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Existence, stability and global attractivity results for nonlinear Riemann-Liouville fractional differential equations

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ABSTRACT

Existence, attractivity, and stability of solutions of a nonlinear fractional differential equation of Riemann-Liouville type are proved using the classical Schauder fixed point theorem and a fixed point result due to Dhage. The results are illustrated with examples.

RESUMEN

Demostramos la existencia, atractividad y estabilidad de soluciones de la ecuación diferencial fraccional no-lineal de tipo Riemann-Liouville usando el clásico teorema de punto fijo de Schauder y un resultado de punto fijo de Dhage. Los resultados se ilustran con ejemplos.

Keywords and Phrases: Fractional differential equation; Asymptotic characterization of solution; Fixed point principle; Existence and stability theorem.

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1 The Problem

Dhage [5, 6, 7] and Dhage *et al.* [10] introduced the class of what they called pulling functions as follows. For $J_\infty = [t_0, \infty)$ with $t_0 \in \mathbb{R}_+ = [0, \infty)$ fixed, a continuous function $g : J_\infty \to (0, \infty)$ is a **pulling function** if $\lim\limits_{t\to\infty} g(t) = \infty$. We will denote the class of all pulling functions on J_∞ by $\mathcal{CRB}(J_\infty)$. We wish to point out that if g is a pulling function, then its reciprocal $\overline{g} = \overline{g}(t) = \frac{1}{g(t)}$ is continuous, bounded, and satisfies $\lim\limits_{t\to\infty} \overline{g}(t) = 0$. Using pulling functions, Dhage [6, 7, 8] proved some attractivity and stability results for nonlinear Caputo fractional differential equations. Instead, in this paper we consider fractional differential equations with a Riemann-Louville fractional derivative and use fixed point techniques, rather than the measure theoretic approach used in Dhage *et al.* [9].

Here we will study the nonlinear fractional differential equation

$$ {}^{RL}D^q_{t_0}\big[a(t)x(t)\big] = f(t, x(t)) \quad \text{a.e.} \quad t \in J_\infty, \tag{1.1}$$

together with the fractional integral initial condition (IC)

$$\lim_{t\to t_0^+} I^{1-q}_{t_0^+}\big[a(t)x(t)\big] = b_0, \tag{1.2}$$

where $a \in \mathcal{CRB}(J_\infty) \cap L^1(J_\infty, \mathbb{R})$ is a pulling function, ${}^{RL}D^q$ is a Riemann-Liouville fractional derivative of order q with $0 < q < 1$, and $f : J_\infty \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Our goal is to characterize the attractivity and stability properties of the solutions of (1.1)–(1.2).

We begin with the following notions from the fractional calculus that are needed in our discussion; these can be found, for example, in Agarwal *et al.* [1], Podlubny [13] or Kilbas *et al.* [12]. Define the function space

$$C(J_\infty, \mathbb{R}) = \big\{x : J_\infty \to \mathbb{R} \mid x \text{ is continuous}\big\},$$

and let $L^1(J_\infty, \mathbb{R})$ denote the class of Lebesgue integrable functions. In what follows, Γ is the usual Euler's gamma function,

$$\Gamma(q) = \int_0^\infty e^{-t}t^{q-1}\,dt,$$

and $[q]$ is the greatest integer less than or equal to q.

Definition 1.1. *Let $J_\infty = [t_0, \infty)$ for some $t_0 \geq 0$ in $\mathbb{R}$. For any $x \in L^1(J_\infty, \mathbb{R})$, the Riemann-Liouville fractional integral of order $q > 0$ is defined as*

$$I^q_{t_0}x(t) = \frac{1}{\Gamma(q)}\int_{t_0}^t \frac{x(s)}{(t-s)^{1-q}}\,ds,\ t \in J_\infty,$$

provided the right hand side is pointwise defined on (t_0, ∞).

Definition 1.2. *If $x \in L^1(J_\infty, \mathbb{R})$, the Riemann-Liouville fractional derivative ${}^{RL}D^q_{t_0}x$ of x of order q is defined as*

$$ {}^{RL}D^q_{t_0}x(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_{t_0}^t (t-s)^{n-q-1}x(s)\,ds,\ n-1<q<n,\ n=[q]+1, $$

provided the right hand side exists.

Note that if a, $x \in L^1(J_\infty, \mathbb{R})$, then ${}^{RL}D^q_{t_0}[a(t)x(t)]$ exists on J_∞.

Definition 1.3. *A function x is called a classical solution of IVP* (1.1)–(1.2) *if*

(i) x is continuous on J_∞, and

(ii) x satisfies (1.1) *and* (1.2).

The fractional differential equation (1.1) is a scalar multiplicative perturbation of the second type obtained by multiplying the unknown function under the Riemann-Liouville derivative by a scalar function. This and other types of perturbations of a differential equation are described in Dhage [3].

2 Properties of solutions

We set our problem (1.1) in the Banach space $BC(J_\infty, \mathbb{R})$ of bounded continuous real-valued functions defined on J_∞ with the usual supremum norm

$$ \|x\| = \sup_{t \in J_\infty} |x(t)|. $$

We take $\mathcal{T} : BC(J_\infty, \mathbb{R}) \to BC(J_\infty, \mathbb{R})$ to be a continuous operator and we study the operator equation

$$ \mathcal{T}x(t) = x(t),\ t \in J_\infty. \tag{2.1} $$

Next, we describe various properties of solutions of the operator equation (2.1) in the space $BC(J_\infty, \mathbb{R})$.

First, we define the concepts of global attractivity and stability of the solutions as given in Banas and Dhage [2].

Definition 2.1. *A solution $x = x(t)$ of (2.1) is called globally attractive if*

$$ \lim_{t\to\infty} (x(t) - y(t)) = 0 \tag{2.2} $$

for each solution $y = y(t)$ of (2.1) in $BC(J_\infty, \mathbb{R})$.

That is, solutions of (2.1) are globally attractive if for arbitrary solutions $x(t)$ and $y(t)$ of (2.1) in $BC(J_\infty, \mathbb{R})$, we have that condition (2.2) is satisfied. If (2.2) is satisfied uniformly in $BC(J_\infty, \mathbb{R})$ in the sense that for every $\epsilon > 0$ there exists $T > 0$ such that, for $t \geq T$,

$$|x(t) - y(t)| \leq \epsilon \tag{2.3}$$

for all solutions x, $y \in BC(J_\infty, \mathbb{R})$ of (2.1), then solutions of (2.1) are said to be *uniformly globally attractive* on J_∞.

Definition 2.2 (Banas and Dhage [2])**.** *A solution $x \in BC(J_\infty, \mathbb{R})$ of equation (2.1) is called asymptotic if $\lim\limits_{t\to\infty} x(t) = 0$. If the limit is uniform with respect to the solution set of the operator equation (2.1) in $BC(J_\infty, \mathbb{R})$ (i.e., for each $\varepsilon > 0$ there exists $T > t_0 \geq 0$ such that $|x(t)| < \varepsilon$ for all solutions x of (2.1) in $BC(J_\infty, \mathbb{R})$ and for all $t \geq T$), we say that solutions of equation (2.1) are uniformly asymptotic on J_∞.*

Definition 2.3. *If all the solutions of the operator equation (2.1) are asymptotic and uniformly globally attractive, we will say that they are uniformly asymptotically attractive or stable on J_∞.*

In order to state the required fixed point techniques to be used in our proofs, we introduce the following concepts.

Definition 2.4 (Dhage [4])**.** *A nondecreasing upper semi-continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function if $\psi(0) = 0$. The class of all $\mathcal{D}$-functions on $\mathbb{R}_+$ is denoted by $\mathfrak{D}$.*

Definition 2.5 (Dhage [4])**.** *Let X be a Banach space with norm $\|\cdot\|$. An operator $\mathcal{T} : X \to X$ is called $\mathcal{D}$-Lipschitz if there exists a $\mathcal{D}$-function $\psi_{\mathcal{T}} \in \mathfrak{D}$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi_{\mathcal{T}}\big(\|x - y\|\big) \tag{2.4}$$

for all x, $y \in X$.

If $\psi_{\mathcal{T}}(r) = kr$, $k > 0$, $\mathcal{T}$ is called a *Lipschitz operator* with Lipschitz constant k. Also, if $0 \leq k < 1$, then $\mathcal{T}$ is called a *contraction* on X and k is referred to as the contraction constant. In addition, if $\psi_{\mathcal{T}}(r) < r$ for $r > 0$, then $\mathcal{T}$ is called a *nonlinear $\mathcal{D}$-contraction* on X, and the set of all nonlinear $\mathcal{D}$-contractions will be denoted by $\mathcal{DN}$.

We say that an operator $\mathcal{T} : X \to X$ is *compact* if $\overline{\mathcal{T}(X)}$ is a compact subset of X. The operator $\mathcal{T}$ is called *totally bounded* if for any bounded subset S of X, $\mathcal{T}(S)$ is a totally bounded subset of X. Moreover, $\mathcal{T}$ is called *completely continuous* if $\mathcal{T}$ is continuous and totally bounded on X. We note that every compact operator is totally bounded, but the converse may not be true; the two notions are equivalent on bounded subsets of X. Additional details on different types of nonlinear contractions and compact and completely continuous operators can be found, for example, in Granas and Dugundji [11].

In an effort to prove our main existence results, we need the following fixed point theorems.

Theorem 2.6 (Schauder [11]). *Let S be a closed, convex, and bounded subset of a Banach space X, and let $\mathcal{T} : S \to S$ be a completely continuous operator. Then the operator equation $\mathcal{T}x = x$ has a solution.*

Theorem 2.7 (Dhage [3]). *Let X be a Banach space and let $\mathcal{T} : X \to X$ be a nonlinear $\mathcal{D}$-contraction. Then the operator equation $\mathcal{T}x = x$ has a unique solution.*

3 Existence, attractivity, and stability of solutions

Definition 3.1. *A function $\beta : J_\infty \times \mathbb{R} \to \mathbb{R}$ is called Carathéodory if*

(i) the map $t \mapsto \beta(t, x)$ is measurable for each $x \in \mathbb{R}$, and

(ii) the map $x \mapsto \beta(t, x)$ is continuous for each $t \in J_\infty$.

The following lemma is often used in the study of nonlinear differential equations.

Lemma 3.2 (Carathéodory). *Let $\beta : J_\infty \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Then the function $t \to \beta(t, x(t))$ is measurable for each $x \in C(J_\infty, \mathbb{R})$.*

We will make use of the following conditions in the remainder of our paper.

(H_1) The function f is bounded on $J_\infty \times \mathbb{R}$ with bound M_f.

(H_2) The function f is Carathédory on $J_\infty \times \mathbb{R}$.

(H_3) There exists a $\mathcal{D}$-function $\psi_f \in \mathfrak{D}$ such that

$$|f(t, x) - f(t, y)| \leq \psi_f(|x - y|)$$

for all x, $y \in \mathbb{R}$ and $t \in J_\infty$.

The following lemma will play an important role in obtaining our existence results.

Lemma 3.3. *For any function $h \in L^1(J_\infty, \mathbb{R})$, the function $x \in BC(J_\infty, \mathbb{R})$ is a solution of the fractional differential equation*

$$^{RL}D^q_{t_0}\big[a(t)x(t)\big] = h(t) \quad a.e. \quad t \in J_\infty, \tag{3.1}$$

satisfying the initial condition

$$\lim_{t \to t_0^+} I^{1-q}_{t_0^+}\big[a(t)x(t)\big] = b_0, \tag{3.2}$$

if and only if x satisfies the nonlinear fractional integral equation

$$x(t) = \frac{b_0}{\Gamma(q)}\frac{(t-t_0)^{q-1}}{a(t)} + \frac{1}{a(t)\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}h(s)\,ds \tag{3.3}$$

for all $t \in J_\infty$.

Proof. Applying the Riemann-Liouville fractional integral operator $I_{t_0}^{q}$ to (3.1), we obtain

$$a(t)x(t) - \frac{I_{t_0}^{1-q}[a(t)x(t)]}{\Gamma(q)}\bigg|_{t=t_0}(t-t_0)^{q-1} = I_{t_0}^{q}h(t) = \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}h(s)\,ds$$

for all $t \in J_\infty$, or

$$x(t) = \frac{b_0}{\Gamma(q)}\frac{(t-t_0)^{q-1}}{a(t)} + \frac{1}{a(t)\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}h(s)\,ds.$$

That is, if $x(t)$ is a solution of (3.1)–(3.2), then $x(t)$ is a solution of (3.3).

Now let $x(t)$ be a solution of (3.3). Then,

$$a(t)x(t) = \frac{b_0}{\Gamma(q)}(t-t_0)^{q-1} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}h(s)\,ds. \tag{3.4}$$

Applying the Riemann-Liouville fractional derivative operator to this expression gives

$${}^{RL}D_{t_0}^{q}\big[a(t)x(t)\big] = {}^{RL}D_{t_0}^{q}\left[\frac{b_0}{\Gamma(q)}(t-t_0)^{q-1}\right] + h(t)$$

since ${}^{RL}D_{t_0}^{q}I_{t_0}^{q}h(t) = h(t)$. Also, since ${}^{RL}D_{t_0}^{q}(t-t_0)^{q-1} = 0$, $x(t)$ satisfies equation (3.1).

From (3.4),

$$I_{t_0}^{1-q}\big[a(t)x(t)\big] = I_{t_0}^{1-q}\left[\frac{b_0}{\Gamma(q)}(t-t_0)^{q-1}\right] + I_{t_0}^{1-q}(I_{t_0}^{q}h(t)).$$

Now $I_{t_0}^{1-q}(I_{t_0}^{q}h) = I_{t_0}h = \int_{t_0}^{t}h(s)ds$ and $\lim_{t\to t_0^+}\int_{t_0}^{t}h(s)ds = 0$. Also, by [1, Proposition 1],

$$\begin{aligned} I_{t_0}^{1-q}\left[\frac{b_0}{\Gamma(q)}(t-t_0)^{q-1}\right] &= \frac{b_0}{\Gamma(q)}I_{t_0}^{1-q}(t-t_0)^{q-1} \\ &= \frac{b_0}{\Gamma(q)}\frac{\Gamma(q)}{\Gamma(q+1-q)}(t-t_0)^{q+1-q-1} = \frac{b_0}{\Gamma(q)}\frac{\Gamma(q)}{\Gamma(1)} = b_0. \end{aligned}$$

Hence,

$$\lim_{t\to t_0^+} I_{t_0}^{1-q}\big[a(t)x(t)\big] = b_0,$$

and so (3.2) is satisfied. This proves the lemma. □

We need to introduce the following class of functions. Let

$$\mathcal{A} = \left\{ f \in C([t_0,\infty),(0,\infty)) : \lim_{t\to t_0} \frac{(t-t_0)^{q-1}}{f(t)} < \infty \text{ and } \lim_{t\to\infty} \frac{t^q}{f(t)} = 0 \right\},$$

and we assume in what follows that the function a in equation (1.1) belongs to the class $\mathcal{A} \cap \mathcal{CRB}(J_\infty)$.

Remark 3.4. *If $a \in \mathcal{CRB}(J_\infty)$, then $\overline{a} \in BC(J_\infty, \mathbb{R}_+)$ and so the number $\|\overline{a}\| = \sup_{t\in J_\infty} \overline{a}(t)$ exists. Also, the function $w : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $w(t) = \overline{a}(t)\, t^q$ is continuous on J_∞ and satisfies the relation $\lim_{t\to\infty} w(t) = 0$, so the number*

$$W = \sup_{t\geq t_0} w(t) \tag{3.5}$$

exists.

Our main existence and global attractivity result is contained in the following theorem.

Theorem 3.5. *Assume that conditions (H$_1$)–(H$_2$) hold. Then (1.1) has a solution defined on J_∞ and the solutions of (1.1) are uniformly globally asymptotically attractive.*

Proof. Since $a(t) \in \mathcal{A} \cap \mathcal{CRB}(J_\infty)$, there exists $d_0 > 0$ such that $\left| \frac{(t-t_0)^{q-1}}{a(t)} \right| \leq d_0$ on J_∞. Set $X = BC(J_\infty, \mathbb{R})$ and define a closed ball $\overline{B}_r(0)$ in X centered at the origin 0 with radius r given by

$$r = \frac{|b_0| d_0}{\Gamma(q)} + \frac{M_f\, W}{\Gamma(q+1)},$$

where M_f is from (H$_1$) and W is given in (3.5). By an application of Lemma 3.3, (1.1) is equivalent to the hybrid fractional integral equation

$$x(t) = \frac{b_0}{\Gamma(q)} \frac{(t-t_0)^{q-1}}{a(t)} + \frac{1}{a(t)\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s))\, ds \tag{3.6}$$

for all $t \in J_\infty$. Define the operator $\mathcal{T}$ on $\overline{B}_r(0)$ by

$$\mathcal{T}x(t) = \frac{b_0}{\Gamma(q)} \frac{(t-t_0)^{q-1}}{a(t)} + \frac{1}{a(t)\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s))\, ds, \quad t \in J_\infty. \tag{3.7}$$

Then (3.6) is transformed into the operator equation

$$\mathcal{T}x(t) = x(t), \;\; t \in J_\infty. \tag{3.8}$$

We will show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.6 with $S = \overline{B}_r(0) \subset BC(J_\infty, \mathbb{R})$. Now from the continuity of the integral, it follows that the function $t \to \mathcal{T}x(t)$ is continuous on J_∞ for each $x \in \overline{B}_r(0)$. Furthermore, by condition (H_1),

$$
\begin{aligned}
|\mathcal{T}x(t)| &\leq \frac{b_0 d_0}{\Gamma(q)} + \frac{M_f}{a(t)\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}\, ds \leq \frac{b_0 d_0}{\Gamma(q)} + \frac{M_f}{\Gamma(q)} |\overline{a}(t)| \int_{t_0}^{t} (t-s)^{q-1} \\
&\leq \frac{b_0 d_0}{\Gamma(q)} + \frac{M_f}{\Gamma(q+1)} |\overline{a}(t)| t^q \leq \frac{b_0 d_0}{\Gamma(q)} + \frac{M_f}{\Gamma(q+1)} W
\end{aligned}
$$

for all $t \in J_\infty$ and all $x \in \overline{B}_r(0)$. Taking the supremum over t,

$$
\|\mathcal{T}x\| \leq \frac{b_0 d_0}{\Gamma(q)} + \frac{M_f W}{\Gamma(q+1)} = r
$$

for all $x \in \overline{B}_r(0)$. As a result, $\mathcal{T}$ maps $\overline{B}_r(0)$ into itself.

To show that $\mathcal{T}$ is a completely continuous operator on $\overline{B}_r(0)$, we first show that it is continuous there. To do this, fix $\epsilon > 0$ and let $\{x_n\}$ be a sequence in $\overline{B}_r(0)$ converging to $x \in \overline{B}_r(0)$. Then,

$$
\begin{aligned}
|(\mathcal{T}x_n)(t) - (\mathcal{T}x)(t)| &\leq \frac{\overline{a}(t)}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \big| f(s, x_n(s)) - f(s, x(s)) \big|\, ds \\
&\leq \frac{\overline{a}(t)}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \Big[|f(s, x_n(s))| + |f(s, x(s))| \Big]\, ds \\
&\leq \frac{2\, M_f \overline{a}(t)}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}\, ds \leq \frac{2\, M_f}{\Gamma(q+1)}\, w(t). \qquad (3.9)
\end{aligned}
$$

Since $a \in \mathcal{A}$, there exists $T > 0$ such that $w(t) \leq \dfrac{\epsilon\Gamma(q+1)}{2M_f}$ for $t \geq T$. Thus, for $t \geq T$, from (3.9), we see that

$$
|(\mathcal{T}x_n)(t) - (\mathcal{T}x)(t)| \leq \epsilon \quad \text{as} \quad n \to \infty.
$$

Let $t \in [t_0, T]$. Then, by the Lebesgue dominated convergence theorem, we obtain

$$
\begin{aligned}
\lim_{n\to\infty} \mathcal{T}x_n(t) &= \lim_{n\to\infty} \left[\frac{b_0}{\Gamma(q)} \frac{(t-t_0)^{q-1}}{a(t)} + \frac{\overline{a}(t)}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x_n(s))\, ds \right] \\
&= \frac{b_0}{\Gamma(q)} \frac{(t-t_0)^{q-1}}{a(t)} + \frac{\overline{a}(t)}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \Big[\lim_{n\to\infty} f(s, x_n(s)) \Big]\, ds \\
&= \mathcal{T}x(t) \qquad (3.10)
\end{aligned}
$$

for all $t \in [t_0, T]$. Moreover, it can be shown as below that $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of functions in X. Now, using arguments similar to those given in Granas *et al.* [11], it follows that $\mathcal{T}$ is a continuous operator on $\overline{B}_r(0)$ into itself.

Next, we show that $\mathcal{T}$ is a compact operator on $\overline{B}_r(0)$. To accomplish this, it suffices to show that every sequence $\{\mathcal{T}x_n\}$ in $\mathcal{T}(\overline{B}_r(0))$ has a convergent subsequence. Similar to what we did above, we can show that $\|\mathcal{T}x_n\| \leq r$ for all $n \in \mathbb{N}$. This shows that $\{\mathcal{T}x_n\}$ is a uniformly bounded sequence in $\mathcal{T}(\overline{B}_r(0))$.

To show that $\{\mathcal{T}x_n\}$ is also an equicontinuous sequence in $\mathcal{T}(\overline{B}_r(0))$, let $\epsilon > 0$ be given. Since $\lim\limits_{t\to\infty} w(t) = 0$, there exists $T_1 > t_0 \geq 0$ such that

$$w(t) < \frac{\epsilon\Gamma(q+1)}{9M_f} \tag{3.11}$$

for all $t \geq T_1$.

Let t, $\tau \in J_\infty$ be arbitrary. If t, $\tau \in [t_0, T_1]$, then we have

$$\begin{aligned}
|\mathcal{T}x_n(t) - \mathcal{T}x_n(\tau)| &\leq \frac{b_0}{\Gamma(q)}\left|\frac{(t-t_0)^{q-1}}{a(t)} - \frac{(\tau-t_0)^{q-1}}{a(\tau)}\right| \\
&\quad + \left|\frac{\overline{a}(t)}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}f(s,x(s))\,ds - \frac{\overline{a}(\tau)}{\Gamma(q)}\int_{t_0}^{\tau}(\tau-s)^{q-1}f(s,x(s))\,ds\right| \\
&\leq \frac{b_0}{\Gamma(q)}\left|\frac{(t-t_0)^{q-1}}{a(t)} - \frac{(\tau-t_0)^{q-1}}{a(\tau)}\right| \\
&\quad + \left|\frac{\overline{a}(t)}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}f(s,x(s))\,ds - \frac{\overline{a}(\tau)}{\Gamma(q)}\int_{t_0}^{t}(\tau-s)^{q-1}f(s,x(s))\,ds\right| \\
&\quad + \left|\frac{\overline{a}(\tau)}{\Gamma(q)}\int_{t_0}^{t}(\tau-s)^{q-1}f(s,x(s))\,ds - \frac{\overline{a}(\tau)}{\Gamma(q)}\int_{t_0}^{\tau}(\tau-s)^{q-1}f(s,x(s))\,ds\right| \\
&\leq \frac{b_0}{\Gamma(q)}\left|\frac{(t-t_0)^{q-1}}{a(t)} - \frac{(\tau-t_0)^{q-1}}{a(\tau)}\right| \\
&\quad + \frac{M_f}{\Gamma(q)}\int_{t_0}^{t}\left|\overline{a}(t)(t-s)^{q-1} - \overline{a}(\tau)(\tau-s)^{q-1}\right|\,ds + \frac{M_f}{\Gamma(q)}\left|\int_{\tau}^{t}\left|\overline{a}(\tau)(\tau-s)^{q-1}\right|ds\right| \\
&\leq \frac{b_0}{\Gamma(q)}\left|\frac{(t-t_0)^{q-1}}{a(t)} - \frac{(\tau-t_0)^{q-1}}{a(\tau)}\right| \\
&\quad + \frac{M_f}{\Gamma(q)}\int_{t_0}^{T}\left|\overline{a}(t)(t-s)^{q-1} - \overline{a}(\tau)(\tau-s)^{q-1}\right|\,ds + \frac{M_f\,\|\overline{a}\|}{\Gamma(q+1)}\left|(\tau-t)^q\right|.
\end{aligned} \tag{3.12}$$

Since the function $t \mapsto \overline{a}(t)(t-s)^{q-1}$ is continuous on the compact interval $[t_0, T_1]$, it is uniformly continuous there. Therefore, for the above ϵ there exist $\delta_1 > 0$ and $\delta_2 > 0$, depending only on ϵ, such that

$$|t-\tau| < \delta_1 \quad \text{implies} \quad \left|\overline{a}(t)(t-s)^{q-1} - \overline{a}(\tau)(\tau-s)^{q-1}\right| < \min\left\{\frac{\epsilon\Gamma(q)}{9b_0}, \frac{\epsilon\Gamma(q)}{9M_fT_1}\right\},$$

and

$$|t-\tau| < \delta_2 \quad \text{implies} \quad |(t-\tau)^q| < \frac{\epsilon\Gamma(q+1)}{9M_f\|\overline{a}\|}.$$

Let $\delta_3 = \min\{\delta_1, \delta_2\}$. Then, if t, $\tau \in [t_0, T_1]$ with $|t-\tau| < \delta_3$, from (3.12) we have

$$|\mathcal{T}x_n(t) - \mathcal{T}x_n(\tau)| < \frac{\epsilon}{3}$$

for all $n \in \mathbb{N}$.

Now, if $t,\, \tau > T_1$, then there is a $0 < \delta_4 < \delta_3$ such that if $|t-\tau| < \delta_4$,

$$\begin{aligned}
|\mathcal{T}x_n(t) - \mathcal{T}x_n(\tau)| &\leq \frac{b_0}{\Gamma(q)}\left|\frac{(t-t_0)^{q-1}}{a(t)} - \frac{(\tau-t_0)^{q-1}}{a(t)}\right| \\
&\quad + \frac{\overline{a}(t)}{\Gamma(q)}\left|\int_{t_0}^{t}(t-s)^{q-1}f(s,x_n(s))\,ds\right| + \frac{\overline{a}(\tau)}{\Gamma(q)}\left|\int_{t_0}^{\tau}(\tau-s)^{q-1}f(s,x_n(s))\,ds\right| \\
&\leq \frac{b_0}{\Gamma(q)}\left|\frac{(t-t_0)^{q-1}}{a(t)} - \frac{(\tau-t_0)^{q-1}}{a(t)}\right| + \frac{M_f}{\Gamma(q+1)}\big[w(t)+w(\tau)\big] \\
&< \frac{\epsilon}{9} + \frac{2\epsilon}{9} = \frac{\epsilon}{3}
\end{aligned}$$

for all $n \in \mathbb{N}$. Similarly, if $t,\, \tau \in \mathbb{R}_+$ with $t < T_1 < \tau$ and $|t-\tau| < \delta < \delta_4$, then

$$|\mathcal{T}x_n(t) - \mathcal{T}x_n(\tau)| \leq |\mathcal{T}x_n(t) - \mathcal{T}x_n(T_1)| + |\mathcal{T}x_n(T_1) - \mathcal{T}x_n(\tau)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}$$

for all $n \in \mathbb{N}$. As a result, $|\mathcal{T}x_n(t) - \mathcal{T}x_n(\tau)| < \epsilon$ for all $t,\, \tau \in J_\infty$ with $|t-\tau| < \delta$ and for all $n \in \mathbb{N}$. This shows that $\{\mathcal{T}x_n\}$ is an equicontinuous sequence in $\overline{B}_r(0)$. An application of the Arzelà-Ascoli theorem implies that $\{\mathcal{T}x_n\}$ has a uniformly convergent subsequence on the compact set $\overline{B}_r(0)$.

Since $\mathcal{T}(\overline{B}_r(0))$ is closed, $\{\mathcal{T}x_n\}$ converges to a point in $\mathcal{T}(\overline{B}_r(0))$, so $\mathcal{T}(\overline{B}_r(0))$ is relatively compact. Therefore, $\mathcal{T}$ is a continuous and compact operator on $\overline{B}_r(0)$. An application of Theorem 2.6 shows that the operator equation $\mathcal{T}x = x$, and hence (1.1), has a solution on J_∞ belonging to $\overline{B}_r(0)$.

To prove the attractivity of solutions, let $x,\, y \in \overline{B}_r(0)$ be any two solutions of (1.1) on J_∞. Then,

$$\begin{aligned}
|x(t)-y(t)| &\leq \frac{\overline{a}(t)}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\big|f(s,x(s)) - f(s,y(s))\big|\,ds \\
&\leq \frac{\overline{a}(t)}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\big[|f(s,x(s))| + |f(s,y(s))|\big]\,ds \leq \frac{2\,M_f}{\Gamma(q+1)}w(t)
\end{aligned}$$

for all $t \in J_\infty$. As in (3.11), for any $\epsilon > 0$ there exists $T_1 > t_0$ such that

$$w(t) < \frac{\epsilon\Gamma(q+1)}{2M_f}$$

for $t \geq T_1$. Thus,

$$|x(t)-y(t)| < \epsilon$$

for all $t \geq T$. Hence, the solutions of (1.1) are uniformly globally attractive on J_∞.

Finally, since a belongs to $\mathcal{A}$, for any $\epsilon > 0$, there exists $T_2 > T_1$ such that

$$\left|\frac{b_0}{\Gamma(q)}\frac{(t-t_0)^{q-1}}{a(t)}\right| < \frac{\epsilon}{2}$$

for $t \geq T_2$. Then for any solution x of (1.1) defined on J_∞,

$$|x(t)| \leq \left|\frac{b_0}{\Gamma(q)}\frac{(t-t_0)^{q-1}}{a(t)}\right| + \frac{\overline{a}(t)}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}|f(s,x(s))|\,ds \leq \frac{\epsilon}{2} + \frac{M_f}{\Gamma(q+1)}w(t) < \epsilon$$

for all $t \geq T_2$, that is, solutions are uniformly globally asymptotically attractive and stable on J_∞. This completes the proof of the theorem. □

In our next theorem, we wish to show that the uniformly globally asymptotically attractive solution of (1.1) obtained from Theorem 3.5 is unique.

Theorem 3.6. *Assume that conditions (H$_1$)–(H$_3$) hold with*

$$\frac{\sup_{t_0\leq t}\overline{a}(t)\,t^q}{\Gamma(q)}\psi_f(r) < r, \quad r > 0. \tag{3.13}$$

Then (1.1) has a unique uniformly stable solution defined on J_∞.

Proof. Set $X = BC(J_\infty, \mathbb{R})$ and define the operator $\mathcal{T} : X \to X$ by (3.7). We want to show that $\mathcal{T}$ is a nonlinear $\mathcal{D}$-contraction on X. Let x, $y \in X$; then by (H$_3$), we obtain

$$\begin{aligned}
|\mathcal{T}x(t) - \mathcal{T}x(t)| &\leq \frac{\overline{a}(t)}{\Gamma(q)}\left|\int_{t_0}^{t}(t-s)^{q-1}\big|f(s,x(s)) - f(s,y(s))\big|\,ds\right| \\
&\leq \frac{\overline{a}(t)}{\Gamma(q)}\left|\int_{t_0}^{t}(t-s)^{q-1}\psi_f\big(|x(s)-y(s)|\big)\,ds\right| \\
&\leq \frac{\overline{a}(t)}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\psi_f\big(|x-y|\big)\,ds \\
&\leq \frac{w(t)}{\Gamma(q+1)}\psi_f\big(|x-y|\big) \leq \frac{W}{\Gamma(q+1)}\psi_f\big(|x-y|\big)
\end{aligned}$$

for all $t \in J_\infty$. Taking the supremum over t in the above inequality yields

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \frac{W}{\Gamma(q+1)}\psi_f\big(|x-y|\big)$$

for all $x, y \in X$, where $\dfrac{W}{\Gamma(q)}\psi_f(r) < r$ for $r > 0$ in view of condition (3.13). This shows that $\mathcal{T}$ is a nonlinear $\mathcal{D}$-contraction on X. By Theorem 2.7, we obtain that the solution of (1.1) obtained in Theorem 3.5 is unique. □

Example 1. *Consider the initial value problem of fractional Riemann-Liouville type*

$$\begin{cases} {}^{RL}D_1^q\big[(t^q+1)e^t x(t)\big] = \dfrac{\ln(|x(t)|+1)}{x^2(t)+2}, \quad t \in J_\infty = [0,\infty), \\ \lim\limits_{t\to 0^+} I_{0^+}^{1-q}\big[(t^q+1)e^t x(t)\big] = 1. \end{cases} \tag{3.14}$$

Here we have $t_0 = 0$, $a(t) = (t^q + 1)e^t$, *and* $f(t,x) = \dfrac{\ln(|x|+1)}{x^2+2}$ *for* $(t,x) \in [0,\infty) \times \mathbb{R}$. *Clearly,* $a(t) \to \infty$ *as* $t \to \infty$ *and* (H_1) *holds with* $M_f = 1$. *It is easy to see that* $\lim\limits_{t\to 0} \dfrac{t^{q-1}}{(t^q+1)e^t} = 0$ *and* $\lim\limits_{t\to\infty} \dfrac{t^q}{(t^q+1)e^t} = 0$, *so* $a \in \mathcal{A}$. *Hence, by Theorem 3.5, (3.15) has a solution and the solutions are uniformly globally asymptotically attractive and stable on* $[0,\infty)$.

Example 2. *Consider the problem*

$$\begin{cases} {}^{RL}D_1^q\big[(t^q+1)e^t x(t)\big] = f(t,x(t)), \quad t \in J_\infty = [0,\infty), \\ \lim\limits_{t\to 0^+} I_{0^+}^{1-q}\big[(t^q+1)e^t x(t)\big] = 1, \end{cases} \tag{3.15}$$

where

$$f(t,x) = \begin{cases} \ln(|x|+1), & \text{if } -5 \le x \le 5, \\ \ln 6, & \text{otherwise.} \end{cases}$$

Now, for $-5 \le x \le 5$,

$$\begin{aligned} |f(t,x) - f(t,y)| &= |\ln(|x|+1) - \ln(|y|+1)| = \ln\frac{|x|+1}{|y|+1} = \ln\frac{1+|y|+|x|-|y|}{|y|+1} \\ &= \ln\left(1 + \frac{|x|-|y|}{|y|+1}\right) \le \ln\left(1 + \frac{|x-y|}{|y|+1}\right) \le \Psi_f(|x-y|). \end{aligned}$$

We can then take our $\mathcal{D}$*-function to be* $\psi_f(r) = \ln(1+r)$ *and* $M_f = \ln 6$. *Since*

$$\frac{\sup_{t\ge t_0} \overline{a}(t)\, t^q}{\Gamma(q)} \psi_f(r) \le \psi_f(r) = \ln(1+r) < r, \quad r > 0, \tag{3.16}$$

condition (3.13) *is satisfied. Therefore by Theorems 3.5 and 3.6, solutions of* (3.15) *exist, are unique, and are uniformly globally asymptotically attractive on* $\mathbb{R}_+$.

Data Availability

Data sharing is not applicable to this article since no datasets were generated or analysed during the current study.

Conflict of Interest

The authors declare that there are no conflicts of interest.

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