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Research article

Schur complement-based infinity norm bounds for the inverse of S-Sparse Ostrowski Brauer matrices

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Abstract: In this paper, we study the Schur complement problem of S-SOB matrices, and prove that the Schur complement of S-Sparse Ostrowski-Brauer (S-SOB) matrices is still in the same class under certain conditions. Based on the Schur complement of S-SOB matrices, some upper bound for the infinite norm of S-SOB matrices is obtained. Numerical examples are given to certify the validity of the obtained results. By using the infinity norm bound, an error bound is given for the linear complementarity problems of S-SOB matrices.

Keywords: S-SOB matrix; Schur complement; infinite norm upper bound
Mathematics Subject Classification: 15A48, 65G50, 90C31, 90C33

1. Introduction

Schur complement of a matrix is widely used and has attracted the attention of many scholars. In 1979, the Schur complement question of a strictly diagonally dominant (SDD) matrix was studied by Carlson and Markham [1]. They certified the Schur complement of SDD matrix is also an SDD matrix. Before long, some renowned matrices such as doubly diagonally dominant matrices and Dashnic-Zusmanovich (DZ) matrices were researched, and the results were analogous [2–5]. In 2020, Li et al. proved that the Schur complements and the diagonal-Schur complements of Dashnic-Zusmanovich type (DZ-type) matrices are DZ-type matrices under certain conditions in [6]. In 2023, Song and Gao [7] proved that the Schur complements and the diagonal-Schur complements of CKV-type matrices are CKV-B-type matrices under certain conditions. Furthermore, there are many conclusions on Schur complements and diagonal-Schur complements for other classes of matrices, see [8–15].

The upper bound of the inverse infinite norm of the non-singular matrix is widely used in mathematics, such as the convergence analysis of matrix splitting and matrix multiple splitting iterative method for solving linear equations. A traditional way to find the upper bound of an infinite norm for the inverse of a nonsingular matrix is to use the definition and properties of a given matrix class, see [16–19] for details. The first work was by Varah [19], who in 1975 gave the upper bound of the infinite norm of the inverse of the SDD matrix. However, in some cases, the bounds of Varah may yield larger values. In 2020, Li [20] obtained two upper bounds of the infinite norm of the inverse of the SDD matrix based on Schur complement, and in 2021, Sang [21] obtained two upper bounds for the infinity norm of DSDD matrices. In 2022, based on the Schur complement, Li and Wang obtained some upper bounds for the infinity norm of the inverse of GDSDD matrices [22].

In this paper, n is a positive integer and $N = \{1, 2, ..., n\}$. Let S be any nonempty subset of N, $S \subset N$, $\overline{S} := N \setminus S$ for the complement of S. $C^{n\times n}$ denotes the set of complex matrices of all $n\times n$. $R^{n\times n}$ denotes the set of all $n \times n$ real matrices. $I \in R^{n\times n}$ is an identity matrix, $A = [a_{ij}] \in C^{n\times n}$, $|A| = [|a_{ij}|] \in R^{n\times n}$ and

$$r_i(A) = \sum_{k\neq i, k\in N} |a_{ik}|,\ r_i^S(A) = \sum_{k\neq i, k\in S} |a_{ik}|,\ i \in N.$$

The matrix A is known as the strictly diagonal dominance SDD matrix, abbreviated as $A \in$ SDD, if

$$|a_{ii}| > r_i(A), i \in N.$$

Definition 1. *[23] Let S be an arbitrary nonempty proper subset of the index set. $A = [a_{ij}] \in C^{n\times n}, n \geq 2$, is called an S-SOB (S-Sparse Ostrowski-Brauer) matrix if*

(i) $|a_{ii}| > r_i^S(A)$ for all $i \in S$;
(ii) $|a_{jj}| > r_j^{\overline{S}}(A)$ for all $j \in S$;
(iii) For all $i \in S$ and all $j \in \bar{S}$ such that $a_{ij} \neq 0$,

$$[|a_{ii}| - r_i^S(A)]|a_{jj}| > r_i^{\overline{S}}(A) r_j(A); \tag{1.1}$$

(iv) For all $i \in S$ and all $j \in \bar{S}$ such that $a_{ji} \neq 0$,

$$[|a_{jj}| - r_j^{\overline{S}}(A)]|a_{ii}| > r_j^S(A) r_i(A). \tag{1.2}$$

Definition 2. *[24] A matrix A is called GDSDD matrix if $J \neq \emptyset$ and there exists proper subsets N_1, N_2 of N such that $N_1 \cap N_2 = \emptyset, N_1 \cup N_2 = N$ and for any $i \in N_1$ and $j \in N_2$,*

$$[|a_{ii}| - r_i^{N_1}(A)][|a_{jj}| - r_j^{N_2}(A)] > r_i^{N_2}(A) r_j^{N_1}(A),$$

where $J := \{i \in N : |a_{ii}| > r_i(A)\}$.

Definition 3. *[25] A matrix A is called an H-matrix, if its comparison matrix $\mu(A) = [\mu_{ij}]$ defined by*

$$\mu_{ii} = |a_{ii}|,\ \mu_{ij} = -|a_{ij}|,\ i, j \in N, i \neq j$$

is an M-matrix, i.e., $[\mu(A)]^{-1} \geq 0$.

It is shown in [1] that if A is an H-matrix, then,

$$[\mu(A)]^{-1} \geq |A^{-1}|. \tag{1.3}$$

Let A be an M-matrix, then $det(A) > 0$.

In addition, it was shown that S-SOB, SDD and GDSDD matrices are nonsingular H-matrix in [23, 26]. Varah [19] gave the following upper bound for the infinity norm of the inverse of SDD matrices:

Theorem 1. [19] *Let $A = [a_{ij}]$ be an SDD matrix. Then,*

$$\left\|A^{-1}\right\|_\infty \le \max_{i\in N} \frac{1}{|a_{ii}| - r_i(A)}. \tag{1.4}$$

Theorem 2. [27] *Let $A = [a_{ij}] \in C^{n\times n}, n \ge 2$, be an S-SOB matrix, where $S \subset N, 1 \le |S| \le n-1$. Then,*

$$\|A^{-1}\|_\infty \le \{\max_{\substack{i\in S:\\ r_i^{\bar{S}}(A)=0}} \frac{1}{|a_{ii}| - r_i^S(A)}, \quad \max_{\substack{j\in\bar{S}:\\ r_j^S(A)=0}} \frac{1}{|a_{jj}|-r_j^{\bar{S}}(A)}, \max_{\substack{i\in S, j\in\bar{S}:\\ a_{ij}\neq 0}} f_{ij}(A,S), \quad \max_{\substack{i\in S, j\in\bar{S}:\\ a_{ji}\neq 0}} f_{ji}(A,\bar{S})\}, \tag{1.5}$$

where

$$f_{ij}(A,S) = \frac{|a_{jj}| + r_i^{\bar{S}}(A)}{[|a_{ii}| - r_i^S(A)]|a_{jj}| - r_i^{\bar{S}}(A) r_j(A)}, \; i \in S, j \in \bar{S}.$$

Theorem 3. [28] *Let $A = [a_{ij}] \in C^{n\times n}$, $n \ge 2$, be an GDSDD matrix, where $S \subset N$, $1 \le |S| \le n-1$. Then,*

$$\|A^{-1}\|_\infty \le \max\{\max_{\substack{i\in N_1,\\ j\in N_2}} \frac{|a_{jj}| - r_j^{N_2}(A) + r_i^{N_2}(A)}{[|a_{ii}| - r_i^{N_1}(A)][|a_{jj}| - r_j^{N_2}(A)] - r_i^{N_2}(A) r_j^{N_1}(A)}, \max_{\substack{i\in N_1,\\ j\in N_2}} \frac{|a_{ii}| - r_i^{N_1}(A) + r_j^{N_1}(A)}{[|a_{ii}| - r_i^{N_1}(A)][|a_{jj}| - r_j^{N_2}(A)] > r_i^{N_2}(A) r_j^{N_1}(A)}\}. \tag{1.6}$$

In this paper, based on the Schur complement, we present some upper bounds for the infinity norm of the inverse of S-SOB matrices, and numerical examples are given to show the effectiveness of the obtained results. In addition, applying these new bounds, a lower bound for the smallest singular value of S-SOB matrices is obtained.

2. The Schur complement of S-SOB matrices

Given a matrix $A = \left(a_{ij}\right) \in C^{n\times n}$ that is nonsingular, $\alpha = \{i_1, i_2, ..., i_k\}$ is any nonempty proper subset of N, $|\alpha|$ is the cardinality of α (the number of elements in α, i.e., $|\alpha| = k$), $\bar{\alpha} = N - \alpha = \{j_1, \cdots, j_l\}$ is the complement of α with respect to N, $A(\alpha, \bar{\alpha})$ is the submatrix of A lying in the rows indexed by α and the columns indexed by $\bar{\alpha}$, $A(\alpha)$ is the leading submatrix of A whose row and column are both indexed by α, and the elements of α and of $\bar{\alpha}$ are both conventionally arranged in increasing order. If $A(\alpha)$ is not singular, the matrix A/α is called the Schur complement of A with respect to $A(\alpha)$. At this point

$$A/\alpha = A(\bar{\alpha}) - A(\bar{\alpha}, \alpha)[A(\alpha)]^{-1} A(\alpha, \bar{\alpha}).$$

Lemma 1. *(Quotient formula [28,29]) Let A be a square matrix. Let B is a nonsingular principal submatrix of A and C is a nonsingular principal submatrix of B. Then, B/C is a nonsingular principal submatrix of A/C and $A/B = (A/C)/(B/C)$, where B/C is the Schur complement of C in matrix B.*

Lemma 2. *Let $A = (a_{ij}) \in C^{n\times n}$ be an S-SOB matrix, $n \geq 2$ and where $\alpha \subseteq S$ or $\alpha \subseteq \bar{S}$. Then, $A(\alpha)$ is an SDD matrix.*

Proof. When $\alpha \subseteq S$, since A is an S-SOB matrix and $|a_{ii}| > r_i^S(A) \geq r_i^\alpha(A) = \sum\limits_{k\neq i, k\in\alpha} |a_{ik}|$ for all $i \in \alpha$, we have $r_i[A(\alpha)] = \sum\limits_{k\neq i, k\in\alpha} |a_{ik}| = r_i^\alpha(A)$ and $|a_{ii}| > r_i[A(\alpha)]$. It is easy to obtain that $A(\alpha)$ is an SDD matrix. Homoplastically, so is $\alpha \subseteq \bar{S}$. □

Lemma 3. *Let $A = (a_{ij}) \in C^{n\times n}$ be an S-SOB matrix, $n \geq 2$ and α be a subset of N. Then, $A(\alpha)$ is an S-SOB matrix.*

Proof. If $S \subset \alpha$, since A is an S-SOB matrix, then,
(i) For all $i \in S$, $|a_{ii}| > r_i^S(A) = r_i^S(A(\alpha))$,
(ii) For all $j \in \bar{S} \cap \alpha$, $|a_{jj}| > r_j^{\bar{S}}(A) > r_j^{\bar{S}\cap\alpha}(A) = r_j^{\bar{S}\cap\alpha}(A(\alpha))$,
(iii) For all $i \in S,\ j \in \bar{S} \cap \alpha$ such that $a_{ij} \neq 0$,

$$\begin{aligned}[|a_{ii}| - r_i^S(A(\alpha))]|a_{jj}| &= [|a_{ii}| - r_i^S(A)]|a_{jj}| > r_i^{\bar{S}}(A) r_j(A) > r_i^{\bar{S}}(A) r_j^{S\cup(\bar{S}\cap\alpha)}(A) \\ &= r_i^{\bar{S}}(A(\alpha)) r_j^{S\cup(\bar{S}\cap\alpha)}(A(\alpha)),\end{aligned}$$

(iv) For all $i \in S,\ j \in \bar{S} \cap \alpha$ such that $a_{ji} \neq 0$,

$$\begin{aligned}[|a_{jj}| - r_j^{\bar{S}\cap\alpha}(A(\alpha))]|a_{ii}| &= [|a_{jj}| - r_j^{\bar{S}\cap\alpha}(A)]|a_{ii}| > r_j^S(A) r_i(A) > r_j^S(A) r_i^{S\cup(\bar{S}\cap\alpha)}(A) \\ &= r_j^S(A(\alpha)) r_i^{S\cup(\bar{S}\cap\alpha)}(A(\alpha)).\end{aligned}$$

Thus, $A(\alpha)$ is an S-SOB matrix and $A(\alpha) \in\{S\text{-SOB}\}$.

In a similar way, if $\bar{S} \subset \alpha$, $A(\alpha)$ is an $\bar{S}$-SOB matrix. Meanwhile, when α is contained neither in S nor in $\bar{S}$, $A(\alpha)$ is an $(S \cap \alpha)$-SOB matrix. Finally, $A(\alpha) \in\{S\text{-SOB}\}$. □

Lemma 4. *Let $A = (a_{ij}) \in C^{n\times n}$ be an S-SOB matrix, $n \geq 2$ and let A be a matrix satisfying $a_{ij} = 0, a_{ii} > r_i(A)$ and $a_{ji} = 0, a_{jj} > r_j(A)$ for $i \in S, j \in \bar{S}$. If $\alpha = \{i_1\} \subset S$, denote*

$$B = (b_{ij}) = \begin{pmatrix} |a_{i_1i_1}| & -r_{i_1}^{S\setminus\alpha}(A) & -r_{i_1}^{\bar{S}}(A) \\ -|a_{j_ti_1}| & |a_{j_tj_t}| - r_{j_t}^{S\setminus\alpha}(A) & -r_{j_t}^{\bar{S}}(A) \\ -|a_{j_si_1}| & -r_{j_s}^{S\setminus\alpha}(A) & |a_{j_sj_s}| - r_{j_s}^{\bar{S}}(A) \end{pmatrix},$$

where $j_t \in (S \setminus \alpha)$, $j_s \in \bar{S}$, then $B \in \{\text{SGDD}_3\}$.

Proof. Since A is an S-SOB matrix, if $S_B = \{1, 2\}$, for all $i \in S_B$, then,

$$\begin{aligned}[|b_{11}| - r_1^{S_B}(B)][|b_{33}| - r_3^{\bar{S}_B}(B)] &= [|a_{i_1i_1}| - r_{i_1}^{S\setminus\alpha}(A)][|a_{j_sj_s}| - r_{j_s}^{\bar{S}}(A)] \\ &= [|a_{i_1i_1}| - r_{i_1}^S(A)][|a_{j_sj_s}| - r_{j_s}^{\bar{S}}(A)].\end{aligned}$$

$$\begin{aligned}[|b_{22}| - r_2^{S_B}(B)][|b_{33}| - r_3^{\bar{S}_B}(B)] &= [|a_{j_tj_t}| - r_{j_t}^{S\setminus\alpha}(A) - ||a_{j_ti_1}|][|a_{j_sj_s}| - r_{j_s}^{\bar{S}}(A)] \\ &= [|a_{j_tj_t}| - r_{j_t}^S(A)][|a_{j_sj_s}| - r_{j_s}^{\bar{S}}(A)].\end{aligned}$$

There exist four different cases.

Case 1. When $|a_{j_s i_1}| \neq 0$, $|a_{i_1 j_s}| \neq 0$.
(i) If $|a_{j_s j_s}| < r_{j_s}(A)$, from Definition 1, we have $|a_{i_1 i_1}| \geq r_{i_1}(A)$,

$$\begin{aligned}[][|b_{11}| - r_1^{S_B}(B)][|b_{33}| - r_3^{\bar{S}_B}(B)] &= [|a_{j_s j_s}| - r_{j_s}^{\bar{S}}(A)]|a_{i_1 i_1}| - [|a_{j_s j_s}| - r_{j_s}^{\bar{S}}(A)]r_{i_1}^{S}(A) \\ &> r_{j_s}^{S}(A) r_{i_1}(A) - r_{j_s}^{S}(A) r_{i_1}^{S}(A) \\ &= r_{j_s}^{S}(A) r_{i_1}^{\bar{S}}(A) > r_{j_s}^{S\setminus\alpha}(A) r_{i_1}^{\bar{S}}(A) = r_1^{\bar{S}_B}(B) r_3^{S_B}(B).\end{aligned}$$

(ii) If $|a_{j_s j_s}| > r_{j_s}(A)$, $|a_{i_1 i_1}| \geq r_{i_1}(A)$, we get

$$\begin{aligned}[][|b_{11}| - r_1^{S_B}(B)][|b_{33}| - r_3^{\bar{S}_B}(B)] &= [|a_{i_1 i_1}| - r_{i_1}^{S}(A)][|a_{j_s j_s}| - r_{j_s}^{\bar{S}}(A)] > r_{j_s}^{S}(A) r_{i_1}^{\bar{S}}(A) \\ &> r_{j_s}^{S\setminus\alpha}(A) r_{i_1}^{\bar{S}}(A) = r_1^{\bar{S}_B}(B) r_3^{S_B}(B).\end{aligned}$$

(iii) If $|a_{j_s j_s}| > r_{j_s}(A)$, $|a_{i_1 i_1}| \leq r_{i_1}(A)$, we obtain

$$\begin{aligned}[][|b_{11}| - r_1^{S_B}(B)][|b_{33}| - r_3^{\bar{S}_B}(B)] &= [|a_{i_1 i_1}| - r_{i_1}^{S}(A)]|a_{j_s j_s}| - [|a_{i_1 i_1}| - r_{i_1}^{S}(A)]r_{j_s}^{\bar{S}}(A) \\ &> r_{i_1}^{\bar{S}}(A) r_{j_s}(A) - r_{i_1}^{\bar{S}}(A) r_{j_s}^{\bar{S}}(A) \\ &= r_{i_1}^{\bar{S}}(A) r_{j_s}^{S}(A) > r_{j_s}^{S\setminus\alpha}(A) r_{i_1}^{\bar{S}}(A) = r_1^{\bar{S}_B}(B) r_3^{S_B}(B).\end{aligned}$$

Case 2. When $|a_{j_s i_1}| \neq 0$, $|a_{i_1 j_s}| = 0$, $|a_{i_1 i_1}| \geq r_{i_1}(A)$ the proof is analogous to (i) and (ii) in Case 1. We obtain

$$[|b_{11}| - r_1^{S_B}(B)][|b_{33}| - r_3^{\bar{S}_B}(B)] > r_1^{\bar{S}_B}(B) r_3^{S_B}(B).$$

Case 3. If $|a_{j_s i_1}| = 0$, $|a_{i_1 j_s}| \neq 0$, then, $|a_{j_s j_s}| > r_{j_s}(A)$. By the same proof method as (ii) and (iii) in Case 1, we have

$$[|b_{11}| - r_1^{S_B}(B)][|b_{33}| - r_3^{\bar{S}_B}(B)] > r_1^{\bar{S}_B}(B) r_3^{S_B}(B).$$

Case 4. If $|a_{j_s i_1}| = 0$, $|a_{i_1 j_s}| = 0$, then, $|a_{i_1 i_1}| > r_{i_1}(A)$, $|a_{j_s j_s}| > r_{j_s}(A)$, and

$$[|b_{11}| - r_1^{S_B}(B)][|b_{33}| - r_3^{\bar{S}_B}(B)] > r_1^{\bar{S}_B}(B) r_3^{S_B}(B).$$

To sum up, the inequality $[|b_{11}| - r_1^{S_B}(B)][|b_{33}| - r_3^{\bar{S}_B}(B)] > r_1^{\bar{S}_B}(B) r_3^{S_B}(B)$ is held. In the same way, the inequality $[|b_{22}| - r_2^{S_B}(B)][|b_{33}| - r_3^{\bar{S}_B}(B)] > r_2^{\bar{S}_B}(B) r_3^{S_B}(B)$ also holds. At last, we obtain $B \in \{\text{GDSDD}_3\}$ and $B = \mu(B)$ is an M-matrix. By Definition 3, we know that $\det B > 0$. The proof is completed. □

Theorem 4. *Let $A = (a_{ij}) \in C^{n\times n}$ be an S-SOB matrix, $n \geq 2$ and let A be a matrix satisfying $a_{ij} = 0$, $a_{ii} > r_i(A)$ and $a_{ji} = 0$, $a_{jj} > r_j(A)$ for $i \in S, j \in \bar{S}$. Denote $A/\alpha = (a'_{j_t j_s})$. If $\alpha \subset S$, then, $A/\alpha \in \{\, GDSDD_{n-k}^{(S\setminus\alpha),\bar{S}}\}$.*

Proof. Note that α contains only one element. If $\alpha = i_1 \subset S$, for all $j_t \in S \setminus \alpha$, $j_s \in \bar{S}$, then we have

$$\begin{aligned}&\left[|a'_{j_t j_t}| - r_{j_t}^{S\setminus\alpha}(A/\alpha)\right]\left[|a'_{j_s j_s}| - r_{j_s}^{\bar{S}}(A/\alpha)\right] - r_{j_t}^{\bar{S}}(A/\alpha) r_{j_s}^{S\setminus\alpha}(A/\alpha) \\ = &\left[|a'_{j_t j_t}| - \sum_{\substack{j_w \in S\setminus\alpha,\\ w\neq t}} |a'_{j_t j_w}|\right]\left[|a'_{j_s j_s}| - \sum_{\substack{j_w \in \bar{S},\\ w\neq s}} |a'_{j_s j_w}|\right] - \sum_{j_w \in \bar{S}} |a'_{j_t j_w}| \sum_{j_w \in S\setminus\alpha} |a'_{j_s j_w}|\end{aligned}$$

$$
\begin{aligned}
&= \left[\left|a_{j_tj_t} - \frac{a_{j_ti_1}a_{i_1j_t}}{a_{i_1i_1}}\right| - \sum_{\substack{j_w\in S\setminus\alpha,\\ w\neq t}}\left|a_{j_tj_w} - \frac{a_{j_ti_1}a_{i_1j_w}}{a_{i_1i_1}}\right|\right] \\
&\times \left[\left|a_{j_sj_s} - \frac{a_{j_si_1}a_{i_1j_s}}{a_{i_1i_1}}\right| - \sum_{\substack{j_w\in \bar{S},\\ w\neq t}}|a_{j_sj_w} - \frac{a_{j_si_1}a_{i_1j_w}}{a_{i_1i_1}}|\right] \\
&- \sum_{j_w\in\bar{S}}\left|a_{j_tj_w} - \frac{a_{j_ti_1}a_{i_1j_w}}{a_{i_1i_1}}\right|\sum_{j_w\in S\setminus\alpha}\left|a_{j_sj_w} - \frac{a_{j_si_1}a_{i_1j_w}}{a_{i_1i_1}}\right| \\
&\geq \left[|a_{j_tj_t}| - r_{j_t}^{S\setminus\alpha}(A) - \frac{|a_{j_ti_1}|r_{i_1}^{S\setminus\alpha}(A)}{|a_{i_1i_1}|}\right]\times\left[|a_{j_sj_s}| - r_{j_s}^{\bar{S}}(A) - \frac{|a_{j_si_1}|r_{i_1}^{\bar{S}}(A)}{|a_{i_1i_1}|}\right] \\
&- \left[r_{j_t}^{\bar{S}}(A) + \frac{|a_{j_ti_1}|r_{i_1}^{\bar{S}}(A)}{|a_{i_1i_1}|}\right]\times\left[r_{j_s}^{S\setminus\alpha}(A) + \frac{|a_{j_si_1}|r_{i_1}^{S\setminus\alpha}(A)}{|a_{i_1i_1}|}\right] \\
&= det[B/\{1\}] = \frac{1}{|a_{i_1i_1}|}detB > 0.
\end{aligned}
$$

We have $A/\{i_1\} \in \{\text{ GDSDD}_{n-1}^{(S\setminus\{i_1\}),\bar{S}}\}$ for any $i_1 \in S$. Consider that α contains more than one element. If $i_1 \in \alpha$, by the quotient formula (in [9] Theorem 2 (ii)), we have $A/\alpha = (A/\{i_1\})/((A(\alpha)/i_1) \in \{\text{GDSDD}_{n-k}^{(S\setminus\alpha),\bar{S}}\}$. The proof is completed. □

Corollary 1. *Let $A = (a_{ij}) \in C^{n\times n}$ be an S-SOB matrix, $n \geq 2$ and let A be a matrix satisfying $a_{ij} = 0,\ a_{ii} > r_i(A)$ and $a_{ji} = 0,\ a_{jj} > r_j(A)$ for $i \in S, j \in \bar{S}$. Denote $A/\alpha = (a'_{j_tj_s})$. If $\alpha \subset \bar{S}$, $j_t \in S,\ j_s \in \bar{S} \setminus \alpha$, then, $A/\alpha \in \{GDSDD_{n-k}^{S,(\bar{S}\setminus\alpha)}\}$.*

Proof. The conclusion can be drawn by using the same proof method as Theorem 4. □

Corollary 2. *Let $A = (a_{ij}) \in C^{n\times n}$ be an S-SOB matrix, $n \geq 2$ and let A be a matrix satisfying $a_{ij} = 0,\ a_{ii} > r_i(A)$ and $a_{ji} = 0,\ a_{jj} > r_j(A)$ for $i \in S, j \in \bar{S}$. Denote $A/\alpha = (a'_{j_tj_s})$. If α is contained neither in S nor in $\bar{S}$, $j_t \in S \setminus \alpha,\ j_s \in \bar{S} \setminus \alpha$, then $A/\alpha \in \{GDSDD_{n-k}^{(S\setminus\alpha),(\bar{S}\setminus\alpha)}\}$.*

Proof. The proof is similar to ([9], Theorem 2 (iii)), so we get $A/\alpha = (A/(S \cap \alpha))/((A(\alpha)/(S \cap \alpha)) \in \{\text{GDSDD}_{n-k}^{(S\setminus\alpha),(\bar{S}\setminus\alpha)}\}$. □

Theorem 5. *Let $A = (a_{ij}) \in C^{n\times n}$ be an S-SOB matrix, $n \geq 2$ and denote $A/\alpha = (a'_{j_tj_s})$. If $\alpha = S$ or $\alpha = \bar{S}$, then A/α is an SDD matrix.*

Proof. If $\{i_1\} = \alpha = S$, for all $j_t \in \bar{\alpha}$, then we have

$$
\begin{aligned}
&|a'_{j_tj_t}| - r_{j_t}(A/\alpha) = |a'_{j_tj_t}| - \sum_{\substack{j_w\in\bar{\alpha},\\ w\neq t}}|a'_{j_tj_w}| \\
&= \left|a_{j_tj_t} - \frac{a_{j_ti_1}a_{i_1j_t}}{a_{i_1i_1}}\right| - \sum_{\substack{j_w\in\bar{\alpha},\\ w\neq t}}\left|a_{j_tj_w} - \frac{a_{j_ti_1}a_{i_1j_w}}{a_{i_1i_1}}\right| \\
&\geq |a_{j_tj_t}| - r_{j_t}^{\bar{\alpha}}(A) - \sum_{j_w\in\bar{\alpha}}\frac{|a_{j_ti_1}a_{i_1j_w}|}{|a_{i_1i_1}|} = |a_{j_tj_t}| - r_{j_t}^{\bar{\alpha}}(A) - \frac{|a_{j_ti_1}|r_{i_1}^{\bar{\alpha}}(A)}{|a_{i_1i_1}|}
\end{aligned}
$$

$$= \quad |a_{j_t j_t}| - r_{j_t}^{\bar{S}}(A) - \frac{|a_{j_t i_1}| r_{i_1}^{\bar{S}}(A)}{|a_{i_1 i_1}|}.$$

If $a_{j_t i_1} = 0$, then we get

$$|a'_{j_t j_t}| - r_{j_t}(A/\alpha) \geq |a_{j_t j_t}| - r_{j_t}^{\bar{S}}(A) - 0 > 0.$$

If $a_{j_t i_1} \neq 0$, then we obtain

$$|a'_{j_t j_t}| - r_{j_t}(A/\alpha) \geq \frac{r_{j_t}^{S}(A) r_{i_1}(A)}{|a_{i_1 i_1}|} - \frac{|a_{j_t i_1}| r_{i_1}^{\bar{S}}(A)}{|a_{i_1 i_1}|} > 0.$$

Hence, for any $\{i_1\} = \alpha = S$, $A/\{i_1\}$ is an SDD matrix. Taking $i_1 \in \alpha = S$ and using the fact that A is SDD, we know its Schur complement is as well. At last, we have $A/\alpha = (A/\{i_1\})/(A(\alpha)/\{i_1\}) \in \{SDD\}$. By the same argument, so is $\alpha = \bar{S}$. □

Corollary 3. *Let $A = (a_{ij}) \in C^{n\times n}$ be an S-SOB matrix, $n \geq 2$ and denote $A/\alpha = (a'_{j_t j_s})$. If $S \subset \alpha$ or $\bar{S} \subset \alpha$, then A/α is an SDD matrix.*

Proof. From Theorem 5, A/S is an SDD matrix, consequently, $A/\alpha = [A/S]/[(A(\alpha)/S] \in \{SDD\}$. Similarly, if $\bar{S} \subset \alpha$, we have $A/\alpha = [A/\bar{S}]/[(A(\alpha)/\bar{S}] \in \{SDD\}$. □

Finally, making a summary of part of the content: if $\alpha \subset S$ or $\alpha \subset \bar{S}$, then $A(\alpha) \in \{SDD\}$, $A/\alpha \in$ { GDSDD}; if $S \subset \alpha$ or $\bar{S} \subset \alpha$, then $A(\alpha) \in$ {S-SOB}, $A/\alpha \in \{SDD\}$; if $S = \alpha$ or $\bar{S} = \alpha$, then $A(\alpha) \in \{SDD\}$, $A/\alpha \in \{SDD\}$; if α is contained neither in S nor in $\bar{S}$, then $A(\alpha) \in$ {S-SOB}, $A/\alpha \in$ {GDSDD}.

3. Schur complement-based infinity bounds for the inverse of S-SOB matrices

In order to obtain the upper bound of the infinite norm of the inverse of the S-SOB matrix, we need to give the definition of a permutation matrix in which every row and every column of it has only one element of 1 and all the other elements are 0. It is easy to see from the definition that permutation matrices are also elementary matrices, so multiplication of any matrix only changes the position of the matrix elements, but does not change the size of the matrix elements.

For a given nonempty proper subset α, there is a permutation matrix P such that

$$P^T A P = \begin{pmatrix} A(\alpha) & A(\alpha, \bar{\alpha}) \\ A(\bar{\alpha}, \alpha) & A(\bar{\alpha}) \end{pmatrix}.$$

We might as well assume that $A(\alpha)$ is nonsingular, let

$$E(P^T A P)F = \begin{pmatrix} A(\alpha) & 0 \\ 0 & A(\bar{\alpha}) - A(\bar{\alpha}, \alpha) A(\alpha)^{-1} A(\alpha, \bar{\alpha}) \end{pmatrix}, \tag{3.1}$$

under the circumstances

$$E = \begin{pmatrix} I_1 & 0 \\ -A(\bar{\alpha}, \alpha) A(\alpha)^{-1} & I_2 \end{pmatrix}$$

and

$$F = \begin{pmatrix} I_1 & -A(\alpha)^{-1} A(\alpha, \bar{\alpha}) \\ 0 & I_2 \end{pmatrix},$$

where I_1 (resp.I_2) is the identity matrix of order l (resp.m). We know that if P is a permutation matrix, then P^T is also a permutation matrix, and $\|P\|_\infty = 1$. From the above we can obtain

$$\|A^{-1}\|_\infty = \|PF(EP^TAPF)^{-1}EP^T\|_\infty,$$

$$\|A^{-1}\|_\infty \leq \|F\|_\infty \|(EP^TAPF)^{-1}\|_\infty \|E\|_\infty. \tag{3.2}$$

Therefore, if the upper bounds of $\|F\|_\infty$, $\|(EP^TAPF)^{-1}\|_\infty$, and $\|E\|_\infty$ can be obtained, the upper bounds of $\|A^{-1}\|_\infty$ can also be obtained, that is, the product of the above three norm bounds needs to be calculated. It's not hard to figure out

$$\|E\|_\infty = 1 + \|A(\bar{\alpha}, \alpha) A(\alpha)^{-1}\|_\infty, \tag{3.3}$$

$$\|F\|_\infty = 1 + \|A(\alpha)^{-1} A(\alpha, \bar{\alpha})\|_\infty, \tag{3.4}$$

and

$$\|(EP^TAPF)^{-1}\|_\infty = \max\{\|A(\alpha)^{-1}\|_\infty, \|(A/\alpha)^{-1}\|_\infty\}. \tag{3.5}$$

In [20], Li gives an upper bound for $\|E\|_\infty$ as follows:

Lemma 5. [20] *Let $A = \left[a_{ij}\right] \in C^{n\times n}$ be nonsingular with $a_{ii} \neq 0$, for $i \in N$, and $\varnothing \neq \alpha \subset N$. If $A(\alpha)$ is nonsingular and*

$$1 > \max_{i\in\alpha} \frac{\max_{j\in\alpha, j\neq i} |a_{ji}|}{|a_{ii}|}(k-1), \tag{3.6}$$

then,

$$\|E\|_\infty \leq \zeta(\alpha) = 1 + k \cdot \max_{i\in\alpha} \frac{\max\limits_{j\in\bar{\alpha}} |a_{ji}|}{|a_{ii}|}\left(1 - \max_{i\in\alpha} \frac{\max_{j\in\alpha, j\neq i} |a_{ji}|}{|a_{ii}|}(k-1)\right)^{-1}. \tag{3.7}$$

Theorem 6. *Let $A = \left[a_{ij}\right] \in C^{n\times n}$ be an S-SOB matrix and $D = [d_{ij}] \in C^{n\times m}$. Then,*

$$\begin{aligned} \|A^{-1}D\|_\infty \leq \max\{ &\max_{i\in S, j\in\bar{S}: a_{ij}\neq 0} \frac{|a_{jj}|R_i(D) + r_i^{\bar{S}}(A)R_j(D)}{[|a_{ii}| - r_i^S(A)]|a_{jj}| - r_i^{\bar{S}}(A)r_j(A)}, \\ &\max_{i\in S, j\in\bar{S}: a_{ji}\neq 0} \frac{|a_{ii}|R_j(D) + r_j^S(A)R_i(D)}{[|a_{jj}| - r_j^{\bar{S}}(A)]|a_{ii}| - r_j^S(A)r_i(A)}, \\ &\max_{i\in S: r_i^{\bar{S}}(A)=0} \frac{R_i(D)}{|a_{ii}| - r_i^S(A)}, \max_{j\in\bar{S}: r_j^S(A)=0} \frac{R_j(D)}{|a_{jj}| - r_j^{\bar{S}}(A)}\}, \end{aligned} \tag{3.8}$$

where $R_i(D) = \sum\limits_{k\in M} |d_{ik}|$.

Proof. Since $A = \left[a_{ij}\right] \in C^{n\times n}$ is an S-SOB matrix, we know from [1] that A is an H-matrix, $[\mu(A)]^{-1} \geq |A^{-1}|$. Let

$$\boldsymbol{\varphi} = |A^{-1}D|e = (\varphi_1, \varphi_2, ..., \varphi_n)^T,$$
$$\boldsymbol{\psi} = (\mu(A))^{-1}|D|e = (\psi_1, \psi_2, ..., \psi_n)^T,$$

and $e = (1, ..., 1)^T$ be an m-dimensional vector, consequently,

$$\boldsymbol{\psi} = \mu(A)^{-1}|D|e \geq |A^{-1}||D|e \geq |A^{-1}D|e = \boldsymbol{\varphi}, \ \textit{and}\ \mu(A)\boldsymbol{\psi} = |D|e.$$

Because of $S \subset N$, $\psi_p = \max\limits_{k\in S}\{\psi_k\},\ \psi_q = \max\limits_{k\in \bar{S}}\{\psi_k\}$, it implies that

$$|a_{ii}|\psi_i - \sum_{k\in N, k\neq i} |a_{ik}|\psi_k = \sum_{k\in M} |d_{ik}|,\ i \in N.$$

If $\psi_p \geq \psi_q$, then,

$$\begin{aligned}
\sum_{k\in M} |d_{pk}| &= |a_{pp}|\psi_p - \sum_{k\in N, k\neq p} |a_{pk}|\psi_k \\
&= |a_{pp}|\psi_p - \sum_{k\in S, k\neq p} |a_{pk}|\psi_k - \sum_{k\in \bar{S}, k\neq p} |a_{pk}|\psi_k \\
&\geq |a_{pp}|\psi_p - \sum_{k\in S, k\neq p} |a_{pk}|\psi_p - \sum_{k\in \bar{S}, k\neq p} |a_{pk}|\psi_q \\
&= [|a_{pp}| - r_p^S(A)]\psi_p - r_p^{\bar{S}}(A)\psi_q.
\end{aligned}$$

That is to say, if $\psi_p \geq \psi_q$, $r_p^{\bar{S}}(A) = 0$, then,

$$\sum_{k\in M} |d_{pk}| \geq [|a_{pp}| - r_p^S(A)]\psi_p,$$

and

$$\begin{aligned}
\|A^{-1}D\|_\infty = \max_{i\in N} \psi_i \leq \psi_p &\leq \frac{\sum\limits_{k\in M} |d_{pk}|}{|a_{pp}| - r_p^S(A)} \\
&\leq \max_{i\in S: r_i^{\bar{S}}(A)=0} \frac{\sum\limits_{k\in M} |d_{ik}|}{|a_{ii}| - r_i^S(A)}.
\end{aligned} \tag{3.9}$$

If $\psi_p \geq \psi_q$, $r_p^{\bar{S}}(A) \neq 0$, then,

$$\sum_{k\in M} |d_{pk}| \geq [|a_{pp}| - r_p^S(A)]\psi_p - r_p^{\bar{S}}(A)\psi_q, \tag{3.10}$$

and

$$\sum_{k\in M} |d_{qk}| = |a_{qq}|\psi_q - \sum_{k\in N, k\neq q} |a_{qk}|\psi_k \geq |a_{qq}|\psi_q - r_q(A)\psi_p. \tag{3.11}$$

By Eq (3.10) $\times|a_{qq}|$ + Eq (3.11)$\times r_p^{\bar{S}}(A)$, we have

$$|a_{qq}|\sum_{k\in M}|d_{pk}|+r_p^{\bar{S}}(A)\sum_{k\in M}|d_{qk}|\geq\{|a_{qq}|[|a_{pp}|-r_p^{S}(A)]-r_p^{\bar{S}}(A)r_q(A)\}\psi_p.$$

Thus,

$$\begin{aligned}||A^{-1}D||_\infty &= \max_{i\in N}\psi_i\leq\psi_p\leq\frac{|a_{qq}|\sum\limits_{k\in M}|d_{pk}|+r_p^{\bar{S}}(A)\sum\limits_{k\in M}|d_{qk}|}{|a_{qq}|[|a_{pp}|-r_p^{S}(A)]-r_p^{\bar{S}}(A)r_q(A)}\\ &\leq \max_{i\in S,j\in\bar{S}:a_{ij}\neq 0}\frac{|a_{jj}|\sum\limits_{k\in M}|d_{ik}|+r_i^{\bar{S}}(A)\sum\limits_{k\in M}|d_{jk}|}{|a_{jj}|[|a_{ii}|-r_i^{S}(A)]-r_i^{\bar{S}}(A)r_j(A)}.\end{aligned}\tag{3.12}$$

If $\psi_q\geq\psi_p$, equally,

$$\begin{aligned}\sum_{k\in M}|d_{qk}| &= |a_{qq}|\psi_q-\sum_{k\in N,k\neq q}|a_{qk}|\psi_k\\ &\geq |a_{qq}|\psi_q-\sum_{k\in\bar{S},k\neq q}|a_{qk}|\psi_q-\sum_{k\in S,k\neq q}|a_{qk}|\psi_p\\ &= [|a_{qq}|-r_q^{\bar{S}}(A)]\psi_q-r_q^{S}(A)\psi_p.\end{aligned}$$

When $r_q^S(A)=0$, $\sum\limits_{k\in M}|d_{qk}|\geq[|a_{qq}|-r_q^{\bar{S}}(A)]\psi_q$.

$$\begin{aligned}||A^{-1}D||_\infty=\max_{i\in N}\psi_i\leq\psi_q &\leq \frac{\sum\limits_{k\in M}|d_{qk}|}{|a_{qq}|-r_q^{\bar{S}}(A)}\\ &\leq \max_{i\in S:r_q^S(A)=0}\frac{\sum\limits_{k\in M}|d_{jk}|}{|a_{jj}|-r_j^{\bar{S}}(A)}.\end{aligned}\tag{3.13}$$

When $r_q^S(A)\neq 0$, then

$$\sum_{k\in M}|d_{pk}|\geq|a_{pp}|\psi_p-r_p(A)\psi_q,\tag{3.14}$$

$$\sum_{k\in M}|d_{qk}|\geq[|a_{qq}|-r_q^{\bar{S}}(A)]\psi_q-r_q^{S}(A)\psi_p.\tag{3.15}$$

Eq (3.14) $\times r_q^S(A)$ + Eq (3.15)$\times|a_{pp}|$, we have

$$r_q^S(A)\sum_{k\in M}|d_{pk}|+|a_{pp}|\sum_{k\in M}|d_{qk}|\geq\{|a_{pp}|[|a_{qq}|-r_q^{\bar{S}}(A)]-r_q^{S}(A)r_q(A)\}\psi_q.$$

Consequently,

$$||A^{-1}D||_\infty = \max_{i\in N}\psi_i\leq\psi_q\leq\frac{r_q^S(A)\sum\limits_{k\in M}|d_{pk}|+|a_{pp}|\sum\limits_{k\in M}|d_{qk}|}{|a_{pp}|[|a_{qq}|-r_q^{\bar{S}}(A)]-r_q^{S}(A)r_q(A)}$$

$$\leq \max_{i\in S,j\in\bar{S}:a_{ji}\neq 0} \frac{|a_{ii}|\sum_{k\in M}|d_{jk}| + r_j^S(A)\sum_{k\in M}|d_{ik}|}{|a_{ii}|[|a_{jj}| - r_j^{\bar{S}}(A)] - r_j^S(A)r_i(A)}. \tag{3.16}$$

The conclusion follows from inequalities Eqs (3.9), (3.12), (3.13) and (3.16). □

Replacing A and D in Theorem 6 with $A(\alpha)$ and $A(\alpha,\bar{\alpha})$, respectively, yields Corollary 4.

Corollary 4. *Let* $A = \left[a_{ij}\right] \in C^{n\times n}$ *be an* S*-SOB matrix and* $\alpha \in N$*, then,* $\|F\|_\infty \leq 1 + \max\{\max_{i\in\alpha} \frac{R_i[A(\alpha,\bar{\alpha})]}{|a_{ii}|-r_i[A(\alpha)]},\ \beta(\alpha),\ \gamma(\alpha), \lambda(\alpha)\}$, *where*

$$\beta(\alpha) = \max\left\{ \max_{\substack{i\in S,\\ j\in(\bar{S}\cap\alpha):a_{ij}\neq 0}} \frac{|a_{jj}|R_i[A(\alpha,\bar{\alpha})] + r_i^{(\bar{S}\cap\alpha)}[A(\alpha)]R_j[A(\alpha,\bar{\alpha})]}{[|a_{ii}| - r_i^S[A(\alpha)]]|a_{jj}| - r_i^{(\bar{S}\cap\alpha)}[A(\alpha)]r_j[A(\alpha)]}, \right.$$
$$\max_{\substack{i\in S,\\ j\in(\bar{S}\cap\alpha):a_{ji}\neq 0}} \frac{|a_{ii}|R_j[A(\alpha,\bar{\alpha})] + r_j^S[A(\alpha)]R_i[A(\alpha,\bar{\alpha})]}{[|a_{jj}| - r_j^{(\bar{S}\cap\alpha)}[A(\alpha)]]|a_{ii}| - r_j^S[A(\alpha)]r_i[A(\alpha)]},$$
$$\left. \max_{\substack{i\in S\\ :r_i^{(\bar{S}\cap\alpha)}[A(\alpha)]=0}} \frac{R_i[A(\alpha,\bar{\alpha})]}{|a_{ii}| - r_i^S[A(\alpha)]},\ \max_{\substack{j\in\bar{S}\cap\alpha\\ :r_j^S[A(\alpha)]=0}} \frac{R_j[A(\alpha,\bar{\alpha})]}{|a_{jj}| - r_j^{(\bar{S}\cap\alpha)}[A(\alpha)]} \right\},$$

$$\gamma(\alpha) = \max\left\{ \max_{\substack{i\in\bar{S},\\ j\in(S\cap\alpha):a_{ij}\neq 0}} \frac{|a_{jj}|R_i[A(\alpha,\bar{\alpha})] + r_i^{(S\cap\alpha)}[A(\alpha)]R_j[A(\alpha,\bar{\alpha})]}{[|a_{ii}| - r_i^{\bar{S}}[A(\alpha)]]|a_{jj}| - r_i^{(S\cap\alpha)}[A(\alpha)]r_j[A(\alpha)]}, \right.$$
$$\max_{\substack{i\in\bar{S},\\ j\in(S\cap\alpha):a_{ji}\neq 0}} \frac{|a_{ii}|R_j[A(\alpha,\bar{\alpha})] + r_j^{\bar{S}}[A(\alpha)]R_i[A(\alpha,\bar{\alpha})]}{[|a_{jj}| - r_j^{(S\cap\alpha)}[A(\alpha)]]|a_{ii}| - r_j^{\bar{S}}[A(\alpha)]r_i[A(\alpha)]},$$
$$\left. \max_{\substack{i\in\bar{S}\\ :r_i^{(S\cap\alpha)}[A(\alpha)]=0}} \frac{R_i[A(\alpha,\bar{\alpha})]}{|a_{ii}| - r_i^{\bar{S}}[A(\alpha)]},\ \max_{\substack{j\in S\\ :r_j^S[A(\alpha)]=0}} \frac{R_j[A(\alpha,\bar{\alpha})]}{|a_{jj}| - r_j^{(S\cap\alpha)}[A(\alpha)]} \right\},$$

$$\lambda(\alpha) = \max\left\{ \max_{\substack{i\in(S\cap\alpha),\\ j\in(\bar{S}\cap\alpha):a_{ij}\neq 0}} \frac{|a_{jj}|R_i[A(\alpha,\bar{\alpha})] + r_i^{(\bar{S}\cap\alpha)}[A(\alpha)]R_j[A(\alpha,\bar{\alpha})]}{[|a_{ii}| - r_i^{(S\cap\alpha)}[A(\alpha)]]|a_{jj}| - r_i^{(\bar{S}\cap\alpha)}[A(\alpha)]r_j[A(\alpha)]}, \right.$$
$$\max_{\substack{i\in(S\cap\alpha),\\ j\in(\bar{S}\cap\alpha):a_{ji}\neq 0}} \frac{|a_{ii}|R_j[A(\alpha,\bar{\alpha})] + r_j^{(S\cap\alpha)}[A(\alpha)]R_i[A(\alpha,\bar{\alpha})]}{[|a_{jj}| - r_j^{(\bar{S}\cap\alpha)}[A(\alpha)]]|a_{ii}| - r_j^{(S\cap\alpha)}[A(\alpha)]r_i[A(\alpha)]},$$
$$\left. \max_{\substack{i\in(S\cap\alpha)\\ :r_i^{(\bar{S}\cap\alpha)}[A(\alpha)]=0}} \frac{R_i[A(\alpha,\bar{\alpha})]}{|a_{ii}| - r_i^{(S\cap\alpha)}[A(\alpha)]},\ \max_{\substack{j\in\bar{S}\cap\alpha\\ :r_j^{(S\cap\alpha)}[A(\alpha)]=0}} \frac{R_j[A(\alpha,\bar{\alpha})]}{|a_{jj}| - r_j^{(\bar{S}\cap\alpha)}[A(\alpha)]} \right\}.$$

Proof. Let $\alpha \subseteq S$ or $\alpha \subseteq \bar{S}$, $A(\alpha)$ be an SDD matrix (from Lemma 2). Thus,

$$\|F\|_\infty = 1 + \|A(\alpha)^{-1}A(\alpha,\bar{\alpha})\|_\infty \leq 1 + \max_{i\in\alpha} \frac{R_i[A(\alpha,\bar{\alpha})]}{|a_{ii}| - r_i[A(\alpha)]}.$$

From Lemma 3, we have

(1) if $S \subset \alpha$, $A(\alpha)$ is an S-SOB matrix, then

$$\|F\|_\infty = 1 + \|A(\alpha)^{-1}A(\alpha,\bar{\alpha})\|_\infty \le 1+ \max\left\{ \max_{\substack{i\in S,\\ j\in(\bar{S}\cap\alpha):a_{ij}\neq 0}} \frac{|a_{jj}|R_i[A(\alpha,\bar{\alpha})] + r_i^{(\bar{S}\cap\alpha)}[A(\alpha)]R_j[A(\alpha,\bar{\alpha})]}{[|a_{ii}| - r_i^S[A(\alpha)]]|a_{jj}| - r_i^{(\bar{S}\cap\alpha)}[A(\alpha)]r_j[A(\alpha)]}, \max_{\substack{i\in S,\\ j\in(\bar{S}\cap\alpha):a_{ji}\neq 0}} \frac{|a_{ii}|R_j[A(\alpha,\bar{\alpha})] + r_j^S[A(\alpha)]R_i[A(\alpha,\bar{\alpha})]}{[|a_{jj}| - r_j^{(\bar{S}\cap\alpha)}[A(\alpha)]]|a_{ii}| - r_j^S[A(\alpha)]r_i[A(\alpha)]}, \max_{\substack{i\in S\\ :r_i^{(\bar{S}\cap\alpha)}[A(\alpha)]=0}} \frac{R_i[A(\alpha,\bar{\alpha})]}{|a_{ii}| - r_i^S[A(\alpha)]}, \max_{\substack{j\in\bar{S}\\ :r_j^S[A(\alpha)]=0}} \frac{R_j[A(\alpha,\bar{\alpha})]}{|a_{jj}| - r_j^{(\bar{S}\cap\alpha)}[A(\alpha)]} \right\}.$$

Hence, $\|F\|_\infty \le 1 + \beta(\alpha)$.

(2) If $\bar{S} \subset \alpha$, $A(\alpha)$ is an $\bar{S}$-SOB matrix, then

$$\|F\|_\infty = 1 + \|A(\alpha)^{-1}A(\alpha,\bar{\alpha})\|_\infty \le 1+ \max\left\{ \max_{\substack{i\in \bar{S},\\ j\in(S\cap\alpha):a_{ij}\neq 0}} \frac{|a_{jj}|R_i[A(\alpha,\bar{\alpha})] + r_i^{(S\cap\alpha)}[A(\alpha)]R_j[A(\alpha,\bar{\alpha})]}{[|a_{ii}| - r_i^{\bar{S}}[A(\alpha)]]|a_{jj}| - r_i^{(S\cap\alpha)}[A(\alpha)]r_j[A(\alpha)]}, \max_{\substack{i\in \bar{S},\\ j\in(S\cap\alpha):a_{ji}\neq 0}} \frac{|a_{ii}|R_j[A(\alpha,\bar{\alpha})] + r_j^{\bar{S}}[A(\alpha)]R_i[A(\alpha,\bar{\alpha})]}{[|a_{jj}| - r_j^{(S\cap\alpha)}[A(\alpha)]]|a_{ii}| - r_j^{\bar{S}}[A(\alpha)]r_i[A(\alpha)]}, \max_{\substack{i\in \bar{S}\\ :r_i^{(S\cap\alpha)}[A(\alpha)]=0}} \frac{R_i[A(\alpha,\bar{\alpha})]}{|a_{ii}| - r_i^{\bar{S}}[A(\alpha)]}, \max_{\substack{j\in S\\ :r_j^{\bar{S}}[A(\alpha)]=0}} \frac{R_j[A(\alpha,\bar{\alpha})]}{|a_{jj}| - r_j^{(S\cap\alpha)}[A(\alpha)]} \right\}.$$

Accordingly, $\|F\|_\infty \le 1 + \gamma(\alpha)$.

(3) If α is contained neither in S nor in $\bar{S}$, $A(\alpha)$ is an $(S\cap\alpha)$-SOB matrix, then we have

$$\|F\|_\infty = 1 + \|A(\alpha)^{-1}A(\alpha,\bar{\alpha})\|_\infty \le 1+ \max\left\{ \max_{\substack{i\in (S\cap\alpha),\\ j\in(\bar{S}\cap\alpha):a_{ij}\neq 0}} \frac{|a_{jj}|R_i[A(\alpha,\bar{\alpha})] + r_i^{(\bar{S}\cap\alpha)}[A(\alpha)]R_j[A(\alpha,\bar{\alpha})]}{[|a_{ii}| - r_i^{(S\cap\alpha)}[A(\alpha)]]|a_{jj}| - r_i^{(\bar{S}\cap\alpha)}[A(\alpha)]r_j[A(\alpha)]}, \max_{\substack{i\in (S\cap\alpha),\\ j\in(\bar{S}\cap\alpha):a_{ji}\neq 0}} \frac{|a_{ii}|R_j[A(\alpha,\bar{\alpha})] + r_j^{(S\cap\alpha)}[A(\alpha)]R_i[A(\alpha,\bar{\alpha})]}{[|a_{jj}| - r_j^{(\bar{S}\cap\alpha)}[A(\alpha)]]|a_{ii}| - r_j^{(S\cap\alpha)}[A(\alpha)]r_i[A(\alpha)]}, \max_{\substack{i\in (S\cap\alpha)\\ :r_i^{(\bar{S}\cap\alpha)}[A(\alpha)]=0}} \frac{R_i[A(\alpha,\bar{\alpha})]}{|a_{ii}| - r_i^{(S\cap\alpha)}[A(\alpha)]}, \max_{\substack{j\in \bar{S}\cap\alpha\\ :r_j^{(S\cap\alpha)}[A(\alpha)]=0}} \frac{R_j[A(\alpha,\bar{\alpha})]}{|a_{jj}| - r_j^{(\bar{S}\cap\alpha)}[A(\alpha)]} \right\} = \lambda(\alpha).$$

Hence, $\|F\|_\infty \le 1 + \lambda(\alpha)$. The proof is completed. □

Lemma 6. *Let $A = \left[a_{ij}\right] \in C^{n\times n}$ be an S-SOB matrix and $x = [\mu(A(\alpha))]^{-1}y^T$, where $\alpha \subseteq S$, or $\alpha \subseteq \bar{S}$. Let $x = (x_1, x_2, \cdots, x_k)$, $y = (y_1, y_2, \cdots, y_k)$, $y_k > 0$, $x_g = \max_{i_k\in\alpha} x_k$, then*

$$0 \le x_k \le \max_{i_v\in\alpha} \frac{y_v}{|a_{i_v i_v}| - r_{i_v}^{\alpha}(A)},\ i_k \in \alpha. \tag{3.17}$$

Proof. Note that $x = [\mu(A(\alpha))]^{-1}y^T$, so $[\mu(A(\alpha))]x = y^T$. For all $\alpha \in S$, or $\alpha \in \bar{S}$, from Lemma 2, $\mu(A(\alpha))$ is an H-matrix, so $[\mu(A(\alpha))]^{-1} \geq 0$ by Eq (3.1). Then

$$y_g = |a_{i_g i_g}|x_g - \sum_{i_v \in \alpha} |a_{i_g i_v}|x_v \geq |a_{i_g i_g}|x_g - \sum_{i_v \in \alpha} |a_{i_g i_v}|x_g,$$

which gives $x_g \leq \frac{y_g}{|a_{i_g i_g}| - \sum\limits_{i_v \in \alpha} |a_{i_g i_v}|} = \frac{y_g}{|a_{i_g i_g}| - r^{\alpha}_{i_g}(A)}$. Consequently, $0 \leq x_k \leq \max\limits_{i_v \in \alpha} \frac{y_v}{|a_{i_v i_v}| - r^{\alpha}_{i_v}(A)}$, $i_k \in \alpha$. □

Lemma 7. *Let $A = \left[a_{ij}\right] \in C^{n\times n}$ be an S-SOB matrix, x, y^T from Lemma 6, if α is contained neither in S nor in $\bar{S}$, $x_g = \max\limits_{i_k \in \alpha} x_k$, then*

$$0 \leq x_k \leq \pi_{y^T}(\alpha), \ i_k \in \alpha, \tag{3.18}$$

where

$$\pi_{y^T}(\alpha) = \max\{\max_{\substack{i\in(S\cap\alpha),\\ j\in(\bar{S}\cap\alpha)}} \frac{|a_{jj}|y_i + r_i^{(\bar{S}\cap\alpha)}[A(\alpha)]y_j}{[|a_{ii}| - r_i^{(S\cap\alpha)}[A(\alpha)]]|a_{jj}| - r_i^{(\bar{S}\cap\alpha)}[A(\alpha)]r_j[A(\alpha)]},$$
$$\max_{\substack{i\in(S\cap\alpha),\\ j\in(\bar{S}\cap\alpha)}} \frac{|a_{ii}|y_j + r_j^{(S\cap\alpha)}[A(\alpha)]y_i}{[|a_{jj}| - r_j^{(\bar{S}\cap\alpha)}[A(\alpha)]]|a_{ii}| - r_j^{(S\cap\alpha)}[A(\alpha)]r_i[A(\alpha)]}\}.$$

Proof. When α is contained neither in S nor in $\bar{S}$, $A(\alpha)$ is an $(S\cap\alpha)$-SOB matrix, so is $\mu(A(\alpha))$. Thus,

$$\|[\mu(A(\alpha))]^{-1}y^T\|_\infty = \|x\|_\infty = \max_{i_k\in\alpha} x_k.$$

Replacing A and D in Theorem 6 with $[\mu(A(\alpha))]^{-1}$ and y^T, respectively, yields

$$\begin{aligned}
\|[\mu(A(\alpha))]^{-1}y^T\|_\infty &\leq \max\{\max_{\substack{i\in(S\cap\alpha),\\ j\in(\bar{S}\cap\alpha)}} \frac{|a_{jj}|y_i + r_i^{(\bar{S}\cap\alpha)}[A(\alpha)]y_j}{[|a_{ii}| - r_i^{(S\cap\alpha)}[A(\alpha)]]|a_{jj}| - r_i^{(\bar{S}\cap\alpha)}[A(\alpha)]r_j[A(\alpha)]},\\
&\max_{\substack{i\in(S\cap\alpha),\\ j\in(\bar{S}\cap\alpha)}} \frac{|a_{ii}|y_j + r_j^{(S\cap\alpha)}[A(\alpha)]y_i}{[|a_{jj}| - r_j^{(\bar{S}\cap\alpha)}[A(\alpha)]]|a_{ii}| - r_j^{S}[A(\alpha)]r_i[A(\alpha)]}\}\\
= \quad &\max\{\max_{\substack{i\in(S\cap\alpha),\\ j\in(\bar{S}\cap\alpha)}} \frac{|a_{jj}|y_i + r_i^{(\bar{S}\cap\alpha)}(A)y_j}{[|a_{ii}| - r_i^{(S\cap\alpha)}(A)]|a_{jj}| - r_i^{(\bar{S}\cap\alpha)}(A)r_j^{\alpha}(A)},\\
&\max_{\substack{i\in(S\cap\alpha),\\ j\in(\bar{S}\cap\alpha)}} \frac{|a_{ii}|y_j + r_j^{(S\cap\alpha)}(A)y_i}{[|a_{jj}| - r_j^{(\bar{S}\cap\alpha)}(A)]|a_{ii}| - r_j^{(S\cap\alpha)}(A)r_i^{\alpha}(A)}\} = \pi_{y^T}(\alpha).
\end{aligned}$$

Which implies that: $0 \leq x_k \leq \pi_{y^T}(\alpha))$, $i_k \in \alpha$. □

For the sake of convenience, assume that the symbol of A/α in this part is the same as in the second part and denote:

$$v_{j_t} = (a_{j_t i_1}, a_{j_t i_2}, \cdots, a_{j_t i_k}), w_{j_s} = (a_{i_1 j_s}, a_{i_2 j_s}, \cdots, a_{i_k j_s})^T,$$
$$|v_{j_t}| = (|a_{j_t i_1}|, |a_{j_t i_2}|, \cdots, |a_{j_t i_k}|), |w_{j_s}| = (|a_{i_1 j_s}|, |a_{i_2 j_s}|, \cdots, |a_{i_k j_s}|)^T.$$

$I = (1, 1, \cdots, 1)^T$ is an k order column vector.

Theorem 7. *Let $A=(a_{ij})\in C^{n\times n}$ be an S-SOB matrix, $n\geq 2$ and A is a matrix satisfying $a_{ij}=0, a_{ii}>r_i(A)$ and $a_{ji}=0, a_{jj}>r_j(A)$ for $i\in S$, $j\in\bar{S}$. Denote $A/\alpha=(a'_{j_tj_s})$. If $\alpha\subset S$, then,*

$$\|A^{-1}\|_\infty\leq\zeta(\alpha)\left[1+\max_{i\in\alpha}\frac{R_i[A(\alpha,\bar{\alpha})]}{|a_{ii}|-r_i[A(\alpha)]}\right]\theta_1(\alpha),$$

where $\theta_1(\alpha)=\max\{\max\limits_{i\in\alpha}\frac{1}{|a_{ii}|-r_i(A(\alpha))},\ \eta_1(\alpha)\}$,

$$\eta_1(\alpha)=$$
$$\max\{\max_{\substack{i\in(S\setminus\alpha),\\ j\in\bar{S}}}\frac{|a_{jj}|-r_j^{\bar{S}}(A)+r_i^{\bar{S}}(A)+\max\limits_{v\in\alpha}\frac{r_v^{\bar{S}}(A)}{|a_{vv}|-r_v^{\alpha}(A)}[r_i^{\alpha}(A)+r_j^{\alpha}(A)]}{h_{i,j}},$$
$$\max_{\substack{i\in(S\setminus\alpha),\\ j\in\bar{S}}}\frac{|a_{ii}|-r_i^{(S\setminus\alpha)}(A)+r_j^{(S\setminus\alpha)}(A)+\max\limits_{v\in\alpha}\frac{r_v^{(S\setminus\alpha)}}{|a_{vv}|-r_v^{\alpha}(A)}[r_i^{\alpha}(A)+r_j^{\alpha}(A)]}{h_{i,j}}\}.$$

$$h_{i,j}=\left[|a_{ii}|-r_i^{(S\setminus\alpha)}(A)-|v_i|[\mu(A(\alpha))]^{-1}\sum_{k\in(S\setminus\alpha)}|w_k|\right]$$
$$\times\left[|a_{jj}|-r_j^{\bar{S}}(A)-|v_j|[\mu(A(\alpha))]^{-1}\sum_{k\in\bar{S}}|w_k|\right]$$
$$-\left[r_i^{\bar{S}}(A)|v_i|[\mu(A(\alpha))]^{-1}\sum_{k\in\bar{S}}|w_k|\right]\times\left[r_j^{\bar{\alpha}}(A)+|v_j|[\mu(A(\alpha))]^{-1}\sum_{k\in(S\setminus\alpha)}|w_k|\right].$$

Proof. By Lemma 2, we know $A(\alpha)$ is an SDD matrix. Applying Varah's bound to $A(\alpha)$, we get

$$\|A(\alpha)^{-1}\|_\infty\leq\max_{i\in\alpha}\frac{1}{|a_{ii}|-r_i(A(\alpha))}.\tag{3.19}$$

By Corollary 4, we have

$$\|F\|_\infty\leq 1+\max_{i\in\alpha}\frac{R_i[A(\alpha,\bar{\alpha})]}{|a_{ii}|-r_i[A(\alpha)]}.\tag{3.20}$$

By Theorem 4, it is easy to know $A/\alpha\in\{\ \mathrm{GDSDD}_{n-k}^{(S\setminus\alpha),\bar{S}}\}$. Therefore, from Theorem 3,

$$\|(A/\alpha)^{-1}\|_\infty\leq$$
$$\max\{\max_{\substack{j_t\in(S\setminus\alpha),\\ j_s\in\bar{S}}}\frac{|a'_{j_sj_s}|-r_{j_s}^{\bar{S}}(A/\alpha)+r_{j_t}^{\bar{S}}(A/\alpha)}{[|a'_{j_tj_t}|-r_{j_t}^{(S\setminus\alpha)}(A/\alpha)][|a'_{j_sj_s}|-r_{j_s}^{\bar{S}}(A/\alpha)]-r_{j_t}^{\bar{S}}(A/\alpha)r_{j_s}^{(S\setminus\alpha)}(A/\alpha)},$$
$$\max_{\substack{j_t\in(S\setminus\alpha),\\ j_s\in\bar{S}}}\frac{|a'_{j_tj_t}|-r_{j_t}^{(S\setminus\alpha)}(A/\alpha)+r_{j_s}^{(S\setminus\alpha)}(A/\alpha)}{[|a'_{j_tj_t}|-r_{j_t}^{(S\setminus\alpha)}(A/\alpha)][|a'_{j_sj_s}|-r_{j_s}^{\bar{S}}(A/\alpha)]-r_{j_t}^{\bar{S}}(A/\alpha)r_{j_s}^{(S\setminus\alpha)}(A/\alpha)}\}.$$

And then

$$[|a'_{j_tj_t}|-r_{j_t}^{(S\setminus\alpha)}(A/\alpha)][|a'_{j_sj_s}|-r_{j_s}^{\bar{S}}(A/\alpha)]-r_{j_t}^{\bar{S}}(A/\alpha)r_{j_s}^{(S\setminus\alpha)}(A/\alpha)$$

$$\geq \left[|a_{j_tj_t}| - r_{j_t}^{(S\setminus\alpha)}(A) - |v_{j_t}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in(S\setminus\alpha)}|w_{j_k}|\right]$$
$$\times\left[|a_{j_sj_s}| - r_{j_s}^{\bar{S}}(A) - |v_{j_s}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in\bar{S}}|w_{j_k}|\right]$$
$$-\left[r_{j_t}^{\bar{S}}(A) + |v_{j_t}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in\bar{S}}|w_{j_k}|\right]\times\left[r_{j_s}^{(S\setminus\alpha)}(A) + |v_{j_s}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in(S\setminus\alpha)}|w_{j_k}|\right] > 0.$$

$$\begin{aligned}
&|a'_{j_sj_s}| - r_{j_s}^{\bar{S}}(A/\alpha) + r_{j_t}^{\bar{S}}(A/\alpha)\\
&\leq |a_{j_sj_s}| - r_{j_s}^{\bar{S}}(A) + r_{j_t}^{\bar{S}}(A) + |v_{j_s}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in\bar{S}}|w_{j_k}| + |v_{j_t}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in\bar{S}}|w_{j_k}|\\
&= |a_{j_sj_s}| - r_{j_s}^{\bar{S}}(A) + r_{j_t}^{\bar{S}}(A) + (|v_{j_s}| + |v_{j_t}|)[\mu(A(\alpha))]^{-1}\sum_{j_k\in\bar{S}}|w_{j_k}|\\
&\leq |a_{j_sj_s}| - r_{j_s}^{\bar{S}}(A) + r_{j_t}^{\bar{S}}(A) + (|v_{j_s}| + |v_{j_t}|)\max_{i_v\in\alpha}\frac{y_v}{|a_{i_vi_v}| - r_{i_v}^{\alpha}(A)}I(by\ \ (3.18))\\
&= |a_{j_sj_s}| - r_{j_s}^{\bar{S}}(A) + r_{j_t}^{\bar{S}}(A) + \max_{i_v\in\alpha}\frac{r_{i_v}^{\bar{S}}(A)}{|a_{i_vi_v}| - r_{i_v}^{\alpha}(A)}[r_{j_t}^{\alpha}(A) + r_{j_s}^{\alpha}(A)].
\end{aligned} \tag{3.21}$$

Similarly,

$$\begin{aligned}
&|a'_{j_tj_t}| - r_{j_t}^{(S\setminus\alpha)}(A/\alpha) + r_{j_s}^{(S\setminus\alpha)}(A/\alpha) \leq |a_{j_tj_t}| - r_{j_t}^{(S\setminus\alpha)}(A) + r_{j_s}^{(S\setminus\alpha)}(A)\\
&+\max_{i_v\in\alpha}\frac{r_{i_v}^{(S\setminus\alpha)}(A)}{|a_{i_vi_v}| - r_{i_v}^{\alpha}(A)}[r_{j_t}^{\alpha}(A) + r_{j_s}^{\alpha}(A)].
\end{aligned} \tag{3.22}$$

Let

$$\begin{aligned}
h_{j_t,j_s} &= \left[|a_{j_tj_t}| - r_{j_t}^{(S\setminus\alpha)}(A) - |v_{j_t}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in(S\setminus\alpha)}|w_{j_k}|\right]\\
&\times\left[|a_{j_sj_s}| - r_{j_s}^{\bar{S}}(A) - |v_{j_s}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in\bar{S}}|w_{j_k}|\right] - \left[|v_{j_t}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in\bar{S}}|w_{j_k}|\right]\\
&\times\left[r_{j_s}^{(S\setminus\alpha)}(A) + |v_{j_s}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in(S\setminus\alpha)}|w_{j_k}|\right] > 0.
\end{aligned} \tag{3.23}$$

Furthermore, by Eqs (3.21)–(3.23), we have

$$\|(A/\alpha)^{-1}\|_\infty \leq$$
$$\max\{\max_{\substack{j_t\in(S\setminus\alpha),\\ j_s\in\bar{S}}}\frac{|a_{j_sj_s}| - r_{j_s}^{\bar{S}}(A) + r_{j_t}^{\bar{S}}(A) + \max_{i_v\in\alpha}\frac{r_{i_v}^{\bar{S}}(A)}{|a_{i_vi_v}|-r_{i_v}^{\alpha}(A)}[r_{j_t}^{\alpha}(A) + r_{j_s}^{\alpha}(A)]}{h_{j_t,j_s}},$$

$$\max_{\substack{j_t\in(S\setminus\alpha),\\ j_s\in\bar{S}}} \frac{|a_{j_tj_t}| - r_{j_t}^{(S\setminus\alpha)}(A) + r_{j_s}^{(S\setminus\alpha)}(A) + \max\limits_{i_v\in\alpha} \frac{r_{i_v}^{(S\setminus\alpha)}(A)}{|a_{i_vi_v}|-r_{i_v}^{\alpha}(A)}[r_{j_t}^{\alpha}(A) + r_{j_s}^{\alpha}(A)]}{h_{j_t,j_s}}\}$$

$$= \max\{\max_{\substack{i\in(S\setminus\alpha),\\ j\in\bar{S}}} \frac{|a_{jj}| - r_j^{\bar{S}}(A) + r_i^{\bar{S}}(A) + \max\limits_{v\in\alpha} \frac{r_v^{\bar{S}}(A)}{|a_{vv}|-r_v^{\alpha}(A)}[r_i^{\alpha}(A) + r_j^{\alpha}(A)]}{h_{i,j}},$$

$$\max_{\substack{i\in(S\setminus\alpha),\\ j\in\bar{S}}} \frac{|a_{ii}| - r_i^{(S\setminus\alpha)}(A) + r_j^{(S\setminus\alpha)}(A) + \max\limits_{v\in\alpha} \frac{r_v^{(S\setminus\alpha)}}{|a_{vv}|-r_v^{\alpha}(A)}[r_i^{\alpha}(A) + r_j^{\alpha}(A)]}{h_{i,j}}\}. \tag{3.24}$$

Finally, by Eqs (3.2), (3.7), (3.19), (3.20) and (3.24), the conclusion follows. □

The following inference can be naturally drawn from Theorem 7:

Corollary 5. *Let $A = (a_{ij}) \in C^{n\times n}$ be an S-SOB matrix, $n \geq 2$ and A be a matrix satisfying $a_{ij} = 0, a_{ii} > r_i(A)$ and $a_{ji} = 0, a_{jj} > r_j(A)$ for $i \in S$, $j \in \bar{S}$. Denote $A/\alpha = (a'_{j_tj_s})$. If $\alpha \subset \bar{S}$, then,*

$$\|A^{-1}\|_\infty \leq \zeta(\alpha)\left[1 + \max_{i\in\alpha} \frac{R_i[A(\alpha,\bar{\alpha})]}{|a_{ii}| - r_i[A(\alpha)]}\right]\theta_2(\alpha),$$

where $\theta_2(\alpha) = \max\{\max\limits_{i\in\alpha} \frac{1}{|a_{ii}|-r_i(A(\alpha))},\ \eta_2(\alpha)\}$,

$$\eta_2(\alpha) =$$

$$\max\{\max_{\substack{i\in S,\\ j\in(\bar{S}\setminus\alpha)}} \frac{|a_{jj}| - r_j^{(\bar{S}\setminus\alpha)}(A) + r_i^{(\bar{S}\setminus\alpha)}(A) + \max\limits_{v\in\alpha} \frac{r_v^{(\bar{S}\setminus\alpha)}(A)}{|a_{vv}|-r_v^{\alpha}(A)}[r_i^{\alpha}(A) + r_j^{\alpha}(A)]}{z_{i,j}},$$

$$\max_{\substack{i\in S,\\ j\in(\bar{S}\setminus\alpha)}} \frac{|a_{ii}| - r_i^{S}(A) + r_j^{S}(A) + \max\limits_{v\in\alpha} \frac{r_v^{S}(A)}{|a_{vv}|-r_v^{\alpha}(A)}[r_i^{\alpha}(A) + r_j^{\alpha}(A)]}{z_{i,j}}\}.$$

$$z_{i,j} = \left[|a_{ii}| - r_i^S(A) - |v_i|[\mu(A(\alpha))]^{-1}\sum_{k\in S}|w_k|\right]$$
$$\times\left[|a_{jj}| - r_j^{(\bar{S}\setminus\alpha)}(A) - |v_j|[\mu(A(\alpha))]^{-1}\sum_{k\in(\bar{S}\setminus\alpha)}|w_k|\right]$$
$$-\left[r_i^{(\bar{S}\setminus\alpha)}(A)|v_i|[\mu(A(\alpha))]^{-1}\sum_{k\in(\bar{S}\setminus\alpha)}|w_k|\right]\times\left[r_j^{\bar{\alpha}}(A) + |v_j|[\mu(A(\alpha))]^{-1}\sum_{k\in S}|w_k|\right].$$

Theorem 8. *Let $A = (a_{ij}) \in C^{n\times n}$ be an S-SOB matrix, $n \geq 2$ and A be a matrix satisfying $a_{ij} = 0, a_{ii} > r_i(A)$ and $a_{ji} = 0, a_{jj} > r_j(A)$ for $i \in S$, $j \in \bar{S}$. Denote $A/\alpha = (a'_{j_tj_s})$. If α is contained neither in S nor in $\bar{S}$, then,*

$$\|A^{-1}\|_\infty \le \zeta(\alpha)\,[1+\lambda(\alpha)]\,\theta_3(\alpha),$$

where $\theta_3(\alpha)=\max\{\delta_1(\alpha),\ \eta_3(\alpha)\}$,

$$\delta_1(\alpha)=\max\{\max_{\substack{i\in(S\cap\alpha),\\ j\in(\bar{S}\cap\alpha)}}\frac{|a_{jj}|+r_i^{(\bar{S}\cap\alpha)}(A(\alpha))}{[|a_{ii}|-r_i^{(S\cap\alpha)}(A(\alpha))]|a_{jj}|-r_i^{(\bar{S}\cap\alpha)}(A(\alpha))r_j(A(\alpha))},$$
$$\max_{\substack{i\in(S\cap\alpha),\\ j\in(\bar{S}\cap\alpha)}}\frac{|a_{ii}|+r_j^{(S\cap\alpha)}[(A(\alpha))]}{[|a_{jj}|-r_j^{(\bar{S}\cap\alpha)}(A(\alpha))]|a_{ii}|-r_j^{(S\cap\alpha)}(A(\alpha))r_i(A(\alpha))}\}.$$

$$\eta_3(\alpha)=\max\{\max_{\substack{i\in(S\setminus\alpha),\\ j\in(\bar{S}\setminus\alpha)}}\frac{|a_{jj}|-r_j^{(\bar{S}\setminus\alpha)}(A)+r_i^{\alpha}(A)+[r_i^{\alpha}(A)+r_j^{\alpha}(A)]\pi_{\mathbf{y_1}}(\alpha)}{f_{i,j}},$$
$$\max_{\substack{i\in(S\setminus\alpha),\\ j\in(\bar{S}\setminus\alpha)}}\frac{|a_{ii}|-r_i^{(S\setminus\alpha)}(A)+r_j^{(S\setminus\alpha)}(A)+[r_i^{\alpha}(A)+r_j^{\alpha}(A)]\pi_{\mathbf{y_2}}(\alpha)}{f_{i,j}}\}.$$

$$\begin{aligned} f_{i,j} &= \left[|a_{ii}|-r_i^{(S\setminus\alpha)}(A)-|v_i|[\mu(A(\alpha))]^{-1}\sum_{k\in(S\setminus\alpha)}|w_k|\right]\\ &\times\left[|a_{jj}|-r_j^{(\bar{S}\setminus\alpha)}(A)-|v_j|[\mu(A(\alpha))]^{-1}\sum_{k\in(\bar{S}\setminus\alpha)}|w_k|\right]\\ &-\left[r_i^{(\bar{S}\setminus\alpha)}(A)+|v_i|[\mu(A(\alpha))]^{-1}\sum_{k\in(\bar{S}\setminus\alpha)}|w_k|\right]\\ &\times\left[r_j^{(S\setminus\alpha)}(A)+|v_j|[\mu(A(\alpha))]^{-1}\sum_{k\in(S\setminus\alpha)}|w_k|\right]. \end{aligned}$$

Proof. By Lemma 3, we know $A(\alpha)$ is an $(S\cap\alpha)$-SOB matrix. Applying the bound of Theorem 2 to $A(\alpha)$, we get

$$\|A(\alpha)^{-1}\|_\infty\le\max\{\max_{\substack{i\in(S\cap\alpha),\\ j\in(\bar{S}\cap\alpha)}}\frac{|a_{jj}|+r_i^{(\bar{S}\cap\alpha)}(A(\alpha))}{[|a_{ii}|-r_i^{(S\cap\alpha)}(A(\alpha))]|a_{jj}|-r_i^{(\bar{S}\cap\alpha)}(A(\alpha))r_j(A(\alpha))},$$
$$\max_{\substack{i\in(S\cap\alpha),\\ j\in(\bar{S}\cap\alpha)}}\frac{|a_{ii}|+r_j^{(S\cap\alpha)}[(A(\alpha))]}{[|a_{jj}|-r_j^{(\bar{S}\cap\alpha)}(A(\alpha))]|a_{ii}|-r_j^{(S\cap\alpha)}(A(\alpha))r_i(A(\alpha))}\}=\delta_1(\alpha). \tag{3.25}$$

By Corollary 4, we have

$$\|F\|_\infty\le 1+\lambda(\alpha). \tag{3.26}$$

By Corollary 2, we know $A/\alpha\in\{\ \mathrm{GDSDD}_{n-k}^{(S\setminus\alpha),(\bar{S}\setminus\alpha)}\}$. Therefore,

$$\|(A/\alpha)^{-1}\|_\infty\le$$

$$\max\{\max_{\substack{j_t\in(S\setminus\alpha),\\ j_s\in(\bar{S}\setminus\alpha)}} \frac{|a'_{j_sj_s}| - r_{j_s}^{(\bar{S}\setminus\alpha)}(A/\alpha) + r_{j_t}^{(\bar{S}\setminus\alpha)}(A/\alpha)}{[|a'_{j_tj_t}| - r_{j_t}^{(S\setminus\alpha)}(A/\alpha)][|a'_{j_sj_s}| - r_{j_s}^{(\bar{S}\setminus\alpha)}(A/\alpha)] - r_{j_t}^{(\bar{S}\setminus\alpha)}(A/\alpha) r_{j_s}^{(S\setminus\alpha)}(A/\alpha)},$$

$$\max_{\substack{j_t\in(S\setminus\alpha),\\ j_s\in(\bar{S}\setminus\alpha)}} \frac{|a'_{j_tj_t}| - r_{j_t}^{(S\setminus\alpha)}(A/\alpha) + r_{j_s}^{(S\setminus\alpha)}(A/\alpha)}{[|a'_{j_tj_t}| - r_{j_t}^{(S\setminus\alpha)}(A/\alpha)][|a'_{j_sj_s}| - r_{j_s}^{(\bar{S}\setminus\alpha)}(A/\alpha)] - r_{j_t}^{(\bar{S}\setminus\alpha)}(A/\alpha) r_{j_s}^{(S\setminus\alpha)}(A/\alpha)}\}. \tag{3.27}$$

And then,

$$\begin{aligned}
&[|a'_{j_tj_t}| - r_{j_t}^{(S\setminus\alpha)}(A/\alpha)][|a'_{j_sj_s}| - r_{j_s}^{(\bar{S}\setminus\alpha)}(A/\alpha)] - r_{j_t}^{(\bar{S}\setminus\alpha)}(A/\alpha) r_{j_s}^{(S\setminus\alpha)}(A/\alpha)\\
&\geq \left[|a_{j_tj_t}| - r_{j_t}^{(S\setminus\alpha)}(A) - |v_{j_t}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in(S\setminus\alpha)}|w_{j_k}|\right]\\
&\times \left[|a_{j_sj_s}| - r_{j_s}^{(\bar{S}\setminus\alpha)}(A) - |v_{j_s}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in(\bar{S}\setminus\alpha)}|w_{j_k}|\right]\\
&- \left[r_{j_t}^{(\bar{S}\setminus\alpha)}(A) + |v_{j_t}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in(\bar{S}\setminus\alpha)}|w_{j_k}|\right]\\
&\times \left[r_{j_s}^{(S\setminus\alpha)}(A) + |v_{j_s}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in(S\setminus\alpha)}|w_{j_k}|\right] > 0.
\end{aligned}$$

$$\begin{aligned}
&|a'_{j_sj_s}| - r_{j_s}^{(\bar{S}\setminus\alpha)}(A/\alpha) + r_{j_t}^{(\bar{S}\setminus\alpha)}(A/\alpha) \leq |a_{j_sj_s}| - r_{j_s}^{(\bar{S}\setminus\alpha)}(A) + r_{j_t}^{(\bar{S}\setminus\alpha)}(A)\\
&+|v_{j_s}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in(\bar{S}\setminus\alpha)}|w_{j_k}| + |v_{j_t}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in(\bar{S}\setminus\alpha)}|w_{j_k}|\\
&= |a_{j_sj_s}| - r_{j_s}^{(\bar{S}\setminus\alpha)}(A) + r_{j_t}^{(\bar{S}\setminus\alpha)}(A) + (|v_{j_s}| + |v_{j_t}|)[\mu(A(\alpha))]^{-1}\sum_{j_k\in(\bar{S}\setminus\alpha)}|w_{j_k}|.
\end{aligned}$$

Let $y^T = \mathbf{y_1} = \sum_{j_k\in(\bar{S}\setminus\alpha)}|w_{j_k}|$, y^T from Lemma 7, we get

$$\begin{aligned}
&|a'_{j_sj_s}| - r_{j_s}^{(\bar{S}\setminus\alpha)}(A/\alpha) + r_{j_t}^{(\bar{S}\setminus\alpha)}(A/\alpha)\\
&\leq |a_{j_sj_s}| - r_{j_s}^{(\bar{S}\setminus\alpha)}(A) + r_{j_t}^{(\bar{S}\setminus\alpha)}(A) + (|v_{j_s}| + |v_{j_t}|)\pi(\alpha)I\\
&= |a_{j_sj_s}| - r_{j_s}^{(\bar{S}\setminus\alpha)}(A) + r_{j_t}^{(\bar{S}\setminus\alpha)}(A) + [r_{j_t}^{\alpha}(A) + r_{j_s}^{\alpha}(A)]\pi_{\mathbf{y_1}}(\alpha).
\end{aligned} \tag{3.28}$$

In like manner, let $y^T = \mathbf{y_2} = \sum_{j_k\in(S\setminus\alpha)}|w_{j_k}|$, y^T from Lemma 7, we get

$$\begin{aligned}
&|a'_{j_tj_t}| - r_{j_t}^{(S\setminus\alpha)}(A/\alpha) + r_{j_s}^{(S\setminus\alpha)}(A/\alpha) \leq |a_{j_tj_t}| - r_{j_t}^{(S\setminus\alpha)}(A) + r_{j_s}^{(S\setminus\alpha)}(A)\\
&+[r_{j_t}^{\alpha}(A) + r_{j_s}^{\alpha}(A)]\pi_{\mathbf{y_2}}(\alpha).
\end{aligned} \tag{3.29}$$

Let

$$f_{j_t,j_s} \quad = \quad \left[|a_{j_tj_t}| - r_{j_t}^{(S\setminus\alpha)}(A) - |v_{j_t}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in(S\setminus\alpha)}|w_{j_k}|\right]$$

$$
\begin{aligned}
&\times \left[|a_{j_sj_s}| - r_{j_s}^{(\bar{S}\setminus\alpha)}(A) - |v_{j_s}|[\mu(A(\alpha))]^{-1} \sum_{j_k\in(\bar{S}\setminus\alpha)} |w_{j_k}|\right] \\
&- \left[r_{j_t}^{(\bar{S}\setminus\alpha)}(A) + |v_{j_t}|[\mu(A(\alpha))]^{-1} \sum_{j_k\in(\bar{S}\setminus\alpha)} |w_{j_k}|\right] \\
&\times \left[r_{j_s}^{(S\setminus\alpha)}(A) + |v_{j_s}|[\mu(A(\alpha))]^{-1} \sum_{j_k\in(S\setminus\alpha)} |w_{j_k}|\right]. \qquad (3.30)
\end{aligned}
$$

Furthermore, by Eqs (3.28)–(3.30), we have

$$
\begin{aligned}
&\|(A/\alpha)^{-1}\|_\infty \le \\
&\max\{\max_{\substack{j_t\in(S\setminus\alpha),\\ j_s\in(\bar{S}\setminus\alpha)}} \frac{|a_{j_sj_s}| - r_{j_s}^{(\bar{S}\setminus\alpha)}(A) + r_{j_t}^{\bar{S}\setminus\alpha}(A) + [r_{j_t}^{\alpha}(A) + r_{j_s}^{\alpha}(A)]\pi_{\mathbf{y_1}}(\alpha)}{f_{j_t,j_s}}, \\
&\max_{\substack{j_t\in(S\setminus\alpha),\\ j_s\in(\bar{S}\setminus\alpha)}} \frac{|a_{j_tj_t}| - r_{j_t}^{(S\setminus\alpha)}(A) + r_{j_s}^{(S\setminus\alpha)}(A) + [r_{j_t}^{\alpha}(A) + r_{j_s}^{\alpha}(A)]\pi_{\mathbf{y_2}}(\alpha)}{f_{j_t,j_s}}\} \\
&= \max\{\max_{\substack{i\in(S\setminus\alpha),\\ j\in(\bar{S}\setminus\alpha)}} \frac{|a_{jj}| - r_{j}^{(\bar{S}\setminus\alpha)}(A) + r_{i}^{(\bar{S}\setminus\alpha)}(A) + [r_{i}^{\alpha}(A) + r_{j}^{\alpha}(A)]\pi_{\mathbf{y_1}}(\alpha)}{f_{i,j}}, \\
&\max_{\substack{i\in(S\setminus\alpha),\\ j\in(\bar{S}\setminus\alpha)}} \frac{|a_{ii}| - r_{i}^{(S\setminus\alpha)}(A) + r_{j}^{(S\setminus\alpha)}(A) + [r_{i}^{\alpha}(A) + r_{j}^{\alpha}(A)]\pi_{\mathbf{y_2}}(\alpha)}{f_{i,j}}\}. \qquad (3.31)
\end{aligned}
$$

Finally, by Eqs (3.2), (3.7), (3.25), (3.26) and (3.31), the conclusion follows. □

Theorem 9. *Let $A = \left[a_{ij}\right] \in C^{n\times n}$ be an S-SOB matrix, $\phi \neq \alpha = S$. If Eq (3.7) holds, then,*

$$
\|A^{-1}\|_\infty \le \zeta(\alpha)\left[1 + \max_{i\in\alpha} \frac{R_i[A(\alpha,\bar{\alpha})]}{|a_{ii}| - r_i[A(\alpha)]}\right]\theta_4(\alpha),
$$

where $\theta_4(\alpha) = \max\{\max_{i\in\alpha} \frac{1}{|a_{ii}|-r_i(A(\alpha))},\ \eta_4(\alpha)\}$,

$$
\eta_4(\alpha) = \max_{j\in\bar{S}} \frac{1}{|a_{jj}| - r_j^{\bar{S}}(A) - |v_j|[\mu(A(\alpha))]^{-1} \sum\limits_{k\in\bar{S}} |w_k|}.
$$

Expressly, when $\phi \neq \alpha = S = \{i\}$,

$$
\|A^{-1}\|_\infty \le \left[1 + \max_{j\in\bar{S}} \frac{|a_{ji}|}{|a_{ii}|}\right]\left[1 + \max_{j\in\bar{S}} \frac{|a_{ji}|}{|a_{ii}|}\right]\theta_4'(\alpha).
$$

$\theta_4'(\alpha) = \max\{\frac{1}{|a_{ii}|},\ \eta_4'(\alpha)\}$,

$$
\eta_4'(\alpha) = \max_{j\in\bar{S}} \frac{1}{|a_{jj}| - r_j^{\bar{S}}(A) - \frac{|a_{ji}|r_i^{\bar{S}}(A)}{|a_{ii}|}}.
$$

Proof. By Lemma 2, we know $A(\alpha)$ is an SDD matrix. $\|A(\alpha)^{-1}\|_\infty$ is the same as Eq (3.19), and $\|F\|_\infty$ is the same as Eq (3.20). By Theorem 5, knowing that A/α is an SDD matrix. Therefore,

$$\begin{aligned}\|(A/\alpha)^{-1}\|_\infty &\le \max_{j_t\in\bar{\alpha}} \frac{1}{|a'_{j_tj_t}| - r_{j_t}(A/\alpha)} \\ &\le \max_{j_t\in\bar{\alpha}} \frac{1}{|a_{j_tj_t}| - r^{\bar{\alpha}}_{j_t}(A) - |v_{j_t}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in\bar{S}}|w_{j_k}|} \\ &= \max_{j_t\in\bar{S}} \frac{1}{|a_{j_tj_t}| - r^{\bar{S}}_{j_t}(A) - |v_{j_t}|[\mu(A(\alpha))]^{-1}\sum_{j_k\in\bar{S}}|w_{j_k}|} \\ &= \max_{j\in\bar{S}} \frac{1}{|a_{jj}| - r^{\bar{S}}_{j}(A) - |v_{j}|[\mu(A(\alpha))]^{-1}\sum_{k\in\bar{S}}|w_{k}|} = \eta_4. \end{aligned} \tag{3.32}$$

Finally, by Eqs (3.2), (3.7), (3.19), (3.20) and (3.32), the conclusion follows. □

A proof similar to Theorem 9 leads to the results.

Corollary 6. *Let $A = \left[a_{ij}\right] \in C^{n\times n}$ be an S-SOB matrix, where $\phi \ne \alpha = \bar{S}$. If Eq (3.7) holds, then,*

$$\|A^{-1}\|_\infty \le \zeta(\alpha)\left[1 + \max_{i\in\alpha}\frac{r_i[A(\alpha,\bar{\alpha})]}{|a_{ii}| - r_i[A(\alpha)]}\right]\theta_5(\alpha),$$

where $\theta_5(\alpha) = \max\{\max\limits_{i\in\alpha}\frac{1}{|a_{ii}|-r_i(A(\alpha))},\ \eta_5(\alpha)\}$,

$$\eta_5(\alpha) = \max_{i\in S}\frac{1}{|a_{ii}| - r^S_i(A) - |v_i|[\mu(A(\alpha))]^{-1}\sum_{k\in S}|w_k|}.$$

Distinguishingly, when $\phi \ne \alpha = \bar{S} = \{i\}$,

$$\|A^{-1}\|_\infty \le \left[1 + \max_{j\in S}\frac{|a_{ji}|}{|a_{ii}|}\right]\left[1 + \max_{j\in S}\frac{|a_{ji}|}{|a_{ii}|}\right]\theta'_5(\alpha).$$

$\theta'_5(\alpha) = \max\{\frac{1}{|a_{ii}|},\ \eta'_5(\alpha)\}$,

$$\eta'_5(\alpha) = \max_{j\in S}\frac{1}{|a_{jj}| - r^S_j(A) - \frac{|a_{ji}|r^S_i(A)}{|a_{ii}|}}.$$

Theorem 10. *Let $A = \left[a_{ij}\right] \in C^{n\times n}$ be an S-SOB matrix, where $S \subset \alpha$. If Eq (3.7) holds, then,*

$$\|A^{-1}\|_\infty \le \zeta(\alpha)[1 + \beta(\alpha)]\theta_6(\alpha),$$

where $\theta_6(\alpha) = \max\{\delta_2(\alpha),\ \eta_6(\alpha)\}$,

$$\delta_2(\alpha) = \max\{\max_{\substack{i\in S, j\in(\bar{S}\cap\alpha),\\ :a_{ij}\ne 0}}\frac{|a_{jj}| + r^{(\bar{S}\cap\alpha)}_i(A(\alpha))}{[|a_{ii}| - r^S_i(A(\alpha))]|a_{jj}| - r^{(\bar{S}\cap\alpha)}_i(A(\alpha))r_j(A(\alpha))},$$

$$\max_{\substack{i\in S, j\in(\bar{S}\cap\alpha),\\ :a_{ji}\neq 0}} \frac{|a_{ii}| + r_j^{(S\cap\alpha)}[(A(\alpha))]}{[|a_{jj}| - r_j^{(\bar{S}\cap\alpha)}(A(\alpha))]|a_{ii}| - r_j^S(A(\alpha))r_i(A(\alpha))},$$

$$\max_{\substack{i\in S, j\in(\bar{S}\cap\alpha)\\ r_i^{(\bar{S}\cap\alpha)}(A(\alpha))=0}} \frac{1}{|a_{ii}| - r_i^S(A(\alpha))}, \max_{\substack{i\in S, j\in(\bar{S}\cap\alpha)\\ :r_j^S(A(\alpha))=0}} \frac{1}{|a_{jj}| - r_j^{(\bar{S}\cap\alpha)}(A(\alpha))}\}.$$

$$\eta_6(\alpha) = \max_{i\in(\bar{S}\backslash\alpha)} \frac{1}{|a_{ii}| - r_i^{(\bar{S}\backslash\alpha)}(A) - |v_i|[\mu(A(\alpha))]^{-1} \sum\limits_{k\in(\bar{S}\backslash\alpha)} |w_k|}.$$

Proof. $A(\alpha)$ is an S-SOB matrix (by Lemma 3). Thus,

$$\|A(\alpha)^{-1}\|_\infty \le \max\{\max_{\substack{i\in S, j\in(\bar{S}\cap\alpha),\\ :a_{ij}\neq 0}} \frac{|a_{jj}| + r_i^{(\bar{S}\cap\alpha)}(A(\alpha))}{[|a_{ii}| - r_i^S(A(\alpha))]|a_{jj}| - r_i^{(\bar{S}\cap\alpha)}(A(\alpha))r_j(A(\alpha))},$$

$$\max_{\substack{i\in S, j\in(\bar{S}\cap\alpha),\\ :a_{ji}\neq 0}} \frac{|a_{ii}| + r_j^{(S\cap\alpha)}[(A(\alpha))]}{[|a_{jj}| - r_j^{(\bar{S}\cap\alpha)}(A(\alpha))]|a_{ii}| - r_j^S(A(\alpha))r_i(A(\alpha))},$$

$$\max_{\substack{i\in S, j\in(\bar{S}\cap\alpha)\\ r_i^{(\bar{S}\cap\alpha)}(A(\alpha))=0}} \frac{1}{|a_{ii}| - r_i^S(A(\alpha))}, \max_{\substack{i\in S, j\in(\bar{S}\cap\alpha)\\ :r_j^S(A(\alpha))=0}} \frac{1}{|a_{jj}| - r_j^{(\bar{S}\cap\alpha)}(A(\alpha))}\} = \delta_2(\alpha). \tag{3.33}$$

From Corollary 4, we know

$$\|F\|_\infty \le 1 + \beta(\alpha). \tag{3.34}$$

By Corollary 3, we obtain A/α is an SDD matrix. Therefore,

$$\|(A/\alpha)^{-1}\|_\infty \le \max_{j_t\in(\bar{S}\backslash\alpha)} \frac{1}{|a_{j_tj_t}| - r_{j_t}^{(\bar{S}\backslash\alpha)}(A) - |v_{j_t}|[\mu(A(\alpha))]^{-1} \sum\limits_{j_k\in(\bar{S}\backslash\alpha)} |w_{j_k}|}$$

$$= \max_{i\in(\bar{S}\backslash\alpha)} \frac{1}{|a_{ii}| - r_i^{(\bar{S}\backslash\alpha)}(A) - |v_i|[\mu(A(\alpha))]^{-1} \sum\limits_{k\in(\bar{S}\backslash\alpha)} |w_k|}. \tag{3.35}$$

Finally, by Eqs (3.2), (3.7), (3.33), (3.34) and (3.35), the conclusion follows. □

According to Theorem 10, the following result will come out naturally.

Corollary 7. *Let $A = [a_{ij}] \in C^{n\times n}$ be an S-SOB matrix, $\bar{S} \subset \alpha$. If Eq (3.7) holds, then*

$$\|A^{-1}\|_\infty \le \zeta(\alpha)[1 + \gamma(\alpha)]\theta_7(\alpha),$$

where $\theta_7(\alpha) = \max\{\delta_3(\alpha),\ \eta_7(\alpha)\}$,

$$\delta_3(\alpha) = \max\{\max_{\substack{i\in(S\cap\alpha),\\ j\in\bar{S}}} \frac{|a_{jj}| + r_i^{\bar{S}}(A(\alpha))}{[|a_{ii}| - r_i^{(S\cap\alpha)}(A(\alpha))]|a_{jj}| - r_i^{\bar{S}}(A(\alpha))r_j(A(\alpha))},$$

$$\max_{\substack{i\in(S\cap\alpha),\\ j\in\bar{S}}} \frac{|a_{ii}| + r_j^{(S\cap\alpha)}[(A(\alpha))]}{[|a_{jj}| - r_j^{\bar{S}}(A(\alpha))]|a_{ii}| - r_j^{(S\cap\alpha)}(A(\alpha))r_i(A(\alpha))},$$
$$\max_{\substack{i\in(S\cap\alpha),\\ j\in\bar{S}}} \frac{1}{|a_{ii}| - r_i^{(S\cap\alpha)}(A(\alpha))}, \max_{\substack{i\in(S\cap\alpha),\\ j\in\bar{S}}} \frac{1}{|a_{jj}| - r_j^{\bar{S}}(A(\alpha))}\}.$$

$\eta_7(\alpha) = \max_{i\in(S\setminus\alpha)} \frac{1}{|a_{ii}|-r_i^{(S\setminus\alpha)}(A)-|v_i|[\mu(A(\alpha))]^{-1}\sum\limits_{k\in(S\setminus\alpha)}|w_k|}.$

Theorem 11. *Let $A = (a_{ij}) \in C^{n\times n}$ be an S-SOB matrix, $n \geq 3$ and let A satisfy that when $a_{ij} = 0, a_{ii} > r_i(A)$ and $a_{ji} = 0, a_{jj} > r_j(A)$ for $i \in S$, $j \in \bar{S}$. Denote $A/\alpha = (a'_{j_t j_s})$, then,*

$$\|A^{-1}\|_\infty \leq \Gamma(A) = \min_{i\in N} \Gamma_i(A).$$

where $\Gamma_i(A) = (1 + \frac{\max\limits_{j\in N, j\neq i}|a_{ji}|}{|a_{ii}|})(1 + \frac{\max\limits_{j\in N, j\neq i}|a_{ij}|}{|a_{ii}|})\tilde{\Gamma}_i(A)$,

$$\tilde{\Gamma}_i(A) = \max\{\frac{1}{|a_{ii}|}, \Gamma'(A)\}.$$

$$\Gamma'(A) = \max\{\max_{\substack{j\in(S\setminus\{i\}),\\ k\in(\bar{S}\setminus\{i\})}} \frac{|c_{kk}| - \sum\limits_{p\in\bar{S}, p\neq k,i}|c_{kp}| + \sum\limits_{p\in\bar{S}, p\neq i}|c_{jp}|}{(|c_{jj}| - \sum\limits_{p\in S, p\neq j,i}|c_{jp}|)(|c_{kk}| - \sum\limits_{p\in\bar{S}, p\neq k,i}|c_{kp}|) - \sum\limits_{p\in S, p\neq i}|c_{kp}|\sum\limits_{p\in\bar{S}, p\neq i}|c_{jp}|},$$
$$\max_{\substack{j\in(S\setminus\{i\}),\\ k\in(\bar{S}\setminus\{i\})}} \frac{|c_{jj}| - \sum\limits_{p\in S, p\neq j,i}|c_{jp}| + \sum\limits_{p\in S, p\neq i}|c_{kp}|}{(|c_{jj}| - \sum\limits_{p\in S, p\neq j,i}|c_{jp}|)(|c_{kk}| - \sum\limits_{p\in\bar{S}, p\neq k,i}|c_{kp}|) - \sum\limits_{p\in S, p\neq i}|c_{kp}|\sum\limits_{p\in\bar{S}, p\neq i}|c_{jp}|}\},$$

and $c_{jk} = a_{jk} - \frac{a_{ji}a_{ik}}{a_{ii}}$.

Proof. Since A is an S-SOB matrix, by Lemma 2 and Theorem 5, we know $A(\alpha)$ and A/α are nonsingular. Therefore, taking $\alpha = \{i\}$, then $A(\alpha) = a_{ii}$, $\bar{\alpha} = N - \{i\}$, and

$$\|A(\alpha)^{-1}\|_\infty \leq \frac{1}{|a_{ii}|}. \tag{3.36}$$

$$\|E\|_\infty = 1 + \frac{\max\limits_{j_s\in\bar{\alpha}}|a_{j_s i}|}{|a_{ii}|} = 1 + \frac{\max\limits_{j\in N, j\neq i}|a_{ji}|}{|a_{ii}|}. \tag{3.37}$$

$$\|F\|_\infty = 1 + \frac{\max\limits_{j_s\in\bar{\alpha}}|a_{ij_s}|}{|a_{ii}|} = 1 + \frac{\max\limits_{j\in N, j\neq i}|a_{ij}|}{|a_{ii}|}. \tag{3.38}$$

Because $A/\alpha = (a'_{j_t j_s})$, let $|a'_{j_t j_s}| = |a_{j_t j_s} - \frac{a_{j_t i} a_{i j_s}}{a_{ii}}| = |c_{j_t j_s}| (j_t, j_s \in (N \setminus \{i\}))$. By calculation, we obtain for $j_t \in (S \setminus \{i\})$, $j_s \in (\bar{S} \setminus \{i\})$,

$$r_{j_t}^{(S\setminus\{i\})}(A/\alpha) = \sum_{\substack{j_p \in (S\setminus\{i\}), \\ j_p \neq j_t}} |c_{j_t j_p}| = \sum_{\substack{j_p \in S, \\ j_p \neq j_t, i}} |c_{j_t j_p}|,$$

$$r_{j_t}^{(\bar{S}\setminus\{i\})}(A/\alpha) = \sum_{j_p \in (\bar{S}\setminus\{i\})} |c_{j_t j_p}| = \sum_{\substack{j_p \in \bar{S}, \\ j_p \neq i}} |c_{j_t j_p}|,$$

$$r_{j_s}^{(\bar{S}\setminus\{i\})}(A/\alpha) = \sum_{\substack{j_p \in (\bar{S}\setminus\{i\}), \\ j_p \neq j_s}} |c_{j_s j_p}| = \sum_{\substack{j_p \in \bar{S}, \\ j_p \neq i}} |c_{j_s j_p}|,$$

$$r_{j_s}^{(S\setminus\{i\})}(A/\alpha) = \sum_{j_p \in (S\setminus\{i\})} |c_{j_s j_p}| = \sum_{\substack{j_p \in S, \\ j_p \neq i}} |c_{j_s j_p}|.$$

By Eq (3.27), we have

$$\|(A/\alpha)^{-1}\|_\infty \leq \max\{ \max_{\substack{j \in (S\setminus\{i\}), \\ k \in (\bar{S}\setminus\{i\})}} \frac{|c_{kk}| - \sum\limits_{\substack{p \in \bar{S}, \\ p \neq k,i}} |c_{kp}| + \sum\limits_{\substack{p \in \bar{S}, \\ p \neq i}} |c_{jp}|}{(|c_{jj}| - \sum\limits_{\substack{p \in S, \\ p \neq j,i}} |c_{jp}|)(|c_{kk}| - \sum\limits_{\substack{p \in \bar{S}, \\ p \neq k,i}} |c_{kp}|) - \sum\limits_{\substack{p \in S, \\ p \neq i}} |c_{kp}| \sum\limits_{\substack{p \in \bar{S}, \\ p \neq i}} |c_{jp}|},$$
$$\max_{\substack{j \in (S\setminus\{i\}), \\ k \in (\bar{S}\setminus\{i\})}} \frac{|c_{jj}| - \sum\limits_{\substack{p \in S, \\ p \neq j,i}} |c_{jp}| + \sum\limits_{\substack{p \in S, \\ p \neq i}} |c_{kp}|}{(|c_{jj}| - \sum\limits_{\substack{p \in S, \\ p \neq j,i}} |c_{jp}|)(|c_{kk}| - \sum\limits_{\substack{p \in \bar{S}, \\ p \neq k,i}} |c_{kp}|) - \sum\limits_{\substack{p \in S, \\ p \neq i}} |c_{kp}| \sum\limits_{\substack{p \in \bar{S}, \\ p \neq i}} |c_{jp}|} \}. \tag{3.39}$$

Finally, by Eqs (3.36), (3.37), (3.38) and (3.39) the conclusion follows. □

We illustrate our results by the following examples:

Example 1. *Consider matrix A as a tri-diagonal $n \times n$ matrix*

$$A = \begin{bmatrix} n + |sin(1)| & bcos(2) & \cdots & bcos(n-1) & bcos(n) \\ sin(2) & n + |sin(2)| & \cdots & bcos(n-1) & bcos(n) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ sin(n-1) & \cdots & sin(n-1) & n + |sin(n-1)| & bcos(n) \\ sin(n) & \cdots & sin(n) & sin(n) & n + |sin(n)| \end{bmatrix}_{n \times n}.$$

Let $b = 1.5$, $n = 10000$. We get that matrix A is an SDD matrix. It is easy to verify matrix A is an SDD matrix, so it is also a S-SOB, $DSDD$, $GDSDD$ and DZ matrix. Therefore, from Theorem 1, we put the result in Table 1.

Actually, $\|A^{-1}\|_\infty = 0.0002$. This example shows that the boundary in Theorem 11 is superior to other theorems in some cases.

Table 1. Upper bounds of matrix A in Example 1.

	$b = 1.5$ $n = 10000$
Bound in Theorem 1	0.2786
Bound in Theorem 2	0.2685
Bound in Theorem 3	0.2485
Bound in [20, Theorem 3]	0.3954
Bound in [31, Corollary 1]	0.2786
Bound in [21, Theorem 1.2]	0.2731
Bound in [21, Corollary 2.6]	0.1937
Bound in Theorem 11	0.1904

Example 2. *Consider matrix*

$$A = \begin{bmatrix} 16.81 & 0.15 & 0.65 & 0.7 & 0.43 & 0.27 & 0.75 & 0.84 & 0.35 & 0.07 \\ 1.9 & 8 & 0.03 & 0.03 & 3.38 & 0.67 & 0.25 & 2.25 & 0.83 & 1.05 \\ 0.12 & 0.95 & 11.84 & 0.27 & 0.76 & 0.65 & 0.5 & 0.81 & 0.58 & 0.53 \\ 0.91 & 0.48 & 0.93 & 12.04 & 0.79 & 0.16 & 0.69 & 0.24 & 0.54 & 0.77 \\ 0.63 & 0.8 & 0.67 & 0.09 & 9.18 & 1.11 & 0.89 & 6.92 & 0.91 & 0.93 \\ 0.09 & 0.14 & 0.75 & 0.82 & 0.48 & 15.49 & 0.95 & 0.35 & 0.28 & 0.12 \\ 0.27 & 0.42 & 0.74 & 0.69 & 0.44 & 0.95 & 12.54 & 0.19 & 0.75 & 0.56 \\ 0.54 & 0.91 & 0.39 & 0.31 & 0.64 & 0.34 & 0.13 & 11.25 & 0.75 & 0.46 \\ 0.95 & 0.79 & 0.65 & 0.95 & 0.70 & 0.58 & 0.14 & 0.61 & 10.38 & 0.01 \\ 0.96 & 0.95 & 0.17 & 0.03 & 0.75 & 0.22 & 0.25 & 0.47 & 0.56 & 17.33 \end{bmatrix}.$$

By computation, the matrix A is an S-SOB matrix and $S = \{2, 3, 5\}$. According to Theorem 2, we obtain

$$\|A^{-1}\|_\infty \le 1.7202.$$

According to Theorem 11, it is easy to get

$$\|A^{-1}\|_\infty \le 0.5061.$$

In practice, $\|A^{-1}\|_\infty = 0.2155$. Obviously, the boundary in Theorem 11 is superior to Theorem 2 in some cases.

Example 3. *Consider matrix*

$$A = \begin{bmatrix} 38 & 1 & 3 & 3 & -4 & 2 & 5 & -1 \\ 1 & 40 & 5 & 4 & 1 & 3 & 1 & -2 \\ 2 & 1 & 36 & 1 & 2 & 1 & -4 & -3 \\ 1 & 3 & 2 & 28 & 3 & 5 & 1 & 2 \\ 4 & 1.5 & -1 & 2 & 31 & -1 & -4 & 4 \\ -8 & 6 & 3 & 5 & 2 & 49 & 2 & 7 \\ 7 & 9 & 1 & -1 & -1 & 7 & 50 & 5 \\ 1 & 13 & 2 & 3 & 6 & 1 & 1 & 44 \end{bmatrix}.$$

Obviously, the matrix A is an SDD matrix, and it's also an S-SOB matrix and $S = \{2, 3, 4, 5, 8\}$. According to Theorem 1, we can obtain

$$\|A^{-1}\|_\infty \leq 0.0909.$$

According to Theorem 2, we can obtain

$$\|A^{-1}\|_\infty \leq 0.0860.$$

According to Theorem 11, we can obtain

$$\|A^{-1}\|_\infty \leq 0.0842.$$

In fact, $\|A^{-1}\|_\infty = 0.0497$. This example shows that the boundary in Theorem 11 is superior to Theorems 1 and 2 in some cases.

4. Error bounds for LCPs of S-SOB matrices

In this section, we will apply the result in Section 3 to the linear complementarity problems (LCPs), to obtain two kinds of error bounds for LCPs of S-SOB matrices. We first need to give some lemmas that would be used in the following theorems:

Lemma 8. [29] *Let $\gamma > 0$ and $\eta \geq 0$, for any $x \in [0, 1]$,*

$$\frac{1}{1 - x + \gamma x} \leq \frac{1}{min\{\gamma, 1\}}, \ \frac{\eta x}{1 - x + \gamma x} \leq \frac{\eta}{\gamma}.$$

Lemma 9. *Suppose that $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ is an S-SOB matrix with positive diagonal entries, let*

$$\tilde{M} = I - D + DM = (\tilde{m}_{ij}), \tag{4.1}$$

then, $\tilde{M}$ is also a real S-SOB matrix with positive diagonal entries, where $D = diag(d_1, \cdots, d_n)$, $d_i \in [0, 1]$.

Proof. Note that

$$\tilde{m}_{ij} = \begin{cases} 1 - d_i + d_i m_{ij}, & i = j, \\ \\ d_i m_{ij}, & i \neq j. \end{cases}$$

Hence, for each $i \in S$, $j \in \bar{S}$,

$$|\tilde{m}_{ii}| = 1 - d_i + d_i m_{ii} \geq d_i m_{ii} > d_i r_i^S(M) = r_i^S(\tilde{M}),$$

$$|\tilde{m}_{jj}| = 1 - d_j + d_j m_{jj} \geq d_i m_{ii} > d_i r_i^{\bar{S}}(M) = r_j^{\bar{S}}(\tilde{M}).$$

Then, for any $i \in S$, $j \in \bar{S}$, $d_i \in (0,1)$, we have

$$\begin{aligned}(|\tilde{m}_{ii}| - r_i^S(\tilde{M}))|\tilde{m}_{jj}| &= (d_i|m_{ii}| - d_i r_i^S(M))d_j|m_{jj}| \\ &= d_i d_j(|m_{ii}| - r_i^S(M))|m_{jj}| \\ &> d_i d_j r_i^{\bar{S}}(M) r_j(M) = r_i^{\bar{S}}(\tilde{M}) r_j(\tilde{M}).\end{aligned}$$

For any $i \in S$, $j \in \bar{S}$, we get

$$\begin{aligned}(|\tilde{m}_{jj}| - r_j^{\bar{S}}(\tilde{M}))|\tilde{m}_{ii}| &= (d_j|m_{jj}| - d_j r_j^{\bar{S}}(M))d_i|m_{ii}| \\ &= d_i d_j(|m_{jj}| - r_j^{\bar{S}}(M))|m_{ii}| \\ &> d_i d_j r_j^S(M) r_i(M) = r_j^S(\tilde{M}) r_i(\tilde{M}).\end{aligned}$$

When $d_i = 0$, $\tilde{m}_{ii} = 1 - d_i + d_i m_{ii} = 1$, we obtain

$$(|\tilde{m}_{ii}| - r_i^S(\tilde{M})))|\tilde{m}_{jj}| = 1 > 0 = r_j^{\bar{S}}(\tilde{M}) r_i(\tilde{M}),$$

$$(|\tilde{m}_{jj}| - r_j^{\bar{S}}(\tilde{M}))|\tilde{m}_{ii}| = 1 > 0 = r_i^S(\tilde{M}) r_j(\tilde{M}).$$

When $d_i = 1$, $\tilde{m}_{ij} = 1 - d_i + d_i m_{ij} = m_{ij}$, then

$$(|\tilde{m}_{ii}| - r_i^S(\tilde{M}))|\tilde{m}_{jj}| = (|m_{ii}| - r_i^S(M))|m_{jj}| > r_j^{\bar{S}}(M) r_i(M) = r_j^{\bar{S}}(\tilde{M}) r_i(\tilde{M}),$$

$$(|\tilde{m}_{jj}| - r_j^{\bar{S}}(\tilde{M}))|\tilde{m}_{ii}| = (|m_{jj}| - r_j^{\bar{S}}(M))|m_{ii}| > r_i^S(M) r_j(M) = r_i^S(\tilde{M}) r_j(\tilde{M}).$$

As $d_i \in [0,1]$, conditions (i)–(iv) in Definition 1 are fulfilled for all $i \in S$ and $j \in \bar{S}$. So the conclusion follows. □

Lemma 9 indicates that $\tilde{M}$ is an S-SOB matrix when M is an S-SOB matrix. We will present an error bound for the linear complementarity problem of S-SOB matrices. The following theorem is one of our main results, which gives an upper bound on the condition constant $\max_{d\in[0,1]^n} \|(I-D+DA)^{-1}\|_\infty$ when A is an S-SOB matrix.

Theorem 12. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an S-SOB matrix with positive diagonal entries, and $\tilde{A} = [\tilde{a_{ij}}] = I - D + DA$, where $D = diag(d_i)$ with $0 \le d_i \le 1$. Then*

$$\max_{d\in[0,1]^n} \|(I-D+DA)^{-1}\|_\infty \le \min_{i\in N} (1 + \max_{\substack{j\in N,\\ j\neq i}}\{\frac{|d_j a_{ji}|}{a_{ii}}, d_j a_{ji})(1 + \max_{\substack{j\in N,\\ j\neq i}}\{\frac{d_i a_{ij}}{a_{ii}}, d_i a_{ij}\}) \max\{\frac{1}{a_{ii}}, 1, \Delta(A), \Delta^{'}(A)\}$$

where

$$\begin{aligned}&\frac{1 + \frac{a_{ki}a_{ij}}{a_{ii}a_{kk}} + \sum_{\substack{p\in\bar{S},\\ p\neq i}} \frac{a_{jp}}{a_{jj}} + \frac{a_{jp}a_{ji}}{a_{ii}a_{jj}}}{\varsigma_j^S(A)\varsigma_j^{\bar{S}}(A) - (\sum \frac{a_{kp}}{a_{kk}} + \sum \frac{a_{ki}a_{ip}}{a_{ii}a_{kk}})(\frac{a_{jp}}{a_{jj}} + \sum \frac{a_{ji}a_{ip}}{a_{ii}a_{jj}})} \\ = \ & \Delta(A),\end{aligned}$$

$$\frac{1+\frac{a_{ji}a_{ik}}{a_{ii}a_{jj}}+\sum\frac{a_{kp}}{a_{kk}}+\frac{a_{kp}a_{ki}}{a_{ii}a_{kk}}}{\varsigma_k^S(A)\varsigma_k^{\bar{S}}(A)-(\sum\frac{a_{kp}}{a_{kk}}+\sum\frac{a_{ki}a_{ip}}{a_{ii}a_{kk}})(\frac{a_{jp}}{a_{jj}}+\sum\frac{a_{ji}a_{ip}}{a_{ii}a_{jj}})}$$
$$= \Delta'(A),$$

and $\varsigma_j^S(A) = \frac{1-d_j+d_ja_{jj}}{1-d_t+d_ta_{tt}} - \frac{a_{ji}a_{ij}}{a_{ii}a_{jj}} - \sum_{\substack{p\in S,\\ p\neq j,i}}\frac{a_{jk}}{a_{jj}} - \sum_{\substack{p\in S,\\ p\neq j,i}}\frac{a_{ji}a_{ik}}{a_{ii}a_{jj}}$.

Proof. Because $\tilde{A} = (\tilde{a_{ij}}) = (I - D + DA)$, we know $\tilde{A}$ is an S-SOB matrix with positive diagonal entries from Lemma 9. By Theorem 11, the following inequality holds

$$\|\tilde{A}\|_\infty \le \max\Gamma(\tilde{A}) = \min_{i\in N}\Gamma_i(\tilde{A}),$$

where $\Gamma_i(\tilde{A}) = (1+\frac{\max_{j\in N, j\neq i}|\tilde{a_{ji}}|}{|\tilde{a_{ii}}|})(1+\frac{\max_{j\in N, j\neq i}|\tilde{a_{ij}}|}{|\tilde{a_{ii}}|})\tilde{\Gamma}_i(\tilde{A})$,

$$\tilde{\Gamma}_i(\tilde{A}) = \max\{\frac{1}{|\tilde{a_{ii}}|}, \Gamma'(\tilde{A})\}.$$

$$\Gamma'(\tilde{A}) = \max\{\max_{\substack{j\in(S\setminus\{i\}),\\ k\in(\bar{S}\setminus\{i\})}}\frac{|\tilde{c_{kk}}| - \sum_{\substack{p\in\bar{S},\\ p\neq k,i}}|\tilde{c_{kp}}| + \sum_{\substack{p\in\bar{S},\\ p\neq i}}|\tilde{c_{jp}}|}{(|\tilde{c_{jj}}| - \sum_{\substack{p\in S,\\ p\neq j,i}}|\tilde{c_{jp}}|)(|\tilde{c_{kk}}| - \sum_{\substack{p\in\bar{S},\\ p\neq k,i}}|\tilde{c_{kp}}|) - \sum_{\substack{p\in S,\\ p\neq i}}|\tilde{c_{kp}}|\sum_{\substack{p\in\bar{S},\\ p\neq i}}|\tilde{c_{jp}}|},$$
$$\max_{\substack{j\in(S\setminus\{i\}),\\ k\in(\bar{S}\setminus\{i\})}}\frac{|\tilde{c_{jj}}| - \sum_{\substack{p\in S,\\ p\neq j,i}}|\tilde{c_{jp}}| + \sum_{\substack{p\in S,\\ p\neq i}}|c_{kp}|}{(|\tilde{c_{jj}}| - \sum_{\substack{p\in S,\\ p\neq j,i}}|\tilde{c_{jp}}|)(|\tilde{c_{kk}}| - \sum_{\substack{p\in\bar{S},\\ p\neq k,i}}|\tilde{c_{kp}}|) - \sum_{\substack{p\in S,\\ p\neq i}}|\tilde{c_{kp}}|\sum_{\substack{p\in\bar{S},\\ p\neq i}}|\tilde{c_{jp}}|}\},$$

and $\tilde{c_{jk}} = \tilde{a_{jk}} - \frac{\tilde{a_{ji}}\tilde{a_{ik}}}{\tilde{a_{ii}}}$.

Since $\tilde{A}$ is a S-SOB matrix, we have $\tilde{a_{ii}} = 1 - d_i + d_ia_{ii}$ and $\tilde{a_{ij}} = d_ia_{ij}$ for all $i, j \in N$.

$$1+\frac{\max_{j\in N, j\neq i}|\tilde{a_{ji}}|}{|\tilde{a_{ii}}|} = 1+\frac{\max_{j\in N, j\neq i}|d_ja_{ji}|}{1-d_i+d_ia_{ii}} \le 1+\frac{\max_{j\in N, j\neq i}|d_ja_{ji}|}{\min\{a_{ii},1\}} \quad (By\ \ Lemma\ \ 8)$$
$$= 1+\max_{\substack{j\in N,\\ j\neq i}}\{\frac{|d_ja_{ji}|}{a_{ii}}, d_ja_{ji}\}. \tag{4.2}$$

Similarly, we have

$$1+\frac{\max_{j\in N, j\neq i}|\tilde{a_{ij}}|}{|\tilde{a_{ii}}|} \le 1+\max\{\frac{d_ia_{ij}}{a_{ii}}, d_ia_{ij}\}. \tag{4.3}$$

By Lemma 8, it is easy to get

$$\frac{1}{\tilde{a_{ii}}} = \frac{1}{1-d_i+d_ia_{ii}} \le \max\{\frac{1}{a_{ii}}, 1\}. \tag{4.4}$$

Denote $1 - d_t + d_t a_{tt} = \max_{i\in N}\{1 - d_i + d_i a_{ii}\}$. From Lemmas 8 and 9, we get

$$
\begin{aligned}
&\frac{|\tilde{c_{kk}}| - \sum\limits_{\substack{p\in\bar{S},\\ p\neq k,i}} |\tilde{c_{kp}}| + \sum\limits_{\substack{p\in\bar{S},\\ p\neq i}} |\tilde{c_{jp}}|}{(|\tilde{c_{jj}}| - \sum\limits_{\substack{p\in S,\\ p\neq j,i}} |\tilde{c_{jp}}|)(|\tilde{c_{kk}}| - \sum\limits_{\substack{p\in\bar{S},\\ p\neq k,i}} |\tilde{c_{kp}}|) - \sum\limits_{\substack{p\in S,\\ p\neq i}} |\tilde{c_{kp}}| \sum\limits_{\substack{p\in\bar{S},\\ p\neq i}} |\tilde{c_{jp}}|} \\
\leq\ & \frac{1 + \frac{a_{ki}a_{ij}}{a_{ii}a_{kk}} + \sum_{\substack{p\in\bar{S},\\ p\neq i}} \frac{a_{jp}}{a_{jj}} + \sum_{\substack{p\in\bar{S},\\ p\neq i}} \frac{a_{jp}a_{ji}}{a_{ii}a_{jj}}}{\varsigma_j^S(A)\varsigma_j^{\bar{S}} - (\sum\limits_{\substack{p\in S,\\ p\neq i}} \frac{a_{kp}}{a_{kk}} + \sum\limits_{\substack{p\in S,\\ p\neq i}} \frac{a_{ki}a_{ip}}{a_{ii}a_{kk}})(\sum\limits_{\substack{p\in\bar{S},\\ p\neq i}} \frac{a_{jp}}{a_{jj}} + \sum\limits_{\substack{p\in\bar{S},\\ p\neq i}} \frac{a_{ji}a_{ip}}{a_{ii}a_{jj}})} \\
=\ & \Delta(A),
\end{aligned}
\tag{4.5}
$$

where $\varsigma_j^S(A) = \frac{1-d_j+d_j a_{jj}}{1-d_t+d_t a_{tt}} - \frac{a_{ji}a_{ij}}{a_{ii}a_{jj}} - \sum\limits_{\substack{p\in S,\\ p\neq j,i}} \frac{a_{jk}}{a_{jj}} - \sum\limits_{\substack{p\in S,\\ p\neq j,i}} \frac{a_{ji}a_{ik}}{a_{ii}a_{jj}}$. In similar way, we know

$$
\begin{aligned}
&\frac{|\tilde{c_{jj}}| - \sum\limits_{\substack{p\in\bar{S},\\ p\neq k,i}} |\tilde{c_{jp}}| + \sum\limits_{\substack{p\in\bar{S},\\ p\neq i}} |\tilde{c_{kp}}|}{(|\tilde{c_{jj}}| - \sum\limits_{\substack{p\in S,\\ p\neq j,i}} |\tilde{c_{jp}}|)(|\tilde{c_{kk}}| - \sum\limits_{\substack{p\in\bar{S},\\ p\neq k,i}} |\tilde{c_{kp}}|) - \sum\limits_{\substack{p\in S,\\ p\neq i}} |\tilde{c_{kp}}| \sum\limits_{\substack{p\in\bar{S},\\ p\neq i}} |\tilde{c_{jp}}|} \\
\leq\ & \frac{1 + \frac{a_{ji}a_{ik}}{a_{ii}a_{jj}} + \sum\limits_{\substack{p\in\bar{S},\\ p\neq k,i}} \sum\limits_{\substack{p\in\bar{S},\\ p\neq k,i}} \frac{a_{kp}}{a_{kk}} + \frac{a_{kp}a_{ki}}{a_{ii}a_{kk}}}{\varsigma_k^S\varsigma_k^{\bar{S}} - (\sum\limits_{\substack{p\in S,\\ p\neq i}} \frac{a_{kp}}{a_{kk}} + \sum\limits_{\substack{p\in S,\\ p\neq i}} \frac{a_{ki}a_{ip}}{a_{ii}a_{kk}})(\sum\limits_{\substack{p\in\bar{S},\\ p\neq i}} \frac{a_{jp}}{a_{jj}} + \sum\limits_{\substack{p\in\bar{S},\\ p\neq i}} \frac{a_{ji}a_{ip}}{a_{ii}a_{jj}})} \\
=\ & \Delta^{'}(A).
\end{aligned}
\tag{4.6}
$$

So, from Eqs (4.2)–(4.6) the conclusion follows. This proof is completed. □

5. Conclusions

Based on the fact that the Schur complement of the S-SOB matrix is a $GDSDD$ matrix, we give an infinity norm bound for the inverse of the S-SOB matrix based on the Schur complement. By using the infinity norm bound for the inverse of the S-SOB matrix, an error bound is given for the linear complementarity problem of the S-SOB matrix.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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