (e^{-t\mathfrak{F}})^{\beta}_{\alpha} Z_\beta.$$

Then, by cancellation

$$\tilde{Z}_{\alpha(0)} = Z_{\alpha(0)} \qquad \text{and} \qquad \frac{d}{dt}_{t=0}\tilde{Z}_{\alpha(t)} = -iE^{\overline{\beta}}_{\alpha(0)}\tilde{Z}_{\overline{\beta}(0)},$$

concluding the proof. $\square$

We next derive some consequences of the integrability conditions

$$\theta([Z_\alpha, Z_\beta]) = 0; \qquad \theta^{\overline{\gamma}}([Z_\alpha, Z_\beta]) = 0, \tag{5.3}$$

which hold along all the deformation, i.e. for all t. We have the following result.

Lemma 5.3. *For all indices α, β, γ we have that*

$$E_{\alpha\beta} = E_{\beta\alpha}; \qquad E^{\overline{\gamma}}_{\alpha,\beta} = E^{\overline{\gamma}}_{\beta,\alpha}.$$

Proof. We differentiate in t the first relation in (5.3), obtaining

$$\begin{aligned} \frac{d}{dt}[Z_\alpha, Z_\beta] &= [\dot{Z}_\alpha, Z_\beta] + [Z_\alpha, \dot{Z}_\beta] \\ &= [F^{\gamma}_{\alpha} Z_\gamma - iE^{\overline{\gamma}}_{\alpha} Z_{\overline{\gamma}}, Z_\beta] + [Z_\alpha, F^{\gamma}_{\beta} Z_\gamma - iE^{\overline{\gamma}}_{\beta} Z_{\overline{\gamma}}], \end{aligned}$$

which by integrability yields

$$0 = \theta\left(\frac{d}{dt}[Z_\alpha, Z_\beta]\right) = -iE_\alpha^{\overline{\gamma}}\theta([Z_{\overline{\gamma}}, Z_\beta]) - iE_\beta^{\overline{\gamma}}\theta([Z_\alpha, Z_{\overline{\gamma}}]).$$

Notice that

$$[Z_{\overline{\gamma}}, Z_\beta] = ih_{\beta\overline{\gamma}}T + \omega_\beta^l(Z_{\overline{\gamma}})Z_l - \omega_{\overline{\gamma}}^{\overline{l}}(Z_\beta)Z_{\overline{l}},$$

which in turn implies

$$0 = -iE_\alpha^{\overline{\gamma}}ih_{\beta\overline{\gamma}} + iE_\beta^{\overline{\gamma}}ih_{\alpha\overline{\gamma}}.$$

Since $h_{\beta\overline{\gamma}} = \delta_\beta^\gamma$, we deduce the first assertion of the lemma. We next differentiate in t the second relation in (5.3) to get

$$0 = \dot{\theta}^{\overline{\gamma}}([Z_\alpha, Z_\beta]) + \theta^{\overline{\gamma}}\left(\frac{d}{dt}[Z_\alpha, Z_\beta]\right).$$

Since

$$\dot{\theta}^{\overline{\gamma}} = iE_l^{\overline{\gamma}}\theta^l - F_{\overline{l}}^{\overline{\gamma}}\theta^{\overline{l}},$$

and

$$[Z_\alpha, Z_\beta] = \omega_\beta^l(Z_\alpha)Z_l - \omega_\alpha^l(Z_\beta)Z_l,$$

we obtain

$$\begin{aligned}
0 &= iE_l^{\overline{\gamma}}\theta^l(\omega_\beta^l(Z_\alpha) - \omega_\alpha^l(Z_\beta)) \\
&\quad + \theta^{\overline{\gamma}}\left([Z_\alpha, F_\beta^\gamma Z_\gamma - iE_\beta^{\overline{\gamma}}Z_{\overline{\gamma}}] + [Z_\alpha, F_\beta^\gamma Z_\gamma - iE_\beta^{\overline{\gamma}}Z_{\overline{\gamma}}]\right) \\
&= -i\omega_\alpha^l(Z_\beta)E_l^{\overline{\gamma}} + i\omega_\beta^l(Z_\alpha)E_l^{\overline{\gamma}} + iE_\alpha^{\overline{l}}\omega_{\overline{l}}^{\overline{\gamma}}(Z_\beta) \\
&\quad + iZ_\beta(E^{\overline{\gamma}})_\alpha - iE_\beta^{\overline{l}}\omega_{\overline{l}}^{\overline{\gamma}}(Z_\alpha) - iZ_\alpha(E_\beta^{\overline{l}}).
\end{aligned}$$

This implies

$$iE_{\alpha,\beta}^{\overline{\gamma}} - iE_{\beta,\alpha}^{\overline{\gamma}} = 0,$$

which is the second assertion. □

Lemma 5.4. *([L2]) For any point $p \in M$, there exists a neighborhood U_p and an admissible co-frame $(\theta^\alpha)_\alpha$ such that $\omega_\alpha^\beta = 0$ at p.*

Lemma 5.5. *For all t, the variation of the torsion is given by*

$$\dot{A}_{\overline{\gamma}}^\alpha = -iE_{\overline{\gamma},0}^\alpha + A_{\overline{l}}^\alpha F_{\overline{\gamma}}^{\overline{l}} - F_l^\alpha A_{\overline{\gamma}}^l, \tag{5.4}$$

while for the variation of the connection we have

$$\begin{aligned}
\dot{\omega}_\beta^\alpha &= \left[i(A_{\overline{\gamma}}^\alpha E_\beta^{\overline{\gamma}} + E_{\overline{\gamma}}^\alpha A_\beta^{\overline{\gamma}}) + F_{\beta,0}^\alpha\right]\theta \\
&\quad + (-iE_{\gamma,\overline{\alpha}}^{\overline{\beta}} - F_{\overline{\alpha},\gamma}^{\overline{\beta}})\theta^\gamma + (-iE_{\overline{\gamma},\beta}^\alpha + F_{\beta,\overline{\gamma}}^\alpha)\theta^{\overline{\gamma}}.
\end{aligned} \tag{5.5}$$

Proof. We start by differentiating in t the relations

$$\theta^\alpha_{(t)}(Z_{\beta(t)}) = \delta^\alpha_\beta; \qquad \theta^\alpha_{(t)}(Z_{\overline{\beta}(t)}) = \theta^\alpha_{(t)}(T) = 0,$$

to get

$$\dot\theta^\alpha(Z_\beta) + \theta^\alpha(\dot Z_\beta) = 0; \qquad \dot\theta^\alpha(Z_{\overline{\beta}}) + \theta^\alpha(\dot Z_{\overline{\beta}}) = 0; \qquad \dot\theta^\alpha(T) = 0.$$

Recalling that by Lemma 5.2 $\dot Z_\alpha = F_\alpha{}^\beta Z_\beta - iE^{\overline{\beta}}_\alpha Z_{\overline{\beta}}$, we obtain

$$\dot\theta^\alpha = -iE^\alpha_{\overline{\beta}}\theta^{\overline{\beta}} - F^\alpha_\beta\theta^\beta. \tag{5.6}$$

Differentiate now in t the structure equation

$$d\theta^\alpha = \theta^\beta_{(t)} \wedge \omega^\alpha_{\beta(t)} + A^\alpha_{\overline{\gamma}(t)}\theta \wedge \theta^{\overline{\gamma}}_{(t)}$$

to get

$$\begin{aligned} d\dot\theta^\alpha &= \dot\theta^\beta \wedge \omega^\alpha_\beta + \theta^\beta \wedge \dot\omega^\alpha_\beta + \dot A^\alpha_{\overline{\gamma}}\theta \wedge \theta^{\overline{\gamma}} + A^\alpha_{\overline{\gamma}}\theta \wedge \dot\theta^{\overline{\gamma}} \\ &= -idE^\alpha_{\overline{\beta}} \wedge \theta^{\overline{\beta}} - iE^\alpha_{\overline{\beta}}d\theta^{\overline{\beta}} - dF^\alpha_\beta \wedge \theta^\beta - F^\alpha_\beta d\theta^\beta. \end{aligned} \tag{5.7}$$

At a given point p we may assume that $\omega^\alpha_\beta = 0$, by Lemma 5.4. Therefore we obtain at p

$$-idE^\alpha_{\overline{\beta}} \wedge \theta^{\overline{\beta}} = -iE^\alpha_{\overline{\beta},0}\theta \wedge \theta^{\overline{\beta}} - iE^\alpha_{\overline{\beta},\gamma}\theta^\gamma \wedge \theta^{\overline{\beta}} - iE^\alpha_{\overline{\beta},\overline{\gamma}}\theta^{\overline{\gamma}} \wedge \theta^{\overline{\beta}};$$

$$d\theta^{\overline{\beta}} = \theta^{\overline{\gamma}} \wedge \omega^{\overline{\beta}}_{\overline{\gamma}} + A^{\overline{\beta}}_\gamma\theta \wedge \theta^\gamma = A^{\overline{\beta}}_\gamma\theta \wedge \theta^\gamma;$$

$$-dF^\alpha_\beta \wedge \theta^\beta - F^\alpha_\beta d\theta^\beta = -F^\alpha_{\beta,0}\theta \wedge \theta^\beta - F^\alpha_{\beta,\gamma}\theta^\gamma \wedge \theta^\beta - F^\alpha_{\beta,\overline{\gamma}}\theta^{\overline{\gamma}} \wedge \theta^\beta - F^\alpha_\beta A^\beta_{\overline{\gamma}}\theta \wedge \theta^{\overline{\gamma}}.$$

Comparing the coefficients of $\theta \wedge \theta^{\overline{\gamma}}$ in (5.7) we deduce the first assertion.

Write next

$$\dot\omega^\alpha_\beta = x^\alpha_\beta\theta + y^\alpha_{\beta\gamma}\theta^\gamma + y^\alpha_{\beta\overline{\gamma}}\theta^{\overline{\gamma}}. \tag{5.8}$$

From the relation $\omega^\beta_\alpha + \omega^{\overline{\alpha}}_{\overline{\beta}} = 0$, which implies $\dot\omega^\beta_\alpha + \dot\omega^{\overline{\alpha}}_{\overline{\beta}} = 0$, we get the system

$$\begin{cases} x^\alpha_\beta + x^{\overline{\beta}}_{\overline{\alpha}} = 0; \\ y^\alpha_{\beta\gamma} = -\overline{y^\beta_{\alpha\overline{\gamma}}} := -y^{\overline{\beta}}_{\overline{\alpha}\gamma}. \end{cases}$$

Substituting (5.6), (5.4) and (5.8) into (5.7) we obtain

$$\begin{cases} x^\alpha_\beta = iE^\alpha_{\overline{\gamma}}A^{\overline{\gamma}}_\beta + iA^\alpha_{\overline{\gamma}}E^{\overline{\gamma}}_\beta + F^\alpha_{\beta,0}; \\ y^\alpha_{\beta\overline{\gamma}} = -iE^\alpha_{\overline{\gamma},\beta} + F^\alpha_{\beta,\overline{\gamma}}, \end{cases} \tag{5.9}$$

which implies in particular $y^\alpha_{\beta\gamma} = -iE^{\overline{\beta}}_{\gamma,\overline{\alpha}} - F^{\overline{\beta}}_{\overline{\alpha},\gamma}$. We also get the relations

$$(y^\alpha_{\beta\gamma} - F^\alpha_{\beta,\gamma})\theta^\beta \wedge \theta^\gamma = 0; \qquad -iE^\alpha_{\overline{\beta},\overline{\gamma}}\theta^{\overline{\gamma}} \wedge \theta^{\overline{\beta}} = 0$$

giving the following constraints on the deformation tensors

$$\begin{cases} y^{\alpha}_{\beta\gamma} - F^{\alpha}_{\beta,\gamma} = y^{\alpha}_{\gamma\beta} - F^{\alpha}_{\gamma,\beta}; \\ E^{\alpha}_{\overline{\beta},\overline{\gamma}} = E^{\alpha}_{\overline{\gamma},\overline{\beta}}. \end{cases}$$

In this way, we obtain the second assertion as well. □

We can now compute the derivative of the curvature tensor with respect to t, together with its contractions.

Proposition 5.6. *For the curvature tensor, the Ricci tensor and the Webster curvature we have the following variation formulas*

$$\begin{aligned} \dot{R}^{\alpha}_{\beta\rho\overline{\sigma}} &= y^{\alpha}_{\beta\overline{\sigma}\rho} - y^{\alpha}_{\beta\rho\overline{\sigma}} + ix^{\alpha}_{\beta}\delta_{\rho\overline{\sigma}} + R^{\alpha}_{\beta l\overline{\sigma}}F^{l}_{\rho} + R^{\alpha}_{\beta\rho\overline{l}}F^{\overline{l}}_{\overline{\sigma}} \\ &\quad + A_{\overline{\alpha\sigma}}E^{\overline{\beta}}_{\rho} + A_{\beta\rho}E^{\alpha}_{\overline{\sigma}} - A_{\overline{\alpha\gamma}}E^{\overline{\gamma}}_{\rho}\delta_{\beta\overline{\sigma}} - A_{\beta\gamma}E^{\gamma}_{\overline{\sigma}}\delta_{\alpha\rho}, \end{aligned} \tag{5.10}$$

$$\dot{R}_{\rho\overline{\sigma}(t)} = iE^{\overline{\gamma}}_{\rho,\overline{\gamma\sigma}} - iE^{\gamma}_{\overline{\sigma},\gamma\rho} - (A^{\overline{\gamma}}_{l}E^{l}_{\overline{\gamma}} + A^{l}_{\overline{\gamma}}E^{\overline{\gamma}}_{l})\delta_{\rho\overline{\sigma}} + R_{l\overline{\sigma}}F^{l}_{\rho} + R_{\rho\overline{l}}E^{\overline{l}}_{\overline{\sigma}}, \tag{5.11}$$

and

$$\dot{R} = \dot{R}_{\alpha\overline{\alpha}} = iE^{\overline{\gamma}}_{l,\overline{\gamma}\overline{l}} - iE^{\gamma}_{\overline{l},\gamma l} - (A^{\overline{\gamma}}_{l}E^{l}_{\overline{\gamma}} + A^{l}_{\overline{\gamma}}E^{\overline{\gamma}}_{l})n + R_{l\overline{\gamma}}F^{l}_{\gamma} + R_{r\overline{\gamma}}F^{\overline{\gamma}}_{\overline{r}}. \tag{5.12}$$

Proof. Differentiate in t the structure equation

$$\begin{aligned} d\omega^{\alpha}_{\beta(t)} - \omega^{\gamma}_{\beta(t)} \wedge \omega^{\alpha}_{\gamma(t)} &= R^{\alpha}_{\beta\rho\overline{\sigma}(t)}\theta^{\rho}_{(t)} \wedge \theta^{\overline{\sigma}}_{(t)} + W^{\alpha}_{\beta\rho(t)}\theta^{\rho}_{(t)} \wedge \theta - W^{\alpha}_{\beta\overline{\rho}}\theta^{\overline{\rho}}_{(t)} \wedge \theta \\ &\quad + i\theta_{\beta(t)} \wedge \tau^{\alpha}_{(t)} - i\tau_{\beta(t)} \wedge \theta^{\alpha}_{(t)} \end{aligned}$$

where, we recall,

$$\tau_{\beta} = A_{\beta\gamma}\theta^{\gamma}; \qquad \tau^{\alpha} = A^{\alpha}_{\overline{\gamma}}\theta^{\overline{\gamma}},$$

and $A^{\alpha}_{\overline{\gamma}} = A_{\overline{\alpha\gamma}}$ since $h_{\alpha\overline{\beta}} = \delta^{\beta}_{\alpha}$. We then deduce

$$\begin{aligned} d\dot{\omega}^{\alpha}_{\beta} - \dot{\omega}^{\gamma}_{\beta} \wedge \omega^{\alpha}_{\gamma} - \omega^{\gamma}_{\beta} \wedge \dot{\omega}^{\alpha}_{\gamma} &= \dot{R}^{\alpha}_{\beta\rho\overline{\sigma}}\theta^{\rho} \wedge \theta^{\overline{\sigma}} + R^{\alpha}_{\beta\rho\overline{\sigma}}(\dot{\theta}^{\rho} \wedge \theta^{\overline{\sigma}} + \theta^{\rho} \wedge \dot{\theta}^{\bar{\sigma}}) \\ &\quad + i\dot{\theta}^{\overline{\beta}} \wedge A_{\overline{\alpha\gamma}}\theta^{\overline{\gamma}} + i\dot{A}_{\overline{\alpha\gamma}}\theta^{\overline{\beta}} \wedge \theta^{\overline{\gamma}} + iA_{\overline{\alpha\gamma}}\theta^{\overline{\beta}} \wedge \dot{\theta}^{\overline{\gamma}} \\ &\quad - iA_{\beta\gamma}\dot{\theta}^{\gamma} \wedge \theta^{\alpha} - i\dot{A}_{\beta\gamma}\theta^{\gamma} \wedge \theta^{\alpha} - iA_{\beta\gamma}\theta^{\gamma} \wedge \dot{\theta}^{\alpha} \\ &\quad \text{mod.} \quad \theta^{\alpha} \wedge \theta \quad \text{and} \quad \theta^{\overline{\alpha}} \wedge \theta. \end{aligned} \tag{5.13}$$

Writing

$$d\dot{\omega}^{\alpha}_{\beta} = x^{\alpha}_{\beta}d\theta + y^{\alpha}_{\beta\gamma\overline{l}}\theta^{\overline{l}} \wedge \theta^{\gamma} + y^{\alpha}_{\beta\gamma\overline{l}}\theta^{\overline{l}} \wedge \theta^{\gamma} + y^{\alpha}_{\beta\overline{\gamma}l}\theta^{l} \wedge \theta^{\overline{\gamma}},$$

keeping only terms of the type $\theta^{\rho} \wedge \theta^{\overline{\sigma}}$ and using $\omega^{\alpha}_{\gamma}(p) = 0$, we get

$$\begin{aligned} \dot{R}^{\alpha}_{\beta\rho\overline{\sigma}} &= y^{\alpha}_{\beta\overline{\sigma}\rho} - y^{\alpha}_{\beta\rho\overline{\sigma}} + ix^{\alpha}_{\beta}\delta_{\rho\overline{\sigma}} + R^{\alpha}_{\beta l\overline{\sigma}}F^{l}_{\rho} + R^{\alpha}_{\beta\rho\overline{l}}F^{\overline{l}}_{\overline{\sigma}} \\ &\quad + A_{\overline{\alpha\sigma}}E^{\overline{\beta}}_{\rho} + A_{\beta\rho}E^{\alpha}_{\overline{\sigma}} - A_{\overline{\alpha\gamma}}E^{\overline{\gamma}}_{\rho}\delta_{\beta\overline{\sigma}} - A_{\beta\gamma}E^{\gamma}_{\overline{\sigma}}\delta_{\alpha\rho}. \end{aligned} \tag{5.14}$$

Recall that, from (5.9),

$$y^{\alpha}_{\beta\overline{\sigma}} = -iE^{\alpha}_{\overline{\sigma},\beta} + F^{\alpha}_{\beta,\overline{\sigma}} \quad \Rightarrow \quad y^{\alpha}_{\beta\overline{\sigma}\rho} = -iE^{\alpha}_{\overline{\sigma},\beta\rho} + F^{\alpha}_{\beta,\overline{\sigma}\rho};$$

$$y^{\alpha}_{\overline{\beta\gamma}} = -iE^{\overline{\beta}}_{\rho,\overline{\alpha}} - F^{\overline{\beta}}_{\overline{\alpha},\rho} \quad \Rightarrow \quad y^{\alpha}_{\beta\rho\overline{\sigma}} = -iE^{\overline{\beta}}_{\rho,\overline{\alpha\sigma}} - F^{\overline{\beta}}_{\overline{\alpha},\rho\overline{\sigma}}$$

and that $x^{\alpha}_{\beta} = i(A^{\overline{\gamma}}_{\beta}E^{\alpha}_{\overline{\gamma}} + A^{\alpha}_{\overline{\gamma}} + A^{\alpha}_{\overline{\gamma}}E^{\overline{\gamma}}_{\beta}) + F^{\alpha}_{\beta,0}$. These last formulas, together with (5.14), yield (5.10).

Contracting then (5.10), for the Ricci tensor $R_{\rho\overline{\sigma}(t)} := R^{\gamma}_{\ \gamma\rho\overline{\sigma}(t)}$ we obtain, after some cancellation that uses $A_{\alpha\beta} = A_{\beta\alpha}$

$$\dot{R}_{\rho\overline{\sigma}(t)} = iE^{\overline{\gamma}}_{\rho,\overline{\gamma\sigma}} - iE^{\gamma}_{\overline{\sigma},\gamma\rho} - (A^{\overline{\gamma}}_{l}E^{l}_{\overline{\gamma}} + A^{l}_{\overline{\gamma}}E^{\overline{\gamma}}_{l})\delta_{\rho\overline{\sigma}} + R_{l\overline{\sigma}}F^{l}_{\rho} + R_{\rho\bar{l}}E^{\bar{l}}_{\overline{\sigma}}. \tag{5.15}$$

We then obtain for the Webster curvature $W_{(t)} = R_{\alpha\overline{\alpha}}$ with a further contraction (recall that $h_{\alpha\overline{\beta}} = \delta_{\alpha\beta}$)

$$\dot{W} = \dot{R}_{\alpha\overline{\alpha}} = iE^{\overline{\gamma}}_{l,\overline{\gamma}\bar{l}} - iE^{\gamma}_{\bar{l},\gamma l} - (A^{\overline{\gamma}}_{l}E^{l}_{\overline{\gamma}} + A^{l}_{\overline{\gamma}}E^{\overline{\gamma}}_{l})n + R_{l\overline{\gamma}}F^{l}_{r} + R_{r\overline{\gamma}}F^{\bar{l}}_{\overline{\gamma}},$$

where we used $F^{\bar{l}}_{\overline{r}} = \overline{F^{l}_{\gamma}} = F^{l}_{\gamma}$. □

We can now pass to the calculation of the second derivative of the Webster curvature with respect to t.

Proposition 5.7. *For the second variation of $W = W_{(t)}$ at $t = 0$ we have the formula*

$$\begin{aligned}\ddot{W} = {} & i\dot{E}^{\overline{\gamma}}_{l,\overline{\gamma}\bar{l}} - A^{\overline{\gamma}}_{l}\dot{E}^{l}_{\overline{\gamma}}n + R_{l\overline{\gamma}}\dot{F}^{l}_{\gamma} - n\dot{A}^{\overline{\gamma}}_{l}E^{l}_{\overline{\gamma}} \\ & - E^{l}_{\overline{\rho}}E^{\overline{\gamma}}_{\rho,\overline{\gamma}l} - E^{l}_{\overline{\gamma}}E^{\overline{\gamma}}_{\rho,l\overline{\rho}} - E^{\overline{\gamma}}_{l}E^{l}_{\overline{\gamma},\rho\overline{\rho}} - E^{\bar{l}}_{\rho}E^{l}_{\overline{\gamma},\gamma\overline{\rho}} - E^{l}_{\overline{\gamma},\overline{\rho}}E^{\overline{\gamma}}_{\rho,l} \\ & - E^{\overline{\gamma}}_{l,\overline{\rho}}E^{l}_{\overline{\gamma},\rho} - E^{\bar{l}}_{\rho,\overline{\rho}}E^{l}_{\overline{\gamma},\gamma} - E^{l}_{\overline{\rho},\rho}E^{\overline{\gamma}}_{l,\overline{\gamma}} + \text{conj..}\end{aligned} \tag{5.16}$$

Proof. From formula (5.12), we see that the contribution in the second variation from $\dot{E}$ and $\dot{F}$ is given by

$$-(A^{\overline{\gamma}}_{l}\dot{E}^{l}_{\overline{\gamma}} + A^{l}_{\overline{\gamma}}\dot{E}^{\overline{\gamma}}_{l})n + R_{l\overline{\gamma}}\dot{F}^{l}_{\gamma} + R_{r\overline{\gamma}}\dot{F}^{\overline{\gamma}}_{\overline{r}},$$

giving the second and third term in the right-hand side of (5.16), plus their conjugates. To compute the remaining terms, we can therefore assume that $\dot{E} = 0$ and $\dot{F} = 0$.

We will need first some preliminary calculation: recall that

$$E^{\overline{\gamma}}_{\alpha,\beta} = Z_{\beta}(E^{\overline{\gamma}}_{\alpha}) - \omega^{l}_{\alpha}(Z_{\beta})E^{\overline{\gamma}}_{l} + \omega^{\overline{\gamma}}_{\bar{l}}(Z_{\beta})E^{\bar{l}}_{\alpha}.$$

Taking the t-derivative and using that $\dot{E} = 0$ at $t = 0$ we get

$$0 = \dot{Z}_{\beta}(E^{\overline{\gamma}}_{\alpha}) - \left[\dot{\omega}^{l}_{\alpha}(Z_{\beta}) + \omega^{l}_{\alpha}(\dot{Z}_{\beta})\right]E^{\overline{\gamma}}_{l} + \left[\dot{\omega}^{\overline{\gamma}}_{\bar{l}}(Z_{\beta}) + \omega^{\overline{\gamma}}_{\bar{l}}(\dot{Z}_{\beta})\right]E^{\bar{l}}_{\alpha}.$$

Recalling that at $t=0$ we can take $F=0$, we have

$$\dot{\omega}^l_\alpha = i(A^l_{\overline{\gamma}}E^{\overline{\gamma}}_\alpha + E^l_{\overline{\gamma}}A^{\overline{\gamma}}_\alpha)\theta - iE^l_{\overline{\gamma},\alpha}\theta^{\overline{\gamma}} - iE^{\overline{\alpha}}_{\gamma,\bar{l}}\theta^\gamma,$$

which after some calculation implies

$$(E^{\overline{\gamma}}_{\alpha,\beta})^{\cdot} = -iE^{\bar{l}}_\alpha E^{\overline{\gamma}}_{\beta,\bar{l}} + iE^{\overline{\beta}}_{\alpha,\bar{l}}E^{\overline{\gamma}}_l + iE^{\bar{l}}_\beta E^{\overline{\gamma}}_{\alpha,\bar{l}}.$$

In a similar manner, from the formula

$$E^{\overline{\gamma}}_{\alpha,\overline{\beta}} = Z_{\overline{\beta}}(E^{\overline{\gamma}}_\alpha) - \omega^l_\alpha(Z_{\overline{\beta}})E^{\overline{\gamma}}_l + \omega^{\overline{\gamma}}_{\bar{l}}(Z_{\overline{\beta}})E^{\bar{l}}_\alpha$$

one finds, for $t=0$

$$(E^{\overline{\gamma}}_{\alpha,\overline{\beta}})^{\cdot} = iE^\rho_{\overline{\beta}}E^{\overline{\gamma}}_{\alpha,\rho} + iE^{\overline{\gamma}}_l E^l_{\overline{\beta},\alpha} + iE^{\bar{l}}_\alpha E^l_{\overline{\beta},\gamma}.$$

We analyze the terms with second-order covariant derivatives. Notice that

$$E^{\overline{\gamma}}_{\alpha,\overline{\beta}\overline{\rho}} = Z_{\overline{\rho}}(E^{\overline{\gamma}}_{\alpha,\overline{\beta}}) - \omega^l_\alpha(Z_{\overline{\rho}})E^{\overline{\gamma}}_{l,\overline{\beta}} - \omega^{\bar{l}}_{\overline{\beta}}(Z_{\overline{\rho}})E^{\overline{\gamma}}_{\alpha,\bar{l}} + \omega^{\overline{\gamma}}_{\bar{l}}(Z_{\overline{\rho}})E^{\bar{l}}_{\alpha,\overline{\beta}}.$$

Since $\omega=0$ at $t=0$ and at a given point p, this implies

$$\begin{aligned}(E^{\overline{\gamma}}_{\alpha,\overline{\beta}\overline{\rho}})^{\cdot} &= \dot{Z}_{\overline{\rho}}(E^{\overline{\gamma}}_{\alpha,\overline{\beta}}) + Z_{\overline{\rho}}(\dot{E}^{\overline{\gamma}}_{\alpha,\overline{\beta}}) \\ &\quad - \dot{\omega}^l_\alpha(Z_{\overline{\rho}})E^{\overline{\gamma}}_{l,\overline{\beta}} - \dot{\omega}^{\bar{l}}_{\overline{\beta}}(Z_{\overline{\rho}})E^{\overline{\gamma}}_{\alpha,\bar{l}} + \dot{\omega}^{\overline{\gamma}}_{\bar{l}}(Z_{\overline{\rho}})E^{\bar{l}}_{\alpha,\overline{\beta}}.\end{aligned}$$

After some straightforward calculation, one then finds

$$\begin{aligned}(E^{\overline{\gamma}}_{\alpha,\overline{\beta}\overline{\rho}})^{\cdot} &= iE^l_{\overline{\rho}}E^{\overline{\gamma}}_{\alpha,\overline{\beta}l} + iE^l_{\overline{\rho},\alpha}E^{\overline{\gamma}}_{l,\overline{\beta}} + iE^l_{\overline{\rho},\gamma}E^{\bar{l}}_{\alpha,\overline{\beta}} - iE^\beta_{\overline{\rho},l}E^{\overline{\gamma}}_{\alpha,\bar{l}} \\ &+ i\left(E^l_{\overline{\beta},\overline{\rho}}E^{\overline{\gamma}}_{\alpha,l} + E^l_{\overline{\beta}}E^{\overline{\gamma}}_{\alpha,l\overline{\rho}} + E^{\overline{\gamma}}_{l,\overline{\rho}}E^l_{\overline{\beta},\alpha} + E^{\overline{\gamma}}_l E^l_{\overline{\beta},\alpha\overline{\rho}} + E^{\bar{l}}_{\alpha,\overline{\rho}}E^l_{\overline{\beta},\gamma} + E^{\bar{l}}_\alpha E^l_{\overline{\beta},\gamma\overline{\rho}}\right).\end{aligned}$$

In particular, taking the trace we obtain after some cancellation

$$\begin{aligned}(E^{\overline{\gamma}}_{\rho,\overline{\beta}\overline{\rho}})^{\cdot} &= i\left(E^l_{\overline{\rho}}E^{\overline{\gamma}}_{\rho,\overline{\gamma}l} + E^l_{\overline{\gamma}}E^{\overline{\gamma}}_{\rho,l\overline{\rho}} + E^{\overline{\gamma}}_l E^l_{\overline{\gamma},\rho\overline{\rho}} + E^{\bar{l}}_\rho E^l_{\overline{\gamma},\gamma\overline{\rho}}\right) \\ &\quad + i\left(E^l_{\overline{\gamma},\overline{\rho}}E^{\overline{\gamma}}_{\rho,l} + E^{\overline{\gamma}}_{l,\overline{\rho}}E^l_{\overline{\gamma},\rho} + E^{\bar{l}}_{\rho,\overline{\rho}}E^l_{\gamma,\overline{\gamma}} + E^l_{\overline{\rho},\rho}E^{\overline{\gamma}}_{l,\overline{\gamma}}\right).\end{aligned}$$

We have that

$$\begin{aligned}\ddot{W} &= i(E^{\overline{\gamma}}_{l,\overline{\gamma}\bar{l}})^{\cdot} - i(E^\gamma_{\bar{l},\gamma l})^{\cdot} - (\dot{A}^{\overline{\gamma}}_l E^l_{\overline{\gamma}} + \dot{A}^l_{\overline{\gamma}}E^{\overline{\gamma}}_l)n \\ &\quad + (\dot{R}_{l\overline{\gamma}}F^l_\gamma + R_{l\overline{\gamma}}\dot{F}^l_\gamma) + (\dot{R}_{\gamma\bar{l}}F^{\bar{l}}_{\overline{\gamma}} + R_{\gamma\bar{l}}\dot{F}^{\bar{l}}_{\overline{\gamma}}).\end{aligned}$$

The second line indeed vanishes since $F^l_\gamma = 0$ at $t=0$, and since $F^l_l = 0$ implies

$$(R_{l\overline{\gamma}} + R_{\gamma\bar{l}})\dot{F}^l_\gamma = \frac{1}{n}(\delta_{l\gamma} + \delta_{\gamma l})\dot{F}^l_\gamma = \frac{2}{n}W\dot{F}^l_l = 0.$$

This concludes the proof. □

Now we can characterize stationarity for $\widetilde{\mathscr{W}}$.

Theorem 5.8. *A pseudo-Hermitian structure is stationary for $\widetilde{\mathscr{W}}$ if and only if it has constant Webster curvature and zero torsion.*

Proof. The restriction of $\widetilde{\mathscr{W}}$ to the conformal class is the renormalized Yamabe functional, and is is well known that stationarity for the Yamabe functional is equivalent to constancy of the Webster curvature.

As for the variation of J,

$$\dot{\widetilde{\mathscr{W}}} = \int_M \dot{R} = \int_M \left(iE^{\overline{\gamma}}_{l,\overline{\gamma}\bar{l}} - iE^{\gamma}_{\bar{l},\gamma l} - (A^{\overline{\gamma}}_{l} E^{l}_{\overline{\gamma}} + A^{l}_{\overline{\gamma}} E^{\overline{\gamma}}_{l})n + R_{l\overline{\gamma}} F^{l}_{\gamma} + R_{r\overline{\gamma}} F^{\overline{\gamma}}_{\overline{r}} \right) =$$

$$= -n \int_M (A^{\overline{\gamma}}_{l} E^{l}_{\overline{\gamma}} + A^{l}_{\overline{\gamma}} E^{\overline{\gamma}}_{l}).$$

By pseudo-Hermitian Bianchi's identities (see [L2]) and Lemma 5.3 A is a valid deformation tensor, and this allows to get the thesis. □

We point out that under that hypothesis that the first Chern class of $T^{(1,0)}M$, $c_1(T^{(1,0)}M)$, is zero, it can be proved (see [ACMY]) that the thesis of Theorem 5.8 can be strengthened and it can be deduced that the pseudo-Hermitian structure is pseudo-Einstein, that is that the traceless pseudo-Hermitian Ricci tensor is zero (see [L2]).

5.2 Second variation of $\widetilde{\mathscr{W}}$

Finally we can compute the second variation of $\widetilde{\mathscr{W}}$.

Lemma 5.9. *For the standard structure $(S^{2n+1}, J_0, \theta_0)$ we have that*

$$\frac{d^2}{dt^2}\Big|_{t=0} \widetilde{\mathscr{W}}(J_{(t)}, \theta_0) = -i\, n \int_{S^{2n+1}} E_{\alpha}{}^{\overline{\gamma}}{}_{,0} E_{\overline{\gamma}}{}^{\alpha} \theta_0 \wedge (d\theta_0)^n + \text{conj.}, \tag{5.17}$$

where $E = 2\frac{d}{dt}|_{t=0} J_{(t)}$.

Proof. Recalling (5.16), we first notice that the terms involving $\dot{E}$ and $\dot{F}$ vanish since they correspond to the first variation of $\widetilde{\mathscr{W}}$ in the direction $\dot{E}$, but we are at a stationary point.

Concerning the quadratic terms in E, we observe that after integrating and using Lemma 5.3, we obtain cancellation in (5.16) of the first with the seventh, of the second with the fifth, of the third with the sixth and of the fourth with the eighth. We are then left with

$$\ddot{\widetilde{\mathscr{W}}} = -n \int_{S^{2n+1}} (\dot{A}^{\overline{\gamma}}_{l} E^{l}_{\overline{\gamma}} + \dot{A}^{l}_{\overline{\gamma}} E^{\overline{\gamma}}_{l}) \theta_0 \wedge (d\theta_0)^n.$$

Recalling formula (5.4) and the fact that we can take $F^{\alpha}_{\beta} = 0$ at $t = 0$, we obtain the desired conclusion. □

In order to study this formula, we introduce a family of vector fields and forms due to Geller ([Ge]):

$$X_{jk} = \overline{z}_j \frac{\partial}{\partial z_k} - \overline{z}_k \frac{\partial}{\partial z_j}, \quad \theta_{jk} = z_j dz_k - z_k dz_j, \qquad j \neq k. \tag{5.18}$$

It does not consist of linearly independent fields and forms, but it permits to express them through simple formulas. In particular every form of type $(0,1)$ can be written as

$$\eta = \sum_{0 \leq j < k \leq n} \eta(\overline{X}_{jk}) \overline{\theta}_{jk}.$$

Similarly, any form of type $(1,0)$ can be written as

$$\eta = \sum_{0 \leq j < k \leq n} \eta(X_{jk}) \theta_{jk},$$

and any vector field of type $(0,1)$ as

$$X = \sum_{0 \leq j < k \leq n} \overline{\theta}_{jk}(X) \overline{Z}_{jk}.$$

Starting with tensor products of objects of the above form, by linearity a tensor S of type $((0,1);(1,0))$ can be written as

$$S = \sum_{j<k, \ell<m} S(\overline{\theta}_{jk}, Z_{\ell m}) \overline{X}_{jk} \otimes \theta_{\ell m}$$

Lemma 5.10. *We have the following relations*

$$\nabla_T Z_{jk} = -i Z_{jk} \quad \textit{for all } j, k.$$

Proof. Since the pseudohermitian torsion is zero and T is parallel

$$\begin{aligned}
\nabla_T Z_{jk} = [T, Z_{jk}] &= \frac{i}{2} \sum \left[z_\alpha \frac{\partial}{\partial z_\alpha} - \overline{z}_\alpha \frac{\partial}{\partial \overline{z}_\alpha}, \overline{z}_j \frac{\partial}{\partial z_k} - \overline{z}_k \frac{\partial}{\partial z_j} \right] \\
&= \frac{i}{2} \left(-\overline{z}_j \frac{\partial}{\partial z_k} + \overline{z}_k \frac{\partial}{\partial z_j} - \overline{z}_j \frac{\partial}{\partial z_k} + \overline{z}_k \frac{\partial}{\partial z_j} \right) \\
&= i \left(-\overline{z}_j \frac{\partial}{\partial z_k} + \overline{z}_k \frac{\partial}{\partial z_j} \right) = -i Z_{jk},
\end{aligned}$$

□

In order to compute the second variation of $\widetilde{\mathscr{W}}$ we note that S^1 acts on $S^{2n+1} \subset \mathbf{C}^{n+1}$ by

$$\rho(e^{i\theta})(z_1, \ldots, z_{n+1}) = (e^{i\theta} z_1, \ldots, e^{i\theta} z_{n+1}),$$

and accordingly on all tensor spaces on S^{2n+1}. Let us denote by Γ_m the space of tensors with only m-th Fourier component with respect to this action, and with $E^{(m)}$ the projection of E on Γ_m.

Theorem 5.11.

$$d_J^2\widetilde{\mathscr{W}}(\theta_0, J_0)[E,E] = \sum_{m\in\mathbf{Z}}(m+4)\int_{S^3}\left|E^{(m)}\right|^2\theta_0\wedge(d\theta_0)^n$$

Proof. We have that

$$\begin{aligned}(\nabla E)(\overline{\theta}_{jk}, Z_{\ell m}; T) &= T(E(\overline{\theta}_{jk}, Z_{\ell m})) - E(\nabla_T\overline{\theta}_{jk}, Z_{\ell m}) - E(\overline{\theta}_{jk}, \nabla_T Z_{\ell m}) = \\ &= T(E(\overline{\theta}_{jk}, Z_{\ell m})) + 2iE(\overline{\theta}_{jk}, Z_{\ell m}).\end{aligned}$$

We notice that if $E \in \Gamma_m$, since the Reeb field for S^{2n+1} is

$$T = \frac{i}{2}\sum_{\alpha=1}^{n+1} z_\alpha\frac{\partial}{\partial z_\alpha} - \overline{z}_\alpha\frac{\partial}{\partial\overline{z}_\alpha},$$

which is $\frac{i}{2}$ times the generator of the action ρ, then

$$T(E(\overline{\theta}_{jk}, Z_{\ell m})) = \frac{m}{2}E(\overline{\theta}_{jk}, Z_{\ell m}).$$

Let

$$E(\overline{\theta}_{jk}, Z_{\ell m}) = \sum_{A,B} E_{jk\ell m}^{A,\overline{B}} z^A\overline{z}^B,$$

with A, B multi-indices.

By orthogonality we can suppose that $|A| - |B| = m$. Then

$$\begin{aligned}&\sum(\nabla E)(\overline{\theta}_{jk}, Z_{\ell m}; T)E(\theta_{\alpha\beta}, \overline{Z}_{\mu\nu})\overline{\theta}_{\mu\nu}(\overline{Z}_{jk})\theta_{\ell m}(Z_{\alpha\beta}) = \\ &= \sum i\left(\frac{m}{2}+2\right)E_{jk\ell m}^{A,\overline{B}} z^A\overline{z}^B\,\overline{E_{\alpha\beta\mu\nu}^{C,\overline{D}}}\,\overline{z}^C z^D\overline{\theta}_{\mu\nu}(\overline{Z}_{jk})\theta_{\ell m}(Z_{\alpha\beta}) = \\ &= i\left(\frac{m}{2}+2\right)\sum E(\theta_{jk}, \overline{Z}_{\ell m})E(\overline{\theta}_{\alpha\beta}, Z_{\mu\nu})\overline{\theta}_{\mu\nu}(\overline{Z}_{jk})\theta_{\ell m}(Z_{\alpha\beta})\end{aligned}$$

so

$$\begin{aligned}\ddot{\mathscr{R}} &= \frac{n}{2}\sum_{m\in\mathbf{Z}}(m+4)\int_{S^3}((E^{(m)})_\alpha{}^{\overline{\gamma}}(E^{(m)})_{\overline{\gamma}}{}^{\alpha} + \text{conj.} = \\ &= n\sum_{m\in\mathbf{Z}}(m+4)\int_{S^3}\left|E^{(m)}\right|^2.\end{aligned}$$

□

Thanks to results of Bland and Duchamp in [Bl, BD], the above formula implies that

- if $n \geq 2$ then $d^2\widetilde{\mathscr{W}}(\theta_0, J_0)$ is negative semidefinite, analogously to the Riemannian case;
- if $n = 1$ then $d^2\widetilde{\mathscr{W}}(\theta_0, J_0)$ is negative semidefinite on the subspace corresponding to embeddable deformations

(see [ACMY] for further comments). We recall that it is known that every pseudoconvex CR manifold of dimension greater or equal to 5 is embeddable in $\mathbf{C}^N$ for some N thanks to a result of Boutet de Monvel, while in dimension three not every CR manifold is embeddable (see [CS]).

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