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# An arithmetic valuative criterion for proper maps of tame algebraic stacks

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**Abstract.** The valuative criterion for proper maps of schemes has many applications in arithmetic, e.g. specializing $\mathbb{Q}_p$-points to $\mathbb{F}_p$-points. For algebraic stacks, the usual valuative criterion for proper maps is ill-suited for these kind of arguments, since it only gives a specialization point defined over an extension of the residue field, e.g. a $\mathbb{Q}_p$-point will specialize to an $\mathbb{F}_{p^n}$-point for some $n$. We give a new valuative criterion for proper maps of tame stacks which solves this problem and is well-suited for arithmetic applications. As a consequence, we prove that the Lang–Nishimura theorem holds for tame stacks.

## 1. Introduction

The well known and extremely useful valuative criterion for properness says, in particular, that if $X \to Y$ is a proper morphism of schemes, $R$ is a DVR with quotient field $K$ and residue field $k$, and we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

there exists a unique lifting $\operatorname{Spec} R \to X$ of $\operatorname{Spec} R \to Y$ extending $\operatorname{Spec} K \to X$. This has many arithmetic applications: most of them use that the statement above ensures the existence of a lifting $\operatorname{Spec} k \to X$ of the composite $\operatorname{Spec} k \subseteq \operatorname{Spec} R \to Y$.

If $X$ and $Y$ are algebraic stacks, and $X \to Y$ is a proper morphism, then this fails, even in very simple examples, unless $X \to Y$ is representable. The correct general statement is that there exists a local extension of DVR $R \to R'$, such that

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if we denote by $K'$ the fraction field of $R'$, the composite $\operatorname{Spec} R' \to \operatorname{Spec} R \to Y$ has a lifting $\operatorname{Spec} R' \to X$ extending the composite $\operatorname{Spec} K' \to \operatorname{Spec} K \to X$ (see for example [18, Tag 0CLY]). For arithmetic applications this is problematic, because the extension $R \subseteq R'$ will typically induce a nontrivial extension of residue fields, so it does not imply that $\operatorname{Spec} k \to Y$ lifts to $\operatorname{Spec} k \to X$, as in the case of schemes.

When $X$ and $Y$ are Deligne–Mumford stacks over a field of characteristic 0 a substitute was found by the first author in 2019, see [4, Appendix B]. In this note we extend this, in a somewhat more precise form, to positive and mixed characteristic. In this context the correct generality is that of *tame stacks*, in the sense of [2]. Tame stacks are algebraic stacks with finite inertia, such that the automorphism group scheme of any object over a field is linearly reductive. In characteristic 0 they coincide with Deligne–Mumford stacks with finite inertia, but in positive and mixed characteristic there are Deligne–Mumford stacks with finite inertia that are not tame, and tame stacks that are not Deligne–Mumford.

In our version the role of the DVR $R'$ above is played by a *root stack* $\sqrt[n]{\operatorname{Spec} R}$; this is not a scheme, but a tame stack with a map $\sqrt[n]{\operatorname{Spec} R} \to \operatorname{Spec} R$, which is an isomorphism above $\operatorname{Spec} K \subseteq \operatorname{Spec} R$ (see the discussion at the beginning of 3). The statement of our main theorem 3.1 is that if $X \to Y$ is a proper morphism of tame algebraic stacks and we have a commutative diagram as above, there exists a unique positive integer $n$ and a unique representable lifting $\sqrt[n]{\operatorname{Spec} R} \to X$ of the composite $\sqrt[n]{\operatorname{Spec} R} \to \operatorname{Spec} R \to Y$ extending $\operatorname{Spec} K \to X$. The key point for arithmetic applications is that the closed point $\operatorname{Spec} k \to \operatorname{Spec} R$ lifts to $\operatorname{Spec} k \to \sqrt[n]{\operatorname{Spec} R}$. This statement is much harder to prove in arbitrary characteristic than in characteristic 0.

Besides the original applications to Grothendieck's section conjecture in [4] [5], this valuative criterion has been applied in [12] to give new proofs and stronger versions of the genericity theorem for essential dimension.

Recall that the Lang–Nishimura theorem states that the property of having a rational point is a birational invariant of smooth proper varieties. Another consequence of our version of the valuative criterion is that the Lang–Nishimura theorem generalizes to tame stacks, see 4.1. Our version of the Lang–Nishimura theorem has an immediate corollary, which we find surprising: if $\mathscr{M}$ is a smooth tame stack which is generically a scheme and $\overline{M} \to M$ is a resolution of singularities of the coarse moduli space $\mathscr{M} \to M$, then a rational point of $M(k)$ lifts to $\mathscr{M}$ if and only if it lifts to $\overline{M}$. This gives a hint of the applications of the Lang–Nishimura theorem to fields of moduli, see [6–11].

Daniel Loughran pointed out to us that, in 2021, J. Ellenberg, M. Satriano and D. Zureick-Brown introduced the related concept of *tuning stack* [13]. With notation as above, if we only assume that $X$ and $Y$ have finite inertia (as opposed to being tame), they prove that there exists a unique algebraic stack $\mathscr{C}$ with finite inertia, coarse moduli space equal to $\operatorname{Spec} R$, a lifting $\operatorname{Spec} K \subset \mathscr{C}$ which is an open embedding and a faithful extension $\mathscr{C} \to X$ of $\operatorname{Spec} K \to X$ unique up to a unique isomorphism. The morphism $\mathscr{C} \to X$ is called a *universal tuning stack* for $\operatorname{Spec} K \to X$. From this point of view our theorem says that, for tame stacks, the universal tuning stack is a root stack and hence it has a rational point in the special fiber. If we drop the tameness assumption, the universal tuning stack might have no rational points in the special fiber, see Examples 4.2 and 4.3.

## 2. Notations and conventions

We will follow the conventions of [14] and [15]; so the diagonals of algebraic spaces and algebraic stacks will be separated and of finite type. In particular, every algebraic space will be *decent*, in the sense of [18, Definition 03I8].

We will follow the terminology of [2]: a *tame stack* is an algebraic stack $X$ with finite inertia, such that its geometric points have linearly reductive automorphism group. This is equivalent to requiring that $X$ is étale locally over its moduli space a quotient by a finite, linearly reductive group scheme [2, Theorem 3.2].

More generally, a morphism $f: X \to Y$ of algebraic stacks is *tame* if the relative inertia group stack $I_{X/Y} \to X$, defined as in [18, Section 050P], is finite and has linearly reductive geometric fibers. See [3, 3].

Using [18, Lemma 0CPK] one can easily prove the following.

**Proposition 2.1.** *Let $f: X \to Y$ be a morphism of algebraic stacks. The following conditions are equivalent.*

(1) *$f$ is tame.*
(2) *If $Z \to Y$ is a morphism, and $Z$ is a scheme, then $Z \times_Y X$ is a tame stack.*
(3) *If $Z \to Y$ is a morphism, and $Z$ a tame stack, then $Z \times_Y X$ is also tame.*

*Furthermore, if $X$ is tame, then the morphism $f$ is also tame.*

## 3. The valuative criterion

A basic example of tame stacks is *root stacks* (see [1, Appendix B2]). Let $S$ be a scheme with an effective Cartier divisor $D \subset S$, and let $\pi \in \mathcal{O}_S(D)(S)$ be a global section defining $D$. If $n$ is a positive integer, we will denote by $\sqrt[n]{S, D}$ the $n^{\text{th}}$ root of $D \subset A$. It is a stack over $S$, such that given a morphism $\phi: T \to S$, the groupoid of liftings $T \to \sqrt[n]{S, D}$ is equivalent to the groupoid whose objects are triples $(L, s, \alpha)$, where $L$ is an invertible sheaf on $T$, $s \in L(T)$ is a global section of $L$, and $\alpha$ is an isomorphism $L^{\otimes n} \simeq \phi^* \mathcal{O}_S(D)$, such that $\alpha(s^{\otimes n}) = \phi^\sharp(\pi)$. This definition does not depend on the choice of $\pi$: if we replace $\pi$ with $\lambda\pi$ for some invertible global section $\lambda \in \mathcal{O}_S^*(S)$ we obtain an isomorphic stack. The morphism $\rho : \sqrt[n]{S, D} \to S$ is an isomorphism on $S \backslash D$. If $S = \operatorname{Spec} R$ is affine and $\mathcal{O}_S(D)$ is trivial, e.g. if $S$ is the spectrum of a DVR, $\sqrt[n]{S, D}$ can be alternatively described as the quotient stack $[\operatorname{Spec} R[t]/(t^n - \pi)/\boldsymbol{\mu}_n]$, where the action of $\boldsymbol{\mu}_n$ on $\operatorname{Spec} R[t]/(t^n - \pi)$ is by multiplication on $t$.

If $R$ is a DVR with residue field $k$ and $\pi \in R$ is a uniformizing parameter, we simply write $\sqrt[n]{\operatorname{Spec} R}$. In this case, the reduced fiber $\rho^{-1}(\operatorname{Spec} k)_{\text{red}}$ is non-canonically isomorphic to the classifying stack $\mathcal{B}_k \boldsymbol{\mu}_n$. In particular the embedding $\operatorname{Spec} k \to \operatorname{Spec} R$ lifts to a morphism $\operatorname{Spec} k \to \sqrt[n]{\operatorname{Spec} R}$.

The following is our version of the valuative criterion for properness.

**Theorem 3.1.** *Let $f: X \to Y$ be a tame, proper morphism of algebraic stacks, $R$ a DVR with quotient field $K$. Suppose that we have a 2-commutative square*

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

*Then there exists a unique positive integer $n$ and a representable lifting $\sqrt[n]{\operatorname{Spec} R} \to X$ of the given morphism $\operatorname{Spec} R \to Y$, making the diagram*

$$\begin{array}{ccccc} & & \operatorname{Spec} K & \longrightarrow & X \\ & \swarrow & \downarrow & \nearrow & \downarrow \\ \sqrt[n]{\operatorname{Spec} R} & \longrightarrow & \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

*2-commutative. Furthermore, the lifting is unique up to a unique isomorphism.*

**Corollary 3.2.** *In the situation above, if $k$ is the residue field of $R$, the composite* $\operatorname{Spec} k \subseteq \operatorname{Spec} R \to Y$ *has a lifting* $\operatorname{Spec} k \to X$.

As one would expect, these statements fail without the tameness hypothesis, even when $Y$ is a scheme and $X$ is a separated Deligne–Mumford stack.

*Example 3.3.* Let $p$ be a prime, $R$ a DVR whose fraction field $K$ has characteristic 0 and contains a $p$-th root of 1, denoted by $\zeta_p$, while its residue field $k$ has characteristic $p$ and is not perfect. An example would be the localization of $\mathbb{Z}[\zeta_p][t]$ at a prime ideal of height 1 containing $p$.

Choose an element $a \in R^*$ whose image in $k$ is not a $p$-th power, and set $R' \stackrel{\text{def}}{=} R(\sqrt[p]{a})$. Then $R'$ is a DVR, since the $R' \otimes_R k = k(\sqrt[p]{\overline{a}})$ is a field (here $\overline{a}$ is the class of $a$ in $k$). Write $K' = K(\sqrt[p]{a})$ for its fraction field and $k' = k(\sqrt[p]{\overline{a}})$ for its residue field.

Call $C_p$ the cyclic group of order $p$ generated by $\zeta_p \in K^*$. The extension $K'/K$ is Galois with cyclic Galois group $C_p$ acting by $\sqrt[p]{a} \mapsto \zeta_p \sqrt[p]{a}$. The action of $C_p$ on $K'$ naturally extends to $R'$.

Let $X$ be the quotient stack $[\operatorname{Spec} R'/C_p]$; this is a separated Deligne–Mumford stack, but it is not tame. Since $(R')^{C_p} = R$ the moduli space of $X$ is $\operatorname{Spec} R$, and we have a natural map $X \to \operatorname{Spec} R$, which is an isomorphism over $\operatorname{Spec} K \subseteq \operatorname{Spec} R$. The morphism $\operatorname{Spec} R' \to X'$ is surjective and the composition $\operatorname{Spec} R' \to X \to \operatorname{Spec} R$ is proper, hence $X \to \operatorname{Spec} R$ is proper as well. Since $k'/k$ is purely inseparable, then $X_k(k)$ is empty: such a $k$-rational point would correspond to a $C_p$-torsor $\operatorname{Spec} A \to \operatorname{Spec} k$ with an equivariant morphism $\operatorname{Spec} A \to \operatorname{Spec} k'$ and thus an embedding of $k'$ in the étale $k$-algebra $A$, which is clearly absurd. In particular, there is no map $\sqrt[n]{\operatorname{Spec} R} \to X$ for any $n$.

**Definition 3.4.** In the situation of Theorem 3.1, we call the integer $n$ the *loop index* of the diagram at the place associated with $R \subseteq K$. If the morphisms $\operatorname{Spec} R \to Y$, $X \to Y$ are implicit in the context (e.g. if $Y$ is $\operatorname{Spec} R$ or the spectrum of a base field) we simply call $n$ the loop index of the morphism $\operatorname{Spec} K \to X$ at the place associated with $R \subseteq K$. If the loop index is 1, we say that $\operatorname{Spec} K \to X$ (more generally, the diagram) is *untangled.*

**Lemma 3.5.** *Let $R \subseteq R'$ be an extension of DVRs with ramification index e, and let $K \subseteq K'$ be the fraction fields of $R$ and $R'$ respectively. If $X$ is a tame stack proper over $R$ and* $\operatorname{Spec} K \to X$ *is a morphism with loop index n, the composite* $\operatorname{Spec} K' \to \operatorname{Spec} K \to X$ *has loop index equal to* $n/\gcd(n, e)$.

*Proof.* Write $m \overset{\text{def}}{=} n/\gcd(n, e)$. The statement follows from the fact that there is a natural representable morphism $\sqrt[m]{\operatorname{Spec} R'} \to \sqrt[n]{\operatorname{Spec} R}$ inducing the given morphism $\operatorname{Spec} K' \to \operatorname{Spec} K$; let us construct it.

Let $\pi, \pi'$ be uniformizing parameters of $R, R'$. Write $\lambda \overset{\text{def}}{=} \pi \cdot \pi'^{-e} \in R'^*$ and $b \overset{\text{def}}{=} e/\gcd(n, e)$, we have $me = nb$. Let $\phi : T \to \operatorname{Spec} R'$ be a scheme over $R'$, we want to give a functor $\sqrt[m]{\operatorname{Spec} R'}(T) \to \sqrt[m]{\operatorname{Spec} R'}(T)$. Let $T \to \sqrt[m]{\operatorname{Spec} R'}$ be a section associated with a triple $(L, s, \alpha)$, where $L$ is an invertible sheaf over $T$, $s \in L(T)$ is a global section and $\alpha$ is an isomorphism $L^{\otimes m} \simeq \mathscr{O}_T$ such that $\alpha(s^{\otimes m}) = \phi^\sharp(\pi')$.

The triple $(L^{\otimes b}, s^{\otimes b}, \phi^\sharp(\lambda) \cdot \alpha^{\otimes e})$ defines a section $T \to \sqrt[n]{\operatorname{Spec} R}$, hence we get a morphism $\sqrt[m]{\operatorname{Spec} R'} \to \sqrt[n]{\operatorname{Spec} R}$. The corresponding morphism $\mathscr{B}\mu_m \to \mathscr{B}\mu_n$ of reduced fibers is faithful since it is associated with the homomorphism $\mu_m \to \mu_n$ defined by $x \mapsto x^b$, which is injective since $b$ is prime with $m$. It follows that $\sqrt[m]{\operatorname{Spec} R'} \to \sqrt[n]{\operatorname{Spec} R}$ is faithful. □

We spend the rest of this section proving Theorem 3.1. Given a DVR $R$ and $\pi \in R$ a uniformizing parameter, write $R^{(n)} \overset{\text{def}}{=} R[t]/(t^n - \pi)$, $K^{(n)} \overset{\text{def}}{=} K[t]/(t^n - \pi)$. We have $\sqrt[n]{\operatorname{Spec} R} = [\operatorname{Spec} R^{(n)}/\boldsymbol{\mu}_n]$.

**Lemma 3.6.** *Let $R$ be a DVR, $m, n$ integers. A morphism* $\sqrt[m]{\operatorname{Spec} R} \to \sqrt[n]{\operatorname{Spec} R}$ *over $R$ exists if and only if $n|m$, and in this case it is unique up to equivalence.*

*Proof.* This follows from the fact that a section $\operatorname{Spec} R^{(m)} \to \sqrt[n]{\operatorname{Spec} R}$ exists if and only if $n|m$, and in this case it is unique up to equivalence. □

**Lemma 3.7.** *Let $R$ be a DVR, $D \subset$* $\operatorname{Spec} R$ *the divisor corresponding to the closed point, $m, n_i, r_i$ for $i = 1, \ldots, m$ positive integers, with $n_i \geq 2$ for every $i$. The fibered product over $R$*

$$X = \prod_{i=1}^{m} \sqrt[n_i]{\operatorname{Spec} R, r_i D}$$

*is normal if and only if $m = 1$ and $r_1 = 1$.*

*Proof.* Notice that $X$ has only two points, an open one and a closed one, and the open one is normal. Hence, $X$ is normal if and only if it is normal at the closed point.

Let $V_{n,r}$ be the scheme $\operatorname{Spec} R[t, s]_s/(t^n s - \pi^r)$, there is an action of $\mathbb{G}_m$ on $V_{n,r}$ given by $(\lambda, t, s) \mapsto (\lambda t, \lambda^{-n} s)$ and $\sqrt[n]{\operatorname{Spec} R, rD} \simeq [V_{n,r}/\mathbb{G}_m]$, see [1, Appendix B]. Consider the fibered product

$$Y = \prod_i V_{n_i, r_i} = \operatorname{Spec} R[t_1, s_1, \ldots, t_m, s_m]_{s_1 \cdots s_m}/(t_i^{n_i} s_i - \pi^{r_i}).$$

The prime ideal $\mathfrak{p} = (t_1, \dots, t_m, \pi)$ is the generic point of the special fiber and has height 1. Since $V_{r,n} \to \sqrt[n]{\operatorname{Spec} R, rD}$ is surjective and smooth, then $Y \to X$ is smooth. Since $\mathfrak{p}$ maps to the closed point of $X$, then $X$ is normal if and only if $Y$ is normal at $\mathfrak{p}$.

Now consider the prime ideal $\mathfrak{p}_0 = (t_1, \dots, t_m, \pi) \subset R[t_1, s_1, \dots, t_m, s_m]$, $\mathfrak{p}$ and $\mathfrak{p}_0$ have equal residue fields and there is a natural surjective linear map $\mathfrak{p}_0/\mathfrak{p}_0^2 \to \mathfrak{p}/\mathfrak{p}^2$. We have that $\mathfrak{p}_0/\mathfrak{p}_0^2$ has dimension $m+1$ generated by the classes of $[t_i]$, $[\pi]$. If $m = 1$ and $r_1 = 1$, then $\mathfrak{p}_0/\mathfrak{p}_0^2$ has dimension 2 and $[\pi]$ is in the kernel of $\mathfrak{p}_0/\mathfrak{p}_0^2 \to \mathfrak{p}/\mathfrak{p}^2$, hence $Y$ is normal at $\mathfrak{p}$.

On the other hand, assume that $Y$ is normal at $\mathfrak{p}$, so that $\mathfrak{p}/\mathfrak{p}^2$ has dimension 1. Since $n_i \geq 2$ for every $i$, the kernel of $\mathfrak{p}_0/\mathfrak{p}_0^2 \to \mathfrak{p}/\mathfrak{p}^2$ is generated by the classes $[\pi^{r_i}]$ and hence has dimension 1 if $r_i = 1$ for some $i$, and dimension 0 otherwise. Since $\mathfrak{p}_0/\mathfrak{p}_0^2$ has dimension $m+1$ and $\mathfrak{p}/\mathfrak{p}^2$ has dimension 1, this implies that $m = 1$ and $r_1 = 1$. □

**Lemma 3.8.** *Let $A$ be a Dedekind domain with fraction field $K$, $D \subset \operatorname{Spec} A$ an effective, reduced divisor. Let $f : X \to \sqrt[n]{(\operatorname{Spec} A, D)}$ be a representable, proper morphism. Every generic section $\operatorname{Spec} K \to X$ of $f$ extends uniquely to a global section $\sqrt[n]{(\operatorname{Spec} A, D)} \to X$.*

*Proof.* Let $Y \subset X$ be the schematic image of a generic section $\operatorname{Spec} K \to X$, we want to prove that $Y \to \sqrt[n]{(\operatorname{Spec} A, D)}$ is an isomorphism. Since the problem is local, we may assume that $A = R$ is a DVR and $D$ is either empty or the closed point. If $D$ is empty, then $\sqrt[n]{\operatorname{Spec} R, D} = \operatorname{Spec} R$ and this is simply the valuative criterion of properness. Suppose that $D$ is the closed point. Consider the flat morphism $\operatorname{Spec} R^{(n)} \to \sqrt[n]{\operatorname{Spec} R}$ and write $X' = X \times_{\sqrt[n]{\operatorname{Spec} R}} \operatorname{Spec} R^{(n)}$, $Y' = Y \times_{\sqrt[n]{\operatorname{Spec} R}} \operatorname{Spec} R^{(n)}$. Thanks to [18, Lemma 0CMK] we have that $Y' \subset X'$ is the schematic image of the induced generic section $\operatorname{Spec} K^{(n)} \to X'$. By the valuative criterion of properness, there is a section $\operatorname{Spec} R^{(n)} \to X'$ which is a closed immersion since $X'$ is representable, this implies that $Y' \to \operatorname{Spec} R^{(n)}$ is an isomorphism. It follows that $Y \to \sqrt[n]{\operatorname{Spec} R}$ is an isomorphism, too. □

**Lemma 3.9.** *Let $R$ be a DVR, $n, m$ positive integers. Assume that $n$ is prime with the residue characteristic of $R$. Consider the $\mu_n$-torsor $\operatorname{Spec} R^{(n)} \to \sqrt[n]{\operatorname{Spec} R}$. There exists a unique way of extending the action of $\mu_n$ to $\sqrt[m]{\operatorname{Spec} R^{(n)}}$, and the quotient $[\sqrt[m]{\operatorname{Spec} R^{(n)}}/\mu_n]$ is isomorphic to $\sqrt[mn]{\operatorname{Spec} R}$.*

*Proof.* We have a natural action $\rho : \sqrt[m]{\operatorname{Spec} R^{(n)}} \times_R \mu_n \to \sqrt[m]{\operatorname{Spec} R^{(n)}}$ induced by the action on $\operatorname{Spec} R^{(n)}$. The action $\rho$ gives a structure of $\mu_n$-torsor to the natural morphism $\sqrt[m]{\operatorname{Spec} R^{(n)}} \to \sqrt[mn]{\operatorname{Spec} R}$. Let $\eta : \sqrt[m]{\operatorname{Spec} R^{(n)}} \times_R \mu_n \to \sqrt[m]{\operatorname{Spec} R^{(n)}}$ be an action such that the diagram

$$\begin{array}{ccc} \sqrt[m]{\operatorname{Spec} R^{(n)}} \times_R \mu_n & \xrightarrow{\eta} & \sqrt[m]{\operatorname{Spec} R^{(n)}} \\ \downarrow & & \downarrow \\ \operatorname{Spec} R^{(n)} \times_R \mu_n & \longrightarrow & \operatorname{Spec} R^{(n)} \end{array}$$

is 2-commutative, we want to show that $\rho$ and $\eta$ are equivalent.

Let $D \subset \operatorname{Spec} R^{(n)} \times_R \mu_n$ be the pullback of the closed point of $\operatorname{Spec} R^{(n)}$, since $\mu_n$ is finite étale over $R$ we have that $D$ is a reduced divisor. Since $\mu_n$ is étale, the natural morphism $\sqrt[m]{\operatorname{Spec} R^{(n)} \times_R \mu_n, D} \to \sqrt[m]{\operatorname{Spec} R^{(n)}} \times_R \mu_n$ is an isomorphism. The scheme $\operatorname{Spec} R^{(n)} \times_R \mu_n$ is finite étale over $\operatorname{Spec} R^{(n)}$, hence it is a disjoint union of Dedekind domains, and $\sqrt[m]{\operatorname{Spec} R^{(n)}} \times_R \mu_n = \sqrt[m]{\operatorname{Spec} R^{(n)} \times_R \mu_n, D}$ is a disjoint union of root stacks over Dedekind domains.

The stack $\underline{\operatorname{Isom}}(\rho, \eta)$ has a proper, representable morphism

$$\underline{\operatorname{Isom}}(\rho, \eta) \to \sqrt[m]{\operatorname{Spec} R^{(n)}} \times_R \mu_n,$$

and for every connected component of $\sqrt[m]{\operatorname{Spec} R^{(n)}} \times_R \mu_n$ there is a generic section. By Lemma 3.8, these generic sections extend to global sections, hence $\eta \simeq \rho$. □

**Corollary 3.10.** *Let $R$ be a DVR, $n, m$ positive integers. Assume that $n$ is prime with the residue characteristic of $R$. Let $X \to \sqrt[n]{\operatorname{Spec} R}$ be a morphism, and assume that the base change of $X$ to* $\operatorname{Spec} R^{(n)}$ *is isomorphic to $\sqrt[m]{\operatorname{Spec} R^{(n)}}$. Then $X \simeq \sqrt[mn]{\operatorname{Spec} R}$.*

**Lemma 3.11.** *Let $R'/R$ be an étale extension of DVRs and $X$ an algebraic stack over $R$, $X' \stackrel{\text{def}}{=} X_{R'}$. If $X' \simeq \sqrt[n]{\operatorname{Spec} R'}$, then $X \simeq \sqrt[n]{\operatorname{Spec} R}$.*

*Proof.* Let $K$ be the residue field of $R$, clearly we have that $X_K \to \operatorname{Spec} K$ is an isomorphism. Write $A = \operatorname{Spec} R' \otimes_R R'$, since $R'$ is étale over $R$ then $A$ is a product of Dedekind domains with a finite number of closed points. Let $D \subset \operatorname{Spec} A$ be the effective, reduced divisor of all closed points and $S \stackrel{\text{def}}{=} \sqrt[n]{\operatorname{Spec} R'} \times_{\sqrt[n]{\operatorname{Spec} R}} \sqrt[n]{\operatorname{Spec} R'}$, it is easy to see that $S \simeq \sqrt[n]{(\operatorname{Spec} A, D)}$.

Let $\phi : \sqrt[n]{\operatorname{Spec} R'} \simeq X' \to X$ be the composite, and consider the two projections $p_1, p_2 : S \to \sqrt[n]{\operatorname{Spec} R'}$. Since $X_{R'}$ is separated, then $X$ is separated, too, and hence $\underline{\operatorname{Isom}}(p_1^*\phi, p_2^*\phi)$ is an algebraic stack with a proper, representable morphism to $S$. There is a generic section $S_K \to \underline{\operatorname{Isom}}(p_1^*\phi, p_2^*\phi)$ which extends to a global section thanks to Lemma 3.8, this gives descent data for a morphism $f : \sqrt[n]{\operatorname{Spec} R} \to X$ (the cocycle condition can be checked on the generic point, where it is obvious). Since the base change to $R'$ of $f$ is an isomorphism, we have that $f$ is an isomorphism, too. □

**Proposition 3.12.** *Let $X$ be a normal, tame stack of finite type over a DVR $R$, and assume that there is a generic section* $\operatorname{Spec} K \to X$ *which is an open, scheme-theoretically dense embedding. Then $X \simeq \sqrt[n]{\operatorname{Spec} R}$ for some $n$.*

*Proof.* Since $X$ is of finite type over $R$, there exists a DVR $R_0 \subset R$ which is the localization of a $\mathbb{Z}$-algebra of finite type and a stack $X_0/R_0$ such that $X \simeq X_{0,R}$. Furthermore, we may assume that the uniformizing parameter of $R_0$ maps to a uniformizing parameter of $R$, so that $\sqrt[n]{\operatorname{Spec} R_0} \times_{R_0} \operatorname{Spec} R \simeq \sqrt[n]{\operatorname{Spec} R}$. Up to replacing $R$, $X$ with $R_0$, $X_0$ we may assume that $R$ is Nagata. Let $k$ be the residue field of $R$ and $p$ its characteristic.

By [2, Theorem 3.2], there exists a DVR $R'$ étale over $R$ and a finite, flat, linearly reductive group scheme $G/R'$ with an action on a scheme $U$ finite over $R'$ such that $X_{R'} \simeq [U/G]$. Up to enlarging $R'$, by [2, Lemma 2.17] there exists a diagonalizable flat, closed subgroup $\Delta \subset G$ such that $H \overset{\text{def}}{=} G/\Delta$ is constant and tame. We may furthermore assume that the degree of $\Delta$ is a power of $p$. Thanks to Lemma 3.11, we may assume $R' = R$.

**Case 1.** $X$ is tame and Deligne-Mumford. Since $\Delta_k$ is connected and $X$ is Deligne-Mumford and generically a scheme, the action of $\Delta$ is free (because otherwise $X$ would have ramified inertia), hence up to replacing $U$ with $U/\Delta$ we may assume that $G$ is constant and tame. Since $X$ is normal and $G$ is constant and tame, then $U$ is normal, too. If $u \in U$ is a geometric point, the stabilizer $G_u$ acts faithfully on the tangent space, hence the automorphism groups of the points of $X$ are cyclic and tame. By [17, Lemma 8.5] and Lemma 3.7, since $X$ is normal we have $X \simeq \sqrt[n]{\operatorname{Spec} R}$ for some $n$.

**Case 2.** $X$ is tame. Let $V \overset{\text{def}}{=} U/\Delta$ and $Y_0 \overset{\text{def}}{=} [V/H]$, we have that $Y_0$ is Deligne-Mumford and there is a natural birational morphism $X \to Y_0$ whose relative inertia is diagonalizable. Let $Y \to Y_0$ be the normalization, it is finite over $Y_0$ since $R$ is Nagata and since $X$ is normal the morphism $X \to Y_0$ lifts to a morphism $X \to Y$. By case 1, there exists an $n$ prime with $p$ and an isomorphism $Y \simeq \sqrt[n]{\operatorname{Spec} R}$. Consider the morphism $\operatorname{Spec} R^{(n)} \to \sqrt[n]{\operatorname{Spec} R}$, it is a $\mu_n$-torsor and hence finite étale since $n$ is prime with $p$, it follows that the base change $X \times_{\sqrt[n]{\operatorname{Spec} R}} \operatorname{Spec} R^{(n)}$ is normal with diagonalizable inertia. By [17, Lemma 8.5] and Lemma 3.7, we have $X \times_{\sqrt[n]{\operatorname{Spec} R}} \operatorname{Spec} R^{(n)} \simeq \sqrt[m]{\operatorname{Spec} R^{(n)}}$ for some integer $m$, hence $X \simeq \sqrt[mn]{\operatorname{Spec} R}$ thanks to Corollary 3.10. □

*Proof of Theorem 3.1.* By base change, we may assume that $Y = \operatorname{Spec} R$ and that $X$ is a tame stack proper over $R$. With an argument similar to the one in the proof of Proposition 3.12, we may reduce to the case in which $R$ is Nagata.

By [17, Theorem B], we may assume that $\operatorname{Spec} K \to X$ is an open, scheme theoretically dense embedding. Since $R$ is Nagata, the normalization $\overline{X}$ is finite and representable over $X$. By Proposition 3.12 we have $\overline{X} = \sqrt[n]{\operatorname{Spec} R}$, hence an extension exists. If $m$ is another integer with a representable extension $\sqrt[m]{\operatorname{Spec} R} \to X$, it factors through $\overline{X} = \sqrt[n]{\operatorname{Spec} R}$ since $\sqrt[m]{\operatorname{Spec} R}$ is normal by Lemma 3.7. We conclude the proof of Theorem 3.1 by Lemma 3.6. □

## 4. The Lang–Nishimura theorem

Here is our version of the Lang–Nishimura theorem for tame stacks.

**Theorem 4.1.** *Let $S$ be a scheme and $X \dashrightarrow Y$ a rational map of algebraic stacks over $S$, with $X$ locally noetherian and integral and $Y$ tame and proper over $S$. Let $k$ be a field, $s\colon \operatorname{Spec} k \to S$ a morphism. Assume that $s$ lifts to a regular point $\operatorname{Spec} k \to X$; then it also lifts to a morphism $\operatorname{Spec} k \to Y$.*

In the standard version of the Lang–Nishimura theorem (see for example [16, Theorem 3.6.11]), which is a standard tool in arithmetic geometry, $X$ and $Y$ are schemes, and $S = \operatorname{Spec} k$. In the applications that we have in mind, the additional flexibility of having a base scheme is important.

*Proof.* According to [15, Théorème 6.3] we can find a smooth morphism $U \to X$ with a lifting $\operatorname{Spec} k \to U$ of $\operatorname{Spec} k \to X$; hence we can replace $X$ by $U$, and assume that $X$ is scheme. Furthermore, if $x$ denotes the image of $\operatorname{Spec} k \to X$ and $k(x)$ its residue field, we have a factorization $\operatorname{Spec} k \to \operatorname{Spec} k(x) \to X$, and we may assume $k = k(x)$. If $x$ has height 0, then $\operatorname{Spec} k$ dominates $X$, and the composite $\operatorname{Spec} k \to X \dashrightarrow Y$ is well defined.

Otherwise, call $U \subseteq X$ the open subset where $f$ is defined. By [12, Lemma 4.3] there exists a DVR $R$ with residue field $k = k(x)$ and a morphism $\operatorname{Spec} R \to X$ that maps the generic point $\operatorname{Spec} K$ of $\operatorname{Spec} R$ into $U$. Thus we get a morphism $\operatorname{Spec} K \to U$, and we apply Corollary 3.2 to the diagram

$$\begin{array}{ccccc} \operatorname{Spec} K & \longrightarrow & U & \xrightarrow{f} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & X & \longrightarrow & S \end{array}$$

thus getting the desired morphism $\operatorname{Spec} k \to Y$. □

The Lang–Nishimura theorem fails for non-tame separated stacks. Let us give two examples, one in mixed characteristic, the other in positive characteristic.

*Example 4.2.* Let $X \to \operatorname{Spec} R$ be the stack constructed in 3.3, it is a non-tame regular Deligne-Mumford stack. Let $k$ be the residue field of $R$. There is a rational map $\operatorname{Spec} R \dashrightarrow X$ and $\operatorname{Spec} R$ has a $k$ rational point, but $X$ has no $k$-rational points.

*Example 4.3.* Let $C_0$ be a smooth, projective curve of positive genus over a finite field $F$ of characteristic $p$ with $C_0(F) \neq \emptyset$. Let $a$ be an indeterminate, write $k \stackrel{\text{def}}{=} F(a)$ and $C \stackrel{\text{def}}{=} C_{0,k}$; since $C_0$ has positive genus $C(k) = C_0(F)$ is finite. Let $f \in k(C)$ be a rational function such that each rational point is a zero of $f$ (this can be easily found using Riemann-Roch). Consider the ramified cover $D \to C$ given by the equation

$$t^p - f^{p-1}t = a;$$

in other words, $D$ is the smooth projective curve associated with the field extension $k(C)[t]/(t^p - f^{p-1}t - a)$. Let $c \in C(k)$ be a rational point, and write $R_c \stackrel{\text{def}}{=} \mathscr{O}_{C,c}[t]/(t^p - f^{p-1}t - a)$, it is a normal domain: if $\overline{R}_c$ is the normalization, both $R_c \otimes k(C) \to \overline{R}_c \otimes k(C)$ and $R_c \otimes k \to \overline{R}_c \otimes k$ are isomorphisms for degree reasons since $R_c \otimes k = k[t]/(t^p - a)$ is a field of degree $p$ over $k$. Hence, $R_c$ is a DVR with residue field $k' \stackrel{\text{def}}{=} k[t]/(t^p - a)$. It follows that $D$ has no $k$-rational points.

The cyclic group $C_p$ acts on $D$ by $t \mapsto t + f$, the field extension $k(D)/k(C)$ is a cyclic Galois cover and $C$ is the quotient scheme $D/C_p$. Let $X$ be the quotient stack $[D/C_p]$, there is a natural birational morphism $X \to C = D/C_p$. A rational point

$\operatorname{Spec} k \to X$ corresponds to a $C_p$-torsor $\operatorname{Spec} A \to \operatorname{Spec} k$ with an equivariant morphism $\operatorname{Spec} A \to D$: since the fibers of $D \to C$ over rational points are isomorphic to $\operatorname{Spec} k'$, a rational point of $X$ gives an embedding of $k'$ in an étale algebra $A$, which is clearly absurd. It follows that $X$ is a proper Deligne-Mumford stack over $k$ with $X(k) = \emptyset$ and a birational map $C \dashrightarrow X$.

As a consequence of Theorem 4.1, we can decide whether a residual gerbe of a tame stack is neutral or not by looking at a resolution of singularities of the coarse moduli space. We find this rather surprising.

**Corollary 4.4.** *Let $X$ be a locally noetherian, regular and integral tame stack with coarse moduli space $X \to M$, and $\overline{M} \to M$ a proper birational morphism, with $\overline{M}$ integral and regular. Assume that there is a lifting* $\operatorname{Spec} k(M) \to X$ *of the generic point* $\operatorname{Spec} k(M) \to M$.

*If $k$ is a field and $m\colon \operatorname{Spec} k \to M$ a morphism, then $m$ lifts to a morphism* $\operatorname{Spec} k \to X$ *if and only if it lifts to a morphism* $\operatorname{Spec} k \to \overline{M}$. ♠

So, for example, if $M$ is regular all morphisms $\operatorname{Spec} k \to M$ lift to $X$, and all residual gerbes are neutral.

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