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H^1 Estimates for Extensions of Holomorphic Functions in Certain Pseudoconvex Domains

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Abstract

Let Ω be some weakly pseudoconvex domain in C^N with C^2-boundary, and V be a one dimensional subvariety in general position in Ω. Then for any function $f \in H^1(V)$, there exists $F \in H^1(\Omega)$ such that $F|_V = f$.

Introduction. Let D be the ellipsoid (real or complex) in C^N, and $\widetilde{M}$ be a subvariety in a neighborhood $\widetilde{D}$ of $\overline{D}$ which intersects ∂D transversally. We set $M = \widetilde{M} \cap D$. In the previous papers ([1], [2]), the author proved that if $f \in H^p(M)$, $1 \leqq p < \infty$, then there exists $F \in H^p(D)$ satisfying $F|_M = f$, where $H^p(G)$ is the Hardy class on a domain G. In the present paper, we study the above problem for some pseudoconvex domain Ω which does not possess the real analytic boundary.

Let $\Psi \in C^2[0, 1]$ be a real function satisfying

(A. 1) $\Psi(0) = 0$, $\Psi(1) = 2$

(A. 2) $\Psi'(t) > 0 \quad (0 < t \leqq 1)$

(A. 3) $\Psi''(t)t + \Psi'(t) > 0 \quad (0 < t < 1)$

(A. 4) there exists $\tau > 0$ such that $\Psi''(t) > 0 \quad (0 < t < \tau)$

(A. 5) $\int_0^1 \log \Psi(t) t^{-1/2} dt > -\infty$.

For $0 < \alpha < \frac{1}{2}$, write $\Psi_\alpha(t) = 2e \cdot \exp(-t^{-\alpha})$.

Then Ψ_α satisfies the above conditions. We set

$$\rho(z) = \sum_{j=1}^{N-1} |z_j|^2 + \Psi(|z_N|^2) - 1, \ \Omega = \{z : \rho(z) < 0\}.$$

Let $\widetilde{V}$ be a one dimensional subvariety in a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$ which intersects $\partial\Omega$ transversally. Suppose that $\widetilde{V}$ is written in the following form

$$\widetilde{V} = \{z \in \Omega : h_i(z) = 0, 1 \leqq i \leqq N-1\}$$

where $h_1, \ldots\ldots, h_{N-1}$ are holomorphic functions in $\widetilde{\Omega}$ satisfying

$$\partial h_1 \wedge \ldots\ldots \wedge \partial h_{N-1} \wedge \partial \rho \neq 0 \text{ on } \tilde{V} \cap \partial \Omega.$$

Let $V = \tilde{V} \cap \Omega$. Then we have

THEOREM. *Let $f \in H^1(V)$. Then there exists a function $H \in H^1(\Omega)$ such that $H(z) = f(z)$ for $z \in V$.*

In what follows we shall adopt the convention of denoting by c any positive constant which does not depend on the relevant parameters in the estimate.

1. Support functions. Now we begin with the following lemma which was proved by Verdera [5].

LEMMA 1. *There exists a constant $\eta = \eta(\Psi) > 0$ such that for $L = \frac{1}{16}$, the following inequality holds*

$$\Psi(|\zeta + v|^2) - \Psi(|\zeta|^2) - 2Re\left(\frac{\partial \Psi}{\partial \zeta}(|\zeta|^2) v\right) \geqq \Psi(L|v|^2),$$
$$\text{for } \zeta,\ v \in C,\ |\zeta| < \eta,\ |v| < \eta.$$

Let U be an open neighborhood of $\bar{\Omega}$ in which ρ is defined. We set

$$F_0(\zeta, z) = \sum_{j=1}^{N} \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) \text{ for } (\zeta, z) \in U \times C^N$$

Then we have

LEMMA 2. *There exist positive constants η, c and M, depending only on Ψ, such that*

$$2Re\ F_0(\zeta, z) \geqq \rho(\zeta) - \rho(z) + c\Psi(M|\zeta - z|^2)$$
$$\text{for } (\zeta, z) \in U \times C^N,\ |\zeta_N| < \eta,\ |\zeta - z| < \eta.$$

PROOF. From lemma 1, we have

$$\Psi(|z_N|^2) - \Psi(|\zeta_N|^2) \geqq 2Re\left(\frac{\partial \Psi}{\partial \zeta_N}(|\zeta_N|^2)(z_N - \zeta_N)\right) + \Psi(L|\zeta_N - z_N|^2)$$

Therefore we obtain

$$\begin{aligned}
\rho(z) &= \rho(\zeta) - 2Re\ F_0(\zeta, z) + (\rho(z) - \rho(\zeta) + 2Re\ F_0(\zeta, z)) \\
&\geqq \rho(\zeta) - 2Re\ F_0(\zeta, z) + \sum_{j=1}^{N-1} |z_j|^2 - \sum_{j=1}^{N-1} |\zeta_j|^2 \\
&\quad + 2Re\left(\frac{\partial \Psi}{\partial \zeta_N}(|\zeta_N|^2)(z_N - \zeta_N)\right) + 2Re\sum_{j=1}^{N-1} \bar{\zeta}_j(\zeta_j - z_j) \\
&\quad + 2Re\left(\frac{\partial \Psi}{\partial \zeta_N}(|\zeta_N|^2)(\zeta_N - z_N)\right) + \Psi(L|z_N - \zeta_N|^2) \\
&= \rho(\zeta) - 2Re\ F_0(\zeta, z) + \sum_{j=1}^{N-1} |z_j - \zeta_j|^2 + \Psi(L|z_N - \zeta_N|^2).
\end{aligned}$$

From the conditions (A. 1), (A. 2) and (A. 4), we have for small $|x|$, $|y|$,

$$\Psi\left(\frac{x^2+y^2}{2}\right)\leqq\frac{1}{2}\Psi(x^2)+\frac{1}{2}\Psi(y^2)\leqq c(x^2+\Psi(y^2)).$$

Therefore we have for some $M>0$,

$$\sum_{j=1}^{N-1}|z_j-\zeta_j|^2+\Psi(L|z_N-\zeta_N|^2)$$
$$\geqq c\Psi\left(\frac{1}{2}\left(\sum_{j=1}^{N-1}|z_j-\zeta_j|^2+L|z_N-\zeta_N|^2\right)\right)\geqq c\Psi(M|z-\zeta|^2).$$

Thus we obtained the required inequality.

We denote by $B(z, r)$ the open ball in C^N with center z and radius r. For ε, $\delta>0$, we set

$$U_\delta=\{z\in U:\rho(z)<\delta\},\ V_\delta=\{z\in U:|\rho(z)|<\delta\},$$
$$U_{\varepsilon,\delta}=\{(\zeta, z)\in V_\delta\times U_\delta:|\zeta-z|<\varepsilon\},$$
$$Z=\{z:z_N=0\}.$$

By applying the technique of Henkin and Cirka [3], we have the following.

LEMMA 3. *There exists ε, δ, $c>0$ depending on Ψ, and continuously differentiable functions*

$\Phi: V_\delta\times U_\delta\to C$, $F: U_{\varepsilon,\delta}\to C$ and $G: U_{\varepsilon,\delta}\to C$ which are holomorphic in $z\in U_\delta$ for each fixed $\zeta\in V_\delta$, such that

(a) *$\Phi=FG$ in $U_{\varepsilon,\delta}$*

(b) *$F(\zeta, \zeta)=0$, $|G|>c$ in $U_{\varepsilon,\delta}$, $|\Phi|>c$ in $(V_\delta\times U_\delta)/U_{\varepsilon,\delta}$*

(c) *For some $M>0$, the following inequality holds*

$$2Re\ F(\zeta, z)\geqq\rho(\zeta)-\rho(z)+c\Psi(M|\zeta-z|^2),\ (\zeta, z)\in U_{\varepsilon,\delta}$$

(d) *$d_\zeta F(\zeta, z)|_{\zeta=z}=\partial\rho(z)$*

(e) *the function Φ can be written in the form*

$$\Phi(\zeta, z)=\sum_{j=1}^{N}(\zeta_j-z_j)P_j(\zeta, z)$$

where $P_j(1\leqq j\leqq N)$ are continuously differentiable in $V_\delta\times U_\delta$, holomorphic in $z\in U_\delta$ for each fixed $\zeta\in V_\delta$.

PROOF. For $a>0$, define

$$W_a=\left\{\zeta:\left|\sum_{i=1}^{N-1}|\zeta_i|^2-1\right|<a,\ |\zeta_N|<a\right\}.$$

We choose a so small that $W_{2a}\subset U$ and $2a<\eta$. Let $\zeta^*\in\partial D/Z$. From the condition (A. 3), ζ^* is a strongly pseudoconvex boundary point of Ω. Then there exists a biholomorphic mapping ϕ of some neighborhood S_{ζ^*} of ζ^* onto some neighborhood W_{ζ^*} of 0 such that $\rho_{\zeta^*}(W)=\rho(\phi^{-1}(W))$ is strictly convex in W_{ζ^*}. From the Taylor expansion, there exist constants $\tilde{\varepsilon}^*>0$ and $\tilde{r}^*>0$ such that

$$2Re\sum_{i=1}^{N}\frac{\partial\rho_{\zeta^*}}{\partial w_i}(w')(w'_i-w_i)\geqq\rho_{\zeta^*}(w')-\rho_{\zeta^*}(w)+\tilde{\gamma}^*|w'-w|^2$$

if $|w'|<\tilde{\varepsilon}$, $|w-w'|<\tilde{\varepsilon}$.

Since the mapping ϕ is a diffeomorphism, there exist ζ, z such that

$$2Re\sum_{i=1}^{N}\frac{\partial\rho_{\zeta^*}}{\partial w_i}(\phi(\zeta))(\phi_i(\zeta)-\phi_i(z))\geqq\rho(\zeta)-\rho(z)+\gamma^*|\zeta-z|^2$$

if $|\zeta-\zeta^*|<\varepsilon^*$ and $|\zeta-z|<\varepsilon^*$.

We set

$$F_{\zeta^*}(\zeta, z)=\sum_{i=1}^{N}\frac{\partial\rho_{\zeta^*}}{\partial w_i}(\phi(\zeta))(\phi_i(\zeta)-\phi_i(z)).$$

Then we obtain

$$(1)\quad \left.\frac{\partial F_{\zeta^*}}{\partial\zeta_k}\right|_{\zeta=z}=\frac{\partial\rho}{\partial z_k}(\zeta)$$

and

$$2Re\ F_{\zeta^*}(\zeta, z)\geqq\rho(\zeta)-\rho(z)+\gamma^*|\zeta-z|^2 \text{ for } |\zeta-\zeta^*|<\varepsilon^*, |\zeta-z|<\varepsilon^*.$$

We select points $\zeta_1^*, \dots, \zeta_p^*$ on $\partial\Omega|Z$ in such a way that the balls $B(\zeta_i^*, \varepsilon_i^*)$, $i=1, 2, \dots, p$, cover $\partial\Omega|W_a$. Let the infinitely differentiable functions λ_i, $i=1, \dots, p$, form a partition of unity in a neighborhood E of $\partial\Omega|W_a$, where supp $\lambda_i\subset B(\zeta_i^*, \varepsilon_i^*)$. We set

$$F_1(\zeta, z)=\sum_{i=1}^{p}\lambda_i(\zeta)F_{\zeta_i^*}(\zeta, z) \text{ for } \zeta\in E, |\zeta-z|<2\varepsilon'=\min_{1\leqq i\leqq p}\varepsilon_i^*.$$

We set $\varepsilon=\min\left(\varepsilon', \frac{\eta}{2}\right)$. Then for $\zeta\in E$, $|\zeta-z|<2\varepsilon$, we have

$$2Re\ F_1(\zeta, z)\geqq\rho(\zeta)-\rho(z)+\gamma|\zeta-z|^2,$$

where $\gamma=\min_{1\leqq i\leqq p}\gamma_i^*$. We choose $\delta>0$ so small that $V_\delta|W_a\subset E$. Let $\chi\in C^\infty(C^N)$ be a function, with support contained in W_{2a}, which is identically 1 on W_a.

We set

$$F(\zeta, z)=\chi(\zeta)F_0(\zeta, z)+(1-\chi(\zeta))F_1(\zeta, z) \text{ for } (\zeta, z)\in U_{2\varepsilon,\delta}.$$

Then we have for $\zeta\in W_a$, $|\zeta-z|<2\varepsilon$,

$$2Re\ F(\zeta, z)=2Re\ F_0(\zeta, z)\geqq\rho(\zeta)-\rho(z)+c\Psi(M|\zeta-z|^2).$$

For $\zeta\in V_\delta|W_a$, $|\zeta-z|<2\varepsilon$. we have

$$2Re\ F(\zeta, z)\geqq\chi(\zeta)(\rho(\zeta)-\rho(z)+c\Psi(M|\zeta-z|^2))+(1-\chi(\zeta))(\rho(\zeta)-\rho(z)+ c|\zeta-z|^2)\geqq\rho(\zeta)-\rho(z)+c\Psi(M|\zeta-z|^2).$$

Thus we obtain (c). Further, using the equality (1),

$$d_\zeta F(\zeta, z)|_{\zeta=z}=\partial\rho.$$

If we choose $\delta>0$ sufficiently small, we have from (c), Re $F(\zeta, z)>0$ for $\zeta\in V_\delta$, $z\in U_\delta$, and $\varepsilon<|\zeta-z|<2\varepsilon$. Let $\mu\in C^\infty(C^N)$ be such that supp $\mu\subset B(0, 2\varepsilon)$, $\mu(z)=1$ for $|z|\leqq\varepsilon$. We define for $(\zeta, z)\in V_\delta\times U_\delta$,

$$\Gamma(\zeta, z)=\begin{cases}\bar{\partial}_z((\log F)\chi(\zeta-z)) & (z\in \text{supp grad}_z\chi(\zeta-z))\\ 0 & (z\notin \text{supp grad}_z\chi(\zeta-z))\end{cases}$$

By $L^2_{(0,1)}(U_\delta)$, we denote the Hilbert space of closed differential forms of type (0, 1) with coefficients belonging to $L^2(U_\delta)$. Since $\Gamma(\zeta, z)\in C^1(V_\delta, L^2_{(0,1)}(U_\delta))$, and U_δ is pseudoconvex, from the Oka-Hörmander theorem there exists the function $C(\zeta, z) \in C^1(V_\delta, L^2(U_\delta))$ with the property $\bar{\partial}_z C(\zeta, z) = \Gamma(\zeta, z)$. We set

$$\Phi(\zeta, z) = \begin{cases} F(\zeta, z) \exp(-C(\zeta, z)) & ((\zeta, z) \in U_{\varepsilon,\delta}) \\ \exp(\log F \cdot \chi(\zeta - z) - C(\zeta, z)) & ((\zeta, z) \in V_\delta \times U_\delta \,|\, U_{\varepsilon,\delta}) \end{cases}$$

Then we have $\bar{\partial}_z \Phi(\zeta, z) = 0$. Thus we proved that

$$\Phi(\zeta, z) \in C^1(V_\delta, O(U_\delta)).$$

We set

$$A(\zeta, z, w) = \Phi(\zeta, z) - \Phi(\zeta, w).$$

Then $A(\zeta, z, w)$ satisfies

$$A(\zeta, z, w) \in C^1(V_\delta, O(U_\delta \times U_\delta)) \text{ and } A(\zeta, z, z) = 0.$$

By Hefer's theorem, taking account of the pseudoconvexity of Ω, there exist functions $Q_i(\zeta, z, w) \in C^1(V_\delta, O(U_\delta \times U_\delta))$ such that

$$A(\zeta, z, w) = \sum_{i=1}^{N} Q_i(\zeta, z, w)(w_i - z_i).$$

Therefore we have

$$\Phi(\zeta, z) = A(\zeta, z, \zeta) = \sum_{i=1}^{N} Q_i(\zeta, z, \zeta)(w_i - \zeta_i).$$

If we set $P_i(\zeta, z) = Q_i(\zeta, z, \zeta)$, we obtain (e).

2. Proof of the theorem. For $\varepsilon > 0$, we set

$$\Omega_\varepsilon = \{z \in \Omega : \rho(z) < \varepsilon\}.$$

Let f^* be the the boundary value of $f \in H^1(V)$. Then $f^* \in L^1(\partial V)$. Then by Hatziafratis [3], we have

LEMMA 4. *For $f \in H^1(V)$, $z \in V$, we have the formula*

$$f(z) = \int_{\partial V} f^*(\zeta) K(\zeta, z)$$

where $K(\zeta, z)$ is written in the following form

$$K(\zeta, z) = \sum_{i=1}^{N} K_i(\zeta, z) d\zeta_i = \sum_{i=1}^{N} \frac{a_i(\zeta, z) d\zeta_i}{\Phi(\zeta, z)},$$

$a_i(\zeta, z)$ being smooth on $V_\delta \times U_\delta$, holomorphic in $z \in U_\delta$.

For $z \in \Omega$, we set

$$H(z) = \int_{\partial V} f^*(\zeta) K(\zeta, z).$$

Then H is holomorphic in Ω and satifies $H|_V = f$. Let $z_0 \in \partial V$. We set

$$\tilde{B} = B\left(z_0, \frac{\delta}{2}\right),\ B = B\left(z_0, \frac{\delta}{4}\right) \text{ and } \tilde{H}(z) = \int_{\partial V \cap B} f^*(\zeta) K(\zeta, z)$$

It is sufficient to show that $\tilde{H} \in H^1(\Omega)$. Let dS_ε be the surface element on $\partial\Omega_\varepsilon$. By Fubini's theorem, we have

$$\int_{\partial\Omega_\varepsilon \cap B} |\tilde{H}(z)|\, dS_\varepsilon(z) \leqq \sum_{j=1}^{N} \int_{\partial V \cap B} \left(\int_{\partial\Omega_\varepsilon \cap B} |K_j(\zeta, z)|\, dS_\varepsilon(z) \right) d\sigma_j(\zeta).$$

Now we estimate

$$I(\zeta)=\int_{\partial\Omega_{\varepsilon}\cap B}\frac{dS_{\varepsilon}(z)}{|\Phi(\zeta, z)|}.$$

There are δ_0, $c>0$ such that for each z sufficiently close to $\partial\Omega$, one can find a smooth (of class C^1) change of coordinates $T(\zeta)=(T_1(\zeta), \ldots, T_N(\zeta))$ satisfying

(i) $T_1(\zeta)=\rho(\zeta)-\rho(z)+\mathrm{iIm}\ F(\zeta, z)$

(ii) $T_j(\zeta)=\zeta_j-z_j \quad (j=2, \ldots, N)$

(iii) $c^{-1}|\zeta-z|\leqq|T(\zeta)|\leqq c|\zeta-z| \quad (\zeta\in B(z, \delta_0))$.

We set

$$t_1=\rho(\zeta)-\rho(z),\ t_2=Im\ F(\zeta, z),\ t=(T_1, \ldots, T_N).$$

Then we have

$$I(\zeta)\leqq c\int_{|t|\leqq c}\frac{dt_2 \ldots dt_{2N}}{\{[\varepsilon+\Psi(M|t|^2)]^2+t_2^2\}^{1/2}}$$

Now we introduce spherical polar coordinates

$$t_2=r\cos\alpha,\ r=(t_2^2+t_3^2+t_4^2)^{1/2}$$

Then we obtain

$$\begin{aligned} I(\zeta)&\leqq c\int_0^c \mathrm{dr}\int_0^{\pi}\frac{r^2\sin\alpha}{\{(\Psi(Mr^2))^2+r^2\cos^2\alpha\}^{1/2}}d\alpha \\ &\leqq c\int_0^c \mathrm{dr}\int_0^1\frac{ds}{\Psi(Mr^2)+s}\leqq c\int_0^c\log\frac{1}{\Psi(r^2)}dr \\ &\leqq c\int_0^1\log\frac{1}{\Psi(t)}t^{-\frac{1}{2}}\mathrm{dt}<\infty. \end{aligned}$$

Thus we have

$$\sup_{\varepsilon}\int_{\partial\Omega_{\varepsilon}\cap B}|\widetilde{H}(z)|\mathrm{dS}_{\varepsilon}(z)<\infty,$$

which completes the proof of the theorem.

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