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Existence of positive solutions for a nonlinear semipositone boundary value problems on a time scale

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ABSTRACT

In this paper, we are concerned with the existence of positive solution of the following semipositone boundary value problem on time scales:

$$(\psi(t)y^{\Delta}(t))^{\nabla}+\lambda_1 g(t,\, y(t))+\lambda_2 h(t,\, y(t))=0,\; t\in[\rho(c),\,\sigma(d)]_{\mathbb{T}},$$

with mixed boundary conditions

$$\alpha y(\rho(c))-\beta\psi(\rho(c))y^{\Delta}(\rho(c))=0,$$
$$\gamma y(\sigma(d))+\delta\psi(d)y^{\Delta}(d)=0,$$

where $\psi : C[\rho(c),\, \sigma(d)]_{\mathbb{T}}$, $\psi(t)>0$ for all $t\in[\rho(c),\,\sigma(d)]_{\mathbb{T}}$; both g and $h : [\rho(c),\,\sigma(d)]_{\mathbb{T}}\times[0,\,\infty)\to\mathbb{R}$ are continuous and semipositone. We have established the existence of at least one positive solution or multiple positive solutions of the above boundary value problem by using fixed point theorem on a cone in a Banach space, when g and h are both superlinear or sublinear or one is superlinear and the other is sublinear for $\lambda_i>0$; $i=1,\,2$ are sufficiently small.

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RESUMEN

En este artículo estudiamos la existencia de soluciones positivas del siguiente problema de valor de frontera semipositón en escalas de tiempo:

$$(\psi(t)y^{\Delta}(t))^{\nabla}+\lambda_1 g(t,\, y(t))+\lambda_2 h(t,\, y(t)) = 0,\ t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}},$$

con condiciones de frontera mixtas

$$\begin{aligned} \alpha y(\rho(c)) - \beta\psi(\rho(c))y^{\Delta}(\rho(c)) &= 0,\\ \gamma y(\sigma(d)) + \delta\psi(d)y^{\Delta}(d) &= 0, \end{aligned}$$

donde $\psi : C[\rho(c),\, \sigma(d)]_{\mathbb{T}}$, $\psi(t) > 0$ para todo $t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}}$; ambas g y $h : [\rho(c),\, \sigma(d)]_{\mathbb{T}} \times [0,\, \infty) \to \mathbb{R}$ son continuas y semipositón. Hemos establecido la existencia de al menos una solución positiva o múltiples soluciones positivas del problema de valor en la frontera anterior usando un teorema de punto fijo en un cono en un espacio de Banach, cuando g y h son ambas superlineales o sublineales o una es superlineal y la otra es sublineal para $\lambda_i > 0$; $i = 1,\, 2$ suficientemente pequeños.

Keywords and Phrases: Positive solutions, boundary value problems, fixed point theorem, cone, time scales.

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1 Introduction

The study of dynamic equations on time scales goes to the seminal work of Stefan Hilger [11] and has received a lot of attention in recent years. Time scales were created to unify the study of continuous and discrete mathematics and particularly used in differential and difference equations. We are interested to prove the results for a dynamic equation where the domain of the unknown function is a time scale $\mathbb{T}$, which is a non-empty closed subset of real numbers $\mathbb{R}$.

We consider the second order semipositone boundary value problem on time scales:

$$(\psi(t)y^{\Delta}(t))^{\nabla} + \lambda_1 g(t,\, y(t)) + \lambda_2 h(t,\, y(t)) = 0,\ t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}}, \tag{1.1}$$

with mixed boundary conditions

$$\begin{aligned} \alpha y(\rho(c)) - \beta\psi(\rho(c))y^{\Delta}(\rho(c)) &= 0, \\ \gamma y(\sigma(d)) + \delta\psi(d)y^{\Delta}(d) &= 0, \end{aligned} \tag{1.2}$$

where λ_1 and λ_2 are positive and

(H_1) $\psi : C[\rho(c),\, \sigma(d)]_{\mathbb{T}}$, $\psi(t) > 0$ for all $t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}}$;

(H_2) $\alpha,\, \beta,\, \gamma,\, \delta, \geq 0$ and $\alpha\delta + \beta\gamma + \alpha\gamma > 0$;

(H_3) g and h : $[\rho(c),\, \sigma(d)]_{\mathbb{T}} \times [0,\, \infty) \to \mathbb{R}$ are continuous satisfying with both g and h are semipositone.

D. R. Anderson and P. Y. Wong [1], have established the existence result for the SL-BVP (1.1) and (1.2) where g is superlinear such that $g(t,\, y) \geq -M$ for some constant $M > 0$ and λ is in some interval of $\mathbb{R}$ with $h(t,\, y) = 0$. They did not establish any results concerning the existence of positive solutions for the boundary value problem (1.1) and (1.2), when g is sublinear. Many findings have also been obtained for the existence of positive solution of the boundary value problem (1.1) and (1.2), when $h(t,\, y) = 0$, but only a few results have been established for the existence of positive solutions when $h(t,\, y) \neq 0$. Motivated by the work of [1] and the references cited therein, we would like to establish the sufficient conditions for the existence of positive solution of the boundary value problem (1.1) and (1.2), when g and h are both superlinear or sublinear or one is superlinear and the other is sublinear for $\lambda_i > 0$; $i = 1,\, 2$ are sufficiently small.

It is worthy of mention that results of this paper not only apply to the set of real numbers or the set of integers but also to more general time scales such as $\mathbb{T} = \mathbb{N}_0^2 = \{t^2 : t \in \mathbb{N}_0\}$, $\mathbb{T} = \{\sqrt{n} : n \in \mathbb{N}_0\}$, etc. For basic notations and concepts on time scale calculus, we refer the readers to monographs [5, 6] and references cited therein. The study of nonlinear, semipositone boundary value problem has considerable importance even in differential equations. In recent years, several researchers studied semipositone boundary value problem on time scales [1, 2, 4, 7, 10, 16, 17]. Semipositone problems arise in many physical and chemical processes such as in chemical reactor theory, astrophysics, gas dynamics and fluidmechanics, relativistic mechanics, nuclear physics, design of suspension bridges, bulking of mechanical systems, combustion and management of natural resources (see [3, 9, 12, 15]). Let a and b such that $0 \leq \rho(a) \leq a < b \leq \sigma(b) < \infty$ and $(\rho(a),\ \sigma(b))_{\mathbb{T}}$ has at least two points.

The plan of the paper is as follows. In Section 2, we provided some preliminary results concerning the Green's function for the homogeneous boundary value problem and some important Lemmas. These results allow us in Section 3 to discuss the existence of at least one or multiple positive solutions. Finally, in Section 4, we illustrate few examples to justify the results obtained in the previous section.

2 Preliminaries

In this section, we have obtained some basic results related to Green's function for the homogeneous boundary value problem and some important Lemmas.

Now let us consider the homogenoeous dynamic boundary value problem

$$(\psi(t)y^{\Delta}(t))^{\nabla} = 0,\ t \in [\rho(c),\ \sigma(d)]_{\mathbb{T}}, \tag{2.1}$$

with boundary conditions (1.2). Green's function $\mathcal{G}(t,\ s)$ (see [7]) for the boundary value problem (2.1) and with the boundary conditions (1.2) is given by

$$\mathcal{G}(t,\ s) = \frac{1}{\varphi}\begin{cases} \left(\beta + \alpha \int_{\rho(c)}^{t} \frac{\nabla\tau}{\psi(\tau)}\right)\left(\delta + \gamma \int_{s}^{\sigma(d)} \frac{\nabla\tau}{\psi(\tau)}\right),\ \rho(c) \leq t \leq s \leq \sigma(d), \\ \\ \left(\beta + \alpha \int_{\rho(c)}^{s} \frac{\nabla\tau}{\psi(\tau)}\right)\left(\delta + \gamma \int_{t}^{\sigma(d)} \frac{\nabla\tau}{\psi(\tau)}\right),\ \rho(a) \leq s \leq t \leq \sigma(b), \end{cases} \tag{2.2}$$

where

$$\varphi = \alpha\delta + \beta\gamma + \alpha\gamma \int_{\rho(c)}^{\sigma(d)} \frac{\nabla\tau}{\psi(\tau)} > 0.$$

Lemma 2.1 ([17])**.** *Assume (H_1) and (H_2) hold. Then the Green function $\mathcal{G}(t,\ s)$ satisfies*

$$(\psi(t)y^{\Delta}(t))^{\nabla} + Q(t) = 0, \quad t \in (\rho(c),\ \sigma(d))_{\mathbb{T}}, \tag{2.3}$$

with mixed boundary conditions (1.2), where $Q \in C_{rd}[\rho(c),\ \sigma(d)]_{\mathbb{T}}$, $Q(t) \geq 0$; then

$$y(t) \geq q(t)\|y\|,\ t \in [\rho(c),\ \sigma(d)]_{\mathbb{T}},\ s \in [a,\ b]_{\mathbb{T}}, \tag{2.4}$$

where $q(t)$ is given by

$$q(t) = \min\left\{ \frac{\beta + \alpha \int_{\rho(c)}^{t} \frac{\nabla\tau}{\psi(\tau)}}{\beta + \alpha \int_{\rho(c)}^{\sigma(d)} \frac{\nabla\tau}{\psi(\tau)}}, \frac{\delta + \gamma \int_{t}^{\sigma(d)} \frac{\nabla\tau}{\psi(\tau)}}{\delta + \gamma \int_{\rho(c)}^{\sigma(d)} \frac{\nabla\tau}{\psi(\tau)}} \right\}.$$

Lemma 2.2 ([1]). *For all $t \in [\rho(c), \sigma(d)]_{\mathbb{T}}$ and $s \in [c, d]_{\mathbb{T}}$, then*

$$q(t)\mathcal{G}(s, s) \leq \mathcal{G}(t, s) \leq \mathcal{G}(s, s), \tag{2.5}$$

where $\mathcal{G}(t, s)$ is given in (2.2) and $q(t)$ is defined as in Lemma 2.1.

Lemma 2.3 ([1]). *Let (H_1) and (H_2) hold and let y_1 be the solution of*

$$(\psi(t)y^{\Delta}(t))^{\nabla} + 1 = 0,\ t \in (\rho(c), \sigma(d))_{\mathbb{T}}, \tag{2.6}$$

with mixed boundary conditions (1.2), then there exists a positive constant C such that

$$y_1(t) \leq C\, q(t),\ t \in [\rho(c), \sigma(d)]_{\mathbb{T}}, \tag{2.7}$$

where

$$C = \frac{1}{\varphi}(\sigma(d) - \rho(c)) \left(\beta + \alpha \int_{\rho(c)}^{\sigma(d)} \frac{\nabla\tau}{\psi(\tau)}\right) \left(\delta + \gamma \int_{\rho(c)}^{\sigma(d)} \frac{\nabla\tau}{\psi(\tau)}\right).$$

Lemma 2.4 ([8]). *Let $\lim_{y\to\infty} \frac{g(t, y)}{y} = \infty$ and define $\mathbb{G} : [0, \infty) \to [0, \infty)$ by*

$$\mathbb{G} = \max_{\rho(c) \leq t \leq \sigma(d),\, 0 \leq y \leq r} g(t, y). \tag{2.8}$$

Then

(I) $\mathbb{G}$ *is non-decreasing;*

(II) $\lim_{r\to\infty} \frac{\mathbb{G}(r)}{r} = \infty$;

(III) there exists $r^* > 0$ such that $\mathbb{G}(r) > 0$ for $r \geq r^*$.

Lemma 2.5 ([8]). *Let $lim_{y\to\infty} \frac{g(t, y)}{y} = 0$ holds. Then $\mathbb{G}$ defined by (2.8) is a nondecreasing function, such that*

$$\lim_{r\to\infty} \frac{\mathbb{G}(r)}{r} = 0.$$

Define a function for $y \in C[\rho(c), \sigma(d)]_{\mathbb{T}}$,

$$\overline{g}(t, y) = \begin{cases} g(t, y),\ y \geq 0, \\ g(t, 0),\ y < 0. \end{cases}$$

and

$$\overline{h}(t, y) = \begin{cases} h(t, y), \ y \geq 0, \\ h(t, 0), \ y < 0. \end{cases}$$

Let us consider the nonlinear boundary value problem:

$$(\psi(t)y^{\Delta})^{\nabla} = -[\lambda_1 \overline{g}(t, y - x) + \lambda_2 \overline{h}(t, y - x) + M], \tag{2.9}$$

with boundary conditions (1.2).

Lemma 2.6. *Assume that $x(t) = My_1(t)$, where $y_1(t)$ is a unique solution of the boundary value problem (2.6) and (1.2). Then $y(t)$ is a solution of the boundary value problem (1.1) and (1.2) if and only if $\overline{y}(t) = y(t) + x(t)$ is a positive solution of the boundary value problem (2.9) and (1.2) with $\overline{y}(t) > x(t)$ for $t \in [\rho(c), \sigma(d)]_{\mathbb{T}}$.*

Proof. Let us assume that $\overline{y}(t)$ is a solution of the boundary value problem (2.9) and (1.2) such that $\overline{y}(t) \geq x(t)$ for any $t \in [\rho(c), \sigma(d)]_{\mathbb{T}}$. Let $y(t) = \overline{y}(t) - x(t) > 0$ on $[\rho(c), \sigma(d)]_{\mathbb{T}}$ as $\overline{y}(t) \geq x(t)$. Now, for any $t \in [\rho(c), \sigma(d)]_{\mathbb{T}}$, we have

$$(\psi(t)\overline{y}^{\Delta}(t))^{\nabla} + [\lambda_1 \overline{g}(t, (\overline{y}(t) - x(t)) + \lambda_2 \overline{h}(t, (\overline{y}(t) - x(t))) + M] = 0,$$

that is,

$$(\psi(t)y^{\Delta}(t))^{\nabla} + (\psi(t)x^{\Delta}(t))^{\nabla} + [\lambda_1 \overline{g}(t, (\overline{y}(t) - x(t)) + \lambda_2 \overline{h}(t, (\overline{y}(t) - x(t))) + M] = 0.$$

By using the definition of y together with the definition of x, we have

$$(\psi(t)y^{\Delta}(t))^{\nabla} + [\lambda_1 \overline{g}(t, y(t)) + \lambda_2 \overline{h}(t, y(t)) + M] + M(\psi(t)y_1^{\Delta}(t))^{\nabla}(t) = 0.$$

Thus,

$$(\psi(t)y^{\Delta}(t))^{\nabla} + \lambda_1 \overline{g}(t, y(t)) + \lambda_2 \overline{h}(t, y(t)) = 0.$$

On the other hand,

$$\begin{aligned}
&\alpha y(\rho(c)) - \beta\psi(\rho(c))y(\rho(c)) \\
&= (\alpha\overline{y}(\rho(c)) - \beta\psi(\rho(c))\overline{y}^{\Delta}(\rho(c))) - (\alpha x(\rho(c)) - \beta\psi(\rho(c))x^{\Delta}(\rho(c))) \\
&= (\alpha\overline{y}(\rho(c)) - \beta\psi(\rho(c))\overline{y}^{\Delta}(\rho(c))) - M(\alpha y_1(\rho(c)) - \beta\psi(\rho(c))y_1^{\Delta}(\rho(c))) = 0,
\end{aligned}$$

and

$$\gamma y(\sigma(d)) + \delta\psi(d)y^{\Delta}(d) = \gamma\overline{y}(\sigma(d)) + \delta\psi(d)\overline{y}^{\Delta}(d) - (\gamma x(\sigma(d)) + \delta\psi(d)x^{\Delta}(d))$$

$$= \gamma\overline{y}(\sigma(d)) + \delta\psi(d)\overline{y}^{\Delta}(d) - M(\gamma y_1(\sigma(d)) + \delta\psi(d)y_1^{\Delta}(d)) = 0.$$

Hence, $y(t)$ is a solution of the boundary value problem (1.1) and (1.2). Hence this completes the proof of the lemma. □

Let us define a Banach space

$$E = \{y : C[\rho(c),\, \sigma(d)]_{\mathbb{T}} \to \mathbb{R}\}$$

endowed with the norm

$$\|y\| = \max\{|y(t)|,\, t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}}\}.$$

Define a cone K on E by

$$K = \{y \in C[\rho(c),\, \sigma(d)]_{\mathbb{T}} : y(t) \geq q(t)\|y\|,\, t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}}\},$$

where $q(t)$ is defined as in Lemma 2.1. Let us define an operator T_λ on K by

$$T_\lambda y(t) = \int_{\rho(c)}^{d} \mathcal{G}(t,\, s)[\lambda_1\overline{g}(s,\, y(s) - x(s)) + \lambda_2\overline{h}(s,\, y(s) - x(s)) + M]\nabla s. \tag{2.10}$$

Lemma 2.7. *Assume that* (H_1)*–*(H_3) *hold. Then* $T_\lambda(K) \subset K$ *and* $T_\lambda : K \to K$ *is a completely continuous operator.*

Proof. First we show that $T_\lambda(K) \subset K$. Let $y \in K$ and $t \in [\rho(a),\ \sigma(b)]_{\mathbb{T}}$. Note that

$$(T_\lambda y)(t) = \int_{\rho(c)}^{d} \mathcal{G}(t,\, s)[\lambda_1\overline{g}(s,\, y(s) - x(s)) + \lambda_2\overline{h}(s,\, y(s) - x(s)) + M]\nabla s,$$

that is,

$$(T_\lambda y)(t) \leq \int_{\rho(c)}^{d} \mathcal{G}(s,\, s)[\lambda_1\overline{g}(s,\, y(s) - x(s)) + \lambda_2\overline{h}(s,\, y(s) - x(s)) + M]\nabla s.$$

Hence,

$$\|T_\lambda y\| \leq \int_{\rho(c)}^{d} \mathcal{G}(s,\, s)[\lambda_1\overline{g}(s,\, y(s) - x(s)) + \lambda_2\overline{h}(s,\, y(s) - x(s)) + M]\nabla s.$$

By use of the Lemma (2.2), we obtain

$$(T_\lambda y)(t) \geq q(t)\int_{\rho(c)}^{d} \mathcal{G}(s,\, s)[\lambda_1\overline{g}(s,\, y(s) - x(s)) + \lambda_2\overline{h}(s,\, y(s) - x(s)) + M]\nabla s,$$

which implies

$$(T_\lambda y)(t) \geq q(t)\|T_\lambda y\|.$$

Thus, $T_\lambda(K) \subset K$. Since f and g are continuous, it shows that T_λ is continuous and by the

Arzelà-Ascoli Theorem [14], it is easy to verify that T_λ is a completely continuous operator. Hence this completes the proof of the lemma □

Lemma 2.8 ([13])**.** *Let E be a real Banach space, and let $K \subset E$ be a cone. Let Ω_1, Ω_2 be two bounded open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$. Assume that $T : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

$$\|Ty\| \leq \|y\| \text{ for all } y \in K \cap \partial\Omega_1 \text{ and and } \|Ty\| \geq \|y\| \text{ for all } y \in K \cap \partial\Omega_2,$$

or

$$\|Ty\| \geq \|y\| \text{ for all } y \in K \cap \partial\Omega_1 \text{ and } \|Ty\| \leq \|y\| \text{ for all } y \in K \cap \partial\Omega_2,$$

then T has at least one fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Let us define the following:

(L_1) $\displaystyle\lim_{y\to\infty} \frac{g(t, y)}{y} = \infty;$

(L_2) $\displaystyle\lim_{y\to\infty} \frac{g(t, y)}{y} = 0;$

(L_3) $\displaystyle\lim_{y\to 0} \frac{g(t, y)}{y} = 0;$

(L_4) $\displaystyle\lim_{y\to 0} \frac{g(t, y)}{y} = \infty;$

(L_5) $\displaystyle\lim_{y\to\infty} \frac{h(t, y)}{y} = \infty;$

(L_6) $\displaystyle\lim_{y\to\infty} \frac{h(t, y)}{y} = 0;$

(L_7) $\displaystyle\lim_{y\to 0} \frac{h(t, y)}{y} = 0;$

(L_8) $\displaystyle\lim_{y\to 0} \frac{h(t, y)}{y} = \infty.$

Note that the limits $(L_i), i \in \mathbb{N}_1^8$, are assumed to be inform with respect t.

We would like to establish the existence of solutions for the boundary value problem (1.1) and (1.2) under the following cases:

(I) L_1 and L_5;

(II) L_1 and L_6;

(III) L_1 and L_7;

(IV) L_2 and L_5;

(V) L_2 and L_6;

(VI) L_2 and L_8;

(VII) L_3 and L_5;

$(VIII)$ L_3 and L_7;

(IX) L_3 and L_8;

(X) L_4 and L_6;

(XI) L_4 and L_7;

(XII) L_4 and L_8.

Remark 2.9. *We fails to apply the Lemma 2.8 for the pairs such as $(XIII)$ L_1 and L_8, (XIV) L_2 and L_7, (XV) L_3 and L_6 & (XVI) L_4 and L_5.*

3 Main Results

Theorem 3.1. *Let (H_1)–(H_3), (L_1) and (L_5) hold. Then the boundary value problem (1.1) and (1.2) has a positive solution for λ_i, $i = 1, 2$ are sufficiently small.*

Proof. Let λ_1 and λ_2 satisfy

$$0 < \lambda_1 + \lambda_2 < \frac{1}{\max\limits_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} g(t, y) + \max\limits_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} h(t, y)}, \tag{3.1}$$

where $r_1 = \max\{(M+1)\|y_1\|, r^*, CM\}$, C and r^* are defined as in Lemma 2.3 and Lemma 2.4, respectively and y_1 be the solution of (1.2) and (2.6). Define $\Omega_{r_1} = \{y \in C[\rho(c), \sigma(d)]_{\mathbb{T}} : \|y\| < r_1\}$. For $y \in K \cap \partial\Omega_{r_1}$, we have

$$\begin{aligned}
(T_\lambda y)(t) &= \int_{\rho(c)}^{d} \mathcal{G}(t, s)[\lambda_1 \overline{g}(s, y-x) + \lambda_2 \overline{h}(s, y-x) + M]\nabla s \\
&\le (\lambda_1+\lambda_2)\left(\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} g(t, y) + \max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} h(t, y)\right)\int_{\rho(c)}^{d} \mathcal{G}(t, s)\nabla s + \int_{\rho(c)}^{d} \mathcal{G}(t, s)M\nabla s \\
&= \left((\lambda_1+\lambda_2)\left(\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} g(t, y) + \max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} h(t, y)\right) + M\right) y_1(t) \\
&\le (1+M)y_1(t) \\
&\le r_1 = \|y\|.
\end{aligned}$$

Thus,

$$\|T_\lambda y\| \le \|y\| \quad \text{for } y \in K \cap \partial\Omega_{r_1}. \tag{3.2}$$

Let us choose a constant $\overline{M} > 0$ such that

$$\frac{1}{2}\overline{M}(\lambda_1+\lambda_2)\mu\left(\min_{t_1\le t\le t_2}\int_{t_1}^{t_2} \mathcal{G}(t, s)\nabla s\right) \ge 1, \tag{3.3}$$

where

$$\mu = \min_{t_1\le s\le t_2} q(s). \tag{3.4}$$

From (L_1) and (L_5), we have for same $\overline{M} > 0$ there exists a constant $l > 0$ such that

$$\begin{aligned}
g(t, y) &\ge \overline{M}y \quad \text{for} \quad y \in [l, \infty), \\
h(t, y) &\ge \overline{M}y \quad \text{for} \quad y \in [l, \infty).
\end{aligned}$$

Now set $r_2 = \max\left\{2r_1, 2CM, \frac{2l_1}{\mu}\right\}$. Define $\Omega_{r_2} = \{y \in C[\rho(c), \sigma(d)]_{\mathbb{T}} : \|y\| < r_2\}$. For $y \in$

$K \cap \partial\Omega_{r_2}$, we have

$$\begin{aligned}
y(s) - x(s) &= y(s) - My_1(s) \\
&\geq y(s) - MCq(s) \\
&\geq y(s) - \frac{CM}{\|y\|}y(s) \\
&\geq y(s) - \frac{CM}{r_2}y(s) \\
&\geq \frac{1}{2}y(s),
\end{aligned}$$

and

$$\begin{aligned}
\min_{t_1 \leq s \leq t_2}(y(s) - x(s)) &\geq \min_{t_1 \leq s \leq t_2} \frac{y(s)}{2} \\
&\geq \min_{t_1 \leq s \leq t_2} \frac{\|y\|}{2} q(s) \\
&= \frac{r_2 \mu}{2} \\
&\geq l.
\end{aligned}$$

For $y \in K \cap \partial\Omega_{r_2}$, we have

$$\begin{aligned}
\min_{t \in [t_1,\, t_2]}(T_\lambda y)(t) &= \min_{t \in [t_1,\, t_2]} \int_{\rho(c)}^{d} \mathcal{G}(t,\, s)[\lambda_1 \overline{g}(s,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M]\nabla s \\
&\geq \min_{t \in [t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s)[\lambda_1 \overline{g}(s,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M]\nabla s \\
&\geq \min_{t \in [t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s)(\lambda_1 + \lambda_2)\overline{M}(y(s) - x(s))\nabla s \\
&\geq \min_{t \in [t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s)(\lambda_1 + \lambda_2)\overline{M}\frac{y(s)}{2}\nabla s \\
&\geq \frac{1}{2}(\lambda_1 + \lambda_2)\overline{M}\mu \min_{t \in [t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s)\|y\|\nabla s \\
&\geq \|y\|.
\end{aligned}$$

Thus,

$$\|T_\lambda y\| \geq \|y\| \quad \text{for} \quad y \in K \cap \partial\Omega_{r_2}. \tag{3.5}$$

By Lemma 2.8, T_λ has a fixed point $\overline{y}$ with $r_1 \leq \|\overline{y}\| \leq r_2$. By use of the Lemma 2.3, it follows that

$$\begin{aligned}\overline{y}(t) &\geq r_1 q(t)\\ &\geq r_1 \frac{y_1(t)}{C}\\ &\geq M y_1(t)\\ &= x(t).\end{aligned}$$

Hence, $y = \overline{y} - x$ is a positive solution of the boundary value problem (1.1) and (1.2). This completes the proof of the theorem. □

Theorem 3.2. *Let* (H_1)–(H_3), (L_4) *and* (L_8) *hold. Then the boundary value problem (1.1) and (1.2) has a positive solution for* λ_i, $i = 1, 2$ *are sufficiently small.*

Proof. The proof of Theorem 3.2 is similar to that of Theorem 3.1, hence it is omitted. □

Theorem 3.3. *Assume that* (H_1), (H_2), (L_2) *and* (L_6) *hold. Let there exist two constant* $D > 0$ *and* $\eta > 0$ *such that*

$$\begin{aligned}g(t, y) &\geq \eta \quad \textit{for} \quad t \in [\rho(c), \sigma(d)],\, y \in [D, \infty),\\ h(t, y) &\geq \eta \quad \textit{for} \quad t \in [\rho(c), \sigma(d)],\, y \in [D, \infty),\end{aligned}$$

then the boundary value problem (1.1) and (1.2) has a positive solution for λ_i, $i = 1, 2$ *are sufficiently small.*

Proof. Set

$$r_1 = \max\left\{\frac{2D}{\mu},\ 2MC\right\}, \tag{3.6}$$

and

$$A = 2r_1 \left(\min_{t_1 \leq t \leq t_2} \int_{t_1}^{t_2} \mathcal{G}(t, s)(\lambda_1 + \lambda_2)\eta \nabla s\right)^{-1}, \tag{3.7}$$

where $\mu = \min\limits_{t_1 \leq s \leq t_2} q(s)$. Our claim is that for $\lambda_i \in [A, \infty)$, $i = 1, 2$, the boundary value problem (1.1) and (1.2) has a positive solution. Define $\Omega_{r_1} = \{y \in C[\rho(c), \sigma(d)]_{\mathbb{T}} : \|y\| < r_1\}$. For $y \in K \cap \partial\Omega_{r_1}$, we have

$$\begin{aligned}y(s) - x(s) &= y(s) - M y_1(s)\\ &\geq y(s) - MCq(s)\\ &\geq y(s) - \frac{CM}{r_3} y(s)\\ &\geq \frac{1}{2} y(s),\end{aligned}$$

and

$$\begin{aligned}\min_{t_1 \le s \le t_2}(y(s)-x(s)) &\ge \min_{t_1 \le s \le t_2}\frac{y(s)}{2}\\ &\ge \min_{t_1 \le s \le t_2}\frac{\|y\|}{2}q(s)\\ &= \frac{r_1\mu}{2}\\ &\ge D.\end{aligned}$$

For $y \in K \cap \partial\Omega_{r_1}$, we have

$$\begin{aligned}\min_{t\in[t_1,\,t_2]}(T_\lambda y)(t) &= \min_{t\in[t_1,\,t_2]}\int_{\rho(c)}^{d}\mathcal{G}(t,\,s)[\lambda_1\overline{g}(s,\,y-x)+\lambda_2\overline{h}(s,\,y-x)+M]\nabla s\\ &\ge \min_{t\in[t_1,\,t_2]}\int_{t_1}^{t_2}\mathcal{G}(t,\,s)[\lambda_1\overline{g}(s,\,y-x)+\lambda_2\overline{h}(s,\,y-x)+M]\nabla s\\ &\ge \min_{t\in[t_1,\,t_2]}\int_{t_1}^{t_2}\mathcal{G}(t,\,s)(\lambda_1+\lambda_2)\eta\nabla s\\ &= r_1 = \|y\|.\end{aligned}$$

Thus,

$$\|T_\lambda y\| \ge \|y\| \quad \text{for} \quad y \in K \cap \partial\Omega_{r_1}. \tag{3.8}$$

From (L_2) and (L_6), we have

$$\begin{aligned}\overline{g}(t,\,y) &\le \epsilon y \quad \text{for} \quad t \in [\rho(c),\,\sigma(d)],\ y \ge l,\\ \overline{h}(t,\,y) &\le \epsilon y \quad \text{for} \quad t \in [\rho(c),\,\sigma(d)],\ y \ge l,\end{aligned}$$

On the other hand, by use of the Lemma 2.4, there exists a $R > 0$ such that

$$R > \max\left\{2r_1,\ \max_{\rho(c)\le t\le\sigma(d)}\int_{\rho(c)}^{d}[\mathcal{G}(t,\,s)M+1]\nabla s\right\}.$$

and ϵ satisfies

$$\max_{\rho(c)\le t\le\sigma(d)}\int_{\rho(c)}^{d}\mathcal{G}(t,\,s)[\epsilon\lambda_1 R+\epsilon\lambda_2 R+M]\nabla s \le R.$$

Let

$$\Omega_R = \{y \in C[\rho(c),\,\sigma(d)]_{\mathbb{T}} : \|y\| < R\}.$$

For $y \in K \cap \partial\Omega_R$, we have

$$\begin{aligned}
T_\lambda y(t) &= \int_{\rho(c)}^{d} \mathcal{G}(t,\, s)[\lambda_1 \overline{g}(s,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M]\nabla s \\
&\leq \int_{\rho(c)}^{d} \mathcal{G}(t,\, s)[\lambda_1 \epsilon R + \epsilon \lambda_2 R + M]\nabla s \\
&\leq R = \|y\|.
\end{aligned}$$

Thus,

$$\|T_\lambda y\| \leq \|y\| \quad \text{for} \quad y \in K \cap \partial\Omega_R. \tag{3.9}$$

By Lemma 2.8, T_λ has a fixed point $\overline{y}$ with $r_1 \leq \|\overline{y}\| \leq R$. It follows that

$$\begin{aligned}
\overline{y}(t) &\geq r_1 q(t) \\
&\geq r_1 \frac{y_1(t)}{C} \\
&\geq 2M y_1(t) \\
&\geq x(t).
\end{aligned}$$

Hence, $y = \overline{y} - x$ is a positive solution of the boundary value problem (1.1) and (1.2). This completes the proof of the theorem. □

Theorem 3.4. *Assume that (H_1)–(H_3), (L_3) and (L_7) hold. Let there exist two constant $D > 0$ and $\eta > 0$ such that*

$$\begin{aligned}
g(t,\, y) &\geq \eta \quad \textit{for} \quad t \in [\rho(c),\, \sigma(d)]\, y \in [D,\, l], \\
h(t,\, y) &\geq \eta \quad \textit{for} \quad t \in [\rho(c),\, \sigma(d)]\, y \in [D,\, l],
\end{aligned}$$

then the boundary value problem (1.1) and (1.2) has a positive solution for λ_i, $i = 1,\, 2$ are sufficiently small.

Proof. The proof of the Theorem 3.4 is similar to that of Theorem 3.3, hence it is omitted. □

Theorem 3.5. *Let (H_1)–(H_3), (L_1) and (L_6) hold. Then the boundary value problem (1.1) and (1.2) has at least two positive solutions for λ_i, $1 = 1,\, 2$ are sufficiently small.*

Proof. If (L_6) holds, then by the Lemma 2.5, there exists a constant $r_1 > 0$ such that

$$\mathbb{G}(r_1) \leq N r_1.$$

Since λ_1 and λ_2 are sufficiently small, we have

$$\left[\lambda_1 \max_{\substack{\rho(c)\le t\le\sigma(d)\\0\le y\le r_1}} g(t,\, y) + \lambda_2 \max_{\substack{\rho(c)\le t\le\sigma(d)\\0\le y\le r_1}} h(t,\, y) + M\right] \int_{\rho(c)}^{d} \mathcal{G}(s,\, s)\nabla s \le r_1.$$

Let $\Omega_{r_1} = \{y \in C[\rho(c),\, \sigma(d)]_{\mathbb{T}} : \|y\| < r_1\}$. For $y \in \partial\Omega_{r_1}$, we have

$$\begin{aligned}(T_\lambda y)(t) &= \int_{\rho(c)}^{d} \mathcal{G}(t,\, s)\left[\lambda_1 \overline{g}(s,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M\right]\nabla s\\
&\le \left(\lambda_1 \max_{\substack{\rho(c)\le t\le\sigma(d)\\0\le y\le r_1}} g(t,\, y) + \lambda_2 \max_{\substack{\rho(c)\le t\le\sigma(d)\\0\le y\le r_1}} h(t,\, y)\right) \int_{\rho(c)}^{d} \mathcal{G}(s,\, s)\nabla s\\
&\quad + \int_{\rho(c)}^{d} \mathcal{G}(s,\, s) M\nabla s \le r_1 = \|y\|.\end{aligned}$$

Thus,

$$\|T_\lambda y\| \le \|y\| \quad \text{for all} \quad y \in K \cap \partial\Omega_{r_1}. \tag{3.10}$$

From (L_1), we have

$$g(t,\, y) > N_1 y \quad \text{for all} \quad y \le l.$$

Let $r_2 = \max\left\{2CM, \frac{2l}{\mu}, 2r_1\right\}$ and $\Omega_{r_2} = \{y \in C[\rho(c),\, \sigma(d)]_{\mathbb{T}} : \|y\| < r_2\}$. For $y \in \partial K \cap \Omega_{r_2}$, we have

$$\begin{aligned}y(s) - x(s) &= y(s) - My_1(s)\\
&\ge y(s) - MCq(s)\\
&\ge y(s) - \frac{CM}{r_5} y(s)\\
&\ge \frac{1}{2} y(s),\end{aligned}$$

and

$$\begin{aligned}\min_{t_1\le s\le t_2} (y(s) - x(s)) &\ge \min_{t_1\le s\le t_2} \frac{y(s)}{2}\\
&\ge \min_{t_1\le s\le t_2} \frac{\|y\|}{2} q(s)\\
&= \frac{r_2\mu}{2}\\
&\ge l.\end{aligned}$$

For $y \in K \cap \partial\Omega_{r_2}$, we have

$$\begin{aligned}
\min_{t\in[t_1,\, t_2]}(T_\lambda y)(t) &= \min_{t\in[t_1,\, t_2]}\int_{\rho(c)}^{d}\mathcal{G}(t,\, s)[\lambda_1\overline{g}(s,\, y-x)+\lambda_2\overline{h}(s,\, y-x)+M]\nabla s\\
&\geq \min_{t\in[t_1,\, t_2]}\int_{t_1}^{t_2}\mathcal{G}(t,\, s)[\lambda_1\overline{g}(s,\, y-x)+\lambda_2\overline{h}(s,\, y-x)+M]\nabla s\\
&\geq \min_{t\in[t_1,\, t_2]}\int_{t_1}^{t_2}\mathcal{G}(t,\, s)\lambda_1 N_1(y(s)-x(s))\nabla s\\
&= r_2 = \|y\|.
\end{aligned}$$

Thus,

$$\|T_\lambda y\| \geq \|y\| \quad \text{for all} \quad y \in K \cap \partial\Omega_{r_2}. \tag{3.11}$$

Let

$$R = \max\left\{\left(\lambda_1 \max_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq R}} g(t,\, y)+\lambda_2 \max_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq R}} h(t,\, y)+M\right)\left(\int_{\rho(c)}^{d}\mathcal{G}(s,\, s)\nabla s\right), 2r_2\right\},$$

then $r_1 < r_2 < R$. Let $\Omega_R = \{y \in C[\rho(c),\, \sigma(d)]_{\mathbb{T}} : \|y\| < R\}$. For $y \in K \cap \Omega_R$, $t \in [\rho(c),\, \sigma(d)]_{\mathbb{T}}$, we have

$$\begin{aligned}
(T_\lambda y)(t) &= \int_{\rho(c)}^{d}\mathcal{G}(t,\, s)\left[\lambda_1\overline{g}(s,\, y-x)+\lambda_2\overline{h}(s,\, y-x)+M\right]\nabla s\\
&\leq \left(\lambda_1 \max_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq R}} g(t,\, y)+\lambda_2 \max_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq R}} h(t,\, y)+M\right)\int_{\rho(c)}^{d}\mathcal{G}(s,\, s)\nabla s\\
&\leq R = \|y\|.
\end{aligned}$$

Thus,

$$\|T_\lambda y\| \leq \|y\| \quad \text{for all} \quad y \in K \cap \partial\Omega_R. \tag{3.12}$$

Thus by the Lemma 2.8, T_λ has at least two fixed points. Hence, the boundary value problem (1.1) and (1.2) has at least two positive solutions. □

Theorem 3.6. *Let (H_1)–(H_3), (L_2) and (L_5) hold. Then the boundary value problem (1.1) and (1.2) has at least two positive solutions for λ_i, $i = 1, 2$ are sufficiently small.*

Proof. The proof of the Theorem 3.6 is similar to that of Theorem 3.5. □

Theorem 3.7. *Let (H_1)–(H_3), (L_4) and (L_7) hold. Then the boundary value problem (1.1) and (1.2) has at least two positive solutions for λ_i, $i = 1, 2$ are sufficiently small.*

Proof. From (L_7), we have

$$\lim_{y\to 0}\frac{h(t,\,y)}{y}=0.$$

For $\epsilon>0$, there exists a $r_1>0$ such that

$$h(t,\,y)\le \epsilon y \quad \text{for} \quad y\in[0,\,r_1).$$

Since λ_1 and λ_2 are sufficiently small, we have

$$\left[(\lambda_1+\lambda_2)\left(\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} g(t,\,y)+\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} h(t,\,y)\right)+M\right]\int_{\rho(c)}^{d}\mathcal{G}(s,\,s)\nabla s\le r_1.$$

Let $\Omega_{r_1}=\{y\in C[\rho(c),\,\sigma(d)]_{\mathbb{T}}:\|y\|<r_1\}$. For $y\in\partial\Omega_{r_1}$, we have

$$\begin{aligned}
(T_\lambda y)(t) &= \int_{\rho(c)}^{d}\mathcal{G}(t,\,s)\left[\lambda_1\overline{g}(s,\,y-x)+\lambda_2\overline{h}(s,\,y-x)+M\right]\nabla s\\
&\le \int_{\rho(c)}^{d}\mathcal{G}(s,\,s)\left(\lambda_1\overline{g}(s,\,y-x)+\lambda_2\overline{h}(s,\,y-x)\right)\nabla s+\int_{\rho(c)}^{d}\mathcal{G}(s,\,s)M\nabla s\\
&\le \left[(\lambda_1+\lambda_2)\left(\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} g(t,\,y)+\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} h(t,\,y)\right)+M\right]\int_{\rho(c)}^{d}\mathcal{G}(s,\,s)\nabla s\\
&\le r_1=\|y\|
\end{aligned}$$

Thus,

$$\|T_\lambda y\|\le\|y\| \quad \text{for all} \quad y\in K\cap\partial\Omega_{r_1}. \tag{3.13}$$

From (L_4), we have

$$g(t,\,y)>N_1 y \quad \text{for all} \quad y\le l.$$

Let $r_2=\max\left\{2CM,\,\frac{2l}{\mu},2r_1\right\}$ and $\Omega_{r_2}=\{y\in C[\rho(c),\,\sigma(d)]_{\mathbb{T}}:\|y\|<r_2\}$. For $y\in K\cap\partial\Omega_{r_2}$, we have

$$\begin{aligned}
y(s)-x(s) &= y(s)-My_1(s)\\
&\ge y(s)-MCq(s)\\
&\ge y(s)-\frac{CM}{r_5}y(s)\\
&\ge \frac{1}{2}y(s),
\end{aligned}$$

and

$$\begin{aligned}\min_{t_1\le s\le t_2}(y(s)-x(s)) &\ge \min_{t_1\le s\le t_2}\frac{y(s)}{2}\\ &\ge \min_{t_1\le s\le t_2}\frac{\|y\|}{2}q(s)\\ &= \frac{r_2\mu}{2}\\ &\ge l.\end{aligned}$$

For $y \in K\cap\partial\Omega_{r_2}$, we have

$$\begin{aligned}\min_{t\in[t_1,\,t_2]}(T_\lambda y)(t) &= \min_{t\in[t_1,\,t_2]}\int_{\rho(c)}^{d}\mathcal{G}(t,\,s)\left[\lambda_1\overline{g}(s,\,y-x)+\lambda_2\overline{h}(s,\,y-x)+M\right]\nabla s\\ &\ge \min_{t\in[t_1,\,t_2]}\int_{t_1}^{t_2}\mathcal{G}(t,\,s)\left[\lambda_1\overline{g}(s,\,y-x)+\lambda_2\overline{h}(s,\,y-x)+M\right]\nabla s\\ &\ge \min_{t\in[t_1,\,t_2]}\int_{t_1}^{t_2}\mathcal{G}(t,\,s)\lambda_1N_1(y(s)-x(s))\nabla s\\ &= r_2=\|y\|.\end{aligned}$$

Thus,

$$\|T_\lambda y\|\ge\|y\| \quad \text{for all} \quad y\in K\cap\partial\Omega_{r_2}. \tag{3.14}$$

Let

$$R=\max\left\{\left(\lambda_1\max_{\substack{\rho(c)\le t\le\sigma(d)\\0\le y\le R}}g(t,\,y)+\lambda_2\max_{\substack{\rho(c)\le t\le\sigma(d)\\0\le y\le R}}h(t,\,y)+M\right)\left(\int_{\rho(c)}^{d}\mathcal{G}(s,\,s)\nabla s\right),2r_2\right\},$$

then $r_1<r_2<R$. Let $\Omega_R=\{y\in C[\rho(c),\,\sigma(d)]_{\mathbb{T}}:\|y\|<R\}$. For $y\in K\cap\Omega_R$, $t\in[\rho(c),\,\sigma(d)]_{\mathbb{T}}$, we have

$$\begin{aligned}(T_\lambda y)(t) &= \int_{\rho(c)}^{d}\mathcal{G}(t,\,s)\Bigg[\lambda_1\overline{g}(s,\,y-x)+\lambda_2\overline{h}(s,\,y-x)+M\Bigg]\nabla s\\ &\le\left[(\lambda_1+\lambda_2)\left(\max_{\substack{\rho(c)\le t\le\sigma(d)\\0\le y\le R}}g(t,\,y)+\max_{\substack{\rho(c)\le t\le\sigma(d)\\0\le y\le R}}h(t,\,y)\right)+M\right]\int_{\rho(c)}^{d}\mathcal{G}(s,\,s)\nabla s\\ &\le R=\|y\|\end{aligned}$$

Thus,

$$\|T_\lambda y\|\le\|y\| \quad \text{for all} \quad y\in K\cap\partial\Omega_R. \tag{3.15}$$

Thus by the Lemma 2.8, T_λ has at least two fixed points. Hence, the boundary value problem (1.1) and (1.2) has at least two positive solutions. □

Theorem 3.8. *Let (H_1)–(H_3), (L_3) and (L_8) hold. Then the boundary value problem (1.1) and (1.2) has at least two positive solutions for λ_i, $i = 1, 2$ are sufficiently small.*

Proof. The proof of the Theorem 3.8 is similar to that of Theorem 3.5. $\square$

Theorem 3.9. *Let (H_1)–(H_3), (L_1) and (L_7) hold. Then the boundary value problem (1.1) and (1.2) has at least one positive solution for λ_i, $i = 1, 2$ are sufficiently small.*

Proof. From (L_1), we have

$$\lim_{y\to\infty} \frac{g(t,\, y)}{y} = \infty.$$

For $k > 0$, there exists a $r_1 > 0$ such that

$$g(t,\, y) \geq ky \quad \text{for} \quad y > r_1.$$

Let $\Omega_{r_1} = \{y \in C[\rho(c),\, \sigma(d)]_{\mathbb{T}} : \|y\| < r_1\}$ and let k satisfy

$$\frac{k\mu}{2}\lambda_1 \min_{t\in[t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s)\nabla s \geq 1.$$

For $y \in K \cap \partial\Omega_{r_1}$, we have

$$\begin{aligned}
\min_{t\in[t_1,\, t_2]} (T_\lambda y)(t) &= \min_{t\in[t_1,\, t_2]} \int_{\rho(c)}^{d} \mathcal{G}(t,\, s) \left[\lambda_1 \overline{g}(s,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M\right] \nabla s \\
&\geq \min_{t\in[t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s)\lambda_1 k(y-x)\nabla s \\
&\geq \frac{k}{2}\lambda_1 \min_{t\in[t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s)\|y\| q(s)\nabla s \\
&\geq \|y\|.
\end{aligned}$$

Thus,

$$\|T_\lambda y\| \geq \|y\| \quad \text{for} \quad y \in K \cap \partial\Omega_{r_1}. \tag{3.16}$$

From (L_7), we have

$$\lim_{y\to 0} \frac{h(t,\, y)}{y} = 0.$$

For $\epsilon > 0$, there exists a $r_2 > 0$ such that

$$h(t,\, y) \leq \epsilon y \quad \text{for} \quad y \in [0,\, \infty).$$

Since λ_1 and λ_2 are sufficiently small, let

$$\left[(\lambda_1+\lambda_2)\left(\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_2}} g(t,\,y)+\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_2}} h(t,\,y)+M\right)\right]\int_{\rho(c)}^{d}\mathcal{G}(s,\,s)\nabla s\le r_2.$$

Let $\Omega_{r_2}=\{y\in C[\rho(c),\,\sigma(d)]_{\mathbb{T}}:\|y\|<r_2\}$. Now for any $y\in K\cap\partial\Omega_{r_2}$, we have

$$\begin{aligned}(T_\lambda y)(t)&=\int_{\rho(c)}^{d}\mathcal{G}(t,\,s)\left[\lambda_1\overline{g}(t,\,y-x)+\lambda_2\overline{h}(s,\,y-x)+M\right]\nabla s\\ &\le\int_{\rho(c)}^{d}\mathcal{G}(s,\,s)\left[\lambda_1\overline{g}(t,\,y-x)+\lambda_2\overline{h}(s,\,y-x)+M\right]\nabla s\\ &\le\left[(\lambda_1+\lambda_2)\left(\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_2}} g(t,\,y)+\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_2}} h(t,\,y)\right)+M\right]\int_{\rho(c)}^{d}\mathcal{G}(s,\,s)\nabla s\\ &\le r_2=\|y\|.\end{aligned}$$

Thus,

$$\|T_\lambda y\|\le\|y\|\quad\text{for}\quad y\in K\cap\partial\Omega_{r_2}. \tag{3.17}$$

Hence, by the Lemma 2.8, T_λ has a fixed point $\overline{y}$ with $r_1<\|\overline{y}\|<r_2$. By the Lemma 2.6, the boundary value problem (1.1) and (1.2) has at least one positive solution. □

Theorem 3.10. *Let (H_1)–(H_3), (L_3) and (L_5) hold. Then the boundary value problem (1.1) and (1.2) has at least one positive solution for λ_i, $i=1,\,2$ are sufficiently small.*

Proof. The proof of the Theorem 3.10 is similar to that of Theorem 3.9. □

Theorem 3.11. *Let (H_1)–(H_3), (L_2) and (L_8) hold. Then the boundary value problem (1.1) and (1.2) has at least one positive solution for λ_i, $i=1,\,2$ are sufficiently small.*

Proof. From (L_2), we have

$$\lim_{y\to\infty}\frac{g(t,\,y)}{y}=0.$$

By Lemma 2.5, there exist $r_1>0$ and $k_1>0$ such that

$$\mathbb{G}(r_1)\le k_1r_1.$$

Let $\Omega_{r_1}=\{y\in C[\rho(c),\,\sigma(d)]_{\mathbb{T}}:\|y\|<r_1\}$. Since λ_1 and λ_2 are sufficiently small, we have

$$\left[(\lambda_1+\lambda_2)\left(\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} g(t,\,y)+\max_{\substack{\rho(c)\le t\le\sigma(d)\\ 0\le y\le r_1}} h(t,\,y)\right)+M\right]\int_{\rho(c)}^{d}\mathcal{G}(s,\,s)\nabla s\le r_1.$$

For any $y \in K \cap \partial\Omega_{r_1}$, we obtain

$$\begin{aligned}
(T_\lambda y)(t) &= \int_{\rho(c)}^{d} \mathcal{G}(t,\, s) \left[\lambda_1 \overline{g}(t,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M\right] \nabla s \\
&\leq \int_{\rho(c)}^{d} \mathcal{G}(s,\, s) \left[\lambda_1 \overline{g}(t,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M\right] \nabla s \\
&\leq \left[(\lambda_1+\lambda_2)\left(\max_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq r_2}} g(t,\, y) + \max_{\substack{\rho(c)\leq t\leq\sigma(d)\\ 0\leq y\leq r_2}} h(t,\, y)\right) + M\right] \int_{\rho(c)}^{d} \mathcal{G}(s,\, s) \nabla s \\
&\leq r_1 = \|y\|.
\end{aligned}$$

Thus,

$$\|T_\lambda y\| \leq \|y\| \quad \text{for} \quad y \in K \cap \partial\Omega_{r_1}. \tag{3.18}$$

From (L_8), we have

$$\lim_{y\to 0} \frac{h(t,\, y)}{y} = \infty.$$

For $k > 0$, there exists a $l > 0$ such that

$$h(t,\, y) \geq ky \quad \text{for} \quad y \in [0,\, l].$$

Let $r_2 = \left\{2cm, \frac{2l}{\mu}, 2r_1\right\}$ and $\Omega_{r_2} = \{y \in C[\rho(c),\, \sigma(d)]_{\mathbb{T}} : \|y\| < r_2\}$. For any $y \in K \cap \partial\Omega_{r_2}$, we have

$$\begin{aligned}
\min_{t\in[t_1,\, t_2]} (T_\lambda y)(t) &= \min_{t\in[t_1,\, t_2]} \int_{\rho(c)}^{d} \mathcal{G}(t,\, s) \left[\lambda_1 \overline{g}(t,\, y-x) + \lambda_2 \overline{h}(s,\, y-x) + M\right] \nabla s \\
&\geq \min_{t\in[t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s) \lambda_2 k(y(s)-x(s)) \nabla s \\
&\geq \frac{k}{2} \lambda_2 \mu \min_{t\in[t_1,\, t_2]} \int_{t_1}^{t_2} \mathcal{G}(t,\, s) \|y\| \nabla s \\
&\geq r_2 = \|y\|.
\end{aligned}$$

Thus,

$$\|T_\lambda y\| \geq \|y\| \quad \text{for} \quad y \in K \cap \partial\Omega_{r_2}. \tag{3.19}$$

Hence, by the Lemma 2.8, T_λ has a fixed point $\overline{y}$ with $r_1 < \|\overline{y}\| < r_2$. By the Lemma 2.6, the boundary value problem (1.1) and (1.2) has at least one positive solution. □

Theorem 3.12. *Let (H_1)–(H_3), (L_4) and (L_6) hold. Then the boundary value problem (1.1) and (1.2) has at least one positive solution for λ_i, $, i = 1,\, 2$ are sufficiently small.*

Proof. The proof of the Theorem 3.12 is similar to that of Theorem 3.11. □

4 Examples

We shall illustrate few examples in different time scales to justify the results obtained in the preceding section.

Example 4.1. *Let us consider the following boundary value problem on time scale* $\mathbb{T} = \mathbb{R}$*,*

$$((1+t^2)y')' + \frac{1}{2}\frac{1+y^2}{52} + \frac{1}{4}\frac{y^2\sin^2 y}{35} = 0, \quad t \in [0,\, 1], \tag{4.1}$$

with boundary conditions

$$\begin{aligned} y(0) - y'(0) &= 0, \\ y(1) + 2y'(1) &= 0, \end{aligned} \tag{4.2}$$

where $\psi(t) = 1+t^2$*,* $M = 1$*,* $\alpha, \beta, \gamma, \delta \geq 0$*,* $g(t,\, y) = \frac{1+y^2}{52}$ *and* $h(t,\, y) = \frac{y^2\sin^2 y}{35}$*. Green's function for the boundary value problem (4.1) and (4.2) is given by*

$$\mathcal{G}(t,\, s) = \frac{1}{2+\frac{\pi}{4}}\begin{cases} \left(1+\tan^{-1} t\right)\left(1+\frac{\pi}{4} - \tan^{-1} s\right), \; t \leq s, \\ \left(1+\tan^{-1} s\right)\left(1+\frac{\pi}{4} - \tan^{-1} t\right), \; s \leq t. \end{cases}$$

All the conditions (H_1)*–*(H_3)*,* (L_1) *and* (L_5) *are satishfied for* $(t,\, y) \in [0,\, 1] \times [0,\, 100]$*. By Theorem 3.1, boundary value problem (4.1) and (4.2) has at least one positive solution for* $\lambda_1 = \frac{1}{2}$ *and* $\lambda_2 = \frac{1}{4}$*.*

Example 4.2. *Let us consider the following boundary value problem on time scale* $\mathbb{T} = \mathbb{Z}$*,*

$$\nabla((1+t)^{-1}y^{\Delta}) + \lambda_1 \sin^2 y + \lambda_2 \sqrt{y}\cos y = 0, \quad t \in [0,\, 3], \tag{4.3}$$

with boundary conditions

$$\begin{aligned} y(0) - \Delta y(0) &= 0, \\ y(3) + \frac{1}{3}\Delta y(2) &= 0, \end{aligned} \tag{4.4}$$

where $\psi(t) = (1+t)^{-1}$*,* $M = 1$*,* $\alpha, \beta, \gamma, \delta \geq 0$*,* $g(t,\, y) = \sin^2 y$ *and* $h(t,\, y) = \sqrt{y}\cos y$*. Green's function for the boundary value problem (4.3) and (4.4) is given by*

$$\mathcal{G}(t,\, s) = \frac{1}{11}\begin{cases} \left(1+\frac{t^2+3t}{2}\right)\left(1+\frac{(3-s)(s+6)}{2}\right), \; t \leq s, \\ \left(1+\frac{s^2+3s}{2}\right)\left(1+\frac{(3-t)(t+6)}{2}\right), \; s \leq t. \end{cases}$$

All the conditions (H_1)*–*(H_3)*,* (L_2) *and* (L_6) *are satishfied for* $(t,\, y) \in [0,\, 3] \times [0,\, 100]$*. Let* $D = 1$

and $\eta = \frac{1}{2}$ *such that* $g(t,\, y) \geq \frac{1}{2}$ *and* $h(t,\, y) \geq \frac{1}{2}$ *for* $t \in [0,\, 3]$, $y \in [1,\, \infty)$. *By Theorem 3.3, boundary value problem (4.3) and (4.4) has at least one positive solution for* λ_i; $i = 1,\, 2$ *are sufficiently small.*

Example 4.3. *Consider the boundary value problem on time scale* $\mathbb{T} = q^{\overline{\mathbb{Z}}} = \{2^k : k \in \mathbb{Z}\} \cup \{0\}$, *where* $q = 2 > 1$,

$$D^q\left((1+t)^{-1} D_q y(t)\right) + \lambda_1 \frac{y^2}{\sin\, y} + \lambda_2 \ln(y) = 0, \quad t \in [0,\, 2], \tag{4.5}$$

with boundary conditions

$$\begin{aligned} y(0) - D_q y(0) &= 0, \\ y(2) + \frac{1}{2} D_q y(1) &= 0, \end{aligned} \tag{4.6}$$

where $\psi(t) = (1+t)^{-1}$, $M = 1$, $\alpha,\, \beta,\, \gamma,\, \delta \geq 0$, $g(t,\, y) = \frac{y^2}{\sin\, y}$ *and* $h(t,\, y) = \ln(y)$. *Green's function for the boundary value problem (4.5) and (4.6) is given by*

$$\mathcal{G}(t,\, s) = \frac{3}{20} \begin{cases} \left(\frac{2t^2+3t+3}{3}\right)\left(\frac{17-3s-2s^2}{3}\right), & t \leq s, \\ \left(\frac{2s^2+3s+3}{3}\right)\left(\frac{17-3t-2t^2}{3}\right), & s \leq t. \end{cases}$$

The conditions (H_1)*–*(H_3), (L_1) *and* (L_6) *are satishfied for* $(t,\, y) \in [0,\, 2] \times [0,\, 100]$. *By Theorem 3.5, boundary value problem (4.5) and (4.6) has at least two positive solutions for* λ_i; $i = 1,\, 2$ *are sufficiently small.*

Example 4.4. *Let us consider the time scale* $\mathbb{T} = \mathbb{P}_{a,\, b} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a] = \mathbb{P}_{1,\, 1} = \bigcup_{k=0}^{\infty} [2k, 2k+1]$, *where* $a = b = 1$. *Consider the following boundary value problem:*

$$\begin{cases} y^{\Delta\nabla} + \lambda_1 \sqrt{y} + \lambda_2\, y\, \ln(1+y), & t \in (0,\, 2), \\ y(0) = 0, \quad y(2) = 0, \end{cases} \tag{4.7}$$

where $\psi(t) = 1$, $M = 1$, $\alpha,\, \beta,\, \gamma,\, \delta \geq 0$, $g(t,\, y) = \sqrt{y}$ *and* $h(t,\, y) = y \ln(y)$. *Green's function for the boundary value problem (4.7) is given by*

$$\mathcal{G}(t,\, s) = \frac{1}{2} \begin{cases} t(1-s), & t \leq s, \\ (1+s)(2-t), & s \leq t. \end{cases}$$

The conditions (H_1)*–*(H_3), (L_4) *and* (L_7) *are satishfied for* $(t,\, y) \in [0,\, 2] \times [0,\, 100]$. *By Theorem 3.7, boundary value problem (4.7) has at least two positive solutions for* λ_i; $i = 1,\, 2$ *are sufficiently small.*

Example 4.5. *Consider the boundary value problem on time scale* $\mathbb{T} = \{\frac{n}{2} : t \in \mathbb{N}_0\}$*:*

$$y^{\Delta\nabla}(t) + \lambda_1 y \ln(1+y) + \lambda_2 \frac{\sqrt{y}\sin y}{6} = 0, \quad t \in \left[0, \frac{3}{2}\right], \tag{4.8}$$

with boundary conditions

$$\begin{aligned} y(0) - y^{\Delta}(0) &= 0, \\ y\left(\frac{3}{2}\right) + y^{\Delta}(1) &= 0, \end{aligned} \tag{4.9}$$

where $\psi(t) = 1$*,* $M = 1$*,* $\alpha, \beta, \gamma, \delta \geq 0$*,* $g(t, y) = y\ln(1+y)$ *and* $h(t, y) = \frac{\sqrt{y}\sin y}{6}$*. Green's function for the boundary value problem (4.8) and (4.9) is given by*

$$\mathcal{G}(t, s) = \frac{2}{7} \begin{cases} (1+s)\left(\frac{5}{2} - t\right), \ t \leq s, \\ (1+t)\left(\frac{5}{2} - s\right), \ s \leq t. \end{cases}$$

The conditions (H_1)*–*(H_3)*,* (L_1) *and* (L_7) *are satisfied for* $(t, y) \in [0, \frac{3}{2}] \times [0, 100]$*. By Theorem 3.9, boundary value problem (4.8) and (4.9) has at least one positive solutions for* λ_i*;* $i = 1, 2$ *are sufficiently small.*

Example 4.6. *Consider the following boundary value problem in time scale* $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$*, where* $h = \frac{1}{2} > 0$*,*

$$\left((1+t)^{-1} y^{\Delta}\right)^{\nabla} + \lambda_1 \sqrt{y}\sin y + \lambda_2 = 0 \quad \textit{for} \quad t \in [0, 2], \tag{4.10}$$

with boundary conditions

$$\begin{aligned} y(0) - y^{\Delta}(0) &= 0, \\ y(2) + \frac{2}{5} y^{\Delta}\left(\frac{3}{2}\right) &= 0, \end{aligned} \tag{4.11}$$

where $\psi(t) = (1+t)^{-1}$*,* $M = 1$*,* $\alpha, \beta, \gamma, \delta \geq 0$*,* $g(t, y) = \sqrt{y}\sin y$ *and* $h(t, y) = 1$*. Green's function for the boundary value problem (4.10) and (4.11) is given by*

$$\mathcal{G}(t, s) = \frac{2}{13} \begin{cases} \left(1 + \frac{s(2s+5)}{4}\right)\left(1 + \frac{(2-t)(2t+9)}{4}\right), \ s \leq t, \\ \left(1 + \frac{t(2t+5)}{4}\right)\left(1 + \frac{(2-s)(2s+9)}{4}\right), \ t \leq s. \end{cases}$$

The conditions (H_1)*–*(H_3)*,* (L_2) *and* (L_8) *are satishfied for* $(t, y) \in [0, 2] \times [0, 100]$*. By Theorem 3.11, boundary value problem (4.10) and (4.11) has at least one positive solutions for* λ_i*;* $i = 1, 2$ *are sufficiently small.*

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