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Infinitely many solutions for a nonlinear Navier problem involving the p-biharmonic operator

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ABSTRACT

In this paper we establish some results of existence of infinitely many solutions for an elliptic equation involving the p-biharmonic and the p-Laplacian operators coupled with Navier boundary conditions where the nonlinearities depend on two real parameters and do not satisfy any symmetric condition. The nature of the approach is variational and the main tool is an abstract result of Ricceri. The novelty in the application of this abstract tool is the use of a class of test functions which makes the assumptions on the data easier to verify.

RESUMEN

En este artículo establecemos algunos resultados sobre la existencia de infinitas soluciones para una ecuación elíptica que involucra los operadores p-biarmónico y p-Laplaciano acoplados con condiciones de borde de Navier, donde las nolinealidades dependen de dos parámetros reales y no satisfacen ninguna condición simétrica. La naturaleza del enfoque es variacional y la herramienta principal es un resultado abstracto de Ricceri. La novedad de la aplicación de esta herramienta abstracta es el uso de una clase de funciones test que hacen que las hipótesis sobre la data sean más fáciles de verificar.

Keywords and Phrases: p-biharmonic operator, p-Laplacian operator, Navier problem, multiplicity.

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1 Introduction

In this paper we investigate the existence of infinitely many solutions to the following p-biharmonic elliptic equation with Navier conditions,

$$\begin{cases} \Delta_p^2 u - \Delta_p u + V(x)|u|^{p-2}u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \Omega \end{cases} \qquad (P_{\lambda,\mu})$$

where $\Omega \subset \mathbb{R}^n$ $(n \geqslant 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $p > \max\left\{1, \frac{n}{2}\right\}$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is the p-biharmonic operator, $\Delta_p u = \nabla(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $V \in C(\overline{\Omega})$ satisfying $\inf_{\overline{\Omega}} V > 0$, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions with suitable behaviors, $\lambda \in \mathbb{R}$ and $\mu > 0$.

In the last years several authors have showed their interest in fourth-order differential problems involving biharmonic and p-biharmonic operators, motivated by the fact that this type of equations finds applications in fields such as the elasticity theory, or more in general, in continuous mechanics. In particular, the fourth-order elliptic equations can describe the static form change of beam or the motion of rigid body, so they are widely applied in physics and engineering. In 1990 Lazer and Mckenna, in a large paper in which they investigated the oscillatory phenomena that led to the collapse of the Tacoma Narrows bridge, considered fourth-order problems with the nonlinearity $(u+1)^+ - 1$; this nonlinearity is useful to study traveling waves in suspension bridges. Anyway the same authors observed that this kind of problems are interesting also when this particular nonlinearity is replaced by a somewhat more general function $F(\cdot, u)$ (see [24, 31, 32]).

As regards fourth-order differential problems involving biharmonic and p-biharmonic operators, a non-negligible part of the literature is devoted to the study of the existence of infinitely many solutions to problems involving only the biharmonic or p-biharmonic operator (see, for instance, [2, 4, 5, 6, 9, 10, 17, 18, 19, 29, 30, 40]) or considering also the presence of Laplacian or p-Laplacian operator ([22, 26, 38, 42, 43]) and/or a term with a potential function ([11, 12, 13, 25, 28]); some authors have also recently considered the case in which a nonlocal term is present ([16, 41]).

Unlike some papers concerning problems set in an unbounded domain (see [2, 4, 11, 12, 13, 18, 19, 30] and above all [25] which inspired us in the choice of this type of problem), most of the literature is devoted to the bounded case. In this case, different approaches have been adopted for obtaining infinitely many solutions. In a lot of papers symmetry conditions on the nonlinearities are assumed together with the use of the symmetric mountain pass theorem of Ambrosetti Rabinowitz (see [26, 40]) or with the use of the fountain theorem ([38, 42, 43]).

In our investigation the approach is variational. More precisely we will apply the following critical point theorem that Ricceri established in 2000 ([34, Theorem 2.5]), recalled below for the reader's convenience.

Theorem 1.1. *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functionals. Assume also that Ψ is (strongly) continuous and coercive. For each $r > \inf_X \Psi$, we put*

$$\varphi(r) := \inf_{x \in \Psi^{-1}(]-\infty, r[)} \frac{\Phi(x) - \inf_{\overline{\Psi^{-1}(]-\infty,r[)}_{\omega}} \Phi}{r - \Psi(x)}$$

where $\overline{\Psi^{-1}(]-\infty, r[)}_w$ is the closure of $\Psi^{-1}(]-\infty, r[)$ in the weak topology. Fixed $\lambda \in \mathbb{R}$, then

a) if $\{r_k\}$ is a real sequence such that $\lim_{k\to\infty} r_k = +\infty$ and $\varphi(r_k) < \lambda$, for each $k \in \mathbb{N}$, the following alternative holds: either $\Phi + \lambda\Psi$ has a global minimum or there exists a sequence $\{x_k\}$ of critical points of $\Phi + \lambda\Psi$ such that $\lim_{k\to\infty} \Psi(x_k) = +\infty$;

b) if $\{s_k\}$ is a real sequence such that $\lim_{k\to\infty} s_k = (\inf_x \Psi)^+$ and $\varphi(s_k) < \lambda$ for each $k \in \mathbb{N}$, the following alternative holds: either there exists a global minimum of Ψ which is a local minimum of $\Phi + \lambda\Psi$ or there exists a sequence $\{x_k\}$ of pairwise distinct critical points of $\Phi + \lambda\Psi$ with $\lim_{k\to\infty} \Psi(x_k) = \inf_X \Psi$, which weakly converges to a global minimum of Ψ.

Since its appearance in 2000 until our days, it has been a powerful tool to get multiplicity results for different kinds of problems. In particular, it has been widely applied to obtain theorems of existence of infinitely many solutions to problems associated with a vast range of differential equations. In each of these applications, in order to guarantee that $\varphi(r_k) < \lambda$ (or $\varphi(s_k) < \lambda$), for each $k \in \mathbb{N}$, and that the functional $\Phi + \lambda\Psi$ has no global minimum, it is necessary to use some sequences of functions defined *ad hoc*. Generally, in these functions the norm of the variable is raised to a suitable power which depends on the nature of the problem and that gives them the requested regularity properties: in some applications the norm is used without power (see, for instance, [3, 7, 14, 15, 23, 27, 39]), in some others it is raised to the second ([9, 10, 29, 33, 35, 36]) or to the third ([22, 28]) or to the fourth power ([1]); in [20, 21] the authors combined the norm with trigonometric functions.

The choice of a particular sequence of functions inside the proof reflects heavily on the assumptions and while there are some cases in which probably the choice is optimal, in some other cases it could happen that a different choice of the sequence would make the result applicable in a greater number of cases. This is the reason we have introduced an abstract class of test functions serving our purpose. We will clarify this fact in Section 3, showing some examples. A similar line of reasoning can be found in [8] and above all in [37] where the author does not choose the test functions arbitrarily during the proof but he uses two generic functions whose properties are described in the statement of his result.

2 Preliminaries

In this section we describe the variational framework in which we will work in our investigations.

To begin with, we denote by $\omega := \pi^{\frac{n}{2}}/\Gamma\left(\frac{n}{2}+1\right)$ the measure of the unit ball in $\mathbb{R}^n$. If X is a Banach space, the symbol $B(x,r)$ stands for the open ball centered at $x \in X$ and of radius $r > 0$.

Let Ω be a bounded smooth domain of $\mathbb{R}^n$, $n \geq 1$, $p > \max\left\{1, \frac{n}{2}\right\}$ and let $V \in C(\overline{\Omega})$ satisfy $\inf_\Omega V > 0$. Put $E = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$; it is a reflexive Banach space when endowed with the standard norm

$$\|u\| = \left(\int_\Omega |\Delta u|^p dx\right)^{\frac{1}{p}}.$$

Moreover, the assumptions on V assure that the position

$$\|u\|_V = \left(\int_\Omega \left(|\Delta u|^p + |\nabla u|^p + V(x)|u|^p\right) dx\right)^{\frac{1}{p}}$$

for any $u \in E$, defines a norm equivalent to the standard one. Being $p > \frac{n}{2}$, the Rellich-Kondrachov theorem assures that E is compactly embedded in $C^0(\overline{\Omega})$; in particular, there exists a constant $c_\infty > 0$ such that

$$\|u\|_\infty \leq c_\infty \|u\| \leq c_\infty \|u\|_V \tag{2.1}$$

for every $u \in E$. Now, motivated by the reasons that we have illustrated in the Introduction, let us introduce the following class of functions. If $\{a_k\}$, $\{b_k\}$, $\{\sigma_k\}$ are three real sequences with $0 < a_k < b_k$ and $\sigma_k > 0$, for each $k \in \mathbb{N}$, let us denote by $\mathcal{H}(\{a_k\}, \{b_k\}, \{\sigma_k\})$ the space of all sequences $\{\chi_k\} \subset W^{2,p}(]a_k, b_k[)$ satisfying

i) $0 \leq \chi_k(x) \leq \sigma_k$ for a.e. $x \in]a_k, b_k[$;

ii) $\lim_{x \to a_k^+} \chi_k(x) = \sigma_k, \quad \lim_{x \to b_k^-} \chi_k(x) = 0$;

iii) $\lim_{x \to a_k^+} \chi_k'(x) = \lim_{x \to b_k^-} \chi_k'(x) = 0$;

iv) for all $j \in \{1, 2\}$ there exists $c_j > 0$, independent of k, such that

$$|\chi_k^{(j)}(x)| \leq c_j \frac{\sigma_k}{(b_k - a_k)^j} \tag{2.2}$$

for a.e. $x \in]a_k, b_k[$ and for all $k \in \mathbb{N}$.

Now, we show how the space $\mathcal{H}(\{a_k\}, \{b_k\}, \{\sigma_k\})$ help us to build some sequences in E that play a crucial role in the proof of the main result.

If $x_0 \in \Omega$, $\{b_k\} \subset]0, +\infty[$ such that $B(x_0, b_k) \subset \Omega$, for each $k \in \mathbb{N}$, and $\{\chi_k\} \in \mathcal{H}(\{a_k\}, \{b_k\}, \{\sigma_k\})$, consider the function $u_k : \Omega \to \mathbb{R}$ defined by setting

$$u_k(x) = \begin{cases} 0 & \text{in } \Omega \setminus B(x_0, b_k), \\ \sigma_k & \text{in } B(x_0, a_k), \\ \chi_k(|x - x_0|) & \text{in } B(x_0, b_k) \setminus B(x_0, a_k) \end{cases}$$

for each $k \in \mathbb{N}$.

Simple computations show that, fixed $k \in \mathbb{N}$, for each $i \in \{1, \ldots, n\}$, we have

$$\frac{\partial u_k}{\partial x_i}(x) = \begin{cases} 0 & \text{in } \Omega \setminus B(x_0, b_k), \\ 0 & \text{in } B(x_0, a_k), \\ \chi_k'(|x - x_0|)\dfrac{x_i - x_i^0}{|x - x_0|} & \text{in } B(x_0, b_k) \setminus B(x_o, a_k) \end{cases}$$

and

$$\frac{\partial^2 u_k}{\partial x_i^2}(x) = \begin{cases} 0 & \text{in } \Omega \setminus B(x_0, b_k), \\ 0 & \text{in } B(x_0, a_k), \\ \chi_k''(|x - x_0|)\dfrac{(x_i - x_i^0)^2}{|x - x_0|^2} + \chi_k'(|x - x_0|)\dfrac{|x - x_0|^2 - (x_i - x_i^0)^2}{|x - x_0|^3} & \text{in } B(x_0, b_k) \setminus B(x_0, a_k) \end{cases}$$

Using these computations together with (2.2), we get the following inequalities

$$|\nabla u_k(x)| \leqslant |\chi_k'(|x - x_0|)| \leq c_1 \frac{\sigma_k}{(b_k - a_k)},$$

and

$$|\Delta u_k(x)| \leqslant |\chi_k''(|x - x_0|)| + |\chi_k'(|x - x_0|)| \frac{(n-1)}{|x - x_0|} \leq c_2 \frac{\sigma_k}{(b_k - a_k)^2} + c_1 \frac{\sigma_k}{(b_k - a_k)} \frac{(n-1)}{a_k}.$$

These inequalities allow us to estimate the norm of the functions u_k as follows

$$\begin{aligned} \|u_k\|_V^p &= \int_\Omega (|\Delta u_k|^p + |\nabla u_k|^p + V(x)|u_k(x)|^p)\, dx \\ &= \int_{B(x_0,b_k)\setminus B(x_0,a_k)} |\Delta u_k(x)|^p dx + \int_{B(x_0,b_k)\setminus B(x_0,a_k)} |\nabla u_k(x)|^p dx + \int_{B(x_0,b_k)} V(x)|u_k(x)|^p dx \\ &\leq \omega \sigma_k^p \left\{ \left[\frac{c_2}{(b_k - a_k)^2} + \frac{c_1(n-1)}{a_k(b_k - a_k)} \right]^p (b_k^n - a_k^n) + \left[\frac{c_1}{(b_k - a_k)} \right]^p (b_k^n - a_k^n) + b_k^n \max_{B(x_0,b_k)} V \right\}. \end{aligned}$$

Let us denote by $\mathcal{C}$ the class of all Carathéodory functions $\eta : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying $\sup_{|t| \leq \xi} |\eta(\cdot, t)| \in L^1(\Omega)$ for all $\xi > 0$ and let $f, g \in \mathcal{C}$.

We say that a function $u \in E$ is a weak solution to $(P_{\lambda,\mu})$ if

$$\int_\Omega \left(|\Delta u|^{p-2}\Delta u \Delta v + |\nabla u|^{p-2}\nabla u \nabla v + V(x)|u|^{p-2}uv\right) dx = \lambda \int_\Omega f(x,u(x))v(x)dx \\ + \mu \int_\Omega g(x,u(x))v(x)dx$$

for each $v \in E$. Obviously the weak solutions to $(P_{\lambda,\mu})$ are exactly the critical points in E of the energy functional defined, for each $u \in E$, by

$$\mathcal{E}(u) := \frac{1}{p}\Psi(u) + \lambda\Phi_F(u) + \mu\Phi_G(u),$$

where

$$\Psi(u) := \|u\|_V^p, \quad \Phi_F(u) := -\int_\Omega F(x,u(x))dx, \quad \Phi_G(u) := -\int_\Omega G(x,u(x))dx,$$

where, for each $(x,t) \in \Omega \times \mathbb{R}$,

$$F(x,t) := \int_0^t f(x,s)ds, \quad G(x,t) := \int_0^t g(x,s)ds.$$

3 Results

The first multiplicity result deals with the case in which f has a global $(m-1)$-sublinear growth, with $m < p$, while different cases are considered for the behaviour of function g.

Theorem 3.1. *Let $V \in C(\overline{\Omega})$ satisfy $\inf_\Omega V > 0$ and let $f, g \in \mathcal{C}$ such that:*

(i_1) there exist $1 < m < p$ and $h \in L^1(\Omega)$ such that $|f(x,t)| \leq h(x)\left(1 + |t|^{m-1}\right)$ for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$,

(i_2) $G(x,t) \geq 0$ for a.e. $x \in \Omega$ and for all $t \geq 0$,

(i_3) there exists $x_0 \in \Omega$ and $\rho > 0$, $p_1, p_2 > 1$ such that $B(x_0,\rho) \subseteq \Omega$ and

$$\liminf_{t\to+\infty} \frac{\int_\Omega \max_{|\xi|\leq t} G(x,\xi)dx}{t^{p_1}} := a < +\infty, \quad \limsup_{t\to+\infty} \frac{\int_{B(x_0,\rho)} G(x,t)dx}{t^{p_2}} := b > 0.$$

Then the following facts hold:

(r_1) if $p_1 < p < p_2$, for all $\lambda \in \mathbb{R}$ and for all $\mu > 0$, the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;

(r_2) if $p_1 < p = p_2$, there exists $\mu_1 > 0$ such that for all $\lambda \in \mathbb{R}$ and for all $\mu > \mu_1$, the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;

(r_3) if $p_1 = p < p_2$, there exists $\mu_2 > 0$ such that for all $\lambda \in \mathbb{R}$ and for all $\mu \in]0, \mu_2[$, the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;

(r_4) if $p_1 = p_2 = p$, there exists $\gamma > 1$ and $C_{V,\gamma,\rho} > 0$ such that, if

$$C_{V,\gamma,\rho} < \frac{b}{\omega c_\infty^p a}, \tag{3.1}$$

(the previous inequality always being satisfied whether $a = 0$ or $b = +\infty$) then $\mu_1 < \mu_2$ and for all $\lambda \in \mathbb{R}$ and for all $\mu \in]\mu_1, \mu_2[$, the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions.

Proof. To prove (r_1), let us apply part $a)$ of Theorem 1.1 choosing $X = E$, Ψ defined as in the Preliminaries and $\Phi = \lambda\Phi_F + \mu\Phi_G$. As we have already observed the critical points of the functional $\Phi + \frac{1}{p}\Psi$ are precisely the weak solution of problem $(P_{\lambda,\mu})$. The functionals Φ and Ψ are sequentially weak lower semicontinuous and moreover Ψ is strongly continuous and coercive. In our case the function φ is defined by setting

$$\varphi(r) = \inf_{\|u\|_V^p < r} \frac{\Phi(u) + \sup_{\|w\|_V^p \le r} (-\Phi)}{r - \|u\|_V^p}$$

for each $r > 0$. Now, we wish to find a sequence $\{r_k\}_{k\in\mathbb{N}}$ such that $\lim\limits_{k\to\infty} r_k = +\infty$ and $\varphi(r_k) < \frac{1}{p}$ for each $k \in \mathbb{N}$. To this aim it suffices to prove that for each $k \in \mathbb{N}$ there exists a function $u_k \in X$, with $\|u_k\|_V^p < r_k$, such that

$$\sup_{\|w\|_V^p \le r_k} \left\{ \lambda \int_\Omega F(x, w(x))dx + \mu \int_\Omega G(x, w(x))dx \right\} - \lambda \int_\Omega F(x, u_k(x))dx +$$

$$-\mu \int_\Omega G(x, u_k(x))dx < \frac{1}{p} \left(r_k - \|u_k\|_V^p\right).$$

Thanks to (i_3), fixed $\overline{a} > a$, for each $k \in \mathbb{N}$ there exists $\alpha_k \ge k$ such that

$$\int_\Omega \max_{|\xi| \le \alpha_k} G(x, \xi)dx \le \overline{a}\alpha_k^{p_1}.$$

Now we choose $u_k = \theta_E$ and

$$r_k = \frac{1}{c_\infty^p} \alpha_k^p.$$

Obviously we have $\lim\limits_{k\to\infty} r_k = +\infty$. Before proving (3), observe that, for each $w \in X$ with $\|w\|_V^p \le r_k$, one has

$$\|w\|_\infty \le c_\infty \|w\|_V \le c_\infty r_k^{\frac{1}{p}} = \alpha_k$$

for each $k \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned}\lambda\int_\Omega F(x,w(x))dx+\mu\int_\Omega G(x,w(x))dx &\le |\lambda|\int_\Omega |h(x)|\left(|w(x)|+\frac{|w(x)|^m}{m}\right)dx\\ &\quad+\mu\int_\Omega \max_{|\xi|\le\alpha_k} G(x,\xi)dx \le |\lambda|\|h\|_1\left(\alpha_k+\frac{\alpha_k^m}{m}\right)+\mu\overline{a}\alpha_k^{p_1}\\ &\le |\lambda|\|h\|_1 c_\infty r_k^{\frac{1}{p}}+\frac{|\lambda|}{m}\|h\|_1 c_\infty^m r_k^{\frac{m}{p}}+\mu\overline{a}c_\infty^{p_1}r_k^{\frac{p_1}{p}}<\frac{1}{p}r_k\end{aligned}$$

for k large enough, being $1<m<p$ and $p_1<p$. So, thanks to part $a)$ of Theorem 1.1, the functional $\Phi+\frac{1}{p}\Psi$ has a global minimum, or there exists a sequence of weak solutions $\{u_k\}\subset E$ such that $\lim_{k\to\infty}\|u_k\|=+\infty$. This part of the proof will end if we show that the functional $\Phi+\frac{1}{p}\Psi$ has no global minimum. To this aim, using (i_3), fixed $0<\overline{b}<b$, we get $\beta_k\in]0,+\infty[$ with $\beta_k\ge k$, such that

$$\int_{B(x_0,\rho)} G(x,\beta_k)dx \ge \overline{b}\beta_k^{p_2}$$

for each $k\in\mathbb{N}$. After choosing $\gamma>1$ such that $B(x_0,\gamma\rho)\subseteq\Omega$ and a sequence $\{\chi_k\}\in\mathcal{H}(\rho,\gamma\rho,\{\alpha_k\})$, we consider

$$w_k(x)=\begin{cases}0, & \text{in } \Omega\setminus B(x_0,\gamma\rho),\\ \beta_k, & \text{in } B(x_0,\rho),\\ \chi_k(|x-x_0|) & \text{in } B(x_0,\gamma\rho)\setminus B(x_0,\rho).\end{cases}$$

Using the estimation of the norm made in the previous section, we get

$$\|w_k\|_V^p\le\omega\beta_k^p\left[\frac{2^{p-1}(\gamma^n-1)}{\rho^{2p-n}(\gamma-1)^{2p}}c_2^p+\frac{(2^{p-1}(n-1)^p+\rho^p)(\gamma^n-1)}{\rho^{2p-n}(\gamma-1)^p}c_1^p+\gamma^n\rho^n\max_{B(x_0,\gamma\rho)}V\right].$$

If we put

$$C_{V,\gamma,\rho}=\frac{2^{p-1}(\gamma^n-1)}{\rho^{2p-n}(\gamma-1)^{2p}}c_2^p+\frac{(2^{p-1}(n-1)^p+\rho^p)(\gamma^n-1)}{\rho^{2p-n}(\gamma-1)^p}c_1^p+\gamma^n\rho^n\max_{B(x_0,\gamma\rho)}V$$

we have

$$\begin{aligned}\Phi(w_k)+\frac{1}{p}\Psi(w_k)&=-\lambda\int_\Omega F(x,w_k(x))dx-\mu\int_\Omega G(x,w_k(x))dx+\frac{1}{p}\|w_k\|_V^p\\ &\le|\lambda|\int_\Omega|h(x)|\left(|w_k(x)|+\frac{|w_k(x)|^m}{m}\right)dx-\mu\int_{B(x_0,\rho)}G(x,\beta_k)dx+\frac{\omega C_{V,\gamma,\rho}}{p}\beta_k^p\\ &\le|\lambda|\|h\|_1\beta_k+|\lambda|\|h\|_1\frac{\beta_k^m}{m}-\mu\overline{b}\beta_k^{p_2}+\frac{\omega C_{V,\gamma,\rho}}{p}\beta_k^p\end{aligned}$$

and, since $1<m<p<d_2$ and $\lim_{k\to\infty}\beta_k=+\infty$, the functional $\Phi+\frac{1}{p}\Psi$ has no global minimum, being $\lim_{k\to\infty}\Phi(w_k)+\dfrac{1}{p}\Psi(w_k)=-\infty$. This concludes the proof of (r_1).

The proof of (r_2) is similar. If $p_1 < p$ and $p_2 = p$, we choose $\mu_1 = \frac{\omega C_{V,\gamma,\rho}}{pb}$ (obviously if $b = +\infty$ we choose $\mu_1 = 0$). Therefore, if $\lambda \in \mathbb{R}$ and $\mu > \mu_1$, choosing $\overline{b}$ such that $\frac{\omega C_{V,\gamma,\rho}}{p\mu} < \overline{b} < b$, in a similar way we have

$$\Phi(w_k) + \frac{1}{p}\Psi(w_k) \leq |\lambda|\|h\|_1\beta_k + |\lambda|\|h\|_1\frac{\beta_k^m}{m} - \left(\mu\overline{b} - \frac{\omega C_{V,\rho,\gamma}}{p}\right)\beta_k^p$$

and, thanks to the choice of $\overline{b}$, also in this case the functional $\Phi + \frac{1}{p}\Psi$ has no global minimum. This concludes (r_2).

As for the proof of (r_3), if $p_1 = p$ and $p_2 > p$, we choose $\mu_2 = \frac{1}{pc_\infty^p a}$ (obviously if $a = 0$ we choose $\mu_2 = +\infty$). Then, fixing $\lambda \in \mathbb{R}$ and $0 < \mu < \mu_2$, we can choose $\overline{a}$ such that $a < \overline{a} < \frac{1}{pc_\infty^p \mu}$. Similar computations give

$$\lambda\int_\Omega F(x,w(x))dx + \mu\int_\Omega G(x,w(x))dx \leq |\lambda|\|h\|_1 c_\infty r_k^{\frac{1}{p}} + \frac{|\lambda|}{m}\|h\|_1 c_\infty^m r_k^{\frac{m}{p}} + \mu\overline{a}c_\infty^p r_k < \frac{1}{p}r_k$$

for k large enough, being $1 < m < p$ and $\mu\overline{a}c_\infty^p < \frac{1}{p}$.

Finally, the proof of (r_4) relies on the considerations made in the previous two cases. We have only to prove that $\mu_1 < \mu_2$, but this is guaranteed by the assumption (3.1). $\square$

Now, we are interested in the existence of infinitely many weak solutions in the case that the nonlinearities f and g have a particular form.

Theorem 3.2. *Let $V \in C(\overline{\Omega})$ satisfy $\inf_\Omega V > 0$, $m < p$, $h \in L^1(\Omega)$, and $r \in L^1(\Omega)\setminus\{0\}$ with $r \geq 0$ a.e. in Ω. Let $s : \mathbb{R} \to \mathbb{R}$ be a continuous function with $\int_0^t s(\xi)d\xi \geq 0$, for all $t \geq 0$. Moreover assume that there exists $p_1, p_2 > 1$, $\alpha, \beta > 0$ and $\{\alpha_k\}$, $\{\beta_k\}$ satisfying $\lim_{k\to\infty}\alpha_k = \lim_{k\to\infty}\beta_k = +\infty$, such that*

$$\max_{|\xi|\leq\alpha_k}\int_0^\xi s(t)dt \leq \alpha\alpha_k^{p_1}, \qquad \int_0^{\beta_k} s(t)dt \geq \beta\beta_k^{p_2}$$

for each $k \in \mathbb{N}$. Then, for the problem

$$\begin{cases} \Delta_p^2 u - \Delta_p u + V(x)|u|^{p-2}u = \lambda h(x)|u|^{m-2}u + \mu r(x)s(u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \Omega \end{cases} \qquad (\overline{P}_{\lambda,\mu})$$

the following facts hold:

$(\overline{r}_1)$ *if $p_1 < p < p_2$, for all $\lambda \in \mathbb{R}$ and for all $\mu > 0$, the problem $(\overline{P}_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*

$(\overline{r}_2)$ *if $p_1 < p = p_2$, there exists $\mu_1 > 0$ such that for all $\lambda \in \mathbb{R}$ and for all $\mu > \mu_1$, the problem $(\overline{P}_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*

$(\overline{r}_3)$ *if* $p_1 = p < p_2$*, there exists* $\mu_2 > 0$ *such that for all* $\lambda \in \mathbb{R}$ *and for all* $\mu \in]0, \mu_2[$*, the problem* $(\overline{P}_{\lambda,\mu})$ *admits a sequence of non-zero weak solutions;*

$(\overline{r}_4)$ *if* $p_1 = p_2 = p$*, there exist* $x_0 \in \Omega$*,* $\rho > 0$*,* $\gamma > 1$ *and* $C_{V,\gamma,\rho} > 0$*, such that, if*

$$C_{V,\gamma,\rho} < \frac{\beta \|r\|_{L^1(B(x_0,\rho))}}{\alpha\omega c_\infty^p \|r\|_{L^1(\Omega)}}, \tag{3.2}$$

then $\mu_1 < \mu_2$ *and for all* $\lambda \in \mathbb{R}$ *and for all* $\mu \in]\mu_1, \mu_2[$*, the problem* $(\overline{P}_{\lambda,\mu})$ *admits a sequence of non-zero weak solutions.*

Proof. We want to apply Theorem 3.1 choosing $f(x,t) = h(x)|t|^{m-2}t$ and $g(x,t) = r(x)s(t)$ for all $(x,t) \in \Omega \times \mathbb{R}$. The hypotheses $(i_1), (i_2)$ are obviously verified. Since $r \not\equiv 0$ we can choose $x_0 \in \Omega$ and $\rho > 0$ such that $B(x_0, \rho) \subset \Omega$ and $r > 0$ in $B(x_0, \rho)$. Then we have:

$$\int_\Omega \max_{|\xi|\le\alpha_k} G(x,\xi)dx = \int_\Omega \max_{|\xi|\le\alpha_k} \left(\int_0^\xi r(x)s(t)dt\right) dx = \|r\|_{L^1(\Omega)} \max_{|\xi|\le\alpha_k} \int_0^\xi s(t)dt \le \|r\|_{L^1(\Omega)}\alpha\alpha_k^{p_1}$$

and

$$\int_{B(x_0,\rho)} G(x,\beta_k)dx = \int_{B(x_0,\rho)} \left(\int_0^{\beta_k} r(x)s(t)dt\right) dx = \|r\|_{L^1(B(x_0,\rho))} \int_0^{\beta_k} s(t)dt \ge \|r\|_{L^1(B(x_0,\rho))}\beta\beta_k^{p_2}.$$

Therefore

$$\liminf_{t\to+\infty} \frac{\int_\Omega \max_{|\xi|\le t} G(x,\xi)dx}{t^{p_1}} \le \|r\|_{L^1(\Omega)}\alpha < +\infty$$

and

$$\limsup_{t\to+\infty} \frac{\int_{B(x_0,\rho)} G(x,t)dx}{t^{p_2}} \ge \|r\|_{L^1(B(x_0,\rho))}\beta > 0.$$

So, (i_3) is also verified with $a = \alpha\|r\|_{L^1(\Omega)}$ and $b = \beta\|r\|_{L^1(B(x_0,\rho))}$. Therefore we can apply the Theorem 3.1 and obtain the conclusions $(\overline{r}_1)$–$(\overline{r}_4)$. $\square$

Now, we want to exhibit two examples. In the first one we present a function s verifying the hypotheses of Theorem 3.2.

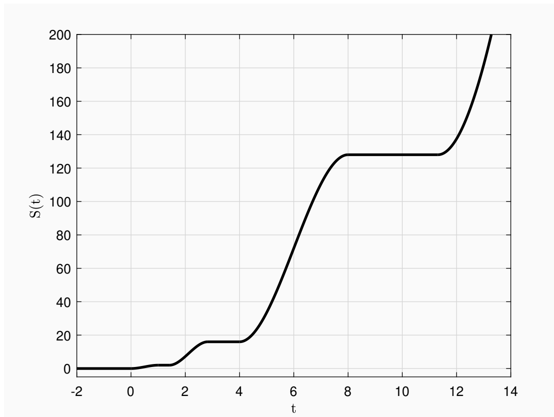
Example 3.3. *Let* $p > 1$*,* $\delta > 1$ *and let* $s : \mathbb{R} \to \mathbb{R}$ *be the function such that*

$$S(t) = \int_0^t s(\xi)d\xi = \begin{cases} 0, & \text{in }]-\infty, 0], \\ -2\delta t^3 + 3\delta t^2, & \text{in }]0, 1], \\ 2^{p(k-1)}\delta^k & \text{in } \left]2^{k-1}\delta^{\frac{k-1}{p}}, 2^{k-1}\delta^{\frac{k}{p}}\right] \quad k \ge 1, \\ A_k t^3 + B_k t^2 + C_k t + D_k & \text{in } \left]2^{k-1}\delta^{\frac{k}{p}}, 2^k\delta^{\frac{k}{p}}\right] \quad k \ge 1 \end{cases}$$

where

$$A_k := -2^{(p-3)k+4}\delta^{\frac{(p-3)k}{p}}\left(\delta - 2^{-p}\right), \quad B_k := 9 \cdot 2^{(p-2)k+2}\delta^{\frac{(p-2)k}{p}}\left(\delta - 2^{-p}\right),$$
$$C_k := -3 \cdot 2^{(p-1)k+3}\delta^{\frac{(p-1)k}{p}}\left(\delta - 2^{-p}\right), \quad D_k := 2^{pk}\delta^k\left(5\delta - 2^{2-p}\right).$$

Using MATLAB by MathWorks, we have plotted the graph of the function S (for $\delta = 2$ and $p = 2$), showed in the following image.



The function s satisfies all the assumption of Theorem 3.2 with $\alpha = 1$, $\beta = \delta$, $\alpha_k = 2^{k-1}\delta^{\frac{k}{p}}$ and $\beta_k = 2^k\delta^{\frac{k}{p}}$, for each $k \in \mathbb{N}$. In particular

$$\max_{|\xi| \le \alpha_k} \int_0^{\xi} s(t)dt = \int_0^{2^{k-1}\delta^{\frac{k}{p}}} s(t)dt = 2^{p(k-1)}\delta^k = \alpha_k^p$$

and

$$\int_0^{\beta_k} s(t)dt = 2^{pk}\delta^{k+1} = \delta\beta_k^p$$

for all $k \in \mathbb{N}$.

In Theorems 3.1 and 3.2, inequalities (3.1) and (3.2) serve to assure that $\mu_1 < \mu_2$; moreover the value of $C_{V,\gamma,\rho}$ depends heavily also on constants c_j and then on the choice of the sequence $\{\chi_k\}$. Obviously, fixed the nonlinearity, the smaller the constant $C_{V,\gamma,\rho}$ the easier the inequalities (3.1) and (3.2) will be verified. The next example is in this direction.

Example 3.4. *Let $p > 1$, $\Omega = B(0,1)$ in $\mathbb{R}^n$, $x_0 = 0$, $r \in L^1(\Omega) \setminus 0$, with $r \ge 0$, $V(x) = |x|^2_{\mathbb{R}^2} + 1$, for all $x \in B(0,1)$, $\rho = \frac{1}{2}$, $\gamma = 2$ and $\{\sigma_k\} \subset]0, +\infty[$ with $\lim_{k\to\infty} \sigma_k = +\infty$. Let $\{\chi_k^1\}, \{\chi_k^2\} \in \mathcal{H}(\frac{1}{2}, 1, \{\sigma_k\})$ the sequences defined by*

$$\chi_k^1(x) = 4\sigma_k(4x^3 - 9x^2 + 6x - 1)$$

and

$$\chi_k^2(x) = \frac{\sigma_k}{2}\cos(\pi(2x-1)+1)$$

for all $x \in]\frac{1}{2}, 1[$ *and for each* $k \in \mathbb{N}$. *We observe that, for each* $x \in]\frac{1}{2}, 1[$,

$$|\chi_k^{1\prime}(x)| \leq 3\sigma_k, \qquad |\chi_k^{1\prime\prime}(x)| \leq 24\sigma_k$$

and then the constants $c_j(\{\chi_k^1\})$, *defined in (2.2), are respectively* $c_1(\{\chi_k^1\}) = \frac{3}{2}$ *and* $c_2(\{\chi_k^1\}) = 6$. *In a similar way, for each* $x \in]\frac{1}{2}, 1[$, *we have*

$$|\chi_k^{2\prime}(x)| \leq \pi\sigma_k, \qquad |\chi_k^{2\prime\prime}(x)| \leq 2\pi^2\sigma_k$$

and, in this case, the constants $c_j(\{\chi_k^2\})$ *are respectively* $c_1(\{\chi_k^2\}) = \frac{\pi}{2}$ *and* $c_2(\{\chi_k^2\}) = \frac{\pi^2}{2}$.
Now let us consider a sequence of functions that, in combination with the norm, raises it to the second power; namely

$$\chi_k^3(x) = \begin{cases} \sigma_k\left(-8x^2+8x-1\right) & in\ ]\frac{1}{2}, \frac{3}{4}[\\ \alpha_k(8x^2-16x+8) & in\ ]\frac{3}{4}, 1[\end{cases} \tag{3.3}$$

for each $k \in \mathbb{N}$. *In this case*

$$|\chi_k^{3\prime}(x)| \leq 4\sigma_k, \quad |\chi_k^{3\prime\prime}(x)| \leq 16\sigma_k$$

and then $c_1(\{\chi_k^3\}) = 2$ *and* $c_2(\{\chi_k^3\}) = 4$. *With respect to these three sequences of test functions the smallest* $C_{V,\gamma,\rho}$ *(among the three) depends on the values of* n *and* p. *For instance, for* $n = 3$ *and* $p = 2$ *the smallest* $C_{V,\gamma,\rho}$ *is the one in correspondence with the sequence* $\{\chi_k^3\}$; *in fact, using MATLAB again to compute these constants, one has*

$$C_{V,\gamma,\rho}(\{\chi_k^1\}) \approx 1270, \quad C_{V,\gamma,\rho}(\{\chi_k^2\}) \approx 969, \quad C_{V,\gamma,\rho}(\{\chi_k^3\}) = 912.$$

But, for instance, for $n = 4$ *and* $p = 3$, *the smallest* $C_{V,\gamma,\rho}$ *is the one in correspondence with the sequence* $\{\chi_k^2\}$ *being*

$$C_{V,\gamma,\rho}(\{\chi_k^1\}) \approx 73737, \quad C_{V,\gamma,\rho}(\{\chi_k^2\}) \approx 53988, \quad C_{V,\gamma,\rho}(\{\chi_k^3\}) = 67262.$$

Obviously if we consider the function s *of Example 3.3, taking a posteriori* $\delta > \frac{\omega c_\infty^p \|r\|_{L^1(\Omega)} C_{V,\gamma,\rho}}{\|r\|_{L^1(B(0,\frac{1}{2})}}$ *the corresponding problem admits a sequence of non-zero weak solutions; but if* δ *is fixed a priori, Theorems 3.1 and 3.2 could be always applied as long as one manages to find an appropriate sequence* $\{\chi_k\}$ *while it is not sure that a generic application of Theorem 1.1 can be applied because the assumptions depends heavily by the particular sequence of test functions fixed during the proof.*

The last theorem concerns the case in which the growth exponent of nonlinearity $f(x,t)$ is exactly $p-1$. In this situation the existence of infinite weak solutions will be obtained not for each $\lambda \in \mathbb{R}$ but in an appropriate interval.

Theorem 3.5. *Let $V \in C(\overline{\Omega})$ satisfy $\inf_\Omega V > 0$ and let $f, g \in \mathcal{C}$ such that (i_2) and (i_3) are verified. Moreover, suppose that:*

$(\tilde{i}_1)$ *there exist $h \in L^1(\Omega)$ such that $|f(x,t)| = h(x)\left(1 + |t|^{p-1}\right)$ for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$.*

Then the following facts hold:

$(\tilde{r}_1)$ *if $p_1 < p < p_2$, for all λ such that $|\lambda| < \frac{1}{\|h\|_1 c_\infty^p}$ (for all λ if $h = 0$) and for all $\mu > 0$, the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*

$(\tilde{r}_2)$ *if $p_1 < p = p_2$, there exists $\mu_1 > 0$ such that, for all $\mu > \mu_1$, there exists $\lambda_\mu > 0$ such that, for all $|\lambda| < \lambda_\mu$, the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*

$(\tilde{r}_3)$ *if $p_1 = p < p_2$, there exists $\mu_2 > 0$ such that, for all $\mu \in]0, \mu_2[$, there exists $\lambda_\mu > 0$ such that, for all $|\lambda| < \lambda_\mu$, the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*

$(\tilde{r}_4)$ *if $p_1 = p_2 = p$, there exists $\gamma > 1$ and $C_{V,\gamma,\rho} > 0$ such that, if*

$$C_{V,\gamma,\rho} < \frac{b}{\omega c_\infty^p a} \tag{3.4}$$

then $\mu_1 < \mu_2$ and for all $\mu \in]\mu_1, \mu_2[$, there exists $\lambda_\mu > 0$ such that, for all $|\lambda| < \lambda_\mu$ the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions.

Proof. The proof is similar to that of Theorem 3.1. In fact, computing the two main evaluations for $m = p$, we get:

$$\lambda \int_\Omega F(x, w(x))dx + \mu \int_\Omega G(x, w(x))dx \leq |\lambda| \|h\|_1 c_\infty r_k^{\frac{1}{p}} + \frac{|\lambda|}{p} \|h\|_1 c_\infty^p r_k + \mu \overline{a} c_\infty^{p_1} r_k^{\frac{p_1}{p}} \tag{3.5}$$

and

$$\Phi(w_k) + \frac{1}{p}\Psi(w_k) \leq |\lambda| \|h\|_1 \beta_k + |\lambda| \|h\|_1 \frac{\beta_k^p}{p} - \mu \bar{b} \beta_k^{p_2} + \frac{\omega C_{V,\gamma,\rho}}{p} \beta_k^p. \tag{3.6}$$

To prove $(\tilde{r}_1)$, fix λ such that $|\lambda| \leq \frac{1}{\|h\|_1 c_\infty^p}$ and $\mu > 0$. Thanks to the choice of λ and to the fact that $p_1 < p$ then, from (3.5) we get

$$\lambda \int_\Omega F(x, w(x))dx + \mu \int_\Omega G(x, w(x))dx < \frac{1}{p} r_k \tag{3.7}$$

for k large enough (remember that $\lim\limits_{k\to\infty} r_k = +\infty$); moreover, from (3.6) we obtain

$$\lim_{k\to\infty} \Phi(w_k) + \frac{1}{p}\Psi(w_k) = -\infty \tag{3.8}$$

because $p < p_2$.

To prove $(\tilde{r}_2)$, it is sufficient to choose $\mu_1 = \frac{\omega C_{V,\gamma,\rho}}{pb}$. Fixed $\mu > \mu_1$ and $\overline{b}$ in a similar way as done in Theorem 3.1, we define $\lambda_\mu = \min\left\{\frac{1}{\|h\|_1 c_\infty^p}, \frac{\mu p\overline{b} - \omega C_{V,\gamma,\rho}}{\|h\|_1}\right\}$. Fixed λ such that $|\lambda| < \lambda_\mu$, obviously, from (3.5), we get (3.7) (for k large enough) because $p_1 < p$ and thanks to the choice of λ. Moreover, using (3.6), the choice of λ and μ guarantees that (3.8) holds.

To prove $(\tilde{r}_3)$, it is sufficient to choose $\mu_2 = \frac{1}{pc_\infty^p a}$. Fixed $\mu \in]0, \mu_2[$ and $\overline{a}$ in a similar way as done in Theorem 3.1, we choose $\lambda_\mu = \frac{1-\mu p c_\infty^p \overline{a}}{\|h\|_1 c_\infty^p}$. Fixed λ such that $|\lambda| < \lambda_\mu$, obviously, from (3.6), we get (3.8) because $p < p_2$. Moreover, using (3.5), the choice of λ and μ guarantees that (3.7) holds.

In the last case, to prove $(\tilde{r}_4)$, we observe that, thanks to (3.4), we have $\mu_1 < \mu_2$. So, fixed $\mu \in]\mu_1, \mu_2[$, and choosing $\overline{a}$ and $\overline{b}$ in a similar way as done in Theorem 3.1, we define $\lambda_\mu = \min\left\{\frac{1-\mu c_\infty^p \overline{a}}{\|h\|_1}, \frac{\mu p\overline{b} - \omega C_{V,\gamma,\rho}}{\|h\|_1}\right\}$. Fixed λ such that $|\lambda| < \lambda_\mu$, obviously, from (3.5), we get (3.7) (for k large enough) because of the choice of λ and μ. Moreover, using (3.6), the choice of λ and μ guarantees that (3.8) holds. □

We conclude with an example related to case $(\tilde{r}_4)$ of Theorem 3.5. In this case we consider the one-dimensional setting, providing an explicit estimate of the constant c_∞ in (3.4).

Example 3.6. *Let $n = 1$, $\Omega =]-1,1[$, $p_1 = p_2 = p = 2$, $V(x) = x^2 + 1$ for all $x \in]-1,1[$, $h \in L^1(]-1,1[)$, $r \in L^1(]-1,1[) \setminus \{0\}$ with $r \geq 0$ in $]-1,1[$ and $\int_{-1/2}^{1/2} r(x)dx > 0$. It is well-known that, for all $u \in W^{2,2}(]-1,1[) \cap W_0^{1,2}(]-1,1[)$, one has*

$$\max_{x\in]-1,1[} |u(x)| \leq \frac{\sqrt{2}}{2} \|u'\|_{L^2(]-1,1[)}$$

and

$$\|u'\|_{L^2(]-1,1[)} \leq \frac{2}{\pi} \|u''\|_{L^2(]-1,1[)},$$

so

$$\max_{x\in]-1,1[} |u(x)| \leq \frac{\sqrt{2}}{\pi} \|u''\|_{L^2(]-1,1[)} \leq \frac{\sqrt{2}}{\pi} \|u\|_V$$

and then $c_\infty = \frac{\sqrt{2}}{\pi}$. Now choosing $x_0 = 0$, $\rho = \frac{1}{2}$, $\gamma = 2$, $\delta > \frac{1064\|r\|_{L^1(]-1,1[)}}{\pi^2\|r\|_{L^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\right)}}$, and $g(t,x) = r(x)s(t)$ (where the function s is that of Example 3.3), assumptions (i_2) and (i_3) are satisfied with $a = \|r\|_{L^1(]-1,1[)}$ and $b = \delta\|r\|_{L^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\right)}$. Using the sequence $\{\chi_k^3\}$ of Example 3.4 as test function, we compute $C_{V,\gamma,\rho} = 266$ (lower than those associated with the other two sequences). It is easy to see that

$$\frac{b}{\omega c_\infty^p a} = \frac{\delta\pi^2\|r\|_{L^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\right)}}{8\|r\|_{L^1(]-1,1[)}} > 266$$

then (3.4) is satisfied and then the fact $(\tilde{r}_4)$ holds. In particular, for all $\mu \in \left]\frac{266}{\delta}, \frac{\pi^2}{4\|r\|_{L^1(]-1,1[)}}\right[$, there exists $\lambda_\mu > 0$ (defined inside the proof of Theorem 3.5) such that, for all $|\lambda| < \lambda_\mu$ the problem $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions.

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