Structural Techniques in Descriptive Complexity

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Abstract

In 2017, Abramsky, Dawar, and Wang published a paper which gave a comonadic characterisation of pebble games, tree-width, and k-variable logic, a key trio of related concepts in Finite Model Theory. In 2018, Abramsky and Shah expanded upon this to give an analogous comonadic characterisation of Ehrenfeucht-Fraisse games, tree-depth, and bounded quantifier rank logic. A key feature of these papers is the connection between two previously distinct subfields of logic in computer science; Categorical Semantics, and Finite Model Theory. This thesis applies the ideas and techniques in these papers to give a categorical account of some cornerstone results of Finite Model Theory, including Rossman's Equirank Homomorphism Preservation Theorem, Courcelle's Theorem (on the model-checking properties of structures of bounded tree-width), and Gaifman's Locality Theorem.

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Chapter 1

Introduction

1.1 Thesis Overview

The pebbling and Ehrenfeucht-Fraisse comonads were introduced by Abramsky, Dawar, and Wang in [1] and by Abramsky and Shah in [5] respectively. In short, this thesis explores those comonads and their application to some topics in Model Theory and Finite Model Theory.

This thesis will consist of four main chapters, the first of which being a technical introduction. In the introduction, we cover the necessary model theoretic background needed to understand the aforementioned comonads and the rest of the thesis, in addition to giving a presentation of the comonads. This material is entirely recapping from both [1] and [5] in addition to some material one may find in a standard source on finite model theory such as [13] or [17]. The sole exception to this is the treatment of constants in the relational signature, and free-variables in a first-order formula, which was initially demonstrated in [26] by the present author.

In more detail, the introduction will cover notions of quantifier rank and variable count for first-order formulae, and the corresponding notions of tree depth and tree width for first-order structures. The relationship between these can be seen by the canonical query of a first-order structure, which allows us to see the relationship between the pre-ordered class of finite first-order structures of bounded tree depth (respectively tree width) and the poset of positive existential first-order formulae of bounded quantifier rank (respectively variable count). In addition, we cover relations between first-order structures that compare them based on which first-order formulae they satisfy, which are essentially graded versions of homomorphisms and elementary equivalence, and see how these relations can be expressed using the Ehrenfeucht-Fraisse and pebble games. Lastly in the introduction we cover the comonads themselves, with a family of comonads corresponding to the Ehrenfeucht-Fraisse games and pebble games respectively. There are two key features of the comonads; firstly, the co-Kleisli morphisms out of the comonads correspond to winning strategies in the games (for one of the players) and hence correspond to relations between first-order structures, and secondly, the co-algebras of the comonads correspond exactly to witnesses for bounded tree depth or bounded tree width (known as forest covers).

The second chapter follows very closely with the paper [26] (written by the present author). The paper seeks to expand on one the key results from [27] known as Rossman's Equirank Homomorphism Preservation Theorem, which says that any first-order formula preserved under homomorphisms is semantically equivalent to a positive existential formula of the same quantifier rank. This is an improvement upon the ordinary Model Theoretic version of the Homomorphism Preservation Theorem which says that a first-order formula which is preserved under homomorphisms is equivalent to a positive existential formula of potentially any quantifier rank. The aims of the paper, and what is presented in the second chapter of this thesis, is to give an account of Rossman's Theorem using the comonadic framework in the introduction, and to work towards Abramsky's conjectured generalisation of Rossman's Theorem, that any first-order formula that is preserved under homomorphisms is equivalent to a positive existential formula of the same quantifier rank and variable count. The conjecture remains unproven, though a special case of it is proved in the second chapter, and we outline a number of possible methods for proving the general case.

The third chapter is work entirely novel to this thesis. The key result of this chapter is the demonstration of a family of adjunctions between first-order structures and a definable subcategory of Kripke structures. We show that one part of the adjunction also tracks first-order formulae of bounded quantifier rank (or bounded variable count) to modal formulae, so that one can express graded versions, or indeed the ordinary version, of elementary equivalence as a form of bisimulation between Kripke structures. We go on to show that the definable subcategory of Kripke structures is actually the Eilenberg-Moore category, the category of coalgebras, for the comonads. This tells us in particular that a pre-image of a first-order structure under the second part of the adjunction is a forest-cover for that structure, and we investigate the relationship between the first-order formulae satisfied by a first-order structure and the modal formulae satisfied by a Kripke structure that is its pre-image. We go on to show that if we view the pre-image as a labelled, directed graph rather than a Kripke structure, we in fact get an exact correspondence between monadic second-order formulae satisfied by the graph and the initial first-order structure. Interestingly, this result provides another approach to proving one of the key steps in Courcelle's Theorem, that any monadic second order formula is linear time checkable on first-order structures of bounded tree-width, which we explain in more detail in the chapter.

The fourth and final chapter is also novel to this thesis, and in it we explore the notion of locality through the Ehrenfeucht-Fraisse comonad. Locality is a notion for first-order structures borrowed from Graph Theory, which bounds the range of quantifiers in a first-order formulae by distance, where distance is defined for a first-order structure via its Gaifman graph. We define a localised subfunctor of the Ehrenfeucht-Fraisse comonad and see how it tells a similar story to the Ehrenfeucht-Fraisse comonad but for localised formulae and a localised version of the Ehrenfeucht-Fraisse game. We also define the notion of "Reachability" and show it characterises precisely the tightest locality bound for which the local and ordinary Ehrenfeucht-Fraisse games coincide. Interestingly, we see the corresponding "Reachability" subfunctor of the Ehrenfeucht-Fraisse comonad is actually a comonad, unlike the general local versions which are simply functors, and hence have identified a non-trivial subfunctor of the Ehrenfeucht-Fraisse comonad which is also a comonad. We go on to consider how the localised subfunctors can also give an account of local forms of elementary equivalence using the construction from the third chapter. This allows us to express a weakened version of Gaifman's Theorem in our framework, which is a key Theorem when considering locality in Model Theory, and essentially says that for any first-order formula there is an equivalent local formula. This expression leads us to conjecture a version of Gaifman's Theorem where we use Reachability rather than ordinary locality.

1.2 Logic and Structures

Throughout this thesis we will be interested primarily in first-order logic (FO), fragments of FO, and their interplay with relational structures. We shall start by showing their duality using canonical queries and term structures, and see how this relationship tracks when restricted to fragments of FO and subcategories of relational structures respectively. We shall then see how particular comonads effectively capture all of the discussion about structures of bounded tree width and tree depth.

We shall fix a finite set σ of relations, which shall be referred to as our vocabulary or signature. The set σ will contain only relations, each with some arity, which is a positive integer saying how many inputs the relation takes. We shall largely be considering FO formulas over σ, constructed using atomic relations, negations, binary connectives $\{\wedge, \vee\}$, and quantifiers $\{\exists x, \forall x\}$ for any variable x. A sentence will be a formula with no variables occurring free in it, and we shall use $\mathcal{L}$ to denote the set of sentences over σ. We will use $\mathcal{L}(l)$ to denote the set of formulas over σ whose free variables occur among the set $x_1, ..., x_l$ (so $\mathcal{L} = \mathcal{L}(0)$). We shall say a formula is positive existential if it is built only using connectives $\{\exists x, \vee, \wedge\}$ and primitive positive if it is built using only $\{\exists x, \wedge\}$, and we shall use superscripts $+$ and *prim* respectively to denote these fragments. Since $\vee$ and $\wedge$ distribute over one another, any positive existential formula can be written as a disjunction of primitive positive ones.

A first-order structure over σ, which we will sometimes call a σ- structure, is a set (sometimes referred to as the universe) A, equipped with relations $R^A \subset A^l$ for each $R \in \sigma$ where l is the arity of R. We often write $A \models R(a_1, ..., a_l)$ instead of $(a_1, ..., a_l) \in R^A$. We will refer to first-order structures using letters A, B, C and use the same symbol to refer to both a structure and its universe. Given two structures A, B a homomorphism (or morphism) $f : A \to B$ is on ordinary set-map with the extra condition that if $A \models R(a_1, ..., a_l)$ then $B \models R(f(a_1), ..., f(a_l))$ for every $R \in \sigma$ and tuple of A. For each l, we define a category $\mathcal{R}_\sigma(l)$, with objects $(A, \bar{a})$, where A is a first-order structure and $\bar{a}$ is an l-tuple of A. Morphisms $f : (A, \bar{a}) \to (B, \bar{b})$ will be first-order morphisms f that also satisfy $f(a_i) = b_i$ for each $i = 1, ..., l$. We will just write $\mathcal{R}_\sigma$ for $\mathcal{R}_\sigma(0)$. We will write $(A, \bar{a}) \models \phi(\bar{x})$ for for the usual satisfaction relation between a first-order structure $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$ and a first-order formula $\phi \in \mathcal{L}(l)$, where the a_i interpret the free variables x_i. We will often leave the free variables in a first-order formula implicit. Given a structure $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$, the substructure induced by some subset $A' \subset A$, is the structure with universe $A' \cup \{a_1, ..., a_l\}$, with relations restricted from A (ie for $\bar{b}$ a tuple of A' of length equal to the arity of some $R \in \sigma$, $A' \models R(\bar{b}) \iff A \models R(\bar{b})$). Note that the inclusion map from an induced substructure to a structure is always a morphism.

Throughout this thesis we will define various fragments of first order logic in a syntactic fashion though we are almost always interested in the semantic content of a formula. Two formulas $\phi, \psi \in \mathcal{L}(l)$ are semantically equivalent (or sometimes said FO equivalent), written $\psi \equiv \phi$, if for every $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$, $(A, \bar{a}) \models \phi \iff (A, \bar{a}) \models \psi$. Often times we may leave semantic equivalence implicit (eg we may say a formula is positive existential when we mean it is semantically equivalent to a positive existential formula).

Both first-order structures and first-order sentences can given a pre-order (many of which are in fact lattices, though we shall not exhibit the details). The canonical query and term structure constructions show us the relationships between these.

We can turn $\mathcal{L}(l)$ into a pre-order as follows (writing $\to$ for the order relation): $\phi \to \psi$ if and only if for every $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$, if $(A, \bar{a}) \models \phi$ then $(A, \bar{a}) \models \psi$, in other words if $\phi\psi$ is a tautology. This is clearly transitive, and further turns $\mathcal{L}(l)$ into a poset since we are interested in formulae only up to semantic equivalence. We write $(A, \bar{a}) \to (B, \bar{b})$ for $(A, \bar{a}), (B, \bar{b}) \in \mathcal{R}_\sigma(l)$ as a shorthand for $\exists f : (A, \bar{a}) \to (B, \bar{b}) \in \mathcal{R}_\sigma(l)$. This is once again a transitive relation, however we do not get a poset this time is $(A, \bar{a}) \leftrightarrows (B, \bar{b})$ does not imply $(A, \bar{a}) \equiv (B, \bar{b})$, let alone equality. In addition, $\mathcal{R}_\sigma(l)$ is a proper class rather than just a set.

The canonical query transports structures to primitive positive first-order sentences as follows:

Definition 1.2.1. *Given $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$, a canonical query for $(A, \bar{a})$ is any $\phi \in \mathcal{L}$ satisfying the following two properties:*

1. $(A, \bar{a}) \models \phi$

2. *For any* $(B, \bar{b}) \in \mathcal{R}_\sigma(l)$, $(B, \bar{b}) \models \phi$ *if and only if* $(A, \bar{a}) \to (B, \bar{b})$

Though we state them separately, condition 1 is clearly implied by condition 2 since there is always the identity map from a structure to itself. The following lemma asserts that every finite structure (ie first-order structure with a finite universe) has a primitive positive first-order canonical query, and taken with the above definition it follows that we can map finite structures onto a sub-poset of first-order formulas.

Lemma 1.2.2. *For any finite* $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$*, there exists* $\phi \in \mathcal{L}^{prim}(l)$ *that is a canonical query for* $(A, \bar{a})$

Proof. For some ordering of the universe of $A := \{a_1, ..., a_l, a_{l+1}, ..., a_m\}$ (where $a_1, ..., a_l$ make up the tuple $\bar{a}$), define $\phi_{(A,\bar{a})}$, to be:

$$\phi_{(A,\bar{a})}(\bar{x}) := \exists x_{l+1}, ..., \exists x_m \bigwedge \{R(x_{i_1}, ..., x_{i_j}) : R \in \sigma,\ (a_{i_1}, ..., a_{i_j}) \in R^A\}$$

Now $\phi_{(A,\bar{a})}$ asserts there are some m (not necessarily distinct elements) of a structure who satisfy at least all of the same atomic relations as the corresponding elements of A. Hence a morphism from $f : (A, \bar{a}) \to (B, \bar{b})$ is precisely the same as finding set of witnesses for $\phi_{(A,\bar{a})}$ in B (via $f(a_i)$:=witness for x_i in B). □

Example 1.2.3. *Suppose* $A = \{a\}$ *is a one element structure that satisfies only a single unary predicate* $P \in \sigma$*. Then* $\phi_A = \exists x_1 P(x_1)$*, and* $\phi_{(A,a)} = P(x_1)$*.*

The construction in the reverse direction is known as a term structure:

Definition 1.2.4. *Given* $\phi \in \mathcal{L}(l)$*, a term structure for* ϕ *is a structure* $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$ *such that* ϕ *is a canonical query for* $(A, \bar{a})$*.*

Any primitive positive formula has a term structure:

Lemma 1.2.5. *Let* $\phi \in \mathcal{L}^{prim}(l)$*. Then* ϕ *has a term structure.*

Proof. Suppose for convenience that variables $x_1, ..., x_l$ do not occur bound in ϕ, and no variable in ϕ is bound more than once. Define C_ϕ to be the structure with universe $\{c_i : x_i \text{ occurs in } \phi\} \cup \{c_1, ..., c_l\}$ to ensure we have a witness for each of the free variables, even if they satisfy no relations, distinguished elements $(c_1, ..., c_l)$, and satisfy atomic relations $R(c_{i_1}, ..., c_{i_m})$ if and only if $R(x_{i_1}, ..., x_{i_m})$ occurs in ϕ. If we use the canonical query construction defined above on C_ϕ we will get back ϕ as required. □

Between the term structure and canonical query construction, we have essentially identified the pre-ordered class of finite first-order structures (under $\to$), with the poset of primitive positive formulae under implication, modulo the following lemma:

Lemma 1.2.6. *If* A *is a term structure for* $\phi \in \mathcal{L}^{prim}(l)$*, then for any* $\psi \in \mathcal{L}^{prim}(l)$*,* $(A, \bar{a}) \models \psi$ *if and only if* $\phi \to \psi$ *is a tautology.*

Proof. Now let $(C, \bar{c}) \in \mathcal{R}_\sigma(l)$ be some arbitrary structure such that $(C, \bar{c}) \models \phi$, we must show $(C, \bar{c}) \models \psi$. By above, there exists some $(B, \bar{b})$ a term structure for ψ, which must satisfy $(B, \bar{b}) \to (A, \bar{a})$ since $(A, \bar{a}) \models \psi$. Now since $(C, \bar{c}) \models \phi$ we have $(A, \bar{a}) \to (C, \bar{c})$, and hence $(B, \bar{b}) \to (C, \bar{c})$, so we can conclude $(C, \bar{c}) \models \psi$. The reverse direction is immediate.

□

1.3 Grading by Variable Count and Quantifier Rank

We will be interested in graded fragments of first-order logic, as they can be more tractable than full first-order logic (see [9] and [18]), and in addition, full first-order logic can be too powerful when considering finite structures (for instance, two finite structures that are elementarily equivalent are actually isomorphic). To that end, we define **quantifier rank** and **variable count**.

Definition 1.3.1. *For a formula $\phi \in \mathcal{L}(l)$, its* **quantifier rank** *is the maximum nesting depth of quantifiers occurring in it:*

- *If ϕ is atomic, it has* **quantifier rank** *0*
- *If ϕ is of form $\chi \wedge \psi$ or $\chi \vee \psi$, it has* **quantifier rank** *equal to the maximum of the* **quantifier rank** *of χ and ψ*
- *If ϕ is of form $\neg\chi$, it has* **quantifier rank** *equal to that of χ*
- *If ϕ is of form $\exists x \chi$ or $\forall x \chi$, then it has* **quantifier rank** *equal to that of χ plus one.*

The **variable count** *of a formula is the total number of distinct variable symbols used in it.*

For natural numbers n, k, let $\mathcal{L}_{n,k}(l)$ be the fragment of $\mathcal{L}(l)$ of consisting of formulas with **quantifier rank** at most n, and **variable count** at most k. Note that the free variables do come out of the total number of allowed variables, and that rebinding variables, including ones that were used freely, with different quantifiers is necessary for full expressivity when limiting the **variable count**. For instance, the existence of a walk of length n in a directed graph can be expressed by a sentence in $\mathcal{L}_{n,2}$, only if one reuses the same variable multiple times. We define $\mathcal{L}_n(l) := \mathcal{L}_{n,n+l}(l)$. The following lemma justifies this definition, by asserting that a formula of **quantifier rank** n can fruitfully use at most $n + l$ variables, one for each quantifier rank plus the number that occur free, so $\mathcal{L}_n(l)$ semantically captures all formulas of **quantifier rank** n, with any number of variables.

Example 1.3.2. *A formula proposing the existence of a walk of length 3 in the language of graphs (ie where σ contains only a single binary relation R) may be written*

$$\exists x_1 \exists x_2 \exists x_3 (R(x_1, x_2) \wedge R(x_2, x_3))$$

which has **quantifier rank** *3 and* **variable count** *3. It could also equivalently written*

$$\exists x_1 ((\exists x_2 R(x_1, x_2)) \wedge (\exists x_2 R(x_2, x_1)))$$

which has **quantifier rank** *2 and* **variable count** *2. In this case, we say both sentences are members of the set $\mathcal{L}_{2,2}$.*

Lemma 1.3.3. *Suppose $\phi \in \mathcal{L}(l)$ has* **quantifier rank** *n. Then there exists $\psi \in \mathcal{L}_n(l)$ semantically equivalent to it.*

Proof. We take for granted some basic properties of semantic equivalence (namely it is preserved under connectives) in order to do an induction on the number of connectives in ϕ. If ϕ is atomic, all of its variables occur free, thus it has at most l of them. If $\phi = \psi \wedge \chi$ or $\psi \vee \chi$, or $\neg\psi$ we simply apply our induction hypothesis to ψ and χ and we are done. If $\phi = \exists x \chi$, we can assume χ is uses at most $n + l - 1$ variables since it has **quantifier rank** at most $n + l - 1$. Hence ϕ uses at most $n + l$ variables. □

We can also grade the existential positive and primitive positive fragments by **quantifier rank** and **variable count**. The dual notions of **quantifier rank** and **variable count** for first-order structures are known as tree depth and tree width respectively, which we will define via (n, k)-covers, which require a somewhat lengthy definition. Similarly to **quantifier rank** and **variable count**, tree depth and tree width are of interest in Finite Model Theory as classes of structures of bounded tree-depth or bounded tree-depth are more tractable than the class of all structures. For instance, see [24] on bounded tree depth, and the discussion in the third chapter on Courcelle's Theorem for bounded tree width. Intuitively, an (n, k)-cover organizes a structure $(A, \bar{a})$ into a tree with $\bar{a}$ in a path at the top, with the extra condition that any elements that occur in some tuple of a relation together are all "visible" to one another (where "visibility" is determined by the structure of the tree).

We will say a **rooted forest** $(F, \bar{r})$ is a pair $(F, \bar{r})$, where F is a graph that forms a forest (ie each connected component of F is a tree), and $\bar{r}$ is a tuple containing exactly one vertex from each component, which we shall refer to as the roots. A **branch** of a rooted forest is a path from a vertex to its root and the **depth** of the forest will be the maximum number of vertices occurring in a single branch.

Definition 1.3.4.

- *The* **Gaifman graph** *of $A \in \mathcal{R}_\sigma$, $G(A)$, is an undirected simple graph with vertex set A, and an edge between a_1, a_2 if they occur together in some guarded tuple of A (that is to say, there is a relation $R \in \sigma$ and tuple $\bar{a} \in R^A$ containing a_1, a_2).*

- *A* **forest cover** *of a graph G is a rooted forest $(F, \bar{r})$ on the same vertex set, such that whenever there is an edge a_1, a_2 in G, they occur together in some branch of F.*

- *A* $\mathbf{k}-$**labelled forest cover** *of a graph G is a forest cover $(F, \bar{r})$ along with a labelling of the vertices $c : F \to \{1, ..., k\}$ such that whenever there is an edge a_1, a_2 in G, and a_1 is an ancestor of a_2 (that is to say, a_1 lies on the unique path from a_2 to its root), then for any a_3 on the unique path from a_1 to a_2, we have $c(a_3) = c(a_1) \implies a_3 = a_1$, or in other words, the label of a_1 is not used again on the path to a_2, including at a_2, whenever there is an edge a_1, a_2 in G.*

Definition 1.3.5.

- *An* $(\mathbf{n}, \mathbf{k})-$**cover** *of A is a k-labelled rooted forest cover of $G(A)$ of depth at most n.*

- *An* $(\mathbf{n}, \mathbf{k})-$**cover** *of $(A, \bar{a})$, $(F, \bar{r})$, is a k-labelled rooted forest cover of $G(A)$ of depth at most $n + l$ with the following two properties:*
  - *F is a tree with unique root a_1*
  - *The unique child of a_i in F is a_{i+1} for $i = 1, ..., l - 1$.*

  Intuitively, F is (n, k)-cover of $G(A) - \{a_1, ..., a_l\}$ that becomes a forest cover of $G(A)$ when $a_1, ..., a_l$ are connected to it as a path.

- *The* **tree depth** *of $(A, \bar{a})$ is the least n such that there exists an (n, k)-cover of $(A, \bar{a})$ for some k (if one exists); or, equivalently, that there exists an (n, n)-cover, since one only needs at most n-labels to label a forest of depth n.*

- *The* **tree width** *of $(A, \bar{a})$ is the minimal k such that there exists a k-labelled forest cover, minus one (subtracting one is a convention so that a tree always has tree width 1).*

Example 1.3.6. *A simple example to understand forest covers is the case of graphs once again. If G is a tree, then (G, r) is a rooted forest cover for any choice of $r \in G$. Labelling the vertices in an alternating fashion (ie the label of r is 1, its children have label 2, their children have label 1 etc) will then provide a 2-labelled forest cover for G, witnessing the fact that any tree does in fact have treewidth 1.*

Tree depth and tree width can be defined in other equivalent ways, using other similar style graph decompositions, recursive definitions, or a coalgebraic definition we give below. Above we said that tree depth and tree width were the analogs of quantifier rank and variable count respectively. This is made explicit with the following lemma:

Lemma 1.3.7. *If $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$ is finite and has an (n, k)-cover, then $(A, \bar{a})$ has a canonical query in ${\mathcal{L}_{n,k}}^{prim}(l)$. Conversely, if $\phi \in {\mathcal{L}_{n,k}}^{prim}(l)$, then ϕ has a term structure with an (n, k)-cover.*

Before we prove this lemma, we briefly discuss the direct scope relation on quantifiers:

Definition 1.3.8. *We will say a variable quantifier q is in the **direct scope** of a quantifier q' in a formula if and only if whenever some other quantifier q'' has q in its scope, it also has q' in its scope. We will say an atomic relation R is in the direct scope of a quantifier q under a similar condition, that if R is in the scope of any other quantifier q' then q is also in the scope q'.*

The point of this definition is the observation that the quantifiers of any formula form a forest under the direct scope relation. We also remark that for a primitive positive formula, it is characterised up to semantic equivalence by this forest, along with a list of the atomic relations in it, and which quantifier each relation is in the direct scope of. This relies on the fact that the $\wedge$ connective is both symmetric and associative. An intuitive way to picture this is an alternative syntax for primitive positive formulae, where rather than written inline, formulae are written on a forest, with each node corresponding to a quantifier and possibly some atomic relations. Now we can proceed with the proof of the lemma:

Proof. (of Lemma 1.3.7). Recall from Lemma 1.2.2, for a given structure $(A, \bar{a})$ with universe $\{a_1, ..., a_m\}$,

$$\phi_{(A,\bar{a})}(\bar{x}) := \exists x_{l+1}, ..., \exists x_m \bigwedge \{R(x_{i_1}, ..., x_{i_j}) : R \in \sigma,\ (a_{i_1}, ..., a_{i_j}) \in R^A\}$$

is a canonical query for it, where the conjunction ranges over all relations between elements of $(A, \bar{a})$. In the first instance, we consider a structure A with no distinguished elements, and then move onto the general case. So suppose A has an (n, k)-cover F, we aim to construct $\psi_A \in \mathcal{L}_{n,k}{}^{\text{prim}}$ that is a canonical query for A. First, we will specify a sentence ψ' by the method described above, giving its forest of quantifiers and the direct scope relation between its atomic relations and quantifiers. Let ψ' have one existential quantifier $\exists x_i$ for each element $a_i \in A$. We specify the direct scope relation between these quantifiers using F, with the rule that x_i is in the direct scope of x_j if and only if a_i is the child of a_j in the forest F. Were we to think of ψ' to be written as a forest, it would simply be the same forest as F. To finish specifying ψ' we need to specify what atomic relations it has and which quantifier it is in the direct scope of. ψ' shall have the same set of atomic relations as ϕ_A, in other words an atomic relation $R(x_{i_1}, ..., x_{i_j})$ whenever $R(a_{i_1}, ..., a_{i_j})$ is in A. Each atomic relation shall be in the direct scope of $\exists x'$ where a' is the element among $a_{i_1}, ..., a_{i_j}$ that occurs furthest from its root in F (we will verify that there is a unique such choice below). We claim that ψ' is actually a sentence, a canonical query for A, and has quantifier rank at most n. The latter claim is immediate, since the maximum nesting depth of quantifiers is the length of the longest chain of quantifiers with each one in the direct scope of the previous, and this is equal to the depth of F (which is n) since the direct scope relation in ψ' is exactly parent child relation in F. It is also clear why ψ' should be a canonical query for A, since

as in Lemma 1.2.2, it contains an existential quantifier for each element of A and an atomic relation for each relation among tuples of A. Finally we need to check ψ' is a sentence, which amounts to checking whenever we have an atomic relation $R(x_{i_1}, ..., x_{i_j})$, it is always in the scope of each of $\exists x_{i_1}, ..., \exists x_{i_j}$. Recall that if $R(a_{i_1}, ..., a_{i_j})$ holds, then by the definition of F, $a_{i_1}, ..., a_{i_j}$ all lie on the same branch of F. Hence, there is a unique a' among them furthest from their common root, and all of them lie on the path from a' to its root. Translating this to the quantifiers, this shows that $\exists x'$ is in the scope of, or equal to one of $\exists x_{i_1}, ..., \exists x_{i_j}$ as required.
Next we use the labelling of F in order to construct ψ_A using at most k variables. Let ψ_A be the formula obtained by replacing each variable of x_i of ψ' by $x_{c(a_i)}$ where $c(a_i)$ is the label of a_i in F. Clearly, this is uses at most k variables (since F uses at most k labels), so what remains to show is that this is equivalent to ψ'. To check this, we need to check that the relabelling of variables, where we use the same variables names multiple times, has not affected the scope relation between quantifiers and atomic relations. So given some relation $R(x_{i_1}, ..., x_{i_j})$ occurring in ψ' we need to check $R(x_{c(i_1)}, ..., x_{c(i_j)})$ is in the scope of the appropriate quantifiers in ψ_A. This might not be the case if one of the variables was reused between one of the original quantifiers and the atomic relation. This amounts to checking that for each a among $a_{i_1}, ..., a_{i_j}$, $c(a)$ is not reused on the path from a to a', where a' is the element furthest from its root among $a_{i_1}, ..., a_{i_j}$. This then follows from the fact that F is an (n, k)-cover, and there is an edge a, a' in $G(A)$ for each choice of a, as witnessed by the relation $R(a_{i_1}, ..., a_{i_j})$. This concludes the case where A is a structure with no distinguished elements. In the case of a structure of form $(A, \bar{a})$ with l distinguished elements, we follow the same construction as above to get a sentence ψ_A. By the definition of an (n, k)-cover of $(A, \bar{a})$, ψ_A will be a sentence using at most l-variables, quantifier rank at most $l + n$, and will lead with a chain of quantifiers $\exists x_{c(a_1)}, ..., \exists x_{c(a_l)}$ each one being the unique quantifier in the direct scope of the previous, since F necessarily begins with the path $a_1, ..., a_l$ with root a_1. By the same argument as above, ψ_A is equivalent to $\exists x_1, ..., \exists x_m \bigwedge\{R(x_{i_1}, ..., x_{i_j}) : R \in \sigma,\ (a_{i_1}, ..., a_{i_j}) \in R^A\}$. However, this is just $\phi_{(A,\bar{a})}$ with extra quantifiers $\exists x_1, ..., \exists x_l$ at the front. Hence, by removing these l quantifiers from the front of each sentence, we see the formula $\phi_{(A,\bar{a})}$ is equivalent to a formula with quantifier rank at most n, and variable count at most l.
Now given a formula $\phi \in \mathcal{L}_{n,k}{}^{\text{prim}}(l)$ recall the construction C_ϕ from Lemma 1.2.5, that gives us a term structure for ϕ, which has an element c_q for each free variable and existential quantifier q in ϕ. Note that in Lemma 1.2.5 we used different variable names for each quantifier for convenience, but we cannot assume that here; so we emphasise C_ϕ has one element for each quantifier occurring in ϕ, not for each variable. We extend the notion of direct scope to the free variables, which for convenience we shall assume are $x_1, ..., x_l$, by setting x_{i+1} to be uniquely in the direct scope of x_i for $i = 1, ..., l-1$, and have the leading quantifiers in ϕ be in the direct scope of x_l. In other words, this is the direct scope relation we would obtain from $\exists x_1, ..., \exists x_l \phi$. Now we claim the forest F

given by the direct scope relation is a forest cover for C_ϕ, and the function f that sends an element c_q to the index of the variable q binds in ϕ, and a free variable x_i to i, is a labelling function. For the former claim, suppose there is an $c_q, c_{q'}$ in $G(C_\phi)$. Then by the definition of C_ϕ there is some atomic relation containing the variables bound by q and q', that is in the scope of both q and q'. Hence q must be in the scope of q' or vice versa, and in either case, they are on the same branch of F as required. Now for the labelling function, we suppose without loss of generality that q is on the unique path from q' to its root, in addition to there being an edge q, q' in $G(C_\phi)$. As we observed before, q' must be in the scope of q, which in particular means the variable bound by q cannot have been reused between q and q', and hence $f(c_q)$ is not reused in the path from q to q', as required. □

Elementary equivalence is an extremely important and well-studied relation in model theory. Two structures $(A, \bar{a}), (B, \bar{b}) \in \mathcal{R}_\sigma(l)$ are elementarily equivalent (written $(A, \bar{a}) \equiv (B, \bar{b})$) if for every $\phi \in \mathcal{L}(l)$, $(A, \bar{a}) \models \phi \iff (B, \bar{b}) \models \phi$. We are interested in various graded fragments of FO, and thus the following relations play a central role much like elementary equivalence does in classical model theory:

Definition 1.3.9. *Let $(A, \bar{a}), (B, \bar{b}) \in \mathcal{R}_\sigma(l)$*

- *We write $(A, \bar{a}) \rightarrow_{n,k} (B, \bar{b})$ if, for every $\phi \in \mathcal{L}^+_{n,k}(l)$, if $(A, \bar{a}) \models \phi$ then $(B, \bar{b}) \models \phi$. We write $(A, \bar{a}) \leftrightarrows_{n,k} (B, \bar{b})$ as a shorthand for $(A, \bar{a}) \rightarrow_{n,k} (B, \bar{b})$ and $(B, \bar{b}) \rightarrow_{n,k} (A, \bar{a})$.*
- *We write $(A, \bar{a}) \equiv_{n,k} (B, \bar{b})$ if, for every $\phi \in \mathcal{L}_{n,k}(l)$, if $(A, \bar{a}) \models \phi$ then $(B, \bar{b}) \models \phi$.*
- *If, in either of the preceding two definitions, $k \geq n + l$ we omit the k and obtain a definition of $\rightarrow_n$ and $\equiv_n$ purely about* **quantifier rank**. *These are special cases of $\rightarrow_{n,k}$ and $\equiv_{n,k}$ which we will occasionally treat separately and occasionally not.*

Clearly, $\equiv_{n,k}$ approximates the elementary equivalence relation, in the sense that if $(A, \bar{a}) \equiv_{n,k} (B, \bar{b})$ for every n, k, then $(A, \bar{a}) \equiv (B, \bar{b})$. The notation $\rightarrow_{n,k}$ is justified as it approximates $\rightarrow$ in the same sense on all structures with positive existential canonical queries (this includes for example, all finite structures which we show below). It is also the case that $\rightarrow$ implies $\rightarrow_{n,k}$ for every pair n, k, since all positive existential formulae are preserved under morphisms. Hence as one would expect, $\rightarrow_{n,k}$ and $\rightarrow_n$ are transitive and reflexive but not symmetric, whereas $\equiv_{n,k}$ and $\equiv_n$ are equivalence relations. The symmetry comes from being able to negate sentences.

Lemma 1.3.10. *If $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$ has a positive existential canonical query the following holds: for any $(B, \bar{b}) \in \mathcal{R}_\sigma(l)$, if $(A, \bar{a}) \rightarrow_{n,k} (B, \bar{b})$ for every n, k then $(A, \bar{a}) \rightarrow (B, \bar{b})$.*

Proof. Let $(A,\bar{a}),(B,\bar{b})$ be as above. By assumption, $(A,\bar{a})$ has some canonical query $\phi \in \mathcal{L}^{+}_{n,k}(l)$ for some n,k. Hence $(B,\bar{b}) \models \phi$, so $(A,\bar{a}) \to (B,\bar{b})$ as required. □

Remark 1.3.11. *The reader may have noticed that we have not defined a relation between structures using the primitive positive fragment. As stated above, any positive existential formula can be expressed as a disjunction of primitive positive ones; hence any such relation would be equivalent to the analogous one defined using existential positive formulae.*

One can give an equivalent characterisation of $\to_{n,k}$ that refers only to structures (which we later refer as the combinatorial characterisation):

Lemma 1.3.12. *For $(A,\bar{a}),(B,\bar{b}) \in \mathcal{R}_\sigma(l)$, $(A,\bar{a}) \to_{n,k} (B,\bar{b})$ if and only if, for every $(C,\bar{c})$ with an (n,k)-cover, if $(C,\bar{c}) \to (A,\bar{a})$ then $(C,\bar{c}) \to (B,\bar{b})$.*

Proof. As remarked it suffices to test only formulas $\phi \in {\mathcal{L}_{n,k}}^{\mathrm{prim}}(l)$. For $\phi \in {\mathcal{L}_{n,k}}^{\mathrm{prim}}(l)$ such that $(A,\bar{a}) \models \phi$, we know it has a term structure $(C,\bar{c})$ that has a (n,k)-cover. We also know $(C,\bar{c}) \to (A,\bar{a})$ since $(A,\bar{a}) \models \phi$. But then $(C,\bar{c}) \to (B,\bar{b})$ by assumption, and hence $(B,\bar{b}) \models \phi$ as desired. For the converse, suppose $(C,\bar{c})$ is some structure with an (n,k)-cover, and $(C,\bar{c}) \to (A,\bar{a})$. We know $(C,\bar{c})$ has a canonical query $\phi \in {\mathcal{L}_{n,k}}^{\mathrm{prim}}(l)$, and that $(A,\bar{a}) \models \phi$ since $(C,\bar{c}) \to (A,\bar{a})$. But then, by assumption we must have $(B,\bar{b}) \models \phi$, and hence $(C,\bar{c}) \to (B,\bar{b})$. □

1.4 Games and Recursive Relations

The relations above can also be defined using a two player game, between players Spoiler and Duplicator. This approach is well studied and well used, dating back to [14] and [15] (see [30] for an overview). To describe these games however, we need first define pseudo-partial homomorphisms and isomorphisms:

Definition 1.4.1. *Let $(A,\bar{a}),(B,\bar{b}) \in \mathcal{R}_\sigma(l)$. We write $(A,\bar{a}) \mapsto (B,\bar{b})$ to denote the assignment where $a_i \mapsto b_i$ for each $i = 1,...,l$. Recall there is an induced substructure on the set $\{\bar{a}\} := \{a_1,...,a_l\}$, by restricting all relations of σ to this set. We say that $(A,\bar{a}) \mapsto (B,\bar{b})$ is a pseudo-partial homomorphism (respectively isomorphism) if the assignment $(A,\bar{a}) \to (B,\bar{b})$ satisfies the homomorphism (respectively isomorphism) condition when restricted to a assignment between induced substructure $\{\bar{a}\} \to \{\bar{b}\}$.*
If the assignment is actually a partial function, then we refer $(A,\bar{a}) \mapsto (B,\bar{b})$ as a partial homomorphism (respectively isomorphism).

The homomorphism condition referred to above is the preservation of relations among the tuple $\bar{a}$ to $\bar{b}$, ie whenever $R(a_{i_1},...,a_{i_j})$ holds so does $R(b_{i_1},...,b_{i_j})$ for any R and tuple $i_1,...,i_j$, which is defining property of a homomorphism. The subtlety in the above definition is that the assignment $a_i \mapsto b_i$ is not necessarily a function; it might be the case that $a_1 = a_2$ but $b_1 \neq b_2$ for example, which would give us a pseudo-partial morphism, rather than an ordinary partial morphism. For our purposes, in the homomorphism case, this will make no difference to the end result for what we define, only serving to make book-keeping easier. However in the isomorphism case, it corresponds to not allowing the equality relation to be used in the language. The reason for this is that equality can be systemically removed from existential positive sentences while maintaining a semantically equivalent sentence, which will correspond to the homomorphism case, but cannot be removed from general sentences, which correspond to the isomorphism case.

We are now ready to define the n-round Ehrenfeucht-Fraisse and (n, k)-pebble games. Each game has a corresponding homomorphism (or "forth") and isomorphism (or "back and forth") game. Since the forth game will be most often used throughout this thesis, after the following definitions we will occasionally leave the word "forth" implicit, only specifying when the back and forth game is being referred to. We shall define the games and then give some remarks on their definitions.

Definition 1.4.2.
- *The n-round forth Ehrenfeucht-Fraisse game, on structures $(A, \bar{a})$ and $(B, \bar{b})$, which we shall write $EF_n((A, \bar{a}), (B, \bar{b}))$, is defined as follows: In round i Spoiler chooses an element α_i of A, and Duplicator responds by choosing an element β_i of B. After n-rounds, Duplicator is the winner if $(A, \bar{a}, \bar{\alpha}) \mapsto (B, \bar{b}, \bar{\beta})$ is a pseudo-partial homomorphism, else Spoiler is the winner.*
- *In the corresponding back and forth game, written $EF_n^{\equiv}((A, \bar{a}), (B, \bar{b}))$, Spoiler can choose to pick an element of either structure each round, and Duplicator must respond by choosing an element from the other structure. This once again creates tuples $\bar{\alpha}$ and $\bar{\beta}$ of A and B respectively. After n-rounds, Duplicator is the winner if $(A, \bar{a}, \bar{\alpha}) \mapsto (B, \bar{b}, \bar{\beta})$ is a partial isomorphism, else Spoiler is the winner.*

Definition 1.4.3.
- *If $k \geq l$, we can define the (n, k)-pebble game, on structures $(A, \bar{a})$ and $(B, \bar{b})$, which we shall write $P_{n,k}((A, \bar{a}), (B, \bar{b}))$, as follows: To start with, Spoiler and Duplicator are given k pebbles each, which shall be placed on elements of A and B throughout the game. Spoiler and Duplicator set up by placing pebbles $1, ..., l$ on $a_1, ..., a_l$ and $b_1, ..., b_l$ respectively. In round i Spoiler chooses a pebble p, and an element α_i of A and Duplicator responds by choosing an element β_i of B. Spoiler's pebble p is placed on α_i (removing it from another element if it was already placed on some element of A), and Duplicator's pebble p is placed on β_i. Let A_i be the structure $(A, \bar{\alpha})$ where α_j is the element pebble j is placed on in A, and define B_i similarly. After the ith round, Spoiler has won if $A_i \mapsto B_i$ is not a pseudo-partial homomorphism, else play continues. Duplicator wins if Spoiler has not won after the nth round.*
- *In the corresponding back and forth game, written $P_{n,k}^{\equiv}((A, \bar{a}), (B, \bar{b}))$, Spoiler can choose to move a pebble on either structure each round, and*

Duplicator must respond by choosing an element from the other structure to move that same pebble onto. This once again creates structures A_i and B_i after each round. After the ith round, Spoiler is the winner if $A_i \mapsto B_i$ is not a partial isomorphism, else play continues. Duplicator is the winner if Spoiler has not won after n rounds.

The key point of these definitions is the following Theorem. In the literature one often sees this Theorem stated in the contrapositive instead, as that is most natural for proving inexpressability results (see e.g. [28]).

Theorem 1.4.4. *Between two structures $(A, \bar{a}), (B, \bar{b})$:*

- *Duplicator has a winning strategy in $EF_n((A, \bar{a}), (B, \bar{b}))$ if and only if $(A, \bar{a}) \rightarrow_n (B, \bar{b})$.*
- *Duplicator has a winning strategy in $EF_n^{\equiv}((A, \bar{a}), (B, \bar{b}))$ if and only if $(A, \bar{a}) \equiv_n (B, \bar{b})$.*
- *Duplicator has a winning strategy in $P_{n,k}((A, \bar{a}), (B, \bar{b}))$ if and only if $(A, \bar{a}) \rightarrow_{n,k} (B, \bar{b})$.*
- *Duplicator has a winning strategy in $P_{n,k}^{\equiv}((A, \bar{a}), (B, \bar{b}))$ if and only if $(A, \bar{a}) \equiv_{n,k} (B, \bar{b})$.*

There are a number of other important remarks to be made about these definitions:

- We used the phrase “Duplicator has a winning strategy” above to mean that no matter the sequence of moves Spoiler uses, Duplicator can always find a winning sequence of moves in reply. Like the outcome of an individual game, in which exactly one of Spoiler or Duplicator wins, it must be the case that exactly one of Duplicator or Spoiler has a winning strategy for pair of structures $(A, \bar{a}), (B, \bar{b})$ and choice of game.
- For the purposes of all of the Ehrenfeucht-Fraisse style games in this thesis, it suffices to consider only deterministic strategies for both Spoiler and Duplicator, so we will implicitly assume all strategies are deterministic going forward.
- There is here a subtle distinction that arises here between considering free variables and constants. Consider some $\phi(\bar{x}) \in \mathcal{L}_n(l)$. For most purposes, we could consider the formula ϕ to be a sentence over $\sigma \cup \{c_1, ..., c_l\}$, where the c_i are constant symbols, since $(A, \bar{a}) \models \phi(\bar{x})$ if and only if $(A, \bar{a}) \models \phi(\bar{c})$ (where of course c_i is interpreted by a_i for each i). However, viewing free variables as constants may decrease the variable count of a formula, so one must be careful when considering formulae of fixed variable count. From the perspective of games, $EF_n((A, \bar{a}), (B, \bar{b}))$ is identical whether the tuples $\bar{a}, \bar{b}$ are thought of as constants or free variables, however $P_{n,k}((A, \bar{a}), (B, \bar{b}))$ would change. If one wished to construct a game

to capture the case where $\bar{a}, \bar{b}$ were interpreting constants, then the tuples $\bar{a}, \bar{b}$ would start with pebbles on them that could not be moved, and did not come out of the original supply of Spoiler and Duplicator's k pebbles.

- We might say a game G (played on two structures, between Spoiler and Duplicator) is equivalent to a game G' when, for any pair of structures $(A, \bar{a}), (B, \bar{b})$, Duplicator has a winning strategy in $G((A, \bar{a}), (B, \bar{b}))$ if and only if Duplicator has a winning strategy in $G'((A, \bar{a}), (B, \bar{b}))$. As the use of the games is derived from the above Theorem, and the Theorem above only cares about the existence of a winning strategy for Duplicator, we may also find it useful to occasionally consider equivalent games rather than the ones described above. A common way to find equivalent games is to apply a small change to the rules and check this does not change who will have a winning strategy out of Spoiler and Duplicator.

- One such example of this, which regards the difference between pseudo and ordinary partial homomorphisms is to define a game $EF'_n((A, \bar{a}), (B, \bar{b}))$ as follows: The game is played as before, except Spoiler may not select an element $a \in A$ that already occurs in the tuple $\bar{a}$ nor an element that has already been selected in a previous round (Duplicator has no restrictions or changes, and the winning conditions for both players are the same). It is immediate that Duplicator has a winning strategy in $EF'_n((A, \bar{a}), (B, \bar{b}))$ if Duplicator has a winning strategy in $EF_n((A, \bar{a}), (B, \bar{b}))$. If Duplicator has a winning strategy in $EF'_n((A, \bar{a}), (B, \bar{b}))$, then Duplicator can use that strategy to find a winning strategy in the game $EF_n((A, \bar{a}), (B, \bar{b}))$ by following the same strategy, except where if Spoiler repeats an already chosen element, Duplicator simply responds by choosing the same element that it chose previously. Another example is a variant of $P_{n,k}((A, \bar{a}), (B, \bar{b}))$ where Spoiler must first place all pebbles before Spoiler is allowed to move any pebbles, which as equivalent to the usual game $P_{n,k}((A, \bar{a}), (B, \bar{b}))$.

- Observe that in the back and forth game, we insist Duplicator must maintain an ordinary partial isomorphism rather than a pseudo-partial isomorphism. Were we to swap the winning condition so that Duplicator must only maintain a pseudo-partial morphism, a winning strategy for Duplicator would entail equivalence in only equality free formulas. This contrasts to the forth game, where swapping the winning condition would make no difference.

- In the EF games, one needs only check the winning condition at the end of the game, unlike in the pebble games, where one must check the winning condition at the end of each round. This is because, in the EF games, the induced substructures are "increasing" (ie the the induced substructures under consideration in round i are subsets of those being considered in round $i + 1$), so checking the winning condition once at the end implies it for all of the earlier rounds.

- The games are all essentially recursive in nature. For example, one can see that Duplicator has a winning strategy in $EF_n(A, B)$, if and only if, for every $a \in A$ there exists some $b \in B$ such that Duplicator has a winning strategy in $EF_{n-1}((A, a), (B, b))$.

The last remark can be thought of as recursive characterisation for the relations $\equiv_{n,k}$ and $\rightarrow_{n,k}$. This is essentially a slightly more formal version of the Ehrenfeucht-Fraisse games presented above. This definition is recursive thus more natural for inductive proofs, however we rely on the game formulation for intuitive purposes. The ith element in the distinguished tuples below correspond to the position of the ith pebble in the corresponding game. In the following we recurse on the quantifier rank n, and we use $\bar{a}[\alpha/a_i]$ to denote the tuple $\bar{a}$ where the ith entry has been swapped with α. We shall briefly introduce this as a pair of new relations $\tilde{\rightarrow}_{n,k}$, and $\tilde{\equiv}_{n,k}$, which we will claim give identical relations to $\rightarrow_{n,k}$ and $\equiv_{n,k}$ respectively. This is a standard approach, and more discussions (and a detailed proof of the relations being identical) can be found in any Finite Model Theory textbook, such as [21] or [13].

Definition 1.4.5.

- *$(A, \bar{a})\tilde{\rightarrow}_{0,k}(B, \bar{b})$ is defined: The assignment $a_i \mapsto b_i$ defines a pseudo-partial morphism from A to B.*
- *If $n > 0$, then $(A, \bar{a})\tilde{\rightarrow}_{n,k}(B, \bar{b})$ is defined:*
  - *The assignment $a_i \mapsto b_i$ defines a pseudo-partial morphism from A to B*
  - *If $l < k$, for every $\alpha \in A$, there exists $\beta \in B$ such that $(A, \bar{a}, \alpha)\tilde{\rightarrow}_{n-1,k}(B, \bar{b}, \beta)$*
  - *For any value of l, for every $\alpha \in A$, $i \in \{1, ..., l\}$, there exists $\beta \in B$ such that $(A, \bar{a}[\alpha/a_i])\tilde{\rightarrow}_{n-1,k}(B, \bar{b}[\beta/b_i])$.*

Definition 1.4.6.

- *$(A, \bar{a})\tilde{\equiv}_{0,k}(B, \bar{b})$ is defined: The assignment $a_i \mapsto b_i$ defines a partial isomorphism from A to B.*
- *If $n > 0$, then $(A, \bar{a})\tilde{\equiv}_{n,k}(B, \bar{b})$ is defined:*
  - *The assignment $a_i \mapsto b_i$ defines a partial isomorphism from A to B*
  - *If $l < k$, for every $\alpha \in A$, there exists $\beta \in B$ such that $(A, \bar{a}, \alpha)\tilde{\equiv}_{n-1,k}(B, \bar{b}, \beta)$, and for every $\beta \in B$, there exists $\alpha \in A$ such that $(A, \bar{a}, \alpha)\tilde{\equiv}_{n-1,k}(B, \bar{b}, \beta)$*
  - *For any value of l, $i \in \{1, ..., l\}$, we have both: for every $\alpha \in A$, there exists $\beta \in B$ such that $(A, \bar{a}[\alpha/a_i])\tilde{\equiv}_{n-1,k}(B, \bar{b}[\beta/b_i])$, and for every $\beta \in B$, there exists $\alpha \in A$ such that $(A, \bar{a}[\alpha/a_i])\tilde{\equiv}_{n-1,k}(B, \bar{b}[\beta/b_i])$.*

In each of the above definitions, the base case (when $n = 0$) just checks the victory condition. After checking for a partial morphism, the recursive step has two further clauses, one for placing a new pebble (adding a new element to the tuple), and one for moving a pebble (replacing an element of the tuple). The recursive definition of $\tilde{\rightarrow}_n$ and $\tilde{\equiv}_n$ are simply defined by deleting the final clause (where l can take any value) of the recursive step above for $\tilde{\rightarrow}_{n,k}$ and $\tilde{\equiv}_{n,k}$ respectively. The following is a generalisation of Theorem 1.4.4 that also includes the recursive versions of each relation:

Theorem 1.4.7. *For any pair of structures $(A, \bar{a}), (B, \bar{b}) \in \mathcal{R}_\sigma(l)$, the following are equivalent:*

1. *Duplicator has a winning strategy in $EF_n((A, \bar{a}), (B, \bar{b}))$*
2. $(A, \bar{a}) \rightarrow_n (B, \bar{b})$
3. $(A, \bar{a}) \tilde{\rightarrow}_n (B, \bar{b})$

The statement above also holds when replacing the triple $(EF_n, \rightarrow_n, \tilde{\rightarrow}_n)$ with any of the triples $(EF_n^{\equiv}, \equiv_n, \tilde{\equiv}_n)$,$(P_{n,k}, \rightarrow_{n,k}, \tilde{\rightarrow}_{n,k})$, or $(P_{n,k}^{\equiv}, \equiv_{n,k}, \tilde{\equiv}_{n,k})$.

The equivalence of the first and second items done by a straightforward induction for each of the triples. We include a standard proof of the equivalence of the second and third items, and once again refer to [21] or [13] for more detail.

Proof. We shall do induction on n. Fix some $(A, \bar{a}), (B, \bar{b}) \in \mathcal{R}_\sigma(l)$.

- If $n = 0$, then it is immediate since the statement becomes "the assignment $a_i \mapsto b_i$ defines a pseudo-partial homomorphism" if and only if "for every $\phi \in \mathcal{L}_0^+(l)$, $(A, \bar{a}) \models \phi \implies (B, \bar{b}) \models \phi$", and the only formulas in $\mathcal{L}_0^+(l)$ are positive boolean combinations of atomic relations between $x_1, ..., x_l$.

- Now suppose $n > 0$ and we can use the inductive hypothesis. We shall also make use of the fact we need only test $\rightarrow_n$ on primitive positive formulae. Suppose we have $(A, \bar{a}) \rightarrow_n (B, \bar{b})$ and some $\alpha \in A$, we need to find some $\beta \in B$ such that $(A, \bar{a}, \alpha) \rightarrow_{n-1} (B, \bar{b}, \beta)$, and then we are done in one direction by the inductive hypothesis. Define

$$\phi := \bigwedge \{\psi \in \mathcal{L}_{n-1}^+(l+1) : (A, \bar{a}, \alpha) \models \psi\}.$$

  We can make this well formed since $\mathcal{L}_n(l)$ is finite up to equivalence for any n, l, which is a fact we prove later on. Observe that if we can find some $\beta \in B$ such that $(B, \bar{b}, \beta) \models \phi$ then we are done since we will have $(A, \bar{a}, \alpha) \rightarrow_{n-1} (B, \bar{b}, \beta)$. But we know $(A, \bar{a}) \models \exists x_{l+1} \phi$ (since $(A, \bar{a}, \alpha) \models \phi$), and hence $(B, \bar{b}) \models \exists x_{l+1} \phi$ since $\exists x_{l+1} \phi \in \mathcal{L}_n(l)$ and $(A, \bar{a}) \rightarrow_n (B, \bar{b})$. Taking β as a witness for x_{l+1} in $\exists x_{l+1} \phi$ therefore works. Now for the reverse, assume we have $(A, \bar{a}) \tilde{\rightarrow}_n (B, \bar{b})$ and let $\phi \in \mathcal{L}_n(l)$ be primitive positive such that $(A, \bar{a}) \models \phi$: we must show $(B, \bar{b}) \models \phi$. We shall also assume the principle connective of ϕ is $\exists x_l$ since we can relabel variables without loss of generality, and if the principle connective was a $\wedge$ we can treat both conjuncts separately. So we write $\phi = \exists x_l \psi$ where $\psi \in \mathcal{L}_{n-1}^+(l+1)$. Let α be A's witness for x_l in ϕ so that $(A, \bar{a}, \alpha) \models \psi$. By $\tilde{\rightarrow}_n$, find $\beta \in B$ such that $(A, \bar{a}, \alpha) \rightarrow_{n-1} (B, \bar{b}, \beta)$. By the inductive hypothesis, we have $(A, \bar{a}, \alpha) \rightarrow_{n-1} (B, \bar{b}, \beta)$ and hence $(B, \bar{b}, \beta) \models \psi$, and hence $(B, \bar{b}) \models \phi$ as required.

□

1.5 Comonads

We now present the comonads $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$. For each definition, we shall start with $\mathbb{E}_n$, the comonad corresponding to quantifier rank and the game EF_n as it is simpler, and then move on to $\mathbb{P}_{n,k}$. Initially we shall work in the category $\mathcal{R}_\sigma$ (ie when $l = 0$), and then move onto arbitrary l.

Intuitively, a strategy for Duplicator in $EF^n(A, B)$ might be thought of as set containing a move for in any situation in the game, so a choice of $b \in B$ for any pair $\bar{a} \in A^l, \bar{b} \in B^l$, (representing the previous moves from the previous l rounds, where $l < n$) and $a \in A$ representing Spoiler's move in the most recent round. We could represent a such a strategy as a function

$$S : ((\bigcup_{l<n} (A \times B)^l) \times A) \to B.$$

Here we consider $(A \times B)^0$ to be some arbitrary 1-element set $\{\star\}$, which represents the position when Spoiler is about to make it's first move. However one can see such strategies contain some unnecessary information. Suppose for some strategy S, we have $S(\star, a_1) = b_1$. In other words, according to the strategy S, if Spoilers first move is a_1 then Duplicator will reply with b_1. Let $b' \neq b_1 \in B$, and $a_2 \in A$. Then S must have a value for $S((a_1, b'), a_1)$. However, if Duplicator is following the strategy S, then the position a_1, b' will never occur! Hence we define a refined strategy, that does not include unnecessary data such as this.

Definition 1.5.1. *A refined strategy for Duplicator in some game $EF^n(A, B)$ is a partial function $S : ((\bigcup_{l<n} (A \times B)^l) \times A) \rightharpoonup B$ that has values defined exactly on positions that can occur according to S. More precisely, $S(\star, a)$ is defined for every $a \in A$, and $S((a_1, b_1, ..., a_l, b_l), a)$ is defined for any $a \in A$, if and only if $S((a_1, b_1, ..., a_{l-1}, b_{l-1}), a_l)$ is defined and equal to b_l.*

It will turn out that refined strategies can be captured exactly by functions when using our comonad. One can see [5] for a more detailed discussion on this than what is presented below.

Definition 1.5.2. *For a structure A, let $\mathbb{E}_n A$ be the set of (non-empty) sequences of elements of A length at most n, and let $\epsilon_A : \mathbb{E}_n A \to A$ be the function that gives the last element of a sequence.*

Lemma 1.5.3. *For any $A, B \in \mathcal{R}_\sigma$, there is a bijective correspondence between:*

- *Refined strategies $S : ((\bigcup_{l<n} (A \times B)^l) \times A) \rightharpoonup B$*
- *Functions $f : \mathbb{E}_n A \to B$.*

Proof. The key observation here is how Duplicator can use a function f to generate a refined strategy. If Spoiler plays a_1 in the first round, Duplicator simply plays $f([a_1])$. In the lth round, if Spoiler has played $a_1, ..., a_{l_1}$ and plays a_l this round, Duplicator will play $f([a_1, ..., a_l])$. This gives the following way of converting a function f to a refined strategy S_f:

- $S_f(\star, a) := f[a]$
- $S_f((a_1, b_1, ..., a_{l-1}, b_{l-1}), a_l)$ is defined only if $S_f((a_1, b_1, ..., a_{l-2}, b_{l-2}), a_{l-1})$ is defined and equal to b_{l-1}, and is set equal to $f([a_1, ..., a_l])$ in this case

S_f is clearly a refined strategy by how it is defined. By recursively applying the second clause in the definition, it must be the case that $b_1 = f([a_1])$, $b_2 = f([a_1, a_2])$ and so on. Given a refined strategy S, we can define a function f_S recursively, by setting:

- $f_S[a] := S(\star, a)$
- $f_S[a_1, ..., a_l] := S(a_1, f([a_1]), ..., a_{l-1}, f([a_1, ..., a_{l-1}]), a_l)$.

Inductively one can see that this will indeed be properly defined. The base case is given by the definition, and for inductive step, recall that $S(a_1, f_S([a_1]), ..., a_{l-1}, f_S([a_1, ..., a_{l-1}]), a_l)$ is defined if and only if $S(a_1, f_S([a_1]), ..., a_{l-1}, f_S([a_1, ..., a_{l-2}]), a_{l-1})$ is defined and equal to $f_S([a_1, ..., a_{l-1}]$. But these follows exactly from the inductive hypothesis and the definition of f_S respectively. It follows quickly from the constructions that for any refined strategy $S_{f_S} = S$, and for any function f, $f_{S_f} = f$, completing the proof of a bijection. □

Strategies for the pebbling game $P_{n,k}$ are almost identical to those in the game EF_n, once we explicitly include the history of the game in the strategies. To exemplify this, consider a game $P_{n,k}(A, B)$ in which Spoiler first places pebble 1 on $a_1 \in A$ in the first round and pebble 2 on $a_2 \in A$ in the second round, and Duplicator responds by placing pebble 1 on $b_i \in B$ and pebble 2 on $b_2 \in B$ in rounds 1 and 2 respectively. Intuitively, one might assume this position may be the same as the position created where the move order was swapped (ie Spoiler plays pebble 2 on a_2 in the first round and so on), however we shall consider these to be different positions to allow a neater representation of strategies as functions. In the game EF_n, the history is implicitly recorded by the order of the tuples $\bar{a}$, $\bar{b}$. If desired, one could consider a “historyless” strategy by considering positions described by sets of elements rather than tuples, though we shall not pursue this here. Once we explicitly take the history of the game to be accounted for in our strategies, the definitions we make parallel the EF game almost exactly, but for keeping track of the pebble that Spoiler has chosen to move with each time. We do not need to keep track of the pebble Duplicator moves as it is required to be the same one. In the following definition we use the shorthand $\mathbf{k} := \{1, ..., k\}$.

Definition 1.5.4. *A strategy for Duplicator in the game $P_{n,k}(A, B)$ is a function*

$$S : ((\bigcup_{l<n} (A \times B \times \mathbf{k})^l)) \times (A \times \mathbf{k})) \to B$$

A refined strategy is a partial function $S : ((\bigcup_{l<n}(A \times B \times \mathbf{k})^l)) \times (A \times \mathbf{k})) \rightharpoonup B$ satisfying:

- *$S(\star, a, i)$ is defined for every $a \in (A, \bar{a})$ and $i \in \mathbf{k}$.*
- *$S((a_1, b_1, i_1, ..., a_{l-1}, b_{l-1}, i_{l-1}), (a_l, i_l))$ is defined only if and only if $S((a_1, b_1, i_1, ..., a_{l-1}, b_{l-2}, i_{l-2}), (a_{l-1}, i_{l-1}))$ is defined and equal to b_{l-1}*

Similarly to before, we define the underlying set form our comonad will take, and see that functions out of it correspond to refined strategies in $P_{n,k}(A, B)$. We omit the proof as it is identical to the case for $E_n(A, B)$ but for carrying along Spoilers choice of pebble.

Definition 1.5.5. *For a structure A, let $\mathbb{P}_{n,k}A$ be the set of non-empty sequences of elements of $A \times \{1, ..., k\}$, of length at most n. Once again, in a slight abuse of notation, let $\epsilon_A : \mathbb{P}_{n,k}A \to A$ denote the first co-ordinate (ie the entry from A) of the last element of a sequence.*

Lemma 1.5.6. *For any structures A, B, in $\mathcal{R}_\sigma$, there is a bijective correspondence between:*

- *Refined strategies $S : ((\bigcup_{l<n}(A \times B \times \mathbf{k})^l)) \times (A \times \mathbf{k})) \rightharpoonup B$*
- *Functions $f : \mathbb{P}_{n,k}A \to B$.*

Our next step is to endow our sets with relational structures, and give some intuition as to why we have done so. We will let, for s_1, s_2 sequences, $s_1 \sqsubset s_2$ denote that s_1 is a prefix of s_2, and $s_1 \sim s_2$ denote $s_1 \sqsubset s_2 \vee s_2 \sqsubset s_1$. For $s \in \mathbb{P}_{n,k}A$, we will refer to the second-coordinate of its last entry as its pebble index.

Definition 1.5.7. *For $A \in \mathcal{R}_\sigma$, $R \in \sigma$:*

- *For $\mathbb{E}_n A$: $(s_1, ..., s_j) \in R^{\mathbb{E}_n A}$ if and only if:*
  1. *$(\epsilon_A(s_1), ..., \epsilon_A(s_j)) \in R^A$.*
  2. *For each $i_1, i_2 \in \{1, ..., j\}$, we have $s_{i_1} \sim s_{i_2}$.*
- *For $\mathbb{P}_{n,k}A$: $(s_1, ..., s_j) \in R^{\mathbb{P}_{n,k}A}$ if and only if:*
  1. *$(\epsilon_A(s_1), ..., \epsilon_A(s_j)) \in R^A$.*
  2. *For each $i_1, i_2 \in \{1, ..., j\}$, we have $s_{i_1} \sim s_{i_2}$.*
  3. *For each $i_1, i_2 \in \{1, ..., j\}$, if $s_{i_1} \sqsubset s_{i_2}$, then the pebble index of s_{i_1} does not occur as a second coordinate in s', where $s_{i_2} = s_{i_1}s'$.*

One can understand these definitions of the relations on $\mathbb{E}_n A$ (or $\mathbb{P}_{n,k}A$), simply as a way of imposing conditions on morphisms $f : \mathbb{E}_n A \to B$. The first condition helps us to ensure that a morphism $f : \mathbb{E}_n A \to B$ entails pseudo-partial morphisms from $\{a_1, ..., a_j\}$ to $\{f([a_1]), ..., f([a_1, ..., a_j])\}$, as would be required in the n-round Ehrenfeucht-Fraisse game. The second condition ensures we are not asking too much of a morphism $f : \mathbb{E}_n A \to B$; if $s_1 \nsim s_2$, then they cannot represent situations that occur in same game, thus there should be no relations between them. For $\mathbb{P}_{n,k}A$ recall that for a sequence s_1, $f(s_1)$ tells us how Duplicator should respond when Spoiler has just moved the pebble index of s_1, pebble i say, to $\epsilon(s_1)$. If $s_1 \sqsubset s_2$, and the pebble index of s_1 occurs again in s_2 (after s_1), that is telling us pebble i has moved, thus there need be no relation between $f([s_1])$ and $f([s_2])$. This is a sketch proof of the following Theorem (a more detailed proof is available in [5]):

Theorem 1.5.8. *For $A, B \in \mathcal{R}_\sigma$, there is a bijection between:*

- *Winning strategies for Duplicator in the n-round Ehrenfeucht-Fraisse game from A to B.*
- *Morphisms $f : \mathbb{E}_n A \to B$.*

Or in the more general case, there is a bijection between:

- *Winning strategies for Duplicator in the (n, k)-pebble game from A to B.*
- *Morphisms $f : \mathbb{P}_{n,k} A \to B$.*

The following states this result in terms of $\to_n$ and $\to_{n,k}$. Stated in this slightly weaker way, the result also follows from the ensuing discussion about how the comonads capture tree width and tree depth respectively.

Theorem 1.5.9. *For A, B in $\mathcal{R}_\sigma$:*

- *$A \to_n B$ if and only if $\mathbb{E}_n A \to B$.*
- *$A \to_{n,k} B$ if and only if $\mathbb{P}_{n,k} A \to B$*

We now display the categorical properties of the assignment $A \mapsto \mathbb{P}_{n,k}A$. Note that $\mathbb{E}_n$ is similar in shape to the non-empty list comonad on the category of sets, prompting us to make the following definition:

Definition 1.5.10. *We turn $\mathbb{E}_n$ into a functor, by pointwise application of functions. In other words for $A, B \in \mathcal{R}_\sigma$ and $f : A \to B$, $\mathbb{E}_n f([a_1, ..., a_j]) := [f(a_1), ..., f(a_j)]$.*
We similarly turn $\mathbb{P}_{n,k}$ into a functor, by pointwise applications and not touching the pebble indices, so $f([(a_1, i_1), ..., (a_j, i_j)]) := [(f(a_1), i_1), ..., (f(a_j), i_j)]$.

As before, we abuse notation by using the same δ for the functors $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$.

Definition 1.5.11. *For $A \in \mathcal{R}_\sigma$, define $\delta : \mathbb{E}_n A \to \mathbb{E}_n \mathbb{E}_n A$ recursively on the length of a sequence by:*

- $\delta_A([a]) := [[a]]$
- $\delta_A(s[a]) := \delta(s)[s[a]]$

In other words, $\delta_A([a_1, ..., a_j]) = [[a_1], [a_1, a_2], ..., [a_1, ..., a_j]]$, the sequence of prefixes of $[a_1, ..., a_j]$. Similar to before, we give an analogous definition for $\mathbb{P}_{n,k}$ keeping the pebble indices fixed:

- $\delta_A([(a,i)]) := [([(a,i)],i)]$
- $\delta_A(s[(a,i)]) := \delta_A(s)[(s[(a,i)],i)]$

The above definition is made with the following Theorem in mind:

Theorem 1.5.12. *$(\mathbb{E}_n, \epsilon, \delta)$ and $(\mathbb{P}_{n,k}, \epsilon, \delta)$ are comonads.*

Proof. Since it is a routine check, in the case of $\mathbb{E}_n$ we refer the reader to [5]. In [5], it also shown $\mathbb{P}_k$ is a comonad where $\mathbb{P}_kA$ is the set containing sequences of any length, rather than being bounded by some n. In other words $\mathbb{P}_kA = \bigcup_{n\in\mathbb{N}} \mathbb{P}_{n,k}A$. The counit ϵ_ω and comultiplication δ_ω for $\mathbb{P}_k$ are defined identically as in the case of $\mathbb{P}_{n,k}$, and hence we may observe that ϵ and δ coincide exactly with the restrictions of ϵ_ω and δ_ω to $\mathbb{P}_{n,k}$. This in turn allows us to verify the comonad diagrams for $\mathbb{P}_{n,k}$ by simply using those for $\mathbb{P}_k$. The only thing needed to verify is that everything we have defined is indeed a morphism in $\mathcal{R}_\sigma$, which follows easily from our definitions. □

We now generalise this to the categories $\mathcal{R}_\sigma(l)$. As commented above, the (n,k)-game between structures $(A,\bar{a})$ and $(B,\bar{b})$ is just the $(n,k+l)$ game where we have fixed Spoiler's and Duplicator's first l moves. Thus we make the following definition:

Definition 1.5.13. *For $(A,\bar{a}) \in \mathcal{R}_\sigma(l)$:*

- *$\mathbb{E}_n(A,\bar{a})$ is the induced substructure of $\mathbb{E}_{n+l}A$ by the subset $\{s \in \mathbb{E}_{n+l}A : s \sim [a_1,...,a_l]\}$. We consider it a structure in $\mathcal{R}_\sigma(l)$ by distinguishing the tuple $([a_1],[a_1,a_2],...,[a_1,...,a_l])$.*
- *$\mathbb{P}_{n,k}(A,\bar{a})$ is the induced substructure of $\mathbb{P}_{n+l,k}A$ on the subset $\{s \in \mathbb{P}_{n+l,k}A : s \sim [(a_1,1),...,(a_l,l)]\}$. We consider it a structure in $\mathcal{R}_\sigma(l)$ by distinguishing the tuple $([(a_1,1)],[(a_1,1),(a_2,2)],...,[(a_1,1),...,(a_l,l)])$.*

We re-use ϵ and δ to denote the obvious natural transformation given by restriction of our previous ϵ and δ maps, and similarly define the action of $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$ on maps by restriction.

This is a slight abuse of notation, though it should always be clear from the context what ϵ and δ refer to. This definition works as one would hope:

Lemma 1.5.14. *$(\mathbb{E}_n, \epsilon, \delta)$ and $(\mathbb{P}_{n,k}, \epsilon, \delta)$ are comonads on $\mathcal{R}_\sigma(l)$*

Proof. Since we have defined these more general comonads simply by restriction, we know that all necessary diagrams commute. We need only observe that all restricted functions land in the appropriate ranges, which follows quickly from the definitions. □

Similarly to above, we will also see:

Lemma 1.5.15. *For $(A,\bar{a}),(B,\bar{b})$ in $\mathcal{R}_\sigma(l)$:*

- $\mathbb{E}_n(A, \bar{a}) \to (B, \bar{b})$ *if and only if* $(A, \bar{a}) \to_n (B, \bar{b})$.
- $\mathbb{P}_{n,k}(A, \bar{a}) \to (B, \bar{b})$ *if and only if* $(A, \bar{a}) \to_{n,k} (B, \bar{b})$.

We can now see how these comonads give us new characterisations of tree depth and tree-width (via (n, k)-covers), as proved in [1] and [2].

Theorem 1.5.16. *There is a one to one correspondence between:*

- *Coalgebras for a structure* $(A, \bar{a})$ *for the comonad* $\mathbb{E}_n$.
- *Forest covers of depth at most* n *of* $(A, \bar{a})$.

There is a one to one correspondence between:

- *Coalgebras for a structure* $(A, \bar{a})$ *for the comonad* $\mathbb{P}_{n,k}$.
- (n, k)*-covers of* $(A, \bar{a})$.

Proof. Once one carefully unpicks the definitions of an (n, k)-cover we will see the following correspondences:

1. Forests on vertex set A with that begin with the path $a_1, ..., a_l$ of depth at most $n + l \leftrightarrow$ Set maps $(A, \bar{a}) \to \mathbb{E}_n(A, \bar{a})$, respecting the distinguished elements, that make the coalgebra diagrams commute.
2. Forest covers of $(A, \bar{a})$ of depth at most $n \leftrightarrow$ coalgebras $(A, \bar{a}) \to \mathbb{E}_n(A, \bar{a})$.
3. (n, k)-covers of $(A, \bar{a}) \leftrightarrow$ coalgebras $(A, \bar{a}) \to \mathbb{P}_{n,k}A$.

Now for the details:

1. Note that the data of a forest can be given by specifying, for each vertex, the unique path to its root. This data can be expressed as a function $F : A \to \mathbb{E}_{n+l}A$ by sending the path $r, a_1, ..., a_m, a$ (from an element a to its root r) to the sequence $s = [r, a_1, ..., a_m, a]$. Certainly not all functions of this type describe forests beginning with the path $a_1, ..., a_l$. The conditions required for the function to be a forest beginning with the path $a_1, ..., a_l$ are precisely that, F restricts to a function $(A, \bar{a}) \to \mathbb{E}_n(A, \bar{a})$ (ie the forest begins with the path $a_1, ..., a_l$), $\epsilon(F(a)) = a$ for each $a \in A$ (ie the unique path to a ends with a), and $\delta(F(a)) = F(F(a))$ for each $a \in A$ (if some b lies on the path on the unique path a, then $F(b)$ is the unique prefix of $F(a)$ ending in b). We have just written inline the commutative diagrams defining a coalgebra, thus we have correspondence 1.
2. Now suppose F satisfies the properties in 1. F is a morphism if and only if, for every tuple $b_1, ..., b_j$ of A, and $R \in \sigma$,

   $$(b_1, ..., b_j) \in R^{(A,\bar{a})} \implies (F(b_1), ..., F(b_j)) \in R^{\mathbb{E}_n(A,\bar{a})}.$$

   Now the consequent is true precisely when $(\epsilon(F(b_1)), ..., \epsilon(F(b_j))) \in R^{(A,\bar{a})}$ and, for each pair b, b' among $b_1, ..., b_j$, $F(b) \sim F(b')$. Now the first condition is met by assumptions on F in (1), since $\epsilon F(b_i) = b_i$ for each i, so we need only check the second condition. We restate it as the following: whenever there is an edge b, b' in $G(A)$ (recall $G(A)$ is the Gaifman graph of A and there is an edge between two vertices if and only there is some tuple $(b_1, ..., b_j)$ containing b, b' and relation $R \in \sigma$ such that $(b_1, ..., b_j) \in R^A$) we have $F(b) \sim F(b')$. Since $F(b)$ describes the unique path from b to its root, we have that $F(b) \sim F(b')$, if and only if b and b' lie on the same branch. Summing up, we have that F is a morphism if and only if whenever there is an edge b, b' in $G(A)$, then b and b' lie on the same branch as the forest described by F, ie if and only if F describes a forest cover of $(A, \bar{a})$. This completes correspondence (2).

3. Given a coalgebra F for $(A, \bar{a})$ over the comonad $\mathbb{E}_n$, we see the correspondence between the data required to turn F into a map to $\mathbb{P}_{n,k}A$, and k-labelling functions of the forest cover described by F is simply sending the label for each element in the forest cover to its pebble index (the unique i that appears in a pair with it in a sequence $s \in \mathbb{P}_{n,k}A$). It is a simple check that we will get a morphism if and only if the labelling turns the forest cover described by F into an (n, k)-cover.

□

We state the following immediate corollary to above:

Corollary 1.5.17. *Any structure of form $\mathbb{P}_{n,k}(A, \bar{a})$ always has a coalgebra provided by δ_A, thus always has an (n, k)-cover.*

This theorem also hints the category of (n, k)-covers is precisely the Eilenberg-Moore category of the comonad $\mathbb{P}_{n,k}$ (more detail is included in the third chapter). It also provides another proof of Theorem 1.5.9:

Theorem 1.5.18. *(restatement of Theorem 1.5.9) For $(A, \bar{a}), (B, \bar{b})$ in $\mathcal{R}_\sigma(l)$:*

- *$(A, \bar{a}) \to_n (B, \bar{b})$ if and only if $\mathbb{E}_n(A, \bar{a}) \to (B, \bar{b})$.*
- *$(A, \bar{a}) \to_{n,k} (B, \bar{b})$ if and only if $\mathbb{P}_{n,k}(A, \bar{a}) \to (B, \bar{b})$.*

Proof. We just prove the second statement using the combinatorial characterisation of $\to_{n,k}$.
Suppose $(A, \bar{a}) \to_{n,k} (B, \bar{b})$. Since $\mathbb{P}_{n,k}(A, \bar{a})$ has an (n, k)-cover, and $\mathbb{P}_{n,k}(A, \bar{a}) \to A$, we then have $\mathbb{P}_{n,k}(A, \bar{a}) \to (B, \bar{b})$.
Now suppose $\mathbb{P}_{n,k}(A, \bar{a}) \to (B, \bar{b})$. Let C have an (n, k)-cover, and $(C, \bar{c}) \to (A, \bar{a})$. Since $(C, \bar{c})$ has an (n, k)-cover, it has a coalgebra, and thus $(C, \bar{c}) \to \mathbb{P}_{n,k}(C, \bar{c})$. By functoriality, $\mathbb{P}_{n,k}(C, \bar{c}) \to \mathbb{P}_{n,k}(A, \bar{a})$ and putting both of these together with the assumption, we get $(C, \bar{c}) \to (B, \bar{b})$. □

As another immediate corollary to the above Theorem, we have the beforementioned categorical description of tree width and tree depth:

Theorem 1.5.19.
- *The tree depth of a structure $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$ (if it exists) is the least n such that $(A, \bar{a})$ has an $\mathbb{E}_n$-coalgebra.*

- *The tree width of a structure* $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$ *(if it exists) is the least* k *such that* $(A, \bar{a})$ *has an* $\mathbb{P}_{n,k}$*-coalgebra (for some* n*).*

Proof. Recall the tree depth of a structure is the least n such that it has an (n, n)-cover, which we have just shown exists if and only if it has an $\mathbb{E}_n$ coalgebra. The tree width case is identical. □

Chapter 2

Towards the Equirank-variable Homomorphism Preservation Conjecture

2.1 Overview

In this chapter we recount the work in [26], in which we proved a special case of Abramsky's Equirank-Variable Homomorphism Preservation Conjecture, and worked towards the general case. In model theory in general, a preservation Theorem is one that relates a syntactic condition of a first-order sentence (eg restricted quantifier rank or being positive existential) to a semantic property (eg satisfaction being preserved under certain types of relations between structures). Before stating some preservation Theorems, we require the following definition:

Definition 2.1.1. *We say a formula $\phi \in \mathcal{L}(l)$ is preserved under some relation $\sim$ if it satisfies the following property: for any $(A, \bar{a}), (B, \bar{b}) \in \mathcal{R}_\sigma(l)$ if $(A, \bar{a}) \models \phi$ and $(A, \bar{a}) \sim (B, \bar{b})$, then $(B, \bar{b}) \models \phi$*

The basic homomorphism preservation Theorem of classical model theory (due to Los, Lyndon and Tarski) is as follows:

Theorem 2.1.2. *For any $\phi \in \mathcal{L}(l)$, ϕ is preserved under homomorphisms, if and only if there is some $\psi \in \mathcal{L}^+(l)$ such that $\phi \equiv \psi$.*

One can observe in this Theorem that the syntactic property (being positive existential) implying the semantic property (being preserved under homomorphism) is almost immediate to prove (it is well known all positive existential formulae are preserved under homomorphisms), but the reverse direction is not obvious. This will be the case for all of the homomorphism preservation theorems we look at.

The above was improved by Rossman in [27], who showed that one can preserve the quantifier rank of the formula in the Theorem:

Theorem 2.1.3. *(Rossman's Equirank Homomorphism Preservation Theorem) Suppose* $\phi \in \mathcal{L}_n(l)$*. Then it is preserved under homomorphisms between first-order structures, if and only if there* $\psi \in \mathcal{L}_n^+(l)$ *such that* $\phi \equiv \psi$*.*

Rossman's proof involved substantial use of structures of bounded tree depth. Owing to this, Abramsky conjectured one could recast the proof in a more categorical style, making use of the comonad $\mathbb{E}_n$, as it naturally creates structures of bounded tree depth. This leads to the idea of generalising to using $\mathbb{P}_{n,k}$ rather than $\mathbb{E}_n$ to obtain the following conjecture where both tree depth and tree width are preserved:

Conjecture 2.1.4. *(Abramsky's Equirank-Variable Preservation Conjecture) Suppose* $\phi \in \mathcal{L}_{n,k}(l)$*. Then it is preserved under homomorphisms between first-order structures, if and only if there* $\psi \in \mathcal{L}_{n,k}^+(l)$ *such that* $\phi \equiv \psi$*.*

The first step towards this conjecture is proving what we refer to as the Mini-HPT, which is the statement that any sentence preserved under $\rightarrow_{n,k}$ is equivalent to an existential positive formula in $\mathcal{L}_{n,k}^+$, directly generalising what Rossman proved solely about quantifier rank. The proof of this is fairly short once we collect some model theoretic facts about $\rightarrow_{n,k}$; namely that $\mathcal{L}_{n,k}^+(l)$ contains only finitely many formulae for any choice of n, k, l, and that $\leftrightarrows_{n,k}$ is therefore a finite index relation on $\mathcal{R}_\sigma(l)$. We also show that we are able to choose a representative set for this finite index relation that consists only of finite structures, each with an (n, k)-cover.

The next section is far more technical. Given the Mini-HPT, it is sufficient to show any first order formula in $\mathcal{L}_{n,k}^+$ that is preserved under homomorphisms is also preserved under $\rightarrow_{n,k}$ in order to prove the Equirank-Variable Conjecture. The idea for proving the former claim is to construct for any $A \in \mathcal{R}_\sigma$, a diagram of form $A \rightarrow B \equiv_{n,k} C \rightarrow \mathbb{P}_{n,k}A$, and building these structures B and C is what the section is devoted to. The structures are built by iteratively gluing copies of the finite representative structures mentioned above to A and $\mathbb{P}_{n,k}A$ respectively, in order that they might achieve a saturation-like property, which we call (n, k)-extendability, directly generalising Rossman's notion of n-extendability. (n, k)-extendability is a purpose-defined property that allows one to infer $\equiv_{n,k}$ from the $\leftrightarrows_{n,k}$ relation, by extending partial isomorphisms between structures.

Unfortunately, along the way we observe that the $\rightarrow_{n,k}$ relation is not preserved under taking coproducts the same way that $\rightarrow_n$ is, which is why we are not able to find structures with the (n, k)-extendability property we require. We do however give a proof modulo a Conjecture that says we can find a finite set of representative structures for which the $\rightarrow_{n,k}$ relation behaves as we would wish with respect to taking coproducts with those structures. We also highlight the case where $n \leq k+2$, for which we prove the Conjecture, giving a slight improvement to Rossman's result, since this includes the case $n = k$, which is simply the Equirank case. Finally, we give some analysis of the construction we have made, with some further suggestions as to how it might be possible to prove the general case.

2.2 Mini-HPT and consequences

In this section, we will prove a weaker preservation Theorem that is a stepping stone to the Equirank-variable preservation conjecture. We generalise the notion of preservation to classes (of structures) also, as it can sometimes be an easier place to work:

Definition 2.2.1. *We say a class of structures $M \subset \mathcal{R}_\sigma(l)$ is closed under a relation $\sim$ if for any $(A, \bar{a}), (B, \bar{b}) \in \mathcal{R}_\sigma(l)$, whenever $(A, \bar{a}) \in M$ and $(A, \bar{a}) \sim (B, \bar{b})$, then $(B, \bar{b}) \in M$.*

This directly extends the notion of a formula being preserved under a relation, by substituting the class of structures which satisfy the formula for M in the above definition (which we shall write $Mod(\phi)$ where ϕ is the formula in question).

We now state what we refer to as the mini-HPT:

Theorem 2.2.2. *For any class $M \subset \mathcal{R}_\sigma(l)$, if M is closed under $\rightarrow_{n,k}$, then there exists $\phi \in \mathcal{L}^+_{n,k}(l)$ such that $M = Mod(\phi)$.*

Proving this Theorem relies on the properties of the comonad $\mathbb{P}_{n,k}$ and the fact that equivalence classes of $\mathcal{R}_\sigma(l)$ under $\leftrightarrows_{n,k}$ are finite. We shall prove the latter and then give a proof of the mini-HPT. This in turn relies on the fact that $\mathcal{L}_{n,k}(l)$ is finite under $\equiv$. This is a well known fact and can be found in textbooks such as [13].

Lemma 2.2.3. *For any n, l, there are finitely many equivalence classes of $\mathcal{L}_n(l)$ under $\equiv$. In other words, there exists some finite set $\phi_1, ..., \phi_j \in \mathcal{L}_n(l)$ such that for any $\psi \in \mathcal{L}_n(l)$, $\psi \equiv \phi_i$ for some i.*

Proof. This is proved by induction on n. If $n = 0$, then $\mathcal{L}_n(l)$ is the set of boolean combinations of atomic relations built using at most l variables, which is finite under $\equiv$, since the relational signature σ is also finite. For the inductive step, note that a formula ϕ of quantifier rank $n+1$ can always be written as a boolean combination of formulae $\phi_1, ..., \phi_m \in \mathcal{L}_n(l)$ and $\exists x_{i_1} \psi_1, ..., \exists x_{i_{m'}} \psi_{m'}$, where each $\psi \in \mathcal{L}_n(l+1)$. By the inductive hypothesis, there are only finitely many formulae of quantifier rank at most n (up to $\equiv$), and since $\equiv$ is preserved under boolean combinations, we can only construct finitely many different formulae of quantifier rank $n+1$ (up to $\equiv$). $\square$

Given this, we now state the corollaries of it that we need for the proof of the mini-HPT.

Corollary 2.2.4. *$\mathcal{L}_n^+(l)$, $\mathcal{L}_{n,k}(l)$, and $\mathcal{L}_{n,k}^+(l)$ are finite up to $\equiv$ for every n, k, l.*

This holds because each of these fragments are subsets of some fragment $\mathcal{L}_n(l)$ for some n, l.

Corollary 2.2.5. *$\mathcal{R}_\sigma(l)$ has finitely many equivalence classes under $\equiv_{n,k}$ and $\leftrightarrows_{n,k}$ for every n, k, l.*

Proof. A structure's equivalence class under $\equiv_{n,k}$ is determined by which formulas it satisfies from $\mathcal{L}_{n,k}(l)$. Since there finitely many of these up to equivalence (say m many), there are finitely many equivalence classes (at most 2^m). The same argument holds for $\leftrightarrows_{n,k}$. □

Another useful fact about the equivalence classes of $\leftrightarrows_{n,k}$ is that each one contains a finite structure with an (n, k)-cover, which we will prove in two lemmas:

Lemma 2.2.6. *For any n, k, l, and structure $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$, there exists a finite $(B, \bar{b}) \in \mathcal{R}_\sigma(l)$ such that $(A, \bar{a}) \leftrightarrows_{n,k} (B, \bar{b})$.*

Proof. Fix n, k, l and let $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$. Recall that the equivalence class of $(A, \bar{a})$ is determined completely by the formulas of $\mathcal{L}_{n,k}^+(l)$ that it satisfies, and there are finitely many formulas up to equivalence. Suppose the formulas $\phi_1, ..., \phi_m$ is a list containing all the formulas of $\mathcal{L}_{n,k}^+(l)$ that $(A, \bar{a})$ satisfies up to equivalence. Now since each of these formulas are positive existential, they are satisfied if and only if they are witnessed by some tuples $\bar{a}^i$ of A (where $\bar{a}^i$ is the tuple witnessing ϕ_i). Now the induced substructure $(A', \bar{a})$ that has universe consisting of the entries of all of these tuples (and the distinguished tuple $\bar{a}$ of A) is certainly finite, and by construction will satisfy each ϕ_i, since it has witnesses for all of them. Hence we will have $(A, \bar{a}) \rightarrow_{n,k} (A', \bar{a})$ by the definition of $\rightarrow_{n,k}$. Since A' is an induced substructure of $(A, \bar{a})$, we have $(A', \bar{a}) \rightarrow (A, \bar{a})$, and hence $(A', \bar{a}) \rightarrow_{n,k} (A, \bar{a})$ as required. □

Lemma 2.2.7. *For any n, k, l, and $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$, $(A, \bar{a}) \leftrightarrows_{n,k} \mathbb{P}_{n,k}(A, \bar{a})$.*

Proof. The co-unit $\epsilon : \mathbb{P}_{n,k}(A, \bar{a}) \rightarrow (A, \bar{a})$ witnesses $\mathbb{P}_{n,k}(A, \bar{a}) \rightarrow (A, \bar{a})$ and hence $\mathbb{P}_{n,k}(A, \bar{a}) \rightarrow_{n,k} (A, \bar{a})$. For the other way around, recall that it suffices to check only primitive positive formulas rather than all positive existential formulas. Suppose some arbitrary $\phi \in \mathcal{L}_{n,k}^+(l)$ is primitive positive and $(A, \bar{a}) \models \phi$. We proved in the introduction that ϕ has a term structure $(C, \bar{c})$ with an (n, k)-cover, and hence a $\mathbb{P}_{n,k}$-coalgebra. Since $(A, \bar{a}) \models \phi$, we get $(C, \bar{c}) \rightarrow (A, \bar{a})$ and hence $\mathbb{P}_{n,k}(C, \bar{c}) \rightarrow \mathbb{P}_{n,k}(A, \bar{a})$ by functoriality. Since $(C, \bar{c})$ has a $\mathbb{P}_{n,k}$-coalgebra, we have $(C, \bar{c}) \rightarrow \mathbb{P}_{n,k}(C, \bar{c})$ and can infer $(C, \bar{c}) \rightarrow \mathbb{P}_{n,k}(A, \bar{a})$ and hence $\mathbb{P}_{n,k}(A, \bar{a}) \models \phi$ as required. □

Putting these two lemmas together along with the fact that $\mathbb{P}_{n,k}(A,\bar{a})$ always has a (n,k)-cover (since it has a $\mathbb{P}_{n,k}$-coalgebra) and is finite whenever $(A,\bar{a})$ is, we can conclude that every equivalence class under $\leftrightarrows_{n,k}$ contains a finite structure with an (n,k)-cover. Henceforth, for each n,k,l, we shall fix a set $F_{n,k,l}$ of representative structures for $\mathcal{R}_\sigma(l)$ under $\leftrightarrows_{n,k}$ each being finite with an (n,k)-cover.

We are now in shape to prove the mini-HPT (re-stated here from above):

Theorem 2.2.8. *For any class $M \subset \mathcal{R}_\sigma(l)$, M is closed under $\to_{n,k}$, if and only there exists $\phi \in \mathcal{L}^+_{n,k}(l)$ such that $M = Mod(\phi)$.*

Proof. As noted in above discussions, the reverse direction is immediate, so we will just prove the forwards direction. Given M, define $F' := F_{n,k,l} \cap M$, and set

$$\phi = \bigvee\{\phi_{(C,\bar{c})} : (C,\bar{c}) \in F'\}.$$

This is a well-formed formula since F' is finite because $F_{n,k,l}$ is. We also know $\phi \in \mathcal{L}^+_{n,k}(l)$ since each $(C,\bar{c})$ has an (n,k)-cover, so has a canonical query in $\mathcal{L}_{n,k}{}^{\mathrm{prim}}(l)$. We now check that ϕ satisfies the required properties. First suppose $(B,\bar{b}) \in Mod(\phi)$, we check $(B,\bar{b}) \in M$. Since $(B,\bar{b}) \models \phi$, we have $(B,\bar{b}) \models \phi_{(C,\bar{c})}$ for some $(C,\bar{c}) \in F'$, and hence $(C,\bar{c}) \to (B,\bar{b})$. Since $(C,\bar{c}) \in M$, we have $(B,\bar{b}) \in M$ by the closure property of M. We now check some $(B,\bar{b}) \in M$ is in $Mod(\phi)$. We know there exists some $(C,\bar{c}) \in F_{n,k,l}$ such that $(B,\bar{b}) \leftrightarrows_{n,k} (C,\bar{c})$. Since $(B,\bar{b}) \in M$, we must have $(C,\bar{c}) \in M$ by the closure property of M and hence $(C,\bar{c}) \in F'$, so $\phi_{(C,\bar{c})}$ is in the disjunction that comprises ϕ. We also know $(B,\bar{b}) \models \phi_{(C,\bar{c})}$ because $(C,\bar{c}) \to_{n,k} (B,\bar{b})$ and $\phi_{(C,\bar{c})} \in \mathcal{L}^+_{n,k}(l)$, and can conclude $(B,\bar{b}) \models \phi$ as required. □

Given the mini-HPT, the equirank variable HPT is now implied by the following conjecture (this would be equivalent to it if we insisted M was of form $Mod(\phi)$ for some formula ϕ):

Conjecture 2.2.9. *Any class $M \subset \mathcal{R}_\sigma(l)$ closed under $\equiv_{n,k}$ and $\to$ is closed under the map $(A,\bar{a}) \to_{n,k} \mathbb{P}_{n,k}(A,\bar{a})$, for every $(A,\bar{a}) \in M$.*

This conjecture implies the equirank variable HPT because being closed under maps of form $(A,\bar{a}) \to_{n,k} \mathbb{P}_{n,k}(A,\bar{a})$ and $\to$, implies being closed under $\to_{n,k}$ because any map of form $(A,\bar{a}) \to_{n,k} (B,\bar{b})$ can be factored into the maps $(A,\bar{a}) \to_{n,k} \mathbb{P}_{n,k}(A,\bar{a}) \to (B,\bar{b})$. We can then use the mini-HPT to find a positive existential formula as desired.

2.3 Proof Strategy: Companion Structures and Extendability

Now we seek to move towards a proof of the previous conjecture. We claim that if for any A, we can find a pair C,D with the following properties, then the conjecture would follow:

1. $A \to D$
2. $C \equiv_{n,k} D$
3. $D \to \mathbb{P}_{n,k} A$

The proof would then follow by, given $A \in M$ and $B \in \mathcal{R}_\sigma$ such that $A \to_{n,k} B$, chasing the following diagram (using the closure properties of M we could then obtain $B \in M$):

$$\begin{array}{ccccc} C & \equiv_{n,k} & D & & \\ \uparrow & & \downarrow & & \\ A & & \mathbb{P}_{n,k} A & \longrightarrow & B \end{array}$$

We will in fact find structures with more properties than this, but we present this as the strategy to clarify the argument. Our aim will be to build the $\equiv_{n,k}$ relation along the $\leftrightarrows_{n,k}$ relation, directly generalising what is called n-extendability in [27]. Consider a pair $A \leftrightarrows_{n,k} B$. By the recursive definition of $\to_{n,k}$ for any $b \in B$ there is some $a \in A$ such that $(B, b) \to_{n-1,k} (A, a)$. If we aim to build a back and forth correspondence, we might hope to find $a \in A$ with the property that $(A, a) \leftrightarrows_{n-1,k} (B, b)$ also, since this will ensure a partial isomorphism between $\{a\}$ and $\{b\}$. If we could do this for any $b \in B$ (and of course relative to some tuple $\bar{b}$ of B), and vice versa, and repeat the process for n steps, then we would have that $A \equiv_{n,k} B$. We state this more formally below:

Definition 2.3.1. *$A \in \mathcal{R}_\sigma$ is (n, k)-extendable if for every:*

- $0 < n' \leq n$
- *Tuple $\bar{a}$ of length $l \leq k$*
- *Structure $(B, \bar{b})$ such that $(A, \bar{a}) \leftrightarrows_{(n',k)} (B, \bar{b})$*
- $\beta \in B$.

When $l < k$ there exists $\alpha \in A$ such that

$$(A, \bar{a}, \alpha) \leftrightarrows_{(n'-1,k)} (B, \bar{b}, \beta).$$

When $l = k$ or $l < k$, for each $i \in \{1, ..., l\}$ there exists $\alpha \in A$ such that

$$(A, \bar{a}[\alpha/a_i]) \leftrightarrows_{(n'-1,k)} (B, \bar{b}[\beta/b_i]).$$

This property of A is telling us we can extend partial isomorphisms between A and any structure B by choosing any element in B (subject to the presence of, and whilst maintaining a $\leftrightarrows_{n,k}$ relation). It follows that:

Lemma 2.3.2. *If $A, B \in \mathcal{R}_\sigma$ are (n,k)-extendable, and $A \leftrightarrows_{n,k} B$, then $A \equiv_{n,k} B$.*

With this definition, we have that if we can construct, for any given A, an (n,k)-extendable $\tilde{A}$ such that $\tilde{A} \leftrightarrows A$, then we can use the above lemma, by setting $C = \tilde{A}$ and $D = (\widetilde{\mathbb{P}_{n,k}A})$ in our diagram above, to obtain the required result. This is of course stronger than strictly necessary, though achievable in some cases.

2.4 Building Extendable Structures

We will build the structure $\tilde{A}$ with extensive use of coproducts, so first we need to investigate how the coproduct interacts with $\rightarrow_{n,k}$. We will sometimes appeal to the intuitive characterisation of $\rightarrow_{n,k}$ to avoid tedious book-keeping of indices. We use $\oplus$ to denote the coproduct of two structures in their respective category. We state without proof the following intuitive characterisation in the most common cases:

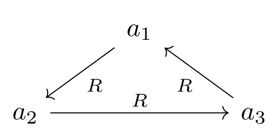
Lemma 2.4.1. *For structures $A, B \in \mathcal{R}_\sigma$, $A \oplus B$ has universe $A \sqcup B$ (the disjoint union of their universes), and relations, for $R \in \sigma$, $R^{A \oplus B} := R^A \cup R^B$. For structures $(A, \bar{a}), (B, \bar{b}) \in \mathcal{R}_\sigma(l)$, in the case that the tuples $\bar{a}$ and $\bar{b}$ have repeated elements in the same places, we can directly construct $(A, \bar{a}) \oplus (B, \bar{b})$, by taking the quotient of $A \oplus B$ under the relation $a_i \sim b_i$ for $i = 1, ..., l$.*

Lemma 2.4.2. *For $A, B, C \in \mathcal{R}_\sigma$, if $A \rightarrow_{n,k} C$ and $B \rightarrow_{n,k} C$ then $A \oplus B \rightarrow_{n,k} C$.*

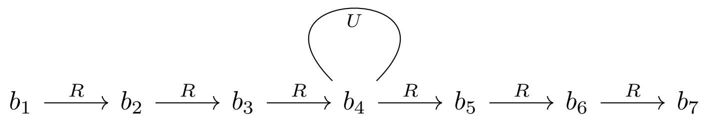
Proof. Intuitively, in the (n,k)-game, Duplicator can just play both strategies for $A \rightarrow_{n,k} C$ and $B \rightarrow_{n,k} C$ simultaneously. If Spoiler plays in A, Duplicator follows the strategy for $A \rightarrow_{n,k} C$, and similarly for B. This corresponds to a map $\mathbb{P}_{n,k}(A \oplus B) \rightarrow \mathbb{P}_{n,k}A \oplus \mathbb{P}_{n,k}B$. For some $s \in \mathbb{P}_{n,k}(A \oplus B)$, if $\epsilon_{A \oplus B}(s) \in B$, we delete all entries of elements of A in the sequence, and if $\epsilon_{A \oplus B}(s) \in A$ we delete all entries of elements of B. □

It is, unfortunately, not the same story in $\mathcal{R}_\sigma(l)$. We provide a counterexample:

Example 2.4.3. *Let σ consist of a single binary relation R, and a unary relation U. Let A be the three cycle on elements $\{a_1, a_2, a_3\}$, pictured below:*



Let B be the path on $\{b_1, ..., b_7\}$ with a single unary relation on b_4.



We see from a simple check that $(A, a_1) \rightarrow_{3,2} (B, b_4)$ (and of course $(B, b_4) \rightarrow_{3,2} (B, b_4)$), but it is not the case that $(A, a_1) \oplus (B, b_4) \rightarrow_{3,2} (B, b_4)$. In the previous case where we worked in $\mathcal{R}_\sigma$, there was no interaction between A and B, and thus Duplicator could run the two strategies simultaneously and independently. However, now we have identified a_1 and b_4 this no longer works.

In the example above we see that if we have at least two more moves than pebbles left to place, then passing $\rightarrow_{n,k}$ through a $\oplus$ fails. The example above was the worst possible in this sense.

Lemma 2.4.4. *Suppose $(A, \bar{a}), (B, \bar{b}), (C, \bar{c}) \in \mathcal{R}_\sigma(l)$, $(A, \bar{a}) \rightarrow_{n,k} (C, \bar{c})$, $(B, \bar{b}) \rightarrow_{n,k} (C, \bar{c})$, and $n \leq k - l + 1$, then $(A, \bar{a}) \oplus (B, \bar{b}) \rightarrow_{n,k} (C, \bar{c})$.*

Proof. First we consider the case $n = k - l$, so we are in fact playing the n-round Ehrenfeucht-Fraisse game, and can make use of the simpler comonad $\mathbb{E}_n$. In terms of strategies, Duplicator can once again run the strategies for $(A, \bar{a})$ and $(B, \bar{b})$ independently. As mentioned in the introduction, it is an equivalent game to not allow Spoiler to place an element where one is already placed, hence we may assume Spoiler will never place one on the identified elements $a_i (= b_i)$, thus Duplicator never has to deal with any conflict between its strategies. Formally, we describe a map $\mathbb{E}_n((A, \bar{a}) \oplus (B, \bar{b})) \rightarrow (\mathbb{E}_n(A, \bar{a}) \oplus \mathbb{E}_n(B, \bar{b}))$ by, for $s \in \mathbb{E}_n((A, \bar{a}) \oplus (B, \bar{b}))$, deleting all entries not in A (respectively B) occurring in s if $\epsilon_{((A,\bar{a}) \oplus (B,\bar{b}))}(s) \in A - \{a_1, ..., a_l\}$ (respectively $B - \{b_1, ..b_l\}$), and if $\epsilon_{((A,\bar{a}) \oplus (B,\bar{b}))}(s)$ occurs among the tuple $\bar{a} (= \bar{b})$, so $\epsilon(s) = a_i$ for some i, then send s to $[a_1, ..., a_i]$.
The case $n = k - l + 1$ is similar to above. Recall again from the introduction we may assume Spoiler cannot move a pebble until all the pebbles have been placed (as this defines an equivalent game). Hence there is only one pebble move after all the pebbles have been placed, so Duplicator can run the strategies for A and B independently with no conflict. □

If $(A, \bar{a}), (B, \bar{b}) \in \mathcal{R}_\sigma(l)$ and $b \in B$, we let $((A, \bar{a}) \oplus (B, \bar{b}), b)$ denote the structure obtained by taking the coproduct $(A, \bar{a}) \oplus (B, \bar{b})$ in $\mathcal{R}_\sigma(l)$, and then distinguishing b in addition to view it as an object of $\mathcal{R}_\sigma(l + 1)$. An important point for later will be when we can conclude $((A, \bar{a}) \oplus (B, \bar{b}), b) \rightarrow_{n,k} (B, \bar{b}, b)$.

Lemma 2.4.5. *If $n \leq k - l + 2$, and $(A, \bar{a}) \rightarrow_{n,k} (B, \bar{b})$, then $((A, \bar{a}) \oplus (B, \bar{b}), b) \rightarrow_{n,k} (B, \bar{b}, b)$.*

Proof. We observe the key difference to the lemma above is that we have replaced $(C, \bar{c})$ with an extra copy of $(B, \bar{b})$, and added an extra distinguished element in B into the conclusion. This allows us to deal with an extra round of pebble movement, hence we may assume $n \leq k - l + 2$ rather than just

$n \leq k - l + 1$. We shall use the identity strategy to witness $(B, \bar{b}) \to_{n,k} (B, \bar{b})$, in other words, Duplicator's strategy in this game is simply to copy Spoiler's moves. We now describe how Duplicator can use this strategy along with a strategy $f : \mathbb{P}_{n,k}(A, \bar{a}) \to (B, \bar{b})$ to find a winning strategy to witness $((A, \bar{a}) \oplus (B, \bar{b}), b) \to_{n,k} (B, \bar{b}, b)$. As before, for the first $k - l$ rounds, we may assume Spoiler places pebbles each round, so as in the lemma above, Duplicator can run both of its strategies independently. In other words, if Spoiler places pebble i_j on an element a_j of A, Duplicator responds by placing pebble i_j, $f[(a_1, i_1), ..., (a_j, i_j)]$ where these are $(a_1, i_1), ..., (a_{j-1}, i_{j-1})$ are the pebbles placements on A. If Spoiler places pebble on b' of B, Duplicator simply responds by placing a pebble on b'. As remarked in the introduction, we may assume Spoiler does not place a pebble on the identified elements $a_1 (= b_1), ..., a_l (= b_l)$ common to both A and B as there are already pebbles placed there in the set up. We also observe this maintains a partial morphism even when the extra distinguished element b is included as Duplicator is following the identity strategy on B, and there are clearly no additional relations created containing any pebbles placed on A and b. After these first $k - l$ pebble placements, there are at most 2 rounds of pebble moves to consider.

Firstly, suppose Spoiler does not move a pebble from identified elements $a_1 (= b_1), ..., a_l (= b_l)$ for its next move. Identically to how we assume Spoiler will not place a pebble onto an element where a pebble is already placed, we may assume Spoiler will not move a pebble where one is already placed. Hence, Spoiler will not be able to move a pebble onto any of the identified elements on the first or second move, as they all already have pebbles on them. So now as before, Duplicator may follow both its strategies in parallel, following the strategy f and the identity strategy, for example if Spoiler moves pebble i_j onto $a_j \in A$, Duplicator responds as before $f[(a_1, i_1), ..., (a_j, i_j)]$ as before, where $(a_1, i_1), ..., (a_j, i_j)$ is the history of moves placed onto A. It does not matter if the pebble was moved from B across to A, since from the perspective of the strategy f it would simply be if Spoiler were placing a new pebble on A.

Thus, the only case left to consider is where Spoiler first moves a pebble off of one the identified elements (without loss of generality we shall assume it is pebble 1 on a_1). Suppose first that Spoiler first moves pebble 1 onto an element of B. In this case, Duplicator simply follows the identity strategy for the first move, placing the pebble onto the same element of B. If Spoiler places the a pebble back onto a_1, or anywhere else in B, Duplicator places the same pebble on that same element, which clearly maintains a partial morphism on both A and B. If Spoiler moves a pebble onto A, Duplicator plays as before using f, which again maintains partial morphisms. The trickiest case to consider is where Spoiler first moves pebble 1 onto $a' \in A$ and then a pebble i onto a_1 (i may or may not be equal to 1), which is the case from the Example 2.4.3. In the example, this caused an issue for Duplicator, since it needed to play $f[(a_1, 1), ..., (a', 1), (a_1, i)]$ to maintain a winning strategy on A, but play $a_1 (= b_1)$ to maintain a winning strategy on B, and these were not equal. The crucial difference is that in this case there is an extra distinguished element b, so there is it least 1 pebble i' already placed b, which is not an element of A. Thus, when Spoiler plays

$(a', 1)$, Duplicator may respond by playing $f[(a_1, 1), ..., (a', i')]$, since i' is not a pebble that was already placed on a. In other words, Duplicator can find a move that maintains a partial morphism that includes a_1, a', and all other elements with pebbles placed on them in A, rather than before where Duplicator could only find a partial morphism that did not necessarily include a_1. Now, if Spoiler plays their last move by moving a pebble onto a_1, Duplicator responds by moving that same pebble onto a_1, which maintains a partial morphism both on A and B. If Spoiler instead played somewhere other than a_1 with its last move, Duplicator can win as before either by following f or the identity strategy, depending on if Spoiler plays on A or B. The only caveat to this is that if Spoiler moves exactly pebble i onto $a'' \in A$, Duplicator responds with $f[(a_1, 1), ..., (a', 1), (a_1, i), (a'', 1)]$ since from the perspective of the strategy f, pebble i was already in use, however Duplicator can make use of pebble 1 now as a_1 no longer has a pebble on it. This has now covered all cases for up to two pebble moves, so we are done.

□

We remark here that the strategy in the above proof cannot be extended for $n > k - l + 2$, as there may not be more than 1 pebble placed on B in the first $k - l$ moves, so we cannot use the same trick again. What is needed in the general case is the following (recall that $F_{n,k,l}$ is a set of representatives for equivalences classes of $\leftrightarrows_{n,k}$ each finite and having an (n, k)-cover):

Conjecture 2.4.6. *There exists a choice of sets $F_{n,k,l}$ for every n, k, l such that for each $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$, and each $(C, \bar{c}, c) \in F_{n,k,l}$ such that $(A, \bar{a}) \to_{n,k} (C, \bar{c})$,and $(C, \bar{c}) \to (A, \bar{a})$, we have $((A, \bar{a}) \oplus (C, \bar{c}), c)) \to_{n,k} (C, \bar{c}, c)$.*

We have this in the case of $n \leq k - l + 2$, indeed, it follows immediately from the more general lemma above that any choice will do, but have been unable to prove it for the general case. We shall see how this would imply the (n, k)-preservation Conjecture, and thus give a proof of the (n, k)-preservation Theorem in the case $n \leq k - l + 2$. We emphasise that the following rely on this conjecture and are thus yet unproven except in this case.

In order to build (n, k)-extendable structures, it is convenient to test (n, k)-extendability on representative structures from $F_{n,k,l}$. Further, as we wish to find structures which are, in a sense, extensions of given structures, we find a test for (n, k)-extendability on structures of form $((A, \bar{a}) \oplus (C, \bar{c}), c)$, where A is fixed and $(C, \bar{c}) \in F_{n,k,l}$ as we will see below:

Lemma 2.4.7. *Assuming Conjecture 2.4.6, a structure A is (n, k)-extendable if for every:*

- *$l < k$,*
- *$0 < n' \leq n - l$,*
- *l-tuple $\bar{a}$ of A,*
- *$(C, \bar{c}, c) \in F_{n',k,l+1}$ such that $(C, \bar{c}) \to (A, \bar{a}) \to_{n'-1,k} (C, \bar{c})$,*

there exists $\alpha \in A$ such that

$$(A, \bar{a}, \alpha) \leftrightarrows_{n'-1,k} ((A, \bar{a}) \oplus (C, \bar{c}), c).$$

Proof. Let $\bar{a}$, n', and l be as above, and let $(B, \bar{b}) \in \mathcal{R}_\sigma(l)$ be such that $(A, \bar{a}) \leftrightarrows_{n,k} (B, \bar{b})$. As in the definition of (n, k)-extendability, we break into cases $l = k$ and $l < k$.

- First let $l < k$ and take some $\beta \in B$. Take $(C, \bar{c}, c) \in F_{n'-1,k,l+1}$ such that $(B, \bar{b}, \beta) \leftrightarrows_{n'-1,k} (C, \bar{c}, c)$. By the recursive characterisation of $\to_{n',k}$, we note that there is $a \in A$ such that $(B, \bar{b}, \beta) \to_{n'-1,k} (A, \bar{a}, a)$. Hence, by transitivity of $\to_{n'-1,k}$, we have $(C, \bar{c}, c) \to_{n'-1,k} (A, \bar{a}, a)$. Since $(C, \bar{c}, c) \in F_{n'-1,k,l+1}$ it has an $(n'-1, k)$-cover, so we get a morphism $(C, \bar{c}) \to (A, \bar{a})$. We also have $(A, \bar{a}) \to_{n'-1,k} (B, \bar{b}) \to (C, \bar{c})$ (the first arrow from the fact $(A, \bar{a}) \to_{n',k} (B, \bar{b})$ and the second from $(B, \bar{b}, \beta) \to_{n'-1,k} (C, \bar{c}, c)$) and can apply the hypothesis of the lemma. So let $\alpha \in A$ be such that $(A, \bar{a}, \alpha) \leftrightarrows_{n'-1,k} ((A, \bar{a}) \oplus (C, \bar{c}), c)$. It is immediate that $(B, \bar{b}, \beta) \to_{n'-1,k} ((A, \bar{a}) \oplus (C, \bar{c}), c)$ since $((A, \bar{a}) \oplus (C, \bar{c}), c)$ is a superstructure of $(C, \bar{c}, c)$. By Conjecture 2.4.6 we have $((A, \bar{a}) \oplus (C, \bar{c}), c) \to_{n'-1,k} (C, \bar{c}, c) \to_{n'-1,k} (B, \bar{b}, b)$. Combining these we get $(A, \bar{a}, \alpha) \leftrightarrows_{n'-1,k} (B, \bar{b}, \beta)$ as required.

- The structure of the second case is very similar. Suppose $l = k$ and let $\beta \in B$, $i \in \{1, ..., l\}$. Without loss of generality we may reorder the tuples $\bar{a}, \bar{b}$ so that $i = l(= k)$. Again let $(C, \bar{c}) \in F_{n',k,l}$ be such that $(B, \bar{b}[\beta/b_i]) \leftrightarrows_{n'-1,k} (C, \bar{c}, c)$. By identical reasoning to the previous case, we see that $(C, \bar{c}) \to (A, \bar{a} - a_l)$ (where $\bar{a} - a_l$ denotes the tuple $\bar{a}$ with the lth entry deleted). The argument now proceeds exactly as in the previous case.

□

We now have the means to build up (n, k)-extendable structures. In the following definition we let $\mathbb{U}$ denote the forgetful functor that sends some $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$ to its underlying structure $A \in \mathcal{R}_\sigma$.

Definition 2.4.8. *For $A \in \mathcal{R}_\sigma$, and a fixed choice of (n, k), let $\Sigma(A)$ be the colimit over all the canonical inclusion maps of form:*

$$A \to \mathbb{U}((A, \bar{a}) \oplus (C, \bar{c}))$$

for $(C, \bar{c}, c) \in F_{n',k,l}$ such that $(C, \bar{c}) \to (A, \bar{a})$, where $0 < n' \leq k - l$. Constructively, one may think of this A and "gluing" all possible structures $(C, \bar{c})$ to it along appropriate tuples.

Lemma 2.4.9. *For $A \in \mathcal{R}_\sigma$, $0 < n' \leq k - l$, $(C, \bar{c}, c) \in F_{n'-1,k,l}$ such that $(C, \bar{c}) \to (A, \bar{a})$, we have*

$$(\Sigma(A), \bar{a}, c) \leftrightarrows ((A, \bar{a}) \oplus (C, \bar{c}), c)$$

Proof. $((A, \bar{a}) \oplus (C, \bar{c}), c) \to (\Sigma(A), \bar{a}, c)$ is just an inclusion map. For the other way around, we note that $\Sigma(A)$ is a colimit so it suffices to give a coherent set of maps out of the objects $(A, \bar{a}') \oplus (C', \bar{c}')$ for appropriate $(C', \bar{c'})$. If $(C', \bar{c'}) = (C, \bar{c})$ and $\bar{a}' = \bar{a}$ we just take the identity map on $((A, \bar{a}) \oplus (C, \bar{c})$, otherwise; take the identity on A along with any map $(C, \bar{c}') \to (A, \bar{a}')$. It is straightforward to see that such maps all commute where necessary and respect the required distinguished elements of $\Sigma(A)$. □

We also have the following:

Lemma 2.4.10. *$\Sigma(A)$ is a co-retract of A, meaning there are maps $A \leftrightarrows \Sigma(A)$ that compose to give the identity on A. In addition, if $\bar{a}$ is any tuple of A, then $(A, \bar{a}) \leftrightarrows (\Sigma(A), \bar{a})$.*

Proof. For the map $A \to \Sigma(A)$, just take the inclusion map. The other way around, for each $(A, \bar{a}) \oplus (C, \bar{c})$, use the map given by the identity on A and any choice of map $(C, \bar{c}) \to (A, \bar{a})$. □

We have, in a sense, that $\Sigma(A)$ satisfies the (n, k)-extendability criteria, but for A rather than for itself. Thus we take the colimit, or countable union, of the structures $\Sigma^i(A)$ (for $i \in \mathbb{N}$) to obtain something akin to a fixed point of Σ, which will be (n, k)-extendable.

Definition 2.4.11. *Let $\Sigma^\omega(A)$ be the colimit over all inclusion maps of form:*

$$\Sigma^i(A) \to \Sigma^{i+1}(A)$$

for $i \in \mathbb{N}$. Or, more constructively, $\Sigma^\omega(A)$is the union $\bigcup_{i \in \mathbb{N}} \Sigma^i(A)$.

$\Sigma^\omega(A)$ enjoys similar properties to $\Sigma(A)$.

Lemma 2.4.12. *$\Sigma^\omega(A)$ is a co-retract of $\Sigma^i(A)$ for any $i \in \mathbb{N}$. In particular, if $\bar{a}$ is any tuple of $\Sigma^i(A)$, then $(\Sigma^i(A), \bar{a}) \leftrightarrows (\Sigma^\omega(A), \bar{a})$.*

Proof. For $\Sigma^i(A) \to \Sigma^\omega(A)$ we take the inclusion map. For the other way around, we need to specify coherent maps $\Sigma^j(A) \to \Sigma^i(A)$ for each $j \in \mathbb{N}$. If $i \leq j$ then just take the inclusion map. If $j > i$ then take compositions of the maps $\Sigma^j(A) \to \Sigma^{j-1}(A)$ as described in Lemma 2.4.10. It is straightforward to see that these maps all commute with the inclusion maps (thus giving us a map out of the colimit), and that the composition of these maps give the identity on $\Sigma^i(A)$. □

We can now show $\Sigma^\omega(A)$ is (n, k)-extendable (modulo Conjecture 2.4.6):

Theorem 2.4.13. *Assuming Conjecture 2.4.6, for any $A \in \mathcal{R}_\sigma$, $\Sigma^\omega(A)$ is (n, k)-extendable,*

Proof. We use Lemma 2.4.7 which gives us the simplest test for (n, k)-extendability. Suppose $0 < n' < k - l$, $l < k$, $\bar{a}$ is an l-tuple of $\Sigma^\omega(A)$, and $(C, \bar{c}, c) \in F_{n'-1,k,l+1}$ is such that $(C, \bar{c}) \to (\Sigma^\omega(A), \bar{a})$. Recall that all structures in $F_{n'-1,k,l+1}$ are finite, so the map $(C, \bar{c}) \to (\Sigma^\omega(A), \bar{a})$ is in fact a map $(C, \bar{c}) \to (\Sigma^i(A), \bar{a})$ for some $i \in \mathbb{N}$. Thus, we have a map $(\Sigma^{i+1}(A), \bar{a}, c) \leftrightarrows (((\Sigma^i(A), \bar{a}) \oplus (C, \bar{c})), c)$. Combining this with the retraction maps from $\Sigma^\omega(A)$, we get

$$(\Sigma^\omega(A), \bar{a}, c) \leftrightarrows (\Sigma^{i+1}(A), \bar{a}, c) \leftrightarrows (((\Sigma^i(A), \bar{a}) \oplus (C, \bar{c})), c) \leftrightarrows (((\Sigma^\omega(A), \bar{a}) \oplus (C, \bar{c})), c)$$

(where the last $\leftrightarrows$ is the retraction maps $\Sigma^i(A) \leftrightarrows \Sigma^\omega(A)$ paired with identity map on C) as required by the conditions of the lemma. □

Thus we may take $\Sigma^\omega(A)$ as $\tilde{A}$ and $\Sigma^\omega(\mathbb{P}_{n,k}A)$ as $\widetilde{\mathbb{P}_{n,k}A}$ in the argument presented at the beginning of the section to obtain a proof of the (n, k)-preservation Conjecture (again, we emphasise this is only in the cases where Conjecture 2.4.6 holds).

2.5 Possible Solutions

We have proved so far that the (n, k)-preservation Conjecture for the case $n \leq k + 2$ holds (and it is also trivially true in the case $k = 1$ for arbitrary n). We also claim it is very easy to extend the work above to formulas with constants. The problem remains open for the general pair. We outline some possible lines to solve the general case:

- Prove Conjecture 2.4.6 for arbitrary (n, k). For the cases $n \leq k - l + 1$ we actually have a more general result which implies it (Lemma 2.4.4), which is not true for arbitrary (n, k). There is some freedom in the choices of sets $F_{n,k,l}$ and it may be that choosing them to have structures of a certain property helps.

- A proof of Conjecture 2.4.6 would in fact give us, for any (n, k), a way of producing an (n, k)-extendable co-retract of any given structure A. As commented at the beginning of the section, this is much stronger than necessary, we only need some pair of co-retracts $\tilde{A}, \widetilde{\mathbb{P}_{n,k}A}$ (of $A, \mathbb{P}_{n,k}A$ respectively) such that $\tilde{A} \equiv_{n,k} \widetilde{\mathbb{P}_{n,k}A}$. It might be possible to tweak the construction shown above to this end. For instance, one can define a structure A to be (n, k)-homomorphically extendable, by replacing $(A, \bar{a}) \to_{n',k} (B, \bar{b})$ with $(A, \bar{a}) \to (B, \bar{b}, b) \to_{n',k} (A, \bar{a})$ everywhere in the hypotheses and conclusion of (n, k)-extendability. One can define (n, k)-cohomomorphically analogously. It will then be the case that if $A \to B \to_{n,k} A$, A is (n, k)-homomorphically extendable and B is (n, k)-cohomomorphically extendable, then $A \equiv_{n,k} B$. We can see in our picture above that $\Sigma^\omega(\mathbb{P}_{n,k}A) \to \Sigma^\omega(A) \to_{n,k} \Sigma^\omega(\mathbb{P}_{n,k}A)$, and the above material shows $\Sigma^\omega(\mathbb{P}_{n,k}A)$ is (n, k)-homomorphically extendable (we avoid the problem of Conjecture 2.4.6, since we work with proper morphisms rather than $\to_{n,k}$). This gives a slightly different challenge of trying to find an (n, k)-cohomomorphically extendable co-retract of A instead.

Chapter 3

A Modal Adjunction

3.1 Overview

A question to ask about the $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$ comonads is how is it they deal so naturally with their respective one-way homomorphism games but not so naturally with the two way isomorphism games. The answer to this question lies in negated atomic relations. It is a simple observation that, given a structure $A \in \mathcal{R}_\sigma$, you describe A completely by giving the interpretations of R in A for each $R \in \sigma$, or dually, by giving the interpretations of $\neg R$ in A for each $R \in \sigma$ (one could in fact formalise this using an opposite category to $\mathcal{R}_\sigma$). However, the structures $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$ do not respect this duality. To see this clearly, suppose $R, S \in \sigma$ are binary relations, and A interprets R, S as negations of one another. Now for any distinct $a_1, a_2, a_3 \in A$, consider sequences $s = [a_1, a_2]$ and $s' = [a_3]$ from $\mathbb{E}_n A$. By the construction of $\mathbb{E}_n$, we know that for any binary relation $R \in \sigma$, $\mathbb{E}_n A \models \neg R(s, s')$, but also $\mathbb{E}_n A \models \neg S(s, s')$, hence it is not possible for $\mathbb{E}_n A$ to respect a negated relation. From another perspective, recall that $\mathbb{E}_n A$ is constructed to model positions in the Ehrenfeucht-Fraisse game. From the perspective of the Ehrenfeucht-Fraisse game, the query $R(s, s')$ is simply an invalid query, as this situation cannot occur in a game since s and s' are not prefixes of one another. However since $\mathbb{E}_n A$ is also a relational structure, it must answer the query with a "no" rather than saying the query is invalid. The problem being is that this "no" is not the negation of some "yes". This works well for the homomorphism game, where Spoiler is not allowed to pose query's involving negated relations, but requires extra considerations in the back and forth game, where negated relations play the same role as non-negated relations.

Whilst back and forth games have been given a treatment in [6] using spans of open morphisms, another natural way to solve this is to consider a different type of structure, where access between elements can be carefully restricted, to prevent the posing of "invalid" queries. We can use a modal, or Kripke, structure for this, making use of the fact that $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$ come ready equipped with a tree-order.

The first section of this chapter builds all of the constituents of an adjunction. First we describe, for any structure A and numbers n, k, modal structures we refer to as $\mathbb{M}_n A$ and $\mathbb{M}_{n,k} A$. The transition relations are built using the natural tree structure of $\mathbb{E}_n A$ and $\mathbb{P}_{n,k} A$, with the latter having i transition relations to account for each of the possible pebble moves. The structures are endowed with propositions that allow each element of them, which each correspond to some $s = [a_1, ..., a_i] \in \mathbb{E}_n A$ or $s \in \mathbb{P}_{n,k} A$, to encode all relations between $a_1, ..., a_i \in A$. We then show that given this definition, if we translate formulas appropriately into modal ones, we can preserve the satisfaction relation between formulas and structures, which allows us to convert the $\equiv_{n,k}$ relation between first-order structures into modal bisimulation.

The next step is to find the adjoint functors to $\mathbb{M}_{n,k}$ and $\mathbb{M}_n$. In order to do this, we first identify an appropriate modally definable subcategory of modal structures to be the domain, which also contains the image of $\mathbb{M}_n$ and $\mathbb{M}_{n,k}$. Once we have the right domain, the needed functor out of it can simply be a forgetful style functor, as modal structures in our subcategory can be viewed as a first-order structure with a forest cover or (n, k)-cover, so we need only forget the cover to find a first-order structure. Finally, we see that these modally definable subcategories are in fact equivalent to the Eilenberg-Moore categories for $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$, by making use of a result from [6], which shows that $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$ arise as adjunctions between the category of first-order structures and a subcategory of the category of modal structures.

In the next section we apply our findings to understand a key part of the proof of Courcelle's Theorem. The problem is to take a first order structure with a (n, k)-cover and output a tree which can interpret MSO formulas on that structure. Now given a structure with an (n, k)-cover, we have some modal structure T in our modally definable subcategory such that the forgetful functor applied to T gives back our original structure, so T is a reasonable candidate. However, we observe that unlike the functor $\mathbb{M}_{n,k}$, our forgetful functor does not preserve the satisfaction relation between formulae and structures for first-order logic, let alone MSO. Instead, we prove that viewing T as a directed, labelled graph, allows us to preserve this satisfaction relation for MSO formulae, if we translate them correctly to be interpreted by the graph. Thus, we find a functorial solution to the problem, which is a step towards potentially giving a full categorical proof of Courcelle's Theorem in the future.

3.2 Building the Adjunction

Throughout this chapter, we shall give definitions and explanations for how both the adjunctions related to $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$ are developed, using the case of $\mathbb{E}_n$ as a simpler example to discuss before working through $\mathbb{P}_{n,k}$, though give proofs only for the $\mathbb{P}_{n,k}$ case.

Definition 3.2.1. *For $A \in \mathcal{R}_\sigma$, we define an associated rooted modal structure*

$\mathbb{M}_n A$. Each such structure will be over the same signature, with a single modality M, and atomic propositions $p_{R,t}$ for every $R \in \sigma$, with t ranging across tuples of length equal to the arity of R, with entries from $1, ..., n$.

- *$\mathbb{M}_n A$ shall have universe $\mathbb{E}_n A \cup \{[\ ]\}$, where $[\ ]$ is thought of as the empty sequence, and will be the root of $\mathbb{M}_n A$.*
- *The transition relation is given by sMs' if and only if $s' = s[a]$ for some $a \in A$. This includes the case $s = [\ ]$, and s' is a sequence of length 1.*
- *We interpret atomic propositions as follows. Firstly, if $s \in \mathbb{M}_n A$ write s_i for the ith entry of s, and write t_i for the ith entry of some tuple t. Now given some $s \in \mathbb{M}_n A$, and proposition $p_{R,t}$, where R is m-ary we have $\mathbb{M}_n A, s \models p_{R,t}$ if and only if $A \models R(s_{t_1}, ..., s_{t_m})$. If some t_i is greater than the length of s, then s_{t_i} will not exist, and in this case we take $\mathbb{M}_n A, s \models \neg p_{R,t}$.*

The notation in the definition above is somewhat fiddly, though the underlying idea is simple: the atomic propositions on $s = [a_1, ..., a_l]$ record all the relations occurring between the elements $a_1, ..., a_l$. To give a concrete example, if $s = [a_1, a_2]$, R is some binary relation, and $t = (2, 2)$, then $\mathbb{M}_n A, s \models p_{R,t}$ if and only if $A \models R(a_2, a_2)$.

We can actually translate formulae as well as structures. Throughout this section, all formulas will be equality free.

Definition 3.2.2. *Given ϕ in $\mathcal{L}_n$, we define $\mathbb{M}_n\phi$ via mapping each part of it as follows:*

- *$R(x_{t_1}, ..., x_{t_m}) \mapsto p_{R,t}$*
- *Connectives $\wedge, \vee, \neg \mapsto \wedge, \vee, \neg$*
- *Quantifiers $\exists x, \forall x \mapsto \Diamond, \Box$.*

We remark here that the translation $\mathbb{M}_n$ for formulas does not depend on the choice of n, we simply use this notation as it reads similarly to previous notations, and to distinguish it from the translations of formulas in the pebbling case.

One slight irritation at this point which is apparent in the definition above is that when an FO formula uses a quantifier one cannot currently distinguish from the translated formula which variable was bound, however this can be solved by insisting the variables are bound in a specific order (so which variable is bound can be recovered by the shape of the formula)

Definition 3.2.3. *Recall a variable quantifier q is in the direct scope of a quantifier q' in a sentence if and only if whenever some other quantifier q'' has q in its scope, it also has q' in its scope.*
We will say a sentence ϕ is "well-written" if and only if the only variable bound

in the scope of no other quantifiers is x_1, and any variable bound in the direct scope of a quantifier binding x_i is x_{i+1}.
A well-written formula $\phi(\bar{x})$ is one that has free variables among $x_1, ..., x_l$ for some l, has x_{l+1} as the only variable bound in the scope of no other quantifiers, and satisfies any variable bound in the direct scope of a quantifier binding x_i is x_{i+1}.

Example 3.2.4. *The formula $\exists x_1((\exists x_2 R(x_1, x_2)) \wedge (\exists x_3 R(x_3, x_1)))$ is not "well-written", since x_3 is in the direct scope of x_1. However it is equivalent to $\exists x_1((\exists x_2 R(x_1, x_2)) \wedge (\exists x_2 R(x_2, x_1)))$ and $\exists x_1((\exists x_2 R(x_1, x_2)) \wedge (\exists x_2 \exists x_3 R(x_3, x_1)))$ which both are well-written. We would like to point out that, despite the name, "well-written" is certainly not a remark on the aesthetics or readability of a formula!*

When a formula is well-written, one could in fact dispense of writing the variable bound in a quantifier, as they are implicit in where the quantifier appears in the formula. It is immediate that every FO formula is equivalent to a well-written formula, simply by relabelling variables, and further that this relabelling preserves **quantifier rank**, though this is not the case if you restrict the number of variables, as shown in an example in the introduction where a single variable can be bound multiple times in the same formula.

Theorem 3.2.5. *If $\phi \in \mathcal{L}_n$ is a well-written, then $A \models \phi \iff \mathbb{M}_n A, [\ ] \models \mathbb{M}_n \phi$.*
If $\phi(\bar{x}) \in \mathcal{L}_n(l)$ is well written, $A, (a_1, ..., a_l) \models \phi \iff \mathbb{M}_{n+l} A, [a_1, ..., a_l] \models \mathbb{M}_n \phi$.

This Theorem can be seen simply by unpacking the relevant definitions, but we will give a proof in the more general case involving $\mathbb{P}_{n,k}$. Insisting on ϕ being well written ensures the atomic relations in ϕ are correctly interpreted by the corresponding atomic propositions in $\mathbb{M}_n \phi$.

We now give the same treatment to $\mathbb{P}_{n,k}$, to define a structure and a translation $\mathbb{M}_{n,k}$. The fundamental ideas are the same, though we require the use of extra modalities to keep track of the pebble indices present in $\mathbb{P}_{n,k}$. Recall that elements $s \in \mathbb{P}_{n,k} A$ are of form $s = [(a_1, e_1), ..., (a_m, e_m)]$ where the a_i are entries from A and the e_i are integers from $1, ..., k$. We call the e_i's pebble indices.

Definition 3.2.6. *For $A \in \mathcal{R}_\sigma$, we define an associated rooted modal structure $\mathbb{M}_{n,k} A$. They shall all be modal structures in the same signature, with modalities $M_1, ..., M_k$, and atomic propositions $p_{R,t}$ for every $R \in \sigma$, with t ranging across tuples of length equal to the arity of R, with entries from $1, ..., k$.*

- *$\mathbb{M}_{n,k} A$ shall have universe $\mathbb{P}_{n,k} A \cup \{[\ ]\}$, where $[\ ]$ is thought of as the empty sequence, and will be the root of $\mathbb{M}_{n,k} A$.*

- *The transition relation is given by sM_is' if and only if $s' = s[(a,i)]$ for some $a \in A$. This includes the case $s = [\,]$, and s' is a sequence of form $[(a,i)]$.*
- *We interpret atomic propositions as follows: Firstly, if $s \in \mathbb{M}_{n,k}A$ write s_i for last A-entry of s with pebble index i (where it exists), and write t_i for the ith entry of some tuple t. Now given some $s \in \mathbb{M}_nA$, and proposition $p_{R,t}$, where R is m-ary we have $\mathbb{M}_nA, s \models p_{R,t}$ if and only if $A \models R(s_{t_1}, ..., s_{t_m})$. If for some t_i, s_{t_i} does not exist, we take $\mathbb{M}_{n,k}A, s \models \neg p_{R,t}$.*

Once again, the notation is fiddly, but the underlying concept is straight foward. Recall for some $s = [(a_1, e_1), ..., (a_m, e_m)]$, it represent the positions of pebbles in a pebble game; s_i is the element of A on which pebble i is placed. The atomic propositions record the relations between the elements of A on which the pebbles are currently placed. For example, if R is a binary relation, and $t = (1,2)$, $p_{R,t}$ asks in words, does the relation R hold between the element on which pebble 1 is currently placed (written s_1) and the element on which pebble 2 is currently placed (written s_2)?

Remark 3.2.7. *In order to see $\mathbb{M}_n$ as a simpler case of $\mathbb{M}_{n,k}$, we can identify structures of form $\mathbb{M}_nA$ as a substructures of $\mathbb{M}_{n,n}A$, by considering only those $s \in \mathbb{M}_{n,n}$ with their pebble indices in ascending order and starting with 1, ie elements of form $[(a_1,1),(a_2,2)...]$. This makes sense of the "well-written" notion for formulae in the case of $\mathbb{M}_n$, where we force the variables quantified to appear in ascending order, and start with 1.*

As promised, there is an analogous Theorem for translation in the pebbling case. This time, a formula will be "well-written" if it only uses variables (both free and bound) from $x_1, ..., x_k$. We do not require them to be bound in a certain order, as the atomic propositions in $\mathbb{M}_{n,k}\phi$ have more flexibility than in the previous case. However, it does make stating the following slightly more complicated due to possible permutations of $1, ..., k$.

Definition 3.2.8. *For a well-written $\phi \in \mathcal{L}_{n,k}$, we define $\mathbb{M}_{n,k}\phi$ identically to $\mathbb{M}_n\phi$, except for the case of translating quantifiers. Here we send $\exists x_i, \forall x_i \mapsto \Diamond_i, \Box_i$*

Theorem 3.2.9. *If $\phi \in \mathcal{L}_{n,k}$ is a well-written, then $A \models \phi \iff \mathbb{M}_{n,k}A, [\,] \models \mathbb{M}_{n,k}\phi$.*
For some $\phi(\bar{x}) \in \mathcal{L}_{n,k}(l)$, with free variables occurring among $x_1, ..., x_k$ (and with l free variables), let its free variables be denoted $x_{t_1}, ..., x_{t_l}$, and recall for some sequence s we write s_i for the last A entry of s with pebble index i. Then $A, (s_{t_1}, ..., s_{t_l}) \models \phi \iff \mathbb{M}_{m,k}A, s \models \mathbb{M}_{n,k}\phi$, so long as $n + length(s) \leq m$ (the length of s being the number of entries from A it has).

Proof. An initial observation to make is that $\mathbb{M}_{n,k}$ preserves boolean combinations of formulae by definition, for instance we have $\mathbb{M}_{n,k}(\phi \wedge \psi) := \mathbb{M}_{n,k}\phi \wedge$

$\mathbb{M}_{n,k}\psi$. Thus, when faced with a formula that is a boolean combination of simpler formulae we need only consider the simpler parts. We will proceed with a proof by induction on the **quantifier rank** of a formula. First, consider a formula with **quantifier rank** 0, ie a formula with no quantifiers, so it is a boolean combination of atomic relations between free variables. By the initial observation above, we need only consider a single such atomic relation. This case now follows directly from the construction of the atomic propositions $p_{R,t}$ (indeed, they were defined exactly by this property!). Now consider a formula ϕ of **quantifier rank** $n+1$. Recall that a formula of of **quantifier rank** $n+1$ can always be written as a boolean combination of formulae of type $\exists x_i \psi$, where the ψ's have **quantifier rank** at most n. Again, since $\mathbb{M}_{n,k}$ preserves boolean combinations, we need only consider the case ϕ is of form $\exists x_i \psi$ for some i and some ψ of **quantifier rank** at most n (in fact we can assume ψ has **quantifier rank** exactly n, else ϕ would have **quantifier rank** at most n and we could simply use the inductive hypothesis). Without loss of generality, for ease of writing the proof, we shall assume the free variables of ψ are $x_1, ..., x_l$ and the new variable bound in ϕ is x_l (we assume here ϕ binds a variable that occurs free in ψ, the other case is trivial). Let $s \in \mathbb{M}_{m,k}A$ satisfy $n + 1 + \text{length}(s) \leq m$, and have entries with pebble indices $1, ..., l-1$, then we must show $A, (s_1, ..., s_{l-1}) \models \phi \iff \mathbb{M}_{m,k}A, s \models \mathbb{M}_{n,k}\phi$. We prove the forward direction as the converse is symmetric. Suppose $A, (s_1, ..., s_{l-1}) \models \phi$. Then there is some $a \in A$ such that $A, (s_1, ..., s_{l-1}, a) \models \psi$. Now consider $s' = s[(a, l)]$. We have $s'_i = s_i$ for $i = 1, ..., l-1$, and $s'_l = a$, so we might write this $A, (s'_1, ..., s'_l) \models \psi$. Since the **quantifier rank** of ψ is n, and $n + length(s') = n + 1 + length(s) \leq m$ we may apply our inductive hypothesis, to obtain $\mathbb{M}_{m,k}A, s' \models \mathbb{M}_{n,k}\psi$. Now since $sM_l s'$, we have $\mathbb{M}_{m,k}A, s \models \Diamond_l \mathbb{M}_{n,k}\psi (= \mathbb{M}_{n,k}\phi)$ as required. □

Now that we have translated formulae we can consider (n-ary) bisimulations. For modal structures T, T' we shall write $T, s \sim_n T', s'$ if for all modal sentences ϕ of **quantifier rank** (where **quantifier rank** for modal sentences is defined as for FO sentences but using $\Box, \Diamond$ in place of $\forall, \exists$) at most n, $T, s \models \phi \iff T', s' \models \phi$. This is actually equivalent to full bisimulation in the cases we are considering since our modal structures are trees of depth at most n. Similarly to the case of $\equiv_n$ and $\equiv_{n,k}$ this can be equivalently described using a recursive definition, or with back and forth games. For an exploration of such games and the general properties of bisimulation, within finite model theory see [25] or [29]. One could prove the following simply using games, but we prefer to use formulae in order to make use of the above properties of the translations we just proved.

Theorem 3.2.10. *For $A, B \in \mathcal{R}_\sigma$:*

- $A \equiv_n B \iff \mathbb{M}_n A, [\,] \sim_n \mathbb{M}_n B, [\,]$. $A \equiv_{n,k} B \iff \mathbb{M}_{n,k}A, [\,] \sim_n \mathbb{M}_{n,k}B, [\,]$.
- *If* $(a_1, ..., a_l)$,$(b_1, ..., b_l)$ *are distinct tuples from* A, B *respectively, then* $A, (a_1, ..., a_l) \equiv_n B, (b_1, ..., b_l) \iff \mathbb{M}_{n+l}A, [a_1, ..., a_l] \sim_n \mathbb{M}_{n+l}B, [b_1, ..., b_l]$.

If s, s' are from $\mathbb{M}_{n,k}A, \mathbb{M}_{n,k}B$ respectively, and there is some l-tuple t such that $s_{t_i} = a_i$ and $s_{t_i} = b_i$ for each $i = 1, ..., l$, and s, s' have no entries with pebble indices except those occurring in t, we have $A, (a_1, ..., a_l) \equiv_{n',k} B, (b_1, ..., b_l) \iff \mathbb{M}_{n,k}A, s \sim_{n',k} \mathbb{M}_{n,k}B, s'$ where n' is the minimum among $n - length(s)$ and $n - length(s')$.

Proof. The backwards direction of each of these follow immediately from the Theorem above, using the translations $\mathbb{M}_n$ and $\mathbb{M}_{n,k}$ respectively (making use of the fact that every FO formula is equivalent to one that is "well-written", so can be fruitfully translated). In the forth direction, we first need guarantee that all modal sentences are in the image of $\mathbb{M}_n$ or $\mathbb{M}_{n,k}$. Whilst this is true, there is small subtlety to be addressed; some interpretations of modal sentences correspond to first order formulae with free variables in them without an interpretation. Consider for instance a unary relation P, and the modal sentence $p_{P,1}$. In the $\mathbb{M}_n$ case this asks of a sequence s whether its first entry satisfies P, and is the translation of $P(x_1)$. However, the fact $\mathbb{M}_nA, [\,] \models \neg p_{P,1}$ would correspond to the statement $A \models \neg P(x_1)$ which is nonsense, as there is no element to interpret x_1. A simple resolution to this is as to allow ourselves to use such "uninterpreted" free variables within our formulas, but just interpret any atomic relations containing them as false. This is clearly equivalent to FO (it just adds a $\perp$ symbol), and the translations, and Theorems proving their correctness, go through as before (as any atomic proposition that would reference an uninterpreted free variable was defined to be false). Now the translations $\mathbb{M}_n$ and $\mathbb{M}_{n,k}$ are surjective onto modal sentences as required. □

Now we have fully completed the translations. Incidentally, one can transport all of FO faithfully into modal sentences, since any FO formula has some finite **quantifier rank** and **variable count**. Elementary equivalence can also be written as a sequence of bisimulations, using $\mathbb{M}_i$ for each $i \in \mathbb{N}$.

We now return to categorical considerations. We claim that $\mathbb{M}_n$ and $\mathbb{M}_{n,k}$ are functors with left adjoints, and the composition of them with their adjoints give rise to the comonads $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$ respectively. Firstly we should state the codomain of each mapping and prove they are functors. In the following, we shall say the **height** of an element s is the number of transitions needed to reach s from $[\,]$.

Definition 3.2.11. *The target codomain $\mathcal{M}_n$ of the mapping $\mathbb{M}_n$ shall have objects that are rooted modal trees $T, [\,]$, with a single modality M, of depth at most n, with atomic propositions of form $p_{R,t}$ as described for structures of form $\mathbb{M}_nA$, satisfying the following two properties, for any tuple t with entries from $1, ..., n$ and R with arity equal to the length of t:*

1. *If $s \in T$ has height m and t contains some $m' > m$, then $T, s \models \neg p_{R,t}$*
2. *If sMs' in T, and no entry of t is greater than the height of s, then $T, s \models p_{R,t} \iff T, s' \models p_{R,t}$.*

Morphisms in $\mathcal{M}_n$ are rooted morphisms of modal trees, in other words, functions $f : T \to T'$ satisfying the following:

- *f sends the root of T to the root of T'*
- *If $T, s \models p$ for some proposition p and $s \in T$ then $T', f(s) \models p$*
- *If sMs' in T then $f(s)Mf(s')$ in T'.*

We remark here that all modal structures of form $\mathbb{M}_n A$ satisfy conditions (1) and (2) and are thus objects in $\mathcal{M}_n$ as required. The category $\mathcal{M}_{n,k}$ is defined very similarly:

Definition 3.2.12. *The target codomain $\mathcal{M}_{n,k}$ of the mapping $\mathbb{M}_{n,k}$ shall have objects that are rooted multi-modal trees $T, [\,]$, with modalities M_i for $i = 1, ..., n$, of depth at most n, with atomic propositions of form $p_{R,t}$ as described for structures of form $\mathbb{M}_{n,k}A$, satisfying the following two properties for any tuple t with entries from $1, ..., n$ and R with arity equal to the length of t:*

1. *If there is no i transition on the unique path from the root to some $s \in T$, and t is a tuple with i as an entry, then $T, s \models \neg p_{R,t}$*
2. *If sM_is' in T, and i is not an entry of t, then $T, s \models p_{R,t} \iff T, s' \models p_{R,t}$*

Morphisms in $\mathcal{M}_n$ are rooted morphisms of (multi-)modal trees, in other words, functions $f : T \to T'$ satisfying the following:

- *f sends the root of T to the root of T'*
- *If $T, s \models p$ for some proposition p and $s \in T$ then $T', f(s) \models p$*
- *For each i, if sM_is' in T then $f(s)M_if(s')$ in T'*

Once again, structures of form $\mathbb{M}_{n,k}$ satisfy conditions (1) and (2), by how the atomic relations are defined on them. In light of an above remark (where we note that we can view structures of form $\mathbb{M}_n A$ as substructures of form $\mathbb{M}_{n,n}A$ where the pebble indices are in ascending order) one can view the conditions (1) and (2) for the objects of $\mathcal{M}_n$ as special cases of the conditions (1) and (2) for the objects of $\mathcal{M}_{n,k}$. At this point, these are simply some arbitrary conditions that are met by objects in the image of $\mathbb{M}_n$ and $\mathbb{M}_{n,k}$. We will see why these conditions are necessary when we consider the adjoints of $\mathbb{M}_n$ and $\mathbb{M}_{n,k}$.

Remark 3.2.13. *Conditions (1) and (2) are also modally definable. We will write out the sentences in the $\mathbb{M}_{n,k}$ case. Let $T, [\,]$ be a modal structure.*

- *Condition (1) can be written as a disjunction $T, [\,] \models \bigwedge \phi$, where ϕ ranges over formulae of type $\Box_{i_1} ... \Box_{i_m} \neg p_{R,t}$, where t contains some entry that does not occur among $i_1, ..., i_m$ (and of course m ranges from $0, ..., n$ and the i's range from $1, ..., k$, and over all appropriate pairs R, t)*

- *Condition (2) can be written as the disjunction* $T, [\,] \models \bigwedge \phi$*, where* ϕ *ranges over formulae of type* $\Box_{i_1} ... \Box_{i_{m-1}}(p_{R,t} \iff \Box_{i_m} p_{R,t})$*, where* i_m *does not occur in* t*. As before,* m *ranges from* $0, ..., n$*, the* i*'s (except for* i_m*) range from* $1, ..., k$*, and we range of all possible pairs* R, t*.*

Pointwise, the action of $\mathbb{M}_n$ and $\mathbb{M}_{n,k}$ on morphisms are identical to those of $\mathbb{E}_n$, and $\mathbb{P}_{n,k}$ (except for mapping the empty sequence also!):

Definition 3.2.14. *Suppose* $f : A \to B$ *is a morphism in* $\mathcal{R}_\sigma$*. In both cases, we give a recursive definition.* $\mathbb{M}_n f$ *is defined by:*

- $\mathbb{M}_n f([\,]) := [\,]$
- *For* $s \in \mathbb{M}_n A$ *that is not the root, then* $s = s'[a]$ *for some* $s' \in \mathbb{M}_n A$ *of lower height. We set* $\mathbb{M}_n f(s) = \mathbb{M}_n f(s')[f(a)]$*.*

Similarly for $\mathbb{M}_{n,k} f$*, we define:*

- $\mathbb{M}_{n,k} f([\,]) := [\,]$
- *For* $s \in \mathbb{M}_{n,k} A$ *that is not the root, then* $s = s'[(a, i)]$ *for some* $s' \in \mathbb{M}_{n,k} A$ *of lower height. We set* $\mathbb{M}_{n,k} f(s) = \mathbb{M}_{n,k} f(s')[(f(a), i)]$*.*

Since these action are pointwise the same as the cases of $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$, we know they will satisfy the functoriality requirements of identity and compositionality. We need only check $\mathbb{M}_n$ and $\mathbb{M}_{n,k}$ are actually morphisms in $\mathcal{M}_n$ and $\mathcal{M}_{n,k}$. We will do the latter case.

Lemma 3.2.15. *If* $f : A \to B$ *is a morphism in* $\mathcal{R}_\sigma$ *then* $\mathbb{M}_{n,k} f : \mathbb{M}_{n,k} A \to \mathbb{M}_{n,k} B$ *is a morphism in* $\mathcal{M}_{n,k}$*.*

Proof. We verify the three conditions of a modal morphism laid out in the definition above. Firstly, $\mathbb{M}_{n,k} f$ sends roots to roots by definition. Secondly, suppose $\mathbb{M}_{n,k} A, s \models p_{R,t}$, we need to show $\mathbb{M}_{n,k} B, \mathbb{M}_{n,k} f(s) \models p_{R,t}$ We will first prove by induction (on the height of s) the following subclaim: for each $i = 1, ...k$, if s_i exists, then $f(s_i) = f(s)_i$ (recall s_i is the last A-entry in the sequence s with pebble index i). If $s = [\,]$, then s_i does not exist for any i, so the statement is vacuously true. Now suppose $s = s'[(a, j)]$. By the inductive hypothesis, we have that $f(s'_i) = f(s')_i$ for every i. Now for $i \neq j$, if s_i exists, then $s_i = s'_i$, so $f(s_i) = f(s'_i) = f(s')_i = (f(s')[(f(a), j])_i = f(s)_i$. If $i = j$ then clearly $f(s_i) = f(s)_i = f(a)$, so the subclaim is proved. Now let us write $t = (t_1, ..., t_m)$. Recall $\mathbb{M}_{n,k} A, s \models p_{R,t}$ if and only if $A \models R(s_{t_1}, ..., s_{t_m})$. Now since f is a homomorphism, we have $B \models R(f(s_{t_1}), ..., f(s_{t_m}))$. By the subclaim, we now have $B \models R(f(s)_{t_1}, ..., f(s)_{t_m})$, which is true if and only if $\mathbb{M}_{n,k} B, f(s) \models p_{R,t}$, as required. Thirdly and finally, we know that $sM_i s'$ in $\mathbb{M}_{n,k} A$, if and only if $s' = s[(a, i)]$, for some $a \in A$. By definition, $f(s') = f(s)[(f(a), i)]$, hence $f(s) M_i f(s')$ as required. ☐

We will now consider maps in the opposite direction. Given a modal tree T, we can simply "forget" its modal structure to create a first order structure.

Although in the case of structures of type $\mathbb{M}_{n,k}A$, each element of a corresponding first order structure is duplicated many times in the modal structure, in the general case we cannot assume this is so, so each element of a modal tree will create a new point in the first order structure. For a formal definition, we first require some notation, which generalises our earlier notation s_i to pick out elements of a sequence s:

Definition 3.2.16. *For $T, [\,] \in \mathcal{M}_n$, and $s \in T$, and i less than or equal to the height of s, write s_i for the ith element along the unique path from $[\,]$ to s (counting $[\,]$ as s_0). For $T, [\,] \in \mathcal{M}_{n,k}$ and $s \in T$, write s_i for the last element on the unique path from $[\,]$ to s that is reached via an M_i transition.*

Definition 3.2.17. *Given a modal tree $T, [\,] \in \mathcal{M}_n$, we create a corresponding structure $\mathbb{U}_n T \in \mathcal{R}_\sigma$. $\mathbb{U}_n T$ shall have universe $T - \{[\,]\}$, and relations given by the following rule: $\mathbb{U}_n T \models R(s^1, ..., s^m)$ if and only:*

- *There is some j such that $s^1, ..., s^m$ all lie along the unique path from $[\,]$ to s^j.*
- *$T, s^j \models p_{R,t}$, where t is the unique tuple of length m such that $s^j_{t_i} = s_i$ for each i.*

For $T, [\,] \in \mathcal{M}_{n,k}$, we define $\mathbb{U}_{n,k} T$ identically (with the same universe, and the same rule for the relations, though note the notation means something slightly different in this case). In the previous case, the existence of a tuple t with the required properties is guaranteed if each $s^1, ..., s^j$ lie on the same branch, however it is not the case here (it is also required that, say, if s^1 is reached via an M_i transition, there are no further M_i transitions used in between s^1 and s^j).

For an intuitive picture of some tree $T, [\,]$, we shall imagine the root to be at the bottom of the tree. Hence, if s' is on the unique path from s to the root, we will refer s' as "below" s, and s as "above" s'. Here we begin to make sense of the conditions (1) and (2) of structures in $\mathcal{M}_n$ and $\mathcal{M}_{n,k}$ (from Definitions 3.2.11 and 3.2.12 respectively). In the case of $\mathbb{U}_n$ for example, we can query some $s \in \mathbb{U}_n A$ for atomic relations between all elements "below" it (ie on the unique path from the root to s). Now, one could equally query any element s' "above" s for the same information (since any elements "below" s would also be below s'). Condition (2) ensures we get the same answer in each case, which will be needed for functoriality.

Definition 3.2.18. *For $f : T \to T'$, define $\mathbb{U}_n f$ by simple restriction. Recall that, as a set, $\mathbb{U}_n T := T - [\,]$. So for $f : T \to T'$ a modal morphism, we set $\mathbb{U}_n f(s) := f(s)$. Note, by the conditions of a modal morphism (and the fact that all our modal structures are assumed to be trees), if $s \neq [\,]$ then $f(s) \neq [\,]$, so this always makes sense as a pointwise map. $\mathbb{U}_{n,k}$ acts on morphisms in exactly the same way.*

In order to show $\mathbb{U}_n$ and $\mathbb{U}_{n,k}$ are functors, we need to check compositionality, identity, and that $\mathbb{U}_n f$ and $\mathbb{U}_{n,k} f$ are always morphisms in $\mathcal{R}_\sigma$. Since $\mathbb{U}_n$ and $\mathbb{U}_{n,k}$ simply act on maps by restricting them, compositionality and identity are straightforward, so we shall only check that morphisms are sent to morphisms. We shall do the $\mathbb{U}_{n,k}$ case.

Lemma 3.2.19. *If $f : T \to T'$ is a modal morphism in $\mathcal{M}_{n,k}$, then $\mathbb{U}_{n,k} f : \mathbb{U}_{n,k} T \to \mathbb{U}_{n,k} T'$ is a first order morphism in $\mathcal{R}_\sigma$.*

Proof. Similarly to functoriality of $\mathbb{M}_{n,k}$, we first prove a subclaim, that for $s \in T$, that if s_i exists, then $f(s_i) = f(s)_i$. We show this via induction, on the height of some $s \in T$. Firstly, if s has height 0, then s_i does not exist for any i, so the claim is vacuously true. If s has height $m+1$, then there exists some unique s', j, where $s' \in T$ has height m and j is from $1, ..., k$, such that $s' M_j s$. Thus, the unique path from the root to s is the unique path from the root to s', and a further j transition. So if $i = j$, $s_i = s$, and if $i \neq j$, $s_i = s'_i$ (and s_i only exists if s'_i does). Since f is a morphism, and $s' M_j s$, we get $f(s') M_j f(s)$. Identically to before, we have that if $i = j$, $f(s_i) = f(s)$, and if $i \neq j$, $f(s_i) = f(s'_i)$ (and $f(s_i)$ only exists if $f(s'_i)$ does). Combining this with the inductive hypothesis, that $f(s'_i) = f(s')_i$ for each i from $1, ..., k$, we obtain that $f(s_i) = f(s)_i$ in either case (that $i = j$ or $i \neq j$). Now in order to show that $\mathbb{U}_{n,k} f$ is a morphism, we must show that if $\mathbb{U}_{n,k} T \models R(s^1, ..., s^m)$, then $\mathbb{U}_{n,k} T' \models R(f(s^1), ..., s^m)$. By the definition of $\mathbb{U}_{n,k} T \models R(s^1, ..., s^m)$, we have that there is some j and tuple t such that $s^j_{t_i} = s^i$ for each i, and $T, s^j \models p_{R,t}$. Now, by the above claim, we get that $f(s^j)_{t_i} = f(s^j_{t_i}) = f(s^i)$ for each i. Also, since f is a morphism, we get that $T', f(s^j) \models p_{R,t}$. Hence, $\mathbb{U}_{n,k} T' \models R(f(s^1), ..., f(s^m))$ as required. □

Our final claims for this section are that $\mathbb{U}_n$ and $\mathbb{U}_{n,k}$ are left-adjoints to $\mathbb{M}_n$ and $\mathbb{M}_{n,k}$ respectively (with composites $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$), and further that $\mathcal{M}_n$ and $\mathcal{M}_{n,k}$ are in fact the Eilenberg-Moore categories (categories of co-algebras) of $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$. Since the second of these two claims is the stronger, we shall simply prove that and state the other as a corollary. The proof of this will use a result from [6] which we discuss first before proving the Theorem later on.

Theorem 3.2.20. *$\mathcal{M}_n$ is equivalent to the Eilenberg-Moore category of $\mathbb{E}_n$, and $\mathcal{M}_{n,k}$ is equivalent to the Eilenberg-Moore category of $\mathbb{P}_{n,k}$.*

Recall that the Eilenberg-Moore category for a comonad (taking $\mathbb{E}_n$ as an example) is the "category of co-algebras". This is a category where the objects are co-algebras $c : A \to \mathbb{E}_n A$ (satisfying the co-algebra conditions, discussed in the introduction), and morphisms are $\mathcal{R}_\sigma$ morphisms that also commute with the co-algebra maps. Concretely, if c, c' are co-algebras for A, B respectively, then a coalgebra morphism is an ordinary $\mathcal{R}_\sigma$ morphism $f : A \to B$ that also satisfies $c' \circ f = \mathbb{E}_n f \circ c$. From general categorical considerations, it is known there is an adjunction between this category and $\mathcal{R}_\sigma$, and further that the composition of the adjunction (in the right direction) is $\mathbb{E}_n$.

In the introduction, we saw that we can characterise co-algebras of $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$ as forest covers or labelled forest covers. We borrow a result from [6] which goes further to characterise the categories of coalgebras. We shall phrase the result in terms of modal morphisms to make it easier for our use. We shall therefore make use of the fact we can view a rooted forest as a modal structure (with no atomic propositions, only a single modality), and a k-labelled forest cover as a modal structure (with no atomic propositions, and k modalities, where an object has unique transition to it M_i rather than the label i).

Lemma 3.2.21. *(from [6]). Let $\mathcal{C}_n$ be the category with objects A, F, where A is a σ structure, and F is a forest cover of A, and morphisms $f : A, F \to B, G$, where f is a $\mathcal{R}_\sigma$ morphism that is also a modal morphism (A and F have the same underlying sets, as do B and G, so it makes sense to ask both if $f : A \to B$ is a $\mathcal{R}_\sigma$ morphism and $f : F \to G$ is a modal morphism). Let $\mathcal{C}_{n,k}$ be defined similarly, but with objects A, T where T is a k-labelled forest cover. Then $\mathcal{C}_n$ and $\mathcal{C}_{n,k}$ are equivalent to the categories of coalgebras for $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$ respectively.*

Now we are given this lemma, it remains to show there is an equivalence between categories $\mathcal{C}_n$ and $\mathcal{M}_n$, and $\mathcal{C}_{n,k}$ and $\mathcal{M}_{n,k}$. We shall just prove the latter.

Proof. (of Theorem 3.2.20). We shall find an essentially surjective, full, and faithful functor $\mathbb{F} : \mathcal{M}_{n,k} \to \mathcal{C}_{n,k}$. Given some $T, [\,] \in \mathcal{M}_{n,k}$, define $\hat{T}$ to be the modal structure on the set $T - \{[\,]\}$, with the same modal relations as T, but without any atomic propositions. Now set $\mathbb{F}T := \mathbb{U}_{n,k}T, \hat{T}$. Immediately, we must show that $\hat{T}$ is a k-labelled forest cover for $\mathbb{U}_{n,k}T$. This is true almost by definition. Firstly, observe $\hat{T}$ is a rooted forest on the elements of $\mathbb{U}_{n,k}T$ (the roots being the elements of height 1). Secondly, we must check the conditions on relations between elements of $\mathbb{U}_{n,k}T$, ie that if $R(s^1, ..., s^m)$ in $\mathbb{U}_{n,k}T$, then $s^1, ..., s^m$ all lie on the same branch of $\hat{T}$, and that for the unique element s^j of greatest height among them, the pebble index of s^i is not repeated on the unique path from s^i to s^j. This condition however is explicitly required whenever $\mathbb{U}_{n,k}T \models R(s^1, ..., s^j)$ in the definition of $\mathbb{U}_{n,k}T$, so there is nothing more to show. $\mathbb{F}$ shall send a modal morphism $f : T \to T'$ to $\mathbb{U}_{n,k}f$. Recall $\mathbb{U}_{n,k}f$ as a set map was simply the restriction of f to $T - \{[\,]\}$. $\mathbb{U}_{n,k}f$ is a $\mathcal{R}_\sigma$ morphism (we saw this when proving $\mathbb{U}_{n,k}$ was a functor), and certainly $\mathbb{U}_{n,k}f$ is a modal morphism from $\hat{T}$ to $\hat{T}'$ as these structures are simply T and T' with the atomic relations forgotten, and f was assumed to be a morphism between the original modal structures T and T'. Trivially, $\mathbb{F}$ will also respect identity and composition of morphisms (in exactly the same way $\mathbb{U}_{n,k}$ did). Now we have a functor, we must show it is an equivalence.

- Essential surjectivity: Given some A, F from $\mathcal{C}_{n,k}$ we must construct some T such that A, F is isomorphic to $\mathbb{U}_{n,k}T, \hat{T}$. Let T be the $\mathcal{M}_{n,k}$ structure, with universe $A \cup \{[\,]\}$, and transition relations given by F (except for putting the root at the bottom). Atomic relations are given identically as they were defined for $\mathbb{M}_{n,k}A$, namely $T, s \models p_{R,t}$ if and only if s_i exists for each i occurring in t and $A \models R(s_{t_1}, ..., s_{t_l})$ (where $t = (t_1, ..., t_l)$). One can see this will satisfy $\mathbb{U}_{n,k}T, \hat{T} = A, F$ immediately by construction. As a side note, one could also pick out T by using the coalgebra associated to F to generate a substructure of $\mathbb{P}_{n,k}A$ and hence a corresponding substructure of $\mathbb{M}_{n,k}T$.

- Full and faithful: Since $\mathbb{F}$ changes morphisms only by restricting them down by a single point (the root), and the image of the root is always determined, it is clear $\mathbb{F}$ is faithful. We need only check now $\mathbb{F}$ is full. Suppose $f' : \mathbb{U}_{n,k}T, \hat{T} \to \mathbb{U}_{n,k}T', \hat{T}'$ is a morphism. We need to check there is some morphism $f : T \to T'$ in $\mathcal{M}_{n,k}$ such that $\mathbb{F}f = f'$. Now by the definition of $\mathbb{F}$ such an f can only be f' (as a set map), that also sends the root of T to the root of T', so we simply need to check such an f is a modal morphism. It is clear such an f will respect the transition structure of T (since it is a modal morphism from $\hat{T} \to \hat{T}'$), so we need to check f preserves atomic relations. So suppose $T, s \models p_{R,t}$, where $t = (t_1, ..., t_j)$. Now since $T \in \mathcal{M}_{n,k}$, we know s_{t_i} exists for each i, and $\mathbb{U}_{n,k}T \models R(s_{t_1}, ..., s_{t_j})$. Now since f' is a morphism, we know $\mathbb{U}_{n,k}T' \models R(f'(s_{t_1}), ..., f'(s_{t_j}))$. But we know $f'(s_{t_i}) = f'(s)_{t_i} = f(s)_{t_i}$ for each i, since f' respects transition relations, and f' is the same function as f. Hence $\mathbb{U}_{n,k}T' \models R(f(s)_{t_1}, ..., f(s)_{t_j})$, so $T', f(s) \models p_{R,t}$ as required.

□

3.3 Courcelle's Theorem Style Application

Courcelle's Theorem [11] is a much celebrated algorithmic meta-theorem. Informally, it states that for any Monadic Second Order sentence ϕ, the model checking problem is linear time over structures of a fixed bounded tree width. Monadic Second Order (MSO) sentences are sentences which allow quantification over unary relations (in addition to all other constructors and connectives present in first order logic). We state it more formally below:

Theorem 3.3.1. *Fix some ϕ an MSO sentence over σ, and some $k \in \mathbb{N}$. Then the decision problem: "Given some $A \in \mathcal{R}_\sigma$ with tree-width at most k, does $A \models \phi$?" can be checked in linear time with respect to the size of the universe of A.*

For an arbitrary MSO formula, its model checking problem over the collection of all structures lives in the Polynomial Hierarchy of complexity. In fact, for any given level of the Polynomial Hierarchy, there exists an MSO formula whose model checking problem is complete for that level (see eg [22] for more detail) . Courcelle's Theorem then starkly contrasts the collection of all structures against collections of structures of bounded tree-width, since it says the model checking problem for any MSO formula is instead linear time over collections of structures with bounded tree-width.

Whilst Courcelle's Theorem is much studied and there a many different proofs (see [7]), the steps of a "standard" proof are as follows:

1. Find some ϕ' (that is an MSO sentence over labelled graphs) such that for any tree decomposition (of width at most k) T of a structure A, $T \models \phi'$ if and only if $A \models \phi$.

2. Given some $A \in \mathcal{R}_\sigma$ with tree-depth at most k it is possible to find a tree decomposition T of width at most k of ϕ in $O(A)$ steps. In addition the universe of T is of size $O(A)$.

3. Using a tree automata (or otherwise), evaluate ϕ' on T in time $o(T)$.

We use here the term "tree decomposition" as opposed to k-labelled forest cover, which is a different construction that also witnesses bounding the tree-width of a particular structure. Tree decompositions are frequently used when discussing tree-width, as in the proof of Courcelle's Theorem (see e.g. [8] for an overview). A tree decomposition (for a structure A) is a tree T whose vertices consists of subsets of A and satisfies certain properties (unlike a k-labelled forest cover whose vertices are exactly A), hence why one must specify the size of the tree decomposition in the overview above. Given a tree decomposition, one can efficiently construct a k-labelled forest cover and vice-versa. We prefer the use of k-labelled forest covers throughout this thesis as they coincide with coalgebras for the $\mathbb{P}_{n,k}$ comonad. In addition, tree decompositions are much more "flexible", in the sense given a tree decomposition one can construct many slightly different versions of it conveying the same information. This flexibility is useful for algorithmic purposes, but not for categorical ones, where uniqueness is important.

The work in the previous section is naturally geared towards step (1) above, as we have methods of translating formulas to be interpreted by modal trees. For our purposes, we will work with a structure and $\mathbb{P}_{n,k}$ coalgebra given, rather than look more deeply into step (2).

Although we know that already we can neatly translate FO formulas well using $\mathbb{M}_{n,k}$, this will not pass over well to MSO formulas. The problem is the lack of correspondence between subsets (which correspond to unary relations) of a structure A and subsets of $\mathbb{M}_{n,k}A$. One would need to write a qualifier to a subset of $\mathbb{M}_{n,k}A$ to insist it is of form $\bigcup S_a$ where S_a are sets of all sequences whose last A entry are a, which is clearly not possible (for instance if a and b satisfied all the same formulae in A, or in other words, had the same type, the structure $\mathbb{M}_{n,k}$ would have no way of telling them apart). In any case, the size of $\mathbb{M}_{n,k}A$ is clearly not $o(A)$, in fact $|\mathbb{M}_{n,k}A| = (k|A|)^n$.

Now given a structure $A \in \mathcal{R}_\sigma$, we know that a k-labelled forest cover for A corresponds to a $\mathbb{P}_{n,k}$ coalgebra (for some n), which we saw in the last section corresponds to a structure $T \in \mathcal{M}_{n,k}$ such that $A = \mathbb{U}_{n,k}FT$. What is the relationship between formulae modelled by such an A and T respectively?

Theorem 3.3.2. *If $T \in \mathcal{M}_{n,k}$ is such that $\mathbb{U}_{n,k}T = A$ then for $\phi \in \mathcal{L}^+_{n,k}$:*

- *If $T \models \mathbb{M}_{n,k}\phi$ then $A \models \phi$*
- *If $A \models \phi$ then there is some ψ such that $\psi \equiv \phi$ and $T \models \mathbb{M}_{n,k}\psi$.*

Proof. We give a sketch for both implications as it gives a very clear idea what is going on; a formal proof would require inductions similar to previous proofs. Recall from the introduction that $\phi \in \mathcal{L}^+_{n,k}$ will be equivalent to a disjunction of primitive positive formulae (formulae that use only existential quantifiers, conjunctions, and atomic relations) of quantifier rank less than or equal to ϕ, so it suffices to prove the above claims for the case that ϕ is primitive positive. Suppose $T \models \mathbb{M}_{n,k}\phi$. This is true if and only if there are witnesses for each $\Diamond$ occurring in the formula, satisfying all the respective subformulae. It is clear from the definition of $\mathbb{U}_{n,k}T$ that those same witnesses will satisfy ϕ in A. Now the same argument does not follow exactly for the converse, as we do not know these witnesses will be accessible to one another in T. For instance, suppose $\phi = \exists x_1 \exists x_2 R(x_1, x_2)$ for some binary $R \in \sigma$, and that ϕ is witnessed by $a_1, a_2 \in A$. In order for these same elements to witness $\Diamond_1\Diamond_2 p_{R,(1,2)}$ it would have to be the case that $[\ ]M_1 a_1 M_2 a_2$, but we have no reason to expect this to be the case. We must re-write ϕ in order that the quantifiers appear in the correct order for T to read. In fact if A is finite, we shall use a lemma (proved below) that does this for all formulas in one go: there exists some ϕ' such that $c_A \equiv \phi'$ (where c_A is the canonical query of A), and $T \models \mathbb{M}_{n,k}\phi'$. Since $A \models \phi$ and ϕ is positive existential, it must be the case that $c_A \implies \phi$ is a tautology, or in other words $c_A \vee \phi \equiv \phi$, and hence $\phi' \vee \phi \equiv \phi$. Since $\mathbb{M}_{n,k}(\phi' \vee \phi) \equiv (\mathbb{M}_{n,k}\phi' \vee \mathbb{M}_{n,k}\phi)$, we get $T \models \mathbb{M}_{n,k}(\phi' \vee \phi)$ (since $T \models \mathbb{M}_{n,k}\phi'$), so setting $\psi = \phi' \vee \phi$ is sufficient to prove the Theorem. If A is not finite, one should consider the (finite) substructure of the elements of A that witness ϕ, and all those elements that occur on the paths from the root to those elements in T, and apply the same line of reasoning to that structure. ☐

Lemma 3.3.3. *Let A be a finite structure with an (n, k)-cover, witnessed by some $T \in \mathcal{M}_{n,k}$ such that $\mathbb{M}_{n,k}T = A$. Then there is a formula ψ such that $\psi \equiv c_A$ (in FO) and $T, [\ ] \models \mathbb{M}_{n,k}\psi$.*

Proof. Recall the canonical query is the existential formula defined by two properties:

1. $A \models c_A$
2. If $A \to B$ then $B \models c_A$.

In the introduction, we realised it by listing all elements of A (using an existential quantifier for each element) and all relations between those elements. This can be done identically to form a canonical query c_T for T satisfying the same two properties, but in $\mathcal{M}_{n,k}$. We shall set $\psi := \mathbb{U}_{n,k}c_T$. By the first implication of the above Theorem (with does not depend on this lemma!), we get that $A \models \psi$ (since $T \models c_T$). Now suppose $A \to B$. Recall $A = \mathbb{U}_{n,k}T$, so since

$\mathbb{U}_{n,k}$ and $\mathbb{M}_{n,k}$ are adjoint, we get $T \to \mathbb{M}_{n,k}B$. Hence $\mathbb{M}_{n,k}B \models c_T$ and hence $B \models \mathbb{U}_{n,k}c_T$ as required (we proved this result in an earlier section). □

A quick consideration shows that there will be no such relationship between satisfaction of formulae when we allow negations. Consider, for instance, the formula $\phi := \exists x_1 \exists x_2 \neg R(x_1, x_2)$. Suppose in some structure $A = \mathbb{U}_{n,k}T$, this is witnessed by a_1, a_2. Without the negation, it is guaranteed by the structure of T that either a_1 occurs on the path from the root to a_2 or vice versa, and it is possible to re-write ϕ to make use of this so it can be read by T, as in the Theorem above. However, since instead it is the case that a_1, a_2 are not necessarily related in A, they could be in different branches of T, and there is no way to re-write ϕ to capture this.

When we introduce universal quantifiers, the relationship between satisfaction of formulae between some T and $\mathbb{U}_{n,k}T$ also fails. Suppose there is no $a \in T$ such that $[\,]M_1 a$. Then $T, [\,] \models \phi$ for any formula of form $\Box_1\psi$ (even if ψ is universally false!). Clearly, this is not the case for $\mathbb{U}_{n,k}T$, hence we no longer have that if $T \models \phi$ then $\mathbb{U}_{n,k}T \models \mathbb{U}_{n,k}\phi$. Notice this example shows how formulae being equivalent over FO is not the same as being equivalent modal formulae. In addition, we cannot hope to find some FO equivalent formula as a proxy for $\mathbb{U}_n\phi$, as $\mathbb{U}_n\phi$ could be universally false in FO.

In order to deal with negations and universal quantifiers, we must extend the power of our quantifiers to be able to search the entire structure. This essentially amounts to viewing our modal structures as if they are labelled directed graphs (ie a directed graph with unary predicates), on which we use ordinary first order (and monadic second order) logic. Now that we are switching away from Modal logic, we can reintroduce equality into our formulae.

Definition 3.3.4. *Given some $T \in \mathcal{M}_{n,k}$ we associate to it a labelled directed tree $T*$ with the universe that of $T - [\,]$. $T*$ shall carry over the exact same atomic propositions as unary predicates, and replace the transitions of T with a single edge relation E and k more unary predicates $c_1, ..., c_k$. We shall have $E(s_1, s_2)$ if and only if $s_1 M_i s_2$ for some i, and $c_l(s)$ if and only if there is some $s' \in T$ such that $s' M_l s$.*

$T*$ carries the exact same data as T, however, we need to replace the transitions with a single edge relation and k unary predicates in order to keep track of the labels of those elements of height 1, as the root, and hence the transitions to those elements, have been removed. Structures of form $T*$ interpret MSO formulae of the vocabulary $\{E, p_{R,t}, c_i\}$ for each appropriate R, t, i. We can now show the following:

Theorem 3.3.5. *Given any MSO formula ϕ over σ, and fixed pair n, k there exists an MSO formula ψ (over $\{E, p_{R,t}, c_i\}$) such that, for any $T \in \mathcal{M}_{n,k}$, we have $\mathbb{U}_{n,k}T \models \phi$ if and only if $T* \models \psi$.*

Proof. Since $\mathbb{U}_{n,k}T$ and $T*$ have the exact same universes, we keep the first and second order quantifiers (and unary predicates coming from second order quantifiers), and connectives present in ϕ the same, we need only change the atomic relations. Suppose $R(s^1, ..., s^l)$ holds in $\mathbb{U}_{n,k}T$, recall this is true if and only if:

1. There is some s (from among $s^1, ..., s^l$) above each $s^1, ..., s^l$ in the tree $T*$.
2. There is some t such that $s_{t_i} = s^i$ for each $i = 1, ..., l$.
3. $T, s \models p_{R,t}$.

Recall that s_j is the element of greatest height below s accessed by a M_j transition in T (or equivalently, satisfies c_j in $T*$), so the first condition does imply the second. Hence the task is to convert the second and third conditions above into a first order formula. Given a variable x, we can write "$x_j = y''$ as follows (this is an unfortunate notation, here we are using "$x_j = y''$ as a proxy for $s_j = s'$ for a sequence s, which clashes with using x_j as a variable name):

$$\begin{aligned} &c_j(y) \wedge \exists P(P(x) \wedge P(y) \wedge \\ &\qquad\qquad \forall z(P(z) \rightarrow \\ &\qquad ((z = y \vee (\neg c_j(z) \wedge \exists! z'(P(z') \wedge E(z', z)))) \\ &\qquad\qquad\qquad \wedge (z = x \vee \exists! z'(P(z') \wedge E(z, z')))) \end{aligned}$$

Here the predicate P is defining a path starting at y and finishing at x, such that no element on the path except y satisfies the predicate c_j (we have made use of the FO definable shorthand $\exists!$ for "there exists a unique"). Intuitively, the formula says P contains x and y, and if z is in P, then it is either y or there is a parent of it in P and it does not have colour c_j, and it is either x or there is a child of it in P. This captures the notion of P being a path from y to x containing no elements except y with colour c_j. Now we can translate an atomic relation $R(x^1, ..., x^l)$ as follows:

$$\theta(R, (x^1, ..., x^l)) := \bigvee_{t,i} (p_{R,t}(x^i) \wedge \bigwedge_j x_{t_j} = x^j).$$

Here t is ranging over all possible tuples of length l with entries among $1, ..., l$ and i is ranging from $1, ..., l$. This formula now asserts that there is some t satisfying conditions (2) and (3) above, so we have that $\mathbb{U}_{n,k}T, (a_1, ..., a_l) \models R(x^1, ..., x^l) \iff T*, (a_1, ..., a_l) \models \theta(R, (x^1, ..., x^l))$ as desired. As suggested above, we now obtain ψ from ϕ by replacing all atomic relations $R(\bar{x})$ with $\theta(R, \bar{x})$ to complete the proof. □

Interestingly, since we are using MSO in the Theorem above we were able to construct ψ without any dependence on n. Unlike in MSO, It is not possible to write "there exists a path from x to y" in ordinary FO, unless the length of the path is bounded. If we restricted ourselves to ordinary FO, we would need to use n as an upper bound for the length of the path. Given ψ did not depend on n, we can rephrase the Theorem in a style more similar to Courcelle's Theorem:

Corollary 3.3.6. *Let ϕ be an MSO formula over σ, and k a natural number. Then there exists an MSO ψ in the language of labelled graphs, such that for any structure A of tree-width at most k, witnessed by and $T \in \mathcal{M}_{n,k}$ for some n, $A \models \phi$ if and only if $T* \models \psi$.*

3.4 Comments and Further Directions

In the first section, we saw how the comonads naturally possessed a modal or transitional structure, and this structure allows us to faithfully translate first order formulae into modal ones. We then proved that this translation was actually an adjunction, via subcategories of modal trees that was equivalent to the Eilenberg-Moore Categories (of $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$ respectively). A natural further direction for this may be to consider other game comonads, such as those in used for the guarded fragment ([3]), or the pebble-relation game ([23]). Is it possible to represent these as modal or transition structures also? In the case of $\mathcal{M}_{n,k}$, we have that the Spoiler's choices are encoded by the transitional part of a modal structure, and the restrictions this places on Duplicator are encoded by the atomic propositions, might this be a general recipe? Perhaps more speculatively using the similar ideas to [2], where a more general categorical treatment is given to graph parameters using discrete density comonads, would it be possible to reconstruct this modal translation in a more systematic style?

Also in the first section, we remarked how we can express elementary equivalence as a sequence of increasing bisimulations; A is elementarily equivalent to B if and only if $A \equiv_n B$ for every n if and only if $\mathbb{M}_n A \sim \mathbb{M}_n B$ for every n. Given that elementary equivalence is so fundamental to first order logic, would it be possible to make use of this modal translation as a method for proving things about first-order logic, by proving an analogous result in the categories $\mathcal{M}_n$? One hurdle to such endeavours is the category $\mathcal{M}_n$ also contains structures not in the image on $\mathbb{M}_n$, which therefore do not correspond to first-order structures. Is there some way to, internally to the category $\mathcal{M}_n$, identify if a modal structure is in the essential image of $\mathbb{M}_n$?

In the second section we focus on $\mathbb{U}_{n,k}$, which is the adjoint of $\mathbb{M}_{n,k}$. We investigated the relationship between satisfaction of formulae in a structure T and satisfaction of corresponding formulae in $\mathbb{U}_{n,k}T$. We saw how this relationship could become an "if and only if" when viewing T as a labelled graph $T*$, and applied this to proving one of the key steps in Courcelle's Theorem. Another key step in Courcelle's Theorem is to use tree automata to decide formulae on trees, and there is a categorical account of tree automata being represented as monads in [19]. Combining with our categorical account of translating an MSO formula into one to a corresponding one in the language of trees, would it be possible to give a fully categorical account of Courcelle's Theorem?

Chapter 4

Locality and Reachability

4.1 Overview

In this section we explore how the notion of locality in first-order logic interacts with the comonad $\mathbb{E}_n$. The notion of locality is studied in Finite Model Theory as, similarly to previous fragments of FO we have discussed, locality constraints on formulas or queries can help make problems more tractable (for an overview, see [20]). In this chapter, we will essentially build a localised version of the three way interaction presented in the introduction between logic, Ehrenfeucht-Fraisse games, and the comonad $\mathbb{E}_n$. We will also be able to express a weakened version of Gaifman's Locality Theorem using localised versions of $\mathbb{E}_n$.

These localised versions are defined simply by considering subfunctors $\mathbb{E}_n^d$ of $\mathbb{E}_n$ that consist only of sequences of elements where each new element is local to the ones already played before, defined by some non-negative parameter d. Unlike in the case of $\mathbb{E}_n$ where we have the co-Kleisli extension, there is no 1-1 correspondence between maps of shape $\mathbb{E}_n^d A \to \mathbb{E}_n^d B$, and maps of shape $\mathbb{E}_n^d A \to B$, which entails that $\mathbb{E}_n^d$ is not a comonad. This is because the former maps represent games in which Spoiler can only play locally and Duplicator can also only play locally, but the latter maps represent games where Spoiler can only play locally but Duplicator is unrestricted, and these games are not equivalent. To complete the picture, we consider localised formulae, which allows us to give a Theorem showing the threefold equivalence of: a winning strategy for Duplicator in a specific Spoiler-Duplicator game, a map out of $\mathbb{E}_n^d$, and a logical relation between two structures. We do this for both shapes of map mentioned above.

In the next section, we find a lower bound on which there is a morphism $\mathbb{E}_n A \to \mathbb{E}_n^d A$, which unfortunately is not a natural transformation. This provides an elegant proof of the fact that for a large enough local bound, Duplicator has a winning strategy in the ordinary Ehrenfeucht-Fraisse game, if and only if, Duplicator has a winning strategy when Spoiler is restricted to only playing locally. The morphism $\mathbb{E}_n A \to \mathbb{E}_n^d A$ effectively tells us that any move too far away from previous moves is a bad move for Spoiler in the Ehrenfeucht-Fraisse game. Next, we define the property of reachability, which exactly characterises those moves which are not bad in this way. Interestingly, it turns out that if Spoiler plays only reachable moves, Duplicator must also only play reachable moves in a winning strategy. Thus, the subfunctor $\mathbb{R}_n$ of $\mathbb{E}_n$, defined as the subset of $\mathbb{E}_n$ consisting of sequences of reachable moves, is actually a comonad, since the aforementioned property allows us to build a co-Kleisli extension for it. This contrasts to the $\mathbb{E}_n^d$ case which had no such property. The final result of this section, and what much of the technical work in this section is devoted to, is proving the existence of a morphism $\mathbb{E}_n A \to \mathbb{R}_n A$ for any A, thus showing that the game where Spoiler is restricted to playing only reachable moves is equivalent to the ordinary Ehrenfeucht-Fraisse game.

Finally, we consider Gaifman's Theorem in the context of our work. In order to do this, we make use of ideas in Chapter 3, giving us an account of local version of $\equiv_n$ using modal structures. We then reformulate a weakened version of Gaifman's Theorem allowing us to state it in our own framework, as the equivalence between bisimulations of two pairs of modal structures, one pair of structures corresponding to $\mathbb{E}_n$, and the other corresponding to $\mathbb{E}_m^d$, for some sufficiently large m, d for a given n. This suggests a conjectured sharpening of it, where we replace $\mathbb{E}_m^d$ with $\mathbb{R}_{m'}$, for some m'.

4.2 Localised Subfunctors of $\mathbb{E}_n$

In an earlier chapter, we proved the following simple lemma (we stated a more general case, for $\to_{n,k}$ before):

Lemma 4.2.1. *For $A, B, C \in \mathcal{R}_\sigma$, if $A \to_n C$ and $B \to_n C$ then $A \oplus B \to_n C$.*

This lemma can be stated at the level of Ehrenfeucht-Fraisse games: "if Duplicator has a winning strategy for $EF_n(A, C)$ and a winning strategy for $EF_n(B, C)$, then it can construct a winning strategy for $EF_n(A \oplus B, C)$". Of course, we can also state it more generally, since, for any structure $A \in \mathcal{R}_\sigma$ we can write it as the coproduct of its connected components $A = \bigoplus A_i$, (connected components of a structure are simply the connected components of its Gaifman graph) so the lemma can be generalised to "Duplicator needs to only know strategies for connected structures, and can combine them together to create strategies for general, possibly disconnected, structures". We can also phrase this discussion at the level of our comonads:

Definition 4.2.2. *For $A \in \mathcal{R}_\sigma$, let $\mathbb{E}_n^\infty A$ be the induced substructure of $\mathbb{E}_n A$, induced by the subset $\{[a_1, ..., a_j] \in \mathbb{E}_n A : a_1, ..., a_j$ all lie in the same component of $A\}$.*

One can see that $\mathbb{E}_n^\infty A$ is isomorphic to $\bigoplus \mathbb{E}_n A_i$ (where the A_i are the connected components of A) since they are clearly bijective as sets and there are certainly no relations between sequences constructed using elements of different components as they are never prefixes of one another. The lemma could then be expressed as a morphism $\mathbb{E}_n A \to \mathbb{E}_n^\infty A$ which we will show exists later.

We use the notation $\mathbb{E}_n^\infty$ as elements in the same connected component of a structure can be thought of as having finite distance to one another. In a sense, the above lemma highlights that Duplicator can treat elements in separate components (ie of infinite distance apart) as irrelevant to one another in the context of an Ehrenfeucht-Fraisse game. Since the Ehrenfeucht-Fraisse games we are considering are of finite rounds, we can naturally consider whether there is some finite distance d for which this is also true. We shall now build up to defining functors $\mathbb{E}_n^d$ and exploring their properties.

Firstly, we must first define the notion of distance in a general first-order structure. This is essentially the graph distance metric applied to the Gaifman graph. The following definitions also apply naturally to structures in $\mathcal{R}_\sigma(l)$ for any l, however the distinguished tuples are irrelevant to the definition so we omit them.

Definition 4.2.3. *For $A \in \mathcal{R}_\sigma$ and $a, b \in A$, we define the distance $D(a, b)$ to be the length of the shortest path from a to b in the Gaifman graph $G(A)$ of A, in the case there is at least one such path (counting the number of edges in the path so eg $S(a, a) = 0$). Else, we write $D(a, b) = \infty$. We define $D(\bar{a}, b)$ to be the minimum among $D(a_i, b)$ where the a_i range among entries of the tuple $\bar{a}$.*

Here we abuse notation by using the same distance function for any structure $A \in \mathcal{R}_\sigma$, though this should not cause any confusion. It is straightforward to see this distance function we have just defined is indeed a metric. As with any metric, we also get a notion of neighbourhood:

Definition 4.2.4. *For d a natural number or $d = \infty$, $A \in \mathcal{R}_\sigma$, and $\bar{a}$ a tuple of A, then the d-neighbourhood of $\bar{a}$ is the substructure of A induced by the set $\{b \in A : D(\bar{a}, b) < d\}$, and will be written $N_d(\bar{a})$.*

Under this metric, we get the useful fact that ordinary first-order homomorphisms are contractions:

Lemma 4.2.5. *If $f : A \to B$ is a morphism in $\mathcal{R}_\sigma$, then f is a contraction. In other words, for any $a, b \in A$, $D(a, b) \geq D(f(a), f(b))$.*

Proof. This follows quickly from the following claim: if a, b are adjacent in $G(A)$, then $f(a), f(b)$ are adjacent in $G(B)$, or $f(a) = f(b)$. It follows because then we would have that any path in the Gaifman graph of A is translated to a path of shorter or equal length in the Gaifman graph of B. To prove the claim, recall a, b are adjacent in $G(A)$ if and only if there exists a tuple $\bar{a}$ of A, and $R \in \sigma$, such that a, b occur as entries in $\bar{a}$ and $A \models R(\bar{a})$. But then, by the definition of a homomorphism, we get $B \models R(f(\bar{a}))$, and of course $f(a), f(b)$ occur among $f(\bar{a})$. Hence $f(a), f(b)$ are adjacent unless $f(a) = f(b)$. □

Clearly, the above also follows for the distance from an element to a tuple, as well as just between two elements. We can now define the functor $\mathbb{E}_n^d$ for arbitrary d. We shall also write the definition for a general l this time.

Definition 4.2.6. *For $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$, natural numbers n, l, d (or $d = \infty$), if $l > 0$ we define $\mathbb{E}_n^d(A, \bar{a}) := \mathbb{E}_n(N_d(\bar{a}), \bar{a})$. We will view this is an induced substructure of $\mathbb{E}_n(A, \bar{a})$ in the natural way. If $l = 0$, we define $\mathbb{E}_n^d A = \{[a_1, ..., a_i] \in \mathbb{E}_n A : a_2, ..., a_i \in N_d(a_1)\}$. In other words, $\mathbb{E}_n^d A$ is the union of sets $\mathbb{E}_{n-1}^d(A, a)$ over all $a \in A$ (not to be confused with the coproduct $\bigoplus_{a \in A} \mathbb{E}_n^d(A, a)$ in the category $\mathcal{R}_\sigma(1)$ where we would identify all sequences of length 1). In either case $\mathbb{E}_n^d$ acts on morphisms by restricting from $\mathbb{E}_n$.*

We must check $\mathbb{E}_n^d$ is always a functor, specifically we must check its action on morphisms. It will certainly respect composition and identity morphisms as $\mathbb{E}_n$ does, though it is not immediate that if $f : (A, \bar{a}) \to (B, \bar{b})$ is a morphism in $\mathcal{R}_\sigma(l)$ then $\mathbb{E}_n^d f$ as defined above actually maps into $\mathbb{E}_n^d(B, \bar{b})$, since a priori it is only a map into $\mathbb{E}_n(B, \bar{b})$. This amounts to checking that for for any $a \in A$, if $a \in N_d(\bar{a})$, then $f(a) \in N_d(f(\bar{a}))$. But we already proved that $D(f(\bar{a}), f(a)) \leq D(\bar{a}, a)$, from which this follows.

We remark that we could also have defined $\mathbb{E}_n^d$ to have some number m of "freely chosen" elements, (ie elements chosen from anywhere in the structure, rather than just ones local to what was already chosen) rather than just 1, or defined a family of functors for each natural number $m \leq n$. We chose to do just one as it keeps the bookkeeping and notation simpler, however all of the following results would follow for any chosen $m \leq n$.

Our next goal is to classify morphisms out of $\mathbb{E}_n^d$, similarly to how we classified morphisms out of $\mathbb{E}_n$. Since $\mathbb{E}_n$ is a comonad, one can move freely between maps of form $f : \mathbb{E}_n A \to B$ and maps of form $g : \mathbb{E}_n A \to \mathbb{E}_n B$. Given a map $g : \mathbb{E}_n A \to \mathbb{E}_n B$, we simply right compose with the counit $\epsilon_B : \mathbb{E}_n B \to B$. Conversely, given a map $\mathbb{E}_n A \to B$, we use the co-Kleisli extension $f \mapsto \mathbb{E}_n f \circ \delta_A$ to get a map from $\mathbb{E}_n A \to \mathbb{E}_n B$. Now clearly $\mathbb{E}_n^d$ comes equipped with a similar natural transformation $\epsilon_A : \mathbb{E}_n^d A \to A$, simply the restriction of the co-unit of $\mathbb{E}_n$, so we can do transformations of the first kind. However, $\mathbb{E}_n^d$ is not a comonad, and does not have a co-Kleisli extension. Consequently, the two different kinds of map will correspond to different logical and game relations.

Firstly, we shall consider the case where $l > 0$ (ie, there is a distinguished tuple from each structure we consider), and discuss what types of Ehrenfeucht-Fraisse games these morphisms relate to:

Lemma 4.2.7. *If $l > 0$, for every n, and $(A, \bar{a}), (B, \bar{b}) \in \mathcal{R}_\sigma(l)$:*

1. *There is a bijective correspondence between:*
   - *Morphisms $\mathbb{E}_n^d(A, \bar{a}) \to (B, \bar{b})$*
   - *Winning strategies for Duplicator in the game $EF_n((N_d(\bar{a}), \bar{a}), (B, \bar{b}))$.*
2. *There is a (non-bijective) correspondence between:*
   - *Morphisms $\mathbb{E}_n^d(A, \bar{a}) \to \mathbb{E}_n^d(B, \bar{b})$*

- *Winning strategies for Duplicator in the game* $EF_n((N_d(\bar{a}),\bar{a}),(N_d(\bar{b}),\bar{b}))$.

Proof. By definition, $\mathbb{E}_n^d(A,\bar{a}) := \mathbb{E}_n(N_d(\bar{a}),\bar{a})$. Item (1) then follows from the lemma in the introduction about the correspondence between morphisms and strategies for Duplicator. For item (2), suppose we have a morphism $f : \mathbb{E}_n^d(A,\bar{a}) \to \mathbb{E}_n^d(B,\bar{b})$. By composition with ϵ_B, we then have a morphism from $\mathbb{E}_n^d(A,\bar{a})$ to $(N_d(\bar{b}),\bar{b})$ and can use (1) to get a winning strategy for Duplicator $EF_n((N_d(\bar{a}),\bar{a}),(N_d(\bar{b}),\bar{b}))$. Note that composition with ϵ_B is not necessarily injective, which breaks the bijectivity of this correspondence. Given a winning strategy for Duplicator in the game $EF_n((N_d(\bar{a}),\bar{a}),(N_d(\bar{b}),\bar{b}))$, we can obtain a map $\mathbb{E}_n(N_d(\bar{a}),\bar{a}) \to (N_d(\bar{b}),\bar{b})$. We can use the co-Kleisli extension of $\mathbb{E}_n$ to produce a map $\mathbb{E}_n(N_d(\bar{a}),\bar{a}) \to \mathbb{E}_n(N_d(\bar{b}),\bar{b})$, which by definition is a map to $\mathbb{E}_n^d(A,\bar{a}) \to \mathbb{E}_n^d(B,\bar{b})$. □

In the first correspondence, we used the game $EF_n((N_d(\bar{a}),\bar{a}),(B,\bar{b}))$ as it is an instance of what we have already defined. We could equally have described this as a variant of an Ehrenfeucht-Fraisse game between $(A,\bar{a})$, and $(B,\bar{b})$, where Spoiler is restricted to playing locally to its starting position, but Duplicator is has access to all of B. In the second correspondence, both Spoiler and Duplicator are restricted to playing locally. Clearly, if Duplicator has a winning strategy when it is restricted to playing only locally, then Duplicator has a winning strategy when it is allowed to use all of B, (semantically this is just composition with the ϵ_B). It is worth a counterexample to see why the reverse fails (and hence that $\mathbb{E}_n^d$ cannot be a comonad, else its co-Kleisli extension would imply the reverse!):

Example 4.2.8. *Suppose $d = \infty$, $A = \{a_1, a_2, a_3\}$ is a path of length 3 (from a_1 to a_3) under some binary relation $R \in \sigma$, and there exist two unary predicates P, P' such that $A \models P(a_1)$ and $A \models P'(a_2)$ (and A satisfies no other relations). Suppose $B = \{b_1, b_2\}$ is such that $B \models R(b_1,b_1)$, $B \models R(b_2,b_2)$, $B \models P(b_1)$, and $B \models P(b_2)$ (and nothing else). Then we will have $(N_d(a_1),a_1) \to_1 (B,b_1)$ but not $(N_d(a_1),a_1) \to_1 (N_d(b_1),b_1)$. This is because Duplicator would need use of b_2 in a winning strategy for the first game (in case Spoiler plays a_3, which is d-local to a_1), but does not have access to it in the second game.*

The idea of the example is that if d is large, Spoiler is still allowed to play "slack" moves (intuitively for now these are just moves that have no chance of interacting with the previous moves, in this case a_3 has no relation to a_1 and there are no other moves remaining so it is "slack" but we will carefully define "slack" later). In a game with no restrictions on Duplicator, if Spoiler plays "slack" move, Duplicator can, and in some cases must, simply treat it as if Spoiler has started a fresh game and play accordingly. However, in the above example, Duplicator is still bound to playing locally to the starting position, so cannot simply play a fresh game. Later on, we shall consider games where Spoiler cannot play "slack" moves.

We also discuss why item (2) in the above lemma is only a correspondence, not a bijective correspondence, as it is relevant to the characterisation of $\mathbb{E}_n^d$

on $\mathcal{R}_\sigma$. The discussion, applies to all the game (endo)functors and comonads discussed so far (ie all functors except for the modal translations in the third), including $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$, and we shall use $\mathbb{E}_n$ as an example. There is a "nice" form of map $\mathbb{E}_n A \to \mathbb{E}_n B$ which are those maps constructed by co-Kleisli extending a map from $\mathbb{E}_n A \to B$. These maps have the property of sending a sequence in $\mathbb{E}_n A$ to a sequence of the same length, in addition to preserving the prefix relation among sequences. However, there are maps not of this form! Consider the following simple example:

Example 4.2.9. *If A is a structure that satisfies no relations, then so is $\mathbb{E}_n A$. Hence, any map out of $\mathbb{E}_n A$ is a morphism.*

Formally, we will say a map of form $\mathbb{E}_n^d(A, \bar{a}) \to \mathbb{E}_n^d(B, \bar{b})$ is "nice" if it sends sequences to sequences of equal length, and preserves prefixes. Notice we could also have defined these maps via a modal translation as in Chapter 2, as these are precisely the maps that preserve the modal (tree) structure of $\mathbb{E}_n$. We now get a bijection between "nice" maps and strategies for $\mathbb{E}_n^d$, which will be stated as a corollary to the analogous claim for $\mathbb{E}_n$, after the following lemma:

Lemma 4.2.10. *Fix n, l (where possibly $l = 0$) and let $f : \mathbb{E}_n(A, \bar{a}) \to (B, \bar{b})$ in $\mathcal{R}_\sigma(l)$. The co-Kleisli extension of f is defined as the map $\hat{f} = \mathbb{E}_n f \circ \delta_A$. We claim that $\hat{f}$ satisfies the following recursive properties:*

1. *If s is of length 1, so $s = [a]$ for some $a \in A$, then $\hat{f}[a] = [f[a]]$ (if $l > 0$ then the only choice for a is a_1, the first entry in the tuple $\bar{a}$).*
2. *If $s = s'[a]$ (for some $s' \in \mathbb{E}_n A$,$a \in A$), then $\hat{f}(s) = \hat{f}(s')[f(s)]$.*

It follows that $\hat{f}$ is "nice".

Proof. Recall δ_A sends a sequence s to its "sequence of prefixes". In other words $\delta_A(s) = [s]$ if s is one element sequence and $\delta_A s = (\delta_A s')[s]$ if $s = s'[a]$ for some $a \in A$, $s' \in \mathbb{E}_n(A, \bar{a})$. Recall also that $\mathbb{E}_n f$ acts pointwise as f, for example $\mathbb{E}_n f[s, s'] = [f(s), f(s')]$. We can now show the properties in the lemma. Firstly, $\hat{f}[a] = \mathbb{E}_n f(\delta_A[a]) = \mathbb{E}_n f[[a]] = [f[a]]$. Next, recall δ_A satisfies the recursive property that if $s = s'[a]$, then $\delta_A(s) = \delta_A(s')[s]$. Hence, $\hat{f}(s) = \mathbb{E}_n f(\delta_A(s)) = \mathbb{E}_n f(\delta_A(s'[a])) = \mathbb{E}_n f((\delta_A(s')[s]) = [\mathbb{E}_n f(\delta_A(s'))][f(s)] = \hat{f}(s')[f(s)]$. The fact that $\hat{f}$ is nice follows immediately by an induction using the recursive property just proved.

□

Lemma 4.2.11. *For fixed n, d, l and $(A, \bar{a}), (B, \bar{b}) \in \mathcal{R}_\sigma(l)$ (including the case $l = 0$). Then there is a bijective correspondence between:*

- *"Nice" maps $\mathbb{E}_n A \to \mathbb{E}_n B$*
- *Winning strategies for Duplicator in the game $EF_n((A, \bar{a}), (B, \bar{b}))$.*

Proof. We know already there is a bijective between winning strategies for Duplicator in the game $EF_n((A, \bar{a}), (B, \bar{b}))$, and maps $\mathbb{E}_n(A, \bar{a}) \to (B, \bar{b})$. We now need to show there is a bijective correspondence between the latter and "nice" maps $\mathbb{E}_n A \to \mathbb{E}_n B$, which we claim is just given by composition with ϵ_B.

- Suppose $f, g : \mathbb{E}_n(A, \bar{a}) \to \mathbb{E}_n(B, \bar{b})$ are two "nice" maps such that $\epsilon_B \circ f = \epsilon_B \circ g$. We will show $f = g$. Let s be a sequence of shortest length such that $f(s) \neq g(s)$. If s has length 1, then write $s = [a]$. Since f, g are "nice" both $f(s)$ and $g(s)$ are sequences of length 1, say $[b], [b']$, but then $b = \epsilon_B f(s) = \epsilon_B g(s) = b'$, so $f(s) = f(s')$, a contradiction. If s has length at most 2, we can write $s = s'[a]$ for some $a \in A$. Since f preserves prefixes, we must have $f(s) = f(s')t$ for some sequence t of B. However, since f also preserves lengths of sequences we get t must be a sequence of length 1, so we can write $f(s) = f(s')[a]$. By the same argument, $g(s) = g(s')[a']$. But by assumption, $f(s') = g(s')$, and $a = \epsilon_B f(s) = \epsilon_B g(s) = a'$, so $f(s) = g(s)$ another contradiction, and we can conclude $f = g$.

- Now we check composition with ϵ_B is surjective. Let $f : \mathbb{E}_n(A, \bar{a}) \to (B, \bar{b})$. We know the co-Kleisli extension $\hat{f}$ is "nice", so it suffices to show $\epsilon_B \circ \hat{f} = f$. We shall do this by induction on the length of a sequence $s \in \mathbb{E}_n(A, \bar{a})$. If s has length 1, write $s = [a]$, then we get $\epsilon_B \hat{f}[a] = \epsilon_B[f(a)] = f(a)$. If s has length greater than 1, we can write $s = s'[a]$. By the above lemma, $\hat{f}[s] = \hat{f}(s'[f(s)])$. Hence $\epsilon_B \hat{f}[s] = f(s)$ as desired.

□

Corollary 4.2.12. *For fixed n, d, l and $(A, \bar{a}), (B, \bar{b}) \in \mathcal{R}_\sigma(l)$ (now where $l > 0$). Then there is a bijective correspondence between:*

- *"Nice" maps $\mathbb{E}_n^d(A, \bar{a}) \to \mathbb{E}_n^d(B, \bar{b})$*
- *Winning strategies for Duplicator in the game $EF_n((N_d(\bar{a}), \bar{a}), (N_d(\bar{b}), \bar{b}))$.*

Proof. We know there is a bijective correspondence between winning strategies for Duplicator in the game $EF_n((N_d(\bar{a}), \bar{a}), (N_d(\bar{b}), \bar{b}))$ and maps $\mathbb{E}_n(N_d(\bar{a}), \bar{a}) \to (N_d(\bar{b}), \bar{b})$, so the result follows by the above lemma. □

Now we consider the case where $l = 0$. In this case, we get the same bijective correspondences as before when we restrict only to "nice" maps, but lose any correspondence when we do not restrict. Consider the following example:

Example 4.2.13. *Suppose we can find a pair A, B and a map $f : \mathbb{E}_n^d A \to \mathbb{E}_n^d B$ such that $f[a] = [b]$ and $f[a, a'] = [b']$ for some distinct $a, a' \in A$, $b, b' \in B$. This could not correspond to any strategy in a local game for Duplicator, since there is no guarantee that by the structure of $\mathbb{E}_n^d$ that b is local to b'.*

We now give the characterisation similar to the case when $l = 0$.

Lemma 4.2.14. *For any natural numbers n, d (or $d = \infty$), and for any $A, B \in \mathcal{R}_\sigma$:*

1. *There is a bijective correspondence between:*
   - *Maps* $\mathbb{E}_n^d A \to B$
   - *Winning strategies for Spoiler in the following game: Spoiler chooses some* $a \in A$, *Duplicator chooses some* $b \in B$, *then the players play* $EF_{n-1}((N_d(a), a), (B, b))$.
2. *There is a bijective correspondence between:*
   - *"Nice" maps* $\mathbb{E}_n^d A \to \mathbb{E}_n^d B$
   - *Winning strategies for Spoiler in the following game: Spoiler chooses some* $a \in A$, *Duplicator chooses some* $b \in B$, *then the players play* $EF_{n-1}((N_d(a), a), (N_d(b), b))$.

Proof. Observe that $\mathbb{E}_n^d A$ is a disjoint union of structures of form $\mathbb{E}_{n-1}^d(A, a)$ over $a \in A$. Thus a map $\mathbb{E}_n^d A \to B$ is a collection of maps of form $\mathbb{E}_{n-1}^d(A, a)$ for each $a \in A$. Each such map is a winning strategy in $EF_{n-1}((N_d(a), a), (B, b))$ for some $b \in B$, hence we get correspondence 1. Correspondence 2 follows in a similar way, after we observe that a map $\mathbb{E}_n^d A \to \mathbb{E}_n^d B$ is "nice" if and only if it is the disjoint union of "nice" maps $\mathbb{E}_{n-1}^d(A, a) \to \mathbb{E}_{n-1}^d(B, b)$ for each $a \in A$. □

We reiterate the point made in the definition of $\mathbb{E}_n^d$ (when $l = 0$), that we have only "freely" chosen one element of a structure A, but we could have just as well freely chosen a tuple. Much like in the case when $l > 0$ we can describe the corresponding games as variants of $EF_n(A, B)$. The game corresponding to maps of form $\mathbb{E}_n^d A \to B$ is the variant of $EF_n(A, B)$ where all of Spoiler's choices after the first must be local to the first choice. The game corresponding to maps of form $\mathbb{E}_n^d A \to \mathbb{E}_n^d B$ is the game $EF_n(A, B)$ but where both Spoiler's and Duplicator's choices after the first must be local to the first choice.

Maps out of $\mathbb{E}_n^d$ can also be characterised using a logical relation, analogously to how maps out of $\mathbb{E}_n$ can be. To do this we use bounded first-order logic.

Definition 4.2.15.
- *The* $(\bar{x}, d)$*-localised existential quantifier is written* $\exists y \in N_d(\bar{x})$, *where* $\exists y \in N_d(\bar{x})\phi(\bar{x}, y)$ *is interpreted by structures*

$$\exists y \ (D(\bar{x}, y) < d \wedge \phi(\bar{x}, y))$$

  In other words for some $(A, \bar{a})$ *in* $\mathcal{R}_\sigma(l)$, $(A, \bar{a}) \models \exists y \in N_d(\bar{x})\phi(\bar{x}, y)$ *if and only if there is some* $a' \in N_d(\bar{a})$ *such that* $(A, \bar{a}, a') \models \phi(\bar{x}, y)$. *Note that sometimes we instead may localise around a subset of the free variables present in a formula rather than all of them.*
- *We define an* $(\bar{x}, d)$*-localised universal quantifier* $\forall y \in N_d(\bar{x}) := \neg(\exists y \in N_d(\bar{x}))\neg$.
- *The d-bounded fragment* $(N_d(\bar{x}), \mathcal{L}(l))$ *(when* $l > 0$*) are formulae constructed using atomic relations, boolean connectives, and* $(\bar{x}, d)$*-localised quantifiers where the only variables occurring free are among* $\bar{x}$ *(ie the*

same as $\mathcal{L}(l)$, except using local quantifiers around $\bar{x}$). When $l = 0$, the d-bounded fragment $(N_d(x_1), \mathcal{L})$ consists of formulae that are boolean combinations of formulae of type $\exists x_1 \phi$ where the $\phi \in (N_d(x_1), \mathcal{L}(1))$. We will sometimes leave the x_1 implicit, simply writing $(N_d, \mathcal{L})$, if it is clear from context what is meant.

- *We use the same superscript $+$ and subscript n to denote existential positive formula and bounded quantifier rank as before.*

We shall treat the localized quantifiers as connectives in their own right, but if $d \neq \infty$ then they are expressible in first order logic. This is because being adjacent in the Gaifman graph is expressible (since σ is finite, one can list all the possible ways two elements could occur in a relation together), and hence one can express there being a path of length d in the Gaifman graph. When $d = \infty$, these quantifiers are certainly not expressible in first order logic, as it is well known there is no query in first-order order logic which expresses "there is some path between two elements", but using the localised quantifiers, given some free variables x, y, one could write $\exists z \in N_\infty(x)(y = z)$ to express exactly that.

Given a formula $\phi(\bar{x}, \bar{y}) \in \mathcal{L}(l)$, we can make a localised formula $\phi_{(d,\bar{x})}$ replacing any instance of a quantifier with a corresponding d-localized quantifier around $\bar{x}$. If we are localizing our quantifiers around all of the free variables in a formula we shall just refer to the localized version as ϕ_d. We are primarily interested in this narrower case, but need the general version to prove the following fact, which is indeed the focus of defining the localized quantifiers in the first place!

Theorem 4.2.16. *Let $(A, \bar{a}, \bar{\alpha}) \in \mathcal{R}_\sigma(l)$, and $\phi(\bar{x}, \bar{y}) \in \mathcal{L}(l)$, where the tuple $\bar{a}, \bar{\alpha}$ interpret the variables $\bar{x}, \bar{y}$ respectively. Then $(A, \bar{a}, \bar{\alpha}) \models \phi_{(d,\bar{x})} \iff (N_d(\bar{a}) \cup \{\bar{\alpha}\}, \bar{a}, \bar{\alpha}) \models \phi$.*

Proof. Intuitively, the idea is that localized quantifiers can only search a given neighbourhood, so from the perspective of a d-bounded fragment, a d-local neighbourhood is the same as an entire structure. Formally we need an induct on n for every value of l:

- If $n = 0$, there are no quantifiers, so $\phi = \phi_{(d,\bar{x})}$ is a sentence only containing free variables, so the result follows immediately from the fact that $N_d(\bar{a}) \cup \{\alpha\}$ is an induced substructure of A.

- Suppose $n = m + 1$ and we have the inductive hypothesis. Since the translation $\phi \mapsto \phi_d$ keeps boolean connectives the same, it suffices to consider only the case $\phi(\bar{x}, \bar{y}) = \exists z \psi(\bar{x}, \bar{y}, z)$, where ψ has quantifier rank at most m. By definition we know that $\phi_{(d,\bar{x})} = \exists z\ (D(\bar{x}, z) < d \wedge \psi_{(d,\bar{x})}(\bar{x}, \bar{y}, z))$. Hence, $(A, \bar{a}, \bar{\alpha}) \models \phi_{(d,\bar{x})}$ if and only if $(A, \bar{a}, \bar{\alpha}) \models \exists z\ (D(\bar{x}, z) < d \wedge \psi_d(\bar{x}, \bar{y}, z))$, which is true if and only if there exists $\beta \in N_d(\bar{a})$ such that $(A, \bar{a}, \bar{\alpha}, \beta) \models \psi_{(d,\bar{x})}$. Applying the inductive hypothesis, this is true if and only if there exists a $\beta \in N_d(\bar{a})$ such that

$(N_d(\bar{a}) \cup \{\bar{\alpha}, \beta\}, \bar{a}, \bar{\alpha}, \beta) \models \psi$. Since by assumption $\beta \in N_d(\bar{a})$, and hence $(N_d(\bar{a}) \cup \{\bar{\alpha}, \beta\}) = N_d(\bar{a}) \cup \{\bar{\alpha}\}$, we know the previous statement is true if and only if $(N_d(\bar{a}) \cup \{\bar{\alpha}\}, \bar{a}, \bar{\alpha}) \models \phi$, and we are done.

□

In terms of Ehrenfeucht-Fraisse games, we now have:

Corollary 4.2.17. *For any given* n, d*, and* $l > 0$*. Given two structures* $(A, \bar{a}), (B, \bar{b}) \in \mathcal{R}_\sigma(l)$*, the following are equivalent:*

1. *Duplicator has a winning strategy in d-local version of* $EF_n((A, \bar{a}), (B, \bar{b}))$ *(where Spoiler and Duplicator can only play locally to* $\bar{a}$*,* $\bar{b}$ *respectively).*
2. *Duplicator has a winning strategy for the game* $EF_n((N_d(\bar{a}), \bar{a}), (N_d(\bar{b}), \bar{b})$*.*
3. $(N_d(\bar{a}), \bar{a}) \rightarrow_n (N_d(\bar{b}), \bar{b})$*.*
4. *For any* $\phi \in (N_d(\bar{x}), \mathcal{L}_n^+(l))$*, if* $(A, \bar{a}) \models \phi$ *then* $(B, \bar{b}) \models \phi$*.*

Proof. (1) and (2) are just two ways of phrasing the same game, one viewed as a game played on the universes of A and B with a restriction, and one viewed as a game played on substructures of A and B. (2) is equivalent to (3) by the standard Ehrenfeucht-Fraisse Theorem discussed in the introduction. (3) is equivalent to (4) in light of the above Theorem, once we observe that every $\phi \mapsto \phi_d$ is a bijection from $\mathcal{L}_n^+(l)$ to $(N_d(\bar{x}), \mathcal{L}_n^+(l))$. It is a bijection since the sets are constructed in exactly the same way, but $\mathcal{L}_n^+(l)$ uses $\exists$ where $(N_d(\bar{x}), \mathcal{L}_n^+(l))$ uses $\exists x \in N_d(\bar{x})$, so replacing a normal quantifier with a local one clearly yields a bijection.

□

The analogous Theorem also holds in the case $l = 0$. Recall that when $l = 0$, the first element is chosen freely and the rest of the game is played locally to that element.

Corollary 4.2.18. *For any given* n, d *and two structures* $A, B \in \mathcal{R}_\sigma$*, the following are equivalent:*

1. *Duplicator has a winning strategy in d-local version of* $EF_n(A, B)$ *(where Spoiler and Duplicator can only play locally to their first chosen element).*
2. *For every* $a \in A$*, there is* $b \in B$ *such that Duplicator has a winning strategy for the game* $EF_{n-1}((N_d(a), a), (N_d(b), b))$*.*
3. *For every* $a \in A$ *there is* $b \in B$ *such that* $(N_d(a), a) \rightarrow_{n-1} (N_d(b), b)$*.*
4. *For any* $\phi \in (N_d, \mathcal{L}_n^+)$*, if* $A \models \phi$ *then* $B \models \phi$*.*

Proof. As in the $l > 0$ case, items (1) and (2) are just two ways of stating the same thing. (2) and (3) are equivalent due to the $l = 1$ case above. □

As promised at the the beginning of the section, we can now fully translate the picture in the introduction into a localized version, though it has many more variations than in the global story. As an example, we state a simple corollary of the above in a style analogous to the introduction:

Theorem 4.2.19. *For any $(A, \bar{a}), (B, \bar{b}) \in \mathcal{R}_\sigma(l)$, the following are equivalent:*

1. $\mathbb{E}_n^d(A, \bar{a}) \to (B, \bar{b})$
2. *For $\phi \in \mathcal{L}_n^+(l)$, if $A \models \phi_d$ then $(B, \bar{b}) \models \phi$.*
3. *Duplicator has a winning strategy in the game $EF_n((N_d(\bar{a}), \bar{a}), (B, \bar{b}))$.*

4.3 Reachability

Next we turn our attention back to one of the motivational questions for this chapter: for what values of d is there a morphism $\mathbb{E}_n A \to \mathbb{E}_n^d A$? We shall consider this question, and use it to motivate the definition of reachability, and an associated comonad.

Firstly we define a function, not necessarily a morphism, f^d:

Definition 4.3.1. *For any $A \in \mathcal{R}_\sigma$, n a natural number, d a natural number or ∞, recursively define $f^d : \mathbb{E}_n A \to \mathbb{E}_n^d A$ as follows:*

- *If $s = [a]$ for some $a \in A$, then $f^d([a]) = [a]$.*
- *If $s = s'[a]$, for some $s' \in \mathbb{E}_n A$ and $a \in A$, then let $b \in A$ be the last entry of s' such that $D(a, b) < \frac{d}{2^{length(s)}}$, and let s'' be the longest prefix of s' whose last entry is b (if $d = \infty$ we take $\frac{d}{2^{length(s)-1}} = \infty$). We then define $f^d(s) := f^d(s'')[a]$. If no such b exists, then $f^d(s) := [a]$.*

Intuitively, this can be viewed as a method for Duplicator to extend from games on d-neighbourhoods of A to games on all of A. If in round i Spoiler plays “close” (this is the first case in recursive step of the above definition, so within distance $\frac{d}{2^i}$ in the ith round) to an element that has already been chosen, Duplicator can treat it as if it is a game being played on that neighbourhood, and play according to its known strategy. If Spoiler plays “far”, then Duplicator can treat it as starting a new game. Halving the distance that is considered “close” each time has two purposes; firstly, it caps the distance between the first and last elements of a sequence of “close” elements to at most $d(1/2+1/4+...+1/2^n) \leq d$, which ensures the range of f^d is indeed within $\mathbb{E}_n^d A$, and secondly, it ensures there is no tension between Duplicators strategies on local games (as newly chosen elements cannot be “close” to two previously chosen elements that were not close before). The latter will become clearer when we consider for which values of d the maps f^d are morphisms. However, this division by 2 at each step does make f^d rather degenerate for small values of d, since at some point the only element close to an element a will be a itself.

We remark here also that in the case $l > 0$ the analogously defined function would be from $\mathbb{E}_n(A, \bar{a}) \to (\mathbb{E}_n^d(A, \bar{a}) \oplus \mathbb{E}_n^d A)$. Everything we prove in the case $l = 0$ extends naturally to the case $l > 0$, but we work in the case $l = 0$ for notational convenience.

Theorem 4.3.2. *If $d \geq 2^n$, then $f^d : \mathbb{E}_n A \to \mathbb{E}_n^d A$ is a morphism for any $A \in \mathcal{R}_\sigma$.*

We will give a proof of this shortly. Unfortunately, this will not produce a natural transformation for any d, since first order morphisms may decrease distance. We provide an explicit example:

Example 4.3.3. *Let $A \in \mathcal{R}_\sigma$ be non-empty, and consider the morphism $p : A \oplus A \to A$. For the sake of the example, we shall encode the coproduct as $A \times \{0\} \cup A \times \{1\}$, so $p(a, i) := a$ for any $a \in A$ and $i = 0, 1$. Now by construction $D((a, 0), (a, 1)) = \infty$ for any $a \in A$. Hence $\mathbb{E}_n^d(p) \circ f^d([(a, 0), (a, 1)]) = \mathbb{E}_n^d(p)[(a, 1)] = [a]$. However, $f^d \circ \mathbb{E}_n(p)[(a, 0), (a, 1)] = f^d[a, a] = [a, a] \neq [a]$.*

We will need the following subclaim before proving Theorem 4.3.2:

Lemma 4.3.4. *Suppose $s, s' \in \mathbb{E}_n A$, and $s \sqsubset s'$ (recall this means s is a prefix of s'), and $D(\epsilon(s), \epsilon(s')) < \frac{d}{2^{length(s')-1}}$. Then $f^d(s) \sqsubset f^d(s')$.*

Proof. We prove this inductively on $m = \text{length}(s')$. If $m = 1$ there is nothing to show, so assume the inductive hypothesis and $m > 1$. First suppose $\epsilon(s)$ is the last entry of s within distance $\frac{d}{2^{m-1}}$ of $\epsilon(s')$. Then by definition, $f^d(s') = f^d(s)[\epsilon(s')]$ and we are done. Otherwise, there is some s'' such that $s \sqsubset s'' \sqsubset s'$ and $D(\epsilon(s''), \epsilon(s')) < \frac{d}{2^{m-1}}$. Now by the properties of distance, we know $D(\epsilon(s), \epsilon(s'')) \leq D(\epsilon(s), \epsilon(s')) + D(\epsilon(s'), \epsilon(s'')) < \frac{d}{2^{m-1}} + \frac{d}{2^{m-1}} = \frac{d}{2^{m-2}} \leq \frac{d}{2^{\text{length}(s'')-1}}$. Hence by the inductive hypothesis we can conclude $f^d(s) \sqsubset f^d(s'')$, and since we know by definition of f^d, $f^d(s'') \sqsubset f^d(s')$ we are done. □

Now we can proceed with the proof of Theorem 4.3.2:

Proof. (of Theorem 4.3.2). We need to show f^d is a morphism, so for some arbitrary $s_1, ..., s_m \in \mathbb{E}_n A$ and $R \in \sigma$ such that $\mathbb{E}_n A \models R(s_1, ..., s_m)$, we must show $\mathbb{E}_n^d A \models R(f^d(s_1), ..., f^d(s_m))$. Recall that $\mathbb{E}_n A \models R(s_1, ..., s_m)$ if and only if we have both $A \models R(\epsilon(s_1), ..., \epsilon(s_m))$ and for each i, j among $1, ..., m$, $s_i \sqsubset s_j$, $s_i = s_j$,or $s_j \sqsubset s_i$. Since $\mathbb{E}_n^d A$ is an induced substructure of $\mathbb{E}_n A$, we have $\mathbb{E}_n^d A \models R(f^d(s_1), ..., f^d(s_m))$ if and only if the same conditions hold for $f^d(s_1), ..., f^d(s_m)$. We know $\epsilon(f^d(s_i)) = \epsilon(s_i)$ for each i, so the first condition holds. Now we must check the second condition for each i, j. Clearly, there is nothing to show if $s_i = s_j$, so without loss of generality assume $s_i \sqsubset s_j$. Since $\epsilon(s_i)$ and $\epsilon(s_j)$ occur in a relation together, we know $D(\epsilon(s_i), \epsilon(s_j)) \leq 1$. We also know $1 \leq \frac{d}{2^n} < \frac{d}{2^{\text{length}(s_j)-1}}$, where the first inequality holds by assumption that $d \geq 2^n$, and the second holds since $\text{length}(s_j) - 1 \leq n - 1 < n$. Hence we can apply the lemma above to obtain $f^d(s_i) \sqsubset f^d(s_j)$ as desired. □

Theorem 4.3.2 essentially tells us that any move further than a distance of 2^n from the moves already in play is too far to be a useful move. We can generalise this as follows:

Definition 4.3.5. *We will say a move α is n-slack from $\bar{a}$, if there exists $H \subset \mathbb{E}_n(A, \bar{a})$ with the following properties:*

- *H is prefix closed.*
- *$[a_1, ..., a_l] \in H$, hence so is all its prefixes, but $[a_1, ..., a_l, \alpha]$ is not in H.*
- *The map $f^B : \mathbb{E}_n(A, \bar{a}) \to H \oplus \mathbb{E}_n A$ is a morphism, where f^H defined below is a generalisation of f^d:*
  - *If $s \sqsubseteq [a_1, ..., a_l]$ then $f^H(s) = s \in H$*
  - *If $s = [a_1, ..., a_l, a]$, and $s \in H$ then $f^H(s) := s \in H$, otherwise $f^H(s) = [a] \in \mathbb{E}_n A$.*
  - *If $s = s'[a]$, if there is prefix s'' of s' such that $f^H(s'')[a] \in H$, then $f^H(s) := f^H(\hat{s})[a] \in H$ where $\hat{s}$ is the longest such prefix. Else, $f^H(s) := f^H(\hat{s})[a] \in \mathbb{E}_n A$, where $\hat{s}$ is the longest prefix of s' such that $f^H(\hat{s}) \in \mathbb{E}_n A$.*

Like before f^H is simply a function that takes a sequence and sorts it into disjoint subsequences, one that is in H and one that is not in H. It is slightly simpler than f^d as we only sort into two subsequences, unlike f^d which sorts a sequence s into many possible subsequences. The definition intuitively captures the notion of Spoiler playing a move α that is too far away to interact with any of the previous moves. In the definition H represents a local game around $a_1, ..., a_n$, so f^H being a morphism exactly represents Duplicator being able to find a winning strategy in the overall game if it can find a winning strategy in a local game represented by H, in which Spoiler cannot start with the move α, and a winning strategy in a "fresh" game, with no elements already chosen, represented by the disjoint copy of $\mathbb{E}_n A$. We can observe that $\mathbb{E}^d_n(A, \bar{a})$ witnesses the fact that any move of distance at most 2^n from the previously played elements is n-slack, since we have just shown f^d is a morphism. However, this is only a sufficient condition, not a necessary one. For example, if A were a structure with 100 elements and satisfying only a single 100 place relation between all its elements, then all elements would be 1-slack from one another, despite being adjacent in the Gaifman graph. We will now constructively define the term "reachable", and an associated comonad, which in fact characterise exactly the moves that are not slack for Spoiler. For the purposes of reachability, we will not distinguish between reachability from tuples and sequences (ie we will say α is n-reachable from $[a_1, ..., a_l]$ and α is n-reachable from $a_1, ..., a_l$ interchangeably).

Definition 4.3.6. *For some given $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$, for $n \geq 1$ we recursively define "α is n-reachable from $\bar{a}$", which we will sometimes write "α is n-reachable from $(A, \bar{a})$" if A is not clear from context:*

1. *There is a relation $R \in \sigma$ and a tuple t (of A) containing α, at least 1 element from among $\bar{a}$, and at most $n-1$ other elements, such that $A \models R(t)$ (a tuple t such that there is an $R \in \sigma$ such that $A \models R(t)$ we will henceforth refer to as a guarded tuple). Or, α occurs among $\bar{a}$.*

2. *There is some β such that α is $(n-1)$-reachable from $\bar{a}, \beta$, and β is $n-1$ reachable from $\bar{a}$.*

As a special case, when $n = 0$, a is n-reachable from $\bar{a}$ only if it occurs among $\bar{a}$.
We will say say $s = [a_1, ..., a_l, b_1, ..., b_m] \in \mathbb{E}_n(A, \bar{a})$ is an n-reachable sequence from $\bar{a}$, if for each i from $1, ..., m$, b_i is $(n-i)$-reachable from $[\bar{a}, b_1, ..., b_{i-1}]$.

When considering reachability, we will often use inductive proofs, and refer to clause (1) and (2) from above. Observe from the above definition is that that 1-reachability (and hence recursively n-reachability) must always be witnessed by (1) (at some point). Hence, when doing an inductive proof, we may first check case (1) (as this covers the case $n = 1$), and then check case (2) assuming an inductive hypothesis. We will often omit the $n = 0$ case as it is trivial. In terms of sequences, we can say α is n-reachable from $\bar{a}$ if and only there is an n-reachable sequence from $\bar{a}$, such that α is $(n - \text{length}(s))$-reachable from s via (1).

Lemma 4.3.7. *If a is n-reachable from $\bar{a}$ in some structure A, then a is not n-slack from $\bar{a}$.*

Proof. We do an induction on n, and start with clause (1) as remarked above, noting it covers the case $n = 1$. If we are in clause (1), let $b_1, ..., b_i$ be at most $n-1$ elements of a such that there is a tuple t containing a, $b_1, ..., b_i$, and at least one element from $a_1, ..., a_l$, a_1 say. Suppose for a contradiction that a is n-slack from $\bar{a}$, and let this be witnessed by some H. Consider the sequence $s = [a_1, ..., a_l, a] \in \mathbb{E}_n(A, \bar{a})$, so by assumption we know $f^H(S) \in \mathbb{E}_n A$. Now $f^H([a_1]) \in H$, hence there can be no relations containing $f^H([a_1])$ and $f^H(s)$, as they occur in different halves of a coproduct. However by assumption, there is a relation containing $[a_1], s, [a_1, ..., a_l, a, b_1], ..., [a_1, ..., a_l, a, b_1, ..., b_i]$, contradicting the fact that f^H is a morphism.
If we are in clause (2) we can now make use of the inductive hypothesis. So we can find a sequence $\beta_1, ..., \beta_i$ such that β_i is $(n-i)$-reachable from $\bar{a}, \beta_1, ..., \beta_{i-1}$, and a is $(n-i)$-reachable from $\bar{a}, \beta_1, ..., \beta_i$ via clause (1) of reachability. This comes from iteratively applying clause (2) of the definition until we reach clause (1). Suppose for a contradiction that a is n-slack from $\bar{a}$, witnessed by some H. Consider the subset $H' \subset H$ being those sequences in H where there are at most $n-1$ elements after a_l. Now observe, $f^{H'}$ acts on $\mathbb{E}_{n-1}(A, \bar{a})$ by restricting f^H, so is a morphism. H' also satisfies the other conditions in the definition of slackness, so we must conclude that $[a_1, ..., a_l, \beta_1] \in H' \subset H$ by the inductive hypothesis, else H' would witness β is $(n-1)$-slack from $\bar{a}$ when it is $(n-1)$-reachable from $\bar{a}$. Next consider the subset of H' consisting of all sequences s such that $s \sqsubseteq [a_1, ..., a_l, \beta_1]$ or $s \sqsupseteq [a_1, ..., a_l, \beta_1]$. By an identical

argument as for β_1, we can conclude once again using the inductive hypothesis that $[a_1, ..., a_l, \beta_1, \beta_2]$ is in this subset, and hence in H. Repeating this, we can see that $[a_1, ..., a_l, \beta_1, ..., \beta_i]$ and all of its prefixes are in H. Now we know a is $(n-i)$-reachable from $a_1, ..., a_l, \beta_1, ..., \beta_i$ via clause (1), so let this be witnessed by $b_1, ..., b_{n-i-1}$. Similarly to before, we know there is a relation among prefixes of $[a_1, ..., a_l, a, \beta_1, ..., \beta_i, b_1, ..., b_{n-i-1}]$ containing $[a_1, ..., a_l, a]$, and at least one other element s among $[a_1], ..., [a_1, ..., a_l, a, \beta_1, ..., \beta_i]$. However, observe $f^H([a_1, ..., a_l, a, \beta_1, ..., \beta_j]) = [a_1, ..., a_l, \beta_1, ..., \beta_j] \in H$ for any j, and by assumption $f^H([a_1, ..., a_l, a]) = [a] \in \mathbb{E}_n A$. Hence, $f^H(s)$ and $f^H([a_1, ..., a_l, a])$ are in separate halves of a coproduct, so there cannot be a relation including them, contradicting the fact that f^H is morphism. □

Further on we prove an analogue of Theorem 4.3.2 but for reachability rather than distance, which shows the converse of the lemma above, that an element being n-slack implies it is not n-reachable. Putting these together we do in fact conclude that reachability is simply a constructive definition for being not slack, which gives a reasonable intuition behind its definition.

Example 4.3.8. *An interesting special case of reachability is that if σ contains only relations of arity 2 or lower, then for any $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$ and $\alpha \in A$, α is n-reachable from $\bar{a}$ if and only if $D(\bar{a}, \alpha) < 2^n$. We can see this inductively. Recall α is 1-reachable if it occurs among $\bar{a}$ or if it creates a new relation when added to $\bar{a}$. The key point is that if σ contains relations of arity 2 or lower, then this is exactly the same as the statement $D(\alpha, \bar{a}) = 0, 1$, ie $D(\alpha, \bar{a}) < 2^1$. For the inductive step, note that if $D(\bar{a}, \alpha) < 2^n$ then there exists α' such that $D(\bar{a}, \alpha') < 2^{n-1}$ and $D(\alpha, \alpha') < 2^{n-1}$. The former statement implies α' is $(n-1)$-reachable from $\bar{a}$ (by the inductive hypothesis). The latter statement implies α is $(n-1)$-reachable from α', hence $(n-1)$-reachable from $\bar{a}, \alpha'$.*

Next we can create a subfunctor of $\mathbb{E}_n$ similar to how we created the subfunctors $\mathbb{E}_n^d$:

Definition 4.3.9. *We define the functor $\mathbb{R}_n$ as follows for $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$:*

- *If $l > 0$, $\mathbb{R}_n(A, \bar{a})$ is the induced substructure $\{s \in \mathbb{E}_n(A, \bar{a}) : s \text{ is an } n\text{-reachable sequence from } \bar{a}\}$.*
- *If $l = 0$, $\mathbb{R}_n A := \bigcup_{a \in A} \mathbb{R}_{n-1}(A, a)$, ie the set of all sequences s such that form an $(n-1)$-reachable sequence from (A, a),where a is the first entry of s.*

In both cases $\mathbb{R}_n$ acts on morphisms by restricting from $\mathbb{E}_n$.

We claim $\mathbb{R}_n$ is in fact a comonad, and that its counit and comultiplications are simply the restrictions of the counit and comultiplication for $\mathbb{E}_n$. Given the maps are simply restrictions, we know they are morphisms and all the necessary diagrams commute, provided all the morphisms land in the desired ranges. In other words, we need to check if $f : (A, \bar{a}) \to (B, \bar{b})$, that the range of $\mathbb{E}_n f$ when restricted to $\mathbb{R}_n(A, \bar{a})$ is within $\mathbb{R}_n(B, \bar{b})$, and we need to check the range of δ

when restricted to $\mathbb{R}_n(A,\bar{a})$ is within $\mathbb{R}_n\mathbb{R}_n(A,\bar{a})$. We will prove these claims via some lemmas:

Lemma 4.3.10. *Let $s = [a_1, ..., a_l, \alpha, \beta] \in \mathbb{E}_n(A,\bar{a})$ for some $(A,\bar{a}) \in \mathcal{R}_\sigma(l)$. Then the map that sends $[a_1, ..., a_l, \alpha] \mapsto s$, sends $s \mapsto [a_1, ..., \alpha]$ and sends s' to s'' where s'' is the sequence s but the α at entry $l+1$ and the β at entry $l+2$ are swapped is a in isomorphism $\mathbb{E}_n(A,\bar{a},\alpha,\beta) \to \mathbb{E}_n(A,\bar{a},\beta,\alpha)$.*

Proof. The fact that the map is a homomorphism follows because it preserves the last element of a sequence and the prefix relation between sequences. It is an isomorphism because the inverse map is also a homomorphism for the same reason. □

Lemma 4.3.11. *Let $(A,\bar{a}),(B,\bar{b}) \in \mathcal{R}_\sigma(l)$, and let α be n-reachable from $\bar{a}$:*

1. *If $f : (A,\bar{a}) \to (B,\bar{b})$ is a morphism in $\mathcal{R}_\sigma(l)$, then if $f(\alpha)$ is n-reachable from $\bar{b}$.*

2. *$[a_1, ..a_l, \alpha]$ is n-reachable from $\delta([a_1, ..., a_l]) = [[a_1], [a_1, a_2], ..., [a_1, ..., a_l]]$.*

Proof. Both of these we prove inductively.

- In either case, if condition (1) of reachability holds because α occurs among $\bar{a}$, then there is nothing to show. If (1) holds otherwise, let it be witnessed by $c_1, ..., c_m \in A$ (where $m \leq n$), ie there is some guarded tuple containing α, $c_1, ..., c_m$, and at least one element of $\bar{a}$. Then $f(c_1), ..., f(c_m)$ witness n-reachability for $f(\alpha)$ from $\bar{b}$, and $[a_1, ..., a_l, \alpha, c_1], [a_1, ..., a_l, \alpha, c_1, c_2], ..., [a_1, ..., a_l, \alpha, c_1, ..., c_m]$ witness n-reachability for α from $\delta([a_1, ..., a_l])$.

- If condition (2) of reachability holds we can use an inductive hypothesis and assume $n > 1$ (noting (1) covers the case $n = 1$). We know there is some β such that α is $(n-1)$-reachable from $\bar{a}, \beta$, and β is $(n-1)$-reachable from $\bar{a}$. By the inductive hypothesis, $f(\alpha)$ is $(n-1)$-reachable from $\bar{b}, f(\beta)$, and $f(\beta)$ is $(n-1)$-reachable from $\bar{b}$, so we can conclude $f(\alpha)$ is n-reachable from $\bar{b}$. We also claim $[a_1, ..., a_l, \alpha, \beta]$ is $(n-1)$-reachable from $\delta([a_1, ..., a_l])$ and $[a_1, ..., a_l, \alpha]$ is $(n-1)$-reachable from $\delta([a_1, ..., a_l])[a_1, ..., a_l, \alpha, \beta]$ (both of which follow from applying the inductive hypothesis, the previous lemma, and the first item of this lemma). From this we can conclude $[a_1, ..., a_l, \alpha]$ is n-reachable from $\delta([a_1, ..., a_l])$ as desired.

□

We now have that $\mathbb{R}_n$ is a comonad by restricting ϵ and δ.

Corollary 4.3.12. *$(\mathbb{R}_n, \epsilon, \delta)$ is a comonad.*

Proof. We know from the above lemma that morphisms preserve reachable sequences, and that the sequence of prefixes of a reachable sequence is also a reachable sequence. Hence if $f : (A,\bar{a}) \to (B,\bar{b})$ is a morphism then so is

$\mathbb{R}_n f := \mathbb{E}_n f|_{\mathbb{R}_n A} : \mathbb{R}_n(A, \bar{a}) \to \mathbb{R}_n(B, \bar{b})$. We also know $\delta|_{\mathbb{R}_n(A,\bar{a})}\mathbb{R}_n(A, \bar{a}) \to \mathbb{R}_n\mathbb{R}_n(A, \bar{a})$ is a morphism, as is $\epsilon|_{\mathbb{R}_n(A,\bar{a})}\mathbb{R}_n A \to (A, \bar{a})$. All the necessary diagrams will commute since they commute for $\mathbb{E}_n$. □

We remark here that the above proof would fail for $\mathbb{E}_n^d$ when trying to restrict the comultiplication. Inherent to the definition of n-reachability is that it is witnessed by a sequence of n elements, hence the property can carry over nicely to structures of form $\mathbb{R}_n A$. However, this does not hold for being within a d-neighbourhood. For instance, if $D(a, b) \geq 2^n$ for some a, b in a structure A then $D([a], [a, b]) = \infty$ in $\mathbb{E}_n A$. Even small distances will still create issues, if they are witnessed by a relation of large arity. For a rather degenerate example, if σ contains a relation of arity $n+1$, then it may be possible to have $D(a, b) = 1$ in some structure A, but $D([a], [a, b]) = \infty$ in $\mathbb{E}_n A$.

We can interpret Ehrenfeucht-Fraisse games using $\mathbb{R}_n$ also:

Definition 4.3.13. *We use an R within EF_n to indicate only n-reachable sequences can be played on a given structure. For example, given $(A, \bar{a}), (B, \bar{b}) \in \mathcal{R}_\sigma(l)$ the game $EF_n(R(A, \bar{a}), (B, \bar{b}))$ is the game where Spoiler only is limited to playing an n-reachable sequence, but Duplicator has no restrictions.*

By analogy to the lemma in the introduction that winning strategies for Duplicator in $EF_n(A, B)$ are the same as maps $\mathbb{E}_n A \to B$, and the same in the case of $\mathbb{E}_n^d$, we claim the following:

Lemma 4.3.14. *For $(A, \bar{a}), (B, \bar{b}) \in \mathcal{R}_\sigma(l)$, there is a bijective correspondence between:*

- *Maps $\mathbb{R}_n(A, \bar{a}) \to (B, \bar{b})$*
- *Winning strategies for Duplicator in the game $EF_n(R(A, \bar{a}), (B, \bar{b}))$.*

There is also a bijective correspondence between:

- *Nice maps $\mathbb{R}_n(A, \bar{a}) \to \mathbb{R}_n(B, \bar{b})$*
- *Winning strategies for Duplicator in the game $EF_n(R(A, \bar{a}), R(B, \bar{b}))$.*

Recall that nice maps $\mathbb{R}_n(A, \bar{a}) \to \mathbb{R}_n(B, \bar{b})$ are those that are prefix preserving, and length preserving, and correspond exactly to those maps which are co-Kleisli extensions of maps of form $\mathbb{R}_n(A, \bar{a}) \to (B, \bar{b})$. In fact, since $\mathbb{R}_n$ is a comonad and has a co-Kleisli extension, the correspondence is actually four way:

Corollary 4.3.15. *For $(A, \bar{a}), (B, \bar{b}) \in \mathcal{R}_\sigma(l)$, there is a bijective correspondence between:*

- *Maps $\mathbb{R}_n(A, \bar{a}) \to (B, \bar{b})$*
- *Winning strategies for Duplicator in the game $EF_n(R(A, \bar{a}), (B, \bar{b}))$*

- *Nice maps* $\mathbb{R}_n(A, \bar{a}) \to \mathbb{R}_n(B, \bar{b})$
- *Winning strategies for Duplicator in the game* $EF_n(R(A, \bar{a}), R(B, \bar{b}))$

Proof. It is sufficient to observe that there is a bijective correspondence between nice maps $\mathbb{R}_n(A, \bar{a}) \to \mathbb{R}_n(B, \bar{b})$ and maps $\mathbb{R}_n(A, \bar{a}) \to B$. Given a map $f : \mathbb{R}_n(A, \bar{a}) \to (B, \bar{b})$, let $\hat{f}$ denote its co-Kleisli extension. By construction of $\hat{f}$ it is clear that $\epsilon_B \hat{f} = f$. Any nice map g is a co-Kleisli extended map, so suppose $g = \hat{f}$ for some map $f : \mathbb{R}_n(A, \bar{a}) \to (B, \bar{b})$. Now $(\widehat{\epsilon_B g}) = \widehat{\epsilon_B \hat{f}} = \hat{f} = g$. Hence the assignments $f \mapsto \hat{f}$ and $g \mapsto \epsilon_B g$ give the bijective correspondence we need. □

In terms of strategies for Duplicator, the above says "if Spoiler plays a reachable sequence of moves, then a winning strategy for Duplicator must also be a reachable sequence of moves."

Example 4.3.16. *For an intuitive picture of this, consider the example above where it was assumed σ had no relations of arity greater than 2, and hence reachability was distance $< 2^n$, and suppose Spoiler starts a game by choosing some $a \in A$ and Duplicator chooses some $b \in B$. Suppose further that Spoiler chooses some a' such that $D(a, a') < 2^{n-1}$ (since there are $n-1$ rounds remaining), but Duplicator chooses some b' such that $D(b, b') \geq 2^{n-1}$. For the next round, Spoiler can choose some a'' with distance $< 2^{n-2}$ from both a, a', but Duplicator cannot find such a b'', so suppose Duplicator picks a b'' of distance $\geq 2^{n-2}$ from b. Spoiler then repeats the same method with a and a'' by choosing an element that bisects their distance. Again, Duplicator will not be able to find such an element. Repeating this process, Spoiler will win after the nth round.*

Next we aim to show the following (when $l = 0$): that Duplicator has a winning strategy in $EF_n(A, B)$ if and only if Duplicator has a winning strategy in $EF_n(R(A), B)$. This will take the form of a morphism $\mathbb{E}_n(A) \to \mathbb{R}_n(A)$ for each $A \in \mathcal{R}_\sigma$ (and the obvious fact there is a morphism in the opposite direction). Once again in the case $l > 0$ we get a morphism $\mathbb{E}_n(A, \bar{a}) \to \mathbb{R}_n(A, \bar{a}) \oplus \mathbb{R}_n A$, and the corresponding result "Duplicator has a winning strategy in $EF_n((A, \bar{a}), (B, \bar{b}))$ if and only if Duplicator has a winning strategy in $EF_n(R(A), B)$ and $EF_n(R(A, \bar{a}), (B, \bar{b}))$" in terms of games. The latter result follows easily from the former, so we will focus on the case $l = 0$.

We also once again remark that we will get a set of morphisms rather than a natural transformation. This is for the same reason as before, that being a reachable sequence is not reflected by morphisms, ie there might be some sequence s that is not an n-reachable sequence, but becomes an n-reachable sequence under a morphism. Similarly to before, we will define a set map $f^r : \mathbb{E}_n A \to \mathbb{R}_n A$ for each $A \in \mathcal{R}_\sigma$, and show it is a morphism.

Definition 4.3.17. *For $A \in \mathcal{R}_\sigma$, we define the map $f^r : \mathbb{E}_n A \to \mathbb{R}_n A$ recursively on some $s \in \mathbb{E}_n A$ as follows:*

- *If $s = [a]$ for some $a \in A$, then $f^r(s) = [a]$.*
- *If $s = s'[a]$, let s'' be the longest prefix of s' such that $f(s'')[a]$ is an n-reachable sequence. Then define $f(s) = f(s'')[a]$. If no such s'' exists, then set $f(s) = [a]$.*

It is worth understanding intuitively what f^r does to a sequence and what it is describing in terms of strategies for Duplicator. Given some sequence s, f^r effectively organises it into disjoint reachable subsequences (though f^r only outputs the reachable subsequence that has the same last element of s). We can see this recursively. Initially, f^r returns a singleton sequence to a singleton sequence. At some step $i+1$, f^r checks the subsequences it has already made for one which $\epsilon(s)$ is reachable from, and adds s to one of those sequences (it chooses the sequence that was added to last if there is more than one to add to). This corresponds directly to a method for Duplicator converting a strategy in $EF_n(R(A), B)$ into a strategy in $EF(A, B)$, by organising it into several separate reachable subgames. At stage i, if Spoiler plays a move that is reachable from one of the previous subgames, then Duplicator moves according to the strategy in that subgame (and picks the last subgame that was played in if there was more than one to choose from). If Spoiler plays a move not reachable from any of the previously started subgames, then Duplicator starts a new subgame. Hence crucial to proving that f^r is a morphism is proving is checking that none of these subgames can possibly overlap, else this would cause a conflict in Duplicators strategy (as the strategies on both subgames may disagree). Before proving this, we will need to collect some facts about reachability:

Lemma 4.3.18. *Let $(A, \bar{a}) \in \mathcal{R}_\sigma(l)$ and let α be n-reachable from $\bar{a}$:*

1. *For any $m \geq n$, and tuple t containing all the entries of $\bar{a}$, α is m-reachable from t.*
2. *Either clause (1) holds from the definition of reachability, or there exists some β such that α is $(n-1)$-reachable from β and β is $(n-1)$-reachable from $\bar{a}$.*
3. *For any $a, b \in A$, a is n-reachable from b if and only if b is n-reachable from a.*
4. *Suppose in addition, there is some β such that β is m-reachable from $\bar{a}, \alpha$, where $m < n$. Then either β is n-reachable from $\bar{a}$ or there exists some α' such that α' is n-reachable from $\bar{a}$, and β is m-reachable from α'.*

Proof. The first item above is immediate from an induction and the definition of reachability, we just include it for reference. Notice for instance it implies that the second statement in the second item is a sufficient condition for α being n-reachable from $\bar{a}$ (as well as just a necessary one, which is what the item claims). We will prove the second, third and fourth claims by induction:

- We start with the second claim. So we are given some α that is n-reachable from $\bar{a}$. This means either clause (1) holds or clause (2) holds from the definition of reachability. Certainly if (1) holds we are done, so we can assume there exists some β such that α is $(n-1)$-reachable from $\bar{a}, \beta$, and β is $(n-1)$-reachable from $\bar{a}$. This gives us two statements to analyse, of which we focus on the first. Since α is $(n-1)$-reachable from $\bar{a}, \beta$, then either (1) or (2) holds. If (1) holds, let t be a guarded tuple that contains α, at least one element of $\bar{a}.\beta$ and at most $n-1$ other elements. If t contains at least one element from $\bar{a}$ then t in fact witnesses that α is n-reachable from $\bar{a}$ via (1) (since it contains at most $n-1$ other elements from A that are not β, so it contains at most n other elements from A possibly including β), so we would be done. If t contains no elements of $\bar{a}$ (hence only β), then t witnesses directly that α is $(n-1)$-reachable from β, and we would also be done (since β would have exactly the property required for the second item of the lemma). Hence, we can assume clause (1) does not hold for the statement that α is $(n-1)$-reachable from $\bar{a}, \beta$, hence by the inductive hypothesis, we may find some γ_1 such that α is $(n-2)$-reachable from γ_1, and γ_1 is $(n-2)$-reachable from $\bar{a}, \beta$. Once again we analyse the former of the two statements. If the fact that γ_1 is $(n-2)$-reachable is witnessed by some guarded tuple t (ie clause (1)), we will either conclude that γ_1 is $(n-1)$-reachable from $\bar{a}$ (in the case t contains at least 1 element of $\bar{a}$) in which case we can use γ_1 to satisfy the conclusion of the lemma (since α is $(n-2)$-reachable, hence $(n-1)$-reachable from γ_1), or we will conclude that γ_1 was $n-2$ reachable from β alone (in the case t contains no elements of $\bar{a}$ so only contains β), and hence α was $(n-1)$-reachable from β alone, and are therefore done. So, once again we may by the inductive hypothesis find some γ_2 such that such that γ_2 is $(n-3)$-reachable from $\bar{a}, \beta$, and γ_1 is $(n-3)$-reachable from γ_2. Observe that since γ_1 is $(n-3)$-reachable from γ_2, and α is $(n-2)$-reachable from γ_1, we know α is $(n-1)$-reachable from γ_2. We then apply the same reasoning as before to the fact that γ_2 is $(n-3)$-reachable from $\bar{a}, \beta$. If it is witnessed by some guarded tuple t via clause (1), then we will either be able to use β or γ_2 to meet the property required by the lemma. If it is witnessed by clause (2), we use the inductive hypothesis again to find some γ_3 that is $(n-4)$-reachable to $\bar{a}, \beta$, and such that γ_2 is $(n-4)$-reachable from γ_3. Similarly to γ_2, we also know that α is $(n-1)$-reachable from γ_3 (since γ_2 is $(n-4)$-reachable from γ_3 is $(n-3)$-reachable from γ_1 is $(n-2)$-reachable from α). We then repeat the same reasoning for γ_3, and so on. At some point, this process must terminate when clause (1) holds for some γ_i, and we will be done.

- For the third item, let $a, b \in A$, and b be n-reachable from a. Either clause (1) holds, in which case it is immediate that a is n-reachable from b, or by item 2 just proven, we may find some c such that b is $(n-1)$-reachable from c, and c is $(n-1)$-reachable from a. Using the inductive hypothesis, we know c is $(n-1)$-reachable from b, and a is $(n-1)$-reachable from c.

Hence, a is n-reachable from b.

- The fourth item we prove by induction on m. Suppose firstly that clause (1) holds for the fact that β is m-reachable from $\bar{a}, \alpha$ (so the covers the base case $m = 1$) hence, it is witnessed by some guarded tuple t. Similarly to above, either t contains at least one entry from $\bar{a}$, in which case the tuple witnesses β is $(m + 1)$-reachable (and hence n-reachable, since $m < n$) from $\bar{a}$, or t contains no entries from $\bar{a}$, and witnesses that β is in fact m-reachable from α. If (1) does not hold, by the second item of lemma we can find some γ such that γ is $(m - 1)$-reachable from $\bar{a}, \alpha$, and β is $(m - 1)$-reachable from γ. Now applying the inductive hypothesis to γ, either γ is n-reachable from $\bar{a}$ (in which case γ satisfies the requirements of the lemma), or there is some β' such that α' is n-reachable from $\bar{a}$ and γ is $(m-1)$-reachable from α'. But then we would also know β is m-reachable from γ so α' would satisfy the requirements of the lemma.

□

We next prove what was set out in the discussion above, which is the final precursor to proving f^r is a morphism (which is analogous to what we used to prove f^d was a morphism for $d > 2^n$ in the previous section):

Lemma 4.3.19. *Suppose that $s \sqsubset s' \in \mathbb{R}_n A$ for some $A \in \mathcal{R}_\sigma$, and suppose a is $(n - \mathit{length}(s'[a]))$-reachable from s. Then $f^r(s) \sqsubset f^r(s'[a])$.*

Proof. In the language of the intuitive description of f^r, this is akin to proving that if two subgames overlap then they are the same subgame. We prove this by induction on the length of a sequence s. If the length is 1, then the claim is trivially true. Suppose now s is of length greater than one, and we can use the inductive hypothesis. By the definition of f^r, $f^r(s'[a]) := f^r(s'')[a]$, where s'' the longest prefix of s' such that a is $(n - \text{length}(s'[a]))$-reachable from s'', assuming there is one such prefix. Since, by assumption, s' is such a prefix, we may assume it is not the longest such, else the claim is immediately true, so let s'' be the longest such prefix, so $s \sqsubset s'' \sqsubseteq s'$, and a is both $(n - \text{length}(s'[a]))$-reachable from both s and s'', and also assume $f^r(s)$ is not a prefix of $f^r(s'')$ (else $f^r(s) \sqsubset f^r(s'') \sqsubset f^r(s'[a])$). As remarked previously, f^r arranges sequences into disjoint, reachable subsequences, so in fact $f^r(s)$ and $f^r(s'')$ can have no overlap at all.

Suppose firstly that $f^r(s'') = [\epsilon(s'')]$. Then a is $(n - \text{length}(s'[a]))$-reachable from $\epsilon(s'')$, hence by the lemma above, $\epsilon(s'')$ is $(n - \text{length}(s'[a]))$-reachable from a. Since a is also $(n - \text{length}(s'[a]))$-reachable from s, we know $\epsilon(s'')$ is in fact $(1 + n - \text{length}(s'[a]))$-reachable from s. But by the inductive hypothesis, we could then conclude $f^r(s) \sqsubset f^r(s'')$ (since $(1 + n - \text{length}(s'[a])) \leq (n - \text{length}(s'')))$, which would contradict $f^r(s)$ and $f^r(s'')$ having no overlap. So we may assume $f^r(s'') = f^r(t)[\epsilon(s'')]$ for some $t \sqsubset s''$. Since $\epsilon(s'')$ is $(n - \text{length}(s''))$-reachable from $f^r(t)$, and a is $(n - \text{length}(s'[a]))$-reachable from $f^r(s'') = f^r(t)[\epsilon(s'')]$, either a is $(n - \text{length}(s'[a]))$-reachable from $f^r(t)$, or there exists some α such that α is $(n - \text{length}(s''))$-reachable from $f^r(t)$ and a is

$(n - \text{length}(s'[a]))$-reachable from α. In the former case, observe we can apply the inductive hypothesis to the sequence $s, t \sqsubset \hat{s}$ where $\hat{s}$ is the sequence s'' but with the last entry swapped for a. This would imply $f^r(t), f^r(s) \sqsubset f^r(\hat{s})$, and hence overlap, but this contradicts the fact that $f^r(t) \sqsubset f^r(s'')$ which has no overlap with $f^r(s)$. In the latter case, set $\hat{s}$ to be the sequence s'' that has instead swapped its last entry for α. As in the case above where $f^r(s'')$ was a one element sequence, this implies that α is also $(n - \text{length}(\hat{s}))$-reachable from $f^r(s)$. But once again, applying the inductive hypothesis to $s, t \sqsubset \hat{s}$ yields $f^r(s)$ and $f^r(t)$ overlap, and hence so do $f^r(s)$ and $f^r(s'')$, contradicting our initial observations. ☐

Now we can finally prove f^r is indeed a morphism:

Theorem 4.3.20. *$f^r : \mathbb{E}_n A \to \mathbb{R}_n A$ is a morphism for any n, and $A \in \mathcal{R}_\sigma$.*

Proof. Suppose $\mathbb{E}_n A \models R(s_1, ..., s_i)$ for some $s_1, ..., s_m \in \mathbb{E}_n A$. This is true if and only if $A \models (\epsilon(s_1), ..., \epsilon(s_m))$, and for each i, j, either $s_i \sqsubset s_j$, $s_j \sqsubset s_i$, or $s_i = s_j$. The second condition is true if and only if we can arrange the s_i into a sequence $s_{i_1} \sqsubset s_{i_2} .... \sqsubset s_{i_m}$. Now clearly f^r commutes with ϵ, so it suffices to show that $f^r(s_{i_j}) \sqsubset f^r(s_{i_{j+1}})$ for each j from $1, ..., m$ in order to conclude $\mathbb{R}_n A \models R(f^r(s_1), ..., f^r(s_m))$. Now by the above lemma, it suffices to show $\epsilon(s_{i_{j+1}})$ is $(n - \text{length}(s_{i_{j+1}}))$-reachable from i. But we know the guarded tuple $(\epsilon(s_1), ..., \epsilon(s_m))$ witnesses exactly this, since it contains at least 1 element from s_{i_j} (namely $\epsilon(s_{i_j})$), and it contains $\epsilon(s_{i_{j+1}})$, and it contains at most $(n - \text{length}(s_{i_{j+1}}))$ other elements not present in s_{i_j} (namely $\epsilon(s_{i_{j+2}}), ..., \epsilon(s_{i_m})$). ☐

4.4 Locality and "Back and Forth" Games

As of yet we have focused on the "forth" local version of Ehrenfeucht-Fraisse games. In this section we briefly discuss local versions of back and forth games from the perspectives of the functors $\mathbb{E}_n^d$ and $\mathbb{R}_n$. In this section, we shall take all locality bounds d to be finite, rather than possibly infinite. Firstly we define the following:

Definition 4.4.1. *For a given pair n, d, and structure A we define*

$$\mathbb{M}_n^d A := \{s \in \mathbb{M}_n A : s = [\,] \ \vee \ s \in \mathbb{E}_n^d A\}.$$

Recall that, as a set, $\mathbb{M}_n A$ was simply $\mathbb{E}_n A \cup \{[\,]\}$, so this is just the subset of $\mathbb{M}_n A$ corresponding to d-local sequences. We similarly define $\mathbb{M}_n^R A := \{s \in \mathbb{M}_n A : s = [\,] \ \vee \ s \in \mathbb{R}_n A\}$.

$\mathbb{M}_n^d$ exhibits similar properties to $\mathbb{M}_n$ and $\mathbb{E}_n^d$.

Lemma 4.4.2.

- *$\mathbb{M}_n^d$, and $\mathbb{M}_n^R$ are functors from $\mathcal{R}_\sigma(l)$ to $\mathcal{M}_n$.*
- *For $\phi \in (N_d, \mathcal{L}_n)$ well written, $A \models \phi$ if and only if $\mathbb{M}_n^d A, [\,] \models \mathbb{M}_n \phi$. Here we are extending the notion of well-written and the action of $\mathbb{M}_n$ to d-local*

sentences in the natural way; namely ϕ is well-written if x_1 is bound by a leading quantifier, and x_{i+1} is the unique variable bound in the scope of x_i (in this case x_i will be bound by a local quantifier if $i > 1$), and $\mathbb{M}_n$ acts on d-local sentences as it did on global sentences, in addition to sending $\exists x \in N_d(x_1), \forall x \in N_d(x_1) \mapsto \Diamond, \Box$.

- *For structures A, B, $A \equiv_n^d B$, if and only if $\mathbb{M}_n^d A \sim_n \mathbb{M}_n^d B$ if and only if $\mathbb{M}_n^d A \sim \mathbb{M}_n^d B$.*

Proof.

- From the perspective of set functions, $\mathbb{M}_n^d$ and $\mathbb{M}_n^R$ act identically to $\mathbb{E}_n^d$ and $\mathbb{R}_n$, except for also mapping roots to roots. Since $\mathbb{E}_n^d$ and $\mathbb{R}_n$ are functors, we will have $\mathbb{M}_n^d$ and $\mathbb{M}_n^R$ are as well.

- Recall Theorem 3.2.5 which was the analogous result for ordinary sentences. Let $\phi \in (N_d, \mathcal{L}_n)$. Noting once again that $\mathbb{M}_n$ preserves boolean connectives, we may assume ϕ is of form $\exists x_1 \psi$ where $\psi \in (N_d(x_1), \mathcal{L}(1))$. Now $A \models \phi$ if and only if there is $a \in A$ such that $(A, a) \models \psi$. By Theorem 4.2.16, this is true if and only if $(N_d(a), a) \models \psi'$, where ψ' is simply ψ with the local quantifiers replaced with ordinary ones (in the language of Theorem 4.2.16, we would have $\psi = (\psi')_d$). Now by Theorem 3.2.5, this is true if and only if $\mathbb{M}_n N_d(a), [a] \models \mathbb{M}_n \psi'$. Recalling that $\mathbb{E}_n^d A = \bigcup_{a \in A} \mathbb{E}_n^d(A, a) = \bigcup_{a \in A} \mathbb{E}_n(N_d(a), a)$, we can observe $\mathbb{M}_n N_d(a), [a]$ is bisimilar to $\mathbb{M}_n^d A, [a]$, since those elements accessible via transitions from $[a]$ in $\mathbb{M}_n^d A$ are precisely the ones from $\mathbb{E}_n^d(A, a) = \mathbb{E}_n N_d(a)$. Hence, $\mathbb{M}_n N_d(a), [a] \models \mathbb{M}_n \psi'$ if and only if $\mathbb{M}_n^d A, [a] \models \mathbb{M}_n \psi'$. Now by definition of how $\mathbb{M}_n$ acts on local formulae, we have $\mathbb{M}_n \psi' = \mathbb{M}_n \psi$. Putting all this together, we have $A \models \phi$ if and only if there exists $a \in A$ such that $\mathbb{M}_n^d A, [a] \models \mathbb{M}_n \psi$. Now the modal transitions from $[\,] \in \mathbb{M}_n^d A$ are exactly to all sequences of form $[a]$ for $a \in A$. Hence there exists $a \in A$ such that $\mathbb{M}_n^d A, [a] \models \mathbb{M}_n \psi$ if and only if $\mathbb{M}_n^d A, [\,] \models \Diamond \mathbb{M}_n \psi$, but $\Diamond \mathbb{M}_n \psi = \mathbb{M}_n \phi$ so we are done.

- This is a direct consequence of the previous item.

□

Rather than proving directly any results about the relationships between local and global back and forth games, instead we see how our functors can be used to state a weakened version of Gaifman's Theorem, and then use that to make one further conjecture.

Gaifman's Theorem (from [16]) might be paraphrased as "any first-order sentence is equivalent to boolean combination of local ones". We state it precisely below:

Theorem 4.4.3. *(Gaifman's Theorem) For any $\phi \in \mathcal{L}$, there exists d such that ϕ is equivalent to a boolean combination formulas of form:*

$$\exists x_1, ..., \exists x_m (\bigwedge_{i \neq j} D(x_i, x_j) > 2d \wedge \bigwedge_i \psi_i(x_i))$$

where each $\psi_i \in (N_d(x_i), \mathcal{L}(1))$.

This is slightly stronger than what we can currently express in our language, as it includes the statement that elements witnessing x_i are a distance at $2d$ apart, ensuring the neighbourhoods $N_d(x_i)$ are all disjoint. We work with a slightly weaker version stated below:

Theorem 4.4.4. *For any* $\phi \in \mathcal{L}$*, there exists* d *such that* ϕ *is equivalent to a boolean combination of local sentences* ψ*, where each* $\psi \in (N_d, \mathcal{L})$.

This is intuitively weaker, because Gaifman's Theorem says that for a given formula ϕ, there exists a d such that for any structure A, it suffices to look at sets of disjoint d-neighbourhoods of A in order to verify whether $A \models \phi$. However our slightly weaker statement, says there exists a d such that it suffices to look at sets of all d-neighbourhoods to verify whether $A \models \phi$, and clearly all sets of neighbourhoods includes all sets of disjoint neighbourhoods. In any case, we shall give a formal proof of Theorem 4.4.4 using Gaifman's Theorem, after proving some small subclaims about $(N_d, \mathcal{L})$.

Lemma 4.4.5. *1. If* $d \leq d'$*, then anything expressible in* $(N_d, \mathcal{L})$*, is expressible in* $(N'_d, \mathcal{L})$*. Or in other words, a* d*-local formula is also a* d'*-local formula.*

2. If $d_1 + d_2 \leq d$*, then* $\exists x_2 \in N_{d_1}(x_1)\psi(x_2)$ *is expressible in* $(N_d(x_1), \mathcal{L})$ *if* $\psi \in (N_{d_2}(x_2), \mathcal{L}(1))$.

Proof. Since of all the fragments above contain boolean connectives, it suffices to show the localised quantifiers of one can be expressed in another, to show a formula of one fragment can be expressed by a formula in another. Concretely, for item (1), it suffices to show $\exists y \in N_d(x)$ can be expressed using d'-local quantifiers, and for item (2), it suffices to show $\exists y \in N_{d_2}(x_2)$ can be expressed using d-local quantifiers around x_1, so long as $x_2 \in N_{d_1}(x_1)$, since it follows from item (1) $\exists x_2 \in N_{d_1}(x_1)$ is also expressible using d-local quantifiers around x_1. As remarked earlier, $y \in N_d(x)$ or equivalently, $D(x, y) < d$ is expressible in ordinary first-order logic for any d. This is because σ is finite, so the statement "x is adjacent to y in the Gaifman graph" is expressible by a disjunction

$$\chi(x, y) := \bigvee \exists \bar{z} R(t)$$

where this disjunction ranges over all possible $R \in \sigma$, and all tuples t containing x,y, and entries from $\bar{z}$. Now the statement $D(x, y) < d$ can be built inductively, where $D(x, y) < 2$ is simply $\chi(x, y)$, and $D(x, y) < 3$ is defined $\chi(x, y) \vee (\exists z \chi(x, z) \wedge \chi(z, y))$, and so on. Now $D(x, y) < d$ is an existential sentence, and all witnesses for quantifiers in it must be at most distance d from x, and hence in $N_{d'}(x)$ if $d \leq d'$. Therefore if we localise the quantifiers in $D(x, y) \leq d$ to $N_{d'}(x)$ we get a semantically equivalent formula expressible using d'-local quantifiers around x. Therefore we can conclude item (1) from above, since the quantifier $\exists y \in N_d(x)$ is equivalent to $\exists y \in N_{d'}(x) \wedge D(x, y) \leq d$ which is built using a d'-local quantifier and a formula expressible using d'-local quantifiers x, as required. Now for the second item, we note since $x_2 \in N_{d_1}(x_1)$, the statement $D(x_2, y) < d_2$, can be localised to $N_d(x_1)$ without changing the meaning of the sentence, as necessarily any witnesses for quantifiers in $D(x_2, y) < d_2$ must be in $N_d(x_1)$, as $d_1 + d_2 \leq d$. Hence we can express $\exists y \in N_{d_2}(x_2)$ as $\exists y \in N_d(x) \land \ D(x_2, y) \leq d_2$, which is expressible using d-local quantifiers around x_1, which concludes item 2. $\square$

Now we can prove Theorem 4.4.4 using Gaifman's Theorem.

Proof. It suffices to show that a formula of form

$$\exists x_1, ..., \exists x_m (\bigwedge_{i \neq j} D(x_i, x_j) > 2d \land \bigwedge_i \psi_i(x_i))$$

can be written as a boolean combination of formulae in $(N_{d'}, \mathcal{L})$ for some d'. Observe the formula above is equivalent to $\bigwedge_i \exists x_i \psi_i'(x_i)$, where

$$\psi_i' := \psi_i(x_i) \land \bigwedge_{j \neq i} \neg \exists y \in N_{2d}(x_i) \ \psi_j(y)$$

The initial formula says there are witnesses for each ψ_i that are distance at most $2d$ apart, and the latter formula says there are witnesses x_i for each ψ_i, and there are no witnesses for ψ_j within a $2d$ radius of x_i for each $i \neq j$, which is a slightly more complicated way of saying the same thing. By the lemma we just proved, the latter formula is expressible as a boolean combination of formulae from $(N_{3d}, \mathcal{L})$, since $d + 2d \leq 3d$. $\square$

Given we saw in the introduction that there are only finitely many (non-equivalent) $\phi \in \mathcal{L}_n$, we can grade Theorem 4.4.4 by quantifier rank:

Lemma 4.4.6. *For any n, there exists a pair m, d such that for every $\phi \in \mathcal{L}_n$, there exists $\psi \in (N_d, \mathcal{L}_m)$ such that $\phi \equiv \psi$.*

This is equivalent to Theorem 4.4.4, since it clearly implies it, and there are only finitely many sentences in $\mathcal{L}_n$ for each n, so we can find an upper bound for the distance and quantifier rank needed to localise them for a given n, which shows it is implied by Theorem 4.4.4. Finally, we are ready to show this can be expressed in terms of n-ary and local n-ary equivalence, and hence in terms of bisimulations using our functors. This will conclude the aim of stating Gaifman's Theorem in our language.

Theorem 4.4.7. *The following are equivalent:*

1. *For any n, there exist m, d such that for every $\phi \in \mathcal{L}_n$, there exists $\psi \in (N_d, \mathcal{L}_m)$ such that $\phi \equiv \psi$.*

2. *For any n, there exists a pair m, d such that the following holds: For any $A, B \in \mathcal{R}_\sigma$, if $A \equiv_m^d B$, then $A \equiv_n B$.*

3. *For any n, there exists a pair m, d such the following holds: For any $A, B \in \mathcal{R}_\sigma$, if $\mathbb{M}^d_m A \sim \mathbb{M}^d_m B$, then $\mathbb{M}_n A \sim \mathbb{M}_n B$.*

Proof. Now (2) and (3) are immediately equivalent from Theorem 3.2.5 and the third item of Theorem 4.4.2. We also claim that (1) implies (2) fairly trivially, since if $A \equiv^d_m B$ and $A \models \phi \in \mathcal{L}_n$, then by simply picking $\psi \in (N_d, \mathcal{L}_m)$ such that $\psi \equiv \phi$, we can conclude $A \models \psi$, hence $B \models \psi$, and hence $B \models \phi$. To prove (2) implies (1) and complete the proof, we require a few subclaims. All of these subclaims apply for both the pairs $\equiv^n, \mathcal{L}_n$ and $\equiv^m_d, (N_d, \mathcal{L}_m)$ as they both have all the properties used to prove the claim (namely being finite up to equivalence, closed under boolean combinations, and finite index equivalence relations), but we will just write out the case of $\equiv^n, \mathcal{L}_n$ for convenience:

- Any equivalence class of $\equiv_n$ is equal to $Mod(\phi)$ for some $\phi \in \mathcal{L}_n$
- Any union of equivalence classes of $\equiv_n$ is equal to $Mod(\phi)$ for some $\phi \in \mathcal{L}_n$
- For any $\phi \in \mathcal{L}_n$, $Mod(\phi)$ is a union of $\equiv_n$ equivalence classes.

For the first item, recall $\mathcal{L}_n$ is finite up to equivalence, so take $\phi_1, ..., \phi_Z$ as a list of representative sentences. Now any equivalence class under $\equiv_n$ is specified exactly by whether for each i from $1, ..., Z$, members of that equivalence class either satisfy ϕ_i or not. Hence, the equivalence class is equal $Mod(\psi)$, where $\psi := \bigwedge q_i \phi_i$, where q_i is either a negation or nothing, depending on whether members of that equivalence class model ϕ or not. For the second item, recall that $\equiv_n$ is finite index, owing to the fact that $\mathcal{L}_n$ is finite up to equivalence. Hence a union of equivalence classes is a union of finitely many equivalence classes, $C_1, ..., C_Y$. Hence, the union of the classes is exactly $Mod(\bigvee \psi_i)$ where $C_i = Mod(\psi_i)$ for each i. For the third item, it is enough to show that if $A \in Mod(\phi)$, and $A \equiv_n B$, then $B \in Mod(\phi)$ to show $Mod(\phi)$ is a union of equivalence classes. But clearly, if $A \in Mod(\phi)$ then $A \models \phi$, so $B \models \phi$, so $B \in Mod(\phi)$.

Now we have proved all of our subclaims, we can show (2) implies (1). Let n, m, d be as in the statement of the lemma, assume (2) and let $\phi \in \mathcal{L}_n$. We seek to show there is $\psi \in (N_d, \mathcal{L}_m)$ equivalent to ϕ. Now (2) tells us that $\equiv^d_m$ is a finer relation $\equiv_n$, so any union of equivalence classes under $\equiv_n$ is a union of equivalence classes under $\equiv^d_m$. So given ϕ, we know $Mod(\phi)$ is a union of $\equiv^n$ equivalence classes, and hence a union of $\equiv^m_d$ equivalence classes. However, we know a union of $\equiv^d_m$ equivalence classes is a equal to $Mod(\psi)$ for some $\psi \in (N_d, \mathcal{L}_m)$, and hence $Mod(\phi) = Mod(\psi)$, or in other words $\phi \equiv \psi$. □

4.5 Remarks and further directions

Clearly the final section in this chapter is brief, and is merely a restatement of a weaker version of Gaifman's Theorem in our own language. It would be interesting to investigate whether one could prove the ordinary statement of Gaifman's Theorem directly using similar techniques to earlier parts of the chapter. In addition, since we saw that one might see $\mathbb{R}_n$ as a "sharpened" version of $\mathbb{E}_n^d$ (for sufficiently large d), we speculate that a direct proof of Gaifman's Theorem in the above framework may yield a proof of the following conjecture:

Conjecture 4.5.1. *For every n, there exists m such that for every $A, B \in \mathcal{R}_\sigma$, $\mathbb{M}_m^R A \sim \mathbb{M}_m^R B$ implies $A \equiv^n B$*

A further observation, is that one may view both reachability and locality as extensions of guardedness (as defined above, a tuple t of a structure A is guarded if there is some $R \in \sigma$ such that $A \models R(t)$). In [3], Abramsky and Marsden gave a comonadic picture of the guarded fragment of first-order logic. Another natural path may be to investigate whether there is any technical connection between the work presented there and and the work in this chapter.

Chapter 5

Closing Remarks

5.1 Overview

Here we round up the main results of the thesis and potential directions for further investigation. The initial chapter introduced the comonads $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$, and the model theoretic background they draw together. With one exception, being the treatment of free variables, this was simply the authors summary of the other materials referenced in the chapter, mostly from [5].

Chapter 2 was based on the paper [26], and considered Rossman's Equirank Homomorphism Theorem from [27]. The paper had two aims: to give a comonadic treatment of it using ideas from the introduction, and to extend it to also consider the variable count of a formula. With regards to the first aim, a more abstract treatment has since been given in [4], however for the second aim the treatment given in the paper has not been improved upon. In it we modestly extended the Equirank HPT to the following result: if ϕ is preserved under homomorphisms and has variable count k and quantifier rank n, where $k \geq n-2$, then there exists a positive existential ψ equivalent to ϕ with the same quantifier rank and variable count. As remarked in the thesis, this extended the Equirank HPT which only preserves quantifier rank, as any sentence with quantifier rank n can always be written with n variables. Potential areas for further investigation would either be to provide a proof for an arbitrary pair n, k or to find a counterexample. We provided potential strategies for extending the proof to the general case at the end of the second chapter. If one believed the conjecture was false for the general case, a reasonable method for looking for a counterexample or contradiction would be to explore the difference between the behaviours of $\to_n$ and $\to_{n,k}$ on the categories $\mathcal{R}_\sigma(l)$. Whilst these relations do behave similarly in most cases, a key difference between them, and the reason we were unable to prove the general case, is that they behave differently when taking coproducts.
A further direction to consider might be to apply the ideas from this section to

Rossman's Finite Homomorphism Preservation Theorem, also from [27], where he showed that a formula preserved under homomorphisms between finite structures is equivalent to a positive existential one over finite structures.

In the Chapter 3, we saw how the structures of form $\mathbb{E}_n A$ and $\mathbb{P}_{n,k} A$ can be viewed as modal structures, and we called the functors resulting from this $\mathbb{M}_n$ and $\mathbb{M}_{n,k}$. We proved that the codomain of these functors is equivalent to the Eilenberg-Moore category for the comonads $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$, and hence $\mathbb{M}_n$ and $\mathbb{M}_{n,k}$ are part of adjunctions that give rise to $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$ respectively. We also saw how, when formulas are written in a particular way, that $\mathbb{M}_n$ and $\mathbb{M}_{n,k}$ preserve the satisfaction relation between formulas and structures. In other words, given some appropriately written first-order formula ϕ, there is a modal formula $\mathbb{M}_n \phi$ such that for any first-order structure A, $A \models \phi$ if and only if $\mathbb{M}_n A \models \mathbb{M}_n \phi$. This allowed us to translate n-ary elementary equivalence to a modal bisimulation. This section poses some interesting questions for further investigation. Firstly, we ask if this process can be generalised. The comonads $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$ are not the only comonads which interpret logics and Ehrenfeucht-Fraisse style games, more are mentioned below, so is it possible to see if those other comonads are modal in nature? If not, what properties of $\mathbb{E}_n$ and $\mathbb{P}_{n,k}$ allow us to do this? A second area for investigation might be to better understand the codomain of the functors $\mathbb{M}_n$ and $\mathbb{M}_{n,k}$, and specifically, where the images of $\mathbb{M}_n$ and $\mathbb{M}_{n,k}$ sit in those codomains. Given that we can translate first-order sentences and n-ary equivalence to those images, is it possible to use this translation to prove anything about first-order logic simply using modal logic?

In the second section we applied the construction in the first to a part of the proof of Courcelle's Theorem. Our goal for the section was to find, given a first-order structure A with an (n, k)-cover, a tree-like structure that can evaluate MSO sentences on behalf of A. From the first section, we had two candidate structures; $\mathbb{M}_{n,k} A$, and a tree T given by the (n, k)-cover, which can be realised as the pre-image of A under the adjoint functor to $\mathbb{M}_{n,k}$. After some analysis, we saw that neither T nor $\mathbb{M}_{n,k}$ worked when viewed as modal structures, however T could be used if viewed as a directed labelled graph rather than a modal structure, where it inherits labels for the vertices from the atomic predicates and the edges from the transition relations. We succeeded in proving one part of Courcelle's Theorem using our comonadic approach, though it remains open whether the proof can be entirely completed in this way. A suitable next step may be to see if the monadic interpretation of tree automata from [19] can be incorporated into our approach.

In the final chapter we "localised" the correspondences between formulae, Ehrenfeucht-Fraisse games and the comonad $\mathbb{E}_n$. To that end, we defined a localised subfunctor of $\mathbb{E}_n$, written $\mathbb{E}_n^d$. Similarly to how maps out of $\mathbb{E}_n$ represent strategies in the ordinary Ehrenfeucht-Fraisse game, maps out of $\mathbb{E}_n^d$ represent strategies in localised versions of Ehrenfeucht-Fraisse games, which in turn correspond to localised versions of the $\to_n$ relation between structures, which use localised formulae. A key difference between $\mathbb{E}_n$ and $\mathbb{E}_n^d$ is the latter is not a comonad, hence we do not have the same correspondence between maps of form $\mathbb{E}_n^d A \to B$ and $\mathbb{E}_n^d A \to \mathbb{E}_n^d B$ as we would in the $\mathbb{E}_n$ case. In terms of games, the former maps correspond to games where Duplicator is allowed to play only locally, and the latter is where Duplicator can play globally, so this lack of correspondence between maps indicates these games are not equivalent.

Next, we showed that for a sufficiently large locality bound, at least 2^n, the local version of the n-round Ehrenfeucht-Fraisse game is in fact equivalent to the ordinary one, which we witnessed with a morphism $\mathbb{E}_n A \to \mathbb{E}_n^d A$ for any A. Unfortunately, it turned out these maps do not form a natural transformation. Next we defined reachability, which was in a sense defined to give the smallest subfunctor of $\mathbb{E}_n$, written $\mathbb{R}_n$, such that there is always a morphism $\mathbb{E}_n A \to \mathbb{R}_n A$. The majority of this section was devoted to building up this morphism, and showing $\mathbb{R}_n$ does in fact form a comonad, unlike $\mathbb{E}_n^d$.

Finally, we investigated how localising interacts with back and forth games. To do this, we provided a modal translation inspired by ideas from Chapter 3, in order to express local n-ary equivalence as a modal bisimulation. We then showed this idea could be used to express a weakened version of Gaifman's Theorem. As a potential direction for further investigation we ask: would it be possible to prove this weakened version, or the ordinary version of Gaifman's Theorem in this setting? Also, would it be possible to prove a sharper version of Gaifman's Theorem where we use $\mathbb{R}_n$ rather than $\mathbb{E}_n^d$?

We would like to point out that this thesis is part of a much larger project with the aim of using semantic techniques to better understand Finite Model Theory. In addition to the work already referenced above, some further results include: capturing generalised quantifiers in [10], generalising Lovasz-type Theorems in [12], and abstracting to arboreal categories in [4].

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Bibliography

[1] Samson Abramsky, Anuj Dawar, and Pengming Wang. The pebbling comonad in finite model theory. In *Logic in Computer Science (LICS), 2017 32nd Annual ACM/IEEE Symposium on*, pages 1–12. IEEE, 2017.

[2] Samson Abramsky, Tomáš Jakl, and Thomas Paine. Discrete density comonads and graph parameters. In *International Workshop on Coalgebraic Methods in Computer Science*, pages 23–44. Springer, 2022.

[3] Samson Abramsky and Dan Marsden. Comonadic semantics for guarded fragments. In *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–13. IEEE, 2021.

[4] Samson Abramsky and Luca Reggio. Arboreal categories and resources. In *48th International Colloquium on Automata, Languages, and Programming (ICALP 2021)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2021.

[5] Samson Abramsky and Nihil Shah. Relating structure and power: Comonadic semantics for computational resources. In *27th EACSL Annual Conference on Computer Science Logic, CSL 2018, September 4-7, 2018, Birmingham, UK*, pages 2:1–2:17, 2018.

[6] Samson Abramsky and Nihil Shah. Relating structure and power: Comonadic semantics for computational resources. *Journal of Logic and Computation*, 31(6):1390–1428, 2021.

[7] Samuel Frederic Barr. Courcelle's theorem: Overview and applications, Batchelor's Project, Oberlin College, 2020.

[8] Hans L Bodlaender. Treewidth: Algorithmic techniques and results. In *International Symposium on Mathematical Foundations of Computer Science*, pages 19–36. Springer, 1997.

[9] Yijia Chen, Jörg Flum, and Xuangui Huang. Slicewise definability in first-order logic with bounded quantifier rank. *arXiv preprint arXiv:1704.03167*, 2017.

[10] Adam Ó Conghaile and Anuj Dawar. Game comonads & generalised quantifiers. In *29th EACSL Annual Conference on Computer Science Logic*, page 1, 2021.

[11] Bruno Courcelle. The monadic second-order logic of graphs. I. Recognizable sets of finite graphs. *Information and computation*, 85(1):12–75, 1990.

[12] Anuj Dawar, Tomáš Jakl, and Luca Reggio. Lovász-type theorems and game comonads. In *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–13. IEEE, 2021.

[13] Heinz-Dieter Ebbinghaus and Jörg Flum. *Finite model theory.* Springer Science & Business Media, 2005.

[14] Andrzej Ehrenfeucht. An application of games to the completeness problem for formalized theories. *Fund. Math*, 49(129-141):13, 1961.

[15] Roland Fraïssé. Sur quelques classifications des relations, basées sur des isomorphismes restreints. *Publ. Sci. Univ. Alger. Sér. A*, 2:273–295, 1955.

[16] Haim Gaifman. On local and non-local properties. In *Studies in Logic and the Foundations of Mathematics*, volume 107, pages 105–135. Elsevier, 1982.

[17] Erich Grädel, Phokion G Kolaitis, Leonid Libkin, Maarten Marx, Joel Spencer, Moshe Y Vardi, Yde Venema, Scott Weinstein, et al. *Finite Model Theory and its applications.* Springer, 2007.

[18] Erich Gradel, Martin Otto, and Eric Rosen. Two-variable logic with counting is decidable. In *Proceedings of Twelfth Annual IEEE Symposium on Logic in Computer Science*, pages 306–317. IEEE, 1997.

[19] Gerco van Heerdt, Tobias Kappe, Jurriaan Rot, Matteo Sammartino, and Alexandra Silva. Tree automata as algebras: Minimisation and determinisation. 2019.

[20] Leonid Libkin. On the forms of locality over finite models. In *Proceedings of Twelfth Annual IEEE Symposium on Logic in Computer Science*, pages 204–215. IEEE, 1997.

[21] Leonid Libkin. Ehrenfeucht-Fraïssé games. In *Elements of Finite Model Theory*, pages 23–43. Springer, 2004.

[22] Leonid Libkin. Monadic second-order logic and automata. In *Elements of Finite Model Theory*, pages 113–137. Springer, 2004.

[23] Yoàv Montacute and Nihil Shah. The pebble-relation comonad in finite model theory. In *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 1–11, 2022.

[24] Jaroslav Nešetřil and Patrice Ossona De Mendez. Tree-depth, subgraph coloring and homomorphism bounds. *European Journal of Combinatorics*, 27(6):1022–1041, 2006.

[25] Martin Otto. Bisimulation invariance and finite models. In *Logic Colloquium*, volume 2, pages 276–298, 2006.

[26] Thomas Paine. A pebbling comonad for finite rank and variable logic, and an application to the equirank-variable homomorphism preservation theorem. *Electronic Notes in Theoretical Computer Science*, 352:191–209, 2020.

[27] Benjamin Rossman. Homomorphism preservation theorems. *Journal of the ACM (JACM)*, 55(3):15, 2008.

[28] Benjamin Rossman. Combining Ehrenfeucht-Fraïssé games. In *2009 24th Annual IEEE Symposium on Logic In Computer Science*, pages 331–331. IEEE, 2009.

[29] Colin Stirling. The joys of bisimulation. In *International Symposium on Mathematical Foundations of Computer Science*, pages 142–151. Springer, 1998.

[30] Wolfgang Thomas. On the Ehrenfeucht-Fraïssé game in theoretical computer science. In *Colloquium on Trees in Algebra and Programming*, pages 559–568. Springer, 1993.