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MORTENSEN OBSERVER FOR A CLASS OF VARIATIONAL INEQUALITIES – LOST EQUIVALENCE WITH STOCHASTIC FILTERING APPROACHES

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Abstract. We address the problem of deterministic sequential estimation for a nonsmooth dynamics governed by a variational inequality. An example of such dynamics is the Skorokhod problem with a reflective boundary condition. For smooth dynamics, Mortensen introduced in 1968 a nonlinear estimator based on likelihood maximisation. Then, starting with Hijab in 1980, several authors established a connection between Mortensen's approach and the vanishing noise limit of the robust form of the so-called Zakai equation. In this paper, we investigate to what extent these methods can be developed for dynamics governed by a variational inequality. On the one hand, we address this problem by relaxing the inequality constraint by penalization: this yields an approximate Mortensen estimator relying on an approximating smooth dynamics. We verify that the equivalence between the deterministic and stochastic approaches holds through a vanishing noise limit. On the other hand, inspired by the smooth dynamics approach, we study the vanishing viscosity limit of the Hamilton-Jacobi equation satisfied by the Hopf-Cole transform of the solution of the robust Zakai equation. In contrast to the case of smooth dynamics, the zero-noise limit of the robust form of the Zakai equation cannot be understood in our case from the Bellman equation on the value function arising in Mortensen's procedure. This unveils a violation of equivalence for dynamics governed by a variational inequality between the Mortensen approach and the low noise stochastic approach for nonsmooth dynamics.

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Résumé. Nous abordons le problème de l'estimation séquentielle déterministe pour une dynamique non lisse régie par une inégalité variationnelle. Un exemple d'une telle dynamique est le problème de Skorokhod avec une condition aux limites réflexive. Pour les dynamiques lisses, Mortensen a introduit en 1968 un estimateur non linéaire fondé sur un principe du maximum de vraisemblance. Puis, à partir de Hijab en 1980, plusieurs auteurs ont établi un lien entre l'approche de Mortensen et la limite de faible bruit de la forme robuste de l'équation dite de Zakai. Dans cet article, nous étudions dans quelle mesure ces méthodes peuvent être développées pour des dynamiques gouvernées par une inégalité variationnelle. D'une part, nous abordons ce problème en relâchant la contrainte d'inégalité par pénalisation : cela donne un estimateur approché de Mortensen reposant sur une dynamique lisse approchée. Nous vérifions que l'équivalence entre les approches déterministe et stochastique est vérifiée quand le bruit tend vers 0. D'autre part, en s'inspirant de du cadre de la dynamique lisse, nous étudions la limite de viscosité évanescente de l'équation de Hamilton-Jacobi satisfaite par la transformée de Hopf-Cole de la solution de l'équation robuste de Zakai. Contrairement au cas de la dynamique lisse, la limite de bruit nul de la forme robuste de l'équation de Zakai ne peut pas être comprise dans notre cas à partir de l'équation de Bellman sur la fonction de valeur résultant de la procédure de Mortensen. Ceci suggère une violation d'équivalence pour la dynamique régie par une inégalité variationnelle entre l'approche de Mortensen et l'approche stochastique à faible bruit pour la dynamique non lisse.

Contents

1. Problem setting

In this paper, we consider the problem of estimating the deterministic state resulting from a nonsmooth dynamical system given an observation. The system state is the solution of a variational inequality, and both the state dynamics and the observation are subjected to disturbances. We aim at finding the "best" deterministic estimate of the state from the observation. The state estimation of various fundamental examples motivates our problem. These include: a) elasto-plasticity (transition from elastic and plastic phases) [14], b) dry friction (transition from static and dynamic phases) [6] or c) impacts (switch of velocity at the instant of contact with an obstacle) [5]. The state variable in these models is non-differentiable at the transition from one phase to another, and variational inequalities are well-suited to describe such situations

As a an example of simple representative nonsmooth dynamics, we study the Skorokhod problem with a reflective boundary condition at 0. We then consider the $\mathbb{R}_+$-valued state variable $x = (x(t))_{t\in[0,T]}$ solution of the variational inequality (VI)

$$\text{for a.e. } t \in [0,T],\ \forall z \geq 0,\ (f(x(t)) + \omega(t) - \dot{x}(t))(z - x(t)) \leq 0, \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function from $\mathbb{R}$ to $\mathbb{R}$, and the state disturbance $\omega : [0,T] \to \mathbb{R}$ is a square integrable function. The map $t \mapsto x(t)$ is continuous and differentiable almost everywhere. Adequate conditions of existence and uniqueness for this classical system are stated in [7]. When $f \equiv 0$, x is solution of the deterministic Skorokhod problem [35, p.231]. Given $\zeta \in \mathbb{R}_+$, the deterministic Skorokhod problem is to find a pair (x, k) satisfying the following four conditions: 1) x is a non negative continuous function with given initial value ζ at $t = 0$, 2) k is a continuous non increasing function vanishing at 0, 3) $x(t) + k(t) = \zeta + \int_0^t \omega(s)\mathrm{d}s$ and 4) k varies only when $x = 0$. For this simple constrained dynamics, the solution is explicit:

$$x(t) + k(t) = \zeta + \int_0^t \omega(s)\,\mathrm{d}s \ \text{ where } \ k(t) := \min_{0\leq s\leq t} \min(0\,;\,\zeta + \int_0^s \omega(\tau)\,\mathrm{d}\tau).$$

Figure 1 illustrates the above trajectory.

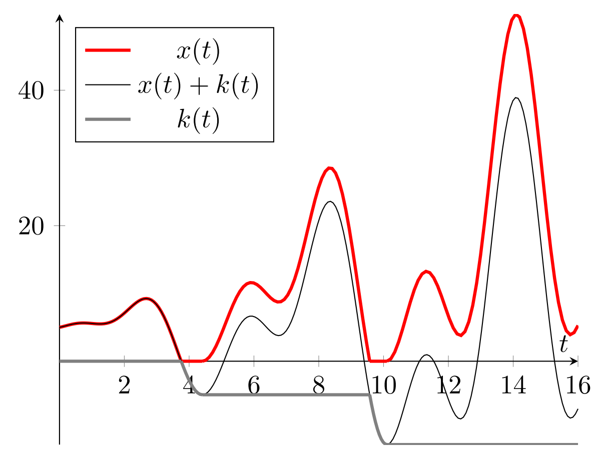

FIGURE 1. Example of a trajectory with an oscillating $\int_0^t \omega(s)ds$. $\zeta + \int_0^t \omega(s)ds$ is represented in black, k in gray and x in red. The "upward push" k ensures that the resulting state variable x stays positive. Pushes occur only when $x = 0$.

To link the dynamics (1) to the available observation, we model the measurement procedure using an observation map $h \in \mathrm{C}^2(\mathbb{R}_+, \mathbb{R})$. The observation is related to dynamics (1) by

$$\forall t \geq 0, \quad \dot{y}(t) = h(x(t)) + \eta(t), \tag{2}$$

where $\eta(t) \in \mathbb{R}$ is the observation disturbance. We follow the usual convention in stochastic filtering, denoting the left hand side (lhs) of (2) by $\dot{y}$. However, in most deterministic observation problems, the lhs of (2) is denoted by y. The interest of the filtering convention will become clear when connecting deterministic and stochastic settings. In the deterministic framework, we introduce the notation $\{\check{x}(t)\}_{t\geq 0}$ to denote the state trajectory (intended to correspond to the actual behavior of a real system). We refer to it as the (partially) observed trajectory, since from it, a measurement procedure produces the observation $\{\dot{y}(t)\}_{t>0}$. From now on we consider that observations are fixed. In this setting, $\eta : t \mapsto \dot{y}(t) - h(\breve{x}(t))$ is a measurement error. Both the state and observation disturbances are unknown but we will assume that they are small in L^2 norm, as detailed below.

Based on the available information $\{\dot{y}(t)\}_{t\geq 0}$, we aim at designing a causal estimator – also called observer – of the partially observed trajectory $\{\breve{x}(t)\}_{t\geq 0}$. The observer should be understood in the sense of [25]. This observer is a causal estimator, in the sense that the estimation at time $t \geq 0$ only depends on the measurements $\{\dot{y}(s)\}_{0\leq s\leq t}$. In other words, the observer is non-anticipative.

For smooth dynamical systems, an "optimal" deterministic approach to non-linear system filtering is proposed by Mortensen [29]. This procedure relies on the minimisation of an energy functional. This energy quantifies the likelihood that the state variable produces a given observation – up to disturbances – on a finite time interval, the final time value of the state being imposed. The lower the energy of a state, the more likely the state is. The Mortensen filter is the minimiser of this energy, also known, since then, as the minimum energy estimator [18,24]. Moreover, for smooth dynamical systems, Mortensen also proposes a differential equation for the dynamics of this estimator. This equation is based on the computation of the energy which is solution, in the viscosity sense, of a Hamilton-Jacobi-Bellman(HJB) dynamics [16,20]. This provides an efficient sequential strategy for estimating nonlinear dynamical systems.

We want to investigate to what extent the Mortensen formalism can be extended to the nonsmooth case of the Skorokhod problem with a reflective boundary condition at 0. Given $(\omega, \zeta) \in \mathrm{L}^2(0,t) \times \mathbb{R}_+$, there exists a continuous function $x_{|\omega,\zeta}$ satisfying (1) with $x(0) = \zeta$, $x_{|\omega,\zeta}$ being differentiable almost everywhere. We then define the finite energy

$$\mathcal{J}(\omega, \zeta, t) := \psi(\zeta) + \int_0^t \ell(\omega(s), x_{|\omega,\zeta}(s), s)\mathrm{d}s,$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Lipschitz and

$$\ell(\omega, x, s) := \frac{1}{2}|\omega|^2 + \frac{1}{2}|\dot{y}(s) - h(x)|^2.$$

If $(\omega^\star, \zeta^\star) \in \mathrm{L}^2(0,t) \times \mathbb{R}_+$ is the unique (respectively one of the) minimizer(s) of $\mathcal{J}$, then the (respectively one of the) most likely state(s) of $\breve{x}$ is $x_{|\omega^\star,\zeta^\star}$. Let us fix the terminal state x at the terminal time t. Given the observation $\{\dot{y}(s), 0 \leq s \leq t\}$, the cost-to-come to the point x at time t is defined by

$$\mathcal{V}(x, t; \dot{y}(.)) := \inf_{(\omega,\zeta)\in\mathcal{A}_{x,t}} \mathcal{J}(\omega, \zeta, t), \tag{3}$$

where the admissible set is defined as

$$\mathcal{A}_{x,t} := \{(\omega, \zeta) \in \mathrm{L}^2(0,t) \times \mathbb{R}_+,\ x_{|\omega,\zeta}(t) = x\}.$$

The Mortensen estimator is then defined as

$$\forall t \geq 0, \quad \hat{x}(t) := \underset{x\in\mathbb{R}}{\operatorname{argmin}}\, \mathcal{V}(x, t; \dot{y}(.)), \text{ when the minimizer is unique.} \tag{4}$$

In the smooth setting, the cost-to-come is proven to be a viscosity solution of a HJB with initial condition ψ. This enables the sequential computation of the cost-to-come, and then of the Mortensen estimator $\hat{x}(t)$ as a minimizer of $x \mapsto \mathcal{V}(x,t)$. From the nonsmooth dynamics given by the variational inequality (1), a Mortensen estimator $t \mapsto \hat{x}(t)$ could still be defined as a minimizer of $x \mapsto \mathcal{V}(x,t)$. However, the HJB equation for the corresponding cost-to-come $\mathcal{V}$ remains unclear.

In the smooth setting, several authors established a connection between the Mortensen approach and the vanishing noise limit of stochastic filtering methods, see e.g. [16,18,20]. The central tool in stochastic filtering is the Zakai equation, whose solution is an unnormalised version of the conditional density of the state given the observation [21,38,39]. The minimum energy approach is then recovered as the vanishing noise limit of the robust form (path-wise form) of the Zakai equation. A proof of this fact can be found in [18], using probabilistic tools from the large deviation theory. The stochastic filtering framework has been similarly applied to the Skorokhod problem, see e.g. [33]. Therefore, it sounds plausible that the HJB equation on the cost-to-come (3) can be obtained as a vanishing noise limit of the stochastic filtering procedure for the Skorokhod problem. However, we will see that the picture is more subtle for this nonsmooth dynamics, because the zero-noise limit of the stochastic approach does not provide the desired equation for the deterministic cost-to-come.

The paper is organized as follows. In Section 2, we propose an approximation of the Mortensen estimator using a penalization approach where the dynamics is smooth. This penalization approach reviews in the same time the general results related to the Mortensen estimator in the smooth case. In Section 3, we start from the non-smooth stochastic filtering procedure and perform a vanishing noise limit similar to the smooth case. This limit provides a candidate for the HJB equation that the cost-to-come (3) should solve. In Section 4, we interpret the solution of this latter HJB equation as the value function of a control problem. We then show that this value function cannot be identified to the cost-to-come (3) related to the nonsmooth dynamics (1). This breaks the equivalence between small noise stochastic filtering and the Mortensen deterministic estimation.

2. The penalized case

We begin our study by considering a smooth dynamics version of the Skorokhod problem where the boundary constraint is penalized. This allows us to review all the basic ingredients that lead to the Mortensen estimator, paving the way for the nonsmooth problem. In addition, the penalized dynamics provides a way to define an approximate Mortensen estimator from measurements associated with the nonsmooth problem, as an alternative to obtain the Mortensen estimator directly from the nonsmooth problem.

2.1. **An approximate Mortensen estimator from nonsmooth dynamics penalization**

We relax the boundary constraint of the underlying dynamics (1) for the energy $\mathcal{V}(x,t;\dot{y}(.))$. The inequality is replaced by a nonlinear equation with a drift penalizing the solution whenever it takes negative values. We then introduce a modified cost-to-come $\mathcal{V}^\kappa(x,t;\dot{y}(.))$ whose definition is similar to $\mathcal{V}(x,t;\dot{y}(.))$ in (3), except that $\mathcal{A}_{x,t}$ is replaced by

$$\mathcal{A}^\kappa_{x,t} := \left\{(\omega,\zeta)\in \mathrm{L}^2(0,t)\times\mathbb{R}_+,\ \exists\, x^\kappa \text{ that satisfies } \dot{x}^\kappa = f^\kappa(x^\kappa)+\omega, \ \text{ a.e. } \ \text{with } x^\kappa(0)=\zeta,\ x^\kappa(t)=x\right\}.$$

Here x^κ is an approximate version in $\mathbb{R}$ of (1) where

$$\begin{cases}\dot{x}^\kappa(t) = f^\kappa(x^\kappa(t))+\omega(t), \ \text{ a.e. } \ t>0,\\ x^\kappa(0)=\zeta,\end{cases} \tag{5}$$

the penalty function f^κ being a C^1 approximation of the Moreau-Yosida regularisation of $f_0^\kappa : x\mapsto \kappa\max(-x,0)+f(x)$. For large enough $\kappa>0$, we require that f^κ agrees with the Moreau-Yosida regularisation over $(-\infty,-\kappa^{-1})\cup\mathbb{R}_+$, and that the slope of f^κ belongs to $[-2\kappa,0]$ for $x\in(-\kappa^{-1},0)$. This is possible because f is Lipschitz continuous. The additional term $\kappa\max(-x,0)$ vanishes as soon as $x\geq 0$, and introduces a drift of strength κ towards the non-negative half-line as soon as $x<-\kappa^{-1}$. This term is also responsible for a drift of strength between 0 and 2κ towards the non-negative half-line when $-\kappa^{-1}<x<0$. As $\kappa\to+\infty$, the solution of (5) can be shown to converge towards the solution x of (1) in the max norm on any finite time interval, using techniques analogous to the Moreau-Yosida regularisation [7].

We then define a relaxed version of the Mortensen estimator as follows:

$$\forall t\geq 0,\quad \hat{x}^\kappa(t) := \underset{x\in\mathbb{R}}{\operatorname{argmin}}\,\mathcal{V}^\kappa(x,t;\dot{y}(.)), \tag{6}$$

under the condition of existence and uniqueness of such a minimizer for the function $x \mapsto \mathcal{V}^\kappa(x,t;\dot{y}(.))$. In $\mathcal{V}^\kappa(x,t;\dot{y}(.))$, we point out that the the given observation $\dot{y}(.)$ was produced – up to measurement errors – from a target system $\breve{x}$ governed by a variational inequality. In other words, the trajectory x^κ, generated from $\dot{y}(.)$ by the penalized dynamics, adds a model error to the already present measurement error. For the ease of reading, we will now write $\mathcal{V}^\kappa(x,t) = \mathcal{V}^\kappa(x,t;\dot{y}(.))$.

2.2. The HJB equation for the cost-to-come with penalized dynamics

If we consider an optimal control pair $(\omega_{|[0,t]},\zeta)$ for the "cost-to-come" problem with terminal state x at time t then for any intermediate time $t-\tau$ between the times 0 and t, the part of this control enclosed by the times 0 and $t-\tau$, namely $\omega_{|[0,t-\tau]}$, remains optimal for the "cost-to-come" problem with terminal state $x^\kappa_{|\omega,\zeta}(t-\tau)$ at time $t-\tau$. This is summarized by the following theorem proved in [20].

Theorem 2.1 (Bellman's principle)**.** *Let $0 \le t_1 \le t_2 \le t$, and choose $(\omega,\zeta) \in \mathcal{A}^\kappa_{x,t}$. Then, we have*

$$\mathcal{V}^\kappa\left(x^\kappa_{|\omega,\zeta}(t_2),t_2\right) \le \mathcal{V}^\kappa\left(x^\kappa_{|\omega,\zeta}(t_1),t_1\right) + \int_{t_1}^{t_2} \ell\left(\omega(s), x^\kappa_{|\omega,\zeta}(s), s\right) \mathrm{d}s.$$

where $\dot{x}^\kappa_{|\omega,\zeta} = f^\kappa(x^\kappa_{|\omega,\zeta}) + \omega$.

We here want to emphasize the importance of the reversibility in time of the penalized problem to properly define the cost-to-come. Indeed, we can consider $x^\kappa_{\mathrm{rev}} : \tau \mapsto x^\kappa(t-\tau)$ following the dynamics $-\dot{x}^\kappa_{\mathrm{rev}}(\tau) = f^\kappa(x^\kappa_{\mathrm{rev}}(\tau)) + \omega(\tau)$ with $x_{\mathrm{rev}}(0) = x$. In this way, we find that $\mathcal{A}^\kappa_{x,t} \neq \emptyset$ and $\mathcal{A}^\kappa_{x,t} = \bigcup_{\omega \in \mathrm{L}^2(0,t)} \{(\omega, x^\kappa_{\mathrm{rev}}(t))\}$. The infinitesimal version of Bellman's principle above becomes (7).

Using the previous definition and Bellman's principle, we obtain, as a direct adaptation of [20], that the dynamics followed by the cost-to-come $\mathcal{V}^\kappa$ is given by the following HJB equation

$$\begin{cases} \partial_t \mathcal{V}^\kappa(x,t) + \mathcal{H}(x,t,\partial_x \mathcal{V}^\kappa(x,t)) = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ \mathcal{V}^\kappa(x,0) = \psi(x), & x \in \mathbb{R} \end{cases} \tag{7}$$

where the Hamiltonian is given by

$$\mathcal{H}(x,t,\lambda) := \max_{\omega \in \mathbb{R}} \left[\lambda(f^\kappa(x) + \omega) - \ell(\omega,x,t)\right] = \frac{1}{2}\lambda^2 + \lambda f^\kappa(x) - \frac{1}{2}\left|\dot{y}(t) - h(x)\right|^2. \tag{8}$$

Clearly, the notion of solution of (7) should be specified and, for the sake of completeness we recall the classical definition of a viscosity solution in $\mathbb{R}$.

Definition 2.2. Let $\mathcal{U} \in \mathrm{C}^0(\mathbb{R}^n \times (0,T);\mathbb{R})$. We say that $\mathcal{U}$ is a viscosity subsolution of (7) provided that for all $\phi \in \mathrm{C}^1(\mathbb{R}^n \times (0,T);\mathbb{R})$, if $\mathcal{U} - \phi$ attains a local maximum at (x,t) then

$$\partial_t \phi(x,t) + \mathcal{H}(x,t,\partial_x \phi(x,t)) \le 0. \tag{9}$$

We say that $\mathcal{U}$ is a viscosity supersolution of (7) provided that for all $\phi \in \mathrm{C}^1(\mathbb{R}^n \times (0,T);\mathbb{R})$, if $\mathcal{U} - \phi$ attains a local minimum at (x,t), then

$$\partial_t \phi(x,t) + \mathcal{H}(x,t,\partial_x \phi(x,t)) \ge 0. \tag{10}$$

If $\mathcal{U}$ is both a viscosity subsolution and supersolution, we say that $\mathcal{U}$ is a viscosity solution of (7).

We then have the following theorem.

Theorem 2.3. *The cost-to-come $(x,t) \mapsto \mathcal{V}^\kappa(x,t)$ defined above is a viscosity solution of* (7).

In the context of the initial nonsmooth dynamics, we would like to understand the initially defined cost-to-come $\mathcal{V}$ as a solution in the viscosity sense of a HJB equation. However, we see that (7) gives little intuition of the potential HJB solution candidate when $\kappa \to \infty$.

2.3. The cost-to-come with penalized dynamics seen as the limit of a stochastic filtering problem

In the context of smooth problems such as our penalized dynamics, bridges between the deterministic problem introduced by Mortensen and the more general stochastic filtering framework, were introduced in [18, 19] and further developed in [20]. In the context of small noise in the stochastic setting, this allows us to understand the solution of the HJB equation (7) as a vanishing viscosity limit of a value function formed from the conditional measure of the state knowing the observation up to the current time. We assume that we can exploit such equivalence bridges to propose a candidate dynamics for our originally defined cost-to-come $\mathcal{V}$ in the case of nonsmooth dynamics.

Let us then introduce a small noise amplitude $\varepsilon > 0$, together with the nonlinear filtering problem in $\mathbb{R}$

$$\begin{cases} \mathrm{d}X_t^{\kappa,\varepsilon} = f^\kappa(X_t^{\kappa,\varepsilon})\mathrm{d}t + \sqrt{\varepsilon}\mathrm{d}B_t^1, \\ \mathrm{d}Y_t^{\kappa,\varepsilon} = h(X_t^{\kappa,\varepsilon})\mathrm{d}t + \sqrt{\varepsilon}\mathrm{d}B_t^2, \\ \text{with the initial condition } (X_0^{\kappa,\varepsilon}, Y_0^{\kappa,\varepsilon}) = (\xi, 0). \end{cases} \tag{11}$$

for independent brownian motions $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$. To give a rigorous meaning to this, consider $\Omega := \mathrm{C}_0^0(\mathbb{R}_+;\mathbb{R}^2)$ the set of continuous functions vanishing at 0, endowed with the topology of uniform convergence on compact sets. Let $\mathcal{F}$ denote the Borel σ-field on Ω. For each $t \geq 0$ and $\omega \in \Omega$, define $B_t(\omega) := \omega(t)$ and set $\mathcal{F}_t := \sigma\{B_s, 0 \leq s \leq t\}$ (the σ algebra generated by B up to time t). In this way, for all $0 \leq s \leq t$, $\mathcal{F}_s \subseteq \mathcal{F}_t$ and $\mathcal{F} = \sigma\left(\cup_{\tau\geq 0}\mathcal{F}_\tau\right)$. We complete the triple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ with the Wiener measure $\mathbb{P}$. We recall that the Wiener measure (see [23]) is the unique probability measure on $(\Omega, \mathcal{F})$ satisfying for all $0 \leq s \leq t$ and $\Gamma \in \mathcal{B}(\mathbb{R}^2)$,

$$\mathbb{P}\left(B_t \in \Gamma | \mathcal{F}_s\right) = \frac{1}{2\pi(s-t)} \int_\Gamma \exp\left(-\frac{\|y - B_s\|^2}{2(t-s)}\right) \mathrm{d}y.$$

Here $\forall \zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2$, $\|\zeta\|^2 := \zeta_1^2 + \zeta_2^2$. Note that since $\{B_0 = 0\} = \Omega$, we have $\mathbb{P}(B_0 = 0) = 1$. Consider now $\varepsilon > 0$, a state $\xi \geq 0$ and a continuous bounded function $h : \mathbb{R} \to \mathbb{R}$, which admits a continuous bounded derivative. To assign a meaning to (11), consider the mapping $\omega(.) \to (x^\kappa(.), y^\kappa(.))$ from $\mathrm{C}_0^0([0,T];\mathbb{R}^2)$ to $\mathrm{C}^0([0,T];\mathbb{R}^2)$ where for every $t \geq 0$,

$$\begin{cases} x^{\kappa,\varepsilon}(t) = \xi + \displaystyle\int_0^t f^\kappa(x^{\kappa,\varepsilon}(s))\mathrm{d}s + \sqrt{\varepsilon}\omega^1(t), \\ y^{\kappa,\varepsilon}(t) = \displaystyle\int_0^t h(x^{\kappa,\varepsilon}(s))\mathrm{d}s + \sqrt{\varepsilon}\omega^2(t) \end{cases}$$

is well-defined and continuous. If we denote this continuous map by $\phi_\xi^{\kappa,\varepsilon}$ then $\mathbb{P}\left(\phi_\xi^{\kappa,\varepsilon}\right)^{-1}$, the push forward measure of $\mathbb{P}$ by $\phi_\xi^{\kappa,\varepsilon}$, is the pathwise law associated with $(X^{\kappa,\varepsilon}, Y^{\kappa,\varepsilon})$ solving (11). The filtering problem now aims to compute the measure-valued process $(\pi_t^{\kappa,\varepsilon})_{t\geq 0}$ defined as

$$\int_\mathbb{R} \varphi \,\mathrm{d}\pi_t^{\kappa,\varepsilon} := \mathbb{E}\left[\varphi(X_t^{\kappa,\varepsilon}) | \sigma\left(Y_s^{\kappa,\varepsilon}\right)_{0\leq s\leq t}\right],$$

for any bounded continuous $\varphi : \mathbb{R} \to \mathbb{R}$, $\sigma\left(Y_s^{\kappa,\varepsilon}\right)_{0\leq s\leq t}$ being the σ-algebra generated by the observation $Y_s^{\kappa,\varepsilon}$ up to time t. This estimate of $\varphi(X_t^{\kappa,\varepsilon})$ is optimal in the least-square sense, given the knowledge of $Y_s^{\kappa,\varepsilon}$ up to time t. An evolution non-linear equation called the Kushner-Stratonovich equation can be derived for $\pi_t^{k,\varepsilon}$ using a sophisticated representation formula involving the innovation process, see for instance [1]. Let's focus on a rather simple approach which relies on the unnormalized conditional measure [1]

$$\int_\mathbb{R} \varphi \,\mathrm{d}\rho_t^{\kappa,\varepsilon} := \mathbb{E}\Big[\exp\Big[\frac{1}{\sqrt{\varepsilon}} \int_0^t h(X_s^{\kappa,\varepsilon})\mathrm{d}Y_s^{\kappa,\varepsilon} - \frac{1}{2\varepsilon} \int_0^t h^2(X_s^{\kappa,\varepsilon})\mathrm{d}s\Big]\varphi(X_t^{\kappa,\varepsilon}) \Big| \sigma\left(Y_s^{\kappa,\varepsilon}\right)_{0\leq s\leq t}\Big],$$

which can be linked to $\pi_t^{\kappa,\varepsilon}$ by the Kallianpur-Striebel formula: for any continuous bounded function φ

$$\int_{\mathbb{R}} \varphi \, \mathrm{d}\pi_t^{\kappa,\varepsilon} = \frac{\displaystyle\int_{\mathbb{R}} \varphi \, \mathrm{d}\rho_t^{\kappa,\varepsilon}}{\displaystyle\int_{\mathbb{R}} \mathrm{d}\rho_t^{\kappa,\varepsilon}}.$$

This formula in this case is an analogous of Bayes' formula, see [1, 22, 34]. The density $q^{\kappa,\varepsilon}(x,t)$ of $\rho_t^{\kappa,\varepsilon}$ with respect to the Lebesgue measure solves the linear stochastic partial differential equation (SPDE)

$$\begin{cases} \mathrm{d}q^{\kappa,\varepsilon}(x,t) = A^*_{\kappa,\varepsilon} q^{\kappa,\varepsilon}(x,t) + \frac{1}{\varepsilon} h(x) q^{\kappa,\varepsilon}(x,t) \mathrm{d}Y_t^{\kappa,\varepsilon}, & (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ q^{\kappa,\varepsilon}(x,0) = q_0^{\kappa,\varepsilon}(x), & x \in \mathbb{R}. \end{cases} \tag{12}$$

This is the Zakai equation, to which a rigours meaning is given in [1, 31, 39]. The operator $A^*_{\kappa,\varepsilon}$ is the formal L^2 adjoint of

$$A_{\kappa,\varepsilon} = \frac{\varepsilon}{2} \partial^2_{xx} + f^\kappa \partial_x.$$

The asymptotic behavior of $q^{\kappa,\varepsilon}(x,t)$ is studied in [20] as $\varepsilon \to 0$. Instead of directly dealing with the Zakai equation, they performed the transform [13, 37]

$$p^{\kappa,\varepsilon}(x,t) = \exp\big(-\frac{1}{\varepsilon} y(t) h(x)\big) q^{\kappa,\varepsilon}(x,t), \tag{13}$$

for a given realisation $(y(t))_{0\le t\le T}$ of $(Y_t^{\kappa,\varepsilon})_{0\le t\le T}$, which leads to the robust form of Zakai equation [2, 9, 11]:

$$\begin{cases} \partial_t p^{\kappa,\varepsilon}(x,t) - \dfrac{\varepsilon}{2} \partial^2_{xx} p^{\kappa,\varepsilon}(x,t) + g^\kappa(x,t) \partial_x p^{\kappa,\varepsilon}(x,t) + \dfrac{1}{\varepsilon} \mathcal{P}^{\kappa,\varepsilon}(x,t) p^{\kappa,\varepsilon}(x,t) = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ p^{\kappa,\varepsilon}(x,0) = q_0^{\kappa,\varepsilon}(x), & x \in \mathbb{R}, \end{cases} \tag{14}$$

where $g^\kappa(x,t) = f^\kappa(x) - y(t) h'(x)$ and

$$\mathcal{P}^{\kappa,\varepsilon}(x,t) = \frac{1}{2} h^2(x) + y(t) A_{\kappa,\varepsilon} h(x) - \frac{1}{2} y^2(t) |h'(x)|^2 + \varepsilon \partial_x (f^\kappa(x) - y(t) h'(x)).$$

Detailed computations can be found in appendix 5.1. By the logarithmic transformation − also known as Hopf-Cole transform −

$$\mathcal{S}^{\kappa,\varepsilon}(x,t) = -\varepsilon \log p^{\kappa,\varepsilon}(x,t), \tag{15}$$

the robust form of Zakai equation can be converted into a HJB equation on $\mathcal{S}^{\kappa,\varepsilon}(x,t)$

$$\begin{cases} \partial_t \mathcal{S}^{\kappa,\varepsilon}(x,t) + \mathcal{H}^{\kappa,\varepsilon}(x,t,\partial_x \mathcal{S}^{\kappa,\varepsilon}) = \dfrac{\varepsilon}{2} \partial^2_{xx} \mathcal{S}^{\kappa,\varepsilon}, & (x,t) \in \mathbb{R} \in \mathbb{R}_+, \\ \mathcal{S}^{\kappa,\varepsilon}(x,0) = \mathcal{S}_0^\kappa(x), & x \in \mathbb{R}, \end{cases} \tag{16}$$

where

$$\mathcal{H}^{\kappa,\varepsilon}(x,t,\lambda) = \lambda g^\kappa(x,t) + \frac{1}{2} \lambda^2 - \mathcal{P}^{\kappa,\varepsilon}(x,t).$$

The $\varepsilon \to 0$ limit of $q^{\kappa,\varepsilon}(x,t)$ is then obtained by studying that of $\mathcal{S}^{\kappa,\varepsilon}(x,t)$. The limit function $\mathcal{S}^\kappa(x,t)$ formally satisfies the HJB equation

$$\begin{cases} \partial_t \mathcal{S}^\kappa(x,t) + \mathcal{H}^\kappa(x,t,\partial_x \mathcal{S}^\kappa) = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ \mathcal{S}^\kappa(x,0) = \mathcal{S}_0^\kappa(x), & x \in \mathbb{R}, \end{cases} \tag{17}$$

where

$$\mathcal{H}^\kappa(x,t,\lambda) = \lambda g^\kappa(x,t) + \frac{1}{2}\lambda^2 - \mathcal{P}^\kappa(x,t),$$
$$\mathcal{P}^\kappa(x,t) = \frac{1}{2}h^2(x) + y(t)h'(x)f^\kappa(x) - \frac{1}{2}y^2(t)|h'(x)|^2.$$

In [20], the authors then establish a link between stochastic and deterministic estimation by proving that

$$\mathcal{V}^\kappa(x,t) = \mathcal{S}^\kappa(x,t) - y(t)h(x),$$

using a uniqueness result for the vanishing viscosity solutions of (17). As a by-product, they obtained the following asymptotic approximation

$$q^{\kappa,\varepsilon}(x,t) \approx \exp\left[-\frac{1}{\varepsilon}\mathcal{V}^\kappa(x,t)\right], \text{ as } \varepsilon \downarrow 0, \tag{18}$$

understood – after taking the logarithm and multiplying by ε – as pathwise uniform convergence over compact sets.

At this point, we recall that the observation $\dot{y}(.)$ was produced – up to measurement errors – from a target system $\check{x}$ governed by a variational inequality. Consequently, the density $q^{\kappa,\varepsilon}(x,t)$ generated from $\dot{y}(.)$ contains a model error due to the finiteness of κ. It sounds reasonable to expect (18) to be true as $\kappa \to +\infty$, when replacing $q^{\kappa,\varepsilon}$ by the density obtained in the nonsmooth setting, and $\mathcal{V}^\kappa$ by the cost-to-come $\mathcal{V}$ in (3). However, the lost equivalence, shown in the next sections, suggests that (18) is no more true in the nonsmooth setting.

3. Vanishing viscosity limit of the stochastic filtering problem related to the constrained dynamics

3.1. The stochastic filtering problem for the constrained dynamics

The stochastic filtering framework provides a way to extend the previous results to the limit case $\kappa \to \infty$ where the dynamics is constrained. This provides a candidate HJB equation that can be explored to define a Mortenten estimator for variational inequality dynamics. Since the full probabilistic framework is much more complicated, we only outline the main ingredients presented in [35] and we set $f = 0$ for the sake of conciseness. The resulting HJB is then rigorously analyzed as such in the next section.

Following [35], let us consider the stochastic variational inequality in $\mathbb{R}_+$

$$\begin{cases} \forall \text{ progressively measurable process } Z, \forall 0 \le s \le t, \\ \int_s^t (Z_r - X_r^\varepsilon)\left(\sqrt{\varepsilon}\mathrm{d}B_r^1 - \mathrm{d}X_r^\varepsilon\right) + \int_s^t \mathcal{I}_{\mathbb{R}_+}(X_r^\varepsilon)\mathrm{d}r \le \int_s^t \mathcal{I}_{\mathbb{R}_+}(Z_r)\mathrm{d}r, \\ \mathrm{d}Y_t^\varepsilon = h(X_t^\varepsilon)\mathrm{d}t + \sqrt{\varepsilon}\mathrm{d}B_t^2, \\ \text{with the initial condition } (X_0, Y_0) = (\zeta, 0), \end{cases} \tag{19}$$

with $\mathcal{I}_{\mathbb{R}_+}$ denoting the convex characteristic function of $\mathbb{R}_+$ (equal to 0 within $\mathbb{R}_+$ and $+\infty$ outside). Following [35, page 239], we say that a triple $(X^\varepsilon, Y^\varepsilon, K^\varepsilon)$, an $\mathbb{R}^3$-valued stochastic process, is a solution of (19), if the following conditions are satisfied $\mathbb{P}$ almost surely (a.s.)

1. $X^\varepsilon, Y^\varepsilon, K^\varepsilon$ are progressively measurable with continuous path and $K_0 = 0$,
2. $\forall t \ge 0, X_t^\varepsilon \ge 0$,
3. $\forall T \ge 0, \|K_T^\varepsilon\| < \infty$,
4. $\forall t \ge 0,\ X_t^\varepsilon + K_t^\varepsilon = \zeta + \sqrt{\varepsilon}B_t^1,$ and $Y_t^\varepsilon = \int_0^t h(X_s^\varepsilon)\mathrm{d}s + \sqrt{\varepsilon}B_t^2,$
5. $\forall 0 \le s \le t,\ \forall z \in \mathbb{R}_+,\ \int_s^t (z - X_r^\varepsilon)\mathrm{d}K_r^\varepsilon \le 0.$

Since the diffusion coefficients in front of B_1 and B_2 are constant, we may fix an arbitrary $\omega \in \Omega$ and regard (19) as a deterministic problem with forcing $\{(B_t^1(\omega), B_t^2(\omega)),\ t \geq 0\}$. Still following [35], we say that a triple $(x^\varepsilon, y^\varepsilon, k^\varepsilon)$ is a solution of the generalized Skorokhod problem $\mathcal{GS}^\varepsilon$, if the following conditions hold:

(1) $x^\varepsilon, y^\varepsilon, k^\varepsilon$ are continuous, $x^\varepsilon(0) = \zeta$ and $k^\varepsilon(0) = 0$,
(2) $\forall t \geq 0, x^\varepsilon(t) \geq 0$,
(3) $k^\varepsilon \in BV_{loc}(\mathbb{R}_+; \mathbb{R})$,
(4) $\forall t \geq 0,\ x^\varepsilon(t) + k^\varepsilon(t) = \zeta + \sqrt{\varepsilon}\omega^1(t)$, and $y^\varepsilon(t) = \int_0^t h(x^\varepsilon(s))\mathrm{d}s + \sqrt{\varepsilon}\omega^2(t)$,
(5) $\forall 0 \leq s \leq t,\ \forall z \in \mathbb{R}_+,\ \int_s^t (z - x^\varepsilon(r))\mathrm{d}k^\varepsilon(r) \leq 0$.

Theorem 3.1. *[35, Theorem 4.17 page 252] Assume h to be sufficiently smooth, $\zeta \in \mathbb{R}_+$ and $\omega(.)$ is a continuous function with $\omega(0) = 0$. Then the $\mathcal{GS}^\varepsilon(\zeta, \omega)$ has a unique solution.*

Theorem 3.2. *[35, Theorem 4.16 page 247] The mapping $(\zeta, \omega) \mapsto (x^\varepsilon, y^\varepsilon) = \mathcal{GS}^\varepsilon(\zeta, \omega)$ is continuous from $\mathbb{R}_+ \times \mathrm{C}^0([0, T]; \mathbb{R}^d) \to \mathrm{C}^0([0, T]; \mathbb{R}^2)$.*

Theorem 3.3. *[35, Theorem 4.18 page 257] The stochastic variational inequality (19) has a unique solution $(X^\varepsilon, Y^\varepsilon, K^\varepsilon)$ progressively measurable with continuous path in the sense of the definition above.*

Remark 3.4. When $f \equiv 0$, finding a solution to the VI (1) with the disturbance function $\omega(.)$ as an input and ζ as initial condition at time 0 is equivalent to finding a solution to the deterministic Skorokhod problem with ζ and the integral of ω as inputs. However, the stochastic Skorokhod problem in (19) is solved with ζ and ω as inputs.This allows to define a continuous mapping as shown in [34, Theorem 4.16 page 247]. With this mapping, a push forward of the Wiener measure is used to establish the solution to the stochastic Skorokhod problem with ζ and two Wiener processes with variance ϵ as inputs.

The stochastic filtering problem of reflected diffusions has been tackled in [2, 27, 28, 30, 33]. As in section 2.3, the unnormalized conditional density $q^\varepsilon(x, t)$ can be defined for the stochastic filtering problem of the constrained dynamics, and it solves the Zakai equation with boundary condition

$$\begin{cases} \mathrm{d}q^\varepsilon(x,t) = \frac{\varepsilon}{2}\partial^2_{xx} q^\varepsilon(x,t) + \dfrac{q^\varepsilon(x,t)}{\varepsilon}\mathrm{d}Y_t^\varepsilon, & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ q^\varepsilon(0,x) = q_0^\varepsilon(x) & x \in \mathbb{R}_+, \\ \partial_x q^\varepsilon(t,0) = 0, & t \in \mathbb{R}_+, \end{cases} \tag{20}$$

for which a rigorous meaning is given in [30–32]. Given a realisation $(y(t))_{0 \leq t \leq T}$ of $(Y_t^\varepsilon)_{0 \leq t \leq T}$, the change of variable

$$p^\varepsilon(x,t) = \exp(-\frac{1}{\varepsilon}y(t)h(x))q^\varepsilon(x,t), \tag{21}$$

now leads to the robust Zakai equation with boundary condition

$$\begin{cases} \partial_t p^\varepsilon(x,t) - y(t)h'(x)\partial_x p^\varepsilon(x,t) + \dfrac{1}{\varepsilon}\mathcal{P}^\varepsilon(x,t)p^\varepsilon(x,t) = \dfrac{\varepsilon}{2}\partial^2_{xx} p^\varepsilon(x,t), & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ \frac{\varepsilon}{2}\partial_x p^\varepsilon(t,0) + \dfrac{y(t)h'(x)}{2}p^\varepsilon(0,t) = 0, & t \in \mathbb{R}_+, \end{cases} \tag{22}$$

where

$$\mathcal{P}^\varepsilon(x,t) = \frac{1}{2}h^2(x) - \frac{\varepsilon}{2}y(t)h''(x) - \frac{1}{2}y^2(t)|h'(x)|^2. \tag{23}$$

Details on this derivation are given in appendix 5.1. By the Hopf-Cole transform

$$\mathcal{S}^\varepsilon(x,t) = -\varepsilon \log p^\varepsilon(x,t), \tag{24}$$

the robust Zakai equation can be converted into the following HJB equation with boundary condition

$$\begin{cases} \partial_t \mathcal{S}^\varepsilon(x,t) + \mathcal{H}^\varepsilon_{\mathcal{S}}\left(x,t,\partial_x \mathcal{S}^\varepsilon(x,t)\right) = \dfrac{\varepsilon}{2}\partial^2_{xx}\mathcal{S}^\varepsilon(x,t), & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ \partial_x \mathcal{S}^\varepsilon(0,t) - y(t)h'(0) = 0, & t \in \mathbb{R}_+, \\ \mathcal{S}^\varepsilon(x,0) = -\varepsilon \log p^\varepsilon(x,0), \end{cases} \tag{25}$$

the Hamiltonian $\mathcal{H}^\varepsilon_{\mathcal{S}}$ being defined in (26) as

$$\mathcal{H}^\varepsilon_{\mathcal{S}} : \begin{cases} \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} & \to \quad \mathbb{R} \\ \qquad (x,t,\lambda) & \mapsto \quad \frac{\lambda^2}{2} - \lambda y(t)h'(x) - \mathcal{P}^\varepsilon(x,t). \end{cases} \tag{26}$$

3.2. **Viscous Hamilton-Jacobi equation on $\mathcal{S}^\varepsilon$**

For the sake of generality, we assume in this section that $f \neq 0$, with only $f(0) = 0$ to avoid additional technical problems at the boundary – see Remark 3.5 for comments on the completely general case.We assume that f and y are bounded C^1 functions with bounded first derivatives, and h is a bounded $xCtwo$ function with bounded derivatives up to order 2 .

Starting from the stochastic filtering problem of the constrained dynamics and inspired by [20], we introduce the Hamilton-Jacobi equation (27) formally satisfied by the Hopf-Cole transform of the solution of the robust Zakai equation as done in the previous section, see (25). We prove a stability result that allows us to recover, in the vanishing viscosity limit, what we will interpret in section 4.1 as a deterministic limit of the stochastic filtering problem. Consider

$$\begin{cases} \partial_t \mathcal{S}^\varepsilon(x,t) + \mathcal{H}^\varepsilon_{\mathcal{S}}(x,t,\partial_x \mathcal{S}^\varepsilon(x,t)) = \dfrac{\varepsilon}{2}\partial^2_{xx}\mathcal{S}^\varepsilon(x,t), & x \in \mathbb{R}^*_+,\ t > 0, \\ -\partial_x \mathcal{S}^\varepsilon(0,t) = -y(t)h'(0), & x = 0,\ t > 0, \\ \mathcal{S}^\varepsilon(x,0) = S_0(x), & x \in \mathbb{R}_+,\ t = 0, \end{cases} \tag{27}$$

for some initial condition: $S_0 \in \mathrm{BUC}(\mathbb{R}_+;\mathbb{R})$ (Bounded Uniformly Continuous), the Hamiltonian $\mathcal{H}^\varepsilon_{\mathcal{S}}$ being defined for $\varepsilon > 0$ as

$$\mathcal{H}^\varepsilon_{\mathcal{S}} : \begin{cases} \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \\ \qquad (x,t,\lambda) \mapsto \dfrac{\lambda^2}{2} + \lambda g_{\mathcal{S}}(x,t) - \left[\dfrac{h(x)^2}{2} + y(t)L_\varepsilon h(x) - \dfrac{1}{2}y(t)^2|h'(x)|^2 + \varepsilon \partial_x g_{\mathcal{S}}(x,t)\right], \end{cases} \tag{28}$$

where by analogy with [20], we set

$$\left| \begin{aligned} g_{\mathcal{S}}(x,t) &:= f(x) - y(t)h'(x), \\ L_\varepsilon &:= \frac{\varepsilon}{2}\partial^2_{xx} + f(x)\partial_x. \end{aligned} \right. \tag{29}$$

Contrary to [20] who started from the stochastic setting, this deterministic equation will be our starting point, not requiring any previous result on the robust Zakai equation with boundary conditions, and defining $\mathcal{S}^\varepsilon$ as solution of (27) rather than as the value function resulting from a dynamic programming approach.

Remark 3.5. If $f(0) \neq 0$, the second line of equation (27) reads instead

$$-\partial_x \mathcal{S}^\varepsilon(0,t) = -y(t)h'(0) - 2f(0), \qquad x = 0, \ \ t > 0.$$

We may add an appropriate smooth, bounded perturbation of bounded derivatives to $\mathcal{S}^\varepsilon$, defining for instance:

$$\bar{\mathcal{S}}^\varepsilon(x,t) := \mathcal{S}^\varepsilon(x,t) - 2xf(0)e^{-x^2},$$

so that $-\partial_x \bar{\mathcal{S}}^\varepsilon(x,t) = -\partial_x \mathcal{S}^\varepsilon(x,t) + 2f(0)$. We thus recover a function satisfying a closely related viscous Hamilton-Jacobi equation whose Hamiltonian can be easily computed. That new Hamiltonian satisfies the same sufficient properties for the rest of the section, and the boundary condition of the new equation does not involve f. Hence, similar results will hold, so to avoid unnecessary technicalities, we choose to take $f(0) = 0$ hereafter.

3.3. **The Vanishing Viscosity Limit Procedure**

We denote the formal limit of $\mathcal{H}_\mathcal{S}^\varepsilon$ as $\varepsilon \to 0$ by

$$\mathcal{H}_\mathcal{S} : \begin{cases} \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} & \to \quad \mathbb{R} \\ (x,t,\lambda) & \mapsto \quad \frac{\lambda^2}{2} + \lambda g_\mathcal{S}(x,t) - \frac{(h(x))^2}{2} - y(t)f(x)h'(x) + \frac{1}{2}(y(t))^2|h'(x)|^2. \end{cases} \tag{30}$$

The main theorem of the section is the following stability result.

Theorem 3.6. *Assume that $f, y \in \mathrm{C}_b^1(\mathbb{R}_+;\mathbb{R})$ and $h \in \mathrm{C}_b^2(\mathbb{R}_+;\mathbb{R})$ are bounded with first (and second for h) bounded derivatives, and $\mathcal{S}^\varepsilon \in \mathrm{BUC}(\mathbb{R}_+;\mathbb{R})$. Then:*

- (i)**:** *for all $\varepsilon > 0$ the second order evolution Hamilton-Jacobi equation (27) admits a unique smooth solution S^ε.*
- (ii)**:** *The Hamiltonian $\mathcal{H}_\mathcal{S}^\varepsilon$ defined in (28) converges locally uniformly as $\varepsilon \to 0$ to the limiting Hamiltonian $\mathcal{H}_\mathcal{S}$ of equation (30),*
- (iii)**:** *$\mathcal{S}^\varepsilon$ converges locally uniformly as $\varepsilon \to 0$ to a continuous function we denote by $\mathcal{S}$,*
- (iv)**:** *$\mathcal{S}$ is the unique viscosity solution of the limiting Hamilton-Jacobi equation below, in the sense of Definition 3.7.*

$$\begin{cases} \partial_t \mathcal{S}(x,t) + \mathcal{H}_\mathcal{S}(x,t,\partial_x \mathcal{S}(x,t)) = 0, & x \in \mathbb{R}_+^*,\ t > 0, \\ -\partial_x \mathcal{S}(0,t) = -y(t)h'(0), & x = 0,\ t > 0, \\ \mathcal{S}(x,0) = \mathcal{S}_0(x), & x \in \mathbb{R}_+,\ t = 0. \end{cases} \tag{31}$$

Let us recall from *e.g.* [3,26] an appropriate notion of solution for the above Hamilton-Jacobi equations with Neumann boundary condition. Consider the first order Hamilton-Jacobi equation on $\mathbb{R}_+$

$$\begin{cases} \partial_t u(x,t) + \mathcal{H}(x,t,u(x,t),\partial_x u(x,t)) = 0, & x \in \mathbb{R}_+^*, t > 0, \\ \mathcal{B}(0,t,u(0,t),\partial_x u(0,t)) = 0 & x = 0, t > 0, \\ u(x,t) = u_0(x), & x \in \mathbb{R}_+, t = 0, \end{cases} \tag{32}$$

for locally Lipschitz $\mathcal{H}$ and $\mathcal{B}$, the latter being strictly increasing with respect to its last variable in the outward normal direction at x: for all $R > 0$, there exists $\nu_R > 0$ such that for all $(x,t,u,\lambda) \in \{0\} \times \mathbb{R}_+ \times [-R,R] \times \mathbb{R}$,

$$\mathcal{B}(x,t,u,\lambda + \alpha n(x)) - \mathcal{B}(x,t,u,\lambda) \geq \nu_R \alpha, \tag{33}$$

where $n(x)$ is the unit outward normal to $\partial\mathbb{R}_+$ at x – so, -1. Note that the satisfaction of this condition is the reason for the $-$ sign preceding $\partial_x \mathcal{S}(0,t)$ in (31), in which:

$$\mathcal{B}(x,t,u,\lambda) = -\lambda + y(t)h'(0).$$

Definition 3.7. A continuous function u is said to be a viscosity subsolution of equation (32) if it satisfies that for all $\phi \in \mathrm{C}^1(\mathbb{R}_+ \times \mathbb{R}_+;\mathbb{R})$, at each maximum point $(x_0,t_0) \in \mathbb{R}_+ \times \mathbb{R}_+$ of $u-\phi$, we have:

$$\begin{cases} \text{If } (x_0,t_0) \in \mathbb{R}_+^* \times \mathbb{R}_+^*, & (\partial_t\phi + \mathcal{H}(\cdot,u,\partial_x\phi))(x_0,t_0) \leq 0, \\ \text{If } (x_0,t_0) \in \{0\} \times \mathbb{R}_+^*, & \min\{\mathcal{B}(\cdot,u,\partial_x\phi)(0,t_0)\ ,\ (\partial_t\phi + \mathcal{H}(\cdot,u,\partial_x\phi))(0,t_0)\} \leq 0, \\ \text{If } (x_0,t_0) \in \mathbb{R}_+^* \times \{0\}, & \min\{u(x_0,0)-u_0(x_0)\ ,\ (\partial_t\phi + \mathcal{H}(\cdot,u,\partial_x\phi))(x_0,0)\} \leq 0, \\ \text{If } (x_0,t_0) = (0,0), & \min\{u(0,0)-u_0(0)\ ,\ \mathcal{B}(\cdot,u,\partial_x\phi)(0,t_0)\ ,\ (\partial_t\phi + \mathcal{H}(\cdot,u,\partial_x\phi))(0,0)\} \leq 0. \end{cases}$$

A continuous function u is said to be a viscosity supersolution of equation (32) if it satisfies that for all $\phi \in \mathrm{C}^2(\bar{\mathbb{R}}_+ \times \mathbb{R}_+;\mathbb{R})$, at each minimum point $(x_0,t_0) \in \mathbb{R}_+ \times \mathbb{R}_+)$ of $u-\phi$, we have:

$$\begin{cases} \text{If } (x_0,t_0) \in \mathbb{R}_+^* \times \mathbb{R}_+^*, & (\partial_t\phi + \mathcal{H}(\cdot,u,\partial_x\phi))(x_0,t_0) \geq 0, \\ \text{If } (x_0,t_0) \in \{0\} \times \mathbb{R}_+^*, & \max\{\mathcal{B}(\cdot,u,\partial_x\phi)(0,t_0)\ ,\ (\partial_t\phi + \mathcal{H}(\cdot,u,\partial_x\phi))(0,t_0)\} \geq 0, \\ \text{If } (x_0,t_0) \in \mathbb{R}_+^* \times \{0\}, & \max\{u(x_0,0)-u_0(x_0)\ ,\ (\partial_t\phi + \mathcal{H}(\cdot,u,\partial_x\phi))(x_0,0)\} \geq 0, \\ \text{If } (x_0,t_0) = (0,0), & \max\{u(0,0)-u_0(0)\ ,\ \mathcal{B}(\cdot,u,\partial_x\phi)(0,t_0)\ ,\ (\partial_t\phi + \mathcal{H}(\cdot,u,\partial_x\phi))(0,0)\} \geq 0. \end{cases}$$

A continuous function u is said to be a viscosity solution of equation (32) if it is both a viscosity subsolution and supersolution.

Theorem 3.8 (Uniqueness and conditional existence of BUC solutions – Theorem 2.1 in [3])**.** *Assume the initial condition u_0 to be bounded and uniformly continuous. Assume $\mathcal{H}$ and $\mathcal{B}$ to be locally Lipschitz continuous, and that $\mathcal{H}$ is locally uniformly Lipschitz continuous, convex and coercive in its last variable. Then, if u and v are respectively a bounded upper semi-continuous (u.s.c.) viscosity subsolution and a bounded lower semi-continuous (l.s.c.) viscosity supersolution of* (32)*, then*

$$u \leq v \qquad \textit{on} \qquad \bar{\Omega} \times [0,T].$$

Moreover, if such u, v exist and $u = v = u_0$ on $\bar{\Omega} \times \{0\}$, then equation (32) *admits a continuous unique viscosity solution.*

Remark 3.9. There are crucial hypotheses of Barles' theorem above that become immediate in our one spatial dimension, first order Hamiltonian setting. First, the open set $\mathbb{R}_+^*$ trivially satisfies that $\partial\mathbb{R}_+^* = \{0\} \in \mathrm{W}^{3,\infty}$. Second, the structure hypotheses labeled $(H1)$, $(H2)$, and $(H3)$ in [3] are clearly satisfied by a first order Hamiltonian H and by our boudary condition, and boil down to classical Lipschitz continuity, convexity and coercivity hypotheses.

To consider homogeneous Neumann conditions, let's work on $w^\varepsilon(x,t) := \mathcal{S}^\varepsilon(x,t) - y(t)h(x)$, instead of directly $\mathcal{S}^\varepsilon$. w^ε is given as the solution of

$$\begin{cases} \partial_t w^\varepsilon(x,t) + \mathcal{H}^\varepsilon(x,t,\partial_x w^\varepsilon(x,t)) = \dfrac{\varepsilon}{2}\partial_{xx}^2 w^\varepsilon(x,t), & x \in \mathbb{R}_+^*,\ t > 0, \\ -\partial_x w^\varepsilon(0,t) = 0, & x = 0,\ t > 0, \\ w^\varepsilon(x,0) = w_0(x), & x \in \mathbb{R}_+,\ t = 0, \end{cases} \tag{34}$$

provided with some locally bounded, Lipschitz initial condition w_0. Local existence and uniqueness is shown in Section 3.4 and global existence and uniqueness in Section 3.5, Corollary 3.14. The Hamiltonian $\mathcal{H}^\varepsilon$ is defined over $(x,t,\lambda) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ as

$$\mathcal{H}^\varepsilon(x,t,\lambda) := \frac{1}{2}\lambda^2 + \lambda f(x) - \frac{1}{2}(h(x))^2 - \varepsilon f'(x) + \frac{\varepsilon}{2}y(t)h''(x) + \dot{y}(t)h(x). \tag{35}$$

Note that this Hamiltonian, its $\varepsilon \to 0$ limit and all the Hamiltonians considered in this paper satisfy the hypotheses of Theorem 3.8.

Remark 3.10. With this point of view, it is possible to directly define w^ε as the solution of equation (34) after proving that it is well-posed, and to introduce $\mathcal{S}^\varepsilon$ as a modification of w^ε. This allows to consider $\mathcal{S}^\varepsilon$ without starting from the general Zakai equation.

Proposition 3.11 (Local uniform convergence of the viscous Hamiltonian)**.** *$\mathcal{H}^\varepsilon$ converges uniformly to $\mathcal{H}$ in $\mathrm{C}^0(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, where:*

$$\mathcal{H}(x,t,\lambda) := \frac{1}{2}\lambda^2 + \lambda f(x) - \frac{1}{2}(h(x))^2 + \dot{y}(t)h(x).$$

In an analogous way, $\mathcal{H}^\varepsilon_{\mathcal{S}}$ defined in equation (28) converges locally uniformly to $\mathcal{H}_{\mathcal{S}}$ defined in equation (30).

Formally, equation (34) tends to the following:

$$\begin{cases} \partial_t w(x,t) + \mathcal{H}(x,t,\partial_x w(x,t)) = 0, & x \in \mathbb{R}_+^*,\ t > 0, \\ -\partial_x w(0,t) = 0, & x = 0,\ t > 0, \\ w(x,0) = w_0(x), & x \in \mathbb{R}_+,\ t = 0, \end{cases} \tag{36}$$

where the boundary condition must be understood in the sense of viscosity solutions, as in Definition 3.7.

Remark 3.12. If w is a viscosity solution of (36), the remark in Section 2 of [3] still holds: using well-chosen test functions, it is possible to prove that the initial condition is satisfied in the classical sense provided w_0 is smooth, as in the present case here. For an extension to nonsmooth initial conditions, we refer to the corresponding chapter of [4].

3.4. **Local existence and uniqueness for the solution of** (34)

We wish to extend equation (34) to $x \in \mathbb{R}$ in a way that guarantees that the restriction to $x \in \mathbb{R}_+$ of the solution of the extended equation $\tilde{w}$ satisfies the Neumann boundary condition. Hence, it is sufficient to construct an extention $\tilde{w}$ that is even, so $\partial_x \tilde{w}$ is odd. Let us proceed by analogy with a reflection method presented in [36, Ch. 3] for the heat equation with Neumann boundary condition:

$$\begin{cases} \partial_t u(x,t) - k\partial^2_{xx} u(x,t) = F(x,t), & x > 0,\ t > 0, \\ \partial_x u(0,t) = 0, & x = 0,\ t > 0, \\ u(x,0) = u_0(x) & x \geq 0,\ t = 0. \end{cases}$$

Let G be the Green heat kernel, defined over $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ as:

$$G_k(x,t) = \frac{1}{\sqrt{4k\pi t}} \exp\left(-\frac{x^2}{4kt}\right).$$

The function

$$\tilde{u}(x,t) := [G_k(\cdot,t) * u_0(|\cdot|)](x) + \int_0^t [G_k(\cdot,t-s) * F(|\cdot|,s)](x)\,\mathrm{d}s$$

is the Duhamel formulation corresponding to the symmetrised equation

$$\begin{cases} \partial_t \tilde{u}(x,t) - k\partial^2_{xx}\tilde{u}(x,t) = F(|x|,t), & x \in \mathbb{R},\ t > 0, \\ \tilde{u}(x,0) = u_0(|x|), & x \in \mathbb{R},\ t = 0, \end{cases}$$

and its restriction to $x \in \mathbb{R}_+$ satisfies the initial Heat equation with Neumann boundary condition.

In an analogous way, we define the symmetrised Hamiltonian $\tilde{\mathcal{H}}$, taking into account that the variable λ will be expected to be an odd function of x:

$$\tilde{\mathcal{H}}^\varepsilon : \begin{cases} \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} & \to \quad \mathbb{R} \\ \quad (x,t,\lambda) & \mapsto \quad \mathcal{H}(|x|, t, \mathrm{sgn}(x)\lambda) = \frac{1}{2}\lambda^2 + \lambda g(x) - \tilde{V}_w^\varepsilon(x,t), \end{cases} \tag{37}$$

with

$$\begin{cases} g(x) := \mathrm{sgn}(x) f(|x|), \\ \tilde{V}_w^\varepsilon(x,t) := \dfrac{1}{2}(h(|x|))^2 - \dot{y}(t)h(|x|) + \varepsilon f'(|x|) - \dfrac{\varepsilon}{2} y(t) h''(|x|). \end{cases} \tag{38}$$

Note that g for $\tilde{w}^\varepsilon$ corresponds to $g_{\mathcal{S}}$ for $\mathcal{S}$, defined in (29). The 'symmetrised' version of equation (34) reads

$$\begin{cases} \partial_t \tilde{w}^\varepsilon(x,t) + \tilde{\mathcal{H}}^\varepsilon(x,t,\partial_x \tilde{w}^\varepsilon(x,t)) = \dfrac{\varepsilon}{2}\partial^2_{xx}\tilde{w}^\varepsilon(x,t), & x \in \mathbb{R},\ t > 0, \\ \tilde{w}^\varepsilon(x,0) = \tilde{w}_0(x), & x \in \mathbb{R},\ t = 0, \end{cases} \tag{39}$$

where we use $\tilde{w}_0 : x \in \mathbb{R} \mapsto w_0(|x|)$. Note that there is no more Neumann boundary condition.

Let us establish the well-posedness of the equation above.

Theorem 3.13 (Local existence and uniqueness of a solution of (39))**.** *Let* $\varepsilon > 0$*. Let* $\tilde{w}_0 \in \mathrm{L}^\infty \cap \mathrm{Lip}$*, and* $\tilde{\mathcal{H}}^\varepsilon \in \mathrm{Lip}_{\mathrm{loc}}$*w.r.t.* λ*. Then there exists* $T > 0$ *such that there exists a unique smooth solution* $\tilde{w}^\varepsilon$ *of equation* (39) *defined on* $\mathbb{R} \times [0,T]$*.*

The proof of Theorem 3.13 is a technical, but relatively standard fixed-point method, so for the sake of conciseness, we will only sketch it.

Proof. We fix ε, and assume, in a first step, that $\tilde{\mathcal{H}}^\varepsilon$ is globally Lipschitz in λ. We prove that for $(x,t) \in \mathbb{R} \times [0,T]$ with T small enough, the mapping of a Picard iterate to the next is a contraction in the norm $\|u\| := \|u\|_{\mathrm{L}^\infty} + \|\partial_x u\|_{\mathrm{L}^\infty}$. We then extend the result to $\tilde{\mathcal{H}}^\varepsilon$ locally Lipschitz in λ by applying a security cylinder method used in [12, Ch.V]. Smoothness follows from that of the Green kernel. □

3.5. Uniform in ε bounds on w^ε

The main result of this section is the global existence, uniqueness and uniform-in-ε boundedness of w^ε stated in Corollary 3.14. For the sake of simplicity, we first prove Theorem 3.15: uniform in ε estimates on the even extension $\tilde{w}^\varepsilon$ to $\mathbb{R} \times \mathbb{R}_+$ of w^ε defined in equation (39).

Our proof strategy in this section is that of [20], with the exceptions that we apply it to $\tilde{w}^\varepsilon$ rather than the extension of $\mathcal{S}^\varepsilon$, that we only have local existence of the solution at fixed ε for now, and that we need to glean a sharper $\mathrm{L}^\infty_{\mathrm{loc}}$ estimate. The global existence of the solution for each ε is a consequence of the uniform bounds (Corollary 3.16), and the exact same proof can then be applied over $[0,T]$.

Corollary 3.14. *Equation* (34) *admits a unique solution* w^ε *defined globally in time, and* w^ε *is locally bounded in* $W^{1,\infty}_x(\mathbb{R}_+; C^{1,1}_t)$ *with analogous bounds to those from Theorem 3.15.*

Theorem 3.15. *Assume* $\tilde{\mathcal{H}}$ *satisfies the assumptions of Theorem 3.8, and let* $T > 0$ *such that equation* (39) *admits a unique smooth solution* $\tilde{w}^\varepsilon$ *over* $\mathbb{R} \times [0,T]$*. Then for every compact subset* $Q \subset \mathbb{R} \times [0,T]$ *there exists* $\varepsilon_0 > 0$ *and* $K > 0$ *such that for all* $0 < \varepsilon < \min(\varepsilon_0, 1)$*, for all* $(x,t),(x,s) \in Q$*,* $\tilde{w}^\varepsilon$ *satisfies:*

$$\begin{aligned} &(i) && |\tilde{w}^\varepsilon(x,t)| \le K, \\ &(ii) && |\partial_x \tilde{w}^\varepsilon(x,t)| \le K, \\ &(iii) && |\tilde{w}^\varepsilon(x,t) - \tilde{w}^\varepsilon(x,s)| \le K\left(|t-s|^{1/2} + |t-s|\right). \end{aligned} \tag{40}$$

Moreover, (i) can be refined into a sharper estimate (iv), where the bound itself does not depend on R. Namely, for all $R \geq \max(8, 16\|g\|_{\mathrm{L}^\infty(\mathbb{R})})$ there exists $\varepsilon_R := \frac{1}{32R^4}$ such that for all $0 < \varepsilon < \varepsilon_R$,

$$(iv) \qquad \|\tilde{w}^\varepsilon\|_{\mathrm{L}^\infty(Q_R)} \leq \|\tilde{w}_0\|_{\mathrm{L}^\infty(\mathbb{R})} + \left[8(1 + \|g\|_{\mathrm{L}^\infty(\mathbb{R})}) + \|V^\varepsilon\|_{\mathrm{L}^\infty(\mathbb{R}\times[0,T])} + 1\right] T + 1. \tag{41}$$

Corollary 3.16. *Equation (39) admits a unique solution $\tilde{w}^\varepsilon$ defined globally in time. Moreover, $\tilde{w}^\varepsilon$ is locally bounded in the norm of Theorem 3.15.*

Proof. Proof of Corollary 3.16 assuming Theorem 3.15

Consider the maximal interval of existence in time of the local solution $\tilde{w}^\varepsilon$. The local uniform boundedness of $\tilde{w}^\varepsilon$ allows to prove that interval is $\mathbb{R}_+$, which implies the existence of a unique solution $\tilde{w}^\varepsilon$ defined globally in time. This in turn allows to apply Theorem 3.15 globally in time, recovering the same bounds over every compact. Uniqueness follows from Theorem 3.13. □

Proof of Corollary 3.14.

Assume Theorem 3.15 and Corollary 3.16 hold. Then the restriction $(x, t) \in \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \tilde{w}^\varepsilon(x, t)$ is well defined globally, bounded locally, and satisfies the equation (34). Here too, uniqueness follows from Theorem 3.13. □

To prove Theorem 3.15, we use the exact same comparison theorem as in [20], relying on the maximum principle for linear parabolic PDE. We denote by $\bar{B}_R \subset \mathbb{R}$ the closed ball centred at 0 with radius $R > 0$, and by $\Gamma_R := \bar{B}_R \times \{0\} \cup \partial\bar{B}_R \times [0, T]$ the parabolic boundary of $Q_R := \bar{B}_R \times [0, T]$, whose interior we denote by $\mathring{Q}_R$.

Lemma 3.17 (Maximum Principle, [17])**.** *Define*

$$\mathscr{L}\varphi := \partial_t\varphi - \frac{\varepsilon}{2}\partial^2_{xx}\varphi + \partial_x\varphi b^\varepsilon,$$

where b^ε is smooth. If $\mathscr{L}\varphi \leq 0$ (respectively, ≥ 0) in $\mathring{Q}_R$, then for all $(x, t) \in Q_R$,

$$\varphi(x, t) \leq \sup_{(z,s)\in\Gamma_R} \varphi(z, s)$$

$$\left(\text{respectively,} \qquad \inf_{(z,s)\in\Gamma_R} \varphi(z, s) \leq \varphi(x, t)\right).$$

Lemma 3.18 (Comparison theorem, [20] Lemma 4.2)**.**

Let $\varepsilon > 0$. Let $\tilde{w}^\varepsilon$ be a solution of (34) over $\mathbb{R} \times [0, T]$ and define

$$\tilde{\mathscr{L}} : v \in \mathrm{C}^1(\mathring{Q}_R; \mathbb{R}) \mapsto \partial_t v - \frac{\varepsilon}{2}\partial^2_{xx} v + g\partial_x v + \frac{1}{2}|\partial_x v|^2 - \tilde{V}^\varepsilon_w,$$

$g(x) = \mathrm{sgn}(x)f(|x|)$ and $\tilde{V}^\varepsilon_w$ being defined in (38). Let $v \in \mathrm{C}^1(\mathring{Q}_R; \mathbb{R})$. If $\tilde{\mathscr{L}}v \geq 0$ (respectively, $\tilde{\mathscr{L}}v \leq 0$) in $\mathring{Q}_R$ and if $\tilde{w}^\varepsilon \leq v$ (resp. $v \leq \tilde{w}^\varepsilon$) on Γ_R, then $\tilde{w}^\varepsilon \leq v$ (resp. $v \leq \tilde{w}^\varepsilon$) in $\mathring{Q}_R$.

Same proof as in [20]: If $\tilde{\mathscr{L}}v \geq 0$, then subtract $\tilde{\mathscr{L}}w^\varepsilon = 0$ and let $\varphi = v - \tilde{w}^\varepsilon$ to get

$$\partial_t\varphi - \frac{\varepsilon}{2}\partial^2_{xx}\varphi + g\partial_x\varphi + \frac{1}{2}\left(|\partial_x v|^2 - |\partial_x\tilde{w}^\varepsilon|^2\right) \geq 0$$

Now $|\partial_x v|^2 - |\partial_x\tilde{w}^\varepsilon|^2 = \partial_x\varphi \cdot (\partial_x v + \partial_x\tilde{w}^\varepsilon)$. Set

$$b^\varepsilon = g + \frac{1}{2}(\partial_x v + \partial_x\tilde{w}^\varepsilon).$$

Then $\mathscr{L}\varphi \geq 0$ and on Γ_R, $\varphi(z, s) \geq 0$. Hence $\varphi(x, t) \geq 0$ for all $(x, t) \in Q_R$ by Lemma 3.17. □

Proof of Theorem 3.15. The proof is very close to that given in [20], inspired by [15]. It relies on the construction of a function v independent of ε such that $\tilde{\mathscr{L}}v \geq 0$ in $\mathring{Q}_R$ and $\tilde{w}^\varepsilon \leq v$ on Γ_R, independent of (sufficiently small) $\varepsilon > 0$, which is achieved by making v tend to ∞ close to the boundary.
Proof of (iv). Let $R \geq 4\max(1, 2\|g\|)$ and $\varepsilon \leq \varepsilon_R := \frac{1}{2R^4}$. Define

$$v(x,t) = \frac{1}{R^2 - |x|^2} + \mu t + M$$

where the constants $\mu > 0, M > 0$ will be adequately chosen later. Then

$$\begin{aligned}\tilde{\mathscr{L}}v =& \mu - \frac{\varepsilon}{2}\left(\frac{2}{(R^2-|x|^2)^2} + \frac{8|x|^2}{(R^2-|x|^2)^3}\right) + \frac{2x}{(R^2-|x|^2)^2}\cdot g + \frac{2|x|^2}{(R^2-|x|^2)^4} - \tilde{V}_w^\varepsilon \\ =& \mu + \frac{1}{(R^2-x^2)^4}\left[x^2 - \varepsilon\left(R^2-x^2\right)\left(4x^2 + (R^2-x^2)\right)\right] + \tilde{\mathscr{E}}_R(x) + \mathscr{G}_R(x) - \tilde{V}_w^\varepsilon(x,t),\end{aligned}$$

where we define:

$$\begin{cases} \tilde{\mathscr{E}}_R(x) := \dfrac{2xg(x)}{(R^2-x^2)^2} \quad \geq \quad -\mathscr{E}_R(x) := -\dfrac{2|x|\|g\|_{\mathrm{L}^\infty(\mathbb{R})}}{(R^2-x^2)^2}, \\ \mathscr{G}_R(x) := \dfrac{x^2}{(R^2-x^2)^4} \geq 0. \end{cases}$$

Hence,

$$\begin{aligned}\tilde{\mathscr{L}}v \geq& \mu + \frac{1}{(R^2-x^2)^4}\left[x^2 - \varepsilon\left(R^4 + 2R^2x^2 - 3x^4\right)\right] - \mathscr{E}_R(x) + \mathscr{G}_R(x) - \tilde{V}_w^\varepsilon(x,t) \\ \geq& \mu + \frac{1}{(R^2-x^2)^4}\left[x^2 - \varepsilon\frac{4R^4}{3}\right] - \mathscr{E}_R(x) + \mathscr{G}_R(x) - \tilde{V}_w^\varepsilon(x,t),\end{aligned}$$

the term $\frac{4R^4}{3}$ being the maximum over $X := x^2 \in \mathbb{R}$ of the second order polynomial in X right above it. For $\varepsilon < \varepsilon_R$: either $|x| \geq 1$ and it follows that $x^2 - \varepsilon\frac{4R^4}{3} \geq 0$; or $|x| < 1$ and

$$\frac{1}{(R^2-x^2)^4}\left[x^2 - \varepsilon\frac{4R^4}{3}\right] \geq \frac{-1}{(R^2-1)^4}.$$

Hence,

$$\tilde{\mathscr{L}}v \geq \mu - \frac{1}{(R^2-1)^4} - \mathscr{E}_R(x) + \mathscr{G}_R(x) - \|\tilde{V}_w^\varepsilon\|_{\mathrm{L}^\infty(\mathbb{R}\times[0,T])}. \tag{42}$$

Claim: For all $(x,t) \in Q_R$,

$$-\mathscr{E}_R(x) + \mathscr{G}_R(x) \geq -8\max(1, \|g\|^2_{\mathrm{L}^\infty(\mathbb{R})}).$$

Proof of the Claim. We will prove the Claim for $x \in [0,R)$ now. *Mutatis mutandis*, the proof for $x \leq 0$ follows with no notable difference. Let $C = 4\max\left(1, \sqrt{\|g\|}\right)$, and $\eta = \frac{1}{C\sqrt{R}}$.

- If $x \leq R - \eta$:

$$\begin{aligned}\mathscr{E}_R(x) \leq \mathscr{E}_R(R-\eta) = 2\|g\|\frac{R-\eta}{\left(R^2-(R-\eta)^2\right)^2} =& \frac{C^2\|g\|}{2}\frac{R - \frac{1}{C\sqrt{R}}}{R - \frac{1}{C\sqrt{R}} + \frac{1}{4C^2R^2}} \\ \leq& \frac{C^2\|g\|}{2} \leq 8\max(1, \|g\|^2).\end{aligned}$$

And since $\mathscr{G}_R(x) \geq 0$, the claimed inequality is satisfied.

- Otherwise, $R-\eta<x<R$:

$$(R^2-x^2)^2\mathscr{G}_R(x)\geq(R^2-(R-\eta)^2)^2\mathscr{G}_R(R-\eta)=\frac{(R-\eta)^2}{(2\eta R-\eta^2)^2}=\frac{C^2}{4}\frac{R^2-\frac{2\sqrt{R}}{C}+\frac{1}{C^2R}}{R-\frac{C}{\sqrt{R}}+\frac{C^2}{4R^2}}.$$

To bound below the last fraction on the right-hand side, observe that since $C>1$ and $R\geq 4$, we have $\frac{2R}{C}<\frac{R^2}{2}$; and since

$$R\sqrt{R}\geq 8R\geq 8\max(4,8\|g\|)>C=4\max(1,\sqrt{\|g\|}),$$

we have:

$$\frac{C}{\sqrt{R}}-\frac{C^2}{4R^2}=\frac{C}{\sqrt{R}}\left[1-\frac{C}{4R\sqrt{R}}\right]\geq 0.$$

We obtain:

$$(R^2-x^2)^2\mathscr{G}_R(x)\geq\frac{C^2R}{8}.$$

Therefore,

$$\mathscr{G}_R(x)-\mathscr{E}_R(x)=(R^2-x^2)^2\left[\mathscr{G}_R(x)-2x\|g\|\right]\geq(R^2-x^2)^2\left[\frac{C^2R}{8}-2R\|g\|\right]\geq 0,$$

because $C^2\geq 16\|g\|$.

This concludes the proof of the Claim. □

From the Claim and equation (42), it is clear that $Lv\geq 0$ over Q_R, provided μ is chosen sufficiently large. Specifically,

$$\mu=\frac{1}{(R^2-1)^4}+8(1+\|g\|\|_{\mathrm{L}^\infty(\mathbb{R})})+\|\tilde{V}^\varepsilon_w\|_{\mathrm{L}^\infty(\mathbb{R}\times[0,T])} \tag{43}$$

suffices. Choose now $M=\|w_0\|_{\mathrm{L}^\infty(\mathbb{R})}$: large enough that

$$w_0(x)\leq M \text{ for all } x\in B_R.$$

Since $v(x,t)\to\infty$ as $|x|\to R$ uniformly in $t\in[0,T]$, it follows from the maximum principle that

$$\tilde{w}^\varepsilon\leq v \quad\text{in}\quad \mathring{Q}_R.$$

Similarly, by considering $-v$ instead of v, we can find a similar upper bound for $\tilde{w}^\varepsilon$.

Since v is continuous in $\mathring{Q}_R$ and $\max_{|x|\leq R/2}\frac{1}{(R^2-x^2)^2}=\frac{4}{3R^2}$, the following bound over $Q_{R/2}$ follows.

$$\|\tilde{w}^\varepsilon\|_{\mathrm{L}^\infty(Q_{R/2})}\leq\frac{4}{3R^2}+\|w_0\|_{\mathrm{L}^\infty}+\mu T, \tag{44}$$

with μ defined in (43). Hence,

$$\|\tilde{w}^\varepsilon\|_{\mathrm{L}^\infty(Q_{R/2})}\leq\|\tilde{w}_0\|_{\mathrm{L}^\infty(\mathbb{R})}+\left[8(1+\|g\|_{\mathrm{L}^\infty(\mathbb{R})})+\|V^\varepsilon\|_{\mathrm{L}^\infty(\mathbb{R}\times[0,T])}+\frac{1}{(R^2-1)^4}\right]T+\frac{4}{3R^2}. \tag{45}$$

The desired estimate follows, concluding the proof of (iv).
Proof of (i). $(iv)\Rightarrow(i)$.

Proof of (ii). The estimate of the partial derivative in x closely follows [20], using a variant of the techniques in [15]. It consists of the following steps.

- Define $Q \subset\subset Q' \subset\subset \mathbb{R} \times (0, T)$, where Q, Q' are open and "$\subset\subset$" means "compactly contained in".
- Choose a smooth function ζ such that $\zeta \equiv 1$ on Q and $\zeta \equiv 0$ near $\partial Q'$, and define
$$z := \zeta^2 |\tilde{w}^\varepsilon|^2 - \lambda \tilde{w}^\varepsilon,$$
where $\lambda > 0$ will be chosen later.
- Apply the maximum principle to z: z reaches its maximum in $\bar{Q}'$. Assume it's reached at $(x_0, t_0) \in Q'$. Then, since z is smooth,
$$\begin{cases} \partial_x z = 0, \\ \quad 0 \le \partial_t z - \dfrac{\varepsilon}{2} \partial_x^2 z. \end{cases}$$
- Writing the previous inequality explicitly in terms of ζ and $\tilde{w}^\varepsilon$ and using the Hamilton-Jacobi equation satisfied by $\tilde{w}^\varepsilon$ yields, for ε sufficiently small, at (x_0, t_0):
$$0 \le -\partial_x \tilde{w}^\varepsilon \cdot \partial_x \left(\zeta^2 |\partial_x \tilde{w}^\varepsilon|^2\right) - g \cdot \partial_x \left(\zeta^2 |\partial_x \tilde{w}^\varepsilon|^2\right) + \frac{\lambda}{2} |\partial_x \tilde{w}^\varepsilon|^2 + C\zeta |\partial_x \tilde{w}^\varepsilon|^3 + C |\partial_x \tilde{w}^\varepsilon|^2 + \lambda C |\partial_x \tilde{w}^\varepsilon| + \lambda C,$$
where we recall that C is a generic constant name. Using now $\partial_x z = 0$ at (x_0, t_0), we have
$$\frac{\lambda}{2} |\partial_x \tilde{w}^\varepsilon|^2 \le C\zeta |\partial_x \tilde{w}^\varepsilon|^3 + C |\partial_x \tilde{w}^\varepsilon|^2 + \lambda C |\partial_x \tilde{w}^\varepsilon| + \lambda C.$$
- Choosing $\lambda = \mu[(\max \zeta)|\partial_x w^\varepsilon| + 1]$, with $\mu > 1$ to be chosen yields:
$$\frac{\mu}{2} |\partial_x \tilde{w}^\varepsilon|^2 \le C |\partial_x \tilde{w}^\varepsilon|^2 + C\lambda\mu.$$
Hence for μ large enough, at (x_0, t_0),
$$|\partial_x \tilde{w}^\varepsilon|^2 \le C\lambda.$$
Hence:
$$z \le C\lambda \text{ in } Q' .$$
- If the max is reached at the boundary, the equation above holds since $\tilde{w}^\varepsilon$ is bounded. From it, [20] recover:
$$\max \zeta^2 |\partial_x \tilde{w}^\varepsilon|^2 \le \max z + C\lambda \le C\lambda$$
and by definition of λ,
$$\max \zeta^2 |\partial_x \tilde{w}^\varepsilon|^2 \le C\mu[\max \zeta |\partial_x \tilde{w}^\varepsilon| + 1],$$
which implies
$$\zeta |\partial_x \tilde{w}^\varepsilon| \le C \quad \text{ in } Q' ,$$
so
$$|\partial_x \tilde{w}^\varepsilon| \le C \quad \text{ in } \bar{Q},$$
concluding the proof.

Proof of (iii). Since $\mathcal{H}$ is locally bounded, the conditions of [10, Lemma 5.2] are met (with $\frac{\varepsilon}{2}$ here, instead of ε), which allows to conclude to the ε-dependent Hölder estimate:
$$\forall (x,t), (x,s) \in Q, \qquad |\tilde{w}^\varepsilon(x,t) - \tilde{w}^\varepsilon(x,s)| \le K \left(\sqrt{\varepsilon}\, |t-s|^{1/2} + |t-s|\right).$$
Since $\varepsilon \in (0, 1)$, taking $\varepsilon = 1$ in the right-hand side concludes the proof.

□

3.6. **Viscosity solution limit – Proof of Theorem 3.6.**

Theorem 3.19. *Assume w_0 is bounded and Lipschitz continuous, and $\mathcal{H}$ satisfies the assumptions of Theorem 3.8. Then there exists a unique viscosity solution of the limiting equation (36), defined over $(x, t) \in \mathbb{R}_+ \times \mathbb{R}_+$, and that solution can be obtained by the vanishing viscosity method.*

Proof. The bounds (i), (ii), and (iii) of Corollary 3.14 and the Arzela-Ascoli theorem (see *e.g.* [26, Theorem 1]) imply that there exists a decreasing subsequence $(\varepsilon_k)_{k\in\mathbb{N}}$ that tends to 0 such that w^{ε_k} converges uniformly over compact sets to a continuous function w. From bound (iv), it follows that w is bounded over $\mathbb{R}_+ \times [0, T]$. Since $\mathcal{H}^\varepsilon$ also converges uniformly over compact sets to $\mathcal{H}$, Proposition 3.11, we may apply the stability result in [3]. Uniqueness results from Theorem 3.8. □

We may now consider, for $0 < \varepsilon < 1$ and for all $(x, t) \in \mathbb{R}_+ \times [0, T]$:

$$\mathcal{S}^\varepsilon(x, t) = w^\varepsilon(x, t) + y(t)h(x).$$

By construction, $\mathcal{S}^\varepsilon$ is smooth and satisfies the second order evolution Hamilton-Jacobi equation (27). The uniqueness of the solution to that equation – point (i) of Theorem 3.6 – is a direct consequence of Corollary 3.14. The local uniform convergence of the Hamiltonian $\mathcal{H}^\varepsilon_{\mathcal{S}}$ to $\mathcal{H}_{\mathcal{S}}$, point (ii), results from Proposition 3.11. Since y and h are bounded and have bounded derivatives, appropriate bounds can be obtained on $\mathcal{S}^\varepsilon$ of the type of those in Corollary 3.14. So (iii), the convergence of $\mathcal{S}^\varepsilon$ to $\mathcal{S}$, follows from that of w^ε to w in the proof of Theorem 3.19.

As for point (iv), the well-posedness of the limit equation follows from that of equation (36): Theorem 3.19. Since y and h are smooth enough, yh may be added or subtracted to any test function, guaranteeing that definition 3.7 applies for w in equation (36) if and only if it applies for $\mathcal{S}$ in equation (31). The vanishing viscosity limit procedure also works in a similar way, concluding the proof of Theorem 3.6.

4. Dynamic programming principle for the HJB limit

At this point, the vanishing viscosity procedure has provided a functional that one could expect to be the cost-to-come associated to the Mortensen estimation of the Skorohod problem. Unfortunately, despite its stochastic interpretation, this function cannot be linked as in [20] to the cost-to-come of the deterministic problem.

4.1. **A control problem interpretation of the limit solution**

The limit $w(x, t) = \mathcal{S}(x, t) - y(t)h(x)$ can be characterised as the unique viscosity solution of the HJB equation (36). Following the method of [20], a backward control process is now built whose cost function $\mathcal{W}$ will be identified to w. Consider the control process associated to the $\mathbb{R}_+$-valued backward trajectories $(z^{x,t}_\omega(s))_{0\le s\le t}$ defined by

$$\begin{cases} \forall \text{ a.e. } s \in [0, t],\ \forall q \geq 0,\ (\dot{z}^{x,t}_\omega(s) - \omega(s))(q - z^{x,t}_\omega(s)) \leq 0 \\ z^{x,t}_\omega(t) = x, \end{cases} \tag{46}$$

Assume this system is partially known through the perturbed observation function $y(t)$ given by

$$\dot{y}(s) = h(z^{x,t}_\omega(s)) + \eta(s),$$

the control parameters η and ω being square-integrable $\mathbb{R}$-valued functions of time. To each such ω can be associated a backward trajectory $z^{x,t}_\omega$. The control problem then consists in minimizing a functional $\psi(z^{x,t}_\omega(0))$ of the arrival point at time 0, together with the L^2 weights of control functions ω and η. The cost rate is thus

$$\tilde{\ell}(\omega, z, s) := \frac{1}{2}\omega^2 + \frac{1}{2}|\dot{y}(s) - h(z)|^2,$$

so that the cost to go (backward in time) from x at time t to time 0 reads

$$\inf_{\omega\in \mathrm{L}^2(0,t)} \psi(z_\omega^{x,t}(0)) + \int_0^t \tilde{\ell}\left(\omega(s), z_\omega^{x,t}(s), s\right) \mathrm{d}s,$$

developing the square $|\dot{y}(s) - h(z(s))|^2$, the term $|\dot{y}(s)|^2$ doesn't affect the minimization problem, and the cost rate can be chosen to be

$$\ell(\omega(s), z(s), s) := \frac{1}{2}\omega^2(s) + \frac{1}{2}h^2(z(s)) - \dot{y}(s)h(z(s)),$$

as required to take the limit in the probabilistic setting. This leads to the functional

$$\mathcal{J}(\omega, x, t) := \psi(z_\omega^{x,t}(0)) + \int_0^t \ell\left(\omega(s), z_\omega^{x,t}(s), s\right) \mathrm{d}s,$$

and the cost function $\mathcal{W}(x,t) := \inf_\omega \mathcal{J}(\omega, x, t)$ will appear to be the desired target function. Note the initial value condition $\mathcal{W}(x,0) = \psi(x)$.

Lemma 4.1 (Principle of Optimality)**.** *Consider a terminal point (x,t). Then for every $0 < \tau < t$*

$$\mathcal{W}(x,t) = \inf_{\omega\in \mathrm{L}^2(t-\tau,t)} \left[\mathcal{W}\left(z_\omega^{x,t}(t-\tau), t-\tau\right) + \int_{t-\tau}^t \ell\left(\omega(s), z_\omega^{x,t}(s), s\right) \mathrm{d}s\right].$$

Proof. Given another control $(\omega'(s))_{0\le s\le t-\tau}$, define the square-integrable control

$$\tilde{\omega}(s) = \begin{cases} \omega'(s) & \text{if } 0 \le s < t-\tau, \\ \omega(s) & \text{if } t-\tau \le s \le t. \end{cases}$$

For $s < t-\tau$ note that $z_{\tilde{\omega}}^{x,t}(s) = z_{\omega'}^{z_\omega^{x,t}(t-\tau), t-\tau}(s)$, so that by definition of $\mathcal{W}$

$$\mathcal{W}(x,t) \le \psi\left(z_{\omega'}^{z_\omega^{x,t}(t-\tau),t-\tau}(0)\right) + \int_0^{t-\tau} \ell\left(\omega'(s), z_{\omega'}^{z_\omega^{x,t}(t-\tau),t-\tau}(s), s\right) \mathrm{d}s + \int_{t-\tau}^t \ell\left(\omega(s), z_\omega^{x,t}(s), s\right) \mathrm{d}s,$$

and taking the infimum over ω' and ω concludes. Equality is achieved by considering a sequence of controls whose costs converge towards the infimum. □

Lemma 4.2 (Uniform terminal continuity)**.** *Consider a terminal point (x,t) and $M > 0$; then $s \mapsto z_\omega^{x,t}(s)$ is continuous at the terminal point $s = t$ uniformly in ω such that $\mathcal{J}(x,\omega,t) \le M$.*

Proof. Consider $\varepsilon > 0$ and a control ω. If $x > 0$, the continuity of $z_\omega^{x,t}$ at t guarantees that

$$\tau_\omega := \sup\left\{\tau > 0\,,\, z_\omega^{x,t}(t-\tau) > 0 \text{ and } |z_\omega^{x,t}(t-\tau) - x| \le \varepsilon\right\} > 0.$$

Considering $0 < \tau < \min(\tau_\omega, 1)$ to make sure that $z_\omega^{x,t}(s) > 0$, one has $\dot{z}_\omega^{x,t}(s) = \omega(s)$ for $t-\tau \le s \le t$ thanks to 46. Thus

$$x - z_\omega^{x,t}(t-\tau) = \int_{t-\tau}^t \dot{z}_\omega^{x,t}(s)\mathrm{d}s = \int_{t-\tau}^t \omega(s)\mathrm{d}s.$$

Using Cauchy-Schwarz inequality

$$|x - z_\omega^{x,t}(t-\tau)| \le \sqrt{2\tau\mathcal{J}(\omega,x,t)} \le \sqrt{2\tau M},$$

and this proves the bound $\tau_\omega \geq \left(\frac{\min(\varepsilon,|x|)}{\sqrt{2M}}\right)^2$, the right-hand side being independent of ω.
In the case $x = 0$, consider

$$\tau_\omega^0 := \sup\left\{\tau > 0\,,\, \forall t-\tau \leq s \leq t,\, z_\omega^{x,t}(s) = 0\right\},$$
$$\tau_\omega^1 := \sup\left\{\tau > \tau_\omega^0\,,\, z_\omega^{x,t}(t-\tau) > 0 \text{ and } z_\omega^{x,t}(t-\tau) \leq \varepsilon\right\}.$$

The continuity of $z_\omega^{x,t}$ indeed guarantees $\tau_\omega^0 < \tau_\omega^1$; since for $\tau_\omega^0 < \tau < \tau' < \tau_\omega^1$

$$z_\omega^{x,t}(t-\tau) - z_\omega^{x,t}(t-\tau') = \int_{t-\tau'}^{t-\tau} \dot{z}_\omega^{x,t}(s)\mathrm{d}s = \int_{t-\tau'}^{t-\tau} \omega(s)\mathrm{d}s,$$

the same reasoning as above gives a positive lower bound for $\tau_\omega^1 - \tau_\omega^0$ which is independent of ω, completing the proof. □

The function $\mathcal{W}$ can now be identified to the previous limit using the HJB equation (36). Note that the Hamiltonian $\mathcal{H}$ can equivalently be defined as

$$\mathcal{H}(x,t,\lambda) = \max_{\omega' \in \mathbb{R}} \lambda\omega' - \ell\left(x,\omega',t\right). \tag{47}$$

Proposition 4.3 (Sub-solution). *The function $\mathcal{W}$ is a viscosity sub-solution of* (36).

Proof. For $x \geq 0$ and $t > 0$, consider a C^1 test function ϕ such that $\mathcal{W} - \phi$ has a local maximum at point (x,t). For any control ω and every $\tau > 0$ small enough, this leads to

$$\mathcal{W}\left(z_\omega^{x,t}(t-\tau), t-\tau\right) - \phi\left(z_\omega^{x,t}(t-\tau), t-\tau\right) \leq \mathcal{W}(x,t) - \phi(x,t),$$

because of $z_\omega^{x,t}(t) = x$ and the continuity of $z_\omega^{x,t}$ at (x,t). Therefore, we have

$$\phi(x,t) - \phi\left(z_\omega^{x,t}(t-\tau), t-\tau\right) \leq \mathcal{W}(x,t) - \mathcal{W}\left(z_\omega^{x,t}(t-\tau), t-\tau\right) \leq \int_{t-\tau}^{t} \ell\left(\omega(s), z_\omega^{x,t}(s), s\right)\mathrm{d}s,$$

using the principle of optimality of Lemma 4.1. Dividing by τ and taking the $\tau \to 0^+$ limit gives

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t} \phi\left(z_\omega^{x,t}(s), s\right) \leq \ell\left(\omega(t), z_\omega^{x,t}(t), t\right),$$

so that

$$\partial_t\phi(x,t) + \dot{z}_\omega^{x,t}(t)\partial_x\phi(x,t) - \ell\left(\omega(t), x, t\right) \leq 0.$$

Then, we have

$$\partial_t\phi(x,t) + \omega(t)\partial_x\phi(x,t) - \ell\left(\omega(t), x, t\right) \leq \partial_x\phi(x,t)\left[\omega(t) - \dot{z}_\omega^{x,t}(t)\right].$$

If $x > 0$ then $\dot{z}_\omega^{x,t}(t) = \omega(t)$ according to (46); else $x = 0$ so that $\omega(t) - \dot{z}_\omega^{x,t}(t) \geq 0$, and one can assume $\partial_x\phi(0,t) \leq 0$ following the definition (3.7). In every case

$$\partial_x\phi(x,t)\left[\omega(t) - \dot{z}_\omega^{x,t}(t)\right] \leq 0.$$

Since this is true for every ω, taking the maximum over $\omega(t)$ allows to recover (47) and

$$\partial_t\phi(x,t) + \mathcal{H}\left(x,t,\partial_x\phi(x,t)\right) \leq 0,$$

as desired. □

Proposition 4.4 (Super-solution). *The function $\mathcal{W}$ is a viscosity super-solution of* (36).

Proof. For $x \geq 0$ and $t > 0$, consider a C^1 test function ϕ such that $\mathcal{W} - \phi$ has a local minimum at point (x,t). Positive numbers $\delta, \delta' > 0$ exist such that

$$|t - t'| \leq \delta \text{ and } |x - x'| \leq h \Rightarrow \mathcal{W}(x', t') - \phi(t', x') \geq \mathcal{W}(x,t) - \phi(x,t)\,. \tag{48}$$

Fix now $\varepsilon > 0$ and $M > \mathcal{W}(x,t)$. By lemma 4.2 $\delta' > 0$ exists such that for every ω with $\mathcal{J}(x, \omega, t) \leq M$

$$0 \leq \tau \leq \delta' \Rightarrow |z_\omega^{x,t}(t-\tau) - x| \leq h.$$

Consider a sequence $(\tau_n)_{n\geq 0}$ which converges to 0 with $0 < \tau_n \leq \min(\delta, \delta')$. In the principle of optimality 4.1 which characterises $\mathcal{W}(x,t)$, it is sufficient to minimize over ω with $\mathcal{J}(\omega, x, t) \leq M$, because $M > \mathcal{W}(x,t)$. Then by definition of the infimum, ω_n with $\mathcal{J}(x, \omega_n, t) \leq M$ exists for every n, satisfying

$$\mathcal{W}(x,t) + \varepsilon\tau_n \geq \mathcal{W}\left(z_{\omega_n}^{x,t}(t-\tau_n), t-\tau_n\right) + \int_{t-\tau_n}^{t} \ell\left(\omega_n(s), z_{\omega_n}^{x,t}(s), s\right) \mathrm{d}s.$$

Using 48, it follows

$$\begin{aligned}\phi(x,t) - \phi\left(z_{\omega_n}^{x,t}(t-\tau_n), t-\tau_n\right) &\geq \mathcal{W}(x,t) - \mathcal{W}\left(z_{\omega_n}^{x,t}(t-\tau_n), t-\tau_n\right) \\ &\geq -\varepsilon\tau_n + \int_{t-\tau_n}^{t} \ell\left(\omega_n(s), z_{\omega_n}^{x,t}(s), s\right) \mathrm{d}s.\end{aligned}$$

The functions ϕ and $z_{\omega_n}^{x,t}$ being differentiable, taking the s-derivative in $\phi\left(z_\omega^{x,t}(s), s\right)$ leads

$$\phi(x,t) - \phi\left(z_{\omega_n}^{x,t}(t-\tau_n), t-\tau_n\right) = \int_{t-\tau_n}^{t} \partial_t\phi\left(z_{\omega_n}^{x,t}(s), s\right) + \dot{z}_{\omega_n}^{x,t}(s)\partial_x\phi\left(z_{\omega_n}^{x,t}(s), s\right) \mathrm{d}s.$$

Therefore, we have

$$\int_{t-\tau_n}^{t} \partial_t\phi\left(z_{\omega_n}^{x,t}(s), s\right) + \dot{z}_{\omega_n}^{x,t}(s)\partial_x\phi\left(z_{\omega_n}^{x,t}(s), s\right) - \ell\left(\omega_n(s), z_{\omega_n}^{x,t}(s), s\right) \mathrm{d}s \geq -\varepsilon\tau_n.$$

Adding $\int_{t-\tau_n}^{t} \partial_x\phi\left(z_{\omega_n}^{x,t}(s), s\right)\omega_n(s)\mathrm{d}s$ to each side,

$$\begin{aligned}&\int_{t-\tau_n}^{t} \partial_t\phi\left(z_{\omega_n}^{x,t}(s), s\right) + \partial_x\phi\left(z_{\omega_n}^{x,t}(s), s\right)\omega_n(s) - \ell\left(\omega_n(s), z_{\omega_n}^{x,t}(s), s\right) \mathrm{d}s \\ &\qquad\qquad \geq -\varepsilon\tau_n + \int_{t-\tau_n}^{t} \partial_x\phi\left(z_{\omega_n}^{x,t}(s), s\right)\left[\omega_n(s) - \dot{z}_{\omega_n}^{x,t}(s)\right] \mathrm{d}s.\end{aligned}$$

Note now that

$$\begin{aligned}\mathcal{H}\left(z_{\omega_n}^{x,t}(s), s, \partial_x\phi\left(z_{\omega_n}^{x,t}(s), s\right)\right) &= \max_{\omega'\in\mathbb{R}} \partial_x\phi\left(z_{\omega_n}^{x,t}(s), s\right)\omega' - \ell\left(\omega', z_{\omega_n}^{x,t}(s), s\right) \\ &\geq \partial_x\phi\left(z_{\omega_n}^{x,t}(s), s\right)\omega_n(s) - \ell\left(\omega_n(s), z_{\omega_n}^{x,t}(s), s\right).\end{aligned}$$

Moreover if $x > 0$, the uniform convergence of lemma 4.2 allows to take n large enough so that $z_{\omega_n}^{x,t}(s) > 0$ for $t - \tau_n \leq s \leq t$ and thus $\dot{z}_{\omega_n}^{x,t}(s) = \omega_n(s)$. If $x = 0$ one can assume $\partial_x\phi(0,t) \geq 0$, and use the fact that $\omega_n(s) - \dot{z}_{\omega_n}^{x,t}(s) \geq 0$ by 46, with equality when $z_{\omega_n}^{x,t}(s) > 0$. In every case

$$\int_{t-\tau_n}^{t} \partial_x\phi\left(z_{\omega_n}^{x,t}(s), s\right)\left[\omega_n(s) - \dot{z}_{\omega_n}^{x,t}(s)\right] \mathrm{d}s \geq 0,$$

for n large enough. Thus

$$\int_{t-\tau_n}^{t} \partial_t \phi\left(z_{\omega_n}^{x,t}(s), s\right) + \mathcal{H}\left(z_{\omega_n}^{x,t}(s), s, \partial_x \phi\left(z_{\omega_n}^{x,t}(s), s\right)\right) \mathrm{d}s \geq -\varepsilon \tau_n.$$

Lemma 4.2 guarantees the continuity of $s \mapsto z_{\omega_n}^{x,t}(s)$ at $s = t$ uniformly in ω_n such that $W_{\omega_n}(t,s) \leq M$, so that dividing by τ_n and taking the $n \to +\infty$ limit gives

$$\partial_t \phi\left(t, x\right) + \mathcal{H}\left(x, s, \partial_x \phi\left(x, t\right)\right) \geq -\varepsilon.$$

Since this hold for every $\varepsilon > 0$, this concludes the proof. □

Theorem 4.5 (Identification)**.** *Using the uniqueness result from Theorem 3.13, it is now possible to identify the solution w of (36) to $\mathcal{W}$, provided that the initial condition is $\psi(x) = w_0(x)$.*

This establishes the desired link between the stochastic filtering problem (19) and the control problem (46). In particular, the limit doesn't allow to compute a recursive estimator, because it stems from a control problem and not a filtering one. The estimation has thus to be done by keeping some approximating noise with (small) amplitude $\varepsilon > 0$, or using the penalized dynamics.

4.2. Lost equivalence with Mortensen's approach

Let's go back to the estimation problem of the constrained dynamics (1) with $f = 0$, namely the Skorokhod problem:

$$\begin{cases} \forall \text{ a.e. } t \in [0,T], \; \forall z \geq 0, \; (\omega(t) - \dot{x}(t))(z - x(t)) \leq 0 \\ x(0) = \zeta. \end{cases} \tag{49}$$

As in Section 2.2, it could be tempting to use a direct deterministic filtering approach base on the cost to come

$$\mathcal{V}(x,t) := \inf_{(\omega,\zeta) \in \mathcal{A}_{x,t}} \left[\psi(\zeta) + \int_0^t \ell(\omega(s), x_{|\omega,\zeta}(s), s) \,\mathrm{d}s \right],$$

where we omit $\dot{y}$ to simplify the notation and the pre-image set can be also defined by

$$\mathcal{A}_{x,t} := \left\{ (\omega, \zeta) \in \mathrm{L}^2(0,t) \times \mathbb{R}_+ : x_{|\omega,\zeta} \text{ follows (49) with } x_{|\omega,\zeta}(0) = \zeta, x_{|\omega,\zeta}(t) = x \right\}.$$

This admissible set is never empty, because it is always possible to reach every $x \geq 0$ at time t starting from any positive $\zeta \geq 0$ by considering a (slow enough) straight line without reflection. However, the dynamics (1) is now well-posed in forward time only: given a value x at time $t > 0$ and a control ω, there's no more well-posedness for the backward in time problem starting from x at time t. This feature is due to the non-reversibility introduced by the reflection and complicates the situation a lot. Furthermore $\mathcal{V}(x,t)$ can no more be easily characterized as the solution of the expected HJB equation (36). Indeed, let's try to show – as done for $\mathcal{W}$ in Proposition 4.3 – that $\mathcal{V}$ is a viscosity sub-solution of equation (36). First of all, one could prove the analogous of Theorem 2.1, which would read here:

$$\mathcal{V}(x,t) = \inf_{(\omega,\zeta) \in \mathcal{A}_{x,t}} \left[\mathcal{V}\left(x_{|\omega,\zeta}(t-\tau), t-\tau\right) + \int_{t-\tau}^t \ell\left(\omega(s), x_{|\omega,\zeta}(s), s\right) \mathrm{d}s \right], \tag{50}$$

Let's now mimic the proof of Proposition 4.3: for $x \geq 0$ and $t > 0$, consider a C^1 test function ϕ such that $\mathcal{V} - \phi$ has a local maximum at point (x,t). For any control ω, any initial condition ζ and every $\tau > 0$ small enough, this leads to

$$\mathcal{V}\left(x_{|\omega,\zeta}(t-\tau), t-\tau\right) - \phi\left(x_{|\omega,\zeta}(t-\tau), t-\tau\right) \leq \mathcal{V}(x,t) - \phi(x,t),$$

because of $x_{|\omega,\zeta}(t) = x$ and the continuity of $x_{|\omega,\zeta}$ at (x,t). We have

$$\phi(x,t) - \phi\left(x_{|\omega,\zeta}(t-\tau), t-\tau\right) \leq \mathcal{V}(x,t) - \mathcal{V}\left(x_{|\omega,\zeta}(t-\tau), t-\tau\right) \leq \int_{t-\tau}^{t} \ell\left(\omega(s), x_{|\omega,\zeta}(s), s\right) \mathrm{d}s,$$

using the principle of optimality given by (50). Dividing by τ and taking the $\tau \to 0^+$ limit gives

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t} \phi\left(x_{|\omega,\zeta}(s), s\right) \leq \ell\left(\omega(t), x_{|\omega,\zeta}(t), t\right),$$

so that

$$\partial_t\phi(x,t) + \dot{x}_{|\omega,\zeta}(t)\partial_x\phi(x,t) - \ell\left(\omega(t), x, t\right) \leq 0.$$

Then

$$\partial_t\phi(x,t) + \omega(t)\partial_x\phi(x,t) - \ell\left(\omega(t), x, t\right) \leq \partial_x\phi(x,t)\left[\omega(t) - \dot{x}_{|\omega,\zeta}(t)\right]. \tag{51}$$

If $x > 0$, then $\dot{x}_{|\omega,\zeta}(t) = \omega(t)$ according to (49). Hence

$$\partial_t\phi(x,t) + \mathcal{H}\left(x, t, \partial_x\phi(x,t)\right) = 0 \leq 0,$$

as desired. However if $x = 0$ then $\omega(t) - \dot{x}_{\kappa|\omega,\zeta}(t) \leq 0$ by definition of the sub-differential dynamics (49). Considering ϕ such that $\partial_x\phi(0,t) \leq 0$, we get

$$\partial_x\phi(x,t)\left[\omega(t) - \dot{x}_{|\omega,\zeta}(t)\right] \geq 0,$$

which, when combined to (51) does not allow to constrain $\partial_t\phi(x,t) + \omega(t)\partial_x\phi(x,t) - \ell\left(\omega(t), x, t\right)$ to be non-positive. The boundary condition appears to be

$$\min\left\{+\partial_x\phi(0,t_0)\ ,\ (\partial_t\phi + H(\cdot, u, \partial_x\phi))(0,t_0)\right\} \leq 0. \tag{52}$$

A similar situation would arise if one tried to prove the super-solution property for $\mathcal{V}$ (the analog of Proposition 4.4). We therefore believe that the connection between the viscosity limit of stochastic filtering and deterministic filtering for dynamics nonreversible in time is broken, and $\mathcal{V}$ cannot be computed from a forward dynamics that appears – from (52) – to be an ill-posed HJB dynamics.

As a result, a recursive estimator of (49) – and similarly for (1) – cannot be the Mortensen estimator computed from $\mathcal{V}$ which does not appear to follow a well-posed HJB equation. As a consequence, to obtain a computable sequential estimator, one must choose between two alternatives:

- approximate the dynamics (1) with the penalized dynamics (5), resulting in an approximate Moretensen estimator;
- define the stochastic filtering problem in terms of (19) and use the tools of stochastic filtering and particle filtering [8] for a small but nonzero value of ε.

5. Appendix

5.1. Derivation of the robust Zakai equation

Lemma 5.1 (Robust Zakai equation)**.** *The random function p^ε satisfies the robust Zakai equation, adding some Robin boundary conditions:*

$$\begin{cases} \partial_t p^\varepsilon(x,t) - y(t)h'(x)\partial_x p^\varepsilon(x,t) + \dfrac{1}{\varepsilon}\mathcal{P}^\varepsilon(x,t)p^\varepsilon(x,t) = \dfrac{\varepsilon}{2}\partial^2_{xx}p^\varepsilon(x,t), & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ \dfrac{\varepsilon}{2}\partial_x p^\varepsilon(0,t) + \dfrac{Y_t h'(x)}{2}p^\varepsilon(0,t) = 0, & t \in \mathbb{R}_+, \end{cases} \tag{53}$$

where

$$\mathcal{P}^\varepsilon(x,t) := \frac{h^2(x)}{2} - \frac{\varepsilon}{2} Y_t h''(x) - \frac{1}{2} Y_t^2 (h'(x))^2.$$

This recovers a result in [11] for robust filtering of reflected diffusion.

Proof. Thanks to Girsanov change of measure (see e.g. [1, 39]), it is sufficient to treat the case where $\left(\frac{Y_t}{\sqrt{\varepsilon}}\right)_{t\geq 0}$ is a standard brownian motion. Then, using Ito's rule for stochastic differential calculus

$$\mathrm{d}p^\varepsilon(x,t) = \exp\left[-\frac{Y_t h(x)}{\varepsilon}\right] \mathrm{d}q^\varepsilon(x,t) + q^\varepsilon(x,t)\mathrm{d}\exp\left[-\frac{Y_t h(x)}{\varepsilon}\right] + \mathrm{d}\left[\exp\left[-\frac{Y_. h(x)}{\varepsilon}\right], q^\varepsilon(x,\cdot)\right]_t,$$

the quadratic cross-variation being given by

$$\mathrm{d}\left[\exp\left[-\frac{Y_. h(x)}{\varepsilon}\right], q^\varepsilon(x,\cdot)\right]_t = -\frac{q^\varepsilon(x,t)h^2(x)}{\varepsilon} \exp\left[-\frac{Y_t h(x)}{\varepsilon}\right].$$

Moreover, by Ito's rule

$$\mathrm{d}\exp\left[-\frac{Y_t h(x)}{\varepsilon}\right] = -\frac{h(x)}{\varepsilon}\exp\left[-\frac{Y_t h(x)}{\varepsilon}\right]\mathrm{d}Y_t + \frac{h^2(x)}{2\varepsilon}\exp\left[-\frac{Y_t h(x)}{\varepsilon}\right]\mathrm{d}t,$$

using (20)

$$\mathrm{d}q^\varepsilon(x,t) = \frac{\varepsilon}{2}\partial^2_{xx} q^\varepsilon(x,t)\mathrm{d}t + \frac{q^\varepsilon(x,t)}{\varepsilon}\mathrm{d}Y_t,$$

this gives

$$\frac{\mathrm{d}}{\mathrm{d}t} p^\varepsilon(x,t) = \frac{\varepsilon}{2}\exp\left[-\frac{Y_t h(x)}{\varepsilon}\right]\partial^2_{xx} p^\varepsilon(x,t) - \frac{h^2(x)}{2\varepsilon} p^\varepsilon(x,t),$$

noticing that

$$\exp\left[-\frac{Y_t h(x)}{\varepsilon}\right]\partial_x q^\varepsilon(x,t) = \partial_x p^\varepsilon(x,t) + \frac{Y_t h'(x)}{\varepsilon} p^\varepsilon(x,t),$$

it is straightforward to obtain that

$$\partial^2_{xx} p^\varepsilon(x,t) = \exp\left[-\frac{Y_t h(x)}{\varepsilon}\right]\partial^2_{xx} q^\varepsilon(x,t) - \frac{2Y_t h'(x)}{\varepsilon} p^\varepsilon(x,t) - p^\varepsilon(x,t)\left[\frac{(Y_t h'(x))^2}{\varepsilon} + \frac{Y_t h''(x)}{\varepsilon}.\right]$$

Gathering everything

$$\frac{\mathrm{d}}{\mathrm{d}t} p^\varepsilon(x,t) = Y_t h'(x)\partial_x p^\varepsilon(x,t) + \frac{p^\varepsilon(x,t)}{\varepsilon}\left(-\frac{h^2(x)}{2} + \frac{(Y_t h'(x))^2}{2} + \frac{\varepsilon}{2} Y_t h''(x)\right) + \frac{\varepsilon}{2}\partial^2_{xx} p^\varepsilon(x,t),$$

which is the desired equation. The boundary conditions are directly obtained from the ones in (20). □

In equation (53), note that the random variable Y_t just behaves as a parameter, which only appears inside the coefficients. This parameter Y_t being defined as the function $\omega \in \Omega \mapsto Y(t,\omega)$, this can be seen as a family of deterministic PDEs indexed by a parameter ω. At this point, it is only necessary to consider given realisations of the trajectory, i.e. continuous deterministic functions $(y(s))_{0\leq s\leq t}$. The remaining question will then be the measurability of the solution in ω, in order to recover a stochastic process $p^\varepsilon(\omega,x,t)$ from solving a deterministic PDE for each $(y(s))_{0\leq s\leq t}$. This question is positively answered by the prominent works [13], [37] which even prove that considering C^1 trajectories $y(t)$ is sufficient. As in the whole paper, this allows to consider $p^\varepsilon(x,t)$ as a deterministic function which depends on a given C^1 trajectory $(y(s))_{0\leq s\leq t}$. The function $p^\varepsilon(x,t)$ is thus the solution of a linear parabolic PDE, for which strong C^2 regularity can be shown using the classical theory.

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References

[1] A. Bain and D. Crisan. Fundamentals of stochastic filtering, volume 60. Springer Science & Business Media, 2008.
[2] T. Banek. Filtering of absorbing and reflecting brownian motions. Systems & control letters, 8(2):153–159, 1986.
[3] G. Barles. Fully non-linear neumann type boundary conditions for second-order elliptic and parabolic equations. Journal of Differential Equations, 106(1):90–106, 1993.
[4] G. Barles. Solutions de viscosité des équations de Hamilton-Jacobi, volume 17 of Mathématiques & Applications (Berlin) [Mathematics & Applications]. Springer-Verlag, Paris, 1994.
[5] J. B., F. Bernardin, and C.-H. Lamarque. Non-smooth deterministic or stochastic discrete dynamical systems. Mechanical Engineering and Solid Mechanics Series. ISTE, London; John Wiley & Sons, Inc., Hoboken, NJ, 2013. Applications to models with friction or impact.
[6] J. Bastien, M. Schatzman, and C.-H. Lamarque. Study of some rheological models with a finite number of degrees of freedom. Eur. J. Mech. A Solids, 19(2):277–307, 2000.
[7] H. Brezis. Ope|rateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. ISSN. Elsevier Science, 1973.
[8] Z. Chen. Bayesian filtering: From Kalman filters to particle filters, and beyond. Statistics, 182(1):1 – 69, 2003.
[9] J. M. C. Clark and D Crisan. On a robust version of the integral representation formula of nonlinear filtering. Probability theory and related fields, 133(1):43–56, 2005.
[10] M. G. Crandall and P. L. Lions. Two approximations of solutions of hamilton-jacobi equations. Mathematics of Computation, 43(167):1–19, 1984.
[11] M. H. A. Davis. On a multiplicative functional transformation arising in nonlinear filtering theory. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, 54(2):125–139, 1980.
[12] J.-P. Demailly. Analyse numérique et équations différentielles. Grenoble Sciences. EDP Sciences, Les Ulis, fourth edition, 2016.
[13] H. Doss. Liens entre équations différentielles stochastiques et ordinaires. In Annales de l'IHP Probabilités et statistiques, volume 13, pages 99–125, 1977.
[14] G. Duvaut and J.-L. Lions. Inequalities in mechanics and physics, volume 219 of Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin-New York, 1976. Translated from the French by C. W. John.
[15] L. C. Evans and H. Ishii. A PDE approach to some asymptotic problems concerning random differential equations with small noise intensities. Ann. Inst. H. Poincaré Anal. Non Linéaire, 2(1):1–20, 1985.
[16] W. H Fleming. Deterministic nonlinear filtering. Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV, 25(3-4):435 – 454 (1998), 1997.
[17] A. Friedman. Partial differential equations of parabolic type. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.
[18] O. Hijab. Minimum Energy Estimation. PhD thesis, University of California, Berkeley, 1980.
[19] O. Hijab. Asymptotic nonlinear filtering and large deviations. Advances in Filtering and Optimal Stochastic Control, pages 170–176, 1982.
[20] M. R. James and J. S. Baras. Nonlinear filtering and large deviations: A PDE-control theoretic approach. Stochastics: An International Journal of Probability and Stochastic Processes, 23(3):391–412, 1988.
[21] A. H. Jazwinsky. Stochastic processes and filtering theory. Academic Press, 1970.
[22] G Kallianpur and C. Striebel. Estimation of stochastic systems: Arbitrary system process with additive white noise observation errors. The Annals of Mathematical Statistics, 39(3):785–801, 1968.
[23] I. Karatzas, I. K. S. Shreve, S. Shreve, and S. E. Shreve. Brownian Motion and Stochastic Calculus, volume 113 of Graduate Texts in Mathematics. Springer New York, 1991.
[24] A. J. Krener. Minimum energy estimation and moving horizon estimation. In 54th IEEE Conference on Decision and 2015, 2015.
[25] A. J. Krener. A Lyapunov theory of nonlinear observers. In G G Yin and Qing Zhang, editors, Stochastic analysis, control, optimization and applications, pages 409–420. Springer, 1998.
[26] P.-L. Lions. Neumann type boundary conditions for Hamilton-Jacobi equations. Duke Mathematical Journal, 52(3), 1985.

[27] P.-L. Lions. Optimal stochastic control with state constraints. In Stochastic Differential Systems Filtering and Control, pages 286–295. Springer, 1985.
[28] J. L. Menaldi. Stochastic control problem for reflected diffusions in a convex bounded domain. In Advances in Filtering and Optimal Stochastic Control, pages 246–255. Springer, 1982.
[29] R. E. Mortensen. Maximum-likelihood recursive nonlinear filtering. J. Optim. Theory Appl., 2(6):386 – 394, 1968.
[30] E. Pardoux. Stochastic partial differential equation for the density of the conditional law of a diffusion process with boundary. In Stochastic analysis (Proceeings of International Conference, Northwestern University, Evanston, Ill., 1978), pages 239–269, 1978.
[31] E. Pardoux. Stochastic partial differential equations and filtering of diffusion processes. 1980.
[32] E. Pardoux. Équations du filtrage non linéaire de la prédiction et du lissage. Stochastics, 6(3-4):193–231, 1982.
[33] E. Pardoux. Filtrage de diffusions avec conditions frontieres: caracterisation de la densite conditionnelle. In Journées de Statistique des Processus Stochastiques, pages 163–188. Springer, 1978.
[34] E. Pardoux. Filtrage non lineaire et equations aux derivees partielles stochastiques associees. In Paul-Louis Hennequin, editor, Ecole d'Eté de Probabilités de Saint-Flour XIX — 1989, pages 68–163, Berlin, Heidelberg, 1991. Springer Berlin Heidelberg.
[35] E. Pardoux and A. Răşcanu. Sdes with multivalued drift. In Stochastic Differential Equations, Backward SDEs, Partial Differential Equations, pages 229–351. Springer, 2014.
[36] W. A. Strauss. Partial differential equations. John Wiley & Sons, Ltd., Chichester, second edition, 2008. An introduction.
[37] H. J. Sussmann. On the gap between deterministic and stochastic ordinary differential equations. The Annals of Probability, pages 19–41, 1978.
[38] J. Xiong. An introduction to stochastic filtering theory, volume 18. OUP Oxford, 2008.
[39] M. Zakai. On the optimal filtering of diffusion processes. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, 11(3):230–243, 1969.