DOI:
10.1016/j.jfa.2023.110158

Journal of Functional Analysis 285 (2023) 110158

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Full Length Article

When are the norms of the Riesz projection and the backward shift operator equal to one?

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A R T I C L E   I N F O

Article history:
Received 14 March 2023
Accepted 28 August 2023
Available online 7 September 2023
Communicated by K. Seip

MSC:
46E15
46E30
47A30
47B35
47B38

Keywords:
Banach function space
Abstract Hardy space
Riesz projection
Backward shift operator

A B S T R A C T

The lower estimate by Gohberg and Krupnik (1968) and the upper estimate by Hollenbeck and Verbitsky (2000) for the norm of the Riesz projection P on the Lebesgue space L^p lead to $\|P\|_{L^p\to L^p} = 1/\sin(\pi/p)$ for every $p \in (1,\infty)$. Hence L^2 is the only space among all Lebesgue spaces L^p for which the norm of the Riesz projection P is equal to one. Banach function spaces X are far-reaching generalisations of Lebesgue spaces L^p. We prove that the norm of P is equal to one on the space X if and only if X coincides with L^2 and there exists a constant $C \in (0,\infty)$ such that $\|f\|_X = C\|f\|_{L^2}$ for all functions $f \in X$. Independently from this, we also show that the norm of P on X is equal to one if and only if the norm of the backward shift operator S on the abstract Hardy space $H[X]$ built upon X is equal to one.

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https://doi.org/10.1016/j.jfa.2023.110158

1. Introduction

For a function $f \in L^1$ on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, let

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{i\theta}\right) e^{-in\theta}\, d\theta, \quad n \in \mathbb{Z}$$

be the Fourier coefficients of f. Let X be a Banach function space on $\mathbb{T}$. We postpone the technical definition until Section 2.1 and mention here only that X is continuously embedded into L^1. Let

$$H[X] := \{g \in X \ : \ \widehat{g}(n) = 0 \quad \text{for all} \quad n < 0\}$$

denote the abstract Hardy space built upon the Banach function space X. In the case $X = L^p$, we will use the standard notation $H^p := H[L^p]$. We will also use the following notation:

$$\mathbf{e}_m(z) := z^m, \quad z \in \mathbb{C}, \quad m \in \mathbb{Z}.$$

It is easy to see that the backward shift operator S, defined by

$$(Sf)(t) := \mathbf{e}_{-1}(t)\left(f(t) - \widehat{f}(0)\right), \quad t \in \mathbb{T},$$

is bounded on the space $H[X]$. Consider the operators $\mathcal{C}$ and P, defined for a function $f \in L^1$ and an a.e. point $t \in \mathbb{T}$ by

$$(\mathcal{C}f)(t) := \frac{1}{\pi i}\, \text{p.v.} \int_{\mathbb{T}} \frac{f(\tau)}{\tau - t}\, d\tau, \quad (Pf)(t) := \frac{f(t) + (\mathcal{C}f)(t)}{2},$$

respectively, where the integral is understood in the Cauchy principal value sense. The operator $\mathcal{C}$ is called the Cauchy singular integral operator and the operator P is called the Riesz projection. The latter term can be explained by the fact that if P is bounded on a Banach function space X, then one has $H[X] = P(X)$ (see [23, Lemma 1.1]).

The lower estimate by Gohberg and Krupnik (see [16, Ch. 9, Theorem 9.1]) and the upper estimate by Hollenbeck and Verbitsky [18] for the norm of the Riesz projection P on the Lebesgue space L^p lead to

$$\|P\|_{L^p \to L^p} = 1/\sin(\pi/p), \quad 1 < p < \infty. \tag{1.1}$$

Hence L^2 is the only space among all Lebesgue spaces L^p for which the norm of the Riesz projection P is equal to one. On the other hand, [5, Theorem 7.7] implies that

$$\|S\|_{H^2 \to H^2} = 1, \quad \|S\|_{H^p \to H^p} > 1 \quad \text{for} \quad p \in (1, \infty) \setminus \{2\}.$$

Thus, for the class of Lebesgue spaces L^p with $1 < p < \infty$, one has

$$\|P\|_{L^p \to L^p} = 1 \quad \Longleftrightarrow \quad \|S\|_{H^p \to H^p} = 1. \tag{1.2}$$

Banach function spaces provide a far-reaching generalisation of Lebesgue spaces. The class of Banach function spaces includes Lebesgue spaces L^p, $1 \le p \le \infty$, Orlicz spaces L^φ, Lorentz spaces $L^{p,q}$, all other rearrangement-invariant spaces (see, e.g., [3, Ch. 2 and 4]), as well as, variable Lebesgue spaces $L^{p(\cdot)}$ (see, e.g., [10]), which are not rearrangement-invariant.

It follows from [21, Theorem 4.5, Corollary 4.6] that if X is a reflexive rearrangement-invariant Banach function space such that $P : X \to X$ is bounded, then

$$\begin{aligned} \|P\|^{\text{ess}}_{X\to X} &:= \inf\{\|P - K\|_{X\to X} : K \text{ is compact on } X\} \\ &\ge \frac{1}{\sin(\pi \min\{p_X, 1 - q_X\})}, \end{aligned} \tag{1.3}$$

where p_X and q_X are the Zippin indices of the space X (see [28, pp. 27-28] for their definition and the proof of the inequalities $0 \le p_X \le q_X \le 1$, which are valid for arbitrary rearrangement-invariant Banach function spaces).

So, if $p_X \ne 1/2$ or $q_X \ne 1/2$, then $\|P\|_{X\to X} \ge \|P\|^{\text{ess}}_{X\to X} > 1$. We note in passing that if X is a rearrangement-invariant Banach function space such that $P : X \to X$ is bounded, then P is maximally noncompact on X, that is,

$$\|P\|_{X\to X} = \|P\|^{\text{ess}}_{X\to X}$$

(see [24, Theorem 1.1]).

Estimate (1.3) does not exclude the possibility of $\|P\|_{X\to X} = 1$ if $p_X = q_X = 1/2$. Note that, for instance, the Lorentz spaces $L^{2,r}$, $1 < r < \infty$, are reflexive rearrangement-invariant Banach function spaces (see, e.g., [3, Ch. 4, Section 4]) with the Zippin indices $p_{L^{2,r}} = q_{L^{2,r}} = 1/2$ (see, e.g., [28, pp. 27–28]), and the operator P is bounded on $L^{2,r}$ for every $1 < r < \infty$ (the latter follows from Calderón's extension of the Marcinkiewicz interpolation theorem [3, Ch. 4, Theorem 4.13]). On the other hand, it follows from Holmstedt's formula (see [19, Theorems 4.2–4.3]) for the K-functional for Lorentz spaces that for $\delta \in (0, 1)$ and $1 \le r \le \infty$, the space $X_{\delta,r} := (L^{2/(1-\delta)}, L^{2/(1+\delta)})_{1/2,r}$, obtained from the Lebesgue spaces $L^{2/(1-\delta)}$ and $L^{2/(1+\delta)}$ by the K-method of real interpolation (see, e.g., [3, Ch. 5]), coincides with the Lorentz space $L^{2,r}$ up to equivalence of the norms. Since the K-method of real interpolation is exact (see, e.g., [3, Ch. 5, Theorem 1.12]), we conclude from (1.1) that

$$\begin{aligned} \|P\|_{X_{\delta,r}\to X_{\delta,r}} &\le \|P\|^{1/2}_{L^{2/(1-\delta)}\to L^{2/(1-\delta)}} \|P\|^{1/2}_{L^{2/(1+\delta)}\to L^{2/(1+\delta)}} \\ &= \frac{1}{\sqrt{\sin\frac{\pi(1-\delta)}{2}}} \frac{1}{\sqrt{\sin\frac{\pi(1+\delta)}{2}}} = \frac{1}{\sin\frac{\pi(1+\delta)}{2}}. \end{aligned}$$

Hence, for every $\varepsilon > 0$ and $r \in [1, \infty]$, one can find $\delta > 0$ such that

$$\|P\|_{X_{\delta,r} \to X_{\delta,r}} \leq \frac{1}{\sin \frac{\pi(1+\delta)}{2}} < 1 + \varepsilon. \tag{1.4}$$

Thus, for every $\varepsilon > 0$ and $r \in [1, \infty]$, one can equip the Lorentz space $L^{2,r}$ with an equivalent norm $\|\cdot\|_{L^{2,r}_\varepsilon}$ such that $\|P\|_{L^{2,r}_\varepsilon \to L^{2,r}_\varepsilon} < 1 + \varepsilon$ (it is enough to take $\|\cdot\|_{L^{2,r}_\varepsilon} := \|\cdot\|_{X_{\delta,r}}$, where δ satisfies (1.4)).

So, the following natural question arises: *can the norm of the Riesz projection P on a (not necessarily rearrangement-invariant) Banach function space X be equal to one if X does not coincide with L^2?* The first main result of the paper gives a negative answer to this question.

Theorem 1.1 *(Main result 1). Let X be a Banach function space such that $\|P\|_{X \to X} = 1$. Then X coincides with L^2 and there exists a constant $C \in (0, \infty)$ such that*

$$\|g\|_X = C\|g\|_{L^2} \quad \textit{for all} \quad g \in X. \tag{1.5}$$

Our second main result deals with the extension of property (1.2) to the setting of Banach function spaces.

Theorem 1.2 *(Main result 2). Let X be a Banach function space. Then $\|P\|_{X \to X} = 1$ if and only if $\|S\|_{H[X] \to H[X]} = 1$.*

The paper is organized as follows. In Section 2, we recall definitions of a Banach function space and its associate space X', of the subspace X_a of all functions of absolutely continuous norm and of the subspace X_b, which is the closure of the set of all simple functions in X. Further, we note that if $X_a = X_b$, then the set of trigonometric polynomials $\mathcal{P}$ is dense in X_b. We also need a few notions from the theory of analytic functions on the open unit disk $\mathbb{D}$, Poisson integrals, and the Hilbert transform H. After these preliminaries, we recall that if $f \in L^p$ is a real-valued function and u is an inner function vanishing at zero, then $H(f \circ u) = (Hf) \circ u$. We conclude Section 2 by recalling several known facts about the Riesz projection scattered in our previous papers. We start Section 3 by proving a property of the norm in a real Hilbert space, and then give a proof of Theorem 1.1.

As far as Theorem 1.2 is concerned, the proof of the implications

$$\begin{aligned} &\|P\|_{X \to X} = 1 \quad \Longrightarrow \quad \|S\|_{H[X] \to H[X]} = 1, \\ &\|S\|_{H[X] \to H[X]} = 1 \quad \Longrightarrow \quad \|Pg\|_X \leq \|g\|_X \quad \text{for all continuous } g \end{aligned} \tag{1.6}$$

is not difficult. The main difficulty lies in extending the estimate $\|Pg\|_X \leq \|g\|_X$ in (1.6) to all $g \in X$ when X is not separable. This difficulty is addressed in Section 4 where we refine [23, Theorem 3.7] and [25, Theorem 3.3] and show that if the Hilbert transform

H is of weak type from the space C of continuous functions to a Banach function space X, then $X_a = X_b$. This implies that if the Riesz projection P is bounded from C to a Banach function space X, then $X_a = X_b$ and $\mathcal{P}$ is dense in X_b. This observation is a key ingredient in the proof of Theorem 1.2 given in Section 5.

Finally, in Section 6, we extend [5, Theorem 7.7] to the setting of Banach function spaces X and show that the norm of P on X can be expressed in terms of Toeplitz operators.

2. Preliminaries

2.1. Banach function spaces and their associate spaces

Let $\mathcal{M}$ be the set of all measurable extended complex-valued functions on $\mathbb{T}$ equipped with the normalized measure $dm(t) = |dt|/(2\pi)$ and let $\mathcal{M}^+$ be the subset of functions in $\mathcal{M}$ whose values lie in $[0, \infty]$.

A mapping $\rho : \mathcal{M}^+ \to [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_n \in \mathcal{M}^+$ with $n \in \mathbb{N}$, and for all constants $a \geq 0$, the following properties hold:

$$\begin{aligned}
&\text{(A1)} \qquad \rho(f) = 0 \Leftrightarrow f = 0 \text{ a.e.}, \ \rho(af) = a\rho(f), \ \rho(f+g) \leq \rho(f) + \rho(g), \\
&\text{(A2)} \qquad 0 \leq g \leq f \text{ a.e.} \ \Rightarrow \ \rho(g) \leq \rho(f) \quad \text{(the lattice property)}, \\
&\text{(A3)} \qquad 0 \leq f_n \uparrow f \text{ a.e.} \ \Rightarrow \ \rho(f_n) \uparrow \rho(f) \quad \text{(the Fatou property)}, \\
&\text{(A4)} \qquad \rho(1) < \infty, \\
&\text{(A5)} \qquad \int_{\mathbb{T}} f(t)\, dm(t) \leq C\rho(f)
\end{aligned}$$

with a constant $C \in (0, \infty)$ that may depend on ρ, but is independent of f. When functions differing only on a set of measure zero are identified, the set X of all functions $f \in \mathcal{M}$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X$, the norm of f is defined by $\|f\|_X := \rho(|f|)$. The set X equipped with the natural linear space operations and this norm becomes a Banach space. If ρ is a Banach function norm, its associate norm ρ' defined on $\mathcal{M}^+$ by

$$\rho'(g) := \sup \left\{ \int_{\mathbb{T}} f(t)g(t)\, dm(t) \ : \ f \in \mathcal{M}^+, \ \rho(f) \leq 1 \right\}, \quad g \in \mathcal{M}^+,$$

is a Banach function norm itself. The Banach function space X' determined by the Banach function norm ρ' is called the associate space (Köthe dual) of X. The associate space X' can be viewed as a subspace of the Banach dual space X^* (see [3, Ch. 1, Sections 1-2]).

2.2. Density of trigonometric polynomials in the subspace X_b

The characteristic (indicator) function of a measurable set $E \subset \mathbb{T}$ is denoted by χ_E. A function f in a Banach function space X is said to have absolutely continuous norm in X if $\|f\chi_{\gamma_n}\|_X \to 0$ for every sequence $\{\gamma_n\}$ of measurable sets such that $\chi_{\gamma_n} \to 0$ almost everywhere as $n \to \infty$. The set of all functions of absolutely continuous norm in X is denoted by X_a. If $X_a = X$, then one says that X has absolutely continuous norm. Let S_0 be the set of all simple functions on $\mathbb{T}$. Let X_b denote the closure of S_0 in the norm of X. We refer to [3, Ch. 1, Section 3] for properties of the subspaces X_a and X_b.

For $n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$, a function of the form $\sum_{k=-n}^{n} \alpha_k \mathbf{e}_k$, where $\alpha_k \in \mathbb{C}$ for all $k \in \{-n, \dots, n\}$, is called a trigonometric polynomial of order n. The set of all trigonometric polynomials is denoted by $\mathcal{P}$.

Lemma 2.1 *([23, Lemma 2.1]). Let X be a Banach function space. If $X_a = X_b$, then the set of trigonometric polynomials $\mathcal{P}$ is dense in X_b.*

2.3. Classes of analytic functions on the open unit disk

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$. Recall that a function F analytic in $\mathbb{D}$ is said to belong to the Hardy space $H^p(\mathbb{D})$, $0 < p \le \infty$, if

$$\|F\|_{H^p(\mathbb{D})} := \sup_{0 \le r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p \, d\theta \right)^{1/p} < \infty, \quad 0 < p < \infty,$$

$$\|F\|_{H^\infty(\mathbb{D})} := \sup_{z \in \mathbb{D}} |F(z)| < \infty.$$

Let g be a measurable function on $\mathbb{T}$ with $\log|g| \in L^1$. An outer function (of absolute value $|g|$) is a function $f = \lambda G$ with $\lambda \in \mathbb{C}$, $|\lambda| = 1$, and

$$G(z) := \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{,\theta} - z} \log|g(e^{i\theta})| \, d\theta \right), \quad z \in \mathbb{D}.$$

The Smirnov class $\mathcal{D}(\mathbb{D})$ consists of all functions f analytic in $\mathbb{D}$, which can be represented in the form $f = f_1/f_2$, where f_2 is outer and $f_1, f_2 \in \bigcup_{0<p\le\infty} H^p(\mathbb{D})$ (see, e.g., [29, Definition 3.3.1]). Recall that an inner function is a function $u \in H^\infty(\mathbb{D})$ such that $|u(e^{i\theta})| = 1$ for a.e. $\theta \in [-\pi, \pi]$.

Lemma 2.2. *If u is an inner function such that $u(0) = 0$, then u is a measure preserving transformation from $\mathbb{T}$ onto itself.*

This lemma goes back to Nordgren (see corollary to [30, Lemma 1] and also [9, Remark 9.4.6], [23, Lemma 2.5], [12, Theorem 5.5]).

For a finite collection $z_1, \dots, z_n \in \mathbb{D}$ and $\gamma \in \mathbb{T}$, the function

$$B(z) = \gamma \prod_{j=1}^{n} \frac{z - z_j}{1 - \overline{z_j} z}$$

is called a finite Blaschke product. As is well known, every finite Blaschke product satisfies

$$|B(z)| < 1 \quad \text{for} \quad z \in \mathbb{D}, \quad |B(\zeta)| = 1 \quad \text{for} \quad \zeta \in \mathbb{T}$$

(see, e.g., [13, Section 3.1]).

2.4. The Hilbert transform and Poisson integrals

The Hilbert transform of a function $f \in L^1$ is defined by

$$(Hf)\left(e^{i\vartheta}\right) := \frac{1}{2\pi} \,\text{p.v.} \int_{-\pi}^{\pi} f\left(e^{i\theta}\right) \cot \frac{\vartheta - \theta}{2} \, d\theta, \quad \vartheta \in [-\pi, \pi].$$

For $\vartheta \in [-\pi, \pi]$ and $r \in [0, 1)$, let

$$P_r(\vartheta) := \frac{1 - r^2}{1 - 2r\cos\vartheta + r^2}, \quad Q_r(\vartheta) := \frac{2r\sin\vartheta}{1 - 2r\cos\vartheta + r^2}$$

be the Poisson kernel and the conjugate Poisson kernel, respectively.

Theorem 2.3. *Let $1 < p < \infty$. If $f \in L^p$ is a real-valued function, then the function defined by*

$$w(re^{i\vartheta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})(P_r + iQ_r)(\vartheta - \theta) \, d\theta, \ \vartheta \in [-\pi, \pi], \ r \in [0, 1), \tag{2.1}$$

belongs to the Hardy space $H^p(\mathbb{D})$ and $\operatorname{Im} w(0) = 0$. Its nontangential boundary values $w(e^{i\vartheta})$ as $z \to e^{i\vartheta}$ exist for a.e. $\vartheta \in [-\pi, \pi]$ and

$$\operatorname{Re} w(e^{i\vartheta}) = f(e^{i\vartheta}), \ \operatorname{Im} w(e^{i\vartheta}) = (Hf)(e^{i\vartheta}) \text{ for a.e. } \vartheta \in [-\pi, \pi]. \tag{2.2}$$

This statement is well known (see, e.g., [27, Ch. I, Section D and Ch. V, Section B.2°]).

The next lemma will play an important role in the proof of Theorem 1.2.

Lemma 2.4. *Let $1 < p < \infty$, $f \in L^p$ be a real-valued function and u be an inner function such that $u(0) = 0$. Then $H(f \circ u) = (Hf) \circ u$.*

Proof. By Lemma 2.2, $u : \mathbb{T} \to \mathbb{T}$ is a measure preserving transformation. Therefore the operator $g \mapsto g \circ u$ is isometric, and hence bounded, on L^p. So, it is sufficient to prove the equality $H(f \circ u) = (Hf) \circ u$ for all f from a dense subset of L^p.

We will suppose that f is Hölder continuous. Then it follows from [17, Ch. IX, §1, Theorem 1] that

$$F(re^{i\vartheta}) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})P_r(\vartheta-\theta)\,d\theta, \quad \vartheta \in [-\pi,\pi], \quad r \in [0,1],$$

is continuous in $\overline{\mathbb{D}}$ and

$$\operatorname{Re} w(e^{i\vartheta}) = F(e^{i\vartheta}) = f(e^{i\vartheta}) \quad \text{for all} \quad \vartheta \in [-\pi,\pi], \tag{2.3}$$

where w is the function defined by (2.1). Further, [17, Ch. IX, §5, Theorem 5] implies that w is Hölder continuous on $\overline{\mathbb{D}}$. Hence $\operatorname{Im} w(e^{i\vartheta})$ is Hölder continuous on $[-\pi,\pi]$. It follows from Theorem 2.3 that $\operatorname{Im} w(e^{i\vartheta}) = (Hf)(e^{i\vartheta})$ for a.e. $\vartheta \in [-\pi,\pi]$. Since f is Hölder continuous, we conclude from Privalov's theorem (see, e.g., [9, Theorem 3.1.1] or [33, Ch. III, Theorem 13.29]) that Hf is Hölder continuous. Since the functions $\operatorname{Im} w(e^{i\vartheta})$ and $(Hf)(e^{i\vartheta})$ are equal almost everywhere and they are continuous, we conclude that they are equal everywhere:

$$\operatorname{Im} w(e^{i\vartheta}) = (Hf)(e^{i\vartheta}) \quad \text{for all} \quad \vartheta \in [-\pi,\pi]. \tag{2.4}$$

Let $W := w \circ u$. Then $W \in H^p(\mathbb{D})$ (see [11, Section 2.6, Corollary to Theorem 2.12])), and $\operatorname{Im} W(0) = \operatorname{Im} w(u(0)) = \operatorname{Im} w(0) = 0$. It follows from (2.3)–(2.4) and $u(e^{i\vartheta}) \in \mathbb{T}$ for a.e. $\vartheta \in [-\pi,\pi]$ that

$$\operatorname{Re} W(e^{i\vartheta}) = \operatorname{Re}(w \circ u)(e^{i\vartheta}) = (f \circ u)(e^{i\vartheta}), \tag{2.5}$$

$$\operatorname{Im} W(e^{i\vartheta}) = \operatorname{Im}(w \circ u)(e^{i\vartheta}) = \big((\operatorname{Im} w) \circ u\big)(e^{i\vartheta}) = \big((Hf) \circ u\big)(e^{i\vartheta}) \tag{2.6}$$

for a.e. $\vartheta \in [-\pi,\pi]$. According to Theorem 2.3,

$$\operatorname{Im} W(e^{i\vartheta}) = (H(\operatorname{Re} W))(e^{i\vartheta}) \tag{2.7}$$

for a.e. $\vartheta \in [-\pi,\pi]$ (see (2.2)). Combining (2.5)–(2.7), we get

$$\big((Hf) \circ u\big)(e^{i\vartheta}) = \operatorname{Im} W(e^{i\vartheta}) = (H(\operatorname{Re} W))(e^{i\vartheta}) = (H(f \circ u))(e^{i\vartheta})$$

for a.e. $\vartheta \in [-\pi,\pi]$. $\square$

2.5. Some known facts on the Riesz projection

In this subsection we list several known facts about the operator P, which will be used in this paper.

Lemma 2.5 *([23, formula (1.4)]). If $f \in L^1$ is such that $Pf \in L^1$, then*

$$(Pf)\hat{}(n) = \begin{cases} \widehat{f}(n), & \text{if} \quad n \geq 0, \\ 0, & \text{if} \quad n < 0. \end{cases}$$

Lemma 2.6 *([22, Lemma 3.1]). Let $f \in L^1$. Suppose there exists $g \in H^1$ such that $\widehat{f}(n) = \widehat{g}(n)$ for all $n \geq 0$. Then $Pf = g$.*

Theorem 2.7 *([25, Theorem 3.4]). Let X be a Banach function space and X' be its associate space. If $P : X_b \to X$ is bounded, then $P : X \to X$ is bounded, P maps X_b into itself,*

$$\|P\|_{X \to X} = \|P\|_{X_b \to X_b}, \tag{2.8}$$

and the adjoint of the bounded operator $P : X_b \to X_b$ is the operator $P : X' \to X'$, which implies that the latter is also bounded.

3. Proof of the first main result

3.1. A property of the norm of a real Hilbert space

Lemma 3.1. *Let $\mathcal{H}$ be a real Hilbert space, ϱ be a norm equivalent to $\|\cdot\|_{\mathcal{H}}$, and ϱ' be the associate norm,*

$$\varrho'(x) := \sup\left\{|\langle y, x\rangle_{\mathcal{H}}| : \ y \in \mathcal{H}, \ \varrho(y) \leq 1\right\}, \quad x \in \mathcal{H},$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ denotes the inner product in $\mathcal{H}$. If

$$\varrho(x)\varrho'(x) = \|x\|^2_{\mathcal{H}} \quad \text{for all} \quad x \in \mathcal{H}, \tag{3.1}$$

then there exists a constant $C \in (0, \infty)$ such that

$$\varrho(x) = C\|x\|_{\mathcal{H}} \quad \text{for all} \quad x \in \mathcal{H}. \tag{3.2}$$

Proof. Fix $a \in \mathcal{H} \setminus \{0\}$ and put

$$C := \frac{\varrho(a)}{\|a\|_{\mathcal{H}}}. \tag{3.3}$$

If $x = 0$, then (3.2) holds trivially. If $x \in \mathcal{H} \setminus \{0\}$ and a are linearly dependent, then there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $x = \lambda a$, and

$$C = \frac{|\lambda|\varrho(a)}{|\lambda|\|a\|_{\mathcal{H}}} = \frac{\varrho(\lambda a)}{\|\lambda a\|_{\mathcal{H}}} = \frac{\varrho(x)}{\|x\|_{\mathcal{H}}},$$

which implies (3.2).

Now suppose that a and $x \in \mathcal{H} \setminus \{0\}$ are linearly independent. Let $\mathcal{L}$ be the two-dimensional subspace spanned by a and x. Choosing an orthonormal basis in $\mathcal{L}$ we can identify $\mathcal{L}$ with $\mathbb{R}^2$ and $\|\cdot\|_{\mathcal{H}}$ with the standard Euclidean norm $\|\cdot\|$ on $\mathbb{R}^2$. With a slight abuse of notation, we denote the norms generated by ϱ and ϱ' on $\mathbb{R}^2$ by the same symbols.

Since ϱ is positively homogeneous of degree 1, it can be represented in the form

$$\varrho(r\cos\theta, r\sin\theta) = r\Phi(\theta), \quad r > 0, \quad \theta \in [0, 2\pi), \tag{3.4}$$

where

$$\Phi(\theta) := \varrho(\cos\theta, \sin\theta), \quad \theta \in [0, 2\pi).$$

Let

$$m := \inf_{\theta \in [0,2\pi)} \Phi(\theta).$$

Then

$$\Phi(\theta) \geq m > 0 \quad \text{for all} \quad \theta \in [0, 2\pi). \tag{3.5}$$

On the other hand, since all norms on $\mathbb{R}^2$ are equivalent, there exists $M \in (0, \infty)$ such that $\varrho(\cdot) \leq M\|\cdot\|$. Then

$$\begin{aligned}
|\Phi(\theta) - \Phi(\theta')| &= |\varrho(\cos\theta, \sin\theta) - \varrho(\cos\theta', \sin\theta')| \\
&\leq \varrho\big((\cos\theta, \sin\theta) - (\cos\theta', \sin\theta')\big) \\
&\leq M\|(\cos\theta, \sin\theta) - (\cos\theta', \sin\theta')\| \\
&= 2M\left|\sin\frac{\theta - \theta'}{2}\right| \leq M|\theta - \theta'|
\end{aligned} \tag{3.6}$$

for all $\theta, \theta' \in [0, 2\pi)$. It follows from (3.5)–(3.6) that $R := 1/\Phi$ is also Lipschitz continuous and hence absolutely continuous.

Take any

$$w \in S_\varrho := \{z \in \mathbb{R}^2 : \ \varrho(z) = 1\} = \{(R(\theta)\cos\theta, R(\theta)\sin\theta) : \ \theta \in [0, 2\pi)\}.$$

It follows from (3.1) and the definition of ϱ' that the function

$$S_\varrho \ni z \mapsto \langle z, w \rangle$$

achieves its maximum on S_ϱ at $z = w$. In other words, for any $\theta_0 \in [0, 2\pi)$, the function

$$\begin{aligned} F(\theta) &:= \big\langle (R(\theta)\cos\theta, R(\theta)\sin\theta), (R(\theta_0)\cos\theta_0, R(\theta_0)\sin\theta_0) \big\rangle_{\mathcal{H}} \\ &= R(\theta_0)R(\theta)(\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0) = R(\theta_0)R(\theta)\cos(\theta - \theta_0) \end{aligned}$$

achieves its maximum at $\theta = \theta_0$. If R is differentiable at θ_0, then

$$0 = F'(\theta_0) = R(\theta_0)\big(R'(\theta)\cos(\theta - \theta_0) - R(\theta)\sin(\theta - \theta_0)\big)\big|_{\theta=\theta_0} = R(\theta_0)R'(\theta_0).$$

Hence R is an absolutely continuous function with $R' = 0$ a.e. So, R is constant. Then it follows from (3.4) that $\varrho(z)/\|z\|$ is constant for $z \in \mathcal{L} \setminus \{0\}$. This observation and (3.3) imply that

$$C = \frac{\varrho(a)}{\|a\|_{\mathcal{H}}} = \frac{\varrho(a)}{\|a\|} = \frac{\varrho(x)}{\|x\|} = \frac{\varrho(x)}{\|x\|_{\mathcal{H}}},$$

which implies (3.2) in the case when x and a are linearly independent. $\square$

3.2. Proof of Theorem 1.1

Since $P : X \to X$ is bounded, we have $X_a = X_b$ (see [23, Theorem 3.7]) and $(X_b)^* = X'$ (see [3, Ch. 1, Corollary 4.2]). Take any $\varepsilon > 0$ and any $g \in X_b$ such that $|g| \geq \varepsilon$ a.e. on $\mathbb{T}$. Put $\log^+ |z| := \max\{0, \log|z|\}$ for $z \in \mathbb{C}$. Since

$$\log|z| = \log^+|z| - \log^+(1/|z|), \quad \log^+|z| \leq |z|, \quad z \in \mathbb{C},$$

it follows from $|g| \geq \varepsilon$ a.e. on $\mathbb{T}$ and $g \in L^1$ that $\log|g| \in L^1$. Then, by Szegő's theorem (see, e.g., [29, Theorem 2.6.1]), the outer function

$$G(z) := \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{-\theta} - z} \log|g(e^{i\theta})|\, d\theta \right), \quad z \in \mathbb{D},$$

belongs to $H^1(\mathbb{D})$ and $|G| = |g|$ a.e. on $\mathbb{T}$. Then $G \in H^1 \cap X = H[X]$. Since P is bounded on X, it follows from Lemma 2.5 and the uniqueness theorem for Fourier series (see, e.g., [26, Ch. 1, Theorem 2.7]) that $PG = G$ a.e. on $\mathbb{T}$. Taking into account that $g \in X_b$ and $|G| = |g|$, we deduce from [3, Ch. 1, Theorem 3.11 and Definition 3.7]) that, in fact, $G \in X_b$. By the Hahn-Banach theorem, there exists $\varphi \in (X_b)^*$ such that $\|\varphi\|_{(X_b)^*} = 1$ and $\varphi(G) = \|G\|_{X_b}$. Since $(X_b)^*$ is isometrically isomorphic to X', there exists $u \in X'$ such that $\|\varphi\|_{(X_b)^*} = \|u\|_{X'}$ and

$$\varphi(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{u(e^{i\theta})}\,d\theta, \quad f \in X_b.$$

Thus $\|u\|_{X'} = 1$ and

$$\|G\|_X = \|G\|_{X_b} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{i\theta})\overline{u(e^{i\theta})}\,d\theta.$$

It follows from Theorem 2.7 that the adjoint operator of $P : X_b \to X_b$ is the operator $P^* = P : X' \to X'$ and

$$\|P\|_{X'\to X'} = \|P\|_{X_b\to X_b} = \|P\|_{X\to X} = 1.$$

So, the function

$$u_+ := Pu \in H[X'] \subset H^1$$

satisfies $\|u_+\|_{X'} \le 1$ and

$$\begin{aligned}\|G\|_X &= \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{i\theta})\overline{u(e^{i\theta})}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} (PG)(e^{i\theta})\overline{u(e^{i\theta})}\,d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{i\theta})\overline{(Pu)(e^{i\theta})}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{i\theta})\overline{u_+(e^{i\theta})}\,d\theta \end{aligned} \tag{3.7}$$

(see [25, Lemma 4.1]). Using Hölder's inequality (see [3, Ch. 1, Theorem 2.4]) and taking into account that $\|u_+\|_{X'} \le 1$, one gets

$$\begin{aligned}\|G\|_X &= \mathrm{Re}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{i\theta})\overline{u_+(e^{i\theta})}\,d\theta\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}\left(G(e^{i\theta})\overline{u_+(e^{i\theta})}\right)\,d\theta \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{i\theta})|\,|u_+(e^{i\theta})|\,d\theta \le \|G\|_X\|u_+\|_{X'} \le \|G\|_X.\end{aligned}$$

Then $\|u_+\|_{X'} = 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}\left(G(e^{i\theta})\overline{u_+(e^{i\theta})}\right)\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{i\theta})|\,|u_+(e^{i\theta})|\,d\theta. \tag{3.8}$$

Since $|G|\,|u_+| - \mathrm{Re}\,(G\overline{u_+}) \ge 0$ a.e. on $\mathbb{T}$, it follows from (3.8) that

$$\operatorname{Re}(G\,\overline{u_+}) = |G|\,|u_+| \quad \text{a.e. on} \quad \mathbb{T}.$$

Hence $\psi := G\,\overline{u_+} \in L^1$ is a nonnegative function and $\log\psi = \log|G| + \log|u_+|$. Since $|G| = |g| \geq \varepsilon$ a.e. on $\mathbb{T}$ and $\|u_+\|_{X'} = 1$, we have $G, u_+ \neq 0$. Taking into account that $G, u_+ \in H^1$, we deduce from [29, Corollary 2.2.3] that $\log|G| \in L^1$ and $\log|u_+| \in L^1$. Thus $\log\psi \in L^1$. By Szegő's theorem (see, e.g., [29, Theorem 2.6.1]), there exists an outer function $\Psi \in H^2(\mathbb{D})$ such that $|\Psi| = \psi^{1/2}$ a.e. on $\mathbb{T}$. So,

$$G\,\overline{u_+} = \psi = |\Psi|^2 = \Psi\overline{\Psi} \quad \text{a.e. on} \quad \mathbb{T},$$

whence

$$\overline{\left(\frac{u_+}{\Psi}\right)} = \frac{\Psi}{G} \quad \text{a.e. on} \quad \mathbb{T}. \tag{3.9}$$

Since $\Psi \in H^2(\mathbb{D})$ and $G \in H^1(\mathbb{D})$ are outer functions, we conclude that Ψ/G belongs to the Smirnov class $\mathcal{D}(\mathbb{D})$. Moreover, $\Psi \in L^2$ and $1/G \in L^\infty$. Hence $\Psi/G \in L^2$. Then, in view of a generalization of Smirnov's theorem (see, e.g., [29, Section 3.3.1(g)] or [11, Theorem 2.11]), $\Psi/G \in H^2(\mathbb{D}) \subset H^1(\mathbb{D})$. Similarly, $\Psi \in H^2(\mathbb{D})$ is an outer function and $u_+ \in H[X'] \subset H^1$. Let us extend u_+ to the unit disk analytically:

$$u_+(z) := \frac{1}{2\pi}\int_{-\pi}^{\pi} u_+(e^{i\theta})\frac{1-r^2}{1-2r\cos(\varphi-\theta)+r^2}\,d\theta, \quad z = re^{i\varphi} \in \mathbb{D}.$$

Then $u_+ \in H^1(\mathbb{D})$. So, $u_+/\Psi \in \mathcal{D}(\mathbb{D})$. On the other hand,

$$\frac{u_+}{\Psi} = \overline{\left(\frac{\Psi}{G}\right)} \in L^2.$$

Hence, applying Smirnov's theorem once again, one gets

$$\frac{u_+}{\Psi} \in H^2(\mathbb{D}) \subset H^1(\mathbb{D}).$$

So, we have shown that $F := u_+/\Psi \in H^2(\mathbb{D})$ and $\overline{F} \in H^2(\mathbb{D})$ (see (3.9)). Taking into account that $\overline{\widehat{F}(n)} = \widehat{\overline{F}}(-n)$ for all $n \in \mathbb{Z}$, we conclude that $\widehat{F}(n) = 0$ for all $n \in \mathbb{Z}\setminus\{0\}$, that is, F is constant. Let us denote this constant by λ. Then (3.9) implies that

$$u_+ = \overline{\lambda}\Psi = |\lambda|^2 G. \tag{3.10}$$

Since $\|u_+\|_{X'} = 1$, one gets $G \in X'$ and $|\lambda|^2 = \|G\|_{X'}^{-1}$. It now follows from (3.7) and (3.10) that

$$\|G\|_X = \frac{|\lambda|^2}{2\pi}\int_{-\pi}^{\pi} G(e^{i\theta})\overline{G(e^{i\theta})}\,d\theta = \frac{1}{\|G\|_{X'}}\frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{i\theta})|^2\,d\theta = \frac{\|G\|_{L^2}^2}{\|G\|_{X'}}.$$

Thus

$$\|G\|_X \|G\|_{X'} = \|G\|_{L^2}^2.$$

So, for every $g \in X_b$ such that $|g| \geq \varepsilon$ a.e. on $\mathbb{T}$, one has $g \in X'$, $g \in L^2$, and

$$\|g\|_X \|g\|_{X'} = \|g\|_{L^2}^2. \tag{3.11}$$

Take any $g \in X_b$ and set

$$\Omega_n := \left\{ \zeta \in \mathbb{T} : \ |g(\zeta)| \geq \frac{1}{n} \right\}, \quad h_n := g\chi_{\Omega_n} + \frac{1}{n}\chi_{\mathbb{T}\setminus\Omega_n}, \quad n \in \mathbb{N}.$$

Then

$$\|g - h_n\|_X \leq \frac{2}{n}\|\chi_{\mathbb{T}\setminus\Omega_n}\|_X \leq \frac{2}{n}\|\mathbf{e}_0\|_X, \quad n \in \mathbb{N}.$$

Hence $\{h_n\}$ converges to g in X as $n \to \infty$.

On the other hand, it is not difficult to see that for all $m, n \in \mathbb{N}$,

$$|h_m - h_n| \leq \frac{2}{\min\{m,n\}} \quad \text{a.e. on} \quad \mathbb{T}.$$

This implies that $\{h_n\}$ is a Cauchy sequence in Y, where Y stands for X' or L^2. Since X and Y are continuously embedded into L^1, one concludes that $\{h_n\}$ converges to g in Y. So,

$$X_b \subseteq X' \cap L^2. \tag{3.12}$$

It follows from the above that (3.11) holds with h_n in place of g. Passing to the limit as $n \to \infty$, one gets (3.11) for every $g \in X_b$.

Take now any $v \in L^2$. Let

$$\chi_0 := \chi_{\{\zeta\in\mathbb{T}:|v(\zeta)|<1\}}, \quad \chi_1 := \chi_{\{\zeta\in\mathbb{T}:|v(\zeta)|\geq 1\}}.$$

Since $|v\chi_0| \leq 1$ a.e. on $\mathbb{T}$, we have $v\chi_0 \in X$. Let us show that $v\chi_1 \in X$. If $\chi_1 = 0$ a.e. on $\mathbb{T}$, then there is nothing to prove. Otherwise, put

$$v_n := \chi_1 \min\{|v|, n\}, \quad n \in \mathbb{N}.$$

Then $v_n \in L^\infty \subset X$ and

$$\|v_n\|_X \|\chi_1\|_{X'} \leq \|v_n\|_X \|v_n\|_{X'} = \|v_n\|_{L^2}^2$$

(see (3.11)). Since $v_n \uparrow |v|\chi_1$ as $n \to \infty$, it follows from [3, Ch. 1, Lemma 1.5] that $v\chi_1 \in X$ and

$$\|v\chi_1\|_X \leq \frac{\|v\chi_1\|_{L^2}^2}{\|\chi_1\|_{X'}} .$$

Therefore $v = v\chi_0 + v\chi_1 \in X$. So, taking into account (3.12), we conclude that

$$X_b \subseteq L^2 \subseteq X.$$

Since X_b, X and L^2 are continuously embedded into L^1, it follows from the closed graph theorem that the embeddings $X_b \subseteq L^2 \subseteq X$ are also continuous and

$$C_1\|g\|_X \leq \|g\|_{L^2} \leq C_2\|g\|_X \quad \text{for all} \quad g \in X_b \tag{3.13}$$

holds with some constants $C_1, C_2 \in (0, \infty)$.

Finally, take any $g \in X$ and set $g_n := \min\{|g|, n\}$. Then $g_n \in L^\infty$ and $g_n \uparrow |g|$ as $n \to \infty$. Hence, by Fatou's lemma (see [3, Ch. 1, Lemma 5.1]), $\|g_n\|_X \uparrow \|g\|_X < \infty$, and it follows from (3.13) that $\|g_n\|_{L^2} \uparrow \varrho < \infty$ for some constant $\varrho \leq C_2\|g\|_X$. So, $|g| \in L^2$, i.e. $g \in L^2$ for every $g \in X$, i.e. $X \subseteq L^2$. We conclude that $X = L^2$ and (3.11), (3.13) hold for all $g \in X$ (cf. [3, Ch. 1, Corollary 1.9]). It is now left to apply Lemma 3.1. $\square$

4. Necessary condition for the boundedness of the Hilbert transform from C to a Banach function space X

4.1. Operators of weak type from C to a Banach function space X

Let $\mathcal{M}_0$ denote the subset of all almost everywhere finite functions in $\mathcal{M}$. It is well known (see, e.g., [14, Theorems 29.4.3 and 29.4.6] or [3, Ch. 1, Exercise 1]) that $\mathcal{M}_0$ can be equipped with a metric d so that $(\mathcal{M}_0, d)$ is a complete linear metric space and the convergence in this metric is equivalent to the convergence in measure. Let X be a Banach function spaces over the unit circle. We say that a linear operator $A : C \to \mathcal{M}_0$ is of weak type (C, X) if there exists a constant $L > 0$ such that for all $\lambda > 0$ and all $f \in C$,

$$\left\|\chi_{\{\zeta\in\mathbb{T}:\ |(Af)(\zeta)|>\lambda\}}\right\|_X \leq L\,\frac{\|f\|_C}{\lambda}. \tag{4.1}$$

We denote the infimum of the constants L satisfying (4.1) by $\|A\|_{C\to X}^{\text{weak}}$ and the set of all operators of weak type (C, X) by $\mathcal{W}(C, X)$.

The proof of the following lemma is the same as that of [23, Lemma 3.1].

Lemma 4.1. *Let X be a Banach function space over the unit circle $\mathbb{T}$. If $A : C \to X$ is bounded, then $A \in \mathcal{W}(C, X)$ and $\|A\|_{C\to X}^{\text{weak}} \leq \|A\|_{C\to X}$.*

4.2. Mapping of a finite family of separated arcs to a single arc

We will say that two open arcs in $\mathbb{T}$ are separated if the distance between them is positive, i.e. if they are disjoint and do not have common endpoints.

Theorem 4.2. *If $E \subset \mathbb{T}$ is a finite union of pairwise separated open arcs,*

$$E = \bigcup_{k=1}^{n} \left(e^{ia_k}, e^{ib_k}\right) \neq \emptyset,$$

and $\ell \subset \mathbb{T}$ is an open arc such that $m(\ell) = m(E)$, then there exists a finite Blaschke product u satisfying $u(0) = 0$ and such that $u^{-1}(\ell) = E$.

Proof. The proof can easily be extracted from the proof of [7, Theorem 7.2] (note that the published version [8] of [7] contains a stronger variant of Theorem 7.2 equipped with a different proof that came from [31, Lemma 5.1]). We provide details here for the sake of completeness as a detailed proof of (4.4) (see below) was omitted in [7].

Take $\omega \in \mathbb{T} \setminus \operatorname{clos} E$ and consider

$$\varphi(z) := i\,\frac{\omega + z}{\omega - z}.$$

This is a conformal homeomorphism of the unit disk $\mathbb{D}$ onto the upper half-plane $\mathbb{C}_+ := \{\zeta \in \mathbb{C} : \operatorname{Im} \zeta > 0\}$ and a diffeomorphism from $\mathbb{T} \setminus \{\omega\}$ onto $\mathbb{R}$. Let

$$K(\zeta) := \prod_{k=1}^{n} \frac{\left|i - \varphi\left(e^{ia_k}\right)\right|}{\left|i - \varphi\left(e^{ib_k}\right)\right|} \cdot \frac{\zeta - \varphi\left(e^{ib_k}\right)}{\zeta - \varphi\left(e^{ia_k}\right)}.$$

Then K maps $\mathbb{C}_+$ into itself, $\mathbb{R} \setminus \left\{\varphi\left(e^{ia_k}\right)\right\}_{k=1}^{n}$ into $\mathbb{R}$, and

$$K^{-1}((-\infty, 0)) = \bigcup_{k=1}^{n} \left(\varphi\left(e^{ia_k}\right), \varphi\left(e^{ib_k}\right)\right)$$

(see [4, Proposition 2.1, Part (3)]).

If ℓ is an arc such that $m(\ell) = m(E)$, then there exists $a \in \mathbb{R}$ such that $\ell = \left(e^{ia}, e^{i(a + 2\pi m(E))}\right)$. Let

$$\psi(v) := e^{ia}\,\frac{v - e^{i\pi m(E)}}{v - e^{-i\pi m(E)}}.$$

Then ψ is a conformal homeomorphism of $\mathbb{C}_+$ onto $\mathbb{D}$ and a diffeomorphism from $\mathbb{R}$ onto $\mathbb{T} \setminus \left\{e^{ia}\right\}$ (see, e.g., [2, Theorem 13.16]). It is easy to see that $\psi^{-1}(\ell) = (-\infty, 0)$.

Let

$$u := \psi \circ K \circ \varphi.$$

Clearly, u is a rational function. It is analytic in $\mathbb{D}$ and maps $\mathbb{D}$ into itself and $\mathbb{T} \setminus \left(\{\omega\} \cup \left\{e^{ia_k}\right\}_{k=1}^n\right)$ into $\mathbb{T} \setminus \left\{e^{ia}\right\}$. The latter implies that u does not have poles in $\mathbb{T}$ and hence is also analytic in a neighbourhood of $\mathbb{T}$. Therefore

$$\lim_{|z|\to 1^-} |u(z)| = 1.$$

It follows from [13, Theorem 3.5.2] (see also [13, Lemma 13.1.4] and [15, Ch. II, Sect. 6]) that u is a finite Blaschke product.

We have

$$\begin{aligned} u^{-1}(\ell) &= \varphi^{-1}\left(K^{-1}\left(\psi^{-1}(\ell)\right)\right) = \varphi^{-1}\left(K^{-1}\left((-\infty, 0)\right)\right) \\ &= \varphi^{-1}\left(\bigcup_{k=1}^n \left(\varphi\left(e^{ia_k}\right), \varphi\left(e^{ib_k}\right)\right)\right) = \bigcup_{k=1}^n \left(e^{ia_k}, e^{ib_k}\right) = E. \end{aligned}$$

It is now left to show that $u(0) = 0$.

Since φ is a fractional linear transformation, it preserves the cross-ratio of any four points (see, e.g., [2, Theorem 13.23]). So,

$$\frac{\left(\varphi(z) - \varphi\left(e^{ia_k}\right)\right)\left(\varphi(0) - \varphi\left(e^{ib_k}\right)\right)}{\left(\varphi(z) - \varphi\left(e^{ib_k}\right)\right)\left(\varphi(0) - \varphi\left(e^{ia_k}\right)\right)} = \frac{\left(z - e^{ia_k}\right)e^{ib_k}}{\left(z - e^{ib_k}\right)e^{ia_k}}, \quad k = 1, \dots, n.$$

Taking the limits as $z \to \omega$, we get

$$\frac{i - \varphi\left(e^{ib_k}\right)}{i - \varphi\left(e^{ia_k}\right)} = \frac{\left(\omega - e^{ia_k}\right)e^{ib_k}}{\left(\omega - e^{ib_k}\right)e^{ia_k}}, \quad k = 1, \dots, n. \tag{4.2}$$

By the inscribed angle theorem, the angle at ω subtended by the arc $\left(e^{ia_k}, e^{ib_k}\right)$ is equal to $(b_k - a_k)/2$. Hence

$$\frac{e^{ib_k} - \omega}{|e^{ib_k} - \omega|}\left(\frac{e^{ia_k} - \omega}{|e^{ia_k} - \omega|}\right)^{-1} = e^{i(b_k - a_k)/2}, \quad k = 1, \dots, n. \tag{4.3}$$

Taking into account (4.2)–(4.3), we get

$$\begin{aligned} K(i) &= \prod_{k=1}^n \frac{\left|i - \varphi\left(e^{ia_k}\right)\right|}{\left|i - \varphi\left(e^{ib_k}\right)\right|} \cdot \frac{i - \varphi\left(e^{ib_k}\right)}{i - \varphi\left(e^{ia_k}\right)} = \prod_{k=1}^n \left|\frac{\omega - e^{ib_k}}{\omega - e^{ia_k}}\right| \cdot \frac{\omega - e^{ia_k}}{\omega - e^{ib_k}}\, e^{i(b_k - a_k)} \\ &= \prod_{k=1}^n \frac{e^{ia_k} - \omega}{|e^{ia_k} - \omega|}\left(\frac{e^{ib_k} - \omega}{|e^{ib_k} - \omega|}\right)^{-1} e^{i(b_k - a_k)} = \prod_{k=1}^n e^{-i(b_k - a_k)/2} e^{i(b_k - a_k)} \end{aligned}$$

$$= \prod_{k=1}^{n} e^{i(b_k - a_k)/2} = \exp\left(i \sum_{k=1}^{n} (b_k - a_k)/2 \right) = e^{i\pi m(E)}. \tag{4.4}$$

Hence

$$u(0) = \psi(K(\varphi(0))) = \psi(K(i)) = \psi\left(e^{i\pi m(E)}\right) = 0,$$

which completes the proof. $\square$

4.3. Estimates for the Hilbert transform of a piecewise linear bump function

In this subsection, we prove a lower estimate for the Hilbert transform of a piecewise linear bump function, which will play an important role in what follows.

Lemma 4.3. *Let* $0 < \beta < \frac{\pi}{2}$, $0 < \varepsilon < \min\left\{\beta, \frac{\pi}{2} - \beta\right\}$, *and*

$$g\left(e^{i\theta}\right) := \begin{cases} (\theta + \pi)/\varepsilon, & -\pi \le \theta \le -\pi + \varepsilon, \\ 1, & -\pi + \varepsilon < \theta < -\beta, \\ -(\theta + \beta - \varepsilon)/\varepsilon, & -\beta \le \theta \le -\beta + \varepsilon, \\ 0, & -\beta + \varepsilon < \theta \le \pi. \end{cases} \tag{4.5}$$

Then

$$\left|(Hg)\left(e^{i\eta}\right)\right| > \frac{1}{\pi}\left|\log\left(\sqrt{2}\sin\frac{\beta + \varepsilon}{2}\right)\right| - \frac{\varepsilon}{2\pi} \quad \text{for all} \quad \eta \in [\pi - \beta, \pi]. \tag{4.6}$$

Proof. Take any $\eta \in [\pi - \beta, \pi]$. Since $\cot\frac{\eta - \theta}{2} \le 0$ for $\theta \in [-\pi, -\beta]$, we have

$$\begin{aligned}
(Hg)(e^{i\eta}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g\left(e^{i\theta}\right)\cot\frac{\eta - \theta}{2}\,d\theta \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{-\pi+\varepsilon} + \int_{-\pi+\varepsilon}^{-\beta} + \int_{-\beta}^{-\beta+\varepsilon}\right) g\left(e^{i\theta}\right)\cot\frac{\eta - \theta}{2}\,d\theta \\
&\le \frac{1}{2\pi}\int_{-\pi+\varepsilon}^{-\beta}\cot\frac{\eta - \theta}{2}\,d\theta + \frac{1}{2\pi}\int_{-\beta}^{-\beta+\varepsilon}\left|\cot\frac{\eta - \theta}{2}\right|d\theta.
\end{aligned} \tag{4.7}$$

An easy calculation shows that

$$\frac{1}{2\pi}\int_{-\pi+\varepsilon}^{-\beta}\cot\frac{\eta - \theta}{2}\,d\theta = \frac{1}{\pi}\log\sin\frac{\eta + \pi - \varepsilon}{2} - \frac{1}{\pi}\log\sin\frac{\eta + \beta}{2}. \tag{4.8}$$

Since

$$\frac{\pi}{2} < \pi - \frac{\beta+\varepsilon}{2} \leq \frac{\eta+\pi-\varepsilon}{2} \leq \pi - \frac{\varepsilon}{2} < \pi$$

for $\eta \in [\pi-\beta, \pi]$, we have

$$\frac{1}{\pi}\log\sin\frac{\eta+\pi-\varepsilon}{2} \leq \frac{1}{\pi}\log\sin\left(\pi - \frac{\beta+\varepsilon}{2}\right) = \frac{1}{\pi}\log\sin\frac{\beta+\varepsilon}{2}. \tag{4.9}$$

Similarly, the inequalities

$$\frac{\pi}{2} \leq \frac{\eta+\beta}{2} \leq \frac{\pi+\beta}{2} < \pi$$

for $\eta \in [\pi-\beta, \pi]$, imply that

$$\frac{1}{\pi}\log\sin\frac{\eta+\beta}{2} \geq \frac{1}{\pi}\log\sin\frac{\pi+\beta}{2} = \frac{1}{\pi}\log\cos\frac{\beta}{2}. \tag{4.10}$$

Taking into account

$$0 < \frac{\beta}{2} < \frac{\beta+\varepsilon}{2} < \frac{\pi}{4},$$

we see that

$$\sin\frac{\beta+\varepsilon}{2} < \frac{1}{\sqrt{2}} < \cos\frac{\beta}{2}.$$

Hence

$$-\frac{1}{\pi}\log\cos\frac{\beta}{2} < \frac{1}{\pi}\log\sqrt{2} < -\frac{1}{\pi}\log\sin\frac{\beta+\varepsilon}{2}. \tag{4.11}$$

Combining (4.8)–(4.11), we arrive at

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi+\varepsilon}^{-\beta}\cot\frac{\eta-\theta}{2}\,d\theta &< \frac{1}{\pi}\log\sin\frac{\beta+\varepsilon}{2} + \frac{1}{\pi}\log\sqrt{2} \\ &= \frac{1}{\pi}\log\left(\sqrt{2}\sin\frac{\beta+\varepsilon}{2}\right) \\ &= -\frac{1}{\pi}\left|\log\left(\sqrt{2}\sin\frac{\beta+\varepsilon}{2}\right)\right|. \end{aligned} \tag{4.12}$$

Since

$$\frac{\pi-\varepsilon}{2} \leq \frac{\eta+\beta-\varepsilon}{2} \leq \frac{\eta-\theta}{2} \leq \frac{\eta+\beta}{2} \leq \frac{\pi+\beta}{2}$$

for $\eta \in [\pi-\beta, \pi]$ and $\theta \in [-\beta, -\beta+\varepsilon]$, the function $\cot\varphi$ is decreasing for $\varphi \in \left[\frac{\pi-\varepsilon}{2}, \frac{\pi+\beta}{2}\right]$, and

$$0 < \cot\frac{\pi-\varepsilon}{2} = -\cot\frac{\pi+\varepsilon}{2} < -\cot\frac{\pi+\beta}{2} = \tan\frac{\beta}{2},$$

we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\beta}^{-\beta+\varepsilon}\left|\cot\frac{\eta-\theta}{2}\right| d\theta &\le \frac{\varepsilon}{2\pi}\max_{\varphi\in\left[\frac{\pi-\varepsilon}{2},\frac{\pi+\beta}{2}\right]}|\cot\varphi| \\ &= \frac{\varepsilon}{2\pi}\tan\frac{\beta}{2} \le \frac{\varepsilon}{2\pi}\tan\frac{\pi}{4} = \frac{\varepsilon}{2\pi}. \end{aligned} \tag{4.13}$$

It follows from (4.7), (4.12), and (4.13) that for $\eta \in [\pi-\beta, \pi]$,

$$-\left|(Hg)(e^{i\eta})\right| \le (Hg)(e^{i\eta}) < -\frac{1}{\pi}\left|\log\left(\sqrt{2}\sin\frac{\beta+\varepsilon}{2}\right)\right| + \frac{\varepsilon}{2\pi},$$

which immediately implies (4.6). $\square$

4.4. Necessary conditions for the Hilbert transform to be of weak type (C, X)

In this subsection, we show that if the Hilbert transform is of weak type (C, X) for some Banach function space X, then $X_a = X_b$.

Let E be a union of pairwise disjoint arcs of small measure. Then

$$F(m(E);\varepsilon) := \frac{1}{\pi}\left|\log\left(\sqrt{2}\sin\left(\pi m(E) + \frac{\varepsilon}{2}\right)\right)\right| - \frac{\varepsilon}{2\pi} \tag{4.14}$$

is large whenever $\varepsilon > 0$ is small. We start by constructing a continuous real-valued function f depending on ε such that $|f| \le 1$ while the modulus of the Hilbert transform of f exceeds $F(m(E);\varepsilon)$ on the set E. This function is the composition of the piecewise linear bump function from Lemma 4.3 and the finite Blaschke product from Theorem 4.2.

Lemma 4.4. *Let $E \subset \mathbb{T}$ be a finite union of pairwise disjoint open arcs such that $0 < m(E) < \frac{1}{4}$. Then for every positive $\varepsilon < 2\pi\min\left\{m(E), \frac{1}{4} - m(E)\right\}$ there exists a continuous function $f : \mathbb{T} \to \mathbb{R}$ such that $|f| \le 1$ and*

$$\left|(Hf)\left(e^{i\eta}\right)\right| > \frac{1}{\pi}\left|\log\left(\sqrt{2}\sin\left(\pi m(E) + \frac{\varepsilon}{2}\right)\right)\right| - \frac{\varepsilon}{2\pi} \text{ for all } e^{i\eta} \in E. \tag{4.15}$$

Proof. If the pairwise disjoint open arcs constituting E are pairwise separated, set $E_0 := E$. Otherwise, let E_0 be the union of E with the set of common endpoints of the non-separated arcs in the family constituting E. In this case, E_0 is obtained from E by merging adjacent open arcs into bigger ones and reducing the total number of arcs. Either way, E_0 is a finite union of pairwise separated open arcs, $E \subseteq E_0$, and $m(E_0) = m(E)$.

Let g be defined by (4.5) with $\beta = 2\pi m(E)$, u be the finite Blaschke product from Theorem 4.2 with $\ell = \{e^{i\theta} \in \mathbb{T} : \theta \in (\pi - 2\pi m(E), \pi)\}$ and with E_0 in place of E. Consider $f := g \circ u$. Since g and u are continuous on $\mathbb{T}$, so is f. Since $Hf = H(g \circ u) = (Hg) \circ u$ in view of Lemma 2.4, it follows from Lemma 4.3 that

$$\left|(Hf)\left(e^{i\eta}\right)\right| = \left|(Hg)\left(u\left(e^{i\eta}\right)\right)\right| > \frac{1}{\pi}\left|\log\left(\sqrt{2}\sin\left(\pi m(E_0) + \frac{\varepsilon}{2}\right)\right)\right| - \frac{\varepsilon}{2\pi}$$

for all $e^{i\eta}$ such that $u\left(e^{i\eta}\right) \in \ell$, i.e. for all $e^{i\eta} \in u^{-1}(\ell) = E_0$. This immediately implies (4.15). $\square$

Corollary 4.5. *Let $E \subset \mathbb{T}$ be a finite union of pairwise disjoint open arcs such that $0 < m(E) < \frac{1}{4}$. Then there exists a continuous function $f : \mathbb{T} \to \mathbb{R}$ such that $|f| \le 1$ and*

$$\left|(Hf)\left(e^{i\eta}\right)\right| > \frac{1}{2\pi}\left|\log\left(\sqrt{2}\sin\left(\pi m(E)\right)\right)\right| \quad \text{for all} \quad e^{i\eta} \in E. \tag{4.16}$$

Proof. Let $F(m(E), \varepsilon)$ be defined by (4.14). Since it is continuous in ε, there exists $\varepsilon > 0$ such that $F(m(E), \varepsilon) - F(m(E), 0) > F(m(E), 0)/2$, whence

$$F(m(E), \varepsilon) > F(m(E), 0)/2.$$

By Lemma 4.4, there exists a continuous real-valued function f such that (4.15) holds. Combining (4.15) with the above inequality, we arrive at (4.16). $\square$

Next we show that if E is a finite union of pairwise disjoint open arcs of small measure and $H \in \mathcal{W}(C, X)$, then $\|\chi_E\|_X = O(1/F(m(E), 0))$.

Lemma 4.6. *Let X be a Banach function space over the unit circle $\mathbb{T}$. If the Hilbert transform H is of weak type (C, X), then for every finite union E of pairwise disjoint open arcs such that $0 < m(E) < \frac{1}{4}$, one has*

$$\|\chi_E\|_X \le \frac{2\pi \|H\|^{\text{weak}}_{C \to X}}{\left|\log\left(\sqrt{2}\,\sin\left(\pi m(E)\right)\right)\right|}. \tag{4.17}$$

Proof. Let

$$\lambda = \frac{1}{2\pi}\left|\log\left(\sqrt{2}\,\sin\left(\pi m(E)\right)\right)\right|.$$

By Corollary 4.5, there exists a function $f \in C$ such that $|f| \le 1$ and

$$\chi_E(\tau) \le \chi_{\{\zeta \in \mathbb{T} \,:\, |(Hf)(\zeta)| > \lambda\}}(\tau), \quad \tau \in \mathbb{T}.$$

Therefore, by the lattice property, taking into account that $H \in \mathcal{W}(C, X)$, we obtain

$$\|\chi_E\|_X \le \left\|\chi_{\{\zeta\in\mathbb{T}\,:\,|(Hf)(\zeta)|>\lambda\}}\right\|_X \le \frac{1}{\lambda}\|H\|^{\rm weak}_{C\to X}\|f\|_C$$
$$\le \frac{2\pi\|H\|^{\rm weak}_{C\to X}}{\left|\log\left(\sqrt{2}\,\sin\left(\pi m(E)\right)\right)\right|},$$

which completes the proof. □

Now we are in a position to prove the main result of this section.

Theorem 4.7. *Let X be a Banach function space over the unit circle $\mathbb{T}$. If the Hilbert transform H is of weak type (C, X), then $X_a = X_b$.*

Proof. Consider a sequence $\{\gamma_j\}_{j\in\mathbb{N}}$ of measurable subsets of $\mathbb{T}$ such that $\chi_{\gamma_j} \to 0$ a.e. on $\mathbb{T}$ as $j \to \infty$. By the dominated convergence theorem,

$$m(\gamma_j) = \int_{\mathbb{T}} \chi_{\gamma_j}(\tau)\,dm(\tau) \to 0 \quad \text{as} \quad j \to \infty.$$

Without loss of generality, one can assume that $0 < m(\gamma_j) < \frac{1}{8}$ for all $j \in \mathbb{N}$. For every $j \in \mathbb{N}$, there exists an open set $\mathcal{E}_j$ such that $\gamma_j \subseteq \mathcal{E}_j$ and $m(\mathcal{E}_j) \le 2m(\gamma_j)$. Each $\mathcal{E}_j$ is the union of an at most countable family of pairwise disjoint open arcs:

$$\mathcal{E}_j = \bigcup_{k=1}^{N_j} \ell_{j,k}, \qquad N_j \in \mathbb{N} \cup \{\infty\}.$$

If N_j is finite, set $E_j := \mathcal{E}_j$. Otherwise, let $\mathcal{E}_j^n = \bigcup_{k=1}^n \ell_{j,k}$. Since $\chi_{\mathcal{E}_j^n} \uparrow \chi_{\mathcal{E}_j}$ a.e. as $n \to \infty$, it follows from the Fatou property (A3) that

$$\left\|\chi_{\mathcal{E}_j^n}\right\|_X \uparrow \left\|\chi_{\mathcal{E}_j}\right\|_X \quad \text{as} \quad n \to \infty.$$

Then there exists $n_j \in \mathbb{N}$ such that

$$\left\|\chi_{\mathcal{E}_j^{n_j}}\right\|_X \ge \frac{1}{2}\left\|\chi_{\mathcal{E}_j}\right\|_X.$$

Set $E_j := \mathcal{E}_j^{n_j}$. Then E_j is a finite union of pairwise disjoint open arcs,

$$\frac{1}{2}m(E_j) \le \frac{1}{2}m(\mathcal{E}_j) \le m(\gamma_j) < \frac{1}{8}, \quad \left\|\chi_{E_j}\right\|_X \ge \frac{1}{2}\left\|\chi_{\mathcal{E}_j}\right\|_X \ge \frac{1}{2}\|\chi_{\gamma_j}\|_X.$$

By Lemma 4.6, for every $j \in \mathbb{N}$,

$$\|\chi_{\gamma_j}\|_X \le 2\|\chi_{E_j}\|_X \le \frac{4\pi\|H\|^{\text{weak}}_{C\to X}}{\left|\log\left(\sqrt{2}\,\sin\left(\pi m(E_j)\right)\right)\right|} \le \frac{4\pi\|H\|^{\text{weak}}_{C\to X}}{\left|\log\left(\sqrt{2}\,\sin\left(2\pi m(\gamma_j)\right)\right)\right|}.$$

Since $m(\gamma_j) \to 0$ as $j \to \infty$, the above estimate implies that $\|\chi_{\gamma_j}\|_X \to 0$ as $j \to \infty$. Thus the constant function 1 has absolutely continuous norm. Then it follows from [3, Ch. 1, Theorem 3.8] that for every measurable set $E \subset \mathbb{T}$, its characteristic function χ_E has absolutely continuous norm. Thus, by [3, Ch. 1, Theorem 3.13], $X_a = X_b$. $\square$

The above theorem and Lemma 4.1 immediately imply the following.

Corollary 4.8. *Let X be a Banach function space over the unit circle $\mathbb{T}$. If the Hilbert transform H is bounded from the space of continuous functions C to a Banach function space X, then $X_a = X_b$.*

5. Proof of the second main result

5.1. Necessary condition for the boundedness of the Riesz projection from C to a Banach function space X

We start this section by rephrasing Corollary 4.8 in terms of the Riesz projection. It improves [23, Theorem 3.7] and [25, Theorem 3.3].

Theorem 5.1. *If the Riesz projection P is bounded from the space of continuous functions C to a Banach function space X, then $X_a = X_b$.*

Proof. If $f \in C \subset L^1$, then

$$Pf := \frac{1}{2}(f + iHf) + \frac{1}{2}\widehat{f}(0) \tag{5.1}$$

(cf. [15, p. 104], [6, Section 1.43] and also [23, formula (1.3)]). Since C is continuously embedded into L^1, the functional $f \mapsto \widehat{f}(0)$ is continuous on the space C. Then it follows from (5.1) that $P : C \to X$ is bounded if and only if $H : C \to X$ is bounded. It follows from this observation and Corollary 4.8 that $X_a = X_b$. $\square$

5.2. A relation between the backward shift and the Riesz projection

The next lemma relates the backward shift operator with the Riesz projection.

Lemma 5.2. *If $f \in H^1$, then*

$$Sf = P(\mathbf{e}_{-1}f). \tag{5.2}$$

Proof. Lemma 2.6 implies that $Pf = f$. Hence

$$\begin{aligned}
(P(\mathbf{e}_{-1}f))(t) &= \frac{(\mathbf{e}_{-1}f)(t) + (\mathcal{C}(\mathbf{e}_{-1}f))(t)}{2} \\
&= \mathbf{e}_{-1}(t)\frac{f(t) + (\mathcal{C}f)(t)}{2} + \frac{1}{2}\left(\mathcal{C}(\mathbf{e}_{-1}f))(t) - \mathbf{e}_{-1}(t)(\mathcal{C}f)(t)\right) \\
&= \mathbf{e}_{-1}(t)(Pf)(t) + \frac{1}{2\pi i}\,\mathrm{p.v.}\int_{\mathbb{T}} \left(\frac{1}{\tau} - \frac{1}{t}\right)\frac{f(\tau)}{\tau - t}\,d\tau \\
&= \mathbf{e}_{-1}(t)f(t) - \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\tau)}{\tau t}\,d\tau = \mathbf{e}_{-1}(t)f(t) - \frac{\mathbf{e}_{-1}(t)}{2\pi}\int_{-\pi}^{\pi} f\left(e^{i\theta}\right)\,d\theta \\
&= \mathbf{e}_{-1}(t)f(t) - \mathbf{e}_{-1}(t)\widehat{f}(0) = \mathbf{e}_{-1}(t)\left(f(t) - \widehat{f}(0)\right) = (Sf)(t). \quad \square
\end{aligned}$$

5.3. Necessary conditions for the backward shift operator to have norm one

In this subsection we point out a consequence of $\|S\|_{H[X]\to H[X]} = 1$.

Lemma 5.3. *If X is a Banach function space and $\|S\|_{H[X]\to H[X]} = 1$, then*

$$\|Pg\|_X \le \|g\|_X \quad \textit{for all} \quad g \in \mathcal{P}. \tag{5.3}$$

Proof. Let $Q := I - P$. For any trigonometric polynomial

$$g(e^{i\theta}) = \sum_{k=-M}^{N} g_k e^{ik\theta},$$

where $M, N \in \mathbb{Z}_+$, one has

$$Pg = P(\mathbf{e}_{-1}\mathbf{e}_1 g) = P(\mathbf{e}_{-1}P(\mathbf{e}_1 g)) + P(\mathbf{e}_{-1}Q(\mathbf{e}_1 g)) = P(\mathbf{e}_{-1}P(\mathbf{e}_1 g)).$$

Since $P(\mathbf{e}_1 g) \in H[X] \cap \mathcal{P} \subset H^1$, it follows from Lemma 5.2 and the above equality that $Pg = SP(\mathbf{e}_1 g)$. Repeating the above argument M times, we get

$$Pg = S^M P(\mathbf{e}_M g).$$

Since $\mathbf{e}_M g \in H[X] \cap \mathcal{P}$, we have $P(\mathbf{e}_M g) = \mathbf{e}_M g$. Hence, taking into account that $\|S\|_{H[X]\to H[X]} = 1$, we get

$$\begin{aligned}
\|Pg\|_X = \|Pg\|_{H[X]} &= \|S^M P(\mathbf{e}_M g)\|_{H[X]} \le \|S^M\|_{H[X]\to H[X]}\|P(\mathbf{e}_M g)\|_{H[X]} \\
&\le \|S\|^M_{H[X]\to H[X]}\|P(\mathbf{e}_M g)\|_{H[X]} = \|\mathbf{e}_M g\|_{H[X]} = \|g\|_X,
\end{aligned}$$

which completes the proof of (5.3). $\square$

5.4. Proof of Theorem 1.2

Take $f \in H[X] \setminus \{0\}$ such that $\widehat{f}(0) = 0$. Then $Sf = \mathbf{e}_{-1}f$, which implies that $|Sf| = |f|$ a.e. on $\mathbb{T}$. Hence $\|Sf\|_{H[X]} = \|f\|_{H[X]}$. On the other hand, $f \in H[X] \subset H^1$. Then it follows from Lemma 2.6 that $Pf = f$. So, one always has

$$\|S\|_{H[X]\to H[X]} \geq 1 \quad \text{and} \quad \|P\|_{X\to X} \geq 1. \tag{5.4}$$

Necessity. Suppose $\|P\|_{X\to X} = 1$. It follows from Lemma 5.2 that $Sf = P(\mathbf{e}_{-1}f)$ for $f \in H[X] \subset H^1$. Hence

$$\|Sf\|_{H[X]} = \|P(\mathbf{e}_{-1}f)\|_X \leq \|P\|_{X\to X}\|\mathbf{e}_{-1}f\|_X = \|f\|_X.$$

Hence $\|S\|_{H[X]\to H[X]} \leq 1$. This inequality and the first inequality in (5.4) imply that $\|S\|_{H[X]\to H[X]} = 1$.

Sufficiency. Suppose that $\|S\|_{H[X]\to H[X]} = 1$. It follows from Lemma 5.3 and from Axioms (A2) and (A4) in the definition of a Banach function space that there exists $k > 0$ such that

$$\|Pg\|_X \leq \|g\|_X \leq k\|g\|_C \quad \text{for all} \quad g \in \mathcal{P}. \tag{5.5}$$

This inequality and the Weierstrass approximation theorem (see, e.g., corollary to [26, Ch. 1, Theorem 2.12]) imply that $P : C \to X$ is bounded. Then Theorem 5.1 implies that $X_a = X_b$. By Lemma 2.1, the set of trigonometric polynomials $\mathcal{P}$ is dense in X_b. Then the first inequality in (5.5) implies that

$$\|Pf\|_X \leq \|f\|_X \quad \text{for all} \quad f \in X_b.$$

Therefore $\|P\|_{X_b\to X} \leq 1$. Then Theorem 2.7 implies that

$$\|P\|_{X\to X} = \|P\|_{X_b\to X} \leq 1. \tag{5.6}$$

Combining the second inequality in (5.4) with (5.6), we arrive at the equality $\|P\|_{X\to X} = 1$. $\square$

Remark 5.4. Let

$$(P_0 f)(t) := f(t) - \widehat{f}(0), \quad t \in \mathbb{T}.$$

Since $|(P_0 f)(t)| = |(Sf)(t)|$, we have $\|P_0\|_{H[X]\to H[X]} = \|S\|_{H[X]\to H[X]}$. Hence it follows from Theorems 1.1 and 1.2 that if $\|P_0\|_{H[X]\to H[X]} = 1$, then X coincides with L^2 and (1.5) holds.

The same is true if $\|P_0\|_{X\to X} = 1$, since $\|P_0\|_{H[X]\to H[X]} = \|P_0\|_{X\to X} = 1$ in this case. Indeed, $P_0\mathbf{e}_1 = \mathbf{e}_1$, whence $\|P_0\|_{H[X]\to H[X]} \geq 1$. On the other hand, $P_0 : H[X] \to H[X]$ is the restriction of $P_0 : X \to X$ to $H[X]$, and $1 \leq \|P_0\|_{H[X]\to H[X]} \leq \|P_0\|_{X\to X} = 1$.

It is easy to see that $P_0 : X \to X$ is a projection onto a subspace of codimension one, and it is instructive to compare the above results to the following ones.

Suppose that X is a real separable Banach function space such that $\|P_0\|_{X\to X} = 1$. It follows from [3, Ch. 1, Corollary 5.6] that $X = X_a$. Then, by [32, Theorem 2] (see also [20, Theorem 4.3]), there exists a positive measurable function w such that

$$\|g\|_X = \left(\int_{\mathbb{T}} g^2(t)w(t)\,dm(t)\right)^{1/2} \quad \text{for all} \quad g \in X.$$

In this case, $\|P_0 g\|_X \leq \|g\|_X$ is equivalent to

$$(\widehat{g}(0))^2 \int_{\mathbb{T}} w(t)\,dm(t) \leq 2\widehat{g}(0) \int_{\mathbb{T}} g(t)w(t)\,dm(t). \tag{5.7}$$

It is easy to see that if w is non-constant, then there exists a simple function g such that $\widehat{g}(0) > 0$ and $\int_{\mathbb{T}} g(t)w(t)\,dm(t) = 0$. For such a function, (5.7) cannot hold. So, w has to be constant, which means that X coincides with L^2 and (1.5) holds.

If X is a real separable rearrangement-invariant Banach function space, and there exists a projection $Q : X \to X$ onto a subspace of finite codimension such that $\|Q\|_{X\to X} = 1$, then X is isometric to L^2 ([32, Theorem 4]), and hence X coincides with L^2 and (1.5) holds (see [1, Theorem 1]).

6. The norm of the Riesz projection in terms of Toeplitz operators

Let X be a Banach function space on which the Riesz projection P is bounded. For $a \in L^\infty$, the Toeplitz operator $T(a)$ on X is defined by

$$T(a)f = P(af), \quad f \in H[X].$$

It is easy to see that

$$\|T(a)\|_{H[X]\to H[X]} \leq \|P\|_{X\to X}\|a\|_{L^\infty}.$$

Note that if P is bounded on X, then in view of Lemma 5.2, the backward shift operator S coincides with the Toeplitz operator $T(\mathbf{e}_{-1})$:

$$Sf = T(\mathbf{e}_{-1})f, \quad f \in H[X].$$

Following [15, Ch. IX, Section 2], let

$$C + H^\infty := \{f + g \ : \ f \in C, \ g \in H^\infty\}.$$

It is well known that $C + H^\infty$ is a closed subalgebra of L^∞ generated by the set $H^\infty \cup \{\mathbf{e}_{-1}\}$ (see, e.g., [15, Ch. IX, Theorem 2.2]).

The following theorem sharpens a part of [5, Theorem 7.7] and extends it from the setting of Lebesgue spaces L^p to the setting of Banach function spaces X.

Theorem 6.1. *Let X be a Banach function space on which the Riesz projection is bounded and*

$$c_X := \sup_{n\in\mathbb{N}} \|T(\mathbf{e}_{-n})\|_{H[X]\to H[X]},$$

$$s_X := \sup_{a\in(C+H^\infty)\setminus\{0\}} \frac{\|T(a)\|_{H[X]\to H[X]}}{\|a\|_{L^\infty}},$$

$$\sigma_X := \sup_{a\in L^\infty\setminus\{0\}} \frac{\|T(a)\|_{H[X]\to H[X]}}{\|a\|_{L^\infty}}.$$

Then

$$c_X = s_X = \sigma_X = \|P\|_{X\to X}.$$

Proof. It is clear that

$$c_X \le s_X \le \sigma_X \le \|P\|_{X\to X}.$$

So, it is sufficient to show that

$$\|P\|_{X\to X} \le c_X. \tag{6.1}$$

By Theorem 2.7, $\|P\|_{X\to X} = \|P\|_{X_b\to X}$. Hence for any $\varepsilon > 0$ there exists $f \in X_b$ such that $\|f\|_X = 1$ and

$$\|Pf\|_X > \|P\|_{X\to X} - \varepsilon.$$

Since $P : X \to X$ is bounded and C is continuously embedded into X (by Axioms (A2) and (A4) in the definition of a Banach function space), we see that $P : C \to X$ is bounded. Hence it follows from Theorem 5.1 that $X_a = X_b$. Then by Lemma 2.1, there exists a trigonometric polynomial

$$g(e^{i\theta}) = \sum_{k=-M}^{N} g_k e^{ik\theta}$$

such that $\|f - g\|_X < \varepsilon$. Then $\|g\|_X < 1 + \varepsilon$ and

$$\|Pg\|_X \geq \|Pf\|_X - \|P(f-g)\|_X > \|P\|_{X\to X} - \varepsilon - \|P\|_{X\to X}\varepsilon.$$

Since $\mathbf{e}_M g \in H[X]$, one has

$$T(\mathbf{e}_{-M})(\mathbf{e}_M g) = P(\mathbf{e}_{-M}\mathbf{e}_M g) = Pg.$$

Therefore

$$\begin{aligned}\|T(\mathbf{e}_{-M})\|_{H[X]\to H[X]} &\geq \frac{\|T(\mathbf{e}_{-M})\big(\mathbf{e}_M g\big)\|_{H[X]}}{\|\mathbf{e}_M g\|_{H[X]}} = \frac{\|Pg\|_{H[X]}}{\|g\|_X} = \frac{\|Pg\|_X}{\|g\|_X} \\ &> \frac{\|P\|_{X\to X} - \varepsilon(1+\|P\|_{X\to X})}{1+\varepsilon}.\end{aligned}$$

Hence

$$c_X > \frac{\|P\|_{X\to X} - \varepsilon(1+\|P\|_{X\to X})}{1+\varepsilon} \quad \text{for all} \quad \varepsilon > 0.$$

Passing to the limit as $\varepsilon \to 0$, we arrive at (6.1). $\square$

Funding

This work is funded by national funds through the FCT - Fundação para a Ciẽncia e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 and UIDP/00297/2020 (Center for Mathematics and Applications).

Data availability

No data was used for the research described in the article.

Acknowledgments

We would like to thank the anonymous referee for useful remarks.

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