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Existence of solutions for higher order $\phi-$Laplacian BVPs on the half-line using a one-sided Nagumo condition with nonordered upper and lower solutions

A. Zerki[1]
K. Bachouche[2]
K. Ait-Mahiout[1, ✉]

[1] *Laboratoire "Théorie du point fixe et Applications", École Normale Supérieure, BP 92, Kouba, 16006, Algiers, Algeria.*

[2] *Departement of Mathematics, Faculty of Sciences, Algiers University 1, Algiers, Algeria and Laboratory of Mathematical Analysis and Applications, Algiers, Algeria.*

a.zerki@ensh.dz
k.bachouche@univ-alger.dz
karima.aitmahiout@g.ens-kouba.dz ✉

ABSTRACT

In this paper, we consider the following $(n+1)$st order bvp on the half line with a $\phi-$Laplacian operator

$$\begin{cases} (\phi(u^{(n)}))'(t) = f(t, u(t), \dots, u^{(n)}(t)), & a.e., \ t \in [0, +\infty), \\ & n \in \mathbb{N} \setminus \{0\}, \\ u^{(i)}(0) = A_i, \ i = 0, \dots, n-2, \\ u^{(n-1)}(0) + au^{(n)}(0) = B, \\ u^{(n)}(+\infty) = C. \end{cases}$$

The existence of solutions is obtained by applying Schaefer's fixed point theorem under a one-sided Nagumo condition with nonordered lower and upper solutions method where f is a L^1-Carathéodory function.

RESUMEN

En este artículo, consideramos el siguiente pvf en la semi-recta de orden $(n+1)$ con un operador $\phi-$Laplaciano

$$\begin{cases} (\phi(u^{(n)}))'(t) = f(t, u(t), \dots, u^{(n)}(t)), & a.e., \ t \in [0, +\infty), \\ & n \in \mathbb{N} \setminus \{0\}, \\ u^{(i)}(0) = A_i, \ i = 0, \dots, n-2, \\ u^{(n-1)}(0) + au^{(n)}(0) = B, \\ u^{(n)}(+\infty) = C. \end{cases}$$

Se obtiene la existencia de soluciones aplicando el teorema de punto fijo de Schaefer bajo una condición unilateral de Nagumo con un método de soluciones inferiores y superiores no-ordenadas donde f es una función L^1-Carathéodory.

Keywords and Phrases: Boundary value problem, One-sided Nagumo condition, Lower and upper solutions, A priori estimates.

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1 Introduction

Differential equations of nth order were studied in many works, with different boundary value conditions by using different methods on bounded and unbounded domains, we quote [5, 6, 8, 9, 10] and references therein.

In this paper we consider the following $\phi-$Laplacian ordinary differential equation of order $n+1$ given by

$$(\phi(u^{(n)}))'(t) = f(t, u(t), \dots, u^{(n)}(t)), \; a.e., \; t \in [0, +\infty), \tag{1.1}$$

where $n \in \mathbb{N} \setminus \{0\}$, ϕ is an increasing homeomorphism satisfying $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$.

Concerning the nonlinearity, we suppose that $f : [0, +\infty) \times \mathbb{R}^{n+1} \to \mathbb{R}$ is a L^1-Carathéodory function.

This equation is subject to the following Sturm-Liouville type boundary conditions:

$$\begin{cases} u^{(i)}(0) = A_i, \; i = 0, \dots, n-2, \\ u^{(n-1)}(0) + au^{(n)}(0) = B, \\ u^{(n)}(+\infty) = C, \end{cases} \tag{1.2}$$

where $a < 0, B, C \in \mathbb{R}$, $A_i \in \mathbb{R}$, $i = 0, 1, \dots, n-2$ and $u^{(n)}(+\infty) = \lim_{t \to +\infty} u^{(n)}(t)$.

To prove the existence of solutions for this problem we use Scheafer's fixed point theorem combined with the upper and lower solutions method with a one-sided Nagumo condition.

The upper and lower solutions method have witnessed qualitative progress in recent years by providing various results, following some papers that use this method [2, 3, 4, 7, 11, 12, 14, 15, 16].

In [12] and [7], the authors study the existence of solutions to the following two problems using the Schauder fixed point theorem with upper and lower solutions method with a one-sided Nagumo condition. The first problem is given by

$$u'''(t) = f\left(t, u(t), u'(t), u''(t)\right), t \in [0, +\infty),$$

$u(0) = A$, $au'(0) + bu''(0) = B$, $u''(+\infty) = C$, with $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is a L^1- Carathéodory function, $a > 0$, $b < 0$, $A, B, C \in \mathbb{R}$. The second problem is

$$u^{(4)}(t) = f\left(t, u(t), u'(t), u''(t), u'''(t)\right), \quad t \in [0, +\infty),$$

where $f : [0, +\infty) \times \mathbb{R}^4 \to \mathbb{R}$ is a L^1-Carathéodory function, and the boundary conditions are of

Sturm-Liouville type,

$$u(0) = A, u'(0) = B, \quad u''(0) + au'''(0) = C, \quad u'''(+\infty) = D,$$

$A, B, C, D \in \mathbb{R}$, $a < 0$ and $u'''(+\infty) := \lim_{t\to+\infty} u'''(t)$.

In the present paper, we have obtained the same results as in [12] and [7], but for a more general problem, where we combine an nth order ordinary differential equation with a $\phi-$Laplacian operator on the half line using non-ordered upper and lower solutions and to compensate the lack of compactness of the interval $[0, +\infty)$ we invoke the Corduneanu lemma (see Lemma 2.6).

This problem has many applications with regards to higher order problems defined on unbounded intervals. We quote, *e.g.*, [14] for $n = 2$. In the case where $\phi(t) = t$, we cite [12] and [7] for the third and fourth order, respectively.

The paper is divided into four sections. Section 2 is devoted to some preliminary definitions and the proof of technical lemmas. In Section 3, we prove the main result and in Section 4, we propose an example where we show the applicability of the main result.

2 Definitions and preliminary results

Let

$$X = \left\{ u \in \mathcal{C}^n[0, +\infty) : \lim_{t\to+\infty} u^{(n)}(t) \text{ exists in } \mathbb{R} \right\}$$

and define the norm $\|u\|_X := \max\{\|u\|_0, \|u'\|_1, \|u''\|_2, \dots, \|u^{(n)}\|_n\}$, where

$$\|u^{(i)}\|_i = \sup_{0\le t<+\infty} \left| \frac{u^{(i)}(t)}{1+t^{n-i}} \right|, \ i = 0, 1, 2, \dots, n.$$

Lemma 2.1. *For each fixed $n \in \mathbb{N} \setminus \{0\}$, let $u \in \mathcal{C}^n([0, +\infty))$. If $\lim_{t\to+\infty} u^{(n)}(t) = \ell$, then*

$$\lim_{t\to+\infty} u^{(n)}(t) = (n-i)! \lim_{t\to+\infty} \frac{u^{(i)}(t)}{1+t^{n-i}}, \quad \textit{for} \quad i \in \{0, 1, \dots, n-1\}.$$

Proof. Let n be fixed in $\mathbb{N}^*$ and $u \in \mathcal{C}^n([0, +\infty))$ such that $\lim_{t\to+\infty} u^{(n)}(t) = \ell$. We have

$$\lim_{t\to+\infty} u^{(n)}(t) - \ell + 1 = 1.$$

So,

$$\lim_{t\to+\infty} u^{(n-1)}(t) - \ell t + t + d_{n-1} = +\infty$$

where d_{n-1} is a real constant. By using L'Hospital's rule, we deduce that

$$\lim_{t\to+\infty}\frac{u^{(n-1)}(t)-\ell t+t+d_{n-1}}{1+t}=\lim_{t\to+\infty}u^{(n)}(t)-\ell+1.$$

Hence,

$$\lim_{t\to+\infty}u^{(n)}(t)=\lim_{t\to+\infty}\frac{u^{(n-1)}(t)}{1+t}=(n-(n-1))!\lim_{t\to+\infty}\frac{u^{(n-1)}(t)}{1+t^{(n-(n-1))}}\cdot$$

In this case, $i=n-1$. To evaluate $\lim_{t\to+\infty}\frac{u^{(n-2)}}{1+t^2}$, we repeat the formula twice:

$$\lim_{t\to+\infty}u^{(n-1)}(t)-\ell t+t+d_{n-1}=+\infty.$$

Then,

$$\lim_{t\to+\infty}u^{(n-2)}(t)-\ell\frac{t^2}{2}+\frac{t^2}{2}+d_{n-1}t+d_{n-2}=+\infty.$$

Using L'Hospital's rule twice, we get

$$\begin{aligned}\lim_{t\to+\infty}\frac{u^{(n-2)}(t)-\frac{\ell}{2}t^2+\frac{1}{2}t^2+d_{n-1}t+d_{n-2}}{1+t^2}&=\lim_{t\to+\infty}\frac{u^{(n-1)}(t)-\ell t+t+d_{n-1}}{2t}\\&=\lim_{t\to+\infty}\frac{u^{(n)}(t)}{2}-\frac{\ell}{2}+\frac{1}{2}\cdot\end{aligned}$$

Then,

$$\lim_{t\to+\infty}u^{(n)}(t)=\lim_{t\to+\infty}2\frac{u^{(n-2)}(t)}{1+t^2}=(n-(n-2))!\lim_{t\to+\infty}\frac{u^{(n-2)}(t)}{1+t^{(n-(n-2))}}\cdot$$

Here $i=n-2$ and d_{n-1},d_{n-2} are real constants. At the order i,

$$\begin{aligned}&\lim_{t\to+\infty}\frac{u^{(i)}(t)}{1+t^{n-i}}-\frac{\ell}{(n-i)!}+\frac{1}{(n-i)!}\\&=\lim_{t\to+\infty}\frac{u^{(i)}(t)-\frac{\ell}{(n-i)!}t^{n-i}+\frac{1}{(n-i)!}t^{n-i}+\frac{1}{(n-i-1)!}t^{n-i-1}+\cdots+d_{i-1}t+d_i}{1+t^{n-i}}\\&=\\&\vdots\\&=\lim_{t\to+\infty}\frac{u^{(n)}(t)}{(n-i)!}-\frac{\ell}{(n-i)!}+\frac{1}{(n-i)!}\cdot\end{aligned}$$

In conclusion,

$$\lim_{t\to+\infty}u^{(n)}(t)=(n-i)!\lim_{t\to+\infty}\frac{u^{(i)}(t)}{1+t^{n-i}}.$$ □

By this Lemma, $(X,\|\cdot\|_X)$ is a Banach space.

The following definition establishes the assumptions assumed on the nonlinearity.

Definition 2.2. *A function $f : [0,+\infty) \times \mathbb{R}^{n+1} \to \mathbb{R}$ is called a L^1-Carathéodory function if it satisfies:*

(i) for each $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$, $t \mapsto f(t, x_0, x_1, \ldots, x_n)$ is measurable on $[0,+\infty)$;

(ii) for almost every $t \in [0,+\infty), (x_0, x_1, \ldots, x_n) \mapsto f(t, x_0, x_1, \ldots, x_n)$ is continuous in $\mathbb{R}^{n+1}$;

(iii) $\forall \rho > 0$, $\exists \varphi_\rho \in L^1[0,+\infty)$, $\forall x \in X$

$$\|x\|_X < \rho \Rightarrow |f(t, x(t), x'(t), \ldots, x^{(n)}(t))| \leq \varphi_\rho(t), \quad a.e., \quad t \in [0,+\infty).$$

Lemma 2.3. *Let $\eta \in L^1[0,+\infty)$. The linear boundary value problem*

$$(\phi(u^{(n)}))'(t) + \eta(t) = 0, \quad a.e., \quad t \in [0,+\infty), \tag{2.1}$$

with boundary conditions (1.2), has a unique solution in X. Moreover, this solution can be expressed as

$$\begin{aligned} u(t) =& A_0 + A_1 t + \cdots + \frac{A_{n-2}}{(n-2)!} t^{n-2} + \frac{B - a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} \eta(s)\, ds\right)}{(n-1)!} t^{n-1} \\ &+ \int_0^t \left(\frac{(t-s)^{n-1}}{(n-1)!}\right) \phi^{-1}\left(\phi(C) + \int_s^{+\infty} \eta(\tau)\, d\tau\right) ds. \end{aligned} \tag{2.2}$$

Proof. We integrate (2.1) from t to $+\infty$,

$$\phi(u^{(n)}(t)) = \phi(C) + \int_t^{+\infty} \eta(\tau)\, d\tau$$

to get

$$u^{(n)}(t) = \phi^{-1}\left(\phi(C) + \int_t^{+\infty} \eta(\tau)\, d\tau\right). \tag{2.3}$$

So,

$$u^{(n)}(0) = \phi^{-1}\left(\phi(C) + \int_0^{+\infty} \eta(\tau)\, d\tau\right). \tag{2.4}$$

By integrating (2.3) on $(0, t]$ and using (1.2) with (2.4),

$$u^{(n-1)}(t) = B - a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} \eta(s)\, ds\right) + \int_0^t \phi^{-1}\left(\phi(C) + \int_s^{+\infty} \eta(\tau)\, d\tau\right) ds. \tag{2.5}$$

Integrating (2.5) on $(0, t]$, we get

$$\begin{aligned} u^{(n-2)}(t) =& A_{n-2} + Bt - a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} \eta(s)\, ds\right) t \\ &+ \int_0^t (t-s)\, \phi^{-1}\left(\phi(C) + \int_s^{+\infty} \eta(\tau)\, d\tau\right) ds. \end{aligned} \tag{2.6}$$

By integrating (2.6) on $(0,t]$,

$$u^{(n-3)}(t) = A_{n-3} + A_{n-2}t + \frac{B - a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} \eta(s)\,ds\right)}{2}t^2 + \int_0^t \frac{(t-s)^2}{2}\phi^{-1}\left(\phi(C) + \int_s^{+\infty} \eta(\tau)\,d\tau\right) ds.$$

Integrating again over $(0,t]$, we find for $i = 0, 1, \ldots, n-1$,

$$\begin{aligned} u^{(i)}(t) = &\sum_{k=i}^{k=n-2} \frac{A_k}{(k-i)!}t^{k-i} + B\,\frac{t^{n-1-i}}{(n-1-i)!} - a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} \eta(s)\,ds\right)\frac{t^{n-1-i}}{(n-1-i)!} \\ &+ \int_0^t \frac{(t-s)^{n-1-i}}{(n-1-i)!}\phi^{-1}\left(\phi(C) + \int_s^{+\infty} \eta(\tau)\,d\tau\right) ds. \end{aligned} \tag{2.7}$$

By (2.7),

$$\begin{aligned} u(t) =& A_0 + A_1 t + \cdots + \frac{A_{n-2}}{(n-2)!}t^{n-2} + \frac{B - a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} \eta(s\,ds)\right)}{(n-1)!}t^{n-1} \\ &+ \int_0^t \left(\frac{(t-s)^{n-1}}{(n-1)!}\right)\phi^{-1}\left(\phi(C) + \int_s^{+\infty} \eta(\tau)\,d\tau\right) ds. \end{aligned}$$

□

Now, we need to have an *a priori* estimate for $u^{(n)}$, for this let $\gamma_i, \Gamma_i \in C[0,+\infty)$, $\gamma_i(t) \le \Gamma_i(t)$, $i = 0, 1, 2, \ldots, n-1$, with $\sup_{t\ge 0} \frac{|\gamma_{n-1}(t)|}{1+t} < +\infty$ and $\sup_{t\ge 0} \frac{|\Gamma_{n-1}(t)|}{1+t} < +\infty$. Define the set

$$E = \left\{(t, x_0, x_1, \cdots, x_n) \in [0,+\infty) \times \mathbb{R}^{n+1} : \gamma_i(t) \le x_i \le \Gamma_i(t), i = 0, 1, 2, \ldots, n-1\right\}.$$

Definition 2.4. A function $f : E \to \mathbb{R}$ is said to satisfy the one-sided Nagumo type growth condition in E if it satisfies either

$$f(t, x_0, x_1, \ldots, x_n) \le \psi(t)h(|x_n|), \quad \forall (t, x_0, x_1, \ldots, x_n) \in E, \tag{2.8}$$

or

$$f(t, x_0, x_1, \ldots, x_n) \ge -\psi(t)h(|x_n|), \quad \forall (t, x_0, x_1, \ldots, x_n) \in E, \tag{2.9}$$

for some positive continuous functions ψ, h, and some $\nu > 1$, such that

$$\sup_{0\le t<+\infty} \psi(t)(1+t)^{\nu} < +\infty, \quad \int^{+\infty} \frac{\phi^{-1}(s)}{h(\phi^{-1}(s))}\,ds = +\infty, \quad \int^{+\infty} \frac{\phi^{-1}(-s)}{h(|\phi^{-1}(-s)|)}\,ds = -\infty. \tag{2.10}$$

Next lemma provides an *a priori* bound.

Lemma 2.5. *Let $f : [0, +\infty) \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a L^1-Carathéodory function satisfying (2.8) with (2.10), or (2.9) with (2.10). Then there exists $R > 0$ such that every solution u of (1.1)- (1.2) satisfying*

$$\gamma_i(t) \leq u^{(i)}(t) \leq \Gamma_i(t), \quad i = 0, \ldots, n-1 \tag{2.11}$$

for $t \in [0, +\infty)$ is such that $\|u^{(n)}\|_n < R$ where R does not depend on the solution u.

Proof. Let u be a solution of (1.1)-(1.2) such that (2.11) holds. Consider $r > 0$ such that

$$r > \max\left\{\left|\frac{B - \Gamma_{n-1}(0)}{a}\right|, \left|\frac{B - \gamma_{n-1}(0)}{a}\right|, |C|\right\}. \tag{2.12}$$

With this inequality we cannot have $|u^{(n)}(t)| > r$ for all $t \in [0, +\infty)$, because

$$|u^{(n)}(0)| = \left|\frac{B - u^{(n-1)}(0)}{a}\right| \leq \max\left\{\left|\frac{B - \Gamma_{n-1}(0)}{a}\right|, \left|\frac{B - \gamma_{n-1}(0)}{a}\right|\right\} < r \tag{2.13}$$

and $|u^{(n)}(+\infty)| = |C| < r$.

In the case where $|u^{(n)}(t)| \leq r$ for all $t \in [0, +\infty)$, it is enough to consider $R > r/2$ to complete the proof:

$$\|u^{(n)}\|_n = \sup_{0 \leq t < +\infty} \left|\frac{u^{(n)}(t)}{2}\right| \leq \frac{r}{2} < R.$$

If there exists $t \in (0, +\infty)$ such that $|u^{(n)}(t)| > r$, then by (2.10), we can take $R > r$ such that

$$\int_{\phi(r)}^{\phi(R)} \frac{\phi^{-1}(s)}{h(\phi^{-1}(s))}\, ds > M\left\{M_1 + \sup_{0 \leq t < +\infty} \frac{|\Gamma_{n-1}(t)|}{1+t} \frac{\nu}{\nu - 1}\right\}$$

and

$$\int_{\phi(-R)}^{\phi(-r)} \frac{\phi^{-1}(s)}{h(|\phi^{-1}(s)|)}\, ds < M\left\{-M_1 + \inf_{0 \leq t < +\infty} \frac{-|\gamma_{n-1}(t)|}{1+t} \frac{\nu}{\nu - 1}\right\}$$

with $M := \sup_{0 \leq t < +\infty} \psi(t)(1+t)^\nu$ and $M_1 := \sup_{0 \leq t < +\infty} \frac{\Gamma_{n-1}(t)}{(1+t)^\nu} - \inf_{0 \leq t < +\infty} \frac{\gamma_{n-1}(t)}{(1+t)^\nu}$.

Assume that the growth condition (2.8) holds. By (2.12), suppose that there exist t_*, $t_+ \in (0, +\infty)$ such that $u^{(n)}(t_*) = r$ and $u^{(n)}(t) > r$ for all $t \in (t_*, t_+]$. Then

$$\begin{aligned}
\int_{\phi(u^{(n)}(t_*))}^{\phi(u^{(n)}(t_+))} \frac{\phi^{-1}(s)}{h(\phi^{-1}(s))}\, ds &= \int_{t_*}^{t_+} \frac{u^{(n)}(s)}{h(u^{(n)}(s))} (\phi(u^{(n)}))'(s)\, ds \\
&= \int_{t_*}^{t_+} \frac{f(s, u(s), u'(s), u''(s), u'''(s), \ldots, u^{(n)}(s))}{h(u^{(n)}(s))} u^{(n)}(s)\, ds \\
&\leq \int_{t_*}^{t_+} \psi(s)\ u^{(n)}(s)\, ds \leq M \int_{t_*}^{t_+} \frac{u^{(n)}(s)}{(1+s)^\nu}\, ds
\end{aligned}$$

$$
\begin{aligned}
&= M \int_{t_*}^{t_+} \left(\left(\frac{u^{(n-1)}(s)}{(1+s)^{\nu}} \right)' + \frac{\nu u^{(n-1)}(s)}{(1+s)^{1+\nu}} \right) ds \\
&= M \left(\frac{u^{(n-1)}(t_+)}{(1+t_+)^{\nu}} - \frac{u^{(n-1)}(t_*)}{(1+t_*)^{\nu}} + \int_{t_*}^{t_+} \frac{\nu u^{(n-1)}(s)}{(1+s)^{1+\nu}} \, ds \right) \\
&\le M \left(M_1 + \sup_{0 \le t < +\infty} \frac{|\Gamma_{n-1}(t)|}{1+t} \int_0^{+\infty} \frac{\nu}{(1+s)^{\nu}} \, ds \right) \\
&< \int_{\phi(r)}^{\phi(R)} \frac{\phi^{-1}(s)}{h(\phi^{-1}(s))} \, ds.
\end{aligned}
$$

So $u^{(n)}(t_+) < R$ and as t_*, t_+ are arbitrary in $(0, +\infty)$, we have $u^{(n)}(t) < R$ for all $t \in [0, +\infty)$.

By the same technique using (2.12), and considering t_- and t_* such that $u^{(n)}(t_*) = -r$, $u^{(n)}(t) < -r$ for all $t \in [t_-, t_*)$, it can be proved that $u^{(n)}(t) > -R$ for all $t \in [0, +\infty)$, therefore $\|u^{(n)}\|_n < R/2 < R$.

If f satisfies (2.9), following similar arguments we get the same conclusion. □

We also need a compactness criterion.

Lemma 2.6 ([1]). *A set $M \subset X$ is relatively compact if the following three conditions hold:*

(1) *all functions from M are uniformly bounded;*

(2) *all functions from M are equicontinuous on any compact interval of $[0, +\infty)$;*

(3) *all functions from M are equiconvergent at infinity, that is, for any given $\epsilon > 0$, there exists a $t_\epsilon > 0$ such that*

$$
\left| \frac{u^{(i)}(t)}{1+t^{n-i}} - \lim_{t \to +\infty} \frac{u^{(i)}(t)}{1+t^{n-i}} \right| < \epsilon, \text{ for all } t > t_\epsilon, u \in M \text{ and } i = 0, 1, 2, 3, \ldots, n.
$$

To end this section, we present the Schaefer Fixed Point Theorem with the definition of lower and upper solutions for our problem (1.1)-(1.2).

Theorem 2.7 ([13]). *Let E be a Banach space and $T : E \to E$ be a completely continuous operator. If the set*

$$
\{x \in E : \ x = \lambda T x \ \ for \ \lambda \in (0,1)\}
$$

is bounded, then T has at least one fixed point.

Definition 2.8. A function $\alpha \in X$ is said to be a lower solution of problem (1.1)-(1.2) if $\phi(\alpha^{(n)}) \in AC[0,+\infty)$ such that

$$(\phi(\alpha^{(n)}))'(t) \geq f(t, \overline{\alpha}(t), \alpha'(t), \ldots, \alpha^{(n)}(t)),\ a.e.,\ t \in [0,+\infty),$$

and

$$\begin{cases} \alpha^{(i)}(0) \leq A_i,\ i = 1, \ldots, n-2, \\ \alpha^{(n-1)}(0) + a\alpha^{(n)}(0) \leq B, \\ \alpha^{(n)}(+\infty) < C, \end{cases} \tag{2.14}$$

where $a < 0,\ B, C \in \mathbb{R},\ A_i \in \mathbb{R},\ i = 0, \ldots, n-2$ and $\overline{\alpha}(t) := \alpha(t) - \alpha(0) + A_0$.

A function $\beta \in X$ where $\phi(\beta^{(n)}) \in AC[0,+\infty)$ is an upper solution if it satisfies the reversed inequalities with $\overline{\beta}(t) := \beta(t) - \beta(0) + A_0$.

3 Main existence result

Theorem 3.1. *Let $f : [0,+\infty) \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a L^1-Carathéodory function, ϕ an increasing homeomorphism satisfying $\phi(0) = 0$, and α, β lower and upper solutions of* (1.1)-(1.2)*, respectively, such that*

$$\alpha^{(n-1)}(t) \leq \beta^{(n-1)}(t), \quad \forall t \in [0,+\infty). \tag{3.1}$$

If f satisfies the one-sided Nagumo condition (2.8)*, or* (2.9)*, on the set*

$$E_* = \Big\{(t, x_0, x_1, \ldots, x_n) \in [0,+\infty) \times \mathbb{R}^{n+1} : \overline{\alpha}(t) \leq x_0 \leq \overline{\beta}(t),\ \alpha'(t) \leq x_1 \leq \beta'(t), \ldots, \\ \alpha^{(n-1)}(t) \leq x_{n-1} \leq \beta^{(n-1)}(t)\Big\}$$

and

$$\begin{aligned} f(t, \overline{\alpha}(t), \alpha'(t), \ldots, \alpha^{(n-2)}(t), x_{n-1}, x_n) &\geq f(t, x_0, \ldots, x_n) \\ &\geq f(t, \overline{\beta}(t), \beta'(t), \ldots, \beta^{(n-2)}(t), x_{n-1}, x_n), \end{aligned} \tag{3.2}$$

for (t, x_{n-1}, x_n) fixed and $\overline{\alpha}(t) \leq x_0 \leq \overline{\beta}(t),\ \alpha'(t) \leq x_1 \leq \beta'(t), \ldots, \alpha^{(n-2)}(t) \leq x_{n-2} \leq \beta^{(n-2)}(t)$, then problem (1.1)-(1.2) *has at least a solution $u \in X$ with $\phi(u^{(n)}) \in AC[0,+\infty)$ and there exists $R > 0$ such that*

$$\overline{\alpha}(t) \leq u(t) \leq \overline{\beta}(t), \alpha'(t) \leq u'(t) \leq \beta'(t), \ldots, \alpha^{(n-1)}(t) \leq u^{(n-1)}(t) \leq \beta^{(n-1)}(t),$$
$$-R < u^{(n)}(t) < R, \quad \forall t \in [0,+\infty).$$

Remark 3.2. *α and β are almost-ordered. In fact, α and β can be chosen such that $\alpha \not\leq \beta$ but we have necessary that $\overline{\alpha} \leq \overline{\beta}$.*

Indeed, from condition (3.1), for all $t \in [0, +\infty)$, we have $\alpha^{(n-1)}(t) \leq \beta^{(n-1)}(t)$. As $\alpha^{(n-2)}(0) \leq A_{n-2} \leq \beta^{(n-2)}(0)$, integrating on $[0, +\infty)$

$$\alpha^{(n-2)}(t) - \alpha^{(n-2)}(0) = \int_0^t \alpha^{(n-1)}(s)\, ds \leq \int_0^t \beta^{(n-1)}(s)\, ds = \beta^{(n-2)}(t) - \beta^{(n-2)}(0).$$

As

$$\alpha^{(n-2)}(t) - \alpha^{(n-2)}(0) + A_{n-2} \leq \beta^{(n-2)}(t) - \beta^{(n-2)}(0) + A_{n-2},$$

then

$$\alpha^{(n-2)}(t) \leq \beta^{(n-2)}(t).$$

By the same technique, one shows that $\alpha^{(i)} \leq \beta^{(i)}$, for $i = 1, 2, \ldots, n-3$, then

$$\alpha(t) - \alpha(0) = \int_0^t \alpha'(s)\, ds \leq \int_0^t \beta'(s)\, ds = \beta(t) - \beta(0),$$

So,

$$\overline{\alpha}(t) \leq \overline{\beta}(t), \quad \forall t \in [0, +\infty).$$

Proof. Consider the j-modified equation for $j = 1, 2$

$$\begin{aligned} (\phi(u^{(n)}))'(t) &= f(t, \delta_0(t, u(t)), \ldots, \delta_{n-1}(t, u^{(n-1)}(t)), \delta_{nj}(t, u^{(n)}(t))) \\ &\quad + \frac{1}{1+t^2} \frac{u^{(n-1)}(t) - \delta_{n-1}(t, u^{(n-1)}(t))}{1 + |u^{(n-1)}(t) - \delta_{n-1}(t, u^{(n-1)}(t))|}, \quad a.e., \ t \in [0, +\infty), \end{aligned} \tag{3.3}$$

where the functions $\delta_i, \delta_{nj} : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$, $i = 0, 1, 2, 3, \ldots, n-1$ and $j = 1, 2$ are given by

$$\delta_0(t, x) = \begin{cases} \overline{\beta}(t), & x > \overline{\beta}(t), \\ x, & \overline{\alpha}(t) \leq x \leq \overline{\beta}(t), \\ \overline{\alpha}(t), & x < \overline{\alpha}(t), \end{cases}$$

$$\delta_i(t, y_i) = \begin{cases} \beta^{(i)}(t), & y_i > \beta^{(i)}(t), \\ y_i, & \alpha^{(i)}(t) \leq y_i \leq \beta^{(i)}(t), \\ \alpha^{(i)}(t), & y_i < \alpha^{(i)}(t), \end{cases} \qquad i = 1, 2, \ldots, n-1,$$

$$\delta_{n1}(t, w) = \begin{cases} N, & w > N, \\ w, & -N \leq w \leq N, \\ -N, & w < -N, \end{cases}$$

where $N > \max\left\{\sup_{0\le t<+\infty} |\alpha^{(n)}(t)|, \sup_{0\le t<+\infty} |\beta^{(n)}(t)|\right\}$, and

$$\delta_{n2}(t,w) = w.$$

For convenience, the proof is divided into three principal steps.

Step 1: Every solution of (3.3)-(1.2), satisfies $\alpha^{(n-1)}(t) \le u_j^{(n-1)}(t) \le \beta^{(n-1)}(t)$ for all $t \in [0,+\infty)$, $j = 1,2$. Let u_j be a solution of the j-modified problem (3.3)-(1.2), $j = 1,2$ and suppose, by contradiction, that there exists $t \in (0,+\infty)$ such that $\alpha^{(n-1)}(t) > u_j^{(n-1)}(t)$, $j = 1,2$. Therefore

$$\inf_{0\le t<+\infty}\left(u_j^{(n-1)}(t) - \alpha^{(n-1)}(t)\right) < 0,\ j = 1,2.$$

By (2.14) this infimum cannot be attained at $+\infty$. In fact,

$$\inf_{0\le t<+\infty}\left(u_j^{(n-1)}(t) - \alpha^{(n-1)}(t)\right) := u_j^{(n-1)}(+\infty) - \alpha^{(n-1)}(+\infty) < 0$$

and

$$u_j^{(n)}(+\infty) - \alpha^{(n)}(+\infty) \le 0.$$

We reach the following contradiction

$$0 \ge u_j^{(n)}(+\infty) - \alpha^{(n)}(+\infty) > C - C = 0.$$

If

$$\inf_{0\le t<+\infty}\left(u_j^{(n-1)}(t) - \alpha^{(n-1)}(t)\right) := u_j^{(n-1)}(0^+) - \alpha^{(n-1)}(0^+) < 0, \quad j = 1,2.$$

Then we have the following contradiction for $j = 1,2$

$$0 \le u_j^{(n)}(0^+) - \alpha^{(n)}(0^+) \le \frac{B - u_j^{(n-1)}(0)}{a} + \frac{\alpha^{(n-1)}(0) - B}{a} = -\frac{1}{a}(u_j^{(n-1)}(0) - \alpha^{(n-1)}(0)) < 0.$$

If there is $t_* \in (0,+\infty)$, we can define for $j = 1,2$

$$\min_{0\le t<+\infty}\left(u_j^{(n-1)}(t) - \alpha^{(n-1)}(t)\right) := u_j^{(n-1)}(t_*) - \alpha^{(n-1)}(t_*) < 0,$$

with $u_j^{(n)}(t_*) = \alpha^{(n)}(t_*)$. Then there exists $\bar{t} > t_*$, such that

$$u_j^{(n-1)}(t) - \alpha^{(n-1)}(t) < 0, \quad u_j^{(n)}(t) - \alpha^{(n)}(t) \ge 0, \quad \text{for all} \quad t \in (t_*, \bar{t}).$$

Therefore by (3.2) and Definition 2.8, we get a contradiction for $j = 1, 2$

$$\begin{aligned}(\phi(u_j^{(n)}))'(t) - (\phi(\alpha^{(n)}))'(t) &= f(t, \delta_0(t, u_j(t)), \ldots, \delta_{n-1}(t, u_j^{(n-1)}(t)), \delta_{nj}(t, u_j^{(n)}(t))) \\ &\quad + \frac{1}{1+t^2} \frac{u_j^{(n-1)}(t) - \delta_{n-1}(t, u_j^{(n-1)}(t))}{1 + |u_j^{(n-1)}(t) - \delta_{n-1}(t, u_j^{(n-1)}(t))|} - (\phi(\alpha^{(n)}))'(t) \\ &= f(t, \delta_0(t, u_j(t)), \ldots, \delta_{n-2}(t, u_j^{(n-2)}(t)), \alpha^{(n-1)}(t), \alpha^{(n)}(t)) \\ &\quad + \frac{1}{1+t^2} \frac{u_j^{(n-1)}(t) - \alpha^{(n-1)}(t)}{1 + |u_j^{(n-1)}(t) - \alpha^{(n-1)}(t)|} - (\phi(\alpha^{(n)}))'(t) \\ &\leq \frac{1}{1+t^2} \frac{u_j^{(n-1)}(t) - \alpha^{(n-1)}(t)}{1 + |u_j^{(n-1)}(t) - \alpha^{(n-1)}(t)|} < 0, \quad a.e. \quad t \in (t_*, \bar{t}).\end{aligned}$$

So, the function $\phi(u_j^{(n)}(t)) - \phi(\alpha^{(n)}(t))$ is decreasing for all $t \in (t_*, \bar{t})$. If $t \in (t_*,\ \bar{t})$,

$$0 = \phi(u_j^{(n)}(t_*)) - \phi(\alpha^{(n)}(t_*)) > \phi(u_j^{(n)}(t)) - \phi(\alpha^{(n)}(t))$$

and $u_j^{(n)}(t) - \alpha^{(n)}(t) < 0$. Therefore $u_j^{(n-1)}(t) - \alpha^{(n-1)}(t)$ is decreasing in $(t_*, \bar{t})$, which is a contradiction. So $u_j^{(n-1)}(t) \geq \alpha^{(n-1)}(t)$, $\forall t \in [0, +\infty)$, $j = 1, 2$. In the same way, we show that $u_j^{(n-1)}(t) \leq \beta^{(n-1)}(t)$, $\forall t \in [0, +\infty)$, $j = 1, 2$.

As $\alpha^{(n-2)}(0) \leq A_{n-2} \leq \beta^{(n-2)}(0)$ and $u_j^{(n-2)}(0) = A_{n-2}$, integrating on $[0, +\infty)$ for $j = 1, 2$,

$$\begin{aligned}\alpha^{(n-2)}(t) - \alpha^{(n-2)}(0) &= \int_0^t \alpha^{(n-1)}(s)\, ds \leq \int_0^t u_j^{(n-1)}(s)\, ds = u_j^{(n-2)}(t) - A_{n-2} \\ &\leq \int_0^t \beta^{(n-1)}(s)\, ds = \beta^{(n-2)}(t) - \beta^{(n-2)}(0).\end{aligned}$$

As

$$\alpha^{(n-2)}(t) - \alpha^{(n-2)}(0) + A_{n-2} \leq u_j^{(n-2)}(t) \leq \beta^{(n-2)}(t) - \beta^{(n-2)}(0) + A_{n-2},$$

then

$$\alpha^{(n-2)}(t) \leq u_j^{(n-2)}(t) \leq \beta^{(n-2)}(t).$$

By the same technique, one shows that $\alpha^{(i)} \leq u_j^{(i)} \leq \beta^{(i)}$, for $i = 1, 2, \ldots, n-3$, $j = 1, 2$, then

$$\alpha(t) - \alpha(0) = \int_0^t \alpha'(s)\, ds \leq \int_0^t u_j'(s)\, ds = u_j(t) - A_0 \leq \int_0^t \beta'(s)\, ds = \beta(t) - \beta(0),$$

$$\overline{\alpha}(t) \leq u_j(t) \leq \overline{\beta}(t), \quad j = 1, 2.$$

Step 2: By Lemma 2.5, if u is a solution of the 2-modified problem (3.3)-(1.2), then there exists $R_1 > 0$, not depending on u, such that

$$\|u^{(n)}\|_n < R_1.$$

Now, we need to consider $N = N_1$, where

$$N_1 > \max\left\{2R_1, \sup_{0\le t<+\infty} |\alpha^{(n)}(t)|, \sup_{0\le t<+\infty} |\beta^{(n)}(t)|\right\}.$$

If the 1-modified problem (3.3)-(1.2) has a solution u, then u is a solution of problem (1.1)-(1.2), where

$$\|u^{(n)}\|_n < R_1 < \frac{N_1}{2} < N_1.$$

Step 3: Problem (3.3)-(1.2) for $j = 1$ has at least one solution. Let us define the operator $T : X \to X$ by

$$\begin{aligned} Tu(t) =& A_0 + A_1 t + \cdots + \frac{A_{n-2}}{(n-2)!}t^{n-2} + \frac{B - a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} F(u(s))\,ds\right)}{(n-1)!}t^{n-1} \\ &+ \int_0^t \left(\frac{(t-s)^{n-1}}{(n-1)!}\right)\phi^{-1}\left(\phi(C) + \int_s^{+\infty} F(u(\tau))\,d\tau\right) ds. \end{aligned}$$

with

$$\begin{aligned} F(u(s)) :=& -f(s, \delta_0(s,u(s)), \ldots, \delta_{n-1}(s, u^{(n-1)}(s)), \delta_{n1}(s, u^{(n)}(s))) \\ &- \frac{1}{1+s^2}\frac{u^{(n-1)}(s) - \delta_{n-1}(s, u^{(n-1)}(s))}{1 + |u^{(n-1)}(s) - \delta_{n-1}(s, u^{(n-1)}(s))|}. \end{aligned}$$

From Lemma 2.3, one can see that the fixed points of T are solutions of the 1-modified (3.3)-(1.2) problem. So it is sufficient to prove that T has a fixed point in X. For this aim, it is enough to prove that the operator T satisfies the condition of the Schaefer fixed point theorem 2.7. The proof is split into three steps.

(1) $T : X \to X$ is well defined. Let $u \in X$. As f is a L^1-Carathéodory function, so, for

$$\rho > \max\{N_1, \|\overline{\alpha}\|_0, \|\overline{\beta}\|_0\} \cup \{\|\alpha^{(i)}\|_i, \|\beta^{(i)}\|_i,\ i = 1, 2, \ldots, n-1\},$$

we obtain

$$\begin{aligned} \int_0^{+\infty} |F(u(s))|\,ds &\le \int_0^{+\infty} \varphi_\rho(s) + \frac{1}{1+s^2}\frac{|u^{(n-1)}(s) - \delta_{n-1}(s, u^{(n-1)}(s))|}{1 + |u^{(n-1)}(s) - \delta_{n-1}(s, u^{(n-1)}(s))|}\,ds \\ &\le \int_0^{+\infty}\left(\varphi_\rho(s) + \frac{1}{1+s^2}\right) ds = M_\rho < +\infty, \end{aligned} \tag{3.4}$$

this means that F is also a L^1-Carathéodory function. Then,

$$\lim_{t\to+\infty}(Tu)^{(n)}(t) = \lim_{t\to+\infty}\phi^{-1}\left(\phi(C)+\int_t^{+\infty}F(u(\tau))\,d\tau\right) = C = \lim_{t\to+\infty}\frac{(Tu)^{(n-1)}(t)}{1+t}$$
$$= 2!\lim_{t\to+\infty}\frac{(Tu)^{(n-2)}(t)}{1+t^2} = 3!\lim_{t\to+\infty}\frac{(Tu)^{(n-3)}(t)}{1+t^3} = \cdots = n!\frac{(Tu)(t)}{1+t^n}\cdot$$

Therefore, $Tu \in X$.

(2) T is continuous. Let $(u_m) \subset X$, such that $u_m \to u$ in X. There exists $r > 0$ such that $\|u_m\|_X < r$, $\forall m \in \mathbb{N}$. We have to prove that $\|Tu_m - Tu\|_X \overset{m\to+\infty}{\longrightarrow} 0$. To this end, we can see that

$$\|Tu_m - Tu\|_0 \overset{m\to+\infty}{\longrightarrow} 0, \|(Tu_m)' - (Tu)'\|_1 \overset{m\to+\infty}{\longrightarrow} 0, \|(Tu_m)'' - (Tu)''\|_2 \overset{m\to+\infty}{\longrightarrow} 0, \dots,$$
$$\|(Tu_m)^{(n)} - (Tu)^{(n)}\|_n \overset{m\to+\infty}{\longrightarrow} 0.$$

We have,

$$\sup_{0\le t<+\infty}|\phi((Tu_m)^{(n)})(t) - \phi((Tu)^{(n)})(t)| = \sup_{0\le t<+\infty}\left|\int_t^{+\infty}F(u_m(\mu))\,d\mu - \int_t^{+\infty}F(u(\mu))\,d\mu\right|$$
$$\le \int_0^{+\infty}\Big|F(u_m(\mu)) - F(u(\mu))\Big|\,d\mu \le 2M_\rho < +\infty.$$

From Lebesgue Dominated Convergence Theorem, $F(u_m(t))$ converges to $F(u(t))$ a.e., $t \in [0, +\infty)$, as $m \to +\infty$, because F is L^1-Carathéodory function, so

$$\int_0^{+\infty}\Big|F(u_m(\mu)) - F(u(\mu))\Big|d\mu \to 0,$$

as $m \to +\infty$, then,

$$\|(Tu_m)^{(n)} - (Tu)^{(n)}\|_n \to 0,$$

as $m \to +\infty$. Moreover, we have that for $i = 0, 1, 2, \dots, n-1$,

$$\sup_{0\le t<+\infty}\left|\frac{(Tu_m)^{(i)}(t)}{1+t^{n-i}} - \frac{(Tu)^{(i)}(t)}{1+t^{n-i}}\right| = \sup_{0\le t<+\infty}\left|-a\frac{\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u_m(s))ds\right)}{1+t^{n-i}}\frac{t^{n-i-1}}{(n-i-1)!}\right.$$
$$+\frac{\int_0^t(t-s)^{n-i-1}\phi^{-1}\left(\phi(C)+\int_s^{+\infty}F(u_m(\tau))d\tau\right)ds}{(1+t^{n-i})(n-i-1)!}$$
$$+a\frac{\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u(s))ds\right)}{1+t^{n-i}}\frac{t^{n-i-1}}{(n-i-1)!}$$
$$\left.-\frac{\int_0^t(t-s)^{n-i-1}\phi^{-1}\left(\phi(C)+\int_s^{+\infty}F(u(\tau))d\tau\right)ds}{(1+t^{n-i})(n-i-1)!}\right|$$

$$
\begin{aligned}
&\leq \sup_{0\leq t<+\infty} \frac{t^{n-i-1}}{(1+t^{n-i})(n-i-1)!} \left| a\phi^{-1}\left(\phi(C)+\int_0^{+\infty} F(u(s))ds\right)\right. \\
&\quad \left. - a\phi^{-1}\left(\phi(C)+\int_0^{+\infty} F(u_m(s))ds\right)\right| \\
&+ \sup_{0\leq t<+\infty} \frac{1}{(1+t^{n-i})(n-i-1)!}\int_0^t (t-s)^{n-i-1} \\
&\quad \left|\phi^{-1}\left(\phi(C)+\int_s^{+\infty} F(u_m(\tau))d\tau\right)\right. \\
&\quad \left. - \phi^{-1}\left(\phi(C)+\int_s^{+\infty} F(u(\tau))d\tau\right)\right| ds \\
&\leq \sup_{0\leq t<+\infty} \frac{2|a|t^{n-i-1}}{(1+t^{n-i})(n-i-1)!}\|(Tu_m)^{(n)}-(Tu)^{(n)}\|_n \\
&+ \sup_{0\leq t<+\infty} \int_0^t \frac{2(t-s)^{n-i-1}}{(1+t^{n-i})(n-i-1)!}\|(Tu_m)^{(n)}-(Tu)^{(n)}\|_n ds \\
&\leq 2|a|\|(Tu_m)^{(n)}-(Tu)^{(n)}\|_n + 2\|(Tu_m)^{(n)}-(Tu)^{(n)}\|_n \to 0,
\end{aligned}
$$

as $m \to +\infty$.

(3) T is compact. Let

$$L_\rho = \max\left\{\phi^{-1}(|\phi(C)|+M_\rho), \left|\phi^{-1}(-|\phi(C)|-M_\rho)\right|\right\}.$$

Let $U \subset X$ be any bounded subset, *i.e.*, there is $r>0$ such that $\|u\|_X < r$ for all $u \in U$. For each $u \in U$, one has for $i = 0, 1, \dots, n-1$.

$$
\begin{aligned}
\|(Tu)^{(i)}\|_i = \sup_{0\leq t<+\infty} &\left| \frac{\sum_{k=i}^{k=n-2}\frac{A_k}{(k-i)!}t^{k-i} + B\frac{t^{n-1-i}}{(n-1-i)!} - a\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u(s))ds\right)\frac{t^{n-1-i}}{(n-1-i)!}}{1+t^{n-i}}\right. \\
&\left. + \int_0^t \frac{(t-s)^{n-1-i}}{(n-1-i)!(1+t^{n-i})}\phi^{-1}\left(\phi(C)+\int_s^{+\infty}F(u(\tau))d\tau\right)ds\right| \\
\leq &\sum_{k=i}^{k=n-2}\frac{|A_k|}{(k-i)!} + \frac{|B|+|a|L_\rho}{(n-1-i)!} + \frac{L_\rho}{(n-i)!} < +\infty,
\end{aligned}
$$

and

$$\|(Tu)^{(n)}\|_n = \frac{1}{2}\sup_{0\leq t<+\infty}\left|\phi^{-1}\left(\phi(C)+\int_t^{+\infty}F(u(\mu))d\mu\right)\right| \leq L_\rho < +\infty.$$

So,

$$\|Tu\|_X \leq |A_0|+|A_1|+|A_2|+\cdots+|A_{n-2}|+|B|+(|a|+1)L_\rho < +\infty.$$

That is, TU is uniformly bounded.

In order to prove that TU is equicontinuous, let $L>0$ and $t_1, t_2 \in [0, L]$ with $t_1 < t_2$.

We have

$$\begin{aligned}\left|\phi((Tu)^{(n)})(t_2)-\phi((Tu)^{(n)})(t_1)\right| &= \left|\phi(C)+\int_{t_2}^{+\infty}F(u(\tau))d\tau-\phi(C)-\int_{t_1}^{+\infty}F(u(\tau))d\tau\right| \\ &= \left|\int_{t_2}^{+\infty}F(u(\tau))d\tau-\int_{t_1}^{+\infty}F(u(\tau))d\tau\right| \\ &= \left|\int_{t_1}^{t_2}F(u(\tau))d\tau\right| \to 0,\end{aligned}$$

as $t_1 \to t_2$. Also,

$$\begin{aligned}\left|\frac{(Tu)^{(n-1)}(t_2)}{1+t_2}-\frac{(Tu)^{(n-1)}(t_1)}{1+t_1}\right| &= \left|\frac{B-a\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u(s))ds\right)}{(1+t_2)}\right. \\ &\quad +\int_0^{t_2}\frac{1}{1+t_2}\phi^{-1}\left(\phi(C)+\int_s^{+\infty}F(u(\tau))d\tau\right)ds \\ &\quad -\frac{B-a\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u(s))ds\right)}{(1+t_1)} \\ &\quad \left.-\int_0^{t_1}\frac{1}{1+t_1}\phi^{-1}\left(\phi(C)+\int_s^{+\infty}F(u(\tau))d\tau\right)ds\right| \\ &\le \left|\frac{B-a\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u(s))ds\right)}{(1+t_2)}\right. \\ &\quad \left.-\frac{B-a\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u(s))ds\right)}{(1+t_1)}\right| \\ &\quad +L_\rho\int_0^{t_1}\left|\frac{1}{1+t_2}-\frac{1}{1+t_1}\right|ds \\ &\quad +L_\rho\int_{t_1}^{t_2}\left|\frac{1}{1+t_2}\right|ds \to 0,\end{aligned}$$

as $t_1 \to t_2$. Moreover, we have that for $i = 0, 1, \ldots, n-2$

$$\begin{aligned}\left|\frac{(Tu)^{(i)}(t_2)}{1+t_2^{n-i}}-\frac{(Tu)^{(i)}(t_1)}{1+t_1^{n-i}}\right| &= \left|\frac{\sum_{k=i}^{k=n-2}\frac{A_k}{(k-i)!}t_2^{k-i}-\left(a\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u(s))ds\right)-B\right)\frac{t_2^{n-1-i}}{(n-1-i)!}}{1+t_2^{n-i}}\right. \\ &\quad -\frac{\sum_{k=i}^{k=n-2}\frac{A_k}{(k-i)!}t_1^{k-i}-\left(a\phi^{-1}\left(\phi(C)+\int_0^{+\infty}F(u(s))ds\right)-B\right)\frac{t_1^{n-1-i}}{(n-1-i)!}}{1+t_1^{n-i}} \\ &\quad +\int_0^{t_2}\frac{(t_2-s)^{n-1-i}}{(n-1-i)!(1+t_2^{n-i})}\phi^{-1}\left(\phi(C)+\int_s^{+\infty}F(u(\tau))d\tau\right)ds \\ &\quad \left.-\int_0^{t_1}\frac{(t_1-s)^{n-1-i}}{(n-1-i)!(1+t_1^{n-i})}\phi^{-1}\left(\phi(C)+\int_s^{+\infty}F(u(\tau))d\tau\right)ds\right|\end{aligned}$$

$$\leq \left| \frac{\sum_{k=i}^{k=n-2} \frac{A_k}{(k-i)!} t_2^{k-i} - \left(a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} F(u(s))ds\right) - B\right) \frac{t_2^{n-1-i}}{(n-1-i)!}}{1+t_2^{n-i}} - \frac{\sum_{k=i}^{k=n-2} \frac{A_k}{(k-i)!} t_1^{k-i} - \left(a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} F(u(s))ds\right) - B\right) \frac{t_1^{n-1-i}}{(n-1-i)!}}{1+t_1^{n-i}} \right|$$
$$+ L_\rho \int_0^{t_1} \left| \frac{(t_2-s)^{n-1-i}}{(n-1-i)!(1+t_2^{n-i})} - \frac{(t_1-s)^{n-1-i}}{(n-1-i)!(1+t_1^{n-i})} \right| ds + L_\rho \int_{t_1}^{t_2} \left| \frac{(t_2-s)^{n-1-i}}{(n-1-i)!(1+t_2^{n-i})} \right| ds \to 0,$$

as $t_1 \to t_2$.

Furthermore, $TU \subset X$ is equiconvergent at infinity. We use that F is L^1-Carathéodory function and the continuity of ϕ^{-1}. From Lemma 2.1, we have that for all $u \in U$, $\lim_{t\to+\infty} (Tu)^{(n)}(t) = C$, then, $\lim_{t\to+\infty} \frac{(Tu)^{(n-i)}(t)}{1+t^i} = \frac{C}{i!}$ for $i \in \{1,\dots,n\}$. So,

$$\left|(Tu)^{(n)}(t) - C\right| = \left|\phi^{-1}\left(\phi(C) + \int_t^{+\infty} F(u(\mu))d\mu\right) - C\right| \to 0,$$

as $t \to +\infty$. Regarding the next derivative, we have

$$\begin{aligned}
\lim_{t\to+\infty} \left| \frac{(Tu)^{(n-1)}(t)}{1+t} - C \right| &= \lim_{t\to+\infty} \left| \frac{1}{1+t} \left(B - a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} F(u(s))\,ds\right) \right.\right. \\
&\qquad \left.\left. + \int_0^t \phi^{-1}\left(\phi(C) + \int_s^{+\infty} F(u(\tau))\,d\tau\right) ds \right) - C \right| \\
&\leq \lim_{t\to+\infty} \left| \frac{1}{1+t} \left(B - a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} F(u(s))\,ds\right)\right)\right| \\
&\qquad + \lim_{t\to+\infty} \left| \frac{1}{1+t} \int_0^t \phi^{-1}\left(\phi(C) + \int_s^{+\infty} F(u(\tau))\,d\tau\right) ds - C \right| \\
&= \lim_{t\to+\infty} \left| \frac{1}{1+t} \left(B - a\phi^{-1}\left(\phi(C) + \int_0^{+\infty} F(u(s))\,ds\right)\right)\right| \\
&\qquad + \lim_{t\to+\infty} \left| \phi^{-1}\left(\phi(C) + \int_t^{+\infty} F(u(\tau))\,d\tau\right) - C \right| = 0.
\end{aligned}$$

By the same technique, one can show that

$$\left| \frac{(Tu)^{(n-i)}(t)}{1+t^i} - \frac{C}{i!} \right| \to 0,$$

as $t \to +\infty$, $i = 2, 3, \dots, n$. So, by Lemma 2.6, the set TU is relatively compact.

Moreover, the set

$$\{u \in X : \ u = \lambda Tu, \ \lambda \in (0,1)\}$$

is bounded, as

$$\|\lambda Tu\|_X \leq |A_0| + |A_1| + |A_2| + \cdots + |A_{n-2}| + |B| + (|a|+1)L_\rho < +\infty, \ \forall \lambda \in (0,1).$$

By Theorem 2.7, T has at least one fixed point $u \in X$ such that

$$\overline{\alpha}(t) \leq u(t) \leq \overline{\beta}(t), \alpha'(t) \leq u'(t) \leq \beta'(t), \ldots, \alpha^{(n-1)}(t) \leq u^{(n-1)}(t) \leq \beta^{(n-1)}(t),$$
$$-2R_1 < u^{(n)}(t) < 2R_1, \quad \forall t \in [0, +\infty).$$ □

Remark 3.3. *If* $n = 1$, *problem* (1.1)-(1.2) *is written as*

$$\begin{cases} (\phi(u'))'(t) = f(t, u(t), u'(t)), \ a.e. \ t \in [0, +\infty), \\ u(0) + au'(0) = B, \\ u'(+\infty) = C. \end{cases}$$

In this case, we cannot consider the functions $\overline{\alpha}$ *and* $\overline{\beta}$. *Moreover, in Theorem 3.1, we do not need to suppose the condition* (3.2) *and the upper and lower solutions are automatically ordered.*

4 Example

Consider the $(n+1)$st order differential equation for a fixed $n \in \mathbb{N} \setminus \{0, 1\}$

$$((u^{(n)})^3)'(t) = f(t, u(t), u'(t), \ldots, u^{(n)}(t)), \ \text{a.e.,} \ \ t \geq 0, \tag{4.1}$$

with the boundary conditions

$$\begin{cases} u(0) = 2, \\ u^{(i)}(0) = 0, \ i = 1, \ldots, n-2, \\ u^{(n-1)}(0) - \dfrac{1}{(n+1)!}u^{(n)}(0) = \dfrac{1}{3}, \\ u^{(n)}(+\infty) = \dfrac{n!}{2}, \end{cases} \tag{4.2}$$

where

$$f(t, x_0, x_1, \ldots, x_n) = \frac{\Big|x_{n-1} - n!t - (n-1)!\Big|(-x_0 + 2) - (x_1 + x_2 + \cdots + x_{n-2})\Big|x_n - n!\Big|}{(1+t^3)(2+t^n)^2}. \tag{4.3}$$

Moreover, the functions $\alpha(t) \equiv 2$ and $\beta(t) = t^n + t^{n-1} + \cdots + t + 1$ are respectively, non-ordered lower and upper solutions for (4.1)-(4.2), with $\overline{\alpha}(t) = 2$ and $\overline{\beta}(t) = t^n + t^{n-1} + \cdots + t + 2$. As,

$\alpha^{(i)} \leq 0 \leq \beta^{(i)}(t)$, $i = 1, \dots, n-2$, $0 = \alpha^{(n)}(+\infty) < \frac{n!}{2} < \beta^{(n)}(+\infty) = n!$ and

$$\beta^{(n-1)}(0) - \frac{\beta^{(n)}(0)}{(n+1)!} = (n-1)! - \frac{n!}{(n+1)!} \geq \frac{1}{3} \geq 0 = \alpha^{(n-1)}(0) - \frac{\alpha^{(n)}(0)}{(n+1)!}.$$

Also, $\beta^{(n-1)}(t) = n!t + (n-1)!$, $\beta^{(n)}(t) = n!$, $((\alpha^{(n)})^3)'(t) = 0$ and $((\beta^{(n)})^3)'(t) = 0$, then, $f(t, \overline{\alpha}(t), \alpha'(t), \dots, \alpha^{(n)}(t)) = 0$ and $f(t, \overline{\beta}(t), \beta'(t), \dots, \beta^{(n)}(t)) = 0$ for all $t \geq 0$. The nonlinearity f satisfies the one-sided Nagumo condition (2.8) with

$$\psi(t) = \frac{1}{1+t^3}, \quad 1 < \nu < 3, \quad h(w) = 2$$

on the set

$$\begin{aligned} E_0 = \Big\{ & (t, x_0, x_1, \dots, x_n) \in [0, +\infty) \times \mathbb{R}^{n+1} : \overline{\alpha}(t) \leq x_0 \leq \overline{\beta}(t), \alpha'(t) \leq x_1 \leq \beta'(t), \\ & \alpha''(t) \leq x_2 \leq \beta''(t), \dots, \alpha^{n-1}(t) \leq x_{n-1} \leq \beta^{(n-1)}(t) \Big\}, \end{aligned}$$

and satisfies the assumptions of Theorem 3.1.

Therefore, there is at least a nontrivial solution u of (4.1)-(4.2), and $R > 0$, such that

$$\begin{gathered} \overline{\alpha}(t) \leq u(t) \leq \overline{\beta}(t), \alpha'(t) \leq u'(t) \leq \beta'(t), \dots, \alpha^{(n-1)}(t) \leq u^{(n-1)}(t) \leq \beta^{(n-1)}(t), \\ -R < u^{(n)}(t) < R, \quad \forall t \in [0, +\infty). \end{gathered}$$

From this, we see that u is a nonnegative function and its derivatives $u^{(i)}$ are nonnegative for $i \in \{1, \dots, n-1\}$ and nondecreasing for $i \in \{1, \dots, n-2\}$.

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References

[1] R. P. Agarwal and D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*. Glasgow, Scotland: Kluwer Academic Publisher, 2001.

[2] S. E. Ariaku, E. C. Mbah, C. C. Asogwa and P. U. Nwokoro, "Lower and upper solutions of first order non-linear ordinary differential equations", *IJSES*, vol. 3, no. 11, pp. 59–61, 2019.

[3] C. Bai and C. Li, "Unbounded upper and lower solution method for third order boundary value problems on the half line", *Electron. J. Differential Equations*, vol. 2009, no. 119, pp. 1–12, 2009.

[4] A. Cabada, J. A. Cid, and L. Sanchez, "Positivity and lower and upper solutions for fourth order boundary value problems", *Nonlinear Anal.*, vol. 67, no. 5, pp. 1599–1612, 2007.

[5] A. Cabada and N. Dimitrov, "Existence of solutions of nth-order nonlinear difference equations with general boundary conditions", *Acta Math. Sci. Ser. B (Engl. Ed.)*, vol. 40, no. 1, pp. 226–236, 2020.

[6] A. Cabada and L. Saavedra, "Existence of solutions for n^{th}-order nonlinear differential boundary value problems by means of fixed point theorems", *Nonlinear Anal. Real World Appl.*, vol. 42, pp. 180–206, 2018.

[7] H. Carrasco and F. Minhós, "Existence of solutions to infinite elastic beam equations with unbounded nonlinearities", *Electron. J. Differential Equations*, vol. 2017, no. 192, pp. 1–11, 2017.

[8] J. R. Graef, L. Kong and F. Minhós, "Higher order boundary value problems with $\phi-$Laplacian and functional boundary conditions", *Comput. Math. Appl.*, vol. 61, no. 2, pp. 236–249, 2011.

[9] M. R. Grossinho, F. Minhós and A. I. Santos, "A note on a class of problems for a higher-order fully nonlinear equation under one-sided Nagumo-type condition", *Nonlinear Anal.*, vol. 70, no. 11, pp. 4027–4038, 2009.

[10] R. Koplatadze, G. Kvinikadze and I. P. Stavroulakis, "Properties A and B of nth order linear differential equations with deviating argument", *Georgian Math. J.*, vol. 6, no. 6, pp. 553–566, 1999.

[11] H. Lian and J. Zhao, "Existence of unbounded solutions for a third-order boundary value problem on infinite intervals", *Discrete Dyn. Nat. Soc.*, vol. 2012, 2012.

[12] F. Minhós and H. Carrasco, "Solvability of higher-order BVPs in the half-line with unbounded nonlinearities", *Discrete Contin. Dyn. Syst.*, vol. 2015, pp. 841–850, 2015.

[13] D. R. Smart; *Fixed point theorems*. London-New York, England-USA: Cambridge University Press, 1974.

[14] A. Zerki, K. Bachouche and K. Ait-Mahiout, "Existence solutions for third order $\phi-$Laplacian bvps on the half-line", *Mediterr. J. Math.*, vol. 19, no. 6, Art. ID 261, 2022.

[15] Q. Zhang, D. Jiang, S. Weng and H. Gao, "Upper and lower solutions for a second-order three-point singular boundary-value problem", *Electron. J. Differential Equations*, vol. 2009, Art. ID 115, 2009.

[16] X. Zhang and L. Liu, "Positive solutions of fourth-order four-point boundary value problems with p-Laplacian operator", *J. Math. Anal. Appl.*, vol. 336, no. 2, pp. 1414–1423, 2007.