Research Article Open Access

Approximate Solutions for Optimal Control of Fixed Boundary Value Problems Using Variational and Minimum Approaches

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Article Info

Received
25/10/2021

Revised
12/03/2023

Accepted
05/04/2023

Published
30/09/2023

ABSTRACT
The optimal control is the process of finding a control strategy that extreme some performance index for a dynamic system (partial differential equation) over the class of admissibility. The present work deals with a problem of fixed boundary with a control manipulated in the structure of the partial differential equation. An attractive computational method for determining the optimal control of unconstrained linear dynamic system with a quadratic performance index is presented. In the proposed method, the difference between every state variable and its initial condition is represented by a finite - term polynomial series. This representation leads to a system of linear algebraic equations which represents the necessary condition of optimality. The linear algebraic system is solved by using two approaches namely the variational iteration method and the minimization approach for unconstrained optimization problem with estimation of gradient and Hessian matrix. These approaches are illustrated by two application examples.

KEYWORDS: Optimal control problems, linear distributed parameter systems, variational method, approximate solutions, polynomial based approximation.

الخلاصة
التحكم الأمثل هو عملية البحث عن استراتيجية تحكم تقوم بتحسين مؤشر أداء معين لنظام ديناميكي (معادلة تفاضلية جزئية) عبر فئة من الحالات المسموح بها. يتناول العمل الحالي مشكلة حدود محددة مع التحكم في هيكل المعادلة التفاضلية الجزئية. يتم تقديم أسلوب حسابي مميز لتحديد التحكم الأمثل لنظام ديناميكي خطي غير مقيد مع مؤشر أداء رباعي الشكل. في الأسلوب المقترح، يتم تمثيل الفرق بين كل متغير حالة وحالته الابتدائية بواسطة سلسلة متعددة الحدود. يؤدي هذا التمثيل إلى نظام من المعادلات الجبرية الخطية التي تمثل الشرط الضروري للأمثلية. يتم حل النظام الجبري الخطي باستخدام طريقين، وهما طريقة التفاضل التكراري والطريقة للتقليل من حالة مشكلة التحسين غير المقيدة مع تقدير للمدى المرجعي ومصفوفة هيسيان. تُوضح هذه الطرق من خلال مثالين تطبيقيين.

INTRODUCTION

Determining the optimal control of linear, distributed parameter models of dynamic systems is one of the principals "state space" design problems. The challenge is to find the optimal trajectories of the control and associated state giving the best tradeoff between performance and cost of the control. Toward this end, variational methods can be used to determine the optimality condition as a two-point boundary value problem. The optimal controller construction using Pontryagin's minimum principle is presented in [5]. The control of linear partial differential equations can be found in [1], on the extension of controllability of nonlinear partial differential equations is presented in [7]. Different based functions are referred to in [6] to approximate the control and state variables such as the Fourier functions, Chebyshev functions, Walsh functions, etc. In this paper one can understand that we mean by formulation of the problem the following concepts: the general description of the problem, the assumptions that needed for holding the problem or to make the problem well defined, the state parameterization, the approximation of the performance index, the approaches of the solution, and finally the discussion of the results.

Description of the Problem

The basic idea of optimal control problem involves finding the control function u(x,t) and the corresponding state function w(x,t) in the suitable rectangular region **Ω** that minimize the quadratic performance index J, where $J = \omega_1 J_1 + \omega_2 J_2$, where ω_1, ω_2 are positive weighting factors, and J_1, J_2 represent two quadratic functionals as shown in details next. The main optimal control problem is:

$$\underset{u\in U}{\text{Min}}\, J(u(x,t)) = \underset{u\in U}{\text{Min}} \int_{\Omega} \begin{pmatrix} \omega_1 J_1(w(x,t),u(x,t),x,t) \\ +\ \omega_2 J_2(w(x,t),u(x,t),x,t) \end{pmatrix} d\Omega \qquad (1)$$

where the region of the solution is:

$$\Omega = \{(x,t) \mid \ x \in [0,d]\ , t \in [0,T],\ T > 0\} \qquad (2)$$

and the control input function u(x,t) is assumed to belong to the class of admissible controls which denoted by U:

$$u \in U = \{ u(x,t) \,|\, u(.\,,.) \in L_2[\Omega] \} \qquad (3)$$

subject to the operator:

$$w_t(x,t)=L(w(x,t),w_x(x,t),w_{xx}(x,t),u(x,t),x,t) \qquad (4)$$

with the initial condition:

$$w(x,0) = \varphi_0(x)\ , \qquad 0 \le x \le d \qquad (5)$$

together with the boundary conditions: (6)-(7)

$$\alpha_1 w(0,t)+\beta_1 w_x(0,t)=f_1(t)\ , \qquad 0 \le t \le T \qquad (6)$$

$$\alpha_2 w(d,t)+\beta_2 w_x(d,t)=f_2(t)\ , \qquad 0 \le t \le T \qquad (7)$$

with the final state:

$$w(x,T) = \varphi_1(x), \qquad 0 \le x \le d \qquad (8)$$

where the functions $f_1(t)$, $f_2(t)$, $\varphi_0(x), \varphi_1(x)$ are suitable smooth given functions. Now, to make the problem is well defined one needs the following hypotheses and assumptions:

MATERIALS AND METHODS

Hypotheses and Basic Assumptions

The main problem (1) – (8) holds when the following assumptions satisfied:

1. L is a linear operator defined on its domain space D(L) as follows:

$$L: D(L) = W_1 \times U_1 \subseteq L_2(\Omega) \ \to\ R(L) = W_2 \times U_2 \subseteq L_2(\Omega) \qquad (9)$$

With

$$\overline{D(L)} = L_2(\Omega) \qquad (10)$$

2. The inner product in $L_2(\Omega)$ is defined by:

$$< w_1, w_2 >_{L_2(\Omega)} = \int_{\Omega} w_1(x,t).\, w_2(x,t)\ d\Omega \qquad (11)$$

where $w_1(x,t),\ w_2(x,t) \in D(L)$.

3. The norm in $L_2(\Omega)$ is defined by:

$$\|w\|_{L_2(\Omega)} = \sqrt{< w, w >} \qquad (12)$$

where $w(x,t) \in D(L)$

4. The function w(x,t) is assumed to be a smooth function of its independent variables x, t.

5. The final time state is assumed to be fixed.

6. The initial and the boundary conditions are assumed to be compatible up to some order.

7. The main quadratic functional J(u(x,t)) in (1) is assumed to be continuous and strictly convex on the compact admissible control space U.

8. The first objective function J_1 is defined as a functional by:

$$J_1[w,u] = \mu_1 \left\| w - w^d \right\|^2_{L_2(\Omega)} + \mu_2 \| w_x \|^2_{L_2(\Omega)} + \mu_3 \| u \|^2_{L_2(\Omega)} \qquad (13)$$

where the norm is given in (12). The above performance index (13) can be explained as follows:

$$\begin{aligned} J_1[w,u] &= \mu_1 < w - w^d, w - w^d >_{L_2(\Omega)} \\ &\quad + \mu_2 < w_x, w_x >_{L_2(\Omega)} + \mu_3 < u, u >_{L_2(\Omega)} \end{aligned} \qquad (14)$$

or another form of Eq. (11) is:

$$J_1[w,u] = \int_{\Omega} \begin{pmatrix} \mu_1 \left[(w(x,t) - w^d(x,t))^T (w(x,t) - w^d(x,t)) \right] \\ + \mu_2 \left[w_x(x,t)^T w_x(x,t) \right] + \mu_3 \left[u(x,t)^T u(x,t) \right] \end{pmatrix} d\Omega \qquad (15)$$

Thus, the quadratic form of above Eq. (15) is:

$$J_1[w,u]=\int_\Omega \begin{pmatrix} \mu_1\left[w(x,t)-w^d(x,t)\right]^2 \\ +\mu_2\left[w_x(x,t)\right]^2+\mu_3\left[u(x,t)\right]^2 \end{pmatrix} d\Omega \quad (16)$$

9. The given function $w^d(x,t)$ in (13), (14), (15), (16) respectively, is the desired state and for the stability point of view, the steady state of the state function and the control function may be inserted in the first objective function J_1.

10. An important point of view is the addition of the $w_x(x,t)$ term to the first objective function J_1 is very useful to ensure that the solution $w(x,t)$ will go to the desired solution such as the steady state solution for all values of the spatial variable x not only at the boundary points when one take only the $w(x,t)$ term in J_1 [8].

11. By substituting the bilinear form (11) into the quadratic functional (16), the Eq. (16) becomes:

$$J_1[w,u]=\int_0^T\int_0^d \begin{pmatrix} \mu_1\left[w(x,t)-w^d(x,t)\right]^2 \\ +\mu_2\left[w_x(x,t)\right]^2+\mu_3\left[u(x,t)\right]^2 \end{pmatrix} dx\,dt \quad (17)$$

12. The second objective function J_2 is defined as a functional in the norm (12) by:

$$J_2[w,u]=\frac{1}{2}\| L(w,u)\|^2_{L_2(\Omega)} - \| f \,.\, L(w,u)\|^2_{L_2(\Omega)} \quad (18)$$

where f is a given function represents the nonhomogeneous part of the PDE (4) or right-hand side of it (if it is existing). J_2 is defined as a functional in the inner product (bilinear) form (11) by:

$$J_2[w,u]=\frac{1}{2}< L(w,u),L(w,u) >_{L_2(\Omega)} - < f,L(w,u) >_{L_2(\Omega)} \quad (19)$$

the above performance index (19) can be explained as follows:

$$J_2[w,u]=\frac{1}{2}\int_\Omega \begin{pmatrix} \left[\left(L(w(x,t),u(x,t))\right)^T . \left(L(w(x,t),u(x,t))\right)\right] \\ -2\left[f(x,t).\left(L(w(x,t),u(x,t))\right)\right] \end{pmatrix} d\Omega \quad (20)$$

Thus, the quadratic form of above functional (20) is:

$$J_2[w,u]=\frac{1}{2}\int_\Omega \begin{pmatrix} \left[L(w(x,t),u(x,t))\right]^2 - \\ 2\left[f(x,t)\,.\,L(w(x,t),u(x,t))\right] \end{pmatrix} d\Omega \quad (21)$$

13. The linear operators L in Eq. s (18) - (21) is taken from the parabolic PDE:

$$w_t(x,t)=\alpha\, w_{xx}(x,t)+\beta u(x,t)+f(x,t) \quad (22)$$

or one can rewrite Eq. (22) in operator form as follows:

$$L(w(x,t),u(x,t))=f(x,t) \quad (23)$$

where the linear operator L is defined by:

$$L(w,u)=\frac{\partial w}{\partial t}-\alpha\frac{\partial^2 w}{\partial x^2}-\beta u \quad (24)$$

by substituting the linear operator (24) into the quadratic functional (21), where the bilinear form $< L(w,u),L(w,u) >_{L^2(\Omega)}$ is defined basically in (11), by using Eq. (23), the quadratic form (21) becomes:

$$J_2[w,u]=\frac{1}{2}\int_0^T\int_0^d \left(\frac{\partial w}{\partial t}-\alpha\frac{\partial^2 w}{\partial x^2}-\beta u\right)^2 - 2\left(f(x,t)\right).\left(\frac{\partial w}{\partial t}-\alpha\frac{\partial^2 w}{\partial x^2}-\beta u\right)dxdt \quad (25)$$

14. The symmetry property of the linear operator L is satisfied with respect to the chosen inner product (bilinear form) (11) in $L_2(\Omega)$, assuming that the non-degeneracy property of the linear operator L is also satisfied in $L_2(\Omega)$ then from [4] one can define the functional J_2 in the form of Eq. (18) or equivalently in the form of Eq. (19).

15. Using Eq. (17) and Eq. (25), the main performance index (1) can be written as:

$$\min_{u\in U} J(u(x,t)) = \min_{u\in U}\int_{\Omega}\left(\begin{array}{l}\omega_1\left(\begin{array}{l}\mu_1\left[w(x,t)-w^d(x,t)\right]^2\\ +\mu_2\left[\dfrac{\partial w(x,t)}{\partial x}\right]^2+\mu_3\left[u(x,t)\right]^2\end{array}\right)\\ +\ \omega_2\left(\begin{array}{l}\left(\dfrac{\partial w}{\partial t}-\alpha\dfrac{\partial^2 w}{\partial x^2}-\beta u\right)^2\\ -2\left(f(x,t)\right).\left(\dfrac{\partial w}{\partial t}-\alpha\dfrac{\partial^2 w}{\partial x^2}-\beta u\right)\end{array}\right)\end{array}\right)d\Omega \tag{26}$$

Where, $\omega_1, \omega_2, \mu_1, \mu_2, \mu_3$ are suitable weighting and presence factors respectively.

State Parameterization

The optimal control problem can be converted to an optimization problem by approximating the state and the control functions by their bases functions, this procedure converts the PDE in infinite dimensional problem into a finite dimensional optimization problem as in the following computational algorithm:

Step 1: Construct the Ritz sequences of bases functions for the state function w(x,t), and for the control function u(x,t) respectively as follows:

$$w_{n1}(x,t) = G(x,t) + \sum_{i=1}^{n1} a_i G_i(x,t) \tag{27}$$

where G(x,t) is a function satisfies the non-homogeneous initial and the compatible boundary conditions given in Eqs. (5) - (7) respectively, and $G_i(x,t),\ i=1,..,n_1$ is a complete sequence of bases functions (a set of linearly independent functions). Next for the control function u(x,t):

$$u_{n2}(x,t) = N(x,t) + \sum_{j=1}^{n2} b_j N_j(x,t) \tag{28}$$

where N(x,t) is a function satisfies the non-homogeneous initial condition for the control function, and $N_j(x,t),\ j=1,..,n_2$ is a complete sequence of bases functions (a set of linearly independent functions).

Step 2: Choose a suitable number n_1 of Ritz bases functions $G_i(x,t),\ i=1,..,n_1$ in $L_2(\Omega)$ such that $G_i(x,t)=0$, for $i=1,..,n_1$ on the corresponding homogenous initial and boundary conditions of the state function, and choose a suitable number n_2 of Ritz bases functions $N_j(x,t)\ \ j=1,..,n_2$ in $L_2(\Omega)$ such that $N_j(x,t)=0$, for $j=1,..,n_2$ on the corresponding homogeneous initial condition for the input control function, and substitute these bases functions into the approximate functions $w_{n1}(x,t),\ u_{n2}(t)$ given in Eqs. (27), (28) respectively with the important conditions:

$$\lim_{n1\to\infty} w_{n1}(x,t) = w(x,t) \tag{29}$$

$$\lim_{n2\to\infty} u_{n2}(x,t) = u(x,t) \tag{30}$$

Step 3: Taking the partial derivatives of Eq. (27) with respect to t and x respectively as follows:

$$\frac{\partial}{\partial t} w_{n1}(x,t) = \frac{\partial}{\partial t} G(x,t) + \sum_{i=1}^{n1} a_i \frac{\partial}{\partial t} G_i(x,t) \tag{31}$$

$$\frac{\partial}{\partial x} w_{n1}(x,t) = \frac{\partial}{\partial x} G(x,t) + \sum_{i=1}^{n1} a_i \frac{\partial}{\partial x} G_i(x,t) \tag{32}$$

$$\frac{\partial^2}{\partial x^2} w_{n1}(x,t) = \frac{\partial^2}{\partial x^2} G(x,t) + \sum_{i=1}^{n1} a_i \frac{\partial^2}{\partial x^2} G_i(x,t) \tag{33}$$

Approximation of the Performance index

Step 4: Substitute Eqs. (27), (28), and the corresponding partial derivatives (31) - (33) into Eq. (26) with some arrangement to get:

$$\min_{u\in U} J[w_{n1}, u_{n2}] = \min_{u\in U}\int_{\Omega}\left(\begin{array}{l}\omega_1\left(\begin{array}{l}\mu_1\left[\left(G(x,t)+\sum_{i=1}^{n1} a_i G_i(x,t)\right)-w^d(x,t)\right]^2\\ +\mu_2\left[\dfrac{\partial}{\partial x}G(x,t)+\sum_{i=1}^{n1} a_i\dfrac{\partial}{\partial x}G_i(x,t)\right]^2\\ +\mu_3\left[N(x,t)+\sum_{j=1}^{n2} b_j N_j(x,t)\right]^2\end{array}\right)\\ +\ \omega_2\left(\begin{array}{l}\left(\begin{array}{l}\left(\dfrac{\partial}{\partial t}G(x,t)-\alpha\dfrac{\partial^2}{\partial x^2}G(x,t)-\beta N(x,t)\right)\\ +\sum_{i=1}^{n1} a_i\left(\dfrac{\partial}{\partial t}G_i(x,t)-\alpha.\dfrac{\partial^2}{\partial x^2}G_i(x,t)\right)\\ -\sum_{j=1}^{n2} b_j\left(\beta\ N_j(x,t)\right)\end{array}\right)^2\\ -2\left(f(x,t)\right).\left(\begin{array}{l}\left(\dfrac{\partial}{\partial t}G(x,t)-\alpha\dfrac{\partial^2}{\partial x^2}G(x,t)-\beta N(x,t)\right)\\ +\sum_{i=1}^{n1} a_i\left(\dfrac{\partial}{\partial t}G_i(x,t)-\alpha.\dfrac{\partial^2}{\partial x^2}G_i(x,t)\right)\\ -\sum_{j=1}^{n2} b_j\left(\beta\ N_j(x,t)\right)\end{array}\right)\end{array}\right)\end{array}\right)d\Omega \tag{34}$$

where f(x,t) is a given function different from one problem to another.

Al-Mustansiriyah Journal of Science
ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 34, Issue 3, 2023 DOI: http://doi.org/10.23851/mjs.v34i3.1104

Solvability of the Problem

Due to the minimization of multi-objective optimization problem one can use two different approaches for solving this problem as follows:

Variational Approach

In this approach to find the approximate optimal control u(x,t), and the corresponding approximate solution of state function w(x,t) one has to find the critical points of the performance index functional given in Eq. (34) for suitable given weighting and presence factors $\omega_1, \omega_2, \mu_1, \mu_2, \mu_3$, of the following functional:

$$J[\vec{a},\vec{b}] = \omega_1 J_1[\vec{a},\vec{b}] + \omega_2 J_2[\vec{a},\vec{b}] \tag{35}$$

Where,

$$\vec{a} = (a_1,\ldots,a_{n1}),\ \vec{b} = (b_1,\ldots,b_{n2}) \tag{36}$$

since both functionals J_1, J_2 are of quadratic type, a linear algebraic solvable system is to be solved for $\vec{a},\vec{b}$ and hence the approximate solution will be obtained. The full description of the method is given next as follows:

<u>Step 5</u>: Differentiate the functional $J[\vec{a},\vec{b}]$ given in above Eq. (35) with respect to the parameters a_p's, $p = 1,\ldots,n_1$ and equal the resulting equations to zero to find the critical points of the functional (34):

$$\frac{\partial J[\vec{a},\vec{b}]}{\partial a_p} = 0, \quad p = 1,\ldots,n_1 \tag{37}$$

Remark (1): For the main functional $J[\vec{a},\vec{b}]$ given in Eq. (35) to assume its minimum at the points $(a_1, a_2, .., a_{n1})$ it is necessary that the derivative of it with respect to a_p's, $p = 1,..,n_1$ is equal to zero, (i.e., (38))

$$\frac{\partial J[\vec{a},\vec{b}]}{\partial a_p} = \omega_1 \frac{\partial J_1[\vec{a},\vec{b}]}{\partial a_p} + \omega_2 \frac{\partial J_2[\vec{a},\vec{b}]}{\partial a_p} = 0, \quad p = 1,..,n_1 \tag{38}$$

$$\frac{\partial J[\vec{a},\vec{b}]}{\partial a_p} = 0 = \int_\Omega \left(\begin{array}{l} \omega_1 \left(\begin{array}{l} 2\mu_1 G_p(x,t)\left[\left(G(x,t) + \sum_{i=1}^{n1} a_i G_i(x,t)\right) - w^d(x,t)\right] \\ + 2\mu_2 \frac{\partial}{\partial x} G_p(x,t)\left[\frac{\partial}{\partial x} G(x,t) + \sum_{i=1}^{n1} a_i \frac{\partial}{\partial x} G_i(x,t)\right] \\ + 0 \end{array} \right) \\ + \omega_2 \left(\begin{array}{l} 2\left(\frac{\partial}{\partial t} G_p(x,t) - \alpha \frac{\partial^2}{\partial x^2} G_p(x,t)\right). \\ \left(\begin{array}{l} \left(\frac{\partial}{\partial t} G(x,t) - \alpha \frac{\partial^2}{\partial x^2} G(x,t) - \beta N(x,t)\right) \\ + \sum_{i=1}^{n1} a_i \left(\frac{\partial}{\partial t} G_i(x,t) - \alpha \frac{\partial^2}{\partial x^2} G_i(x,t)\right) \\ - \sum_{j=1}^{n2} b_j\left(\beta N_j(x,t)\right) - f(x,t) \end{array} \right) \end{array} \right) \end{array} \right) d\Omega \tag{39}$$

<u>Step 6</u>: Differentiate the functional $J[\vec{a},\vec{b}]$ given in above Eq. (35) with respect to the parameters b_q's, $q = 1,\ldots,n_2$; and equal the resulting equations to zero to find the critical points of the functional (35):

$$\frac{\partial J[\vec{a},\vec{b}]}{\partial b_q} = 0, \quad q = 1,\ldots,n_2 \tag{40}$$

Remark (2): For the functional $J[\vec{a},\vec{b}]$ given in Eq. (35) to assume its minimum at the points $(b_1, b_2, .., b_{n2})$ it is necessary that the derivative of it with respect to b_q's, $q = 1,..,n_2$ is equal to zero. i.e: (41)

$$\frac{\partial J[\vec{a},b]}{\partial b_q} = \omega_1 \frac{\partial J_1[\vec{a},b]}{\partial b_q} + \omega_2 \frac{\partial J_2[\vec{a},b]}{\partial b_q} = 0, \quad q = 1,\ldots,n_2 \tag{41}$$

$$\frac{\partial J[\vec{a},\vec{b}]}{\partial b_q} = 0 = \int_\Omega \left(\begin{array}{l} \omega_1 \left(\begin{array}{l} \mu_1(0) + \mu_2(0) \\ + 2\mu_3 N_q(x,t)\left[N(x,t) + \sum_{j=1}^{n2} b_j N_j(x,t)\right] \end{array} \right) \\ + \omega_2 \left(2\beta N_q(x,t) \left(\begin{array}{l} \left(\frac{\partial}{\partial t} G(x,t) - \alpha \frac{\partial^2}{\partial x^2} G(x,t) - \beta N(x,t)\right) \\ + \sum_{i=1}^{n1} a_i \left(\frac{\partial}{\partial t} G_i(x,t) - \alpha \frac{\partial^2}{\partial x^2} G_i(x,t)\right) \\ - \sum_{j=1}^{n2} b_j\left(\beta N_j(x,t)\right) - f(x,t) \end{array} \right) \right) \end{array} \right) d\Omega \tag{42}$$

Step 7: Rearranging above Eqs. (39), (42) for a_p's, $p=1,...,n_1$ and for b_q's, $q=1,...,n_2$ to get n × n system of algebraic equations CA=D, for the unknowns $A=[a_1,a_2,..,a_{n_1},b_1,b_2,..,b_{n_2}]^T$, where $n=n_1+n_2$ as follows: For p=1,2,.., n_1, i= 1,2,.., n_1, n_1+1,.., n_1+n_2.

$$e_{ip}(x,t)=2\omega_1\begin{pmatrix}\mu_1 G_p(x,t)\,G_i(x,t)+\\ \mu_2\frac{\partial}{\partial x}G_p(x,t)\,\frac{\partial}{\partial x}G_i(x,t)\end{pmatrix}+2\omega_2\left(\frac{\partial}{\partial t}G_p(x,t)-\alpha\frac{\partial^2}{\partial x^2}G_p(x,t)\right).\left(\frac{\partial}{\partial t}G_i(x,t)-\alpha\frac{\partial^2}{\partial x^2}G_i(x,t)-\beta N_j(x,t)\right)\quad(43)$$

$$k_p(x,t)=2\omega_1\begin{pmatrix}\mu_1 G_p(x,t)\,w^d(x,t)-\\ \mu_1 G_p(x,t)\,G(x,t)-\\ \mu_2\frac{\partial}{\partial x}G_p(x,t)\,\frac{\partial}{\partial x}G(x,t)\end{pmatrix}+2\omega_2\begin{pmatrix}\left(\frac{\partial}{\partial t}G_p(x,t)-\alpha\frac{\partial^2}{\partial x^2}G_p(x,t)\right)\\ \begin{pmatrix}f(x,t)-(\frac{\partial}{\partial t}G(x,t)-\\ \alpha\frac{\partial^2}{\partial x^2}G(x,t)-\beta N(x,t))\end{pmatrix}\end{pmatrix}\quad(44)$$

For q = n_1+1,.., n_1+n_2 , i= 1,2,.., n_1, n_1+1,.., n_1+n_2:

$$e_{iq}(x,t)=2\omega_1\mu_3\left(N_q(x,t)\,N_j(x,t)\right)+2\omega_2\,\beta N_q(x,t)\left(\frac{\partial}{\partial t}G_i(x,t)-\alpha\frac{\partial^2}{\partial x^2}G_i(x,t)-\beta N_j(x,t)\right)\quad(45)$$

and

$$k_q(x,t)=2\omega_1\mu_3\left(N_q(x,t)\,N(x,t)\right)+2\omega_2\,\beta N_q(x,t)\begin{pmatrix}f(x,t)-\\ (\frac{\partial}{\partial t}G(x,t)-\alpha\frac{\partial^2}{\partial x^2}G(x,t)-\beta N(x,t))\end{pmatrix}\quad(46)$$

using the Eqs. (43) – (46) the n × n system of algebraic equations CA=D with C is n × n constant matrix defined as:

$$C=\begin{bmatrix}\int_0^T\int_0^d e_{11}(x,t)\,dxdt & \int_0^T\int_0^d e_{12}(x,t)\,dxdt & \cdots & \int_0^T\int_0^d e_{1n}(x,t)\,dxdt\\ \int_0^T\int_0^d e_{21}(x,t)\,dxdt & \int_0^T\int_0^d e_{22}(x,t)\,dxdt & \cdots & \int_0^T\int_0^d e_{2n}(x,t)\,dxdt\\ \vdots & \vdots & \cdots & \vdots\\ \int_0^T\int_0^d e_{n1}(x,t)\,dxdt & \int_0^T\int_0^d e_{1n}(x,t)\,dxdt & \cdots & \int_0^T\int_0^d e_{nn}(x,t)\,dxdt\end{bmatrix}\quad(47)$$

and the right-hand side n×1 vector D is defined by:

$$D=\begin{bmatrix}\int_0^T\int_0^d k_1(x,t)\,dx\,dt\\ \vdots\\ \int_0^T\int_0^d k_{n1}(x,t)\,dx\,dt\\ \int_0^T\int_0^d k_{n1+1}(x,t)\,dx\,dt\\ \vdots\\ \int_0^T\int_0^d k_n(x,t)\,dx\,dt\end{bmatrix}\quad(48)$$

Step 8: By using MATLAB program for the linear n × n algebraic system, one can solve the problem for n= n_1+n_2 unknown parameters a_p's, $p=1,..,n_1$ and b_q's, $q=1,..,n_2$, as will be show in illustrative examples next.

Minimum Approach

Optimization problems represent finding parameters that minimize or maximize an objective function with or without satisfying constraints. A search for extremum (a minimum or maximum) of a scalar valued function is an optimization problem, it is nothing else than the search for the zeros of the gradient of that function. There are two types of optimization problems, these are constrained optimization and unconstrained optimization [9], in this work only unconstrained optimization is deal with. The minimization approach for unconstrained optimization problem with estimation of gradient and Hessian matrix is to find the optimal control $u\in U$ that minimize the performance index $J(u(x,t))$ subject to w(x,t) is the solution of the problem (22) and this approach is continuous until the optimized solution of the problem (1) – (8) is achieved. In optimization, Quasi-Newton methods are algorithms for finding local maxima or minima of functions, these methods are based on

Al-Mustansiriyah Journal of Science
ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 34, Issue 3, 2023 DOI: http://doi.org/10.23851/mjs.v34i3.1104

Newton's method to find the stationary point of a function when the gradient is zero.

Newton's method assumes that the function can be locally approximated as a quadratic in the region around the optimum and uses the first and second derivatives to find the stationary point. In higher dimensions Newton's method uses the gradient and the Hessian matrix of second derivatives of the function to be minimized. In Quasi-Newton methods, the Hessian matrix does not be computed. The Hessian is updated by analyzing successive gradient vector instead. Quasi-Newton methods are a generalization of the secant method to find the root of the first derivative for multidimensional problems. One of the chief advantages of Quasi-Newton methods over Newton's method is that the Hessian matrix does not need to be inverted. It is usually an estimate of the inverse matrix directly [2].

RESULTS AND DISCUSSIONS

All the above details of the explained methods will be applied to the following two simulation problems. Each problem will be solved by using the two explained approaches as follows:

Illustrative Problem I

Consider the Main optimal control problem Eq. (1) or equivalently Eq. (26):

$$\min_{u\in U} J(u(x,t)) = \min_{u\in U}\left(\omega_1 J_1 + \omega_2 J_2\right) \quad (49)$$

with the weight factors $\omega_1 = \omega_2 = 0.5$ and the first objective function J_1 as in Eq. (16) is the functional:

$$J_1(w(x,t),u(x,t)) = \int_\Omega \begin{pmatrix} \mu_1\left[w^d(x,t)-w(x,t)\right]^2 \\ + \ \mu_3\left[u(x,t)\right]^2 \end{pmatrix} d\Omega \quad (50)$$

with the presence factors $\mu_1 = \mu_3 = 0.5$, $\mu_2 = 0$, and the second objective function J_2 as in Eq. (25) is:

$$J_2(w(x,t),u(x,t)) = \int_\Omega \begin{pmatrix} \frac{1}{2}\left[w_t(x,t)-\alpha w_{xx}(x,t)-\beta u(x,t)\right]^2 \\ - \ \left[f(x,t).\left[w_t(x,t)-\alpha w_{xx}(x,t)-\beta u(x,t)\right]\right] \end{pmatrix} d\Omega \quad (51)$$

where the region of the solution is $\Omega = \{(x,t) \mid x\in[0,1], t\in[0,1]\}$, and $f(x,t) = 2w^d(x,t)$, $\alpha=\beta=1$.

Subject to:

$$w_t(x,t) = w_{xx}(x,t) + 2w^d(x,t) + u(x,t) \quad \text{on}\ \Omega \quad (52)$$

with the initial condition:

$$w(x,0) = 0 \qquad 0 \le x \le 1 \quad (53)$$

together with boundary conditions:

$$w(0,t) = 0, \qquad 0 \le t \le 1 \quad (54)$$

$$w(1,t) = 0, \qquad 0 \le t \le 1 \quad (55)$$

with the final state is:

$$w(x,1) = -0.6146\, x(x-1), \quad 0 \le x \le 1 \quad (56)$$

and the desired state is:

$$w^d(x,t) = t^2 x(1-x) \quad (57)$$

The solution steps are:

<u>Step1</u>: From (27) set $w_{n1}(x,t) = G(x,t) + \sum_{i=1}^{n1} a_i G_i(x,t)$ with $n_1 = 5$ where $G(x,t) = 0$, $G_1(x,t) = t\,x(1-x)[t]$, $G_2(x,t) = t\,x(1-x)[t\,x]$, $G_3(x,t) = t\,x(1-x)[t^2 x]$, $G_4(x,t) = t\,x(1-x)[t^2 x^2]$, $G_5(x,t) = t\,x(1-x)[t^2 x^3]$

Thus,

$$w_5(x,t) = \sum_{i=1}^{5} a_i G_i(x,t) \quad (58)$$

where $G_i(x,t) = 0$, $i = 1,..,5$ on the corresponding homogeneous conditions of the given initial and boundary conditions (53) – (55), and no restrictions are given on the solution at the final state (56) at t = 1. And from (28) set $u_{n2}(x,t) = N(x,t) + \sum_{j=1}^{n2} b_j N_j(x,t)$ with $n_2 = 5$ where $N(x,t) = 0$, $N_1(x,t) = t\,x(1-x)[t]$, $N_2(x,t) = t\,x(1-x)[t\,x]$, $N_3(x,t) = t\,x(1-x)[t^2 x]$, $N_4(x,t) = t\,x(1-x)[t^2 x^2]$, $N_5(x,t) = t\,x(1-x)[t^2 x^3]$

thus,

$$u_5(x,t)=\sum_{j=1}^{5} b_j N_j(x,t) \quad (59)$$

Step2: Define the linear operator L by:
$L(w,u)=w_t(x,t)-\alpha w_{xx}(x,t)-u(x,t) \text{ on } \Omega$ (60)
where the conditions (9) – (12) are held for this problem. To find the classical solution w(x,t) which means continuous function in the closed domain Ω and satisfy the PDE (60) in the open domain Ω and is equal to zero on the boundary $\partial\Omega, \forall u(x,t) \in U$ which is given in Eq. (3), and $f(x,t)=2w^d(x,t)$.

Results of Variational Approach

The problem of finding the approximate optimal control u(x,t) and the corresponding approximate solution of the state function w(x,t) is converted to find the critical points of the functional (35), set $n_1 = n_2 = 5$ this means: $\vec{a}=(a_1,a_2,a_3,a_4,a_5), \vec{b}=(b_1,b_2,b_3,b_4,b_5)$ (61) and differentiate the functional (35) with respect to $\vec{a}$ by using Eq. (38) with p=1,..,5 to get Eqs. (43) and (44) and differentiate the functional (35) with respect to $\vec{b}$ by using Eq. (41) with q=1,...,5 to get Eqs. (45) and (46). All above leads to a system of n×n dimension with Eqs. (47) and (48) and finally using step 8 of the computational algorithm for finding the approximate solutions to obtain the values of $\vec{a}$, $\vec{b}$ as shown next in tables after the explanation of applying the second approach to the same illustrative problem I for the sake of comparison.

Results of Minimum Approach

The details are explained in 6.2, with setting $n_1 = n_2 = 5$ and using Eq. (61).

***Remark (3)*:** Not only the result values of $\vec{a}$, $\vec{b}$ for two approaches are shown next but also the result values of $\vec{a}$, $\vec{b}$ of applying Quasi – Newton method to illustrative problem I and illustrative problem II are also listed in the Tables 1 and 2 of results to enrich the comparison details of the modified approaches.

Table 1: The Approximate Values of $\vec{a}$, $\vec{b}$ for Illustrative Problem I of Eqs. (49) - (57)

Parameter name	Variational approach	Minimum approach	Quasi – Newton method
a_1	0.1030	0.1031	0.1031
a_2	0.0652	0.0641	0.0653
a_3	0.0368	0.0366	0.0359
a_4	- 0.1143	- 0.1113	- 0.1125
a_5	0.0008	- 0.0007	- 0.0002
b_1	- 0.3340	- 0.2924	- 0.3063
b_2	0.8771	0.7023	0.8418
b_3	- 0.1750	- 0.2760	- 0.3744
b_4	- 1.8910	- 1.3405	- 1.3947
b_5	1.2053	0.9202	0.9026

Table 2: The Absolute Error of the Four Functionals Constructing the Illustrative Problem I of Eqs. (49) - (57).

Functional name	Variational approach	Minimum approach	Quasi – Newton method
J_1	1.3912×10^{-4}	4.6397×10^{-5}	4.6463×10^{-5}
J_2	1.7090×10^{-4}	1.7078×10^{-4}	1.7092×10^{-4}
J_3	2.6193×10^{-5}	2.6198×10^{-5}	2.6186×10^{-5}
Main J	4.0839×10^{-7}	9.7867×10^{-6}	9.7881×10^{-6}

Figures 1 and 2 demonstrate the values of the J1 and J2 functions for illustrative Problem I using the First and Second approaches, as well as the Quasi-Newton method. Figure 3 displays the approximations and desired solutions in the variational, minimum, and Quasi-Newton approaches, respectively. Figure 4 presents the main J values for illustrative Problem I using the First and Second approaches, along with the Quasi-Newton method.

Al-Mustansiriyah Journal of Science
ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 34, Issue 3, 2023 DOI: http://doi.org/10.23851/mjs.v34i3.1104

Figure 1: J1 values of illustrative problem I for First and second approaches and Quasi-Newton method respectively.

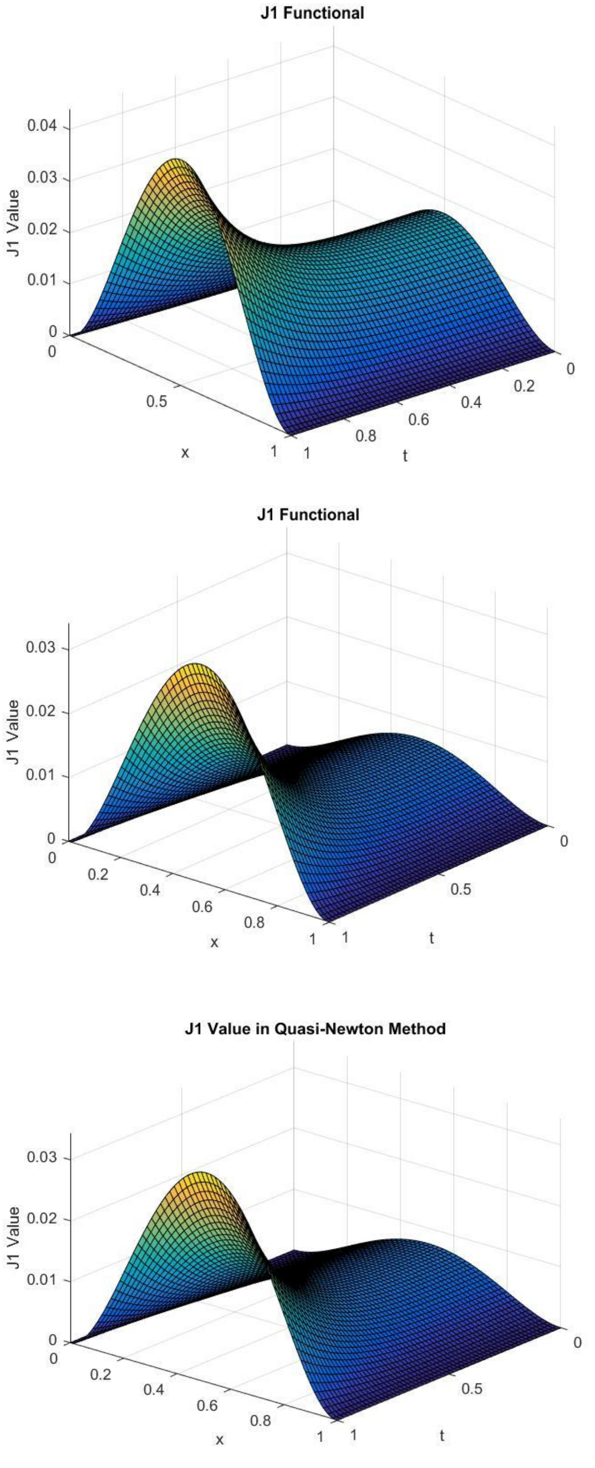


Figure 2: J2 values of illustrative problem I for First and second approaches and Quasi-Newton method respectively

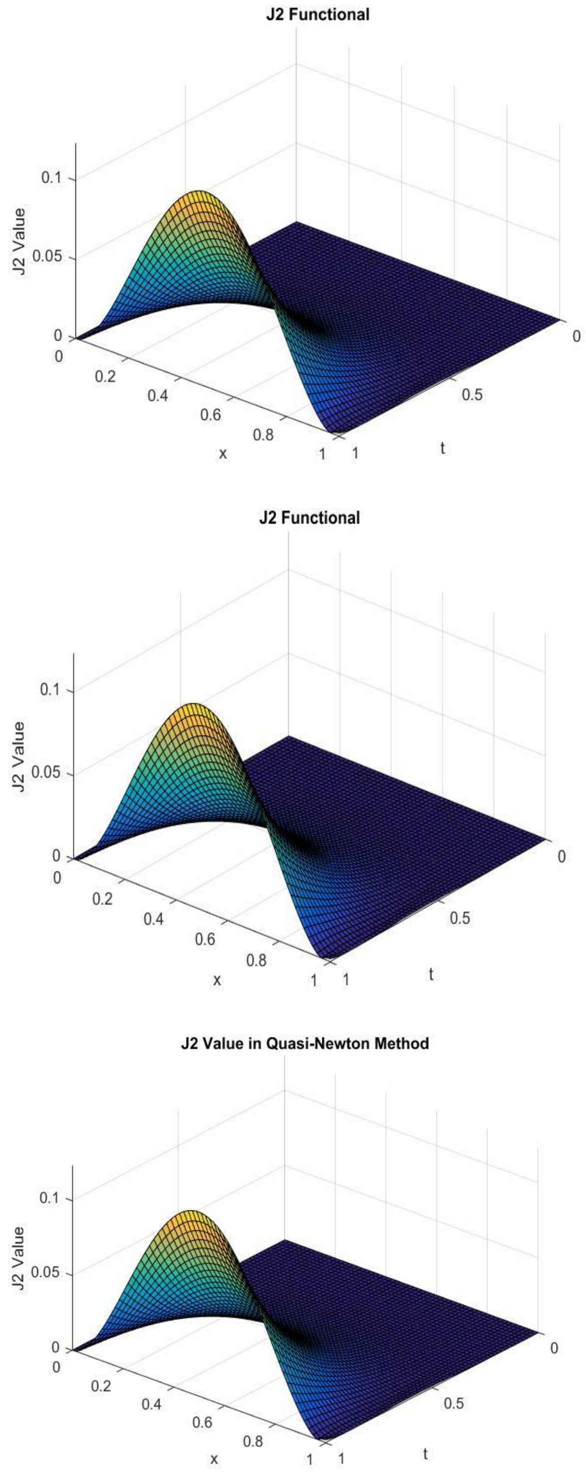


Figure 3: J3 values of illustrative problem I for First and second approaches and Quasi-Newton method respectively

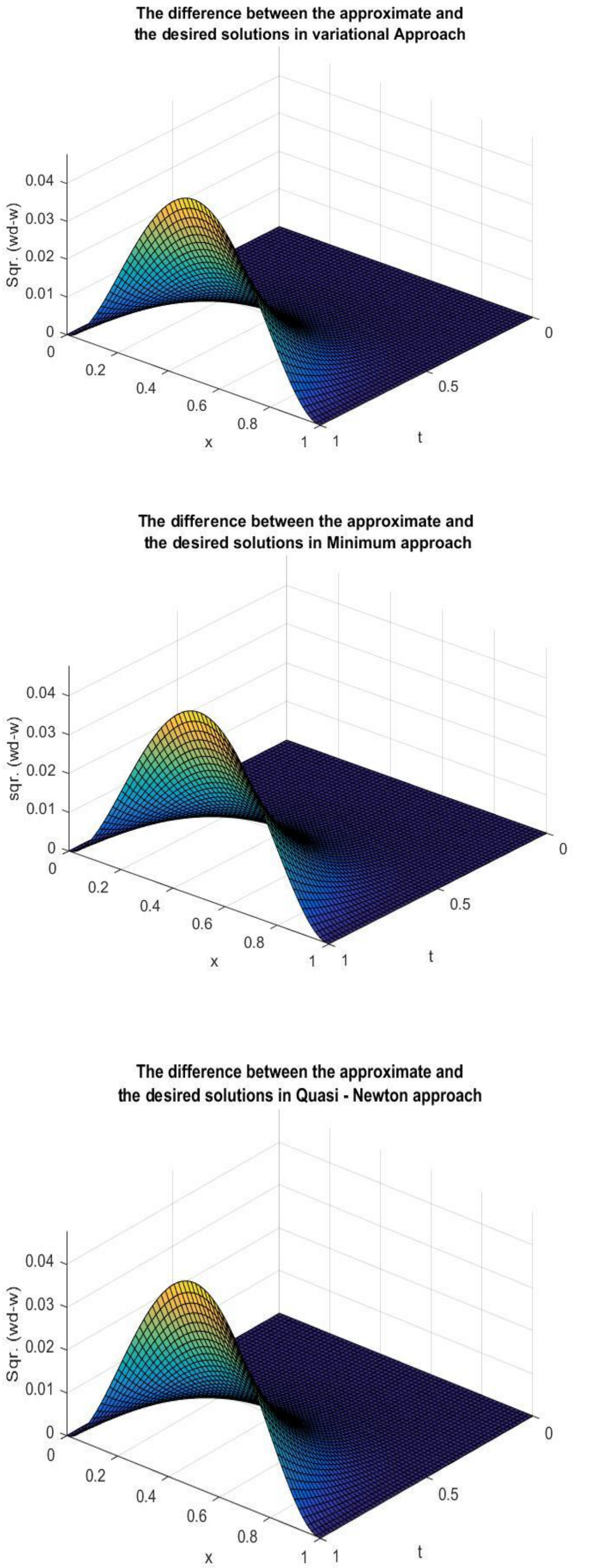
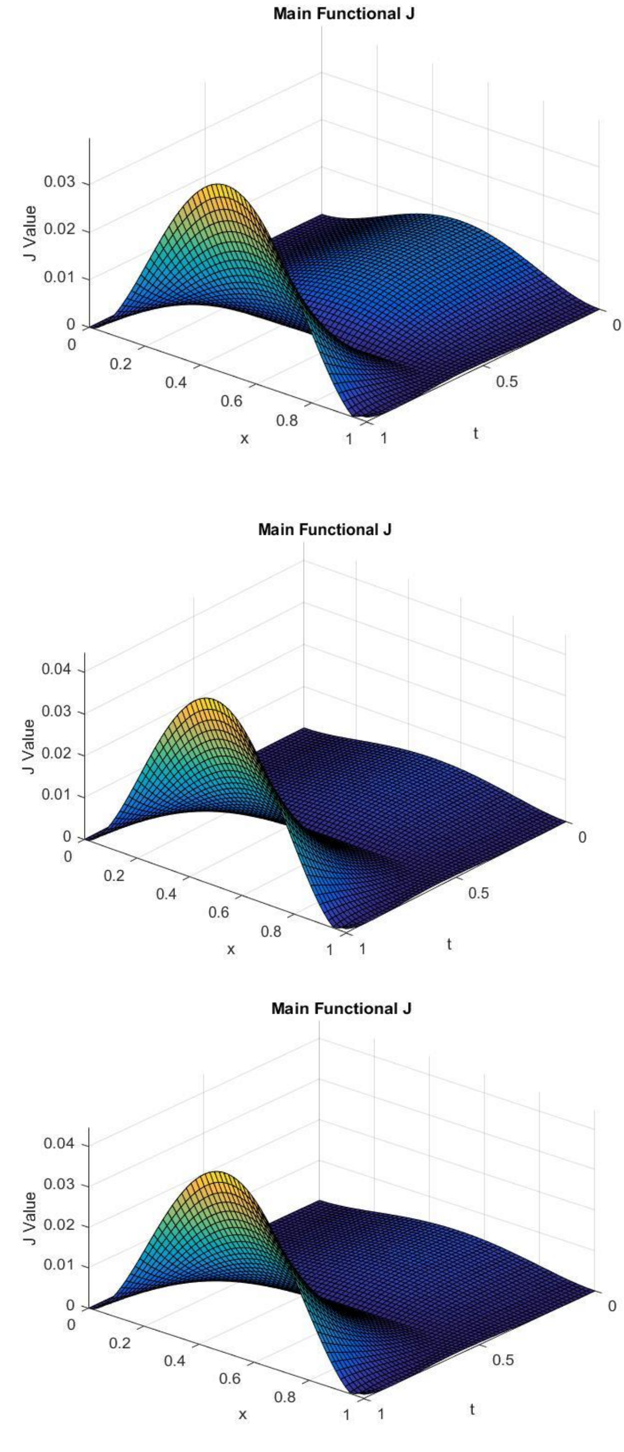


Figure 4: Main J values of illustrative problem I for First and second approaches and Quasi-Newton method respectively.



Illustrative Problem II

Consider the Main optimal control problem Eq. (1) or equivalently Eq. (26):

$$\min_{u\in U} J(u(x,t)) = \min_{u\in U} \left(\omega_1 J_1 + \omega_2 J_2\right) \tag{62}$$

with the weight factors $\omega_1 = \omega_2 = 0.5$ and the first objective function J_1 as in Eq. (16) is the functional:

$$J_1(w(x,t),u(x,t)) = \int_\Omega \left(\mu_1 \left[w^d(x,t) - w(x,t)\right]^2 + \mu_3 \left[u(x,t)\right]^2 \right) d\Omega \quad (63)$$

with the presence factors $\mu_1 = \mu_3 = 0.5$, $\mu_2 = 0$, and the second objective function J_2 as in Eq. (25) is:

$$J_2(w(x,t),u(x,t)) = \int_\Omega \left(\begin{array}{l} \frac{1}{2}[w_t(x,t) - \alpha w_{xx}(x,t) - \beta u(x,t)]^2 \\ - [f(x,t).[w_t(x,t) - \alpha w_{xx}(x,t) - \beta u(x,t)]] \end{array} \right) d\Omega \quad (64)$$

where the region of the solution is $\Omega = \{(x,t) \mid x \in [0,1], t \in [0,0.5]\}$, and $f(x,t) = 0$, $\alpha = x(x-1)$ $\beta = 1$. Subject to:

$$w_t(x,t) = x(x-1)\, w_{xx}(x,t) + u(x,t) \quad \text{on}\, \Omega \quad (65)$$

with the initial condition:

$$w(x,0) = 0, \qquad 0 \le x \le 1 \quad (66)$$

together with boundary conditions:

$$w(0,t) = t^2, \qquad 0 \le t \le 0.5 \quad (67)$$

$$w(1,t) = 0, \qquad 0 \le t \le 0.5 \quad (68)$$

with the final state:

$$w(x,0.5) = 0, \qquad 0 \le x \le 1 \quad (69)$$

and the desired state is:

$$w^d(x,t) = t^2(1-x) \quad (70)$$

The solution steps are:

Step1: From (27) set $w_{n1}(x,t) = G(x,t) + \sum_{i=1}^{n1} a_i G_i(x,t)$ with $n_1 = 5$ where $G(x,t) = 0$, $G_1(x,t) = t\,x(1-x)[t]$, $G_2(x,t) = t\,x(1-x)[t\,x]$, $G_3(x,t) = t\,x(1-x)[t^2 x]$, $G_4(x,t) = t\,x(1-x)[t^2 x^2]$, $G_5(x,t) = t\,x(1-x)[t^2 x^3]$

Thus $w_5(x,t) = \sum_{i=1}^{5} a_i G_i(x,t)$ Where $G_i(x,t) = 0$, $i = 1,..,5$ on the corresponding homogeneous conditions of the given initial and boundary conditions (66) – (68) and no restrictions are given on the solution at the final state (69) at t = 0.5 and from (28) set $u_{n2}(x,t) = N(x,t) + \sum_{j=1}^{n2} b_j N_j(x,t)$ with $n_2 = 5$ where $N(x,t) = 0$, $N_1(x,t) = t\,x(1-x)[t]$, $N_2(x,t) = t\,x(1-x)[t\,x]$, $N_3(x,t) = t\,x(1-x)[t^2 x]$, $N_4(x,t) = t\,x(1-x)[t^2 x^2]$, $N_5(x,t) = t\,x(1-x)[t^2 x^3]$ as in Eq. (59) Thus $u_5(x,t) = \sum_{j=1}^{5} b_j N_j(x,t)$

Step2: Define the linear operator L by:

$$L(w,u) = w_t(x,t) - x(x-1) w_{xx}(x,t) - u(x,t) \quad \text{on}\, \Omega \quad (71)$$

Results of Variational Approach

The explanation of the first approach of illustrative problem II is typical the same as explained in illustrative problem I.

Results of Minimum Approach

The explanation of the second approach of illustrative problem II is typical the same as explained in illustrative problem I.

Table 3: The Approximate Values of $\vec{a}$, $\vec{b}$ for Illustrative Problem II of Eqs. (62) - (70).

Parameter name	Variational approach	Minimum approach	Quasi – Newton method
a_1	- 0.4373	- 0.4062	- 0.4169
a_2	0.4616	0.3667	0.3583
a_3	2.2624	2.1910	2.1516
a_4	- 5.4233	- 5.0502	- 4.9142
a_5	2.6245	2.4655	2.5230
b_1	0.6906	0.7545	0.6317
b_2	- 2.1003	-3.6014	- 4.6038
b_3	-2.9774	1.8418	7.6747
b_4	-10.9517	- 1.0861	5.0385
b_5	61.2323	31.2719	5.4363

Table 4: The Absolute Error of the Four Functionals Constructing the Illustrative Problem II of Eqs. (62) - (70).

Functional name	Variational approach	Minimum approach	Quasi – Newton method
J_1	4.4686×10^{-5}	4.4970×10^{-5}	4.5286×10^{-5}
J_2	2.8957×10^{-7}	2.8020×10^{-7}	2.8652×10^{-7}
J_3	2.8827×10^{-6}	2.9024×10^{-6}	2.9166×10^{-6}
Main J	1.3042×10^{-5}	1.3087×10^{-5}	1.3194×10^{-5}

Figures 5, 6, 7, and 8 demonstrate the values of the J1, J2, J3 functions, and the main J values, respectively, for illustrative problem II using the first and second approaches and Quasi-Newton methods.

Figure 5: J1 values of illustrative problem II for First and second approaches and Quasi-Newton method respectively.

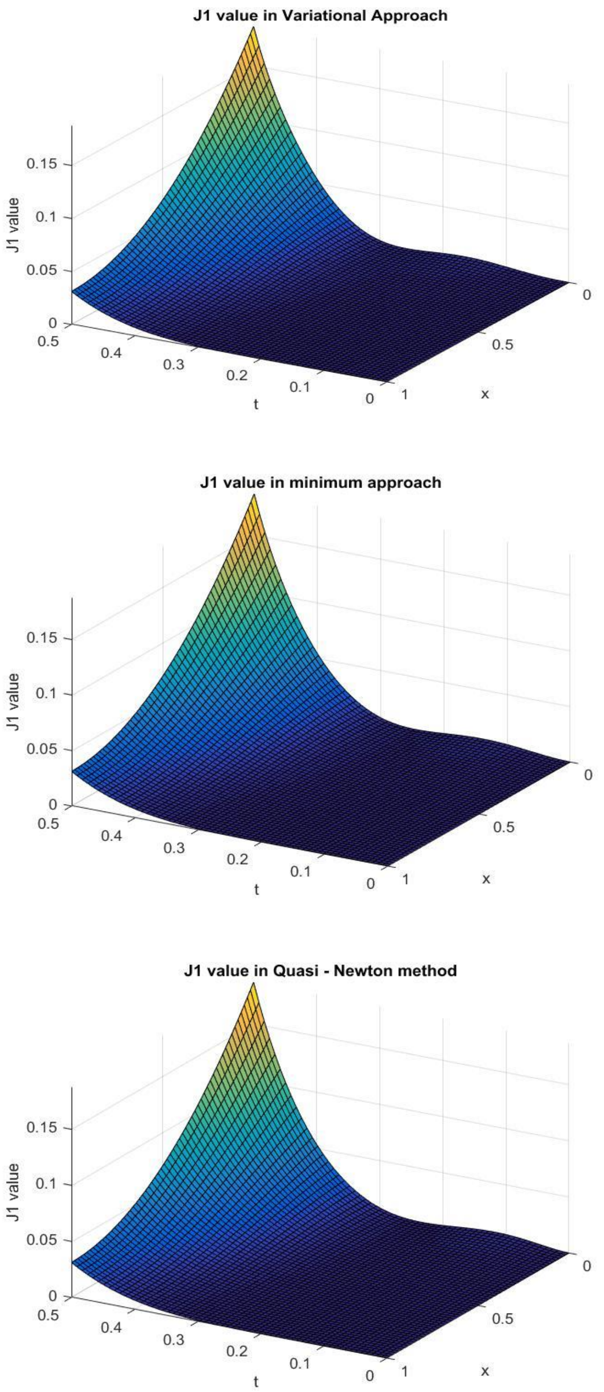
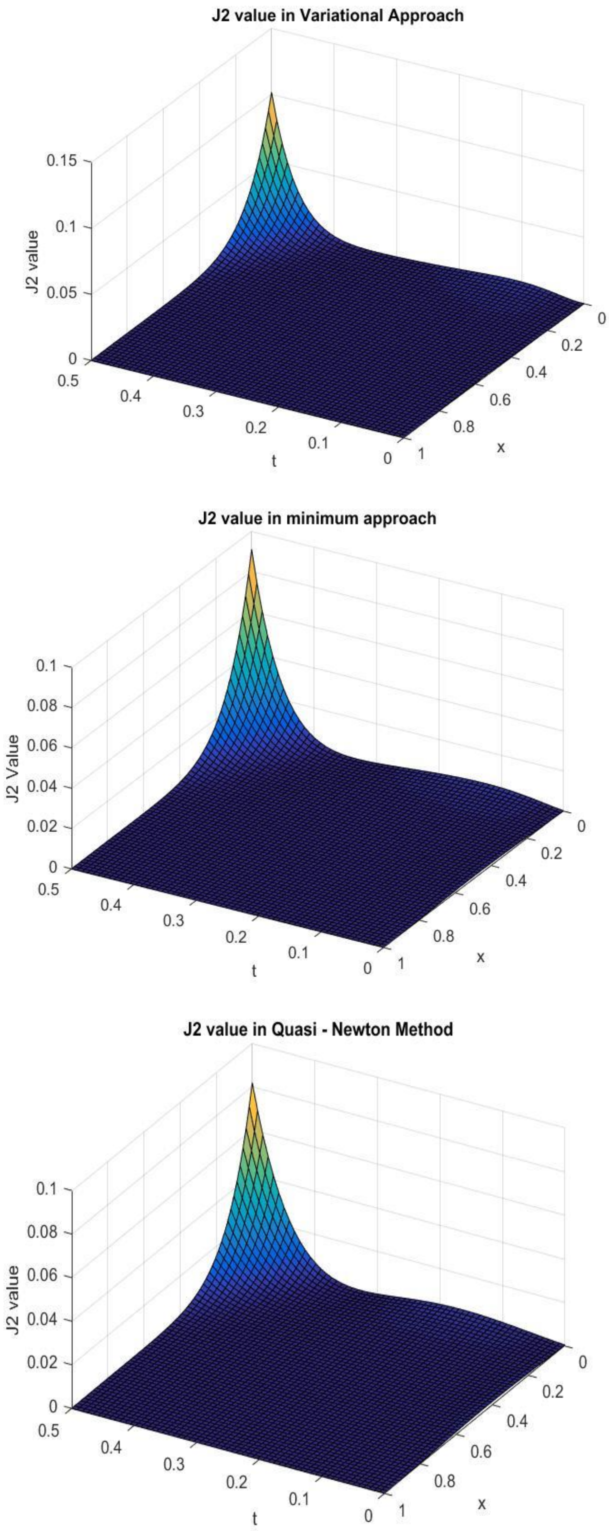


Figure 6: J2 values of illustrative problem II for First and second approaches and Quasi-Newton method respectively.



Al-Mustansiriyah Journal of Science
ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 34, Issue 3, 2023 DOI: http://doi.org/10.23851/mjs.v34i3.1104

Figure 7: J3 values of illustrative problem II for First and second approaches and Quasi-Newton method respectively.

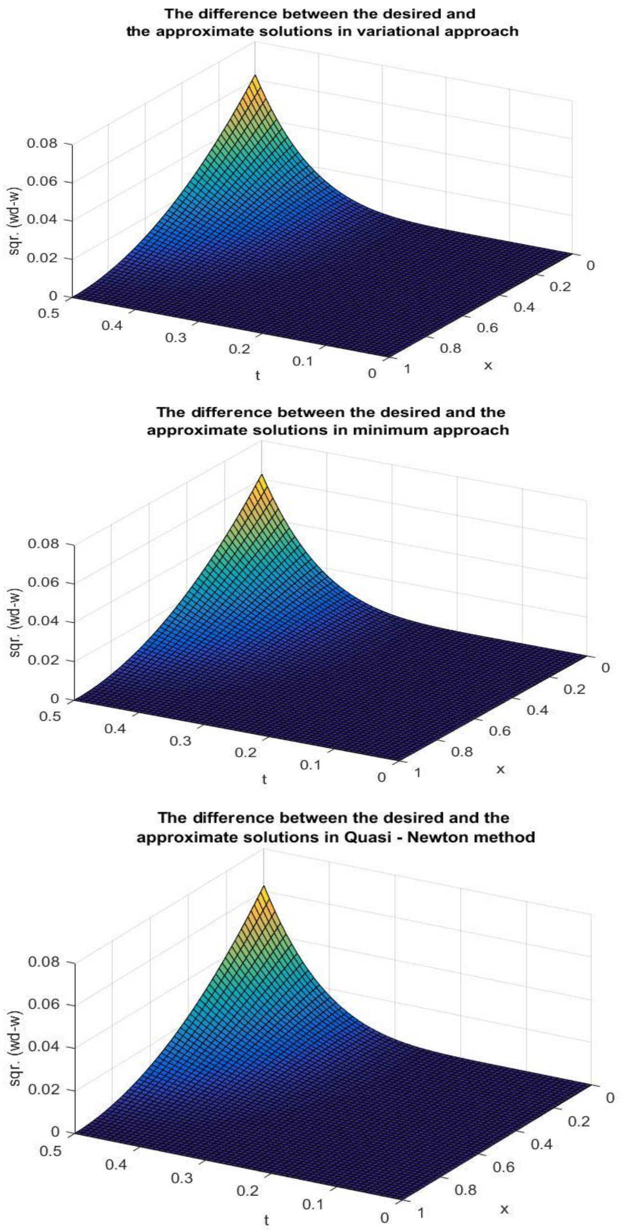
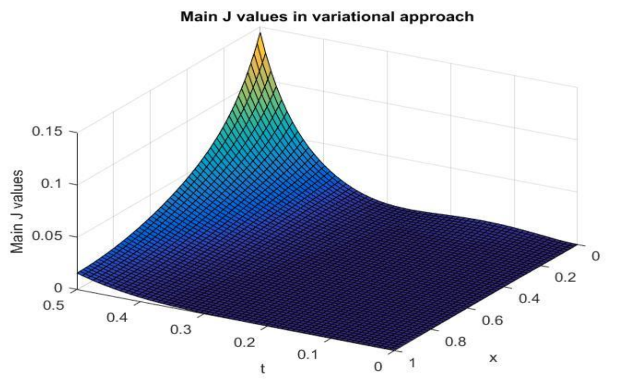
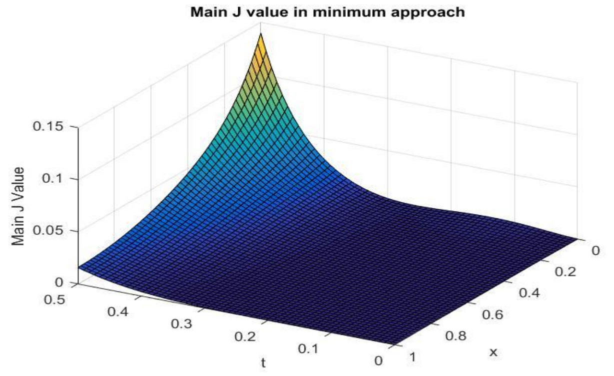
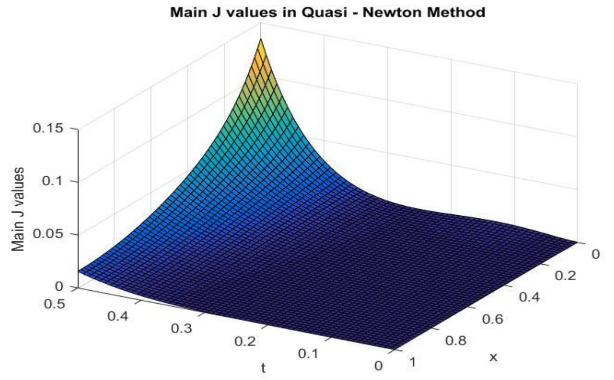


Figure 8: Main J values of illustrative problem II for First and second approaches and Quasi-Newton method respectively







CONCLUSIONS

We obtained good approximate solutions and accepted error values for the control problems with fixed boundaries using the first and second approaches, as we compared them with the quasi-Newton method. In all the problems we solved, we utilized a small number of basis functions, typically five or fewer, and achieved satisfactory results. However, it is worth noting that the solution approaches can be further improved by incorporating a greater number of basis functions. All the problems we tackled demonstrated a high degree of agreement in their solutions. The general structure of functional (35) may vary or take on a different form from one problem to another, depending on changes in the form of Equation (22).

Disclosure and Conflicts of Interest: The authors advertise that they have no conflicts of interest.

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How to Cite

A. S. Hameed and R. A. . Zaboon, "Approximate Solutions for Optimal Control of Fixed Boundary Value Problems Using Variational and Minimum Approaches", *Al-Mustansiriyah Journal of Science*, vol. 34, no. 3, pp. 72–85, Sep. 2023.