CONSTRUCTING HOMOGENEOUS GORENSTEIN IDEALS

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ABSTRACT. In 1983 Kustin and Miller introduced a construction of Gorenstein ideals in local Gorenstein rings, starting from smaller such ideals. We review and modify their construction in the case of graded rings and discuss it within the framework of Gorenstein liaison theory. We determine invariants of the constructed ideal. Concerning the problem of when a given Gorenstein ideal can be obtained by the construction, we derive a necessary condition and exhibit a Gorenstein ideal that can not be obtained using the construction.

1. INTRODUCTION

In [9] Kustin and Miller introduce a construction that produces, for given Gorenstein ideals $\mathfrak{b} \subset \mathfrak{a}$ with grades g and g-1, respectively, in a Gorenstein local ring R, a new Gorenstein ideal I of grade g in a larger Gorenstein ring R[v]. Here v is a new indeterminate. In [10] they give an interpretation for their construction via liaison theory. These beautiful results prompted us to review their construction for homogeneous Gorenstein ideals in a graded Gorenstein ring. Instead of introducing a new indeterminate, we use a suitable homogeneous element. The construction in [9] does not quite reveal the conditions on that homogeneous element. Therefore, we reverse the steps. We use two direct Gorenstein links to produce a new Gorenstein ideal and to describe a generating set of it. Then we adapt the original Kustin-Miller construction suitably in order to produce a graded free resolution of the new Gorenstein ideal that is often minimal. We also consider the question of when the process can be reversed, that is, when can a Gorenstein ideal be obtained using the construction.

This paper is organized as follows. In Section 2 we recall some results from liaison theory and the mapping cone procedure. In Section 3 we present a construction of homogeneous Gorenstein ideals via liaison theory. Given two homogeneous Gorenstein ideals $\mathfrak{b} \subset \mathfrak{a}$ of grades of g-1 and g in a graded Gorenstein ring R, by choosing an appropriate homogeneous element f in R we construct a homogeneous Gorenstein ideal $I = \mathfrak{b} +$ $(\alpha_{g-1}^* + (-1)^g f a_g^*)$ in the original ring R. Here α_{g-1}^* and a_g^* are row vectors derived from comparing the resolutions of \mathfrak{a} and \mathfrak{b} and the second ideal is generated by the entries of the specified row vector (see Theorem 3.1).

Using liaison theory, one also gets a graded free resolution of I. However, this resolution is never minimal. Adapting the original Kustin-Miller construction and its proof we obtain a smaller resolution that is often minimal (see Theorem 4.1). The key is a short exact sequence, which also allows us to interpret the linkage construction in Section 2 as an elementary biliaison from \mathfrak{a} on \mathfrak{b} . Furthermore, we obtain a necessary condition on \mathfrak{a} for constructing a given Gorenstein ideal I by such a biliaison (see Corollary 4.4).

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The original Kustin-Miller construction has been used to produce many interesting classes of Gorenstein ideals. In birational geometry it is known as unprojection (see, e.g., [14, 15, 3]). We illustrate the flexibility of our homogeneous construction by producing examples in Section 5. These include the Artinian Gorenstein ideals with socle degree two as classified by Sally [17] and the ideals of submaximal minors of a generic square matrix that are resolved by the Gulliksen-Negard complex. We also consider some Tom unprojections as studied in [3]. We conclude with an example of a homogeneous Gorenstein ideal that can not be obtained using the construction of Theorem 3.1 with a strictly ascending biliaison.

2. LIAISON AND MAPPING CONES

We frequently use ideas from liaison theory and mapping cones. We briefly recall some relevant concepts in this section.

Throughout this note R denotes a commutative Noetherian ring that is either local with maximal ideal \mathfrak{m} or graded. In the latter case we assume that $R = \bigoplus_{j\geq 0} [R]_j$ is generated as $[R]_0$ -algebra by $[R]_1$ and $[R]_0$ is a field. We denote by \mathfrak{m} its maximal homogenous ideal $\bigoplus_{j\geq 0} [R]_j$. If R is a graded ring, we consider only homogeneous ideals of R.

Assume that R is a Gorenstein ring, that is, it has finite injective dimension as an R-module. A Gorenstein ideal of R is a perfect ideal \mathfrak{c} such that R/\mathfrak{c} is Gorenstein. An ideal $I \subset R$ is said to be (directly) linked to an ideal $J \subset R$ by a Gorenstein ideal $\mathfrak{c} \subset R$ if $\mathfrak{c} \subset I \cap J$ and $\mathfrak{c} : I = J$ and $\mathfrak{c} : J = I$. Symbolically, we write $I \sim_{\mathfrak{c}} J$. Liaison is the equivalence relation generated by linkage. The equivalence classes are called liaison classes. Since we allow Gorenstein ideals in order to link, this is also referred to as Gorenstein liaison. We always work in this generality. For a comprehensive introduction to liaison theory we refer to [13].

It is not difficult to show that all complete intersections of a fixed grade are in the same liaison class. Much more is true.

Theorem 2.1. All Gorenstein ideals of R of grade g are in the same liaison class.

This has been shown in [5] for non-Artinian homogeneous Gorenstein ideals in a polynomial ring. However, the arguments work in this generality.

From now on we focus on graded rings as the results hold analogously for local rings if one forgets the grading.

Let R be a graded Gorenstein ring, and let M be a graded R-module. For any integer s, the module M(s) is the module M with the shifted grading given by $[M(s)]_j = [M]_{s+j}$. The R-dual of M is the graded module $M^* = \operatorname{Hom}_R(M, R)$. The dual of M with respect to the field $[R]_0$ is denoted by M^{\vee} . It is also a graded R-module.

The canonical module of M, denoted by ω_M , is the $[R]_0$ -dual of the local cohomology module $H^{\dim M}_{\mathfrak{m}}(M)$. By local duality, there is a graded isomorphism $\omega_M \cong \operatorname{Ext}^g_R(M, R)(s)$, where $g = \dim R - \dim M$ and s is the integer such that $\omega_R \cong R(s)$.

Now we recall a very useful short exact sequence. If the ideals I and J are linked by a Gorenstein ideal \mathfrak{c} , there is a short exact sequence

(1)
$$0 \to \mathfrak{c} \hookrightarrow I \to \omega_{R/J}(-s) \to 0,$$

where s is the integer such that $\omega_{R/\mathfrak{c}} \cong R/\mathfrak{c}(s)$ (see, e.g., [12, Lemma 3.5]).

For example, this sequence implies that R/J is Cohen-Macaulay if R/I has this property, and that the mapping cone procedure can be used to derive a free resolution of $\omega_{R/J}$ from the resolutions of I and \mathfrak{c} , and thus of J by dualizing (see [16]). Because of its importance we recall the mapping cone procedure (see, e.g., [18]). Lemma 2.2. Let

$$0 \longrightarrow M \xrightarrow{\alpha} N \longrightarrow K \longrightarrow 0$$

be a short exact sequence of graded R-modules, and let

 $\mathbb{F}: 0 \longrightarrow F_n \xrightarrow{d_n^M} F_{n-1} \longrightarrow \dots \longrightarrow F_i \xrightarrow{d_i^M} \dots \longrightarrow F_1 \xrightarrow{d_1^M} F_0 \longrightarrow M \longrightarrow 0$ and

 $\mathbb{G}: 0 \longrightarrow G_n \xrightarrow{d_n^N} G_{n-1} \longrightarrow \dots \longrightarrow G_i \xrightarrow{d_i^N} \dots \longrightarrow G_1 \xrightarrow{d_1^N} G_0 \longrightarrow N \longrightarrow 0$ be graded free resolutions of M and N, respectively. Then α induces a comparison map $\varphi: \mathbb{F} \to \mathbb{G}$. Its mapping cone is the following graded free resolution of K:

$$0 \longrightarrow F_n \xrightarrow{\partial_g} G_n \oplus F_{n-1} \longrightarrow \dots \longrightarrow G_i \oplus F_{i-1} \xrightarrow{\partial_i} \dots$$
$$\longrightarrow G_1 \oplus F_0 \xrightarrow{\partial_1} G_0 \longrightarrow K \longrightarrow 0,$$
$$\partial_i = \begin{bmatrix} d_i^N & \varphi_{i-1} \\ 0 & -d^M \end{bmatrix} \text{for } i = 1, 2, \dots, n.$$

where $\partial_i = \begin{bmatrix} i & i & i \\ 0 & -d_{i-1}^M \end{bmatrix}$ for i = 1, 2, ..., n. An analysis of the mapping cone procedure implies the following result by Buchsbaum-

Lemma 2.3. Let \mathfrak{c} be a Gorenstein ideal of R. Then

Eisenbud [1] and Peskine-Szpiro [16].

- (a) If R/I is Gorenstein and $\mathfrak{c} \subsetneq I$ with $\operatorname{grade}(\mathfrak{c}) = \operatorname{grade}(I)$, then $J = \mathfrak{c} : I$ is perfect with at most one more minimal generator than \mathfrak{c} .
- (b) Let J ⊂ R be a perfect ideal such that c ⊊ J, grade(c) = grade(J), and all minimal generators of c are also minimal generators of J. If J has one more minimal generator than c, then I = c : J is a Gorenstein ideal.

In Case (b), if \mathfrak{c} is a complete intersection, then J is an almost complete intersection, that is, I has g + 1 minimal generators, where $g = \operatorname{grade} I$.

3. Gorenstein Ideals obtained by two links

In this section we use liaison to produce a homogeneous Gorenstein ideal starting from two given homogeneous Gorenstein ideals. This also allows us to relate the Hilbert functions of the involved ideals.

Let R be a graded Gorenstein ring. Let \mathfrak{a} and $\mathfrak{b} \subset \mathfrak{a}$ be homogeneous Gorenstein ideals in R of grade g and g-1, respectively. Let

$$\mathbb{A}: 0 \longrightarrow A_g = R(-v) \xrightarrow{a_g} A_{g-1} \xrightarrow{a_{g-1}} \dots \longrightarrow A_1 \xrightarrow{a_1} R \longrightarrow 0$$

and

$$\mathbb{B}: 0 \longrightarrow B_{g-1} = R(-u) \xrightarrow{b_{g-1}} \dots \longrightarrow B_1 \xrightarrow{b_1} R \longrightarrow 0$$

be graded minimal free resolutions of R/\mathfrak{a} and R/\mathfrak{b} respectively. The embedding $\mathfrak{b} \hookrightarrow \mathfrak{a}$ induces the following commutative diagram:

Fixing bases for all the free modules, we identify the maps with their coordinate matrices. Using these assumptions and notation, the main result of this section is: **Theorem 3.1.** Assume $d = u - v \ge 0$. Let $y \in \mathfrak{a}$ be a homogeneous element such that $\mathfrak{b} : y = \mathfrak{b}$. The embedding $\mu : (\mathfrak{b}, y) \hookrightarrow \mathfrak{a}$ induces an *R*-module homomorphism $\omega_{R/\mathfrak{a}} \to \omega_{R/(\mathfrak{b},y)}$ that is multiplication by some homogeneous element $\omega \in R$. Its degree is $d + \deg y$.

Assume there is a homogeneous element $f \in R$ of degree d such that $\mathfrak{b} : (\omega + fy) = \mathfrak{b}$. Consider the ideal I obtained from \mathfrak{a} by the two links

$$\mathfrak{a} \sim_{(\mathfrak{b},y)} J \sim_{(\mathfrak{b},\omega+fy)} I,$$

that is, $I = (\mathfrak{b}, \omega + fy) : [(\mathfrak{b}, y) : \mathfrak{a}]$. Then I is a Gorenstein ideal with the same grade as \mathfrak{a} . It can be written as

$$I = \mathfrak{b} + (\alpha_{g-1}^* + (-1)^g f a_g^*) = (\mathfrak{b}, \alpha_{g-1}^* + (-1)^g f a_g^*),$$

where α_{g-1}^* and a_g^* are interpreted as row vectors and "+" indicates their component-wise sum whose entries, together with generators of \mathfrak{b} , generate I.

Proof. As in the proof of the Lemma 2.3, we use the mapping cone procedure repeatedly. Multiplication by y induces a short exact sequence

$$0 \to R/\mathfrak{b}(-e) \to R/\mathfrak{b} \to R/(\mathfrak{b}, y) \to 0,$$

where $e = \deg y$. Thus, we obtain a minimal graded free resolution \mathbb{B}' of (\mathfrak{b}, y) :

$$\mathbb{B}' : 0 \to R(-u-e) \xrightarrow{d_g} \begin{array}{c} R(-u) & B_1 \\ \oplus & \oplus \\ B_{g-2}(-e) & R(-e) \end{array} \xrightarrow{d_1} (\mathfrak{b}, y) \to 0,$$

where

$$d_1 = \begin{bmatrix} b_1 & y \end{bmatrix}, \ d_g = \begin{bmatrix} (-1)^{g-1}y \\ b_{g-1} \end{bmatrix}, \ \text{ and } d_i = \begin{bmatrix} b_i & (-1)^{i-1}yI_{m_{i-1}} \\ 0 & b_{i-1} \end{bmatrix} \text{ if } 2 \le i < g.$$

Here I_{m_i} denotes the identity matrix with $m_i = \operatorname{rank} B_i$ rows.

Using this resolution, the embedding $\mu : (\mathfrak{b}, y) \hookrightarrow \mathfrak{a}$ induces the following commutative diagram

(3)

where the maps have the form

$$\mu_i = \begin{bmatrix} \alpha_i & r_i \end{bmatrix}.$$

It shows that the map μ_g is multiplication by an element of degree d + e. By assumption, we may assume that ω is this element.

The mapping cone $C(\mu)$ of $\mu : \mathbb{B}' \to \mathbb{A}$ is

where the maps are

$$\partial_1 = \begin{bmatrix} a_2 & \mu_1 \end{bmatrix} = \begin{bmatrix} a_2 & \alpha_1 & r_1 \end{bmatrix}, \quad \partial_g = \begin{bmatrix} \mu_g \\ -d_g \end{bmatrix} = \begin{bmatrix} r_g \\ (-1)^g y \\ -b_{g-1} \end{bmatrix},$$

and $\partial_i = \begin{bmatrix} a_{i+1} & \mu_i \\ 0 & -d_i \end{bmatrix} = \begin{bmatrix} a_{i+1} & \alpha_i & r_i \\ 0 & -b_i & (-1)^i y I_{m_{i-1}} \\ 0 & 0 & -b_{i-1} \end{bmatrix}$ if $2 \le i < g$

The Sequence (1) shows that $C(\mu)$ gives a free resolution of a shift of the canonical module of R/J. Hence, the dualized and shifted complex $C(\mu)^*(-u-e)$ provides a graded free resolution of the ideal $J = (\mathfrak{b}, y, \omega)$. We do not claim that the stated generating set of Jis minimal.

By assumption, there is a homogeneous element $f \in R$ of degree $u - v = d \ge 0$ such that $z := \omega + fy$ is regular in R/\mathfrak{b} . Hence, (\mathfrak{b}, z) is a Gorenstein ideal of grade g in J. Consider now the second link

$$J \sim_{(\mathfrak{b},\omega+fy)} I.$$

As in the case of the ideal (\mathfrak{b}, y) , a mapping cone gives a free resolution of (\mathfrak{b}, z) . Thus, the embedding $\xi : (\mathfrak{b}, z) \hookrightarrow J$ induces the following commutative diagram:

where the maps are

$$t_1 = \begin{bmatrix} b_1 & z \end{bmatrix}, \ t_g = \begin{bmatrix} (-1)^{g-1}z \\ b_{g-1} \end{bmatrix}, \ \text{ and } t_i = \begin{bmatrix} b_i & (-1)^{i-1}zI_{m_{i-1}} \\ 0 & b_{i-1} \end{bmatrix} \ \text{if } 2 \le i < g.$$

Since $J = (\mathfrak{b}, y, \omega + fy)$, we can choose the following coordinate matrix for ξ_1 :

$$\xi_{1} = \begin{array}{ccccc} 1 & \dots & m_{1} & m_{1} + 1 \\ 1 & 0 & 1 \\ 2 & 0 & (-1)^{g} f \\ 3 & & 0 \\ \vdots & & 0 \\ \vdots & & 0 \\ m_{1} + 2 & & 0 \end{array}$$

where the matrix γ_1 is invertible.

By Sequence (1), the mapping cone $C(\xi)$ gives a free resolution of (a shift of) the canonical module R/I. Using the self-duality of the free resolutions A and B, $C(\xi)$ can

be re-written as

where

$$l_1 = \begin{bmatrix} \partial_g^* & \xi_1 \end{bmatrix}.$$

Since the matrix γ_1 and the upper right entry of ξ_1 are invertible, the cokernel of l_1 is isomorphic to coker \bar{l}_1 , where

$$\bar{l}_1 = \begin{bmatrix} \alpha_{g-1}^* + (-1)^g f a_g^* & b_{g-1}^* \end{bmatrix} : A_1(-\deg z) \oplus B_1(-\deg y) \oplus B_2 \to R(-\deg y).$$

It follows that the canonical module of R/I has only one minimal generator. Hence, I is a Gorenstein ideal and coker $\bar{l}_1 \cong (R/I)(-\deg y)$. The latter implies the claimed description of a generating set of the ideal I.

Notice that a sufficiently general choice of the element f always gives a desired element $\omega + fy$ in Theorem 3.1, at least if the field $k = R/\mathfrak{m}$ is infinite.

We illustrate the result by a simple example.

Example 3.2. Consider the complete intersections $\mathfrak{a} = (x, y, z)$ and $\mathfrak{b} = (x^2 - z^2, y^2 - z^2)$ in the polynomial ring K[x, y, z], where K is a field of characteristic zero. Linking \mathfrak{a} by $\mathfrak{b}+(z^2)$, we get as residual $J = \mathfrak{b}+(z^2, xyz)$. Choosing f = 5z, we link J by $\mathfrak{b}+(xyz+fz^2)$ to

$$I = \mathfrak{b} + (xf + yz, yf + xz, zf + xy) = (x^2 - z^2, y^2 - z^2, xz, yz, xy + 5z^2).$$

Observe that for the second link we cannot take f = z because $xyz + z^3$ is a zero divisor modulo \mathfrak{b} .

Using basic properties of links, we conclude this section by relating the Hilbert function of I to the Hilbert functions of \mathfrak{a} and \mathfrak{b} . Recall that the Hilbert function of a graded R-module M is defined by $\mathfrak{h}_M(j) = \dim_k[M]_j$.

Corollary 3.3. Adopt the notation and assumptions of Theorem 3.1. Then, for all integers j, the Hilbert function of R/I is given by

$$\mathfrak{h}_{R/I}(j) = \mathfrak{h}_{R/\mathfrak{a}}(j-d) + \mathfrak{h}_{R/\mathfrak{b}}(j) - \mathfrak{h}_{R/\mathfrak{b}}(j-d).$$

Proof. In the proof of Theorem 3.1 we have seen that the mapping cone (4) gives the following short exact sequence;

$$0 \longrightarrow (R/I)(-e) \longrightarrow R/(\mathfrak{b},z) \longrightarrow R/J \longrightarrow 0,$$

where deg z = d + e. Furthermore, by symmetry of liaison, the first link provides (\mathfrak{b}, y) : $\omega = \mathfrak{a}$. This implies the short exact sequence

$$0 \longrightarrow (R/\mathfrak{a})(-e-d) \longrightarrow R/(\mathfrak{b},y) \longrightarrow R/J \longrightarrow 0.$$

Combining the above two sequences we deduce

$$\begin{split} \mathfrak{h}_{R/I}(j) &= \mathfrak{h}_{R/(\mathfrak{b},z)}(j+e) - \mathfrak{h}_{R/J}(j+e) \\ &= \mathfrak{h}_{R/(\mathfrak{b},z)}(j+e) - \mathfrak{h}_{R/(\mathfrak{b},y)}(j+e) + \mathfrak{h}_{R/\mathfrak{a}}(j-d) \\ &= -\mathfrak{h}_{R/\mathfrak{b}}(j-d) + \mathfrak{h}_{R/\mathfrak{b}}(j) + \mathfrak{h}_{R/\mathfrak{a}}(j-d), \end{split}$$

as claimed.

4. A VARIATION OF THE KUSTIN-MILLER CONSTRUCTION

In the previous section we have seen that the Complex (4) provides a free resolution of the Gorenstein ideal I, constructed in Theorem 3.1. However, this resolution is not minimal if $g \ge 3$. In this Section we construct a smaller resolution of I by modifying the approach of Kustin and Miller in [9].

Theorem 4.1. Adopt the notation and assumptions of Theorem 3.1. Then there is an short exact sequence of graded R-modules

$$0 \longrightarrow (\mathfrak{a}/\mathfrak{b})(-d) \longrightarrow R/\mathfrak{b} \longrightarrow R/I \longrightarrow 0.$$

Moreover, the ideal I has a graded free resolution of the form

where the maps are described in the proof below.

Proof. We follow the approach in [9], but adjust it suitably. Thus, we focus on the needed modifications and refer for more details to [9].

First, the mapping cone \mathbb{M} of $\alpha : \mathbb{B} \to \mathbb{A}$ gives the exact sequence:

(5)

$$\mathbb{M}: 0 \to \bigoplus_{B_{g-1}}^{A_g} \to \ldots \to \bigoplus_{B_j}^{A_{j+1}} \xrightarrow{\begin{bmatrix} a_{j+1} & \alpha_j \\ 0 & -b_j \end{bmatrix}} \bigoplus_{\substack{B_j}}^{A_{j+1}} \to \ldots \to \bigoplus_{B_1}^{A_2} \xrightarrow{\begin{bmatrix} a_2 & \alpha_1 \end{bmatrix}} A_1 \to \mathfrak{a}/\mathfrak{b}.$$

Second, by [1, Proposition 1.1], the resolutions \mathbb{A} and \mathbb{B} admit a DGC algebra structure. These induce perfect pairings $B_i \times B_{g-1-i} \to B_g$ and $A_i \times A_{g-i} \to A_g$. We use the former to define degree d homomorphisms $\beta_i : A_i \to B_{i-1}(d)$ by mapping $x_i \in A_i$ on the unique element $\beta_i(x_i)$ such that, for all $z_{g-i} \in B_{g-i}$,

(6)
$$\beta_i(x_i) \cdot z_{g-i} = (-1)^{i+1} x_i \cdot \alpha_{g-i}(z_{g-i})$$

in $A_g = B_{g-1}(d)$. It follows that

(7)
$$\beta_1(x_1) = x_1 \cdot \alpha_{g-1}(1_{B_{g-1}})$$

and that β_g is multiplication by the unit $(-1)^{g+1}$. Using the perfect pairings on \mathbb{A} , we also get

(8)
$$\beta_i \circ a_{i+1} = b_i \circ \beta_{i+1}.$$

Third, [9, Lemma 1.1] shows that Diagram (2) induces a graded homomorphism of complexes $\xi : \mathbb{B} \otimes \mathbb{B} \to \mathbb{A}[1]$ such that, for all $z_i \in B_i$:

(i) $B_i \otimes B_j \to A_{i+j+1}$ is defined if $i, j \ge 0$,

- (ii) $\xi(z_i \otimes z_j) = (-1)^i \xi(z_j \otimes z_i),$
- (iii) $\xi(z_i \otimes z_i) = 0$ if *i* is odd,
- (iv) $\xi(z_0 \otimes z_i) = 0$, and
- (v) $\alpha_{i+j}(z_i z_j) \alpha_i(z_i) \cdot \alpha_j(z_j) = \xi(b_i(z_i) \otimes z_j) + (-1)^i \xi(z_i \otimes b_j(z_j)) + a_{i+j+1}(\xi(z_i \otimes z_j)).$

Finally, we define a degree d homomorphism of complexes $h : \mathbb{B} \to \mathbb{B}(d)$ by mapping $z_i \in B_i$ on the unique element $h_i(z_i)$ such that, for all $z_{g-1-i} \in B_{g-1-i}$,

$$h_i(z_i) \cdot z_{g-1-i} = (-1)^{i+1} \xi(z_i \otimes z_{g-1-i})$$

Notice that the above Condition (iv) implies $h_0 = h_{g-1} = 0$ and that Condition (v) yields (9) $\beta_i \circ \alpha_i = h_{i-1} \circ b_i + b_i \circ h_i.$

Consider now the following diagram with exact rows \mathbb{M} and \mathbb{B} , respectively: (10)

$$0 \to A_g \oplus B_{g-1} \to \dots \to A_{j+1} \oplus B_j \to \dots \to A_2 \oplus B_1 \xrightarrow{[a_2,\alpha_1]} A_1 \to \mathfrak{a}/\mathfrak{b}$$

$$\downarrow^{[\beta_g,(-1)^g f \,\mathrm{id}]} \qquad \downarrow^{[\beta_{j+1},h_j+(-1)^{j+1}f \,\mathrm{id}]} \qquad \downarrow^{[\beta_2,h_1+f \,\mathrm{id}]} \qquad \downarrow^{\beta_1+fa_1}$$

$$0 \to B_{g-1}(d) \to \dots \to B_j(d) \xrightarrow{b_j} \dots \to B_1(d) \xrightarrow{b_1} R(d) \to (R/\mathfrak{b})(d)$$

Using Equations (8) and (9), we see that all the squares commute. It follows that $\beta_1 + fa_1$ induces a homomorphism $\varphi : \mathfrak{a}/\mathfrak{b} \to (R/\mathfrak{b})(d)$ such that the resulting right-most square in the above diagram also becomes commutative. Thus, the mapping cone gives the chain complex

where the maps are

$$l_1 = \begin{bmatrix} b_1 & \beta_1 + fa_1 \end{bmatrix}, \ l_2 = \begin{bmatrix} b_2 & \beta_2 & h_1 + f \text{ id} \\ 0 & -a_2 & -\alpha_1 \end{bmatrix}, \ l_g = \begin{bmatrix} \beta_g & h_{g-1} + (-1)^g f \text{ id} \\ -a_g & -\alpha_{g-1} \\ 0 & b_{g-1} \end{bmatrix},$$

and

$$l_i = \begin{bmatrix} b_i & \beta_i & h_{i-1} + (-1)^i f \text{ id} \\ 0 & -a_i & -\alpha_{i-1} \\ 0 & 0 & b_{i-1} \end{bmatrix} \text{ if } 3 \le i \le g-1.$$

Using Equation (7), it follows that

$$\operatorname{Im} l_1 = \mathfrak{b} + (\alpha_{g-1}^* + f a_g^*)$$

All this remains true if we replace f by $(-1)^g f$. Then Theorem 3.1 shows that $I = \mathfrak{b} + (\alpha_{g-1}^* + (-1)^g f a_g^*)$ is a Gorenstein ideal, and Diagram (10) yields that I fits into an exact sequence

$$(\mathfrak{a}/\mathfrak{b})(-d) \xrightarrow{\varphi} R/\mathfrak{b} \longrightarrow R/I \longrightarrow 0.$$

It allows us to compute the Hilbert function of ker φ . Comparing with Corollary 3.3, we deduce that the kernel of φ is trivial. Hence, we obtain the desired short exact sequence

$$0 \longrightarrow (\mathfrak{a}/\mathfrak{b})(-d) \xrightarrow{\varphi} R/\mathfrak{b} \longrightarrow R/I \longrightarrow 0.$$

Now it follows that the above complex \mathbb{L} gives a free resolution of I(d). Since β_g is multiplication by a unit, we can split off the isomorphic free modules A_g and $B_{g-1}(d)$ in the map l_g . After this cancellation we get a complex that is, up to a degree shift, the claimed free resolution of I.

The free resolution of I we just derived is smaller than the one obtained from the Complex (4). In fact, it is often minimal.

Corollary 4.2. If the polynomial f is not a unit and each map α_i is minimal whenever $1 \leq i \leq g-1$, that is, $\operatorname{Im} \alpha_i \subset \mathfrak{m} A_i$, then the resolution of I described in Theorem 4.1 is a graded minimal free resolution of I.

Proof. Since the maps α_i are minimal, the definition of β_i (see Equation (6)) implies that also β_i is a minimal map whenever $1 \leq i \leq g-1$. Now the description of the maps in the free resolution obtained in Theorem 4.1 shows that all its maps between free modules are minimal. Hence, it is a minimal resolution.

The short exact sequence in Theorem 4.1 allows us to re-interpret Theorem 3.1 in terms of liaison theory. To this end we recall the following definition.

Suppose $J \subset I \cap K$ are homogeneous ideals in R with grade(I) = grade(J) + 1, and J is Cohen-Macaulay and generically Gorenstein. If there is an isomorphism of graded R-modules

$$I/J(-s) \cong K/J,$$

then it is said that K is obtained from I by an *elementary biliaison* on J. It has the same grade as I. (See [11, 13, 8] for more details.)

Using this concept, we get:

Proposition 4.3. The homogeneous Gorenstein ideal $I = (\mathfrak{b}, \alpha_{g-1}^* + (-1)^g f a_g^*)$ in Theorem 3.1 is obtained from \mathfrak{a} by an elementary biliaison on \mathfrak{b} .

Proof. Theorem 4.1 provides the short exact sequence

$$0 \longrightarrow (\mathfrak{a}/\mathfrak{b})(-d) \xrightarrow{\varphi} R/\mathfrak{b} \longrightarrow R/I \longrightarrow 0$$

Thus, we get an isomorphism $\mathfrak{a}/\mathfrak{b}(-d) \cong I/\mathfrak{b}$. Since \mathfrak{b} is Gorenstein the claim follows directly from the definition of an elementary biliaison.

So far we have studied the construction of a new homogeneous Gorenstein ideal Iof grade g from smaller homogeneous Gorenstein ideals $\mathfrak{b} \subset \mathfrak{a}$ of grades g-1 and g, respectively. It is natural to ask when this construction can be reversed. One more precise version of this problem is whether, for given homogeneous Gorenstein ideals I and \mathfrak{a} of grade g, there is a homogeneous Gorenstein ideal $\mathfrak{b} \subset \mathfrak{a}$ of grade g-1 such that Ican be obtained from \mathfrak{a} by an elementary biliaison on \mathfrak{b} . This question has already been considered in the local case in [9]. We now derive a necessary condition in the graded case. Recall that the Castelnuovo-Mumford regularity of a homogeneous Gorenstein ideal $I \subset R$ is

$$\operatorname{reg} I = \min\{m \mid [H^{i}_{\mathfrak{m}}(I)]_{j} = 0 \text{ whenever } i + j > m\},\$$

where $H^i_{\mathfrak{m}}(I)$ denotes the *i*-th local cohomology module with support in \mathfrak{m} . If *I* has finite projective dimension over *R*, then its regularity can also be computed from a minimal free resolution as

$$\operatorname{reg} I = \min\{m \mid [\operatorname{Tor}_i^R(I, R/\mathfrak{m})]_j = \text{ whenever } i - j > m\}.$$

Corollary 4.4. Let I and \mathfrak{a} be homogeneous Gorenstein ideals of grade g. If reg I – reg \mathfrak{a} is not even, then there is no homogeneous Gorenstein ideal $\mathfrak{b} \subset \mathfrak{a}$ such that I can be obtained from \mathfrak{a} by an elementary biliaison on \mathfrak{b} .

Proof. From the degree shift of the last free module in the minimal free resolution of \mathfrak{a} , we see (using the notation in Diagram (2)) that

$$\operatorname{reg} \mathfrak{a} = v - g + 1.$$

If I is obtained from \mathfrak{a} and \mathfrak{b} as in Theorem 3.1, then the free resolution of I described in Theorem 4.1 gives

$$\operatorname{reg} I = u + d - g + 1.$$

It follows that

$$\operatorname{reg} I - \operatorname{reg} \mathfrak{a} = u - v + d = 2d$$

This implies the assertion.

Using our description of the minimal free resolution in Theorem 4.1, we show in Example 5.5 below that there is a Gorenstein ideal I that cannot be obtained by the construction in Theorem 3.1 if reg $\mathfrak{a} < \operatorname{reg} I$. In general, it is open when a given Gorenstein ideal can be produced by an elementary biliaison as in Theorem 3.1.

5. Examples

We describe various examples for the construction in Theorem 3.1. We conclude with exhibiting an ideal that can not be produced using this construction.

We begin with the easiest case, where \mathfrak{a} and \mathfrak{b} are complete intersection ideals. It extends Example 3.2. This case has also been discussed in the spirit of the original Kustin-Miller construction in [14, Section 4].

Example 5.1. Let R be a graded Gorenstein ring, and let h_1, \ldots, h_g and p_1, \ldots, p_{g-1} be regular sequences of homogeneous elements such that

$$\mathfrak{b} = (p_1, \dots, p_{g-1}) \subset (h_1, \dots, h_g) = \mathfrak{a}.$$

Then there is a homogeneous $g \times (g-1)$ matrix M such that (as matrices)

$$\begin{pmatrix} p_1 & \dots & p_{g-1} \end{pmatrix} = \begin{pmatrix} h_1 & \dots & h_g \end{pmatrix} \cdot M.$$

Setting $u = \sum \deg p_i$ and $v = \sum \deg h_j$, we get the following comparison map between the graded minimal free resolutions of R/\mathfrak{a} and R/\mathfrak{b}

Denote by M_i the square matrix obtained by deleting row i of M. Then, by Theorem 3.1, for a sufficiently general $f \in R$ of degree $d = v - u \ge 0$, the ideal

$$I = (p_1, \ldots, p_{g-1}, \det M_1 + fh_1, \ldots, \det M_g + fh_g)$$

is a homogeneous Gorenstein ideal of grade g. Moreover, if no entry of the matrix M is a unit, then the graded free resolution of I described in Theorem 4.1 is minimal. In particular, then I has 2g - 1 minimal generators.

We can be more explicit in the following special case. Assume x_1, x_2, \dots, x_g is a regular sequence of homogeneous elements in R. Consider $\mathfrak{b} = (x_1^{m_1}, x_2^{m_2}, \dots, x_{g-1}^{m_{g-1}}) \subset$

 $(x_1^{n_1}, x_2^{n_2}, \cdots, x_g^{n_g}) = \mathfrak{a}$ and assume $d := \sum_{i=1}^{g-1} m_i - \sum_{i=1}^{g} n_i \ge 0$. Then, for a sufficiently general $f \in R$ of degree d,

$$I = (x_1^{m_1}, \cdots, x_{g-1}^{m_{g-1}}, fx_1^{n_1}, \cdots, fx_{g-1}^{n_{g-1}}, c + fx_g^{n_g})$$

is a Gorenstein ideal, where $c = \prod_{j=1}^{g-1} x_j^{m_j - n_j}$. Moreover, if $m_j > n_j$ for each $j = 1, \ldots, g-1$, then the resolution in Theorem 4.1 is a minimal free resolution of I.

In the next example we show that all the Gorenstein ideals with socle degree two can be obtained by one elementary biliaison from a complete intersection.

Example 5.2. Consider the Artinian Gorenstein ideals $I \subset R = K[x_1, \ldots, x_n]$ with *h*-vector (1, n, 1), where K is a field. These ideals have been classified by Sally in [17, Theorem 1.1]. Each such ideal is of the form

$$I = (x_i x_j \mid 1 \le i < j \le n) + (x_1^2 - c_1 x_n^2, \dots, x_{n-1}^2 - c_{n-1} x_n^2),$$

where $c_1, \ldots, c_{n-1} \in K$ are suitable units. It can be obtained by an elementary biliaison as in Theorem 3.1 from $\mathfrak{a} = (x_1, \ldots, x_n)$ on $\mathfrak{b}R$, where \mathfrak{b} is such a Sally ideal in n-1variables. More precisely, define the ideal \mathfrak{b} as

$$\mathbf{b} = (x_i x_j \mid 1 \le i < j \le n-1) + (x_1^2 - c_1 x_n^2, \dots, x_{n-2}^2 - c_{n-2} x_n^2).$$

Then it is not too difficult to see that there are the following links

$$\mathfrak{a} \sim_{(\mathfrak{b}, x_n)} (\mathfrak{b}, x_n, x_{n-1}^2) \sim_{(\mathfrak{b}, x_{n-1}^2 - c_{n-1} x_n^2)} I$$

Note that $(\mathfrak{b}, x_n, x_{n-1}^2) = (x_1, \dots, x_{n-1})^2 + (x_n).$

The following classical example has been studied from various points of view.

Example 5.3. Let $M = (x_{ij})$ be a generic $n \times n$ matrix, where $n \geq 2$. The ideal $I = I_{n-1}(M)$ in K[M], generated by the submaximal minors of M is a Gorenstein ideal of grade four. Its graded minimal free resolution is given by the Gulliksen-Negard complex (see [6]):

$$0 \to R(-2n) \to R^{n^2}(-n-1) \to R^{2(n^2-1)}(-n) \to R^{n^2}(-n+1) \to I \to 0.$$

Kustin and Miller show that this resolution can be obtained by using their original construction (see [9, Example 2.4]). Gorla [7] studies these ideals from a liaison-theoretic point of view. Here we make the linkage steps more explicit.

If n = 2, then I is a complete intersection. Assume $n \ge 3$, and let N be the generic $(n-1) \times (n-1)$ obtained from M by deleting its last row and column. Its $(n-2) \times (n-2)$ minors generate a homogeneous Gorenstein ideal $\mathfrak{a} = I_{n-2}(N)$ of grade 4. Denote by $M_{i,j}$ the $(n-1) \times (n-1)$ minor of M obtained by deleting row i and column j. The ideal

$$\mathbf{b} = (M_{1,n}, M_{2,n}, \cdots, M_{n-1,n}, M_{n,1}, \cdots, M_{n,n-1})$$

is a Gorenstein ideal of grade three (see, e.g, [9, Example 2.4]). Sylvester's identity implies that (see, the proof of Theorem 3.1 in [7]):

$$N_{1,1} \cdot I + \mathfrak{b} = M_{1,1} \cdot \mathfrak{a} + \mathfrak{b}.$$

It follows that there are the following links

$$\mathfrak{a} \sim_{(\mathfrak{b},N_{1,1})} (\mathfrak{b},N_{1,1},M_{1,1}) \sim_{(\mathfrak{b},M_{1,1})} I.$$

Hence I can be obtained from \mathfrak{a} by an ascending biliaison on \mathfrak{b} as described in Theorem 3.1. Repeating the construction, we see that I can be obtained from the complete intersection $(x_{11}, x_{12}, x_{21}, x_{22})$ by (n-2) such ascending biliaisons.

Now we consider some Gorenstein ideals with 9 generators and 16 syzygies. Such Gorenstein ideals are investigated in depth from the point of view of unprojections in [3].

Example 5.4. Let R = K[a, b, c, d, e, f, x, y, z] be a polynomial ring in 9 variables over a field K. Consider a generic 3×3 symmetric matrix A and a generic skew-symmetric matrix B:

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}.$$

Then, for $\lambda \neq 0$ in K, define a 6×6 skew-symmetric matrix $N = \begin{bmatrix} B & A \\ -A & \lambda B \end{bmatrix}$. It is called "extrasymmetric" in [3, 4] because it is obtained from a generic skew-symmetric matrix by specializing some of the variables. The ideal \mathfrak{a} generated by the 4×4 Pfaffians of N

$$\begin{split} \mathfrak{a} =& (b^2 - ad + \lambda x^2, bc - ae + \lambda xy, c^2 - af + \lambda y^2, cd - be + \lambda xz, ce - bf + \lambda yz, \\ e^2 - df + \lambda z^2, cx - by + az, ex - dy + bz, fx - ey + cz). \end{split}$$

It is the defining ideal of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ into \mathbb{P}^8 and a typical case of a Tom unprojection (see [3, 4, 15]). In particular, \mathfrak{a} is equal to the ideal generated by the 2 × 2 minors of a 3 × 3 generic matrix $A + \sqrt{-\lambda}B$. Hence, the Gulliksen and Negard complex gives its minimal free resolution:

$$0 \longrightarrow R(-6) \xrightarrow{a_4} R^9(-4) \xrightarrow{a_3} R^{16}(-3) \xrightarrow{a_2} R^9(-2) \xrightarrow{a_1} \mathfrak{a} \longrightarrow 0.$$

In order to perform the construction of Theorem 3.1, we choose the first three listed generators of \mathfrak{a} to define a complete intersection

$$\mathbf{b} = (b^2 - ad + \lambda x^2, bc - ae + \lambda xy, c^2 - af + \lambda y^2)$$

inside \mathfrak{a} . Then we link as follows:

is a homogeneous Gorenstein ideal of grade 4:

$$\mathfrak{a} \sim_{(\mathfrak{b}, cd - be + \lambda xz)} (\mathfrak{b}, cd - be + \lambda xz, ax) \sim_{(\mathfrak{b}, ax + (cd - be + \lambda xz))} I.$$

Explicitly, the resulting ideal I is

$$I = (e^{2} - df - cx + by + az + \lambda z^{2}, ce - bf + ay + \lambda yz, cd - be + ax + \lambda xz, c^{2} - af + \lambda y^{2}, bc - ae + \lambda xy, ac + \lambda fx - \lambda ey + \lambda cz, b^{2} - ad + \lambda x^{2}, ab + \lambda ex - \lambda dy + \lambda bz, a^{2} + \lambda cx - \lambda by + \lambda az).$$

It has the same Betti table as \mathfrak{a} . In fact, I is again an example of a Tom unprojection. This time the extrasymmetric matrix is

$$M = \begin{bmatrix} 0 & x & y & a & b & c \\ -x & 0 & \frac{1}{\lambda}a + z & b & d & e \\ y & -\frac{1}{\lambda}a - z & 0 & c & e & f \\ -a & -b & -c & 0 & \lambda x & \lambda y \\ -b & -d & -e & -\lambda x & 0 & a + \lambda z \\ -c & -e & -f & -\lambda y & -a - \lambda z & 0 \end{bmatrix},$$

so $I = Pf_4(M)$.

We conclude with an example of a Gorenstein ideal that cannot be produced using the construction of Theorem 3.1 with a strictly ascending biliaison.

Example 5.5. Let *I* be a generic Artinian Gorenstein ideal in $R = K[x_1, \ldots, x_5]$ with *h*-vector (1, 5, 5, 1), where *K* is an infinite field. It has the least possible Betti numbers. More precisely, its graded minimal free resolution is pure and has the form

(12)
$$0 \to R(-8) \to R^{10}(-6) \to R^{16}(-5) \to R^{16}(-3) \to R^{10}(-2) \to I \to 0.$$

We claim that there are no Gorenstein ideals \mathfrak{a} and \mathfrak{b} to produce I using a biliaison as in Theorem 3.1 that is strictly ascending, i.e., d > 0 or, equivalently, \mathfrak{a} has smaller regularity than I.

Indeed, to see this assume such ideals \mathfrak{a} and \mathfrak{b} do exist. Since reg I = 3, this forces reg $\mathfrak{a} = 1$ by Corollary 4.4. It follows that the *h*-vector of R/\mathfrak{a} must be (1,1). Hence, possible after a change of coordinates, we may assume

$$\mathfrak{a} = (x_1, x_2, x_3, x_4, x_5^2).$$

Thus, Corollary 3.3 gives that R/\mathfrak{b} has *h*-vector (1, 4, 1), that is, \mathfrak{b} is a Sally ideal (see Example 5.2). Its graded minimal free resolution has the form

$$0 \to R(-7) \to R^{6}(-5) \to R^{5}(-3) \oplus R^{5}(-4) \to R^{6}(-2) \to \mathfrak{b} \to 0.$$

By Theorem 4.1, we have the following short exact sequence of graded R-modules

$$0 \longrightarrow (\mathfrak{a}/\mathfrak{b})(-1) \longrightarrow R/\mathfrak{b} \longrightarrow R/I \longrightarrow 0.$$

Consider now the comparison map between the resolutions of $(\mathfrak{a}/\mathfrak{b})(-1)$ and R/\mathfrak{b} in homological degree two. Using the notation of the proof of Theorem 4.1, it is

$$A_3(-1) \oplus B_2(-1) \xrightarrow{[\beta_3, h_2 - f \, \mathrm{id}]} B_2 = R^5(-4) \oplus R^5(-5)$$

Since deg f = 1, the map $h_2 - f$ id is minimal. Moreover, notice that $A_3(-1) = R^4(-4) \oplus R^6(-5)$. Considering the map β_3 in degree 4, the mapping cone procedure implies that $[\operatorname{Tor}_2^R(R/I, K)]_4 \neq 0$. Hence I does not have a pure resolution as in (12), which completes the argument.

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