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A derivative-type operator and its application to the solvability of a nonlinear three point boundary value problem

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ABSTRACT

In this paper we introduce an operator that can be thought as a derivative of variable order, *i.e.* the order of the derivative is a function. We prove several properties of this operator, for instance, we obtain a generalized Leibniz's formula, Rolle and Cauchy's mean theorems and a Taylor type polynomial. Moreover, we obtain its inverse operator. Also, with this derivative we analyze the existence of solutions of a nonlinear three-point boundary value problem of "variable order".

RESUMEN

En este artículo introducimos un operador que puede ser pensado como una derivada de orden variable, *i.e.* el orden de la derivada es una función. Demostramos varias propiedades de este operador, por ejemplo, obtenemos una fórmula generalizada de Leibniz, teoremas de valor medio de Rolle y Cauchy y un polinomio de tipo Taylor. Más aún, obtenemos su operador inverso. También con esta derivada analizamos la existencia de soluciones de un problema no lineal de valor en la frontera de tres puntos de "orden variable".

Keywords and Phrases: Fractional Derivative, boundary value problem, Hammerstein-Volterra integral equation.

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1 Motivation

Derivatives of non-integer order have been studied since the celebrated question of L'Hospital to Leibniz about the meaning of $\frac{d^n f}{dx^n}$ when $n = 1/2$. There are several definitions of derivatives of fractional order, *e.g.*, derivative of *Riemann-Louville, Caputo, Hadamard, Erdélyi-Kober, Grünwald-Letnikov* and *Riesz*, among others. Typically, these derivatives are defined using an integral form of the classical derivative, as a consequence of it, some basic properties of the usual derivative, as the product rule and chain rule are lost. For a more comprehensible information about these notions we recommend [17, 20, 30].

Despite of the lack of some properties, derivatives of fractional order appear in many real world applications as, for instance, in *memory effects and future dependence, control theory of dynamical systems, nanotechnology, viscoelasticity* and *financial modeling* see, *e.g.*, [8, 12, 18, 19, 21, 24, 25, 31, 32]. Thus, due to this development, in the last decades a lot of research has been devoted to the study of the existence of solutions for several kinds of boundary value problems of fractional type, see, for instance, [2, 3, 4, 5, 9, 11, 26, 28, 29] and references therein.

In order to overcome the limitations of the classical derivative, in [16] it is introduced a new limit-based definition of derivative, the so-called *conformable fractional derivative*, which can be seen as a natural extension of the fractional derivative, although as it is stated in [7], *it is best to consider the conformable derivative in its own right, independent of fractional derivative theory.* Some of the basic properties, physical interpretation and some boundary value problems for conformable differential equations can be found in [1, 6, 10, 14, 15, 33, 34] and its references.

In this article, based on the idea of conformable fractional derivative and in ideas from [13], we consider an extension of the conformable fractional derivative of order α and develop some of its properties. Additionally, we study the existence and uniqueness of solutions for a nonlinear three-point boundary value problem in this new setting.

2 Derivative of variable order

We now introduce the notion of (φ, ω)-derivative.

Definition 2.1. *Let $f : [a,b] \longrightarrow \mathbb{R}$. The (φ, ω)-derivative at the point $x \in (a,b)$ $(\varphi(x) \neq 0)$ is defined as*

$$D^{\varphi}_{\omega} f(x) = D^{\varphi}_{\omega}(f)(x) = D^{(\varphi,1)}_{\omega} f(x) = \lim_{h\to 0} \frac{f(x + h\varphi(x)) - f(x)}{\omega(x+h) - \omega(x)}. \tag{2.1}$$

Where ω is a strictly increasing function and φ is a function. At the point $x \in (a,b)$ such that

$\varphi(x) = 0$ *we define the* (φ, ω)*-derivative as*

$$D_\omega^\varphi f(x) = \lim_{\xi \to x} D_\omega^\varphi f(\xi),$$

when the limit exists.

Taking $\varphi(x) = x^{1-\alpha}$ and $\omega(x) = x$ we obtain the conformable fractional derivative of order α, *cf.* [16].

Theorem 2.2. *Let* f, g *be* (φ, ω)*-differentiable. Then:*

(a) The function f *is continuous.*

(b) $D_\omega^\varphi(a) = 0$*,* a *is a constant.*

(c) $D_\omega^\varphi(af + g) = aD_\omega^\varphi(f) + D_\omega^\varphi(g)$*.*

(d) $D_\omega^\varphi(fg) = fD_\omega^\varphi(g) + fD_\omega^\varphi(g)$*.*

(e) $D_\omega^\varphi\left(\dfrac{f}{g}\right) = \dfrac{fD_\omega^\varphi(g) - fD_\omega^\varphi(g)}{g^2}$*.*

(f) If f *and* ω *are differentiable, we have*

$$D_\omega^\varphi(f)(t) = \varphi(t)\frac{f'(t)}{\omega'(t)}. \tag{2.2}$$

(g) If f, g *and* ω *are differentiable, we have*

$$D_\omega^\varphi(f \circ g)(t) = f'(g(t)) \cdot D_\omega^\varphi(g)(t).$$

Proof. It is a matter of direct calculations. □

Formula (2.2) enables us to calculate in a straightforward way some (φ,ω)-derivatives. For example, letting $\varphi(x) = \sin(x)$, $f(x) = \cos(x)$, and $\omega(x) = x$, we have

$$D_\omega^\varphi \varphi(x) = \sin(x)\cos(x), \quad D_\omega^\varphi f(x) = -\sin^2(x),$$

whereas taking φ and f as above with $\omega(x) = e^x - 1$ we get

$$D_\omega^\varphi \varphi(x) = \frac{\sin(x)\cos(x)}{e^x}, \quad D_\omega^\varphi f(x) = \frac{-\sin^2(x)}{e^x}.$$

We now introduce the n-iterated (φ, ω)-derivative.

Definition 2.3. *By* $D_\omega^{(\varphi,n)}f(x)$ *we define the* n*-iterated* (φ,ω)*-derivative of the function* f*, i.e.*

$$D_\omega^{(\varphi,n)}f(x) = D_\omega^\varphi\left(D_\omega^{(\varphi,n-1)}f\right)(x),$$

with the convention $D_\omega^{(\varphi,0)}f(x) := f(x)$*.*

Theorem 2.4 (Generalized Leibniz's formula)**.** *We have*

$$D_\omega^{(\varphi,n)}(f_1f_2\cdots f_m) = \sum_{\substack{i_1+i_2+\cdots+i_m=n\\ i_j=\overline{0,n}}} n!\frac{D_\omega^{(\varphi,i_1)}(f_1)D_\omega^{(\varphi,i_2)}(f_2)\cdots D_\omega^{(\varphi,i_m)}(f_m)}{i_1!i_2!\cdots i_m!}, \tag{2.3}$$

where we suppose that all is well-defined.

Proof. For $m=2$, equation (2.3) is obtained by induction on n and using the formula for the (φ,ω)-derivative of the product, in this case we obtain

$$D_\omega^{(\varphi,n)}(f_1f_2) = \sum_{j=0}^{n}\binom{n}{j}D_\omega^{(\varphi,n-j)}(f_1)D_\omega^{(\varphi,j)}(f_2). \tag{2.4}$$

By the well-known method to prove the multinomial theorem from the binomial theorem we can, in the same way, obtain (2.3) from (2.4). □

Theorem 2.5 (Fermat's Theorem)**.** *Let* $f:[a,b]\longrightarrow\mathbb{R}$ *have a local maximum or minimum at* $x=c\in(a,b)$ *and* $D_\omega^\varphi(f)(c)$ *exists. Then* $D_\omega^\varphi(f)(c)=0$*.*

Proof. Let us suppose, without loss of generality, that $x=c$ is a minimum of f. We have, for sufficiently small $h\neq 0$, that

$$\operatorname{sgn}(h\varphi(c))\frac{f(c+h\varphi(c))-f(c)}{\omega(c+h)-\omega(c)} \geqslant 0. \tag{2.5}$$

From (2.5) and the hypothesis of the existence of $D_\omega^\varphi(f)(c)$ the result follows. □

Theorem 2.6 (Rolle's theorem)**.** *Let* $f:[a,b]\longrightarrow\mathbb{R}$ *be a continuous function in* $[a,b]$ *and* (φ,ω)*-differentiable in* (a,b) *such that* $f(a)=f(b)=0$*. Then there exists* $c\in(a,b)$ *such that* $D_\omega^\varphi(f)(c)=0$*.*

Proof. Supposing, without loss of generality, that there exists $\xi\in(a,b)$ such that $f(\xi)\geq 0$. Then by Weierstraß theorem, there exists $c\in(a,b)$ which is a maximum. Invoking Fermat's theorem 2.5 we end the proof. □

Theorem 2.7 (Cauchy mean-value theorem)**.** *Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be both continuous on the closed interval $[a, b]$ and (φ, ω)-differentiable in the open interval (a, b). Then there exists a number $\xi \in (a, b)$ such that*

$$[f(b) - f(a)]D_\omega^\varphi(g)(\xi) = [g(b) - g(a)]D_\omega^\varphi(f)(\xi). \tag{2.6}$$

Proof. The proof follows, as in the classical case, from Rolle's theorem 2.6 applied to the function

$$F(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)].$$ □

3 Integration of variable order

Definition 3.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$. We define the (φ, ω)-integral of the function f as*

$$I_\omega^\varphi(f)(t) = \int_a^t \frac{f(\xi)}{\varphi(\xi)}\, \mathrm{d}\omega(\xi), \tag{3.1}$$

where the integral is understood in the Lebesgue-Stieltjes sense.

Notice that for $f \in L^\infty([a, b])$ and $\frac{1}{\varphi} \in L^1([a, b], \mathrm{d}w)$ the integral (3.1) is finite.

When f, φ and ω' are continuous functions, it is straightforward the relation $D_\omega^\varphi(I_\omega^\varphi f)(t) = f(t)$, since

$$D_\omega^\varphi(I_\omega^\varphi f)(t) = \frac{\varphi(t)}{\omega'(t)} D\left(\int_a^t \frac{f(\xi)}{\varphi(\xi)}\, \mathrm{d}\omega(\xi)\right)(t) = f(t),$$

using (2.2).

In the case φ and ω' are continuous functions, the following *Lagrange mean-value theorem*

$$D_\omega^\varphi(f)(\xi) = \frac{f(b) - f(a)}{I_\omega^\varphi(1)(b) - I_\omega^\varphi(1)(a)}, \quad \xi \in (a, b) \tag{3.2}$$

is valid, when $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on the closed interval $[a, b]$ and (φ, ω)-differentiable in the open interval (a, b). The equation (3.2) follows from (2.6) taking $g(x) = I_\omega^\varphi(1)(x)$ (note that $I_\omega^\varphi(1)(a) = 0$, but we leave it in (3.2) just for keeping with the parallel in the classical case).

By $I_\omega^{(\varphi,n)}\varphi(x)$ we define the n-iterated (φ, ω)-integral of the function f, *i.e.*

$$I_\omega^{(\varphi,n)} f(x) = I_\omega^\varphi(I_\omega^{(\varphi,n-1)} f)(x),$$

with the convention $I_\omega^{(\varphi,0)} f(x) := f(x)$.

4 Taylor formula

In this section we will obtain a Taylor type formula using the (φ,ω)-derivative with a remainder which generalizes well-know remainders, *i.e.* Cauchy, Lagrange, Peano, Schlömilch, among others, *cf.* [22, 23, 27] for similar remainders for the classical derivative.

Theorem 4.1. *Let $f : I \longrightarrow \mathbb{R}$ be a continuous function in the open interval I and n-times (φ,ω)-differentiable function in I. We also require that φ, ω' and $I_\omega^{(\varphi,j)}(1)(x)$ are continuous functions, for $j = \overline{1,n}$. Moreover, let $g : I \longrightarrow \mathbb{R}$ be a n-times (φ,ω)-differentiable function such that $D_\omega^{(\varphi,j)} g(a) = 0$ for $j = \overline{1,n-1}$ and $D_\omega^{(\varphi,k)} g(y) \neq 0$ for all y different from a and x and $j = \overline{1,n-1}$. Then, for all $x \in I$ we have*

$$f(x) = \sum_{j=0}^{n} D_\omega^{(\varphi,j)}(f)(a)\; I_\omega^{(\varphi,j)}(1)(x) + R_n(x), \tag{4.1}$$

with

$$R_n(x) = \frac{g(x) - g(a)}{D_\omega^{(\varphi,n)} g(\xi)} \left(D_\omega^{(\varphi,n)}(f)(\xi) - D_\omega^{(\varphi,n)}(f)(a) \right), \tag{4.2}$$

where $x \neq a$ and ξ is between a and x.

Proof. We first note that, since

$$D_\omega^{(\varphi,n)}(I_\omega^{(\varphi,j)} 1)(x) = \begin{cases} I_\omega^{(\varphi,j-n)}(1)(x), & j > n, \\ 1, & n = j, \\ 0, & n > j. \end{cases}$$

we have

$$R_n(a) = D_\omega^{(\varphi,1)}(R_n)(a) = \cdots = D_\omega^{(\varphi,n-1)}(R_n)(a) = 0. \tag{4.3}$$

By the Cauchy type finite increment formula (2.6), relations (4.3) and the hypothesis on g we have

$$\begin{aligned} \frac{R_n(x) - R_n(a)}{g(x) - g(a)} &= \frac{D_\omega^{(\varphi,1)}(R_n)(\theta_1) - D_\omega^{(\varphi,1)}(R_n)(a)}{D_\omega^{(\varphi,1)}(g)(\theta_1) - D_\omega^{(\varphi,1)}(g)(a)} = \ldots = \frac{D_\omega^{(\varphi,n-1)}(R_n)(\theta_{n-1}) - D_\omega^{(\varphi,n-1)}(R_n)(a)}{D_\omega^{(\varphi,n-1)}(g)(\theta_{n-1}) - D_\omega^{(\varphi,n-1)}(g)(a)} \\ &= \frac{D_\omega^{(\varphi,n)}(R_n)(\xi)}{D_\omega^{(\varphi,n)}(g)(\xi)}, \end{aligned} \tag{4.4}$$

where $\xi := \theta_n$. On the other hand, (φ,ω)-differentiating the equality (4.1) n-times we obtain $D_\omega^{(\varphi,n)}(f)(x) - D_\omega^{(\varphi,n)}(f)(a) = D_\omega^{(n)}(R_n)(x)$ which, together with (4.4), entails (4.2). $\square$

5 Three-point boundary value problems of variable order

Inspired in [10], we are interested in the use of the (φ, ω)-derivative to study the solutions of the following nonlinear boundary value problem

$$D_\omega^\varphi (D+\lambda) x(t) = f(t, x(t)), \qquad t \in [0,1] \tag{5.1}$$

$$x(0) = 0, \quad x'(0) = \alpha, \quad x(1) = \beta x(\eta), \tag{5.2}$$

where D_ω^φ is the derivative of variable order, D is the ordinary derivative, $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a known function, β, λ and α are real numbers, $\lambda \neq 0$ and $\eta \in (0,1)$. Notice that in virtue of Theorem 2.2, a sufficient condition for the well posedness of equation (5.1) is, by considering $\omega \in C^1[0,1]$, and $x \in C^2[0,1]$. Thus, in the sequel we consider these conditions on the functions ω and x. In addition, in order to use the (φ, ω)-integral, we are going to assume that φ is continuous and bounded away from zero.

From the conditions on the functions ω and φ we conclude that the following non negative numbers are finite

$$\Omega := \sup_{t\in[0,1]} \omega'(t) < \infty, \quad M := \sup_{t\in[0,1]} \left| \frac{1}{\varphi(t)} \right| < \infty.$$

We will use these numbers in the sequel to establish the existence results.

First, as usual, we will consider the linear boundary value problem:

$$D_\omega^\varphi (D+\lambda) x(t) = g(t), \qquad t \in [0,1], \quad g \in C[0,1] \tag{5.3}$$

$$x(0) = 0, \quad x'(0) = \alpha, \quad x(1) = \beta x(\eta), \quad \alpha, \beta, \lambda \in \mathbb{R}, \quad \lambda \neq 0, \quad \eta \in (0,1). \tag{5.4}$$

To obtain a solution for the boundary value problem, we apply the (φ, ω)-integral to equation (5.3):

$$(D+\lambda)x(t) + (D+\lambda)x(0) = I_\omega^\varphi(g)(t), \tag{5.5}$$

where, using the boundary condition (5.4), $(D+\lambda)x(0) = \alpha$. Then, equation (5.5) simplifies as

$$(D+\lambda)x(t) + \alpha = I_\omega^\varphi(g)(t). \tag{5.6}$$

Let $y(t) = e^{\lambda t} x(t)$, Then we rewrite (5.6) as

$$Dy(t) = e^{\lambda t} I_\omega^\varphi(g)(t) - \alpha e^{\lambda t}.$$

Integrating from 0 to t we obtain

$$y(t) - y(0) = \int_0^t e^{\lambda s} I_\omega^\varphi(g)(s)\, \mathrm{d}s - \frac{\alpha}{\lambda}(e^{\lambda t} - 1)$$

$$y(t) = \int_0^t e^{\lambda s} \int_0^s \frac{g(r)}{\varphi(r)} \omega'(r)\,\mathrm{d}r\,\mathrm{d}s - \frac{\alpha}{\lambda}(e^{\lambda t} - 1), \qquad (y(0) = x(0) = 0).$$

Now, notice that

$$\int_0^t e^{\lambda s} \int_0^s \frac{g(r)}{\varphi(r)} \omega'(r)\,\mathrm{d}r\,\mathrm{d}s = \frac{e^{\lambda t}}{\lambda} \int_0^t \frac{g(s)}{\varphi(s)} \omega'(s)\,\mathrm{d}s - \frac{1}{\lambda} \int_0^t \frac{e^{\lambda s} g(s)}{\varphi(s)} \omega'(s)\,\mathrm{d}s, \quad 0 \le s \le t \le 1.$$

From here we have that

$$y(t) = \frac{e^{\lambda t}}{\lambda} \int_0^t \frac{g(s)}{\varphi(s)} \omega'(s)\,\mathrm{d}s - \frac{1}{\lambda} \int_0^t \frac{e^{\lambda s} g(s)}{\varphi(s)} \omega'(s)\,\mathrm{d}s - \frac{\alpha}{\lambda}(e^{\lambda t} - 1), \quad 0 \le s \le t \le 1.$$

Thus,

$$x(t) = \frac{1}{\lambda} \int_0^t \frac{g(s)}{\varphi(s)} \omega'(s)\,\mathrm{d}s - \frac{e^{-\lambda t}}{\lambda} \int_0^t \frac{e^{\lambda s} g(s)}{\varphi(s)} \omega'(s)\,\mathrm{d}s + \frac{\alpha}{\lambda}(e^{-\lambda t} - 1).$$

Finally, from the condition $\beta x(\eta) = x(1)$ we get

$$\frac{\beta}{\lambda} \int_0^\eta \frac{g(s)}{\varphi(s)} (1 - e^{\lambda(s-\eta)}) \omega'(s)\,\mathrm{d}s + \frac{\alpha\beta}{\lambda}(e^{-\lambda\eta} - 1) - \frac{1}{\lambda} \int_0^1 \frac{g(s)}{\varphi(s)} (1 - e^{\lambda(s-1)})\,\mathrm{d}s - \frac{\alpha}{\lambda}(e^{-\lambda} - 1) = 0.$$

Therefore, introducing this equality into the formula of function x above, we obtain the following expression for x satisfying boundary value problem (5.3)–(5.4)

$$\begin{aligned} x(t) = {} & \frac{1}{\lambda} \int_0^t \frac{g(s)}{\varphi(s)} (1 - e^{\lambda(s-t)}) \omega'(s)\,\mathrm{d}s + \frac{\beta}{\lambda} \int_0^\eta \frac{g(s)}{\varphi(s)} \left(1 - e^{\lambda(s-\eta)}\right) \omega'(s)\,\mathrm{d}s \\ & - \frac{1}{\lambda} \int_0^1 \frac{g(s)}{\varphi(s)} \left(1 - e^{\lambda(s-1)}\right) \omega'(s)\,\mathrm{d}s + \frac{\alpha}{\lambda} \left(e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta\right). \end{aligned}$$

Notice that actually we just proved the following result.

Theorem 5.1. *The linear boundary value problem* (5.3)–(5.4) *has a unique solution given by*

$$\begin{aligned} x(t) = {} & \frac{1}{\lambda} \int_0^t \frac{g(s)}{\varphi(s)} \omega'(s) k(s,t)\,\mathrm{d}s + \frac{\beta}{\lambda} \int_0^\eta \frac{g(s)}{\varphi(s)} \omega'(s) k(s,\eta)\,\mathrm{d}s - \frac{1}{\lambda} \int_0^1 \frac{g(s)}{\varphi(s)} k(s,1) \omega'(s)\,\mathrm{d}s \\ & + \frac{\alpha}{\lambda} \left(e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta\right). \end{aligned}$$

where, $k(s,t) = 1 - e^{\lambda(s-t)}$.

Now, we are going to analyze the existence of solutions for the nonlinear boundary value problem:

$$D_\omega^\varphi (D + \lambda) x(t) = f(t, x(t)), \quad t \in [0,1], \quad \lambda \in (-1, \infty) \setminus \{0\} \tag{5.7}$$

$$x(0) = 0, \quad x'(0) = \alpha, \quad x(1) = \beta x(\eta). \tag{5.8}$$

As in Theorem 5.1, we can transform boundary value problem (5.7)–(5.8) into the nonlinear

Hammerstein-Volterra integral equation

$$x(t) = \frac{1}{\lambda}\int_0^t \frac{f(s,x(s))}{\varphi(s)}k(s,t)\omega'(s)\,\mathrm{d}s + \frac{\beta}{\lambda}\int_0^\eta \frac{f(s,x(s))}{\varphi(s)}k(s,\eta)\omega'(s)\,\mathrm{d}s$$
$$-\frac{1}{\lambda}\int_0^1 \frac{f(s,x(s))}{\varphi(s)}k(s,1)\omega'(s)\,\mathrm{d}s + \frac{\alpha}{\lambda}\left(e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta\right).$$

where, $k(s,t) = 1 - e^{\lambda(s-t)}$.

In order to investigate the existence of a solution for this integral equation, we analyze it as a fixed point problem; that is, letting

$$\begin{array}{rcl} T : (C^2[0,1], \|\cdot\|_\infty) & \longrightarrow & (C^2[0,1], \|\cdot\|_\infty) \\ x(t) & \longmapsto & Tx(t) \end{array}$$

$$Tx(t) := \frac{1}{\lambda}\int_0^t \frac{f(s,x(s))}{\varphi(s)}k(s,t)\omega'(s)\,\mathrm{d}s + \frac{1}{\lambda}\int_0^1 \frac{f(s,x(s))}{\varphi(s)}\left(\chi_{(0,\eta)}(s)\beta k(s,\eta) - k(s,1)\right)\omega'(s)\,\mathrm{d}s$$
$$+\frac{\alpha}{\lambda}\left(e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta\right), \tag{5.9}$$

(with $\chi_{(0,\eta)}(s)$ the characteristic function of the interval $(0,\eta)$), we have that the existence of the solution of the integral equation is equivalent to the existence of a fixed point of the operator T. To assure that the operator T applies $C^2[0,1]$ into itself, we assume that $f(t,x(t))$ is continuous and differentiable in the first variable.

We are going to use metric fixed point theory (Banach's contraction principle) to provide conditions to guarantee that the boundary value problem (5.7)–(5.8) has a unique solution.

Theorem 5.2. *Let $f : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and differentiable in the first variable function satisfying that*

$$|f(t,x) - f(t,y)| \le K|x-y|, \quad K > 0, \quad \textit{for all } t \in [0,1], \quad x, y \in \mathbb{R}.$$

Then, the nonlinear boundary value problem (5.7)–(5.8) *has a unique solution provide that*

$$\frac{(|\beta|+1)MK\Omega}{|\lambda|} < \frac{1}{4},$$

where $M := \sup_{t\in[0,1]} \dfrac{1}{|\varphi(t)|}$ and $\Omega := \sup_{t\in[0,1]} w'(t)$.

Proof. As we saw, it is sufficient to show that the operator T defined by the formula (5.9) has a unique fixed point. Let x and y be two functions in $C^2[0,1]$. Then,

$$\begin{aligned}|Tx(t)-Ty(t)| &= \Bigg|\frac{1}{\lambda}\int_0^t \frac{(f(s,x(s))-f(s,y(s))}{\varphi(s)}k(s,t)\omega'(s)\,\mathrm{d}s \\ &\quad +\frac{1}{\lambda}\int_0^1 \frac{(f(s,x(s))-f(s,y(s))}{\varphi(s)}\Big(\chi_{(0,\eta)}(s)\beta k(s,\eta)-k(s,1)\Big)\omega'(s)\,\mathrm{d}s\Bigg| \\ &\leq \frac{1}{|\lambda|}\int_0^t \frac{|k(s,t)|}{|\varphi(s)|}|f(s,x(s))-f(s,y(s))|\omega'(s)\,\mathrm{d}s \\ &\quad + \frac{1}{|\lambda|}\int_0^1 \frac{\Big|\chi_{(0,\eta)}(s)\beta k(s,\eta)-k(s,1)\Big|}{|\varphi(s)|}|f(s,x(s))-f(s,y(s))|\omega'(s)\,\mathrm{d}s.\end{aligned}$$

On the other hand,

$$|k(s,t)| = |1-e^{\lambda(s-t)}| \leq 1+e^{\lambda(s-t)}.$$

Notice that for $-1<\lambda<0$, the inequality $|e^x-1|<7/4|x|$, for $0<|x|<1$, gives us the estimate

$$|1-e^{\lambda(s-t)}| < \frac{7}{4}\lambda(s-t) < \frac{7}{4} < 2.$$

Then,

$$\sup_{s\in[0,t]}(1+e^{\lambda(s-t)}) \leq 2, \quad -1<\lambda<0.$$

Now, for $\lambda>0$,

$$\sup_{s\in[0,t]}(1+e^{\lambda(s-t)}) = 1+e^{-\lambda t} \leq 2, \qquad \text{for any } t\in[0,1].$$

Therefore, we obtain the following bound

$$|k(s,t)| \leq 2. \tag{5.10}$$

Moreover,

$$\begin{aligned}\Big|\chi_{(0,\eta)}(s)\beta k(s,\eta)-k(s,1)\Big| &= \Big|-\chi_{(0,\eta)}(s)\beta e^{\lambda(s-\eta)}+e^{\lambda(s-1)}+\chi_{(0,\eta)}(s)\beta-1\Big| \\ &\leq |-\chi_{(0,\eta)}(s)\beta e^{\lambda(s-\eta)}|+|e^{\lambda(s-1)}|+|\beta|+1,\end{aligned}$$

where, for $-1<\lambda<0$, we have that

$$\sup_{s\in[0,\eta]} e^{\lambda(s-\eta)} = 1, \quad \sup_{s\in[0,1]} e^{\lambda(s-1)} = 1.$$

In the case $\lambda>0$, we get

$$\sup_{s\in[0,\eta]} e^{\lambda(s-\eta)} = e^{-\lambda\eta} \leq 1, \quad \sup_{s\in[0,1]} e^{\lambda(s-1)} = e^{-\lambda} \leq 1.$$

With these bounds we obtain the following estimation

$$\left|\chi_{(0,\eta)}(s)\beta k(s,\eta) - k(s,1)\right| \leq 2(|\beta|+1). \tag{5.11}$$

We introduce the bounds (5.10) and (5.11) into the difference $|Tx(t) - Ty(t)|$:

$$\begin{aligned} |Tx(t) - Ty(t)| \leq & \frac{2}{|\lambda|}\int_0^t \frac{1}{|\varphi(s)|}|f(s,x(s)) - f(s,y(s))|\omega'(s)\,\mathrm{d}s \\ & + \frac{2(|\beta|+1)}{|\lambda|}\int_0^1 \frac{1}{|\varphi(s)|}|f(s,x(s)) - f(s,y(s))|\omega'(s)\,\mathrm{d}s. \end{aligned} \tag{5.12}$$

Since $f(s,x(s))$ is Lipschitz in the second variable, then

$$\begin{aligned} |Tx(t) - Ty(t)| &\leq \frac{2}{|\lambda|}\int_0^t \frac{K}{|\varphi(s)|}|x(s) - y(s)|\omega'(s)\,\mathrm{d}s + \frac{2(|\beta|+1)}{|\lambda|}\int_0^1 \frac{K}{|\varphi(s)|}|x(s) - y(s)|\omega'(s)\,\mathrm{d}s \\ &\leq \frac{2(|\beta|+1)}{|\lambda|}\int_0^t \frac{K}{|\varphi(s)|}|x(s) - y(s)|\omega'(s)\,\mathrm{d}s \\ &+ \frac{2(|\beta|+1)}{|\lambda|}\int_0^1 \frac{K}{|\varphi(s)|}|x(s) - y(s)|\omega'(s)\,\mathrm{d}s. \end{aligned}$$

Taking the maximum over $t \in [0,1]$ we obtain

$$\|Tx - Ty\|_\infty \leq 2\frac{2(|\beta|+1)}{|\lambda|}KM\Omega\|x-y\|_\infty.$$

Therefore, T is a contraction operator, since $\mu = 2\frac{2(|\beta|+1)}{|\lambda|}KM\Omega < 1$. Thus from the Banach contraction principle, T has a unique fixed point as desired. $\square$

Now, we are going to use topological fixed point theory, more precisely Schaefer's fixed point theorem, to establish the existence of at least one solution of boundary value problem (5.7)–(5.8), dropping the Lipschitzianity of the function f.

First, we prove that the operator T is compact.

Theorem 5.3. *The operator $T : (C^2[0,1], \|\cdot\|_\infty) \longrightarrow (C^2[0,1], \|\cdot\|_\infty)$ is compact.*

Proof. We start by proving the continuity of T. Let $(x_n) \subset C^2[0,1]$, $x \in C^2[0,1]$ be such that $\|x_n - x\|_\infty \to 0$. We have to show that $\|Tx_n - Tx\|_\infty \to 0$. Fixed $\varepsilon > 0$, there exists $K \geq 0$ such that

$$\begin{aligned} \|x_n\|_\infty &\leq K, \qquad \forall n \in \mathbb{N} \\ \|x\|_\infty &\leq K. \end{aligned}$$

Since $f : [0,1] \times [-K,K] \longrightarrow \mathbb{R}$ is continuous, then it is uniformly continuous on $[0,1] \times [-K,K]$.

Thus there exists $\delta(\varepsilon) > 0$ such that

$$|f(s_1, x(s_1)) - f(s_2, y(s_2))| \leq \varepsilon,$$

for every $(s_1, x(s_1)), (s_2, y(s_2)) \in [0,1] \times [-K, K]$ such that $\|(s_1 - s_2, x(s_1) - y(s_2))\|_2 < \delta(\varepsilon)$.

From the fact that $\|x_n - x\|_\infty \to 0$, it follow that there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\sup_{t\in[0,1]} |x_n(t) - x(t)| < \delta,$$

for every $n \geq N(\varepsilon)$. Consequently, from (5.12),

$$\begin{aligned}
\|Tx_n - Tx\|_\infty &= \sup_{t\in[0,1]} |Tx_n(t) - Tx(t)| \\
&\leq \sup_{t\in[0,1]} \left\{ \frac{2}{|\lambda|} \int_0^t \frac{|f(s, x_n(s)) - f(s, x(s))|}{|\varphi(s)|} \omega'(s)\,\mathrm{d}s \right. \\
&\quad + \left. \frac{2(|\beta|+1)}{|\lambda|} \int_0^1 \frac{|f(s, x_n(s)) - f(s, x(s))|}{|\varphi(s)|} \omega'(s)\,\mathrm{d}s \right\} \\
&< \frac{2|\beta|+4}{|\lambda|} M\Omega\varepsilon, \qquad M := \sup_{t\in[0,1]} \frac{1}{|\varphi(t)|}, \quad \Omega := \sup_{t\in[0,1]} \omega'(t).
\end{aligned}$$

Therefore, the operator T is continuous. To prove the compactness we consider a bounded set $X \subset C^2[0,1]$ and we will show that $T(X)$ is relatively compact in $(C^2[0,1], \|\cdot\|_\infty)$ by using the Arzela-Ascoli theorem. Let $K \geq 0$ be such that

$$\|x\|_\infty \leq K,$$

for every $x \in X$.

From the bounds (5.10) and (5.11) we have

$$\begin{aligned}
|Tx(t)| \leq & \frac{1}{|\lambda|} \int_0^t \frac{|f(s, x(s))|}{|\varphi(s)|} |k(s,t)| \omega'(s)\,\mathrm{d}s + \frac{1}{|\lambda|} \int_0^1 \frac{|f(s, x(s))|}{|\varphi(s)|} |\chi_{(0,\eta)}\beta k(s,\eta) - k(s,1)| \omega'(s)\,\mathrm{d}s \\
& + \left|\frac{\alpha}{\lambda}\right| |e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta| \\
\leq & \frac{2M\Omega}{|\lambda|} \int_0^t |f(s, x(s))|\,\mathrm{d}s + \frac{2(|\beta|+1)}{|\lambda|} M\Omega \int_0^1 |f(s, x(s))|\,\mathrm{d}s \\
& + \left|\frac{\alpha}{\lambda}\right| |e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta| \\
\leq & \frac{2|\beta|+4}{|\lambda|} M\Omega \int_0^1 |f(s, x(s))|\,\mathrm{d}s + \left|\frac{\alpha}{\lambda}\right| |e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta|.
\end{aligned}$$

We obtain a upper bound for $|e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta|$, namely

$$|e^{-\lambda t} - e^{-\lambda} + \beta e^{-\lambda\eta} - \beta| \leq \Delta, \quad t \in [0,1],$$

where

$$\Delta := \begin{cases} (|\beta|+1)e^{-\lambda}, \ -1 < \lambda < 0 \\ 2(|\beta|+1), \ \lambda > 0. \end{cases}$$

On the other hand, since the function f is uniformly continuous on the compact set $[0,1]\times[-K,K]$, then there exists, and it is finite, the positive number

$$R_K = \|f\|_\infty = \sup_{x\in X} \sup_{s\in[0,1]} |f(s,x(s))| < \infty, \quad (s,x(s)) \in [0,1]\times[-K,K].$$

Thus, we have

$$\|Tx\|_\infty \leq \frac{2|\beta|+4}{|\lambda|} M\Omega R_K + \left|\frac{\alpha}{\lambda}\right| \Delta, \tag{5.13}$$

for every $x \in X$. That means, the set $T(X)$ is bounded in $C^2[0,1]$. Now, if $t_1, t_2 \in [0,1]$, are such that $t_1 \leq t_2$ and satisfy $|t_1 - t_2| < \delta$, then

$$\begin{aligned} |Tx(t_1) - Tx(t_2)| &= \left|\frac{1}{\lambda}\int_0^{t_1} \frac{f(s,x(s))}{\varphi(s)}\omega'(s)\,\mathrm{d}s - \frac{1}{\lambda}\int_0^{t_2} \frac{f(s,x(s))}{\varphi(s)}\omega'(s)\,\mathrm{d}s - \frac{\alpha}{\lambda}e^{-\lambda t_1} + \frac{\alpha}{\lambda}e^{-\lambda t_2}\right| \\ &= \left|\frac{1}{\lambda}\int_{t_1}^{t_2} \frac{f(s,x(s))}{\varphi(s)}\omega'(s)\,\mathrm{d}s + \frac{\alpha}{\lambda}(e^{-\lambda t_2} - e^{-\lambda t_1})\right| \\ &\to 0, \quad \text{as } |t_1 - t_2| \to 0 \end{aligned}$$

for every $x \in X$, so the set $T(X) \subset C^2[0,1]$ satisfies the hypotheses of Arzela-Ascoli's theorem, so $T(X)$ is relatively compact in $C^2[0,1]$. Therefore, the operator T is compact. □

Now, we establish the following existence result.

Theorem 5.4. *Let $f : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and differentiable in the first variable function, and let us assume that there exist $C, D \geq 0$ and $q \in (0,1)$ such that*

$$|f(s,r)| \leq C|r|^q + D.$$

For every $(s,r) \in [0,1]\times\mathbb{R}$. Then, the nonlinear boundary value problem (5.7)–(5.8) *has at least one solution.*

Proof. The theorem is proved once we assure the existence of at least a fixed point of the operator T. Let

$$\mathcal{S} = \{x \in C^2[0,1] \ : \ \exists\sigma \in [0,1] \text{ such that } x = \sigma Tx\}.$$

To apply Schaefer's fixed point theorem we should show that $\mathcal{S}$ is bounded. Let $x \in \mathcal{S}$,

$$\|x\|_\infty = \sigma \|Tx\|_\infty.$$

Now, from (5.13) we have

$$|Tx(t)| \leq \frac{2|\beta|+4}{|\gamma|} \int_0^1 \frac{|f(s,x(s))|}{|\varphi(s)|}\omega'(s)\,\mathrm{d}s + \left|\frac{\alpha}{\beta}\right|\Delta \leq \frac{2|\beta|+4}{|\gamma|} M\Omega(C\|x\|_\infty^q + D) + \left|\frac{\alpha}{\beta}\right|\Delta < \infty.$$

Then

$$\|x\|_\infty = \sigma\|Tx\|_\infty \leq \sigma\frac{2|\beta|+4}{|\gamma|} M\Omega(C\|x\|_\infty^q + D) + \left|\frac{\alpha}{\beta}\right|\Delta\sigma < \infty.$$

This inequality and the fact $q \in (0,1)$ shows that $\mathcal{S}$ is bounded. Thus, from Schaefer's fixed point theorem, the operator T has a fixed point, which implies that boundary value problem (5.7)–(5.8) has a solution. □

Notice from the proof of the theorem above, that we can use the functions φ and ω given in the definition of the (φ,ω)-derivative to rewrite Theorem 5.4 as:

Theorem 5.5. *Let $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and differentiable in the first variable function, and let us assume that there exist $C, D \geq 0$ and $q \in (0,1)$ such that*

$$\frac{|f(s,r)|}{|\varphi(s)|}\omega'(s) \leq C|r|^q + D.$$

For every $(s,r) \in [0,1] \times \mathbb{R}$. Then, the nonlinear boundary value problem (5.7)–(5.8) has at least one solution.

Schaefer's theorem is a consequence of the Schauder fixed point theorem, which is a localization fixed point result. We will use Schauder's theorem to give a localization result for the solutions of boundary value problem (5.7)–(5.8).

Theorem 5.6. *Let $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and differentiable in the first variable function and, in addition, let us assume that $f \in L^1([0,1] \times \mathbb{R})$. Let $\overline{B}(r)$ be the closed ball with radius r. Then, the nonlinear boundary value problem (5.7)–(5.8) has at least one solution for every closed ball $\overline{B}(r)$ such that*

$$r \geq \frac{2|\beta|+4}{|\lambda|} M\Omega\|f\|_1 + \left|\frac{\alpha}{\lambda}\right|\Delta, \tag{5.14}$$

with,

$$\Delta := \begin{cases} (|\beta|+2)e^{-\lambda}, \ -1 < \lambda < 0 \\ 2(|\beta|+1), \ \lambda > 0 \end{cases}, \quad M := \sup_{t\in[0,1]}\left|\frac{1}{\varphi(t)}\right|, \quad \Omega = \sup_{t\in[0,1]} \omega'(t).$$

Proof. Since the operator T is continuous and compact, we can apply Schauder's fixed point theorem, once we prove that $T(\overline{B}(r)) \subset \overline{B}(r)$.

From (5.13) and the hypotheses, we have

$$\begin{aligned} |Tx(t)| &\leq \frac{2|\beta|+4}{|\gamma|}\int_0^1 \frac{|f(s,x(s))|}{|\varphi(s)|}\omega'(s)\,\mathrm{d}s + \left|\frac{\alpha}{\beta}\right|\Delta \\ &\leq \frac{2|\beta|+4}{|\gamma|}M\Omega\|f\|_1 + \left|\frac{\alpha}{\beta}\right|\Delta \\ &\leq r. \end{aligned} \tag{5.15}$$

Thus, $\|Tx\|_\infty \leq r$. Consequently $T(\overline{B}(r)) \subset \overline{B}(r)$. Finally, from Schauder's theorem, T has a fixed point, as so boundary value problem (5.7)–(5.8) has at least one solution, for every closed ball $\overline{B}(r)$ with radius r as in (5.14). □

We can control the growth behavior of the nonlinear function f and still guarantee the existence of solutions for BVP (5.7)–(5.8). Some of these behaviors, as we will see, can be controlled in terms of the functions φ and ω given in the definition of the (φ,ω)-derivative, which can be interpreted as behaviors scaled for the (φ,ω)-derivative.

The main idea is to replace the integral term (5.15) with some condition which allows found a bound for it. For instance if we assume that f is uniformly bounded by $A>0$ on $[0,1]\times\mathbb{R}$, then use the estimate $|f(s,x(s))|\leq A$ in (5.15) and obtain the radius $r \geq \frac{2|\beta|+4}{|\lambda|}M\Omega AK + \left|\frac{\alpha}{\lambda}\right|\Delta$.

If we assume that $|f(s,y)| \leq A\frac{|\varphi(s)|}{w'(s)}$, for some $A>0$, for all $(s,x)\in[0,1]\times\mathbb{R}$, the integral term is less or equal to A and the radius is $r \geq \frac{2|\beta|+4}{|\lambda|}A + \left|\frac{\alpha}{\lambda}\right|\Delta$.

Finally, if $|f(s,x(s))| \leq \dfrac{|\gamma|}{2(|\beta|+2)M\Omega}s|x(s)|$, and we assume that

$$\left|\frac{\alpha}{\lambda}\right|\Delta \leq \frac{r}{2}, \quad \text{for each } r>0 \text{ given.}$$

Then, estimate (5.15) is rewrite as

$$|Tx(t)| \leq \frac{\|x\|_\infty}{2} + \left|\frac{\alpha}{\beta}\right|\Delta \leq \frac{r}{2}+\frac{r}{2} = r.$$

This proves that T applies any ball of radius r into itself. Therefore, we conclude that BVP (5.7)–(5.8) as at leat one solution on each ball of radius r.

In similar fashion it can be proved that for

$$|f(s,y)| \leq \frac{|\gamma|}{4(|\beta|+2)M\Omega(\frac{|a_n|}{n+1}+\cdots+|a_0|)}|a_ns^n+\cdots+a_0|)|x(s)|,$$

and

$$\left|\frac{\alpha}{\lambda}\right| \Delta \leq \frac{r}{2}, \quad \text{for each } r > 0,$$

the same conclusion holds.

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