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SOME APPLICATION OF A GENERALIZED DISTRIBUTION SERIES ON CERTAIN CLASS OF ANALYTIC FUNCTIONS

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Abstract. In this search, we investigate a relation between generalized distribution series and particular subclasses of univalent functions. Further, we obtain the sufficient conditions for generalized distribution series $\mathcal{N}_{\psi}(\tau, z)$ and $\mathcal{M}^{*}_{\psi}(\eta, \tau, z)$ belongs to $\mathfrak{L}^{\theta}(A, B; \gamma)$. Also, we investigate some mapping properties for this class. Finally, we obtain some corollaries and consequences of the main results.

1 Introduction and Auxiliary results

Let $\mathcal{A}$ be the family of all analytic functions f of the form:

$$f(z) = z + \sum_{\kappa=2}^{\infty} c_{\kappa} z^{\kappa}, \tag{1.1}$$

in $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ with normalization condition $f(0) = f'(0) - 1 = 0$.

Definition 1. *[6] For $f(z) \in \mathcal{A}$, given by (1.1), and $g(z) \in \mathcal{A}$ of the form*

$$g(z) = z + \sum_{\kappa=2}^{\infty} b_{\kappa} z^{\kappa},$$

then Hadamard product (or convolution) of two power series $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z + \sum_{\kappa=2}^{\infty} c_{\kappa} b_{\kappa} z^{\kappa} = (g * f)(z).$$

Let $\mathcal{S}$ or $\mathcal{S}^{*}$ and $\mathcal{K}$ denote the subclasses of starlike and convex functions, respectively (see [6, 11]). Kanas and Wisniowska [14, 15] defined the classes $\mathcal{TS}(\ell)$ and $\mathcal{UCV}(\ell)$ that are uniformly starlike functions and uniformly convex functions, respectively, as following:

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Definition 2. *[14, 15] A function $f(z) \in \mathcal{A}$ of the form (1.1) is in the class $\mathcal{TS}(\ell)$ if it satisfies the following condition:*

$$Re\left\{1+\frac{zf'(z)}{f(z)}\right\} \geq \ell\left|\frac{zf'(z)}{f(z)}\right| \quad (\ell \geq 0;\ z \in \mathbb{U})$$

Definition 3. *[14, 15] A function $f(z)$ of the form (1.1) is in the class $\mathcal{UCV}(\ell)$ if it satisfies the following condition:*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \ell\left|\frac{zf''(z)}{f'(z)}\right| \quad (\ell \geq 0;\ z \in \mathbb{U}).$$

Aouf [2, with $p = 1$] defined and studied the class $\mathfrak{L}^{\theta}(A, B; \gamma)$ as follows:

Definition 4. *A function $f(z) \in \mathcal{A}$ is in $\mathfrak{L}^{\theta}(A, B; \gamma)$ if it satisfies the following inequality:*

$$\left|\frac{e^{i\theta}(f'(z)-1)}{Be^{i\theta}f'(z)-[Be^{i\theta}+(A-B)(1-\gamma)\cos\theta]}\right| < 1,$$

with $-1 \leq A < B \leq 1$, $|\theta| < \frac{\pi}{2}$, $0 \leq \gamma < 1$, $z \in \mathbb{U}$.

Note that

(i) $\mathfrak{L}^{\theta}(-1, 1; \gamma) = \mathfrak{L}^{\theta}(\gamma)$ (see [13]);

(ii) $\mathfrak{L}^{\theta}(A, B; 0) = \mathfrak{L}^{\theta}(A, B)$ (see [25]);

(iii) $\mathfrak{L}^{0}(-\beta, \beta; \gamma) = \mathfrak{L}(\beta, \gamma)$ $(0 < \beta \leq 1)$ (see [12]);

(iv) $\mathfrak{L}^{0}(-\beta, \beta; 0) = \mathfrak{L}(\beta)$ $(0 < \beta \leq 1)$ (see [4, 18]).

The applications of hypergeometric functions [19, 26], generalized hypergeometric functions [10], generalized Bessel functions [3, 7, 22], Wright function [24] and Fox-Wright function [5] are play an important role in geometric function theory. Porwal [21] (see also [1, 8, 9, 17]) defined Poisson distribution series and investigated a necessary and sufficient conditions for classes of univalent functions. Lately, Porwal [20] defined and studied the generalized distribution series. Also, he obtained some necessary and sufficient conditions belonging to the certain classes of univalent functions. After the appearance of this paper several researchers begin to study a relation between generalized distribution series and particular subclasses of univalent functions (see [16, 23]).

Now, we recall the definition of generalized distribution as follows:

Definition 5. *The generalized distribution series is defined as*

$$T = \sum_{\kappa=0}^{\infty} t_{\kappa}, \qquad (t_{\kappa} \geq 0).$$

The series T converges.

The generalized discrete probability distribution with its probability mass function is given by

$$\chi(\kappa)=\frac{t_\kappa}{T},\qquad (\kappa\in\mathbb{N}\cup\{0\},\ \mathbb{N}=\{1,2,3,...\}).$$

Obviously $\chi(\kappa)\geq 0$ and $\sum_\kappa \chi(\kappa)=1$. Also, he defined the series

$$\psi(x)=\sum_{\kappa=0}^{\infty} t_\kappa x^\kappa.$$

The above equation can be verified easily that it is convergent when $-1<x\leq 1$.

Note that:

$$\psi'(x)=\sum_{\kappa=1}^{\infty}\kappa t_\kappa x^{(\kappa-1)},$$

and

$$\psi''(x)=\sum_{\kappa=2}^{\infty}\kappa(\kappa-1)t_\kappa x^{(\kappa-2)}.$$

In [20], Porwal defined a power series whose coefficients are probabilities of the generalized distribution as follows:

$$\mathcal{G}_\psi(z)=z+\sum_{\kappa=2}^{\infty}\frac{t_{\kappa-1}}{T}z^\kappa. \tag{1.2}$$

Now, we defined the functions $\mathcal{N}_\psi(\tau,z)$ and $\mathcal{M}^*_\psi(\eta,\tau,z)$ as follows:

$$\begin{aligned}\mathcal{N}_\psi(\tau,z) &= (1-\tau)\mathcal{G}_\psi(z)+\tau z\left(\mathcal{G}_\psi(z)\right)' \\ &= z+\sum_{\kappa=2}^{\infty}[1+\tau(\kappa-1)]\frac{t_{\kappa-1}}{T}z^\kappa \qquad (\tau\geq 0,\ z\in\mathbb{U}),\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}\mathcal{M}^*_\psi(\eta,\tau,z) &= (1-\tau+\eta)\mathcal{G}_\psi(z)+(\tau-\eta)z(\mathcal{G}_\psi(z))'+\tau\eta z^2(\mathcal{G}_\psi(z))'' \\ &= z+\sum_{\kappa=2}^{\infty}[1+(\kappa-1)(\tau-\eta+\kappa\tau\eta)]\frac{t_{\kappa-1}}{T}z^\kappa \\ &\qquad (\tau,\ \eta\geq 0,\ \tau\geq\eta,\ z\in\mathbb{U}).\end{aligned} \tag{1.4}$$

Also, we defined the linear operator $\mathcal{W}_\psi:\mathcal{A}\to\mathcal{A}$ as follows:

$$\begin{aligned}\mathcal{W}_\psi(z) &= \mathcal{G}_\psi(z)*f(z) \\ &= z+\sum_{\kappa=2}^{\infty}\frac{t_{\kappa-1}}{T}c_\kappa z^\kappa, \qquad (z\in\mathbb{U}).\end{aligned} \tag{1.5}$$

To prove our results, we will need the following lemmas.

Lemma 6. *[2, Theorem 4, with $p = 1$] A function $f(z) \in \mathcal{A}$ defined by (1.1) belongs to the class $\mathfrak{L}^{\theta}(A, B; \gamma)$ if*

$$\sum_{\kappa=2}^{\infty} \frac{\kappa(1+|B|)}{(B-A)\cos\theta}|c_{\kappa}| \leq 1-\gamma, \quad (-1 \leq A < B \leq 1;\, |\theta| < \pi/2;\, 0 \leq \gamma < 1).$$

Lemma 7. *[2, Theorem 1, with $p = 1$] A function $f(z) \in \mathcal{A}$ defined by (1.1) belongs to the class $\mathfrak{L}^{\theta}(A, B; \gamma)$ if*

$$|c_{\kappa}| \leq \frac{(B-A)(1-\gamma)\cos\theta}{\kappa} \quad (\kappa \geq 2).$$

Lemma 8. *[14] For some ℓ and $f(z) \in \mathcal{A}$, if the inequality*

$$\sum_{\kappa=2}^{\infty}(\kappa + \ell(\kappa-1))|c_{\kappa}| \leq 1,$$

holds, then $f \in \mathcal{TS}(\ell)$.

Lemma 9. *[15] For some ℓ and $f(z) \in \mathcal{A}$, if the inequality*

$$\sum_{\kappa=2}^{\infty} \kappa(\kappa-1)|c_{\kappa}| \leq \frac{1}{\ell+2},$$

holds, then $f \in \mathcal{UCV}(\ell)$.

In this paper, we establish a relation between subclasses of univalent functions and generalized distribution series. The main aim of the present investigation is to obtain some conditions for generalized distribution series belongs to $\mathfrak{L}^{\theta}(A, B; \gamma)$. Then, we investigate some mapping properties for this class. Finally, we obtain some corollaries and consequences of the main results.

2 Main Results

Unless otherwise mentioned, we assume that $0 \leq \gamma < 1$, $\ell \geq 0$, $|\theta| < \frac{\pi}{2}$, $-1 \leq A < B \leq 1$, $\tau, \eta \geq 0$ and $\tau \geq \eta$.

Theorem 10. *The sufficient condition for $\mathcal{N}_{\psi}(\tau, z)$ to be in the class $\mathfrak{L}^{\theta}(A, B; \gamma)$ is*

$$\left(\frac{1+|B|}{T}\right)\left[\tau\psi''(1) + (1+2\tau)\psi'(1) + \psi(1) - 1\right] \leq (B-A)(1-\gamma)\cos\theta. \tag{2.1}$$

Proof. Since

$$\mathcal{N}_{\psi}(\tau, z) = z + \sum_{\kappa=2}^{\infty}[1+\tau(\kappa-1)]\frac{t_{\kappa-1}}{T}z^{\kappa}, \quad (z \in \mathbb{U}).$$

By applying Lemma 6, we need to prove that

$$I_1 = \sum_{\kappa=2}^{\infty} \kappa(1+|B|)[1+\tau(\kappa-1)]\frac{t_{\kappa-1}}{T} \leq (B-A)(1-\gamma)\cos\theta.$$

Thus,

$$\begin{aligned} I_1 &= \sum_{\kappa=2}^{\infty} \kappa(1+|B|)[1+\tau(\kappa-1)]\frac{t_{\kappa-1}}{T} \\ &= \left(\frac{1+|B|}{T}\right)\left[\sum_{\kappa=2}^{\infty}\tau(\kappa-1)(\kappa-2)t_{\kappa-1} + \sum_{\kappa=2}^{\infty}(1+2\tau)(\kappa-1)t_{\kappa-1} + \sum_{\kappa=2}^{\infty} t_{\kappa-1}\right] \\ &= \left(\frac{1+|B|}{T}\right)\left[\sum_{\kappa=1}^{\infty}\tau\kappa(\kappa-1)t_{\kappa} + \sum_{\kappa=1}^{\infty}(1+2\tau)\kappa t_{\kappa} + \sum_{\kappa=1}^{\infty} t_{\kappa}\right] \\ &= \left(\frac{1+|B|}{T}\right)\left[\tau\psi''(1) + (1+2\tau)\psi'(1) + \psi(1) - 1\right]. \end{aligned}$$

But the last equation is bounded by $(B-A)(1-\gamma)\cos\theta$ if (2.1) holds. □

Theorem 11. *The sufficient condition for $\mathcal{M}^*_\psi(\eta,\tau,z)$ to be in the class $\mathfrak{L}^\theta(A,B;\gamma)$ is*

$$\left(\frac{1+|B|}{T}\right)[\tau\eta\psi'''(1)+(\tau-\eta+5\tau\eta)\psi''(1)+(2\tau-2\eta+1+4\tau\eta)\psi'(1)+\psi(1)-1] \leq (B-A)(1-\gamma)\cos\theta. \tag{2.2}$$

Proof. Since

$$\mathcal{M}^*_\psi(\eta,\tau,z) = z + \sum_{\kappa=2}^{\infty}[1+(\kappa-1)(\tau-\eta+\kappa\tau\eta)]\frac{t_{\kappa-1}}{T}z^\kappa, \quad (z\in\mathbb{U}).$$

By applying Lemma 6, we need to prove that

$$I_2 = \sum_{\kappa=2}^{\infty} \kappa(1+|B|)[1+(\kappa-1)(\tau-\eta+\kappa\tau\eta)]\frac{t_{\kappa-1}}{T} \leq (B-A)(1-\gamma)\cos\theta.$$

Thus,

$$\begin{aligned}
I_2 &= \sum_{\kappa=2}^{\infty} \kappa(1+|B|)[1+(\kappa-1)(\tau-\eta+\kappa\tau\eta)]\frac{t_{\kappa-1}}{T} \\
&= \left(\frac{1+|B|}{T}\right)\left[\sum_{\kappa=2}^{\infty}\tau\eta(\kappa-1)(\kappa-2)(\kappa-3)t_{\kappa-1}+\sum_{\kappa=2}^{\infty}(\tau-\eta+5\tau\eta)(\kappa-1)(\kappa-2)t_{\kappa-1}\right. \\
&\left.+\sum_{\kappa=2}^{\infty}(1+2\tau-2\eta+4\tau\eta)(\kappa-1)t_{\kappa-1}+\sum_{\kappa=2}^{\infty}t_{\kappa-1}\right] \\
&= \left(\frac{1+|B|}{T}\right)\left[\sum_{\kappa=1}^{\infty}\tau\eta\kappa(\kappa-1)(\kappa-2)t_{\kappa}+\sum_{\kappa=1}^{\infty}(\tau-\eta+5\tau\eta)\kappa(\kappa-1)t_{\kappa}\right. \\
&\left.+\sum_{\kappa=1}^{\infty}(1+2\tau-2\eta+4\tau\eta)\kappa t_{\kappa}+\sum_{\kappa=1}^{\infty}t_{\kappa}\right] \\
&= \left(\frac{1+|B|}{T}\right)\left[\tau\eta\psi'''(1)+(\tau-\eta+5\tau\eta)\psi''(1)+(1+2\tau-2\eta+4\tau\eta)\psi'(1)+\psi(1)-1\right]
\end{aligned}$$

But this last equation is bounded by $(B-A)(1-\gamma)\cos\theta$ if (2.2) holds. □

3 Inclusion Properties

Theorem 12. *If the following condition*

$$\left(\frac{1+|B|}{T}\right)[\psi''(1)+3\psi'(1)+\psi(1)-1] \le (B-A)(1-\gamma)\cos\theta \tag{3.1}$$

holds, then $\mathcal{W}_\psi(z)$ *maps the class* $\mathcal{S}$ *or* $(\mathcal{S}^*)$ *to the class* $\mathfrak{L}^\theta(A,B;\gamma)$.

Proof. Since

$$\mathcal{W}_\psi(z) = z + \sum_{\kappa=2}^{\infty}\frac{t_{\kappa-1}}{T}c_\kappa z^\kappa, \quad (z\in\mathbb{U}).$$

By applying Lemma 6, we need to prove that

$$\sum_{\kappa=2}^{\infty}\kappa(1+|B|)\frac{t_{\kappa-1}}{T}|c_\kappa| \le (B-A)(1-\gamma)\cos\theta.$$

Using $f(z) \in \mathcal{S}$, then the inequality $|c_\kappa| \leq \kappa$ holds, we obtain that

$$\begin{aligned}
I_3 &= \sum_{\kappa=2}^{\infty} \kappa(1+|B|)\frac{t_{\kappa-1}}{T}|c_\kappa| \leq \sum_{\kappa=2}^{\infty} \kappa^2(1+|B|)\frac{t_{\kappa-1}}{T} \\
&\leq \left(\frac{1+|B|}{T}\right)\left[\sum_{\kappa=2}^{\infty}(\kappa-1)(\kappa-2)t_{\kappa-1} + \sum_{\kappa=2}^{\infty} 3(\kappa-1)t_{\kappa-1} + \sum_{\kappa=2}^{\infty} t_{\kappa-1}\right] \\
&\leq \left(\frac{1+|B|}{T}\right)\left[\sum_{\kappa=1}^{\infty}\kappa(\kappa-1)t_\kappa + \sum_{\kappa=1}^{\infty} 3\kappa t_\kappa + \sum_{\kappa=1}^{\infty} t_\kappa\right] \\
&\leq \left(\frac{1+|B|}{T}\right)\left[\psi''(1) + 3\psi'(1) + \psi(1) - 1\right].
\end{aligned}$$

But the last equation is bounded by $(B-A)(1-\gamma)\cos\theta$ if (3.1) holds. □

Theorem 13. *If the condition*

$$\left(\frac{1+|B|}{T}\right)\left[\psi'(1) + \psi(1) - 1\right] \leq (B-A)(1-\gamma)\cos\theta$$

satisfies, then $\mathcal{W}_\psi(z)$ maps the class $\mathcal{K}$ to the class $\mathfrak{L}^\theta(A,B;\gamma)$.

Proof. Since

$$\mathcal{W}_\psi(z) = z + \sum_{\kappa=2}^{\infty} \frac{t_{\kappa-1}}{T} c_\kappa z^\kappa, \quad (z \in \mathbb{U}).$$

By applying Lemma 6, we need to prove that

$$\sum_{\kappa=2}^{\infty} \kappa(1+|B|)\frac{t_{\kappa-1}}{T}|c_\kappa| \leq (B-A)(1-\gamma)\cos\theta.$$

Using $f(z) \in \mathcal{K}$, then the inequality $|c_\kappa| \leq 1$ holds, we obtain the required result. □

Theorem 14. *If the condition*

$$\left(\frac{(B-A)(1-\gamma)\cos\theta}{T}\right)\psi'(1) \leq \frac{1}{\ell+2} \tag{3.2}$$

holds, then $\mathcal{W}_\psi(z)$ maps the class $\mathfrak{L}^\theta(A,B;\gamma)$ to the class $\mathcal{UCV}(\ell)$.

Proof. Since

$$\sum_{\kappa=2}^{\infty} \kappa(1+|B|)\frac{t_{\kappa-1}}{T}|c_\kappa| \leq (B-A)(1-\gamma)\cos\theta.$$

By applying Lemma 9, we need to prove that

$$\sum_{\kappa=2}^{\infty} \kappa(\kappa-1)\frac{t_{\kappa-1}}{T}|c_\kappa| \leq \frac{1}{\ell+2}.$$

Thus,

$$\begin{aligned} I_4 &= \sum_{\kappa=2}^{\infty} \kappa(\kappa-1)\frac{t_{\kappa-1}}{T}|c_\kappa| \\ &\leq \sum_{\kappa=2}^{\infty} (\kappa-1)\frac{t_{\kappa-1}}{T}(B-A)(1-\gamma)\cos\theta \\ &= \left(\frac{(B-A)(1-\gamma)\cos\theta}{T}\right)\sum_{\kappa=1}^{\infty}\kappa t_\kappa \\ &= \left(\frac{(B-A)(1-\gamma)\cos\theta}{T}\right)\psi'(1). \end{aligned}$$

But the last equation is bounded by $\frac{1}{\ell+2}$ if (3.2) is holds. □

4 Special cases

By specializing A, B, τ, γ and θ in the above theorems, we will obtain new results for different subclasses mentioned in the introduction.

Corollary 15. *Let $\tau = 0$ in Theorem 10, then the sufficient condition for $\mathcal{G}_\psi(z)$ to be in the class $\mathfrak{L}^\theta(A, B; \gamma)$ is*

$$\left(\frac{1+|B|}{T}\right)\left[\psi'(1)+\psi(1)-1\right] \leq (B-A)(1-\gamma)\cos\theta.$$

Corollary 16. *Let $A = -1$ and $B = 1$ in Theorem 10, then the sufficient condition for $\mathcal{N}_\psi(\tau, z)$ to be in the class $\mathfrak{L}^\theta(\gamma)$ is*

$$\left(\frac{1}{T}\right)\left[\tau\psi''(1)+(1+2\tau)\psi'(1)+\psi(1)-1\right] \leq (1-\gamma)\cos\theta.$$

Corollary 17. *Let $\gamma = 0$ in Theorem 10, then the sufficient condition for $\mathcal{N}_\psi(\tau, z)$ to be in the class $\mathfrak{L}^\theta(A, B)$ is*

$$\left(\frac{1+|B|}{T}\right)\left[\tau\psi''(1)+(1+2\tau)\psi'(1)+\psi(1)-1\right] \leq (B-A)\cos\theta.$$

Corollary 18. *Let $A = -\beta$, $B = \beta\,(0 < \beta \leq 1)$, $\theta = 0$ and $\gamma = 0$ in Theorem 10, then the sufficient condition for $\mathcal{N}_\psi(\tau, z)$ to be in the class $\mathfrak{L}(\beta)$ is*

$$\left(\frac{1+|\beta|}{T}\right)\left[\tau\psi''(1)+(1+2\tau)\psi'(1)+\psi(1)-1\right] \leq 2\beta.$$

Corollary 19. *Let $A = -\beta$, $B = \beta\,(0 < \beta \leq 1)$ and $\theta = 0$ in Theorem 10, then the sufficient condition for $\mathcal{N}_\psi(\tau, z)$ to be in the class $\mathfrak{L}(\beta;\gamma)$ is*

$$\left(\frac{1+|\beta|}{T}\right)\left[\tau\psi''(1) + (1+2\tau)\psi'(1) + \psi(1) - 1\right] \leq 2\beta(1-\gamma).$$

Corollary 20. *Let $A = -1$ and $B = 1$ in Theorem 11, then the sufficient condition for $\mathcal{M}^*_\psi(\eta, \tau, z)$ to be in the class $\mathfrak{L}^\theta(\gamma)$ is*

$$\left(\frac{1}{T}\right)[\tau\eta\psi'''(1)+(\tau-\eta+5\tau\eta)\psi''(1)+(2\tau-2\eta+1+4\tau\eta)\psi'(1)+\psi(1)-1] \leq (1-\gamma)\cos\theta.$$

Corollary 21. *Let $\gamma = 0$ in Theorem 11, then the sufficient condition for $\mathcal{M}^*_\psi(\eta, \tau, z)$ to be in the class $\mathfrak{L}^\theta(A, B)$ is*

$$\left(\frac{1+|B|}{T}\right)[\tau\eta\psi'''(1)+(\tau-\eta+5\tau\eta)\psi''(1)+(2\tau-2\eta+1+4\tau\eta)\psi'(1)+\psi(1)-1] \leq (B-A)\cos\theta.$$

Corollary 22. *Let $A = -\beta$, $B = \beta\,(0 < \beta \leq 1)$, $\theta = 0$ and $\gamma = 0$ in Theorem 11, then the sufficient condition for $\mathcal{M}^*_\psi(\eta, \tau, z)$ to be in the class $\mathfrak{L}(\beta)$ is*

$$\left(\frac{1+|\beta|}{T}\right)[\tau\eta\psi'''(1)+(\tau-\eta+5\tau\eta)\psi''(1)+(2\tau-2\eta+1+4\tau\eta)\psi'(1)+\psi(1)-1] \leq 2\beta.$$

Corollary 23. *Let $A = -\beta$, $B = \beta\,(0 < \beta \leq 1)$ and $\theta = 0$ in Theorem 11, then the sufficient condition for $\mathcal{M}^*_\psi(\eta, \tau, z)$ to be in the class $\mathfrak{L}(\beta;\gamma)$ is*

$$\left(\frac{1+|\beta|}{T}\right)[\tau\eta\psi'''(1)+(\tau-\eta+5\tau\eta)\psi''(1)+(2\tau-2\eta+1+4\tau\eta)\psi'(1)+\psi(1)-1] \leq 2\beta(1-\gamma).$$

Corollary 24. *Let $A = -1$ and $B = 1$ in Theorem 12, then $\mathcal{W}_\psi(z)$ maps the class $\mathcal{S}$ or $(\mathcal{S}^*)$ to the class $\mathfrak{L}^\theta(\gamma)$ if*

$$\left(\frac{1}{T}\right)[\psi''(1) + 3\psi'(1) + \psi(1) - 1] \leq (1-\gamma)\cos\theta$$

holds.

Corollary 25. *Let $\gamma = 0$ in Theorem 12, then $\mathcal{W}_\psi(z)$ maps the class $\mathcal{S}$ or $(\mathcal{S}^*)$ to the class $\mathfrak{L}^\theta(A, B)$ if*

$$\left(\frac{1+|B|}{T}\right)[\psi''(1) + 3\psi'(1) + \psi(1) - 1] \leq (B-A)\cos\theta$$

holds.

Corollary 26. *Let* $A=-\beta$, $B=\beta\,(0<\beta\leq 1)$, $\theta=0$ *and* $\gamma=0$ *in Theorem 12, then* $\mathcal{W}_{\psi}(z)$ *maps the class* $\mathcal{S}$ *or* $(\mathcal{S}^{*})$ *to the class* $\mathfrak{L}(\beta)$ *if*

$$\left(\frac{1+|\beta|}{T}\right)[\psi''(1)+3\psi'(1)+\psi(1)-1]\leq 2\beta$$

holds.

Corollary 27. *Let* $A=-\beta$, $B=\beta\,(0<\beta\leq 1)$ *and* $\theta=0$ *in Theorem 12, then* $\mathcal{W}_{\psi}(z)$ *maps the class* $\mathcal{S}$ *or* $(\mathcal{S}^{*})$ *to the class* $\mathfrak{L}(\beta;\gamma)$ *if*

$$\left(\frac{1+|\beta|}{T}\right)[\psi''(1)+3\psi'(1)+\psi(1)-1]\leq 2\beta(1-\gamma)$$

holds.

Corollary 28. *Let* $A=-1$ *and* $B=1$ *in Theorem 13, then* $\mathcal{W}_{\psi}(z)$ *maps the class* $\mathcal{K}$ *to the class* $\mathfrak{L}^{\theta}(\gamma)$ *if the condition*

$$\left(\frac{1}{T}\right)\left[\psi'(1)+\psi(1)-1\right]\leq(1-\gamma)\cos\theta$$

holds.

Corollary 29. *Let* $\gamma=0$ *in Theorem 13, then* $\mathcal{W}_{\psi}(z)$ *maps the class* $\mathcal{K}$ *to the class* $\mathfrak{L}^{\theta}(A,B)$ *if*

$$\left(\frac{1+|B|}{T}\right)\left[\psi'(1)+\psi(1)-1\right]\leq(B-A)\cos\theta$$

holds.

Corollary 30. *Let* $A=-\beta$, $B=\beta\,(0<\beta\leq 1)$, $\theta=0$ *and* $\gamma=0$ *in Theorem 13, then* $\mathcal{W}_{\psi}(z)$ *maps the class* $\mathcal{K}$ *to the class* $\mathfrak{L}(\beta)$ *if*

$$\left(\frac{1+|\beta|}{T}\right)\left[\psi'(1)+\psi(1)-1\right]\leq 2\beta$$

holds.

Corollary 31. *Let* $A=-\beta$, $B=\beta\,(0<\beta\leq 1)$ *and* $\theta=0$ *in Theorem 13, then* $\mathcal{W}_{\psi}(z)$ *maps the class* $\mathcal{K}$ *to the class* $\mathfrak{L}(\beta;\gamma)$ *if*

$$\left(\frac{1+|\beta|}{T}\right)\left[\psi'(1)+\psi(1)-1\right]\leq 2\beta(1-\gamma)$$

holds.

Corollary 32. *Let $A=-1$ and $B=1$ in Theorem 14, then $\mathcal{W}_\psi(z)$ maps the class $\mathfrak{L}^\theta(\gamma)$ to the class $\mathcal{UCV}(\ell)$ if condition*

$$\left(\frac{2(1-\gamma)\cos\theta}{T}\right)\psi'(1)\leq\frac{1}{\ell+2}$$

holds.

Corollary 33. *Let $\gamma=0$ in Theorem 14, then $\mathcal{W}_\psi(z)$ maps the class $\mathfrak{L}^\theta(A,B)$ to the class $\mathcal{UCV}(\ell)$ if*

$$\left(\frac{(B-A)\cos\theta}{T}\right)\psi'(1)\leq\frac{1}{\ell+2}$$

holds.

Corollary 34. *Let $A=-\beta$, $B=\beta$, $\theta=0$ and $\gamma=0$ in Theorem 14, then $\mathcal{W}_\psi(z)$ maps the class $\mathfrak{L}(\beta)$ to the class $\mathcal{UCV}(\ell)$ if*

$$\left(\frac{2\beta}{T}\right)\psi'(1)\leq\frac{1}{\ell+2}$$

holds.

Corollary 35. *Let $A=-\beta$, $B=\beta$ and $\theta=0$ in Theorem 14, then $\mathcal{W}_\psi(z)$ maps the class $\mathfrak{L}(\beta;\gamma)$ to the class $\mathcal{UCV}(\ell)$ if*

$$\left(\frac{2\beta(1-\gamma)}{T}\right)\psi'(1)\leq\frac{1}{\ell+2}$$

holds.

Remark 36. *Taking $t_\kappa=m^\kappa/\kappa!$, $(m>0)$ in above Theorems and Corollaries, we obtain the results that obtained by [8].*

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