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Research article

Degenerate r-truncated Stirling numbers

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Abstract: For any positive integer r, the r-truncated (or r-associated) Stirling number of the second kind $S_2^{(r)}(n,k)$ enumerates the number of partitions of the set $\{1,2,3,\dots,n\}$ into k non-empty disjoint subsets, such that each subset contains at least r elements. We introduce the degenerate r-truncated Stirling numbers of the second kind and of the first kind. They are degenerate versions of the r-truncated Stirling numbers of the second kind and of the first kind, and reduce to the degenerate Stirling numbers of the second kind and of the first kind for $r = 1$. Our aim is to derive recurrence relations for both of those numbers.

Keywords: degenerate r-associated Stirling numbers of the second kind; degenerate r-truncated Stirling numbers of the first kind; degenerate r-truncated Bell polynomials
Mathematics Subject Classification: 11B73, 11B83

1. Introduction and preliminaries

Explorations for the degenerate versions of some special numbers and polynomials have become lively interests for some mathematicians in recent years, which began from the pioneering work of Carlitz (see [1,2]). These have been done by employing various methods, such as generating functions, combinatorial methods, p-adic analysis, umbral calculus, operator theory, differential equations, special functions, probability theory and analytic number theory (see [5,9–13,16,17] and the references therein).

The Stirling number of the second kind $S_2(n,k)$ enumerates the number of partitions of the set $[n] = \{1,2,\dots,n\}$ into k nonempty disjoint sets, while the r-truncated(or r-associated) Stirling number of the second kind $S_2^{(r)}(n,k)$ counts the number of partitions of the set $[n]$ into k non-empty disjoint subsets, such that each subset contains at least r elements, for any positive integer r. The reader refers to [7] for further details on the r-associated Stirling numbers of the first kind and of the second kind.

Our aim is to introduce the degenerate r-truncated Stirling numbers of the second kind and the first kind and to derive their recurrence relations. They are degenerate versions of the r-truncated Stirling numbers of the second kind and of the first kind, and reduce to the degenerate Stirling numbers of the second kind and of the first kind for $r = 1$. Here, we mention that the degenerate Stirling numbers of both kinds appear very frequently when one studies various degenerate versions of some special numbers and polynomials.

The outline of this paper is as follows. In Section 1, we recall the definitions and recurrence relations of the classical, the degenerate and the r-truncated Stirling numbers of both kinds. Section 2 is the main result of this paper. We introduce the degenerate r-truncated Stirling numbers of the second kind and express the degenerate r-truncated Bell polynomials in terms of the degenerate r-truncated Stirling numbers of the second kind. In Theorem 2, we derive a recurrence relation for the degenerate Stirling numbers of the second kind. We also introduce the degenerate r-truncated Stirling numbers of the first kind and deduce a recurrence relation for those numbers in Theorem 4. For the rest of this section, we recall the facts that are needed throughout this paper.

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponentials are defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \frac{(x)_{n,\lambda}}{n!} t^n, \quad e_\lambda(t) = e_\lambda^{(1)}(t), \quad \text{(see [1,2])}, \tag{1.1}$$

where

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda), \quad (n \geq 1), \quad \text{(see [10,11])}. \tag{1.2}$$

Let $\log_\lambda t$ be the compositional inverse of $e_\lambda(t)$, called the degenerate logarithm, such that $\log_\lambda(e_\lambda(t)) = e_\lambda(\log_\lambda(t)) = t$.

Then, we have

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty} \frac{(1)_{n,1/\lambda}\lambda^{n-1}}{n!} t^n, \quad \text{(see [9])}. \tag{1.3}$$

Note that $\lim_{\lambda\to 0} e_\lambda(t) = e^t$, $\lim_{\lambda\to 0} \log_\lambda(1+t) = \log(1+t)$.

For $k \geq 0$, the Stirling numbers of the first kind are defined by

$$\frac{1}{k!}\big(\log(1+t)\big)^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \quad (k \geq 0), \quad \text{(see [3, 13, 14, 17])}. \tag{1.4}$$

The Stirling numbers of the second kind are given by

$$\frac{1}{k!}(e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad \text{(see [5, 6, 18, 19, 20])}. \tag{1.5}$$

From (1.4) and (1.5), we have for $n, k \geq 0$, with $n \geq k$ (see [10,15,18])

$$\begin{aligned} S_1(n+1,k) &= S_1(n,k-1) - nS_1(n,k), \\ S_2(n+1,k) &= S_2(n,k-1) + kS_2(n,k). \end{aligned} \tag{1.6}$$

Recently, the degenerate Stirling numbers of the first kind and of the second kind were respectively defined by

$$\frac{1}{k!}\big(\log_\lambda(1+t)\big)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}, \tag{1.7}$$

and

$$\frac{1}{k!}(e_\lambda(t)-1)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!},\quad \text{(see [9])}, \tag{1.8}$$

where k is a non-negative integer.

By (1.7) and (1.8), we get

$$\begin{aligned} S_{1,\lambda}(n+1,k)&=S_{1,\lambda}(n,k-1)+(k\lambda-n)S_{1,\lambda}(n,k),\\ S_{2,\lambda}(n+1,k)&=S_{2,\lambda}(n,k-1)+(k-n\lambda)S_{2,\lambda}(n,k), \end{aligned} \tag{1.9}$$

where $n,k\geq 0$ with $n\geq k$ (see [9]).

Note that $\lim_{\lambda\to 0}S_{1,\lambda}(n,k)=S_1(n,k),\ \lim_{\lambda\to 0}S_{2,\lambda}(n,k)=S_2(n,k)$.

The generating function of the r-truncated Stirling numbers of the second kind is given by

$$\frac{1}{k!}\left(e^t-\sum_{l=0}^{r-1}\frac{t^l}{l!}\right)^k=\sum_{n=rk}^{\infty}S_2^{(r)}(n,k)\frac{t^n}{n!},\quad \text{(see [4, 6, 7, 8])}. \tag{1.10}$$

Thus, by (1.10), we obtain the recursion formula of $S_2^{(r)}(n,k)$, for $n,k\geq 0$ with $n\geq rk$, which is given by

$$S_2^{(r)}(n,k)=kS_2^{(r)}(n-1,k)+\binom{n-1}{r-1}S_2^{(r)}(n-r,k-1), \tag{1.11}$$

with the initial condition $S_2^{(r)}(n,k)=0$ if $n<kr$ and $S_2^{(r)}(n,k)=\frac{(rk)!}{k!(r!)^k}$ if $n=rk$.

2. Degenerate r-truncated (or r-associated) Stirling numbers

As degenerate versions of the r-truncated Stirling numbers of the second kind, we consider the degenerate r-truncated Stirling numbers of the second kind given by

$$\frac{1}{k!}\left(e_\lambda(t)-\sum_{l=0}^{r-1}\frac{(1)_{l,\lambda}}{l!}t^l\right)^k=\sum_{n=kr}^{\infty}S_{2,\lambda}^{(r)}(n,k)\frac{t^n}{n!}, \tag{2.1}$$

where $r\in\mathbb{N}$ and k is a non-negative integer.

From (2.1), we note that

$$\begin{aligned} S_{2,\lambda}^{(r)}(n,k)&=\frac{1}{k!}\sum_{\substack{l_1+\cdots+l_k=n\\ l_i\geq r}}\frac{n!(1)_{l_1,\lambda}(1)_{l_2,\lambda}\cdots(1)_{l_k,\lambda}}{l_1!l_2!\cdots l_k!}\\ &=\frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}(-1)^m\sum_{l_1,l_2,\dots,l_m=0}^{r-1}\frac{n!\left(\prod_{i=1}^{m}(1)_{l_j,\lambda}\right)(k-m)_{n-l_1-l_2-\cdots-l_m,\lambda}}{l_1!l_2!\cdots l_m!(n-l_1-l_2-\cdots-l_m)!}, \end{aligned} \tag{2.2}$$

where $n\geq kr\geq 0,\ k,r\geq 0$.

Note that $S_{2,\lambda}^{(r)}(n,k)=0$ if $n<kr$ and $S_{2,\lambda}^{(r)}(rk,k)=\frac{(rk)!((1)_{r,\lambda})^k}{k!(r!)^k}$.

We define the degenerate r-truncated Bell polynomials as

$$e^{x\left(e_\lambda(t)-\sum_{l=0}^{r-1}\frac{(1)_{l,\lambda}}{l!}t^l\right)}=\sum_{n=0}^{\infty}\phi_{n,\lambda}^{(r)}(x)\frac{t^n}{n!},\ (r\in\mathbb{N}). \tag{2.3}$$

Thus, by (2.1) and (2.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}&=\sum_{k=0}^{\infty}x^k\frac{1}{k!}\Big(e_\lambda(t)-\sum_{l=0}^{r-1}\frac{(1)_{l,\lambda}}{l!}t^l\Big)^k\\&=\sum_{k=0}^{\infty}x^k\sum_{n=kr}^{\infty}S_{2,\lambda}^{(r)}(n,k)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{r}]}x^kS_{2,\lambda}^{(r)}(n,k)\frac{t^n}{n!},\end{aligned}$$

where $[x]$ denotes the greatest integer not exceeding x. Therefore, we obtain the following theorem.

Theorem 1. *For any integers n, r with $n \geq 0, r \geq 1$, we have*

$$\phi_{n,\lambda}^{(r)}(x)=\sum_{k=0}^{[\frac{n}{r}]}x^kS_{2,\lambda}^{(r)}(n,k).$$

Now, we want to find a recursion formula for the degenerate r-truncated Stirling numbers of the second kind.

Taking the derivative with respect to t on both sides of (2.1), we obtain

$$\begin{aligned}\sum_{n=kr-1}^{\infty}S_{2,\lambda}^{(r)}(n+1,k)\frac{t^n}{n!}&=\sum_{n=kr}^{\infty}S_{2,\lambda}^{(r)}(n,k)\frac{t^{n-1}}{(n-1)!}\\&=\frac{d}{dt}\frac{1}{k!}\Big(e_\lambda(t)-\sum_{l=0}^{r-1}\frac{(1)_{l,\lambda}}{l!}t^l\Big)^k.\end{aligned}\tag{2.4}$$

Here we note that

$$\begin{aligned}&\frac{d}{dt}\frac{1}{k!}\Big(e_\lambda(t)-\sum_{l=0}^{r-1}\frac{(1)_{l,\lambda}}{l!}t^l\Big)^k\\&=\frac{k}{k!}\Big(e_\lambda(t)-\sum_{l=0}^{r-1}\frac{(1)_{l,\lambda}}{l!}t^l\Big)^{k-1}\Big(e_\lambda^{1-\lambda}(t)-\sum_{l=1}^{r-1}\frac{(1)_{l,\lambda}}{l!}lt^{l-1}\Big)\\&=\frac{1}{(k-1)!}\Big(e_\lambda(t)-\sum_{l=0}^{r-1}\frac{(1)_{l,\lambda}}{l!}t^l\Big)^{k-1}\Big(e_\lambda(t)-(1+\lambda t)\sum_{l=1}^{r-1}\frac{(1)_{l,\lambda}}{(l-1)!}t^{l-1}\Big)\frac{1}{1+\lambda t}.\end{aligned}\tag{2.5}$$

For the derivation of (2.6), we introduce the following notation

$$E_{\lambda,k-1}^{r}=\frac{1}{(k-1)!}\Big(e_\lambda(t)-\sum_{l=0}^{r-1}\frac{(1)_{l,\lambda}}{l!}t^l\Big)^{k-1}.$$

Then, by (2.4) and (2.5), we get

$$\sum_{n=kr-1}^{\infty}\Big\{S_{2,\lambda}^{(r)}(n+1,k)+n\lambda S_{2,\lambda}^{(r)}(n,k)\Big\}\frac{t^n}{n!}=(1+\lambda t)\sum_{n=kr-1}^{\infty}S_{2,\lambda}^{(r)}(n+1,k)\frac{t^n}{n!}\tag{2.6}$$

$$
\begin{aligned}
&= E^{r}_{\lambda,k-1}\Big(e_{\lambda}(t) - \sum_{l=0}^{r-2}\frac{(1)_{l+1,\lambda}}{l!}t^{l} - \lambda\sum_{l=1}^{r-1}\frac{(1)_{l,\lambda}}{(l-1)!}t^{l}\Big)\\
&= E^{r}_{\lambda,k-1}\Big(e_{\lambda}(t) - \sum_{l=0}^{r-2}\frac{(1)_{l,\lambda}}{l!}(1-\lambda l)t^{l} - \lambda\sum_{l=1}^{r-1}\frac{(1)_{l,\lambda}}{(l-1)!}t^{l}\Big)\\
&= E^{r}_{\lambda,k-1}\Big(e_{\lambda}(t) - \sum_{l=0}^{r-1}\frac{(1)_{l,\lambda}}{l!}t^{l}\Big) + E^{r}_{\lambda,k-1}\Big(\frac{1}{(r-1)!}(1)_{r-1,\lambda}t^{r-1}\Big)\\
&+ E^{r}_{\lambda,k-1}\Big(\lambda\sum_{l=0}^{r-2}\frac{(1)_{l,\lambda}}{l!}lt^{l} - \lambda\sum_{l=0}^{r-1}\frac{(1)_{l,\lambda}}{l!}lt^{l}\Big)\\
&= \frac{k}{k!}\Big(e_{\lambda}(t) - \sum_{l=0}^{r-1}\frac{(1)_{l,\lambda}}{l!}t^{l}\Big)^{k} + E^{r}_{\lambda,k-1}\Big(\frac{(1)_{r-1,\lambda}t^{r-1}}{(r-1)!}\Big)\\
&- \lambda E^{r}_{\lambda,k-1}\Big(\frac{(1)_{r-1,\lambda}}{(r-1)!}(r-1)t^{r-1}\Big)\\
&= k\sum_{n=kr}^{\infty}S^{(r)}_{2,\lambda}(n,k)\frac{t^{n}}{n!} + \sum_{n=r(k-1)}^{\infty}S^{(r)}_{2,\lambda}(n,k-1)\frac{t^{n}}{n!}\frac{(1)_{r-1,\lambda}}{(r-1)!}t^{r-1}\\
&- \lambda\frac{(r-1)(1)_{r-1,\lambda}}{(r-1)!}\sum_{n=r(k-1)}^{\infty}S^{(r)}_{2,\lambda}(n,k-1)\frac{t^{n}}{n!}t^{r-1}
\end{aligned}
$$

$$
\begin{aligned}
= \sum_{n=kr-1}^{\infty}\Big\{&kS^{(r)}_{2,\lambda}(n,k) + (1)_{r-1,\lambda}\binom{n}{r-1}S^{(r)}_{2,\lambda}(n-r+1,k-1)\\
&- \lambda(r-1)(1)_{r-1,\lambda}\binom{n}{r-1}S^{(r)}_{2,\lambda}(n-r+1,k-1)\Big\}\frac{t^{n}}{n!}.
\end{aligned}
$$

Comparing the coefficients on both sides of (2.6), we have the following theorem.

Theorem 2. *For $n,k \geq 0$ with $n \geq kr-1$, we have*

$$
\begin{aligned}
S^{(r)}_{2,\lambda}(n+1,k) = (k-n\lambda)S^{(r)}_{2,\lambda}(n,k) &+ (1)_{r-1,\lambda}\binom{n}{r-1}S^{(r)}_{2,\lambda}(n-r+1,k-1)\\
&- \lambda(r-1)(1)_{r-1,\lambda}\binom{n}{r-1}S^{(r)}_{2,\lambda}(n-r+1,k-1).
\end{aligned}
$$

Corollary 3. *If $r = 1$ in Theorem 2, then we have*

$$
S^{(1)}_{2,\lambda}(n+1,k) = (k-n\lambda)S^{(1)}_{2,\lambda}(n,k) + S^{(1)}_{2,\lambda}(n,k-1), \tag{2.7}
$$

where $n,k \geq 0$ with $n \geq k-1$. So our result agrees with the fact in (1.9)*, as $S^{(1)}_{2,\lambda}(n,k) = S_{2,\lambda}(n,k)$.*

Now, we define the degenerate r-truncated Stirling numbers of the first kind as

$$
\frac{1}{k!}\Big(\log_{\lambda}(1+t) - \sum_{l=1}^{r-1}\frac{(1)_{l,1/\lambda}\lambda^{l-1}}{l!}t^{l}\Big)^{k} = \sum_{n=kr}^{\infty}S^{(r)}_{1,\lambda}(n,k)\frac{t^{n}}{n!}, \tag{2.8}
$$

where k is a nonnegative integer and $r \geq 1$.

Taking the derivative with respect to t on both sides of (2.8), we get

$$\sum_{n=kr-1}^{\infty} S_{1,\lambda}^{(r)}(n+1,k)\frac{t^n}{n!} = \sum_{n=kr}^{\infty} S_{1,\lambda}^{(r)}(n,k)\frac{t^{n-1}}{(n-1)!} \tag{2.9}$$
$$= \frac{d}{dt}\frac{1}{k!}\bigg(\log_\lambda(1+t) - \sum_{l=1}^{r-1}\frac{(1)_{l,1/\lambda}}{l!}\lambda^{l-1}t^l\bigg)^k.$$

Here we observe that

$$\frac{d}{dt}\frac{1}{k!}\bigg(\log_\lambda(1+t) - \sum_{l=1}^{r-1}\frac{(1)_{l,1/\lambda}}{l!}\lambda^{l-1}t^l\bigg)^k \tag{2.10}$$
$$= \frac{k}{k!}\bigg(\log_\lambda(1+t) - \sum_{l=1}^{r-1}\frac{(1)_{l,1/\lambda}}{l!}\lambda^{l-1}t^l\bigg)^{k-1}\bigg(\frac{(1+t)^\lambda}{1+t} - \sum_{l=1}^{r-1}\frac{(1)_{l,1/\lambda}}{(l-1)!}\lambda^{l-1}t^{l-1}\bigg).$$

For the derivation of (2.11), we introduce the following notation

$$L_{\lambda,k-1}^{r} = \frac{1}{(k-1)!}\bigg(\log_\lambda(1+t) - \sum_{l=1}^{r-1}\frac{(1)_{l,1/\lambda}}{l!}\lambda^{l-1}t^l\bigg)^{k-1}.$$

Then, by (2.9) and (2.10), we get

$$\sum_{n=kr-1}^{\infty} S_{1,\lambda}^{(r)}(n+1,k)\frac{t^n}{n!}(1+t) \tag{2.11}$$
$$= L_{\lambda,k-1}^{r}\bigg((1+t)^\lambda - \sum_{l=1}^{r-1}\frac{(1)_{l,1/\lambda}}{(l-1)!}\lambda^{l-1}t^{l-1}(1+t)\bigg)$$
$$= L_{\lambda,k-1}^{r}\bigg((1+t)^\lambda - \sum_{l=0}^{r-2}\frac{(1)_{l+1,1/\lambda}}{l!}\lambda^{l}t^{l} - \sum_{l=1}^{r-1}\frac{(1)_{l,1/\lambda}}{l!}l\lambda^{l-1}t^l\bigg)$$
$$= L_{\lambda,k-1}^{r}\bigg((1+t)^\lambda - \sum_{l=0}^{r-2}(1)_{l,1/\lambda}\Big(1-\frac{l}{\lambda}\Big)\lambda^{l}\frac{t^{l}}{l!} - \sum_{l=1}^{r-1}\frac{(1)_{l,1/\lambda}}{l!}\lambda^{l-1}lt^l\bigg)$$
$$= L_{\lambda,k-1}^{r}\bigg((1+t)^\lambda - 1 - \lambda\sum_{l=1}^{r-2}\frac{\lambda^{l-1}(1)_{l,1/\lambda}}{l!}t^l + \sum_{l=0}^{r-2}(1)_{l,1/\lambda}\lambda^{l-1}l\frac{t^l}{l!}\bigg)$$
$$- L_{\lambda,k-1}^{r}\bigg(\sum_{l=1}^{r-1}\frac{(1)_{l,1/\lambda}}{l!}\lambda^{l-1}lt^l\bigg)$$
$$= \lambda L_{\lambda,k-1}^{r}\bigg(\log_\lambda(1+t) - \sum_{l=1}^{r-1}\frac{(1)_{l,1/\lambda}}{l!}\lambda^{l-1}t^l\bigg) + \lambda^{r-1}\frac{(1)_{r-1,1/\lambda}}{(r-1)!}L_{\lambda,k-1}^{r}t^{r-1}$$
$$- \frac{(r-1)(1)_{r-1,1/\lambda}}{(r-1)!}\lambda^{r-2}L_{\lambda,k-1}^{r}t^{r-1}$$

$$
\begin{aligned}
&= k\lambda \sum_{n=kr}^{\infty} S_{1,\lambda}^{(r)}(n,k)\frac{t^n}{n!} + \lambda^{r-1}(1)_{r-1,1/\lambda} \sum_{n=kr-1}^{\infty} \binom{n}{r-1} S_{1,\lambda}^{(r)}(n-r+1,k-1)\frac{t^n}{n!} \\
&\quad - (r-1)(1)_{r-1,1/\lambda}\lambda^{r-2} \sum_{n=kr-1}^{\infty} \binom{n}{r-1} S_{1,\lambda}^{(r)}(n-r+1,k-1)\frac{t^n}{n!} \\
&= \sum_{n=kr-1}^{\infty} \bigg\{ k\lambda S_{1,\lambda}^{(r)}(n,k) + \lambda^{r-1}(1)_{r-1,1/\lambda}\binom{n}{r-1} S_{1,\lambda}^{(r)}(n-r+1,k-1) \\
&\quad - (r-1)(1)_{r-1,1/\lambda}\lambda^{r-2}\binom{n}{r-1} S_{1,\lambda}^{(r)}(n-r+1,k-1) \bigg\} \frac{t^n}{n!}.
\end{aligned}
$$

On the other hand, by simple calculation, we get

$$
\begin{aligned}
&\sum_{n=kr-1}^{\infty} S_{1,\lambda}^{(r)}(n+1,k)\frac{t^n}{n!}(1+t) \\
&= \sum_{n=kr-1}^{\infty} S_{1,\lambda}^{(r)}(n+1,k)\frac{t^n}{n!} + \sum_{n=kr}^{\infty} S_{1,\lambda}^{(r)}(n,k) n \frac{t^n}{n!} \\
&= \sum_{n=kr-1}^{\infty} (S_{1,\lambda}^{(r)}(n+1,k) + nS_{1,\lambda}^{(r)}(n,k))\frac{t^n}{n!}.
\end{aligned} \tag{2.12}
$$

From (2.11) and (2.12), we obtain the following theorem.

Theorem 4. *Let $r \in \mathbb{N}$ with $r \geq 1$. Then, for $n, k \geq 0$ with $n \geq kr - 1$, we have*

$$
\begin{aligned}
&S_{1,\lambda}^{(r)}(n+1,k) + nS_{1,\lambda}^{(r)}(n,k) \\
&\quad = k\lambda S_{1,\lambda}^{(r)}(n,k) + (\lambda - r + 1)(1)_{r-1,1/\lambda}\lambda^{r-2}\binom{n}{r-1} S_{1,\lambda}^{(r)}(n-r+1,k-1).
\end{aligned}
$$

Corollary 5. *If $r = 1$ in Theorem 4, then we have*

$$
S_{1,\lambda}^{(1)}(n+1,k) = (k\lambda - n)S_{1,\lambda}^{(1)}(n,k) + S_{1,\lambda}^{(1)}(n,k-1), \tag{2.13}
$$

where $n, k \geq 0$ with $n \geq k - 1$. So our result agrees with the fact in (1.9)*, as $S_{1,\lambda}^{(1)}(n,k) = S_{1,\lambda}(n,k)$.*

3. Conclusions

In recent years, studying degenerate versions of some special numbers and polynomials have drawn the attention of many mathematicians with their regained interests, not only in combinatorial and arithmetical properties but also in applications to differential equations, identities of symmetry and probability theory.

In this paper, we introduced the degenerate r-truncated Stirling numbers of the second kind and the first kind and derived their recurrence relations. They are degenerate versions of the r-truncated Stirling numbers of the second kind and the first kind, and reduce to the degenerate Stirling numbers of the second kind and the first kind for $r = 1$.

As one of our future research projects, we would like to continue to explore degenerate versions of some special numbers and polynomials and their applications to physics, science and engineering.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Taekyun Kim and Dae San Kim are the Guest Editors of special issue "Number theory, combinatorics and their applications: theory and computation" for AIMS Mathematics. Taekyun Kim and Dae San Kim were not involved in the editorial review and the decision to publish this article.

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