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ON THE PLURICLOSED FLOW ON OELJEKLAUS-TOMA MANIFOLDS

ELIA FUSI AND LUIGI VEZZONI

ABSTRACT. We investigate the pluriclosed flow on Oeljeklaus-Toma manifolds. We parametrize left-invariant pluriclosed metrics on Oeljeklaus-Toma manifolds and we classify the ones which lift to an algebraic soliton of the pluriclosed flow on the universal covering. We further show that the pluriclosed flow starting from a left-invariant pluriclosed metric has a long-time solution ω_t which once normalized collapses to a torus in the Gromov-Hausdorff sense. Moreover the lift of $\frac{1}{1+t}\omega_t$ to the universal covering of the manifold converges in the Cheeger-Gromov sense to $(\mathbb{H}^s \times \mathbb{C}^s, \tilde{\omega}_\infty)$ where $\tilde{\omega}_\infty$ is an algebraic soliton.

1. INTRODUCTION

Oeljeklaus-Toma manifolds are a very interesting class of complex manifolds introduced and firstly studied in [17]. These manifolds are defined as compact quotients of the type

$$M = \frac{\mathbb{H}^r \times \mathbb{C}^s}{U \rtimes \mathcal{O}_{\mathbb{K}}}$$

where $\mathbb{H} \subseteq \mathbb{C}$ is the upper half-plane, $\mathcal{O}_{\mathbb{K}}$ is the ring of algebraic integers of an algebraic extension \mathbb{K} of \mathbb{Q} satisfying $[\mathbb{K} : \mathbb{Q}] = r+2s$ and U is a free subgroup of rank r of $\mathcal{O}_{\mathbb{K}}^{s,+}$ satisfying some compatible conditions. The action of $U \rtimes \mathcal{O}_{\mathbb{K}}$ on $\mathbb{H}^r \times \mathbb{C}^s$ is defined via some embeddings of \mathbb{K} in \mathbb{R} and \mathbb{C} . Oeljeklaus-Toma manifolds have a rich geometric structure. For instance, they have a natural structure of \mathbb{T}^{r+2s} -torus bundle over a \mathbb{T}^r and a structure of solvmanifold [13], i.e. they are always compact quotients of a solvable Lie group by a lattice. The Poincaré metric¹ $\omega_{\mathbb{H}^r} = \sqrt{-1} \sum_{a=1}^r \frac{dz_a \wedge d\bar{z}_a}{4(\Im z_a)^2}$ induces a degenerate metric ω_∞ on M which has a central role in the study of geometric flows on these manifolds. The pair (r, s) is called the *type* of the manifold. The case of type $(r, s) = (1, 1)$ corresponds to the Inoue-Bombieri surfaces [11].

In [2, 7, 29, 32] the Chern-Ricci flow [10, 28] on Oeljeklaus-Toma manifolds M of type $(r, 1)$ is studied. Accordingly to the results in [2, 7, 29, 32], under some assumptions on the initial Hermitian metric, the flow has a long-time solution ω_t such that $(M, \frac{\omega_t}{1+t})$ converges in the Gromov-Hausdorff sense to an r -dimensional torus \mathbb{T}^r as $t \rightarrow \infty$. The result can be adapted to Oeljeklaus-Toma manifolds of arbitrary type by assuming the initial metric to be left-invariant with respect to the structure of solvmanifold. Moreover, a result of Lauret in [14] allows us to give a characterization of left-invariant Hermitian metrics on an Oeljeklaus-Toma manifold which lift to an algebraic soliton of the Chern-Ricci flow on the universal covering of the manifold (see Proposition 4.1 in the present paper).

Following the same approach, we focus on the pluriclosed flow on Oeljeklaus-Toma manifolds when the initial pluriclosed Hermitian metric is left-invariant. The pluriclosed flow is a geometric flow of pluriclosed metrics, i.e. of Hermitian metrics having the fundamental form $\partial\bar{\partial}$ -closed, introduced by Streets and Tian in [21]. The flow belongs to the family of the Hermitian curvature flows [20] and evolves an initial pluriclosed metric along the $(1, 1)$ -component of the Bismut-Ricci form. Namely, on a Hermitian manifold (M, ω) there always exists a unique metric connection ∇^B , called the *Bismut connection*, preserving the complex structure and such that

$$\omega(T^B(\cdot, \cdot), J\cdot) \quad \text{is a 3-form,}$$

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¹In the whole paper we identify a Hermitian metric with its fundamental form.

where T^B is the torsion of ∇^B . The *Bismut-Ricci form* of ω is then defined as

$$\rho_B(X, Y) := \sqrt{-1} \sum_{i=1}^n R_B(X, Y, X_i, \bar{X}_i),$$

where R_B is the curvature tensor of ∇^B and $\{X_i\}$ is a unitary frame of ω . ρ_B is always a closed real form. Given a pluriclosed Hermitian metric ω on M , the *pluriclosed flow* is then defined as the geometric flow of pluriclosed metrics governed by the equation

$$\partial_t \omega_t = -\rho_B^{1,1}(\omega_t), \quad \omega|_{t=0} = \omega.$$

The pluriclosed flow was deeply studied in literature, see for instance [3, 5, 6, 9, 12, 19, 22, 23, 24, 25, 26, 27] and the references therein.

Our main result is the following

Theorem 1.1. *Let ω be a left-invariant pluriclosed Hermitian metric on an Oeljeklaus-Toma manifold M . Then the pluriclosed flow starting from ω has a long-time solution ω_t such that $(M, \frac{\omega_t}{1+t})$ converges in the Gromov-Hausdorff sense to (\mathbb{T}^s, d) . Moreover, ω lifts to an expanding algebraic soliton on the universal covering of M if and only if it is diagonal and the first s diagonal components coincide. Finally, $(\mathbb{H}^s \times \mathbb{C}^s, \frac{\omega_t}{1+t})$ converges in the Cheeger-Gromov sense to $(\mathbb{H}^s \times \mathbb{C}^s, \tilde{\omega}_\infty)$ where $\tilde{\omega}_\infty$ is an algebraic soliton.*

Here we recall that a left-invariant Hermitian metric ω on a Lie group G with a left-invariant complex structure is an *algebraic soliton* for a geometric flow of left-invariant Hermitian metrics if $\omega_t = c_t \varphi_t^*(\omega)$ solves the flow, where $\{c_t\}$ is a positive scaling and $\{\varphi_t\}$ is a family of automorphisms of G preserving the complex structure. Moreover the distance d in the statement is the distance induced by $3\omega_\infty$ on the torus base of M . Now we describe the condition *diagonal* appearing in the statement of Theorem 1.1. The existence of a pluriclosed metric on an Oeljeklaus-Toma manifold imposes some restrictions, see [1, Corollary 3]. In particular, the manifold has type (s, s) and admits a left-invariant $(1, 0)$ -coframe $\{\omega^1, \dots, \omega^s, \gamma^1, \dots, \gamma^s\}$ satisfying

$$\begin{cases} d\omega^k = \frac{\sqrt{-1}}{2} \omega^k \wedge \bar{\omega}^k & k = 1, \dots, s, \\ d\gamma^i = \sum_{k=1}^s \lambda_{ki} \omega^k \wedge \gamma^i - \sum_{k=1}^s \lambda_{ki} \bar{\omega}^k \wedge \gamma^i & i = 1, \dots, s, \end{cases}$$

with

$$\Im \lambda_{ki} = -\frac{1}{4} \delta_{ik}.$$

By ω *diagonal* we mean that it takes a diagonal form with respect to such a coframe. The first part of Theorem 1.1 in the case of the Inoue-Bombieri surfaces is proved in [5, Corollary 3.18].

Theorem 1.1 provides a description of the long-time behavior of the solution ω_t to the pluriclosed flow as $t \rightarrow \infty$. For the definition of the convergence in the Gromov-Hausdorff sense we refer to Section 3 in the preset paper, while here we briefly recall the definition of convergence in the Cheeger-Gromov sense: a sequence of pointed riemannian manifolds (M_k, g_k, p_k) *converges in the Cheeger-Gromov sense to a pointed riemannian manifold (M, g, p)* if there exists a sequence of open subsets A_k of M so that every compact subset of M eventually lies in some A_k , and a sequence of smooth maps $\phi_k: A_k \rightarrow M_k$ which are diffeomorphisms onto some open set of M_k which satisfy $\phi_k(p_k) = p$, such that

$$\phi_k^*(g_k) \rightarrow g \quad \text{smoothly on every compact subset, as } k \rightarrow \infty.$$

See [15, Section 6] for a deep analysis of Cheeger-Gromov convergence both in the general case and in the homogeneous one and [14, Section 5.1] for the case of Hermitian Lie groups.

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2. DEFINITION OF OELJEKLAUS-TOMA MANIFOLDS

We briefly recall the construction of Oeljeklaus-Toma manifolds [17].

Let $\mathbb{Q} \subseteq \mathbb{K}$ be an algebraic number field with $[\mathbb{K} : \mathbb{Q}] = r + 2s$ and $r, s \geq 1$. Let $\sigma_1, \dots, \sigma_r: \mathbb{K} \rightarrow \mathbb{R}$ be the real embeddings of \mathbb{K} and $\sigma_{r+1}, \dots, \sigma_{r+2s}: \mathbb{K} \rightarrow \mathbb{C}$ be the complex embeddings of \mathbb{K} satisfying $\sigma_{r+s+i} = \bar{\sigma}_{r+i}$, for every $i = 1, \dots, s$. We denote by $\mathcal{O}_{\mathbb{K}}$ the ring of algebraic integers of \mathbb{K} and by $\mathcal{O}_{\mathbb{K}}^*$ the group of units of $\mathcal{O}_{\mathbb{K}}$. Let

$$\mathcal{O}_{\mathbb{K}}^{*,+} = \{u \in \mathcal{O}_{\mathbb{K}}^* \mid \sigma_i(u) > 0, \text{ for every } i = 1, \dots, r\}$$

be the group of totally positive units of $\mathcal{O}_{\mathbb{K}}$. The groups $\mathcal{O}_{\mathbb{K}}$ and $\mathcal{O}_{\mathbb{K}}^{*,+}$ act on $\mathbb{H}^r \times \mathbb{C}^s$ as

$$a \cdot (z_1, \dots, z_r, w_1, \dots, w_s) = (z_1 + \sigma_1(a), \dots, z_r + \sigma_r(a), w_1 + \sigma_{r+1}(a), \dots, w_s + \sigma_{r+s}(a)), \text{ for all } a \in \mathcal{O}_{\mathbb{K}}$$

and

$$u \cdot (z_1, \dots, z_r, w_1, \dots, w_s) = (\sigma_1(u)z_1, \dots, \sigma_r(u)z_r, \sigma_{r+1}(u)w_1, \dots, \sigma_{r+s}(u)w_s), \text{ for every } u \in \mathcal{O}_{\mathbb{K}}^{*,+}.$$

There always exists a free subgroup U of rank r of $\mathcal{O}_{\mathbb{K}}^{*,+}$ such that $\text{pr}_{\mathbb{R}^r} \circ l(U)$ is a lattice of rank r in \mathbb{R}^r , where $l: \mathcal{O}_{\mathbb{K}}^{*,+} \rightarrow \mathbb{R}^{r+s}$ is the logarithmic representation of units

$$l(u) = (\log \sigma_1(u), \dots, \log \sigma_r(u), 2 \log |\sigma_{r+1}(u)|, \dots, 2 \log |\sigma_{r+s}(u)|)$$

and $\text{pr}_{\mathbb{R}^r}: \mathbb{R}^{r+s} \rightarrow \mathbb{R}^r$ is the projection on the first r coordinates. The action of $U \rtimes \mathcal{O}_{\mathbb{K}}$ on $\mathbb{H}^r \times \mathbb{C}^s$ is free, properly discontinuous and co-compact. An *Oeljeklaus-Toma manifold* is then defined as the quotient

$$M := \frac{\mathbb{H}^r \times \mathbb{C}^s}{U \rtimes \mathcal{O}_{\mathbb{K}}}$$

and it is a compact complex manifold having complex dimension $r + s$.

The structure of torus bundle of an Oeljeklaus-Toma manifold can be seen as follows: we have

$$\frac{\mathbb{H}^r \times \mathbb{C}^s}{\mathcal{O}_{\mathbb{K}}} = \mathbb{R}_+^r \times \mathbb{T}^{r+2s}$$

and that the action of U on $\mathbb{H}^r \times \mathbb{C}^s$ induces an action on $\mathbb{R}_+^r \times \mathbb{T}^{r+2s}$ such that, for every $x \in \mathbb{R}_+^r$ and $u \in U$, the induced map

$$u: (x, \mathbb{T}^{r+2s}) \mapsto (\sigma_1(u)x_1, \dots, \sigma_r(u)x_r, \mathbb{T}^{r+2s})$$

is a diffeomorphism. Hence

$$M = \frac{\mathbb{R}_+^r \times \mathbb{T}^{r+2s}}{U}$$

inherits the structure of a \mathbb{T}^{r+2s} -bundle over \mathbb{T}^r . We denote by π and F the projections

$$\pi: \mathbb{H}^r \times \mathbb{C}^s \rightarrow M, \quad F: M \rightarrow \mathbb{T}^r.$$

From the viewpoint of Lie groups, the universal covering of an Oeljeklaus-Toma manifold M has a natural structure of solvable Lie group G and the complex structure on M lifts to a left-invariant complex structure [13]. Therefore, Oeljeklaus-Toma manifolds can be seen as compact solvmanifolds with a left-invariant complex structure. The solvable structure on the universal covering of M can be described in terms of the existence of a left-invariant $(1, 0)$ -coframe $\{\omega^1, \dots, \omega^r, \gamma^1, \dots, \gamma^s\}$ such that

$$(1) \quad \begin{cases} d\omega^k = \frac{\sqrt{-1}}{2} \omega^k \wedge \bar{\omega}^k & k = 1, \dots, r, \\ d\gamma^i = \sum_{k=1}^r \lambda_{ki} \omega^k \wedge \gamma^i - \sum_{k=1}^r \lambda_{ki} \bar{\omega}^k \wedge \gamma^i & i = 1, \dots, s, \end{cases}$$

where

$$\lambda_{ki} = \frac{\sqrt{-1}}{4} b_{ki} - \frac{1}{2} c_{ki}$$

and $b_{ki}, c_{ki} \in \mathbb{R}$ depend on the embeddings σ_j as

$$(2) \quad \sigma_{r+i}(u) = \left(\prod_{k=1}^r (\sigma_k(u))^{\frac{b_{ki}}{2}} \right) e^{\sqrt{-1} \sum_{k=1}^r c_{ki} \log \sigma_k(u)},$$

for any $u \in U$, $k = 1, \dots, r$ and $i = 1, \dots, s$. Since $U \subseteq \mathcal{O}_{\mathbb{K}}^*$, it is easy to see that

$$l(U) \subseteq \left\{ x \in \mathbb{R}^{r+s} \mid \sum_{i=1}^{r+s} x_i = 0 \right\}.$$

This fact together with (2) implies that, for every $u \in U$,

$$\sum_{i=1}^r \log \sigma_i(u) \left(1 + \sum_{k=1}^s b_{ik} \right) = 0,$$

which, since $\text{pr}_{\mathbb{R}^r} \circ l(U)$ is a lattice of rank r in \mathbb{R}^r , is equivalent to

$$(3) \quad \sum_{k=1}^s b_{ik} = -1, \quad \text{for all } i = 1, \dots, r.$$

The dual frame $\{Z_1, \dots, Z_r, W_1, \dots, W_s\}$ to $\{\omega^1, \dots, \omega^r, \gamma^1, \dots, \gamma^s\}$ satisfies the following structure equations:

$$[Z_k, \bar{Z}_k] = -\frac{\sqrt{-1}}{2}(Z_k + \bar{Z}_k), \quad [Z_k, W_i] = -\lambda_{ki}W_i, \quad [Z_k, \bar{W}_i] = \bar{\lambda}_{ki}\bar{W}_i,$$

for $k = 1, \dots, r$, $i = 1, \dots, s$. Consequently the Lie algebra \mathfrak{g} of the universal covering of M splits as vector space as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{J}$$

where \mathfrak{J} is an abelian ideal and \mathfrak{h} is a subalgebra isomorphic to $\underbrace{\mathfrak{f} \oplus \dots \oplus \mathfrak{f}}_{r\text{-times}}$, where \mathfrak{f} is the *filiform* Lie

algebra $\mathfrak{f} = \langle e_1, e_2 \rangle$, $[e_1, e_2] = -\frac{1}{2}e_1$. The complex structure J induced on \mathfrak{g} preserves both \mathfrak{h} and \mathfrak{J} and its restriction $J_{\mathfrak{h}}$ on \mathfrak{h} satisfies

$$J_{\mathfrak{h}} = \underbrace{J_{\mathfrak{f}} \oplus \dots \oplus J_{\mathfrak{f}}}_{r\text{-times}},$$

where $J_{\mathfrak{f}}$ is the complex structure on \mathfrak{f} defined by $J_{\mathfrak{f}}(e_1) = e_2$. Moreover

$$[\mathfrak{h}^{1,0}, \mathfrak{J}^{0,1}] \subseteq \mathfrak{J}^{0,1}.$$

3. CONVERGENCE IN THE GROMOV-HAUSDORFF SENSE

We briefly recall Gromov-Hausdorff convergence of metric spaces. The *Gromov-Hausdorff distance* between two metric spaces (X, d_X) , (Y, d_Y) is the infimum of all positive ϵ for which there exist two functions $F: X \rightarrow Y$, $G: Y \rightarrow X$, not necessarily continuous, satisfying the following four properties

$$\begin{aligned} |d_X(x_1, x_2) - d_Y(F(x_1), F(x_2))| &\leq \epsilon, & d_X(x, G(F(x))) &\leq \epsilon, \\ |d_Y(y_1, y_2) - d_X(G(y_1), G(y_2))| &\leq \epsilon, & d_Y(y, F(G(y))) &\leq \epsilon, \end{aligned}$$

for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$. If $\{d_t\}_{t \in [0, \infty)}$ is a 1-parameter family of distances on X , (X, d_t) converges to (Y, d_Y) in the *Gromov-Hausdorff sense* if the Gromov-Hausdorff distance between (X, d_t) and (Y, d) tends to 0 as $t \rightarrow \infty$.

Let $\{\omega_t\}_{t \in [0, \infty)}$ be a smooth curve of Hermitian metrics on an Oeljeklaus-Toma manifold and let d_t be the induced distance on M . For a smooth curve γ on M , let $L_t(\gamma)$ be the length of γ with respect to ω_t . We further denote by \mathcal{H} the foliation induced by \mathfrak{h} on M .

Proposition 3.1. *Let $\{\omega_t\}_{t \in [0, \infty)}$ be a smooth curve of Hermitian metrics on an Oeljeklaus-Toma manifold such that*

$$\lim_{t \rightarrow \infty} \omega_t = \omega_{\infty}$$

pointwise. Assume that there exist $T \in (0, \infty)$ and $C > 0$ such that

1. $L_t(\gamma) \leq CL_0(\gamma)$, for every smooth curve γ in M ;
2. $L_t(\gamma) \leq (C/\sqrt{t})L_0(\gamma)$, for every smooth curve γ in M such that $\dot{\gamma} \in \ker \omega_{\infty}$.

Assume further

3. for every $\epsilon, \ell > 0$, there exists $T > 0$ such that $|L_t(\gamma) - L_\infty(\gamma)| < \epsilon$, for every $t > T$ and every curve γ in M tangent to \mathcal{H} and such that $L_\infty(\gamma) < \ell$.

Then (M, d_t) converges in the Gromov-Hausdorff sense to (\mathbb{T}^r, d) , where d is the distance induced by ω_∞ onto \mathbb{T}^r .

Proof. We follow the approach in [29, Section 5] and in [32, Proof of Theorem 1.1]. Let M be an Oeljeklaus-Toma manifold. Consider the structure of M as \mathbb{T}^{r+2s} -bundle over a \mathbb{T}^r . Let $F: M \rightarrow \mathbb{T}^r$ be the projection onto the base and let $G: \mathbb{T}^r \rightarrow M$ be an arbitrary map such that $F \circ G = \text{Id}_{\mathbb{T}^r}$. We show that, for every $\epsilon > 0$, there exists $T > 0$ such that

$$(4) \quad |d_t(p, q) - d(F(p), F(q))| \leq \epsilon,$$

$$(5) \quad |d(a, b) - d_t(G(a), G(b))| \leq \epsilon,$$

$$(6) \quad d_t(p, G(F(p))) \leq \epsilon,$$

$$(7) \quad d(a, F(G(a))) \leq \epsilon,$$

for every $t \geq T$, $p, q \in M$, $a, b \in \mathbb{T}^r$ which implies the statement.

Note that (7) is trivial since

$$d(a, F(G(a))) = 0,$$

for every $a \in \mathbb{T}^r$.

Then, we show that (6) is satisfied. Let $p, q \in M$ be two points in the same fiber over \mathbb{T}^r . Assume $p = \pi(z, w)$. We denote with $\mathcal{L}_{(z, w)}$ the leaf of the foliation $\ker \omega_\infty$ on the universal covering of M passing through (z, w) . We easily see that, for all $(z, w) \in \mathbb{H}^r \times \mathbb{C}^s$, $\mathcal{L}_{(z, w)} = \{z\} \times \mathbb{C}^s$. In view of [30, Section 2], for every $z \in \mathbb{H}^r$, $\pi(\{z\} \times \mathbb{C}^s)$ is the leaf of the foliation $\ker \omega_\infty$ on M passing through p and it is dense in the fiber $F^{-1}(F(p))$. Let B_R be the standard ball in \mathbb{C}^s about the origin having radius R . We can choose R so that every point in $F^{-1}(F(p))$ has distance with respect to d_0 less than $\epsilon/2C$ to $\pi(\{z\} \times \bar{B}_R)$. On the other hand, given two points in $\pi(\{z\} \times \bar{B}_R)$, they can be joined with a curve γ in $F^{-1}(F(p))$ which is tangent to $\ker \omega_\infty$. Hence, for any such curve, condition 2. implies

$$L_t(\gamma) \leq \frac{C'}{\sqrt{t}},$$

for a uniform constant C' depending only on R . Let $p_0 = \pi(z, 0)$, γ_1 be a curve in $F^{-1}(F(p))$ connecting p with p_0 tangent to $\ker \omega_\infty$ and γ_2 be a curve connecting p_0 with q having minimal length with respect to d_0 . Hence, by using 1., for t sufficiently large, we have

$$d_t(p, q) \leq L_t(\gamma_1) + L_t(\gamma_2) \leq \frac{C'}{\sqrt{t}} + CL_0(\gamma_2) \leq \frac{C'}{\sqrt{t}} + \frac{\epsilon}{2} \leq \epsilon,$$

i.e.

$$d_t(p, q) \leq \epsilon$$

and (6) follows.

Next we show (4) and (5). First of all, we denote with g the riemannian metric on \mathbb{T}^r induced by ω_∞ , for an explicit expression of g see [32, Section 2], and we observe that

$$(8) \quad L_g(F(\gamma)) \leq L_\infty(\gamma), \text{ for every curve } \gamma \text{ in } M,$$

and the equality holds if and only if

$$\dot{\gamma} \in \mathcal{Y} = \text{span}_{\mathbb{C}} \left\{ \frac{1}{2\sqrt{-1}} (Z_i - \bar{Z}_i) \mid i = 1, \dots, r \right\}.$$

Let $p, q \in M$. We can find a curve γ in M connecting p with a point \tilde{q} in the \mathbb{T}^{r+2s} -fiber containing q which is tangent to \mathcal{Y} and such that $F(\gamma)$ is a minimal geodesic on (\mathbb{T}^r, g) , see for instance [29, Proof of Theorem 5.1] or [32, Proof of Theorem 1.1]. By applying 3. we have

$$d_t(p, q) \leq d_t(p, \tilde{q}) + d_t(\tilde{q}, q) \leq d_t(p, \tilde{q}) + \epsilon \leq L_t(\gamma) + \epsilon \leq L_\infty(\gamma) + 2\epsilon = L_g(F(\gamma)) + 2\epsilon = d(F(p), F(q)) + 2\epsilon,$$

for t big enough, i.e.

$$(9) \quad d_t(p, q) - d(F(p), F(q)) \leq 2\epsilon,$$

for t sufficiently large.

Next, using again (8), we obtain, for $p, q \in M$,

$$d(F(p), F(q)) \leq L_g(F(\gamma)) \leq L_\infty(\gamma) \leq L_t(\gamma) + \epsilon = d_t(p, q) + \epsilon,$$

for t big enough, where γ is curve which realizes the distance $d_t(p, q)$. Hence we obtain

$$(10) \quad d(F(p), F(q)) - d_t(p, q) \leq \epsilon.$$

By substituting $p = G(a)$ and $q = G(b)$ in (9) and (10) we infer

$$-\epsilon \leq d_t(G(a), G(b)) - d(a, b) \leq 2\epsilon$$

and (4) and (5) follow. \square

4. THE LEFT-INVARIANT CHERN-RICCI FLOW ON OELJEKLAUS-TOMA MANIFOLDS

Given a Hermitian manifold (M, ω) , the Chern connection of ω is the unique connection ∇ on (M, ω) preserving both ω and the complex structure such that the $(1, 1)$ -component of its torsion tensor is vanishing. The *Chern-Ricci form* of ω is the real closed $(1, 1)$ -form

$$\rho_C(X, Y) := \sqrt{-1} \sum_{i=1}^n R_C(X, Y, X_i, \bar{X}_i),$$

where R_C is the curvature tensor of ∇ and $\{X_i\}$ is a unitary frame of ω . The *Chern-Ricci flow* is then defined as the geometric flow

$$\partial_t \omega_t = -\rho_C(\omega_t), \quad \omega|_{t=0} = \omega.$$

In this section we prove the following

Proposition 4.1. *Let ω be a left-invariant Hermitian metric on an Oeljeklaus-Toma manifold M . Then ω lifts to an expanding algebraic soliton for the Chern-Ricci flow on the universal covering of M if and only if it takes the following expression with respect to the coframe $\{\omega^1, \dots, \omega^r, \gamma^1, \dots, \gamma^s\}$ satisfying (1):*

$$(11) \quad \omega = \sqrt{-1} \left(A \sum_{i=1}^r \omega^i \wedge \bar{\omega}^i + \sum_{i,j=1}^s g_{r+ir+j} \gamma^i \wedge \bar{\gamma}^j \right).$$

Moreover, the Chern-Ricci flow starting from ω has a long-time solution $\{\omega_t\}$ such that $(M, \frac{\omega_t}{1+t})$ converges as $t \rightarrow \infty$ in the Gromov-Hausdorff sense to (\mathbb{T}^r, d) , where d is the distance induced by ω_∞ onto \mathbb{T}^r . Finally, $(\mathbb{H}^r \times \mathbb{C}^s, \frac{\omega_t}{1+t})$ converges in the Cheeger-Gromov sense to $(\mathbb{H}^r \times \mathbb{C}^s, \tilde{\omega}_\infty)$ where $\tilde{\omega}_\infty$ is an algebraic soliton.

The proof of Proposition 4.1 is based on the following Theorem of Lauret

Theorem 4.2 (Lauret [14]). *Let (G, J) be a Lie group with a left-invariant complex structure. Then the Chern-Ricci form of a left-invariant Hermitian metric ω on (G, J) does not depend on the Hermitian metric. Moreover, if $P \neq 0$ is the endomorphism associated to ρ_C with respect to ω , then the following are equivalent:*

- (1) ω is an algebraic soliton of the Chern-Ricci flow,
- (2) $P = cI + D$, for some $D \in \text{Der}(\mathfrak{g})$,
- (3) The eigenvalues of P are either 0 or c , for some $c \in \mathbb{R}$ with $c \neq 0$, $\ker P$ is an abelian ideal of the Lie algebra of G and $(\ker P)^\perp$ is a subalgebra.

Proof of Proposition 4.1. Let M be an Oeljeklaus-Toma manifold. Since the Chern-Ricci form does not depend on the choice of the left-invariant Hermitian metric, it is enough to compute ρ_C for the “canonical metric”

$$(12) \quad \omega = \sqrt{-1} \left(\sum_{i=1}^r \omega^i \wedge \bar{\omega}^i + \sum_{j=1}^s \gamma^j \wedge \bar{\gamma}^j \right).$$

We recall that the Chern-Ricci form of a left-invariant Hermitian metric $\omega = \sqrt{-1} \sum_{a=1}^n \alpha^a \wedge \bar{\alpha}^a$ on a Lie group G^{2n} with a left-invariant complex structure takes the following algebraic expression:

$$(13) \quad \rho_C(X, Y) = - \sum_{a=1}^n (\omega([X, Y]^{0,1}, X_a, \bar{X}_a) + \omega([X, Y]^{1,0}, \bar{X}_a, X_a)),$$

for every left-invariant vector fields X, Y on G , where $\{\alpha^i\}$ is a left-invariant unitary $(1, 0)$ -coframe with dual frame $\{X_a\}$ (see e.g. [31]). By applying (13) to the canonical metric (12) we have

$$\begin{aligned} \rho_C(X, Y) = & - \sum_{a=1}^r \{ \omega([X, Y]^{0,1}, Z_a, \bar{Z}_a) + \omega([X, Y]^{1,0}, \bar{Z}_a, Z_a) \} \\ & - \sum_{b=1}^s \{ \omega([X, Y]^{0,1}, W_b, \bar{W}_b) + \omega([X, Y]^{1,0}, \bar{W}_b, W_b) \}. \end{aligned}$$

Clearly,

$$\rho_C(Z_i, \bar{Z}_j) = 0, \quad \text{for all } i \neq j, \quad \rho_C(W_i, \bar{W}_j) = 0, \quad \text{for every } i, j = 1, \dots, s.$$

Moreover, since \mathfrak{J} is an abelian ideal and ω makes \mathfrak{J} and \mathfrak{h} orthogonal, we have:

$$\rho_C(Z_i, \bar{W}_j) = 0, \quad \text{for all } i = 1, \dots, r, \quad j = 1, \dots, s.$$

Moreover we have

$$\omega([Z_i, \bar{Z}_i]^{0,1}, Z_a, \bar{Z}_a) = \frac{\sqrt{-1}}{4} \delta_{ia}, \quad \omega([Z_i, \bar{Z}_i]^{1,0}, \bar{Z}_a, Z_a) = \frac{\sqrt{-1}}{4} \delta_{ia}$$

and

$$\omega([Z_i, \bar{Z}_i]^{0,1}, W_b, \bar{W}_b) = \frac{1}{2} \lambda_{ib}, \quad \omega([Z_i, \bar{Z}_i]^{1,0}, \bar{W}_b, W_b) = -\frac{1}{2} \bar{\lambda}_{ib}$$

which imply

$$\rho_C(Z_i, \bar{Z}_i) = -\sqrt{-1} \left(\frac{1}{2} + \sum_{b=1}^s \Im(\lambda_{ib}) \right) = -\frac{\sqrt{-1}}{4}.$$

and, consequently,

$$\rho_C = -\omega_\infty.$$

In general, we have that

$$P_i^j = (\rho_C)_{i\bar{k}} g^{\bar{k}j} = \begin{cases} -\frac{1}{4} g^{\bar{i}j} & \text{if } i \in \{1, \dots, r\}, \\ 0 & \text{otherwise} . \end{cases}$$

Then, part (3) of Theorem 4.2 readily implies that any left-invariant Hermitian metrics of the form (11) lifts to an expanding algebraic soliton on the universal covering of M with cosmological constant $c = \frac{1}{4A}$. Conversely, let ω be an algebraic soliton for the Chern-Ricci flow. Then, thanks to part (2) of Theorem 4.2, we have that

$$P - cI \in \text{Der}(\mathfrak{g}).$$

On the other hand, we can easily see that, if $D \in \text{Der}(\mathfrak{g})$, then $\mathfrak{h} \subseteq \ker D$, see proof of Corollary 5.4 in the present paper for the details. This readily implies that

$$-\frac{1}{4} g^{\bar{i}i} = -\frac{1}{4} g^{\bar{j}j} = c, \quad \text{for all } i, j = 1, \dots, r, \quad g^{\bar{i}j} = 0, \quad \text{for all } i \in \{1, \dots, r\}, j \neq i,$$

from which the claim follows.

Moreover, the Chern-Ricci flow evolves an arbitrary left-invariant Hermitian metric ω as $\omega_t = \omega + t\omega_\infty$ and $\frac{\omega_t}{1+t} \rightarrow \omega_\infty$ as $t \rightarrow \infty$. In order to obtain the claim regarding the Gromov-Hausdorff convergence, we show that $\frac{\omega_t}{1+t}$ satisfies conditions 1,2,3 in Proposition 3.1. Here we denote by $|\cdot|_t$ the norm induced by ω_t .

Condition 2 is trivially satisfied since $\omega_{t|\mathfrak{J} \oplus \mathfrak{J}} = \omega_0$, for every $t \geq 0$, and

$$L_t(\gamma) = \frac{1}{\sqrt{1+t}} L_0(\gamma),$$

for every curve γ in M tangent to $\ker \omega_\infty$.

On the other hand, for a vector $v \in \mathfrak{h}$, we have

$$\frac{1}{\sqrt{1+t}} |v|_t \leq C |v|_0,$$

for a constant $C > 0$ independent on v . This, together with condition 2, guarantees condition 1.

In order to prove condition 3, let $\epsilon, \ell > 0$ and $T > 0$ be such that

$$\left| \frac{|v|_t}{\sqrt{1+t}} - |v|_\infty \right| \leq \frac{\epsilon}{\ell},$$

for every $v \in \mathfrak{h}$ and $t \geq T$. Let γ be a curve in M tangent to \mathcal{H} which is parametrized by arclength with respect to ω_∞ and such that $L_\infty(\gamma) < \ell$. Then

$$|L_t(\gamma) - L_\infty(\gamma)| \leq \int_0^b \left| \frac{1}{\sqrt{1+t}} |\dot{\gamma}|_t - |\dot{\gamma}|_\infty \right| da \leq \frac{\epsilon}{\ell} b \leq \epsilon,$$

since $b \leq \ell$.

For the last statement, we identify ω_t with its pull-back onto $\mathbb{H}^r \times \mathbb{C}^s$ and we fix as base point the identity element of $\mathbb{H}^r \times \mathbb{C}^s$. Firstly, we observe that the endomorphism D represented with respect to the frame $\{Z_1, \dots, Z_r, W_1, \dots, W_s\}$ by the following matrix:

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I}_\mathfrak{J} \end{pmatrix}$$

is a derivation of \mathfrak{g} . Moreover, we can construct

$$\exp(s(t)D) = \begin{pmatrix} \mathbf{I}_\mathfrak{h} & 0 \\ 0 & e^{s(t)\mathbf{I}_\mathfrak{J}} \end{pmatrix} \in \text{Aut}(\mathfrak{g}, J), \quad \text{for every } t \geq 0,$$

where $s(t) = \log(\sqrt{1+t})$ and define the 1-parameter family $\{\varphi_t\} \subseteq \text{Aut}(\mathbb{H}^r \times \mathbb{C}^s, J)$ such that

$$d\varphi_t = \exp(s(t)D), \quad \text{for every } t \geq 0.$$

Trivially, we see that

$$\begin{aligned} \varphi_t^* \frac{\omega_t}{1+t}(Z_i, \bar{Z}_j) &= \sqrt{-1} \frac{1}{1+t} \left(g_{i\bar{j}} + \frac{t}{4} \delta_{ij} \right) \rightarrow \frac{\sqrt{-1}}{4} \delta_{ij} \quad \text{as } t \rightarrow \infty, \\ \varphi_t^* \frac{\omega_t}{1+t}(Z_i, \bar{W}_j) &= \sqrt{-1} \frac{e^{s(t)}}{1+t} g_{i\bar{r}+j} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \\ \varphi_t^* \frac{\omega_t}{1+t}(W_i, \bar{W}_j) &= \sqrt{-1} \frac{e^{2s(t)}}{1+t} g_{r+i\bar{r}+j} \rightarrow \sqrt{-1} g_{r+i\bar{r}+j} \quad \text{as } t \rightarrow \infty. \end{aligned}$$

These facts guarantee that

$$\varphi_t^* \frac{\omega_t}{1+t} \rightarrow \omega_\infty + \omega_{\mathfrak{J} \oplus \mathfrak{J}} \quad \text{as } t \rightarrow \infty,$$

hence, the assertion follows. \square

5. PROOF OF THE MAIN RESULT

In this section we prove Theorem 1.1.

The existence of pluriclosed metrics on Oeljeklaus-Toma manifolds was studied in [1], [8] and [18]. In particular from [1] it follows the following result.

Theorem 5.1 (Corollary 3, [1]). *An Oeljeklaus-Toma manifold of type (r, s) admits a pluriclosed metric if and only if $r = s$ and*

$$(14) \quad \sigma_j(u) |\sigma_{r+j}(u)|^2 = 1, \quad \text{for every } j = 1, \dots, s \text{ and } u \in U.$$

Condition (14) in the previous Theorem can be rewritten in terms of the structure constants appearing in (1). Indeed, (1) together with (14) forces $b_{ki} \in \{0, -1\}$ and $b_{ki}b_{li} = 0$, for every $i, k, l = 1, \dots, s$ with $k \neq l$. In particular, using (3), for every fixed index $k \in \{1, \dots, s\}$, there exists a unique $i_k \in \{1, \dots, s\}$ such that

$$b_{ki_k} = -1, \quad b_{ki} = 0,$$

for all $i \neq i_k$ and, if $k \neq l$, then $i_k \neq i_l$. Hence, up to a reorder of the γ_j 's, we may and do assume, without loss of generality, $i_k = k$, for every $k \in \{1, \dots, s\}$, i.e.

$$(15) \quad \lambda_{ki} = \begin{cases} -\frac{1}{2}c_{ki} & \text{if } i \neq k, \\ -\frac{1}{2}c_{kk} - \frac{\sqrt{-1}}{4} & \text{if } i = k. \end{cases}$$

Proposition 5.2 (Characterization of left-invariant pluriclosed metrics on Oeljeklaus-Toma manifolds). *A left-invariant metric ω on an Oeljeklaus-Toma manifold admitting pluriclosed metrics is pluriclosed if and only if it takes the following expression with respect to a coframe $\{\omega^1, \dots, \omega^s, \gamma^1, \dots, \gamma^s\}$ satisfying (1) and (15):*

$$(16) \quad \omega = \sqrt{-1} \sum_{i=1}^s A_i \omega^i \wedge \bar{\omega}^i + B_i \gamma^i \wedge \bar{\gamma}^i + \sqrt{-1} \sum_{r=1}^k (C_r \omega^{p_r} \wedge \bar{\gamma}^{p_r} + \bar{C}_r \gamma^{p_r} \wedge \bar{\omega}^{p_r})$$

for some $A_1, \dots, A_s, B_1, \dots, B_s \in \mathbb{R}_+$, $C_1, \dots, C_k \in \mathbb{C}$, where $\{p_1, \dots, p_k\} \subseteq \{1, \dots, s\}$ are such that

$$\lambda_{jp_i} = 0, \quad \text{for all } j \neq p_i, \quad \text{for all } i = 1, \dots, k.$$

Proof. We assume $s > 1$ since the case $s = 1$ is trivial. Let

$$\omega = \sqrt{-1} \sum_{p,q=1}^s A_{p\bar{q}} \omega^p \wedge \bar{\omega}^q + B_{p\bar{q}} \gamma^p \wedge \bar{\gamma}^q + C_{p\bar{q}} \omega^p \wedge \bar{\gamma}^q + \bar{C}_{p\bar{q}} \gamma^q \wedge \bar{\omega}^p$$

be an arbitrary real left-invariant $(1, 1)$ -form on M , with $A_{p\bar{p}}, B_{p\bar{p}} \in \mathbb{R}$, for every $p = 1, \dots, s$, $A_{p\bar{q}}, B_{p\bar{q}} \in \mathbb{C}$, for all $p, q = 1, \dots, s$ with $p \neq q$, and $C_{p\bar{q}} \in \mathbb{C}$, for every $p, q = 1, \dots, s$.

From the structure equations (1), it easily follows

$$(17) \quad \begin{cases} \partial\bar{\partial}(\omega^p \wedge \bar{\omega}^q) \in \langle \omega^p \wedge \omega^p \wedge \bar{\omega}^p \wedge \bar{\omega}^q \rangle \\ \partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) \in \langle \omega^i \wedge \omega^j \wedge \bar{\omega}^l \wedge \bar{\gamma}^m \rangle \\ \partial\bar{\partial}(\gamma^p \wedge \bar{\gamma}^q) \in \langle \omega^i \wedge \bar{\omega}^j \wedge \gamma^l \wedge \bar{\gamma}^m \rangle \end{cases}$$

and that ω is pluriclosed if and only if the following three conditions are satisfied

$$(18) \quad \sum_{p,q=1}^s A_{p\bar{q}} \partial\bar{\partial}(\omega^p \wedge \bar{\omega}^q) = 0;$$

$$(19) \quad \sum_{p,q=1}^s B_{p\bar{q}} \partial\bar{\partial}(\gamma^p \wedge \bar{\gamma}^q) = 0;$$

$$(20) \quad \sum_{p,q=1}^s C_{p\bar{q}} \partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) = 0.$$

The first relation in (17) yields that (18) is satisfied if and only if

$$A_{p\bar{q}} = 0, \text{ for all } p \neq q.$$

Next we focus on (19). We have

$$\partial\bar{\partial}(\gamma^p \wedge \bar{\gamma}^q) = \partial \left(- \sum_{\delta=1}^s \lambda_{\delta p} \bar{\omega}^\delta \wedge \gamma^p \wedge \bar{\gamma}^q - \gamma^p \wedge \sum_{\delta=1}^s \bar{\lambda}_{\delta q} \bar{\omega}^\delta \wedge \bar{\gamma}^q \right)$$

and

$$\partial\bar{\partial}(\gamma^p \wedge \bar{\gamma}^q) = \sum_{\delta=1}^s (\bar{\lambda}_{\delta q} - \lambda_{\delta p}) (\partial\bar{\omega}^\delta \wedge \gamma^p \wedge \bar{\gamma}^q - \bar{\omega}^\delta \wedge \partial\gamma^p \wedge \bar{\gamma}^q + \bar{\omega}^\delta \wedge \gamma^p \wedge \partial\bar{\gamma}^q)$$

which implies

$$\begin{aligned} \partial\bar{\partial}(\gamma^p \wedge \bar{\gamma}^q) &= \sum_{\delta=1}^s \frac{\sqrt{-1}}{2} (\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \omega^\delta \wedge \bar{\omega}^\delta \wedge \gamma^p \wedge \bar{\gamma}^q - \sum_{\delta=1}^s (\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \bar{\omega}^\delta \wedge \left(\sum_{a=1}^s \lambda_{ap} \omega^a \wedge \gamma^p \right) \wedge \bar{\gamma}^q \\ &\quad + \sum_{\delta=1}^s (\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \bar{\omega}^\delta \wedge \gamma^p \wedge \left(- \sum_{a=1}^s \bar{\lambda}_{aq} \omega^a \wedge \bar{\gamma}^q \right) \\ &= \sum_{\delta=1}^s \frac{\sqrt{-1}}{2} (\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \omega^\delta \wedge \bar{\omega}^\delta \wedge \gamma^p \wedge \bar{\gamma}^q + \sum_{\delta, a} (\lambda_{ap} - \bar{\lambda}_{aq}) (\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \omega^a \wedge \bar{\omega}^\delta \wedge \gamma^p \wedge \bar{\gamma}^q. \end{aligned}$$

Finally, we get

$$\begin{aligned} \partial\bar{\partial}(\gamma^p \wedge \bar{\gamma}^q) &= \sum_{\delta=1}^s (\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \left(\frac{\sqrt{-1}}{2} + \lambda_{\delta p} - \bar{\lambda}_{\delta q} \right) \omega^\delta \wedge \bar{\omega}^\delta \wedge \gamma^p \wedge \bar{\gamma}^q \\ &\quad + \sum_{\delta \neq a} (\lambda_{ap} - \bar{\lambda}_{aq}) (\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \omega^a \wedge \bar{\omega}^\delta \wedge \gamma^p \wedge \bar{\gamma}^q \end{aligned}$$

and that condition (19) is equivalent to

$$B_{p\bar{q}} \left(\sum_{\delta=1}^s (\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \left(\frac{\sqrt{-1}}{2} + \lambda_{\delta p} - \bar{\lambda}_{\delta q} \right) \omega^\delta \wedge \bar{\omega}^\delta + \sum_{\delta \neq a} (\lambda_{ap} - \bar{\lambda}_{aq}) (\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \omega^a \wedge \bar{\omega}^\delta \right) = 0,$$

for every $p, q = 1, \dots, s$.

By using our conditions on the b_{ki} 's, it is easy to show that the quantity

$$\sum_{\delta=1}^s (\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \left(\frac{\sqrt{-1}}{2} + \lambda_{\delta p} - \bar{\lambda}_{\delta q} \right) \omega^\delta \wedge \bar{\omega}^\delta + \sum_{\delta \neq a} (\lambda_{ap} - \bar{\lambda}_{aq}) (\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \omega^a \wedge \bar{\omega}^\delta$$

is vanishing for $p = q$ and, consequently, there are no restrictions on the $B_{q\bar{q}}$'s. Now we observe that the real part of

$$(\bar{\lambda}_{pq} - \lambda_{pp}) \left(\frac{\sqrt{-1}}{2} + \lambda_{pp} - \bar{\lambda}_{pq} \right)$$

is different from 0, for every p, q with $p \neq q$, which forces $B_{p\bar{q}} = 0$, for $p \neq q$. Indeed, we have

$$\begin{aligned} \bar{\lambda}_{\delta q} - \lambda_{\delta p} &= \frac{1}{2} (c_{\delta p} - c_{\delta q}) - \frac{\sqrt{-1}}{4} (b_{\delta p} + b_{\delta q}), \\ \frac{\sqrt{-1}}{2} + \lambda_{\delta p} - \bar{\lambda}_{\delta q} &= -\frac{1}{2} (c_{\delta p} - c_{\delta q}) + \frac{\sqrt{-1}}{2} \left(1 + \frac{b_{\delta p} + b_{\delta q}}{2} \right) \end{aligned}$$

which implies

$$(21) \quad \Re \left((\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \left(\frac{\sqrt{-1}}{2} + \lambda_{\delta p} - \bar{\lambda}_{\delta q} \right) \right) = -\frac{(c_{\delta p} - c_{\delta q})^2}{4} + \frac{1}{4} \left(\frac{b_{\delta p} + b_{\delta q}}{2} \right) \left(1 + \frac{b_{\delta p} + b_{\delta q}}{2} \right).$$

Since $p \neq q$, we have

$$b_{pp} = -1, \quad b_{pq} = 0,$$

and so (21) computed for $\delta = q$ gives

$$\Re \left((\bar{\lambda}_{pq} - \lambda_{pp}) \left(\frac{\sqrt{-1}}{2} + \lambda_{pq} - \bar{\lambda}_{pp} \right) \right) = \frac{1}{4} \left(-(c_{pp} - c_{pq})^2 - \frac{1}{4} \right) \neq 0,$$

as required. Therefore equation (19) is satisfied if and only if

$$B_{p\bar{q}} = 0, \quad \text{for all } p \neq q.$$

Next we focus on (20). We have

$$\partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) = \partial \left(\frac{\sqrt{-1}}{2} \omega^p \wedge \bar{\omega}^p \wedge \bar{\gamma}^q - \omega^p \wedge \left(\sum_{\delta=1}^s \bar{\lambda}_{\delta q} \bar{\omega}^\delta \wedge \bar{\gamma}^q \right) \right)$$

and

$$\begin{aligned} \partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) &= \frac{\sqrt{-1}}{2} \left(-\frac{\sqrt{-1}}{2} \omega^p \wedge \omega^p \wedge \bar{\omega}^p \wedge \bar{\gamma}^q + \omega^p \wedge \bar{\omega}^p \wedge \left(-\sum_{\delta=1}^s \bar{\lambda}_{\delta q} \omega^\delta \wedge \bar{\gamma}^q \right) \right) \\ &\quad + \sum_{\delta=1}^s \frac{\sqrt{-1}}{2} \bar{\lambda}_{\delta q} \omega^p \wedge \omega^\delta \wedge \bar{\omega}^\delta \wedge \bar{\gamma}^q + \sum_{\delta=1}^s \bar{\lambda}_{\delta q} \omega^p \wedge \bar{\omega}^\delta \wedge \left(\sum_{a=1}^s \bar{\lambda}_{a q} \omega^a \wedge \bar{\gamma}^q \right). \end{aligned}$$

Hence we get

$$\begin{aligned} \partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) &= \sum_{\substack{\delta=1 \\ \delta \neq p}}^s \frac{\sqrt{-1}}{2} \bar{\lambda}_{\delta q} \omega^p \wedge \bar{\omega}^p \wedge \omega^\delta \wedge \bar{\gamma}^q + \sum_{\substack{\delta=1 \\ \delta \neq p}}^s \frac{\sqrt{-1}}{2} \bar{\lambda}_{\delta q} \omega^p \wedge \omega^\delta \wedge \bar{\omega}^\delta \wedge \bar{\gamma}^q \\ &\quad + \sum_{\substack{\delta, a \\ a \neq p}} \bar{\lambda}_{\delta q} \bar{\lambda}_{a q} \omega^p \wedge \bar{\omega}^\delta \wedge \omega^a \wedge \bar{\gamma}^q \end{aligned}$$

and

$$\begin{aligned} \partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) &= \sum_{\substack{\delta=1 \\ \delta \neq p}}^s \frac{\sqrt{-1}}{2} \bar{\lambda}_{\delta q} \omega^p \wedge \bar{\omega}^p \wedge \omega^\delta \wedge \bar{\gamma}^q + \sum_{\substack{a=1 \\ a \neq p}}^s \bar{\lambda}_{p q} \bar{\lambda}_{a q} \omega^p \wedge \bar{\omega}^p \wedge \omega^a \wedge \bar{\gamma}^q \\ &\quad + \sum_{\substack{\delta=1 \\ \delta \neq p}}^s \frac{\sqrt{-1}}{2} \bar{\lambda}_{\delta q} \omega^p \wedge \omega^\delta \wedge \bar{\omega}^\delta \wedge \bar{\gamma}^q + \sum_{\substack{\delta, a \\ \delta \neq p \\ a \neq p}} \bar{\lambda}_{\delta q} \bar{\lambda}_{a q} \omega^p \wedge \bar{\omega}^\delta \wedge \omega^a \wedge \bar{\gamma}^q. \end{aligned}$$

Therefore

$$\begin{aligned} \partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) &= \sum_{\substack{\delta=1 \\ \delta \neq p}}^s \bar{\lambda}_{\delta q} \left(\frac{\sqrt{-1}}{2} + \bar{\lambda}_{p q} \right) \omega^p \wedge \bar{\omega}^p \wedge \omega^\delta \wedge \bar{\gamma}^q + \\ &\quad \sum_{\substack{\delta=1 \\ \delta \neq p}}^s \bar{\lambda}_{\delta q} \left(\frac{\sqrt{-1}}{2} - \bar{\lambda}_{\delta q} \right) \omega^p \wedge \omega^\delta \wedge \bar{\omega}^\delta \wedge \bar{\gamma}^q + \sum_{\substack{\delta \neq a \\ \delta \neq p \\ a \neq p}} \bar{\lambda}_{\delta q} \bar{\lambda}_{a q} \omega^p \wedge \bar{\omega}^\delta \wedge \omega^a \wedge \bar{\gamma}^q \end{aligned}$$

and (20) is equivalent to

$$C_{p\bar{q}} \left(\sum_{\substack{\delta=1 \\ \delta \neq p}}^s \bar{\lambda}_{\delta q} \left(\frac{\sqrt{-1}}{2} + \bar{\lambda}_{p q} \right) \bar{\omega}^p \wedge \omega^\delta + \sum_{\substack{\delta=1 \\ \delta \neq p}}^s \bar{\lambda}_{\delta q} \left(\frac{\sqrt{-1}}{2} - \bar{\lambda}_{\delta q} \right) \omega^\delta \wedge \bar{\omega}^\delta + \sum_{\substack{\delta \neq a \\ \delta \neq p \\ a \neq p}} \bar{\lambda}_{\delta q} \bar{\lambda}_{a q} \bar{\omega}^\delta \wedge \omega^a \right) = 0,$$

for every $p, q = 1, \dots, s$. Since

$$\lambda_{pq} \neq \pm \frac{\sqrt{-1}}{2}, \quad \text{for all } p, q = 1, \dots, s,$$

the quantity

$$E_{p\bar{q}} := \sum_{\substack{\delta=1 \\ \delta \neq p}}^s \bar{\lambda}_{\delta q} \left(\frac{\sqrt{-1}}{2} + \bar{\lambda}_{pq} \right) \bar{\omega}^p \wedge \omega^\delta + \sum_{\substack{\delta=1 \\ \delta \neq p}}^s \bar{\lambda}_{\delta q} \left(\frac{\sqrt{-1}}{2} - \bar{\lambda}_{\delta q} \right) \omega^\delta \wedge \bar{\omega}^\delta + \sum_{\substack{\delta \neq a \\ \delta \neq p \\ a \neq p}} \bar{\lambda}_{\delta q} \bar{\lambda}_{a q} \bar{\omega}^\delta \wedge \omega^a$$

is vanishing if and only if

$$\lambda_{\delta q} = 0, \quad \text{for all } \delta \neq p.$$

Since $\lambda_{qq} \neq 0$, it follows

$$E_{p\bar{q}} \neq 0, \quad \text{for every } p, q \text{ with } p \neq q$$

and

$$E_{p\bar{p}} = 0 \text{ if and only if } c_{\delta p} = 0, \text{ for all } \delta \neq p.$$

Hence the claim follows. \square

Proposition 5.3. *Let*

$$(22) \quad \omega = \sqrt{-1} \sum_{i=1}^s A_i \omega^i \wedge \bar{\omega}^i + B_i \gamma^i \wedge \bar{\gamma}^i + \sqrt{-1} \sum_{r=1}^k (C_r \omega^{p_r} \wedge \bar{\gamma}^{p_r} + \bar{C}_r \gamma^{p_r} \wedge \bar{\omega}^{p_r})$$

be a left-invariant pluriclosed Hermitian metric on an Oeljeklaus-Toma manifold, where the components are with respect to a coframe $\{\omega^1, \dots, \omega^s, \gamma^1, \dots, \gamma^s\}$ satisfying (1) and (15) and $\{p_1, \dots, p_k\} \subseteq \{1, \dots, s\}$ are such that

$$\lambda_{j p_i} = 0, \text{ for all } j \neq p_i, \text{ for all } i = 1, \dots, k.$$

Then, the (1,1)-part of the Bismut-Ricci form of ω takes the following expression:

$$\begin{aligned} \rho_B^{1,1} &= -\sqrt{-1} \sum_{r=1}^k \frac{3}{4} \left(1 + \frac{|C_r|^2}{A_{p_r} B_{p_r} - |C_r|^2} \right) \omega^{p_r} \wedge \bar{\omega}^{p_r} - \sqrt{-1} \sum_{i \notin \{p_1, \dots, p_k\}} \frac{3}{4} \omega^i \wedge \bar{\omega}^i \\ &- \sqrt{-1} \sum_{r=1}^k \left(-\frac{3}{16} - \frac{c_{p_r p_r}^2}{4} - \frac{\sqrt{-1} c_{p_r p_r}}{4} \right) \frac{B_{p_r} C_r}{A_{p_r} B_{p_r} - |C_r|^2} \omega^{p_r} \wedge \bar{\gamma}^{p_r} + \text{conjugates}. \end{aligned}$$

Proof. We recall that the Bismut-Ricci form of a left-invariant Hermitian metric $\omega = \sqrt{-1} \sum_{a,b=1}^n g_{a\bar{b}} \alpha^a \wedge \bar{\alpha}^b$ on a Lie group G^{2n} with a left-invariant complex structure takes the following algebraic expression:

$$(23) \quad \rho_B(X, Y) = - \sum_{a,b=1}^n g^{a\bar{b}} \omega([X, Y]^{1,0}, X_a, \bar{X}_b) + g^{\bar{a}b} \omega([X, Y]^{0,1}, \bar{X}_a, X_b) + \sqrt{-1} \sum_{a,b=1}^n g^{a\bar{b}} \omega([X, Y], J[X_a, \bar{X}_b]),$$

for every left-invariant vector fields X, Y on G , where $\{\alpha^i\}$ is a left-invariant (1,0)-coframe with dual frame $\{X_a\}$ and $(g^{\bar{a}a})$ is the inverse matrix to $(g_{i\bar{j}})$ (see e.g. [31]). We apply (23) to a left-invariant Hermitian metric on an Oeljeklaus-Toma manifold of the form (22).

We have

$$g^{\bar{i}s+i} = \begin{cases} 0 & \text{if } i \notin \{p_1, \dots, p_k\}, \\ -\frac{C_i}{A_i B_i - |C_i|^2} & \text{otherwise,} \end{cases} \quad g^{\bar{i}i} = \frac{B_i}{A_i B_i - |C_i|^2}, \quad g^{\overline{s+i}s+i} = \frac{A_i}{A_i B_i - |C_i|^2}$$

and taking into account that the ideal \mathcal{J} is abelian, we have

$$\rho_B(X, Y) = - \sum_{i=1}^4 \rho_i(X, Y),$$

where

$$\begin{aligned}\rho_1(X, Y) &= \sum_{a=1}^s g^{a\bar{a}} (\omega([X, Y]^{1,0}, Z_a, \bar{Z}_a) - \frac{\sqrt{-1}}{2} \omega([X, Y], Z_a - \bar{Z}_a) + \omega([X, Y]^{0,1}, \bar{Z}_a, Z_a)), \\ \rho_2(X, Y) &= \sum_{a=1}^s g^{s+a\bar{s}+a} (\omega([X, Y]^{1,0}, W_a, \bar{W}_a) + \omega([X, Y]^{0,1}, \bar{W}_a, W_a)), \\ \rho_3(X, Y) &= \sum_{r=1}^k g^{p_r \overline{s+p_r}} (\omega([X, Y]^{1,0}, Z_{p_r}, \bar{W}_{p_r}) - \omega([X, Y], [Z_{p_r}, \bar{W}_{p_r}])) + g^{\overline{p_r s+p_r}} \omega([X, Y]^{0,1}, \bar{Z}_{p_r}, W_{p_r}), \\ \rho_4(X, Y) &= \sum_{r=1}^k g^{s+p_r \bar{p}_r} (\omega([X, Y]^{1,0}, W_{p_r}, \bar{Z}_{p_r}) + \omega([X, Y], [W_{p_r}, \bar{Z}_{p_r}])) + g^{\overline{s+p_r p_r}} \omega([X, Y]^{0,1}, \bar{W}_{p_r}, Z_{p_r}).\end{aligned}$$

Next we focus on the computation of $\rho_B(Z_i, \bar{Z}_j)$. Thanks to (1), we easily obtain that

$$\rho_B(Z_i, \bar{Z}_j) = 0, \quad \text{for every } i, j = 1, \dots, s, \quad i \neq j.$$

On the other hand,

$$\rho_1(Z_i, \bar{Z}_i) = -\frac{\sqrt{-1}}{2} \sum_{a=1}^s g^{a\bar{a}} \left(-\frac{\sqrt{-1}}{2} \omega(Z_i + \bar{Z}_i, Z_a - \bar{Z}_a) \right) = \frac{\sqrt{-1}}{2} g^{i\bar{i}} A_i = \frac{\sqrt{-1}}{2} \left(\frac{A_i B_i}{A_i B_i - |C_i|^2} \right).$$

Moreover, we have

$$\begin{aligned}\rho_2(Z_i, \bar{Z}_i) &= -\frac{\sqrt{-1}}{2} \sum_{a=1}^s g^{s+a\bar{s}+a} (\omega([Z_i, W_a], \bar{W}_a) + \omega([\bar{Z}_i, \bar{W}_a], W_a)) \\ &= -\sqrt{-1} \sum_{a=1}^s g^{s+a\bar{s}+a} \Re \omega([Z_i, W_a], \bar{W}_a).\end{aligned}$$

Using (1), we have

$$\begin{aligned}\omega([Z_i, W_a], \bar{W}_a) &= -\sqrt{-1} \lambda_{ia} B_a, \\ \Re \omega([Z_i, W_a], \bar{W}_a) &= \frac{B_a b_{ia}}{4} = -\frac{B_a}{4} \delta_{ia}.\end{aligned}$$

Then,

$$\rho_2(Z_i, \bar{Z}_i) = \sqrt{-1} \frac{g^{s+i\bar{s}+i} B_i}{4} = \frac{\sqrt{-1}}{4} \frac{A_i B_i}{A_i B_i - |C_i|^2}.$$

Next we observe that

$$\rho_3(Z_i, \bar{Z}_i) + \rho_4(Z_i, \bar{Z}_i) = 0$$

which implies

$$(24) \quad \rho_B(Z_i, \bar{Z}_i) = \begin{cases} -\sqrt{-1} \frac{3}{4} \left(1 + \frac{|C_r|^2}{A_{p_r} B_{p_r} - |C_r|^2} \right) & \text{if there exists } r = 1, \dots, k \text{ such that } i = p_r, \\ -\sqrt{-1} \frac{3}{4} & \text{if } i \notin \{p_1, \dots, p_k\}. \end{cases}$$

We have

$$\begin{aligned}\rho_3(Z_i, \bar{Z}_i) &= \sum_{j=1}^k g^{p_j \overline{s+p_j}} \omega([Z_i, \bar{Z}_i], [Z_{p_j}, \bar{W}_{p_j}]) = -\frac{\sqrt{-1}}{2} \sum_{j=1}^k g^{p_j \overline{s+p_j}} \bar{\lambda}_{p_j p_j} \omega(Z_i + \bar{Z}_i, \bar{W}_{p_j}) \\ &= \begin{cases} 0 & \text{if } i \notin \{p_1, \dots, p_k\}, \\ \frac{1}{2} g^{i\bar{s}+i} \bar{\lambda}_{ii} C_i & \text{otherwise.} \end{cases}\end{aligned}$$

We compute the three addends in the expression of ρ_4 separately:

$$\begin{aligned}\omega([Z_i, \bar{Z}_i]^{1,0}, W_{p_j}, \bar{Z}_{p_j}) &= -\frac{1}{2}\lambda_{ip_j}\bar{C}_{p_j} = \begin{cases} 0 & \text{if } i \notin \{p_1, \dots, p_k\} \text{ or } i \neq p_j, \\ -\frac{1}{2}\lambda_{ii}\bar{C}_i & \text{otherwise;} \end{cases} \\ \omega([Z_i, \bar{Z}_i], [W_{p_j}, \bar{Z}_{p_j}]) &= \frac{1}{2}\lambda_{p_j p_j}g_{is+p_j} = \begin{cases} 0 & \text{if } i \notin \{p_1, \dots, p_k\} \text{ or } i \neq p_j, \\ \frac{1}{2}\lambda_{ii}\bar{C}_i & \text{otherwise;} \end{cases} \\ \omega([Z_i, \bar{Z}_i]^{0,1}, \bar{W}_{p_j}, Z_{p_j}) &= \frac{1}{2}\bar{\lambda}_{ip_j}g_{s+p_j p_j} = \begin{cases} 0 & \text{if } i \neq p_j, \\ \frac{1}{2}\bar{\lambda}_{ii}C_i & \text{otherwise.} \end{cases}\end{aligned}$$

It follows

$$\rho_3(Z_i, \bar{Z}_i) = \rho_4(Z_i, \bar{Z}_i) = 0 \quad \text{if } i \notin \{p_1, \dots, p_k\},$$

and, for $i \in \{p_1, \dots, p_k\}$,

$$\rho_3(Z_i, \bar{Z}_i) + \rho_4(Z_i, \bar{Z}_i) = -\frac{1}{2}g^{is+i}\bar{\lambda}_{ii}C_i - g^{s+i}\frac{1}{2}\lambda_{ii}\bar{C}_i + g^{s+i}\frac{1}{2}\lambda_{ii}\bar{C}_i + g^{s+i}\frac{1}{2}\bar{\lambda}_{ii}C_i = 0.$$

Now, we focus on the calculation of $\rho_B(Z_i, \bar{W}_j)$. We have

$$\begin{aligned}\rho_1(Z_i, \bar{W}_j) &= \sum_{a=1}^s g^{a\bar{a}}\bar{\lambda}_{ij} \left(-\frac{\sqrt{-1}}{2}\omega(\bar{W}_j, Z_a - \bar{Z}_a) + \omega([\bar{W}_j, \bar{Z}_a], Z_a) \right) \\ &= \begin{cases} 0 & \text{otherwise,} \\ \sqrt{-1}g^{i\bar{i}}C_i\bar{\lambda}_{ii} \left(\frac{\sqrt{-1}}{2} - \bar{\lambda}_{ii} \right) & \text{if } i = j \in \{p_1, \dots, p_k\}, \end{cases}\end{aligned}$$

and since \mathfrak{J} is abelian

$$\rho_2(Z_i, \bar{W}_j) = 0.$$

Furthermore

$$\begin{aligned}\rho_3(Z_i, \bar{W}_j) &= \sum_{j=1}^k g^{\bar{p}_j s + p_j} \omega([Z_i, \bar{W}_j]^{0,1}, \bar{Z}_{p_j}, W_{p_j}) = -\sqrt{-1} \sum_{j=1}^k g^{\bar{p}_j s + p_j} \bar{\lambda}_{ij} \bar{\lambda}_{p_j p_j} g_{s+\bar{j} s + p_j} \\ &= \begin{cases} 0 & \text{otherwise,} \\ -\sqrt{-1}\bar{\lambda}_{jj}^2 g^{\bar{j} s + j} B_j & \text{if } i = j \in \{p_1, \dots, p_k\} \end{cases}\end{aligned}$$

and

$$\begin{aligned}\rho_4(Z_i, \bar{W}_j) &= \sum_{j=1}^k g^{s+p_j \bar{p}_j} \omega([Z_i, \bar{W}_j], [W_{p_j}, \bar{Z}_{p_j}]) = \sqrt{-1} \sum_{j=1}^k g^{s+p_j \bar{p}_j} \bar{\lambda}_{ij} \lambda_{p_j p_j} g_{s+\bar{j} s + p_j} \\ &= \begin{cases} 0 & \text{otherwise,} \\ \sqrt{-1}g^{s+j\bar{j}}\bar{\lambda}_{jj}\lambda_{jj}B_j & \text{if } i = j \in \{p_1, \dots, p_k\}. \end{cases}\end{aligned}$$

It follows that $\rho_B(Z_i, \bar{W}_j) \neq 0$ if and only if $i = j \in \{p_1, \dots, p_k\}$. In such a case, we have

$$\rho_B(Z_j, \bar{W}_j) = -\sqrt{-1} \left(g^{s+j\bar{j}} B_j (|\lambda_{jj}|^2 - \bar{\lambda}_{jj}^2) + g^{j\bar{j}} C_j \bar{\lambda}_{jj} \left(\frac{\sqrt{-1}}{2} - \bar{\lambda}_{jj} \right) \right).$$

Since

$$g^{s+j\bar{j}} B_j = -\frac{B_j C_j}{A_j B_j - |C_j|^2} \quad \text{and} \quad g^{j\bar{j}} C_j = \frac{B_j C_j}{A_j B_j - |C_j|^2},$$

we infer

$$\rho_B(Z_j, \bar{W}_j) = -\sqrt{-1} \left(\bar{\lambda}_{jj} \left(\frac{\sqrt{-1}}{2} - \bar{\lambda}_{jj} \right) - (|\lambda_{jj}|^2 - \bar{\lambda}_{jj}^2) \right) \frac{B_j C_j}{A_j B_j - |C_j|^2}.$$

Taking into account that $\lambda_{jj} = -\frac{\sqrt{-1}}{4} - \frac{c_{jj}}{2}$, we obtain

$$\rho_B(Z_j, \bar{W}_j) = -\sqrt{-1} \left(-\frac{3}{16} - \frac{c_{jj}^2}{4} - \frac{\sqrt{-1}c_{jj}}{4} \right) \frac{B_j C_j}{A_j B_j - |C_j|^2}$$

and the claim follows. \square

Corollary 5.4. *Let ω be a left-invariant pluriclosed Hermitian metric on an Oeljeklaus-Toma manifold M . Then ω lifts to an algebraic expanding soliton of the pluriclosed flow on the universal covering of M if and only if it takes the following diagonal expression with respect to a coframe $\{\omega^1, \dots, \omega^s, \gamma^1, \dots, \gamma^s\}$ satisfying (1) and (15):*

$$(25) \quad \omega = \sqrt{-1} \sum_{i=1}^s A\omega^i \wedge \bar{\omega}^i + B_i \gamma^i \wedge \bar{\gamma}^i.$$

Proof. Let ω be a pluriclosed left-invariant metric on an Oeljeklaus-Toma manifold M . In view of [14, Section 7], ω lifts to an algebraic expanding soliton of the pluriclosed flow on the universal covering of M if and only if

$$\rho_B^{1,1}(\cdot, \cdot) = c\omega(\cdot, \cdot) + \frac{1}{2}(\omega(D\cdot, \cdot) + \omega(\cdot, D\cdot)),$$

for some $c \in \mathbb{R}_-$ and some derivation D of \mathfrak{g} such that $DJ = JD$.

Assume that ω takes the expression in formula (25). Proposition 5.3 implies that ρ_B is represented with respect to the basis $\{Z_1, \dots, Z_s, W_1, \dots, W_s\}$ by the matrix

$$P = -\frac{3}{4A} \begin{pmatrix} I_b & 0 \\ 0 & 0 \end{pmatrix}.$$

Since

$$\frac{3}{4A} \begin{pmatrix} 0 & 0 \\ 0 & I_\gamma \end{pmatrix}$$

induces a symmetric derivation on \mathfrak{g} , ω lifts to an algebraic expanding soliton of the pluriclosed flow on the universal covering of M and the first part of the claim follows.

In order to prove the second part of the statement, we need some preliminary observations on derivations D of \mathfrak{g} that commute with J , i.e. such that

$$D(\mathfrak{g}^{1,0}) \subseteq \mathfrak{g}^{1,0}, \quad D(\mathfrak{g}^{0,1}) \subseteq \mathfrak{g}^{0,1}.$$

We can write

$$DZ_i = \sum_{j=1}^s k_j^i Z_j + m_j^i W_j \quad \text{and} \quad D\bar{Z}_i = \sum_{j=1}^s l_j^i \bar{Z}_j + r_j^i \bar{W}_j.$$

Since D is a derivation, we have, for all $i = 1, \dots, s$,

$$D[Z_i, \bar{Z}_i] = [DZ_i, \bar{Z}_i] + [Z_i, D\bar{Z}_i].$$

On the other hand

$$\begin{aligned} D[Z_i, \bar{Z}_i] &= -\frac{\sqrt{-1}}{2} \left(\sum_{j=1}^s k_j^i Z_j + l_j^i \bar{Z}_j + m_j^i W_j + r_j^i \bar{W}_j \right), \\ [DZ_i, \bar{Z}_i] &= -\frac{\sqrt{-1}}{2} k_i^i (Z_i + \bar{Z}_i) - \sum_{j=1}^s m_j^i \lambda_{ij} W_j, \\ [Z_i, D\bar{Z}_i] &= -\frac{\sqrt{-1}}{2} l_j^i (Z_i + \bar{Z}_i) + \sum_{j=1}^s r_j^i \bar{\lambda}_{ij} \bar{W}_j \end{aligned}$$

and

$$\begin{aligned} 0 &= D[Z_i, \bar{Z}_i] - [DZ_i, \bar{Z}_i] - [Z_i, D\bar{Z}_i] \\ &= -\frac{\sqrt{-1}}{2} \sum_{j \neq i} k_j^i Z_j + l_j^i \bar{Z}_j + \frac{\sqrt{-1}}{2} l_i^i Z_i + \frac{\sqrt{-1}}{2} k_i^i \bar{Z}_i + \sum_{j=1}^s m_j^i \left(\lambda_{ij} - \frac{\sqrt{-1}}{2} \right) W_j - r_j^i \left(\frac{\sqrt{-1}}{2} + \bar{\lambda}_{ij} \right) \bar{W}_j \end{aligned}$$

which forces $DZ_i, D\bar{Z}_i = 0$, for all $i = 1, \dots, s$. It follows that $D|_{\mathfrak{h}} = 0$.

Moreover, for all $I, I' \in \mathfrak{J}$, we have

$$0 = D[I, I'] = [DI, I'] + [I, DI'],$$

which implies

$$[DI, I'] = -[I, DI'].$$

Assume

$$DW_i = \sum_{j=1}^s k_j^{s+i} Z_j + m_j^{s+i} W_j \quad \text{and} \quad D\bar{W}_i = \sum_{j=1}^s l_j^{s+i} \bar{Z}_j + r_j^{s+i} \bar{W}_j,$$

then

$$[DW_i, \bar{W}_i] = \sum_{j=1}^s k_j^{s+i} [Z_j, \bar{W}_i] \in \mathfrak{J}^{0,1} \quad \text{and} \quad [W_i, D\bar{W}_i] = \sum_{j=1}^s l_j^{s+i} [W_i, \bar{Z}_j] \in \mathfrak{J}^{1,0}.$$

This implies

$$DW_i = \sum_{j=1}^s m_j^{s+i} W_j, \quad D\bar{W}_i = \sum_{j=1}^s r_j^{s+i} \bar{W}_j,$$

i.e. $D(\mathfrak{J}) \subseteq \mathfrak{J}$. Moreover, for all $i = 1, \dots, s$, we have that

$$D[Z_i, W_i] = -\lambda_{ii} DW_i = -\sum_{j=1}^s \lambda_{ij} m_j^{s+i} W_j,$$

while $[DZ_i, W_i] = 0$ and

$$[Z_i, DW_i] = -\sum_{j=1}^s m_j^{s+i} \lambda_{ij} W_j.$$

Using again the fact that D is a derivation, we have

$$DW_i = \sum_{j \in J_i} m_j W_j$$

where

$$J_i = \{j \in \{1, \dots, s\} \mid \lambda_{ii} = \lambda_{ij}\}.$$

With analogous computations, we infer

$$D\bar{W}_i = \sum_{j \in J_i} r_j^{s+i} \bar{W}_j.$$

Clearly, $i \in J_i$. On the other hand, for all $i = 1, \dots, s$, we know that $\Im(\lambda_{ii}) \neq 0$, while, for all $i \neq j$, $\lambda_{ij} \in \mathbb{R}$. This guarantees that, for all $i = 1, \dots, s$,

$$J_i = \{i\}.$$

This allows us to write

$$DW_i = m_i^{s+i} W_i, \quad D\bar{W}_i = r_i^{s+i} \bar{W}_i.$$

From the relations above, we obtain that

$$\text{Der}(\mathfrak{g})^{1,0} = \{E \in \text{End}(\mathfrak{g})^{1,0} \mid \mathfrak{h} \subseteq \ker(E), E(\langle W_i \rangle) \subseteq \langle W_i \rangle, \text{ for all } i = 1, \dots, s\}.$$

First of all, we suppose that ω is a pluriclosed Hermitian metric which takes the following diagonal expression with respect to a coframe $\{\omega^1, \dots, \omega^s, \gamma^1, \dots, \gamma^s\}$ satisfying (1) and (15):

$$\omega = \sqrt{-1} \sum_{i=1}^s A_i \omega^i \wedge \bar{\omega}^i + B_i \gamma^i \wedge \bar{\gamma}^i.$$

such that there exist $i, j \in \{1, \dots, s\}$ such that $A_i \neq A_j$ and we suppose that ω is an algebraic soliton. Thanks to the facts regarding derivations proved before, we have that

$$\begin{aligned} -\sqrt{-1}\frac{3}{4} &= \rho_B(Z_i, \bar{Z}_i) = c\omega(Z_i, \bar{Z}_i) + \frac{1}{2}(\omega(DZ_i, \bar{Z}_i) + \omega(Z_i, D\bar{Z}_i)) = \sqrt{-1}cA_i, \\ -\sqrt{-1}\frac{3}{4} &= \rho_B(Z_j, \bar{Z}_j) = c\omega(Z_j, \bar{Z}_j) + \frac{1}{2}(\omega(DZ_j, \bar{Z}_j) + \omega(Z_j, D\bar{Z}_j)) = \sqrt{-1}cA_j, \end{aligned}$$

which is impossible, since $A_i \neq A_j$.

Now suppose that ω is a pluriclosed metric on M which is not diagonal. So, we suppose that there exists $\bar{j} = 1, \dots, s$ such that $C_{\bar{j}} \neq 0$. Then, assume that there exist a constant $c \in \mathbb{R}$ and $D \in \text{Der}(\mathfrak{g})$ such that

$$(\rho_B)^{1,1}(\cdot, \cdot) = c\omega(\cdot, \cdot) + \frac{1}{2}(\omega(D\cdot, \cdot) + \omega(\cdot, D\cdot)), \quad DJ = JD.$$

On the other hand

$$\begin{aligned} 0 &= \rho_B(W_{\bar{j}}, \bar{W}_{\bar{j}}) = c\omega(W_{\bar{j}}, \bar{W}_{\bar{j}}) + \frac{1}{2}(\omega(DW_{\bar{j}}, \bar{W}_{\bar{j}}) + \omega(W_{\bar{j}}, D\bar{W}_{\bar{j}})) = \sqrt{-1}cB_{\bar{j}} + \frac{\sqrt{-1}}{2}(r_{\bar{j}}^{s+\bar{j}} + m_{\bar{j}}^{s+\bar{j}})B_{\bar{j}}, \\ \rho_B(Z_{\bar{j}}, \bar{W}_{\bar{j}}) &= c\omega(Z_{\bar{j}}, \bar{W}_{\bar{j}}) + \frac{1}{2}(\omega(DZ_{\bar{j}}, \bar{W}_{\bar{j}}) + \omega(Z_{\bar{j}}, D\bar{W}_{\bar{j}})) = \sqrt{-1}cC_{\bar{j}} + \frac{\sqrt{-1}}{2}r_{\bar{j}}^{s+\bar{j}}C_{\bar{j}}, \\ \rho_B(\bar{Z}_{\bar{j}}, W_{\bar{j}}) &= c\omega(\bar{Z}_{\bar{j}}, W_{\bar{j}}) + \frac{1}{2}(\omega(D\bar{Z}_{\bar{j}}, W_{\bar{j}}) + \omega(\bar{Z}_{\bar{j}}, DW_{\bar{j}})) = -\sqrt{-1}c\bar{C}_{\bar{j}} - \frac{\sqrt{-1}}{2}m_{\bar{j}}^{s+\bar{j}}\bar{C}_{\bar{j}}, \end{aligned}$$

which implies that

$$c = -\frac{1}{2}(r_{\bar{j}}^{s+\bar{j}} + m_{\bar{j}}^{s+\bar{j}}),$$

On the other hand,

$$\rho_B(Z_{\bar{j}}, \bar{W}_{\bar{j}}) = \sqrt{-1}K C_{\bar{j}},$$

where

$$K = \left(\frac{3}{16} + \frac{c_{\bar{j}\bar{j}}^2}{4} + \frac{\sqrt{-1}c_{\bar{j}\bar{j}}}{4} \right) \frac{B_{\bar{j}}}{A_{\bar{j}}B_{\bar{j}} - |C_{\bar{j}}|^2}.$$

Then,

$$K = c + \frac{1}{2}r_{\bar{j}}^{s+\bar{j}} = -\frac{1}{2}m_{\bar{j}}^{s+\bar{j}}$$

and

$$\bar{K} = c + \frac{1}{2}m_{\bar{j}}^{s+\bar{j}} = -\frac{1}{2}r_{\bar{j}}^{s+\bar{j}}.$$

From this we obtain that

$$c = K + \bar{K} = 2\Re c(K) > 0.$$

On the other hand, we have

$$-\sqrt{-1}\frac{3}{4} \left(1 + \frac{|C_{\bar{j}}|^2}{A_{\bar{j}}B_{\bar{j}} - |C_{\bar{j}}|^2} \right) = \rho_B(Z_{\bar{j}}, \bar{Z}_{\bar{j}}) = c\omega(Z_{\bar{j}}, \bar{Z}_{\bar{j}}) + \frac{1}{2}(\omega(DZ_{\bar{j}}, \bar{Z}_{\bar{j}}) + \omega(Z_{\bar{j}}, D\bar{Z}_{\bar{j}})) = \sqrt{-1}cA_{\bar{j}},$$

which implies that c must be negative. From this the claim follows. \square

Corollary 5.5. *Let ω be a pluriclosed Hermitian metric on an Oeljeklaus-Toma manifold which takes the form (16). Then the pluriclosed flow starting from ω is equivalent to the following system of ODEs:*

$$(26) \quad \begin{cases} A'_i = \frac{3}{4} & \text{if } i \notin \{p_1, \dots, p_k\}, \\ A'_{p_r} = \frac{3}{4} \left(1 + \frac{|C_r|^2}{A_{p_r}B_{p_r} - |C_r|^2} \right) & \text{for all } r = 1, \dots, k, \\ B'_j = 0 & \text{for all } j = 1, \dots, s, \\ C'_r = - \left(\frac{3}{16} + \frac{c_{p_r p_r}^2}{4} + \frac{\sqrt{-1}c_{p_r p_r}}{4} \right) \frac{B_{p_r}C_r}{A_{p_r}B_{p_r} - |C_r|^2} & \text{for all } r = 1, \dots, k. \end{cases}$$

Moreover, $|C_r|$ is bounded, for all $r = 1, \dots, k$, the solution exists for all $t \in [0, +\infty)$ and $A_i \sim \frac{3}{4}t$, as $t \rightarrow +\infty$, for all $i = 1, \dots, s$.

In particular,

$$\frac{\omega_t}{1+t} \rightarrow 3\omega_\infty$$

as $t \rightarrow \infty$.

Proof. Observe that, for every $r \in \{1, \dots, k\}$,

$$(|C_r|^2)' = - \left(\frac{3}{8} + \frac{c_{jj}^2}{2} \right) \frac{B_{p_r} |C_r|^2}{A_{p_r} B_{p_r} - |C_r|^2} \leq 0,$$

which guarantees that $|C_r|^2$ is bounded. On the other hand, denote, for all $r = 1, \dots, k$,

$$u_r = A_{p_r} B_{p_r} - |C_r|^2.$$

We have that

$$u_r' = A_{p_r}' B_{p_r} - (|C_r|^2)' = \frac{3}{4} B_{p_r} + \left(\frac{9}{8} + \frac{c_{p_r p_r}^2}{2} \right) \frac{B_{p_r} |C_r|^2}{A_{p_r} B_{p_r} - |C_r|^2} \geq 0.$$

This guarantees

$$A_{p_r}' = \frac{3}{4} \left(1 + \frac{|C_r|^2}{A_{p_r} B_{p_r} - |C_r|^2} \right) \leq \frac{3}{4} \left(1 + \frac{K}{u_r(0)} \right),$$

where $K > 0$ such that $|C_r|^2 \leq K$, for all $t \geq 0$. This implies the long-time existence. As regards the last part of the statement, it is sufficient to prove that

$$\lim_{t \rightarrow +\infty} \frac{|C_r|^2}{u_r} = 0.$$

But,

$$u_r' \geq \frac{3}{4} B_{p_r}.$$

So,

$$u_r \geq \frac{3}{4} B_{p_r} t + u_r(0) \rightarrow +\infty, \quad t \rightarrow +\infty.$$

Then,

$$\lim_{t \rightarrow +\infty} u_r(t) = +\infty,$$

and, since $|C_r|^2$ is bounded, the assertion follows. \square

Proof of Theorem 1.1. Let ω be a left-invariant pluriclosed metric on an Oeljeklaus-Toma manifold. Corollary 5.5 implies that pluriclosed flow starting from ω has a long-time solution ω_t such that

$$\frac{\omega_t}{1+t} \rightarrow 3\omega_\infty \quad \text{as } t \rightarrow \infty.$$

We show that $\frac{\omega_t}{1+t}$ satisfies conditions 1,2,3 in Proposition 3.1. Here we denote by $|\cdot|_t$ the norm induced by ω_t .

Taking into account that

$$\omega_t|_{\mathfrak{g} \oplus \mathfrak{g}} = \omega_0|_{\mathfrak{g} \oplus \mathfrak{g}},$$

condition 2 follows.

Thanks to the fact that condition 2 holds,

$$\omega_t|_{\mathfrak{h} \oplus \mathfrak{h}} = \sum_{i=1}^s A_i(t) \omega^i \wedge \bar{\omega}^i$$

with $\frac{A_i(t)}{1+t} \rightarrow \frac{3}{4}$ as $t \rightarrow \infty$ and there exist $C, T > 0$ such that, for every vector $v \in \mathfrak{h}$,

$$\frac{1}{\sqrt{1+t}} |v|_t \leq C |v|_0,$$

for every $t \geq T$, condition 1 is satisfied.

In order to prove Condition 3, let $\epsilon, \ell > 0$ and let γ be a curve in M tangent to \mathcal{H} which is parametrized by arclength with respect to $3\omega_\infty$ and such that $L_\infty(\gamma) < \ell$. Let $v = \dot{\gamma}$ and $T > 0$ such that

$$\left| \frac{A_i(t)}{1+t} - \frac{3}{4} \right| \leq \frac{3\epsilon^2}{4\ell^2},$$

for $t \geq T$. Then

$$\left| \frac{1}{1+t}|v|_t^2 - |v|_\infty^2 \right| \leq \sum_{i=1}^s \left| \frac{A_i(t)}{1+t} - \frac{3}{4} \right| |v_i|^2 \leq \frac{\epsilon^2}{\ell^2}$$

and

$$|L_t(\gamma) - L_\infty(\gamma)| \leq \int_0^b \left| \frac{1}{\sqrt{1+t}} |\dot{\gamma}|_t - |\dot{\gamma}|_\infty \right| da \leq \frac{\epsilon}{\ell} b \leq \epsilon,$$

since $b \leq \ell$.

Now we show the last part of the statement, using the same argument as in Proposition 4.1, and we prove that $(\mathbb{H}^s \times \mathbb{C}^s, \frac{\omega_t}{1+t})$ converges in the Cheeger-Gromov sense to $(\mathbb{H}^s \times \mathbb{C}^s, \tilde{\omega}_\infty)$ where $\tilde{\omega}_\infty$ is an algebraic soliton. Again, here we are identifying ω_t with its pull-back onto $\mathbb{H}^s \times \mathbb{C}^s$ and we are fixing as base point the identity element of $\mathbb{H}^s \times \mathbb{C}^s$. It is enough to construct a 1-parameter family of biholomorphisms $\{\varphi_t\}$ of $\mathbb{H}^s \times \mathbb{C}^s$ such that

$$\varphi_t^* \frac{\omega_t}{1+t} \rightarrow \tilde{\omega}_\infty.$$

As we already observed, since \mathfrak{J} is abelian the endomorphism represented by the matrix

$$D = \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathfrak{J}} \end{pmatrix}$$

is a derivation of \mathfrak{g} that commutes with the complex structure J . Then, we can consider

$$d\varphi_t = \exp(s(t)D) = \begin{pmatrix} I_{\mathfrak{h}} & 0 \\ 0 & e^{s(t)}I_{\mathfrak{J}} \end{pmatrix} \in \text{Aut}(\mathfrak{g}, J)$$

where $s(t) = \log(\sqrt{1+t})$. Using $d\varphi_t$, we can define

$$\varphi_t \in \text{Aut}(\mathbb{H}^s \times \mathbb{C}^s, J).$$

For $i = 1, \dots, s$ we have

$$\begin{aligned} \frac{1}{1+t}(\varphi_t^* \omega_t)(Z_i, \bar{Z}_i) &= \frac{1}{1+t} \omega_t(Z_i, \bar{Z}_i) \rightarrow \frac{3}{4} \sqrt{-1}, \quad \text{as } t \rightarrow \infty, \\ \frac{1}{1+t}(\varphi_t^* \omega_t)(Z_i, \bar{W}_i) &= \frac{1}{\sqrt{1+t}} \omega_t(Z_i, \bar{W}_i) \rightarrow 0, \quad \text{as } t \rightarrow \infty, \\ \frac{1}{1+t}(\varphi_t^* \omega_t)(W_i, \bar{W}_i) &= \omega_t(W_i, \bar{W}_i) = \sqrt{-1} B_i(0). \end{aligned}$$

Then,

$$\frac{1}{1+t} \varphi_t^* \omega_t \rightarrow \tilde{\omega}_\infty, \quad \text{as } t \rightarrow \infty,$$

where

$$\tilde{\omega}_\infty = 3\omega_\infty + \omega_{|\mathfrak{J} \oplus \mathfrak{J}}.$$

Notice that $\tilde{\omega}_\infty$ is an algebraic soliton diagonal since $\omega_{|\mathfrak{J} \oplus \mathfrak{J}}$ is diagonal in view of Proposition 5.2. \square

6. A GENERALIZATION TO SEMIDIRECT PRODUCT OF LIE ALGEBRAS

From the viewpoint of Lie groups, the algebraic structure of Oeljeklaus-Toma manifolds is quite rigid and some of the results in the previous sections can be generalized to semidirect product of Lie algebras.

In this section we consider a Lie algebra \mathfrak{g} which is a semidirect product of Lie algebras

$$\mathfrak{g} = \mathfrak{h} \ltimes_{\lambda} \mathfrak{J},$$

where $\lambda: \mathfrak{h} \rightarrow \text{Der}(\mathfrak{J})$ is a representation. We further assume that \mathfrak{g} has a complex structure of the form

$$J = J_{\mathfrak{h}} \oplus J_{\mathfrak{J}}$$

where $J_{\mathfrak{h}}$ and $J_{\mathfrak{J}}$ are complex structures on \mathfrak{h} and \mathfrak{J} , respectively.

The following assumptions are all satisfied in the case of an Oeljeklaus-Toma manifold:

- i. \mathfrak{h} has a $(1, 0)$ -frame such that $\{Z_1, \dots, Z_r\}$ such that $[Z_k, \bar{Z}_k] = -\frac{\sqrt{-1}}{2}(Z_k + \bar{Z}_k)$, for all $k = 1, \dots, r$ and the other brackets vanish;
- ii. \mathfrak{J} is a $2s$ -dimensional abelian Lie algebra and $J_{\mathfrak{J}}$ is a complex structure on \mathfrak{J} ;
- iii. $\lambda(\mathfrak{h}^{1,0}) \subseteq \text{End}(\mathfrak{J})^{1,0}$;
- iv. \mathfrak{J} has a $(1, 0)$ -frame $\{W_1, \dots, W_s\}$ such that $\lambda(Z) \cdot \bar{W}_r = \lambda_r(Z) \bar{W}_r$, for every $r = 1, \dots, s$, where $\lambda_r \in \Lambda^{1,0}(\mathfrak{h})$;
- v. $\sum_{a=1}^s \Im(\lambda_a(Z_i))$ is constant on i .
- vi. \mathfrak{J} has a $(1, 0)$ -frame $\{W_1, \dots, W_s\}$ such that $\lambda(Z) \cdot W_r = \lambda'_r(Z) W_r$, for every $r = 1, \dots, s$, where $\lambda'_r \in \Lambda^{1,0}(\mathfrak{h})$ and $\sum_{a=1}^s \Im(\lambda'_a(Z_i))$ is constant on i .

Note that condition i. is equivalent to require that $\mathfrak{h} = \underbrace{\mathfrak{f} \oplus \dots \oplus \mathfrak{f}}_{r\text{-times}}$ equipped with the complex structure

$J_{\mathfrak{h}} = \underbrace{J_{\mathfrak{f}} \oplus \dots \oplus J_{\mathfrak{f}}}_{r\text{-times}}$, while in condition iv. the existence of $\{W_r\}$ and λ_r is equivalent to require that

$$\lambda(Z) \circ \lambda(Z') = \lambda(Z') \circ \lambda(Z),$$

for every $Z, Z' \in \mathfrak{h}^{1,0}$.

The computations in Section 5 can be used to study solutions to the flow

$$(27) \quad \partial_t \omega_t = -\rho_B^{1,1}(\omega_t)$$

in semidirect products of Lie algebras (this flow coincides to the pluriclosed flow only when the initial metric is pluriclosed). We have the following

Proposition 6.1. *Let $\mathfrak{g} = \mathfrak{h} \ltimes_{\lambda} \mathfrak{J}$ be a semidirect product of Lie algebras equipped with a splitting complex structure $J = J_{\mathfrak{h}} \oplus J_{\mathfrak{J}}$ and let ω be a Hermitian metric on \mathfrak{g} making \mathfrak{h} and \mathfrak{J} orthogonal. Then the Bismut Ricci-form of ω satisfies $\rho_{B|\mathfrak{h} \oplus \mathfrak{J}}^{1,1} = \rho_{B|\mathfrak{J} \oplus \mathfrak{J}}^{1,1} = 0$.*

If i-iv hold and $\omega|_{\mathfrak{h} \oplus \mathfrak{h}}$ is diagonal with respect to the frame $\{Z_i\}$ then the $(1, 1)$ -component of the Bismut-Ricci form of ω does not depend on ω and the solution to the flow (27) starting from ω takes the following expression

$$\omega_t = \omega - t \rho_B^{1,1}(\omega).$$

If i-iv and vi hold and $\omega|_{\mathfrak{h} \oplus \mathfrak{h}}$ is a multiple of the canonical metric with respect to the frame $\{Z_i\}$, then ω is a soliton for flow (27) with cosmological constant $c = \frac{1}{2} + \sum_{a=1}^s \Im(\lambda'_a(Z_i))$.

The previous Proposition does not cover the case when properties i-iv are satisfied and the restriction to $\mathfrak{h} \oplus \mathfrak{h}$ of the initial Hermitian inner product

$$\omega = \sqrt{-1} \sum_{a,b=1}^r g_{a\bar{b}} \omega^a \wedge \bar{\omega}^b + \sqrt{-1} \sum_{a,b=1}^s g_{r+a, r+b} \gamma^a \wedge \bar{\gamma}^b$$

is not diagonal with respect to $\{Z_i\}$. In this case flow (27) evolves only the components $g_{i\bar{i}}$ of ω along $\omega^i \wedge \bar{\omega}^i$ via the ODE

$$\partial_t g_{i\bar{i}} = \frac{1}{4} \sum_{a=1}^r g^{\bar{a}a} \Re g_{i\bar{a}} - \frac{1}{2} \sum_{c,d=1}^s g^{\bar{r}+dr+c} \{ \omega([Z_i, W_c], \bar{W}_d) + \omega([\bar{Z}_i, \bar{W}_c], W_d) \}$$

where $g_{i\bar{i}}$ depends on t . Note that the quantities $-\frac{1}{2} \sum_{c,d=1}^s g^{\bar{r}+dr+c} \{ \omega([Z_i, W_c], \bar{W}_d) + \omega([\bar{Z}_i, \bar{W}_c], W_d) \}$ appearing in the evolution of $g_{i\bar{i}}$ are independent on t .

The same computations as in Section 4 imply the following

Proposition 6.2. *Let $\mathfrak{g} = \mathfrak{h} \ltimes_{\lambda} \mathfrak{J}$ be a semidirect product of Lie algebras equipped with a splitting complex structure $J = J_{\mathfrak{h}} \oplus J_{\mathfrak{J}}$. Assume that properties i, ii, iii are satisfied and let ω be a left-invariant Hermitian metric on \mathfrak{g} . Then*

$$\rho_{C|\mathfrak{J} \oplus \mathfrak{J}} = \rho_{C|\mathfrak{h} \oplus \mathfrak{J}} = 0,$$

while $\rho_{C|\mathfrak{h} \oplus \mathfrak{h}}$ is diagonal with respect to $\{Z_1, \dots, Z_r\}$.

If further also iv. holds, then

$$\rho_C(Z_i, \bar{Z}_i) = -\sqrt{-1} \left(\frac{1}{2} - \sum_{a=1}^s \Im(\lambda_a(Z_i)) \right), \quad \text{for all } i = 1, \dots, r.$$

If, in addition, v. holds, then ω is a soliton for the Chern-Ricci flow with cosmological constant $c = \frac{1}{2} - \sum_{a=1}^s \Im(\lambda_a(Z_i))$ if and only if $\omega_{\mathfrak{h} \oplus \mathfrak{h}}$ is a multiple of the canonical metric on \mathfrak{h} with respect to the frame $\{Z_i\}$ and $\omega_{\mathfrak{h} \oplus \mathfrak{J}} = 0$.

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