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Research article

On commutativity of quotient semirings through generalized derivations

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Abstract: This research article aims to study quotient MA-semirings determined by the prime ideals. Derivations are important tools to study algebraic structures. We establish some theorems on commutativity of quotient MA-semirings under certain differential identities. Results of this paper are extensions of many well known facts of this topic.

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1. Introduction and preliminaries

In general commutators are undefined in semirings. However, this notion is well defined in a special class of semirings called MA-semirings, therefore Lie theory and other related notions can be studied in semirings. Javed et al. [1] defined MA-semiring as an additive inverse semiring S with absorbing zero $'0'$ satisfying A_2 condition of Bandlet and Petrich [2] that is $u+u' \in Z(S)$ for all $u \in S$, where u' is the pseudo inverse of $u \in S$, and $Z(S)$ is the center of S. The class of MA-semirings properly contains the class of rings, meaning every ring is an MA-semiring while converse may not be true (for more detail, and examples of MA-semirings, we refer reader to [3–7]).

Theory of ideals has become a notion of special worth in certain algebras and ring theory. Several ideals, such as prime ideals, k-ideals, Q-ideals, Quasi-ideals, Jordan ideals, Lie ideals have been defined and studied for different types of algebraic structures (for ready reference one can see [8–13]). An ideal I of a semiring S is called prime if for $a, b \in S$, $aSb \subseteq I$ implies either $a \in I$ or $b \in I$. An ideal I of a semiring S is said to be a Q-ideal if there exists a partitioning subset Q of S such that $S = \bigcup\{q + I : q \in Q\}$ and if $q_1, q_2 \in Q$, then $(q_1 + I)\bigcap(q_2 + I) \neq \phi$ if and only if $q_1 = q_2$ (see [9]). An ideal I of a semiring S is said to be k-ideal if $a+b \in I$ and $b \in I$, then $a \in I$. In fact, every Q-ideal is k-ideal but converse may not be true in general (see [8]). Throughout the sequel by I_Q, we mean prime Q-ideal unless mentioned otherwise.

Derivations and its generalized formats defined for any algebraic structure have become a very rich area of investigation and researchers have contributed a remarkable work on it (for reference, one can see [1, 3, 14–18]). A derivation is an additive mapping $d : S \longrightarrow S$ such that $d(ut) = d(u)t + ud(t)$. An additive mapping $G_d : S \longrightarrow S$ fulfilling $G_d(tu) = G_d(t)u + td(u)$ is a generalized derivation associated with a derivation d. The commutator and Jordan product of $t, u \in S$ are respectively defined as $[t, u] = tu + u^{'}t$ and $t \circ u = tu + ut$. For the sequel, we state some useful identities of MA-semirings: For all $t, u, v \in S$, we have $[tu, v] = t[u, v] + [t, v]u$, $[t, tu] = t[t, u]$, $[t, u] + [u, t] = u(t + t^{'}) = t(u + u^{'})$, $[t, uv] = [t, u]v + u[t, v]$, $[t, u]^{'} = [t, u^{'}] = [t^{'}, u]$, $(tu)^{'} = t^{'}u = tu^{'}$, $t \circ (u + v) = t \circ u + t \circ v$. For more on MA-semirings, we refer [1, 3].

A ring R is commutative if and only if $[u, v] = 0$, for all $u, v \in R$. However, in the case of an MA-semiring, the sufficiency of the above said statement is valid while the necessity part may not be true in general. One can find many reasonable examples in support of this statement. To make it more easy to understand: in rings $[u, u] = 0$ but in MA-semirings $[u, u] = 0$ may not hold in general. Nadeem and others [19] proved some results on commutativity of Quotient rings enforced by generalized derivations satisfying different identities on prime ideals. In this paper we prove these results for MA-semirings.

2. Main results

Unless otherwise specified, in the sequel, we refer to G_d as a generalized derivation associated with a derivation d.

Theorem 2.1. *Let $(S, +, .)$ be an MA-semiring and I_Q be a prime Q-ideal of S. Then the set $S/I_Q = \{u + I_Q : u \in S\}$ forms an MA-semiring with respect to the addition $\oplus$ and multiplication $\odot$ defined by*

(1) $(u + I_Q) \oplus (t + I_Q) = u + t + I_Q$,

(2) $(u + I_Q) \odot (t + I_Q) = u.t + I_Q$,

for all $u, t \in S$.

Proof. We only verify A_2 axiom of Bandlet and Petrich [2], the other axioms of the definition of MA-semiring are straightforward. For this let $u + I_Q, t + I_Q \in S/I_Q$. Then

$$\begin{aligned} & [(u + I_Q) \oplus (u + I_Q)^{'}] \odot (t + I_Q) \\ &= [(u + I_Q) \oplus (u^{'} + I_Q)] \odot (t + I_Q) \\ &= [(u + u^{'} + I_Q)] \odot (t + I_Q) \\ &= (u + u^{'})t + I_Q = t(u + u^{'}) + I_Q \\ &= (t + I_Q) \odot [(u + I_Q) \oplus (u^{'} + I_Q)] \\ &= (t + I_Q) \odot [(u + I_Q) \oplus (u + I_Q)^{'}]. \end{aligned}$$

Thus, $(S/I_Q, \oplus, \odot)$ is an MA-semiring. □

The following theorem is an extension of Lemma 1.2 of [19].

Theorem 2.2. *Let I_Q be a prime Q-ideal of an MA-semiring S. If*

(1) $[t, u] \in I_Q$ *or*

(2) $t \circ u \in I_Q$,

for all $t, u \in S$, then S/I_Q is a commutative MA-semiring.

Proof. (1). Based on the hypothesis, we have $[t, u] \in I_Q$, for all $t, u \in S$. As $0 \in I_Q \subseteq S$, therefore $[t, u] + I_Q = I_Q$ and therefore by the addition and multiplication in S/I_Q, we can write $I_Q = (t + I_Q)(u + I_Q) + (u + I_Q)(t^{'} + I_Q)$. Hence $(t + I_Q)(u + I_Q) = (u + I_Q)(t + I_Q)$. This shows that S/I_Q is commutative. (2). Using the same reasoning as above, we have

$$(t + I_Q)(u + I_Q) + (u + I_Q)(t + I_Q) = I_Q, \tag{2.1}$$

which further implies

$$(t + I_Q)(u + I_Q) = (u + I_Q)(t^{'} + I_Q). \tag{2.2}$$

In (2.1) substituting ur for u, we get $[(t+I_Q)(u+I_Q)](r+I_Q)+(u+I_Q)[(r+I_Q)(t+I_Q)] = I_Q$ and using (2.2), we obtain $[(t+I_Q)(u+I_Q)](r+I_Q)+[(u+I_Q)(t^{'}+I_Q)](r+I_Q) = I_Q$. Therefore $[(t+I_Q), (u+I_Q)](r+I_Q) = I_Q$. As I_Q is prime and $I_Q \neq S$, therefore $[t, u] + I_Q = [(t + I_Q), (u + I_Q)] = I_Q$. By the first part, we conclude that S/I_Q is a commutative MA-semiring. □

Corollary 2.1. *Let S be a prime MA-semiring S. If*

(1) $[t, u] = 0$ *or*

(2) $t \circ u = 0$,

for all $t, u \in S$, then S is a commutative MA-semiring.

Remark 2.1. *If S/I_Q is commutative, then it has no nonzero zero divisors. Indeed, suppose that $I_Q = (t + I_Q)(u + I_Q)$. Then $I_Q = tu + I_Q$ which implies that $tu \in I_Q$. As I_Q is prime, either $t \in I_Q$ or $u \in I_Q$ and therefore either $t + I_Q = I_Q$ or $u + I_Q = I_Q$.*

Following result is an extension of Proposition 1.3 of [19].

Theorem 2.3. *Let I_Q be a prime Q-ideal of an MA-semiring S. If G_d is a generalized derivation satisfying*

$$[t, G_d(t)] \in I_Q \tag{2.3}$$

for all $t \in S$, then either S/I_Q is a commutative or $d(S) \subseteq I_Q$.

Proof. Linearizing (2.3), we get $[t, G_d(u)] + [t, G_d(t)] + [u, G_d(t)] + [u, G_d(u)] \in I_Q$. As I_Q is prime Q-ideal, again using (2.3) and the fact that I_Q is a k-ideal, we obtain

$$[t, G_d(u)] + [u, G_d(t)] \in I_Q \tag{2.4}$$

for all $t, u \in S$. Substituting ut for u in (2.4), we get $[t, G_d(ut)]+[ut, G_d(t)] \in I_Q$ and using MA-semiring identities, we can write

$$[t, G_d(u)]t + [t, u]d(t) + u[t, d(t)] + u[t, G_d(t)] + [u, G_d(t)]t \in I_Q,$$

which further implies

$$[t,u]d(t) + u[t,d(t)] \in I_Q. \tag{2.5}$$

Substituting vu for u in (2.5) and again using (2.5), we obtain $[t,v]ud(t) \in I_Q$ (i.e $[t,v]Sd(t) \subseteq I_Q$). As I_Q is prime, we have either $[t,v] \in I_Q$ for all $t,v \in S$ or $d(t) \in I_Q$ for all $t \in S$. By Theorem 2.2, we have either S/I_Q is commutative or $d(S) \subseteq I_Q$. □

Theorems 2.4–2.8 are the extended version of the corresponding results proved in [19].

Theorem 2.4. *Let I_Q be a prime Q-ideal of an MA-semiring S. If G_d is a generalized derivation of S such that*

$$G_d(tu) + G_d(t)G_d(u) \in I_Q, \tag{2.6}$$

for all $t,u \in S$. Then either S/I_Q is commutative or $d(S) \subseteq I_Q$.

Proof. In (2.6) substituting uw for u, we get $G_d(tuw)+G_d(t)G_d(uw) = G_d(tu)w+tud(w)+G_d(t)G_d(u)w+G_d(t)ud(w) \in I_Q$ and using (2.6) again and the fact that I_Q is a k-ideal, we have

$$tud(w) + G_d(t)ud(w) \in I_Q. \tag{2.7}$$

In (2.7) substituting ts for t, we obtain $tsud(w) + G_d(ts)ud(w) \in I_Q$ and therefore , which can be further written as

$$tsud(w) + G_d(t)sud(w) + td(s)ud(w) \in I_Q. \tag{2.8}$$

In (2.7) substituting su for u, we obtain $tsud(w) + G_d(t)sud(w) \in I_Q$ and therefore

$$tsud(w) + G_d(t)sud(w) \in I_Q. \tag{2.9}$$

As I_Q is Q-ideal, therefore using (2.8) and (2.9), we obtain $td(s)ud(w) \in I_Q$ and therefore $[t,r]d(s)Sd(w) \in I_Q$. As I_Q is prime ideal, therefore either $[t,r]d(s) \in I_Q$ or $d(w) \in I_Q$. If $d(w) \in I_Q$, then $d(S) \subseteq I_Q$. On the other hand, if $[t,r]d(s) \in I_Q$, then using commutator identities, we find $[t,r]Sd(s) \in I_Q$. Using the same arguments as before, we find either $d(S) \subseteq I_Q$ or $[t,r] \in I_Q$ and employing Theorem 2.2, we obtain S/I_Q is commutative. □

We can establish Theorem 2.5 by similar reasoning used in the proof of Theorem 2.4.

Theorem 2.5. *Let I_Q be a prime Q-ideal of an MA-semiring S. If G_d is a generalized derivation of S such that*

$$G_d(tu) + G_d(t)(G_d(u))' \in I_Q,$$

for all $t,u \in S$, then either S/I_Q is commutative or $d(S) \subseteq I_Q$.

Theorem 2.6. *Let I_Q be a prime Q-ideal of an MA-semiring S. If G_d is a generalized derivation of S such that*

$$G_d(tw) + G_d(w)(G_d(t))' \in I_Q, \tag{2.10}$$

for all $t,w \in S$, then either S/I_Q is commutative or $d(S) \subseteq I_Q$.

Proof. In (2.10) writing tw in place of t, we obtain $G_d(tw)w+twd(w)+G_d(w)G_d(t)w^{'}+G_d(w)t^{'}d(w) \in I_Q$. As I_Q is Q-ideal, therefore

$$twd(w) + G_d(w)t^{'}d(w) \in I_Q. \tag{2.11}$$

In (2.11) substituting st for t, we obtain $stwd(w) + G_d(w)st^{'}d(w) \in I_Q$ and therefore $stwd(w) + I_Q + G_d(w)st^{'}d(w) + I_Q = I_Q$, which further implies

$$stwd(w) + I_Q = G_d(w)std(w) + I_Q. \tag{2.12}$$

Left multiplying (2.11) by s, we obtain

$$stwd(w) + I_Q + sG_d(w)t^{'}d(w) + I_Q = I_Q. \tag{2.13}$$

Using (2.12) in (2.13), we get $[G_d(w), s]td(w) + I_Q = I_Q$, therefore $[G_d(w), s]S\,d(w) \subseteq I_Q$. As I_Q is prime, we get either $[G_d(w), s] \in I_Q$ or $d(S) \subseteq I_Q$. As a result of Theorem 2.3 S/I_Q is commutative or $d(S) \subseteq I_Q$. □

Using the proof of Theorem 2.6, we can establish the following result.

Theorem 2.7. *Let I_Q be a prime Q-ideal of an MA-semiring S. If G_d is a generalized derivation of S such that*

$$G_d(tw) + G_d(w)G_d(t) \in I_Q,$$

for all $t, w \in S$, then either S/I_Q is commutative or $d(S) \subseteq I_Q$.

Following result provide a generalized form of Theorem 1.5 of [19].

Theorem 2.8. *Let I_Q be a prime Q-ideal of an MA-semiring S. If G_d is a generalized derivation meeting one of the conditions given below:*

(1) $G_d(tw) + tw \in I_Q$,

(2) $G_d(tw) + t^{'}w \in I_Q$,

(3) $G_d(tw) + wt \in I_Q$,

(4) $G_d(tw) + w^{'}t \in I_Q$,

(5) $G_d(t)G_d(w) + tw \in I_Q$,

(6) $G_d(t)G_d(w) + t^{'}w \in I_Q$,

(7) $G_d(t)G_d(w) + wt \in I_Q$,

(8) $G_d(t)G_d(w) + w^{'}t \in I_Q$,

for all $t, w \in S$, then either S/I_Q is commutative or $d(S) \subseteq I_Q$.

Proof. (1). Based on the hypothesis, we have

$$G_d(tw) + tw \in I_Q, \text{ for all } t, w \in S. \tag{2.14}$$

If $G_d = 0$, then from (2.14), we have $tw \in I_Q$ and interchanging t and w, we get $wt \in I_Q$. From the last two expressions, we have $t \circ w \in I_Q$. Employing Theorem 2.2, we conclude that S/I_Q is commutative. Next, we consider the case when $G_d \neq 0$. In (2.14), writing ws in place of w, we get $(G_d(tw) + tw)s + twd(s) \in I_Q$. As I_Q is Q-ideal, therefore using (2.14), we obtain $twd(s) \in I_Q$ and therefore substituting wr for w, we obtain

$$twrd(s) \in I_Q. \tag{2.15}$$

In (2.15) interchanging t and w, we get

$$wtrd(s) \in I_Q. \tag{2.16}$$

From (2.15) and (2.16), we can write $(t \circ w)S\,d(s) \subseteq I_Q$ and by the primeness of I_Q, we obtain either $t \circ w \in I_Q$ or $d(S) \subseteq I_Q$. Again by the Theorem 2.2, we conclude either S/I_Q is commutative or $d(S) \subseteq I_Q$.

By the similar arguments, we can prove (2).

(3). An appeal to the hypothesis, we have

$$G_d(tw) + wt \in I_Q \text{ for all } t, w \in S \tag{2.17}$$

For $G_d = 0$, we have $wt \in I_Q$. We can prove that S/I_Q is commutative using the same reasoning as in the proof of (1). Next we consider the case when $G_d \neq 0$. In (2.17) substituting wt for t, we obtain $(G_d(tw) + wt)t + twd(t) \in I_Q$ and using (2.17) again, we obtain $twd(t) \in I_Q$. Remaining part follows through similar arguments of the proof of (1).

Part (4) can be followed similarly as part (3).

(5). Based on the hypothesis

$$G_d(t)G_d(w) + tw \in I_Q, \tag{2.18}$$

for all $t, w \in S$. The case when $G_d = 0$ is straightforward. For the second case when $G_d \neq 0$, substituting ws for w in (2.18), we get $(G_d(t)G_d(w) + tw)s + G_d(t)wd(s) \in I_Q$. As I_Q is Q-ideal, using (2.18) again, $G_d(t)wd(s) \in I_Q$. Since I_Q is prime, therefore either $G_d(t) \in I_Q$ or $d(S) \subseteq I_Q$. If

$$G_d(t) \in I_Q. \tag{2.19}$$

In (2.19) substituting tw for t, we get $G_d(t)w + td(w) \in I_Q$. As I_Q is Q-ideal, using (2.19) we obtain $td(w) \in I_Q$ and writing $[r, s]t$ in place of t, we further get $[r, s]S\,d(w) \subseteq I_Q$. As I_Q is prime, we obtain $[r, s] \in I_Q$ or $d(w) \in I_Q$. In view of Theorem 2.2, we conclude that S/I_Q is commutative or $d(S) \subseteq I_Q$.

Proof of (6) is not quite different from the proof of (5).

(7). Based on the hypothesis, we have

$$G_d(t)G_d(w) + wt \in I_Q \text{ for all } t, w \in S. \tag{2.20}$$

The case when $G_d = 0$ is straightforward. We consider the case when $G_d \neq 0$. In (2.20) replacing w by wt, we obtain $(G_d(t)G_d(w) + wt)t + G_d(t)wd(t) \in I_Q$. As I_Q is Q-ideal, using (2.20) again, we obtain $G_d(t)wd(t) \in I_Q$. Remaining part can be followed by the similar arguments of the proof of part (5).

On the similar lines of (7), we can establish (8). □

The following is an extension of Theorem 1.6 of [19].

Theorem 2.9. *Let I_Q be a prime Q-ideal of an MA-semiring S. If G_d is a generalized derivation meeting one of the conditions below:*

(1) $G_d(tw) + [t, w]' \in I_Q$,

(2) $G_d(tw) + [t, w] \in I_Q$,

(3) $G_d(tw) + t' \circ w \in I_Q$,

(4) $G_d(tw) + t \circ w \in I_Q$,

(5) $G_d(t)G_d(w) + [t, w]' \in I_Q$,

(6) $G_d(t)G_d(w) + [t, w] \in I_Q$,

(7) $G_d(t)G_d(w) + t' \circ w \in I_Q$,

(8) $G_d(t)G_d(w) + t \circ w \in I_Q$,

for all $t, w \in S$, then S/I_Q is a commutative.

Proof. (1). Based on the hypothesis, we have

$$G_d(tw) + [t, w]' \in I_Q, \text{ for all } t, w \in S. \tag{2.21}$$

If $G_d = 0$, then by Theorem 2.2, we obtain the required result. Suppose that $G_d \neq 0$. In (2.21) substituting ws for w and using MA-semiring identities, we obtain $(G_d(tw)+[t, w]')s+twd(s)+w[t, s]' \in I_Q$. As I_Q is Q-ideal, therefore using (2.21) again, we get

$$twd(s) + w[t, s]' \in I_Q. \tag{2.22}$$

In (2.22) writing rw in place of w, we obtain

$$trwd(s) + rw[t, s]' \in I_Q. \tag{2.23}$$

From (2.22) we can also write $trwd(s) + I_Q + rw[t, s]' + I_Q = I_Q$, which further implies

$$trwd(s) + I_Q = rw[t, s] + I_Q. \tag{2.24}$$

Multiplying (2.22) by r from the left, we obtain $rtwd(s) + rw[t, s]' \in I_Q$ which can further written as

$$rtwd(s) + I_Q + rw[t, s]' + I_Q = I_Q. \tag{2.25}$$

Using (2.24) into (2.25), we obtain $[r, t]wd(s) + I_Q = I_Q$ and therefore $[r, t]S d(s) \subseteq I_Q$. As I_Q is prime, therefore either $[r, t] \in I_Q$ or $d(S) \subseteq I_Q$. For the first possibility, employing Theorem 2.2, we conclude that S/I_Q is commutative. On the other hand if $d(S) \subseteq I_Q$, then from (2.23), we have $rw[t, s] \in I_Q$ and hence $[t, s]S[t, s] \subseteq I_Q$. In view of the primeness of I_Q, using Theorem 2.2, we get the required result.

Part (2) can be established on the similar lines of Part (1).

(3). Based on the hypothesis, we have

$$G_d(tw) + t' \circ w \in I_Q \text{ for all } t, w \in S. \tag{2.26}$$

For $G_d = 0$, from Theorem 2.2, we obtain the required result. Assume that $G_d \neq 0$. In (2.26) substituting wr for w, we obtain $G_d(tw)r + twd(r) + t^{'}wr + wrt^{'} \in I_Q$ and using the definition of MA-semiring, we can write

$$\begin{aligned} G_d(tw)r + twd(r) + t^{'}wr + wrt^{'} + wr(t + t^{'}) &= G_d(tw)r + twd(r) + t^{'}wr + wrt^{'} + w(t + t^{'})r \\ &= G_d(tw)r + twd(r) + t^{'}wr + wt^{'}r + wtr + wrt^{'} \\ &= G_d(tw)r + twd(r) + (t^{'} \circ w)r + w[t, r]. \end{aligned}$$

Therefore $G_d(tw)r + (t^{'} \circ w)r + twd(r) + w[t, r] \in I_Q$. As I_Q is Q-ideal, using (2.26), we obtain

$$twd(r) + w[t, r] \in I_Q. \tag{2.27}$$

In (2.27) substituting sw for w and using MA-semiring identities, we obtain $tswd(r) + sw[t, r] \in I_Q$, which further gives $tswd(r) + I_Q + sw[t, r] + I_Q = I_Q$ and therefore

$$t^{'}swd(r) + I_Q = sw[t, r] + I_Q. \tag{2.28}$$

Multiplying (2.27) by s from the left, we obtain $stwd(r) + sw[t, r] \in I_Q$ and therefore

$$stwd(r) + I_Q + sw[t, r] + I_Q = I_Q. \tag{2.29}$$

Using (2.28) into (2.29), we get $stwd(r) + I_Q + t^{'}swd(r) + I_Q = I_Q$ and therefore

$$[s, t]wd(r) \in I_Q. \tag{2.30}$$

As I_Q is prime, therefore either $[s, t] \in I_Q$ or $d(S) \subseteq I_Q$. Repeating the same arguments as above, we obtain the required result.

Part (4) can be proved on the similar lines of Part (3).

(5). Based on the hypothesis, we have

$$G_d(t)G_d(w) + [t, w]^{'} \in I_Q \text{ for all } t, w \in S. \tag{2.31}$$

In view of Theorem 2.2, the case when $G_d = 0$, is straightforward. Assume that $G_d \neq 0$. In (2.31) substituting wv for w and using MA-semiring commutator identities, we get $G_d(t)G_d(w)v + G_d(t)wd(v) + w[t, v]^{'} + [t, w]v^{'} \in I_Q$. Using (2.31) again, we obtain

$$G_d(t)wd(v) + w[t, v]^{'} \in I_Q. \tag{2.32}$$

In (2.32), substituting sw for w, we obtain $G_d(t)swd(v) + sw[t, v]^{'} \in I_Q$, which further implies $G_d(t)swd(v) + I_Q + sw[t, v]^{'} + I_Q = I_Q$ and therefore

$$G_d(t)swd(v) + I_Q = sw[t, v] + I_Q. \tag{2.33}$$

Multiplying (2.32) by s from the left, we obtain $sG_d(t)wd(v) + sw[t, v]^{'} \in I_Q$ which further gives

$$sG_d(t)wd(v) + I_Q + sw[t, v]^{'} + I_Q = I_Q. \tag{2.34}$$

Using (2.33) in (2.34), we get $[s, G_d(t)]wd(v) + I_Q = I_Q$, therefore $[s, G_d(t)]S\,d(v) \subseteq I_Q$. By the primeness of I_Q, we have either $[s, G_d(t)] \in I_Q$ or $d(S) \subseteq I_Q$. If $d(S) \subseteq I_Q$, then from (2.32), we have

$$w[t, v] \in I_Q. \tag{2.35}$$

In (2.35) substituting $[t, v]w$ for w, we obtain $[t, v]S[t, v] \subseteq I_Q$. As I_Q is prime, $[t, v] \in I_Q$ and hence by Theorem 2.2, S/I_Q is commutative. Second, suppose $[s, G_d(t)] \in I_Q$. By Theorem 2.3, we have either S/I_Q is commutative or $d(S) \subseteq I_Q$. If $d(S) \subseteq I_Q$, then by the same arguments, we again find S/I_Q is commutative.

Part (6) can be followed on the similar lines of Part (5).

(7). Based on the hypothesis, we have

$$G_d(t)G_d(w) + t^{'} \circ w \in I_Q \text{ for all } t, w \in S. \tag{2.36}$$

If $G_d = 0$, then $t \circ w \in I_Q$, for all $t, w \in S$ and hence by Theorem 2.2 S/I_Q is commutative. Next, we suppose that $G_d \neq 0$. In (2.36) writing ws in place of w, we obtain

$$G_d(t)G_d(w)s + G_d(t)wd(s) + t^{'}ws + wst^{'} \in I_Q. \tag{2.37}$$

By the definition of MA-semiring, we have

$$\begin{aligned} G_d(t)G_d(w)s + G_d(t)wd(s) + t^{'}ws + wst^{'} &= G_d(t)G_d(w)s + G_d(t)wd(s) + t^{'}ws + ws(t^{'} + t + t^{'}) \\ &= G_d(t)G_d(w)s + G_d(t)wd(s) + t^{'}ws + w(t^{'} + t)s + wst^{'} \\ &= G_d(t)G_d(w)s + G_d(t)wd(s) + (t^{'} \circ w)s + w[t, s] \\ &= (G_d(t)G_d(w) + (t^{'} \circ w))s + G_d(t)wd(s) + w[t, s]. \end{aligned}$$

Therefore (2.37) becomes

$$(G_d(t)G_d(w) + (t^{'} \circ w))s + G_d(t)wd(s) + w[t, s] \in I_Q.$$

As I_Q is a Q-ideal, therefore using (2.36) again, we

$$G_d(t)wd(s) + w[t, s] \in I_Q. \tag{2.38}$$

Remaining part can be followed through the similar arguments of Part (5).

Part (8) can be followed on the same lines of the proof of Part (7). □

3. Conclusions

We discussed some notions about generalized derivations and prime Q-ideals of MA-semirings. Some interesting results about the commutativity of quotient MA-semirings under certain differential identities are proved. Taking $I_Q = \{0\}$, one can obtain important consequences of the results for prime MA-semirings.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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