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On stability of nonlocal neutral stochastic integro differential equations with random impulses and Poisson jumps

Sahar M. A. Maqbol[1], ✉
R. S. Jain[1]
B. S. Reddy[1]

[1] *School of Mathematical Sciences, Swami Ramanand Teerth Marathwada University, Nanded-431606, India.*
saharmohmad465@gmail.com ✉

ABSTRACT

This article aims to examine the existence and Hyers-Ulam stability of non-local random impulsive neutral stochastic integrodifferential delayed equations with Poisson jumps. Initially, we prove the existence of mild solutions to the equations by using the Banach fixed point theorem. Then, we investigate stability via the continuous dependence of solutions on the initial value. Next, we study the Hyers-Ulam stability results under the Lipschitz condition on a bounded and closed interval. Finally, we give an illustrative example of our main result.

RESUMEN

Este artículo examina la existencia y estabilidad de Hyers-Ulam de ecuaciones integrodiferenciales con retardo no-locales aleatorias impulsivas neutrales estocásticas con saltos de Poisson. Inicialmente probamos la existencia de soluciones mild de las ecuaciones usando el teorema del punto fijo de Banach. Luego, investigamos la estabilidad a través de la dependencia continua de las soluciones respecto del valor inicial. A continuación, estudiamos resultados acerca de la estabilidad de Hyers-Ulam bajo la condición de Lipschitz en un intervalo cerrado y acotado. Finalmente, damos un ejemplo ilustrativo de nuestro resultado principal.

Keywords and Phrases: Existence of mild solutions, Hyers-Ulam (HU) stability, random impulsive, stochastic integro differential equations, time delays.

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1 Introduction

A model to represent the system with the occurrence of a sudden change in state at some time points is provided by impulsive differential equations. Differential equations (DEs) with fixed time impulses have been studied by many authors [7, 15, 22]. However, in the real world, impulses frequently occur at unpredictable times. Wu and Meng [21] introduced the generic DEs with random impulses, where the impulsive moments are random variables and any solution of the equations is a stochastic process, to better depict this phenomenon in reality. Examples of integer-order DEs with random impulses that have moderate solutions have been mentioned in [9, 18, 19]. The stochastic differential equations (SDEs) with random impulse involving fractional derivatives also have been studied in [10, 20, 24].

Poisson jumps are now a common modelling element in the fields of physics, biology, medicine, economics, and finance. A jump term must naturally be included in the SDEs. Furthermore, many real-world systems (such those that experience abrupt price changes or jumps as a result of stock market crashes, earthquakes, epidemics, etc.) could experience some jump-type stochastic disturbances. Since these system's sample pathways are not continuous, stochastic processes with jumps are a better fit for describing these models. These jump models typically come from Poisson random measurements. Such system's sample pathways (abbreviated c'adl'ag) are right continuous and have left limits. For more details, see the monographs [1, 23] and references therein.

On the other hand, impulsive differential equations also caught the interest of researchers see [2, 11, 12, 13]. Differential equations with fixed moments of impulses have become a natural framework for modeling processes in economics, physics, and population dynamics. The impulses usually exist at deterministic or random points. The properties of fixed-type random impulses are investigated in many articles [18, 19]. A. Anguraj *et al.* [4] established the existence and HU stability of random impulsive stochastic functional integrodifferential equations with finite delays. Moreover, Lang, Wenxuan, *et al.* [16] investigated the existence and HU stability of solutions for SDEs with random impulses. D. Chalishajar *et al.* [6] studied the existence, uniqueness, and stability of non-local random impulsive neutral stochastic differential equations with Poisson jumps. Recently, D. Baleanu, *et al.* [5] discussed the existence and stability results of mild solutions for random impulsive stochastic integro-differential equations (RISIDEs) with noncompact semigroups and resolvent operators in Hilbert spaces. R. Kasinathan *et al.* [14] investigated the existence and stability results of mild solutions for RISIDEs with noncompact semigroups via resolvent operators.

In A. Anguraj *et al.* [3] have been studied the existence and UH stability of SDEs with random impulse driven by Poisson jumps of the type

$$\begin{aligned} d(z(t)) &= f(t,z_t) + g(t,z_t)dW(t) + \int_{\mathfrak{U}} h(t,z_t,z)\tilde{K}(ds,dz), \quad t \geq t_0, \quad t \neq t_q, \\ z(\sigma_q) &= b_q(\delta_q)z(\sigma_q^-), \ q = 1,2,\ldots \\ z_{t_0} &= \sigma = \{\sigma(\theta) : -\delta \leq \theta \leq 0\}. \end{aligned}$$

Motivated by the above works, this paper aims to fill this gap by investigating the existence, stability and HU stability of non-local random impulsive neutral stochastic integrodifferential delayed equations (NRINSIDEs) and Poisson jumps.

The considered following NRINSIDEs with Poisson jumps of the type

$$\begin{aligned} d[z(t) + h(t,z_t)] = \left[f(t,z_t) + \int_0^t k(t,s,z_s)\,ds\right] dt + g(t,z_t)\,dW(t) \qquad (1.1) \\ + \int_{\mathfrak{U}} P(t,z_t,z)\tilde{K}(ds,dz), \quad t \geq t_0, \quad t \neq t_q, \end{aligned}$$

$$z(\sigma_q) = b_q(\delta_q)z(\sigma_q^-), \quad q = 1,2,\ldots, \qquad (1.2)$$

$$z_{t_0} + r(z) = z_0 = \sigma = \{\sigma(\theta) : -\delta \leq \theta \leq 0\}, \qquad (1.3)$$

where δ_q is a random variable defined from Ω to $\mathcal{D}_q \overset{def}{=} (0, d_q)$ for $q = 1,2,\ldots$, where $0 < d_q < \infty$. Moreover, suppose that $\delta_\imath$ and $\delta_\jmath$ are independent of each other as $\imath \neq \jmath$ for $\imath, \jmath = 1,2\ldots$ Here $f : [t_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, $h : [t_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, $g : [t_0, \mathcal{T}] \times \mathfrak{C} \times \to \mathbb{R}^{d\times m}$, $k : [t_0, \mathcal{T}] \times [t_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, $r : \mathfrak{C} \to \mathfrak{C}$ and $b_q : \mathcal{D}_q \to \mathbb{R}^{d\times d}$ are Borel measurable functions, and z_t is $\mathbb{R}^d$-valued stochastic process such that

$$z_t = \{z(t+\theta) : -\delta \leq \theta \leq 0\}, \quad z_t \in \mathbb{R}^d.$$

We assume that $\sigma_0 = t_0$ and $\sigma_q = \sigma_{q-1} + \delta_q$ for $q = 1,2,\ldots$ Obviously, $\{\sigma_q\}$ is a process with independent increments. The impulsive moments σ_q from a strictly increasing sequence, *i.e.* $\sigma = \sigma_0 < \sigma_1 < \sigma_2 < \cdots < \lim_{k\to\infty} \sigma_q = \infty$, and $z(\sigma_q^-) = \lim_{t\to\sigma_q - 0} z(t)$. Denote by $\{\mathbb{G}(t), t \geq 0\}$ the simple counting process generated by $\{\sigma_q\}$, and $\{\mathbb{K}(t), t \geq 0\}$ is a given m-dimensional Wiener process, and denote $\mathfrak{F}_t^{(1)}$ the σ-algebra generated by $\{\mathbb{G}_t, t \geq 0\}$, and denote $\mathfrak{F}_t^{(2)}$ the σ-algebra generated by $\{\mathbb{K}_t, t \geq 0\}$. We assume that $\mathfrak{F}_\infty^{(2)}, \mathfrak{F}_\infty^{(2)}$ and σ are mutually independent. In (1.1)-(1.3), $\tilde{K}(dt,dz) = K(dt,dz) - dt\,\nu(du)$ denotes the compensated Poisson measure independent of $W(t)$ and $\tilde{K}(dt,dz)$ represents the Poisson counting measure associated with a characteristic measure ν.

Highlights:

(1) This work extends the work of A. Anguraj *et al.* [3].

(2) Time delay of NRINSIDEs and Poisson jumps is taken care of by the prescribed phase space $\mathcal{B}$.

The structure of this article is as follows: In section 2, we mention some concepts and principles. Section 3 is devoted to studying the existence of mild solutions of the system (1.1)-(1.3). In section 4, the stability of the mild solution of the equations (1.1)-(1.3) is studied. In section 5, we investigate the HU stability of the system (1.1)-(1.3). An example is given to illustrate the theory in section 6. At the end, the last section deals with the conclusion and acknowledgement.

2 Preliminaries

Suppose that $(\Omega, \mathfrak{F}_t, \mathcal{P})$ is a probability space with filtration $\{\mathfrak{F}_t\},\ t \geq 0$ fulfilling $\mathfrak{F}_t = \mathfrak{F}_t^{(1)} \cup \mathfrak{F}_t^{(2)}$. Let $\pounds^p = (\Omega, \mathbb{R}^d)$ be the collection of all strongly measurable, p^{th} integrable, $\mathfrak{F}_t$ measurable, $\mathbb{R}^d$-random variables in z with the norm $\|z\|_{\pounds_p} = (\mathbb{E}\|z\|_t^p)^{1/p}$. Let $\delta > 0$ and denote the Banach space of all piecewise continuous $\mathbb{R}^d$-valued stochastic process $\{\sigma(t),\ t \in [-\delta, 0]\}$ by $\mathfrak{C}([-\delta, 0], \pounds(\Omega, \mathbb{R}^d))$ equipped with the norm

$$\|\psi\|_{\mathfrak{C}} = \left(\sup_{-\delta \leq \theta \leq 0} \mathbb{E}\|\psi(\theta)\|_t^p\right)^{1/p}.$$

The initial data

$$z_{t_0} + r(z) = z_0 = \sigma = \{\sigma(\theta) : -\delta \leq \theta \leq 0\}, \tag{2.1}$$

is an $\mathfrak{F}_{t_0}$ measurable, $[-\delta, 0]$ to $\mathbb{R}^d$-valued random variable such that $\mathbb{E}\|\sigma\|^p < \infty$.

2.1 Poisson jump process

Let $(p(t))_{t\geq 0}$ be an $\mathcal{H}$-valued, σ-finite stationary $\mathfrak{F}_t$-adapted Poisson point process on $(\Omega, \mathfrak{F}, (\mathfrak{F}_t), \mathcal{P})$. The counting random measure K defined by

$$K((t_1, t_2] \times \mathfrak{U})(w) = \sum_{t_1 < s \leq t_2} I_{\mathfrak{U}}(p(s)(w)),$$

for any $\mathfrak{U} \in \mathcal{B}_\sigma(\mathcal{H})$ is called the Poisson random measure associated to the Poisson point process p. This measure ν is said to be a Levy measure. Then the measure $\hat{K}$ is defined by

$$\hat{K}((0, t] \times \mathfrak{U}) = K((0, t] \times \mathfrak{U}) - t\nu(\mathfrak{U}).$$

This measure $\hat{K}(dt,du)$ is called the compensated Poisson random measure, and $dt\,\nu(\mathfrak{U})$ is called the compensator.

Definition 2.1. *For a given $\mathcal{T} \in (t_0,\infty)$, a $\mathbb{R}^d$-valued stochastic process $z(t)$ on $t_0-\delta \le t \le \mathcal{T}$ is called the solution to equation (1.1)-(1.3) with the initial data (2.1), if for each $t_0 \le t \le \mathcal{T}$, $z_{t_0} = \sigma$, $\{z_{t_0}\}_{t_0\le t\le \mathcal{T}}$ is $\mathfrak{F}_t$-adapted and*

$$\begin{aligned} z(t) \;=\; & \sum_{q=0}^{\infty}\Bigg[\prod_{\imath=1}^{q} b_\imath(\delta_\imath)\sigma(0) - r(z) + h(0,\sigma) - \prod_{\imath=1}^{q} b_\imath(\delta_\imath)h(t,z_t) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath} f(s,z_s)\,ds + \int_{\sigma_q}^{t} f(s,z_s)\,ds \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_0^s k(s,\varsigma,z_\varsigma)\,d\varsigma ds + \int_{\sigma_q}^{t}\int_0^s k(s,\varsigma,z_\varsigma)\,d\varsigma ds \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath} g(s,z_s)\,dW(s) + \int_{\sigma_q}^{t} g(s,z_s)\,dW(s) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_{\mathfrak{U}} P(s,z_s,u)\tilde{K}(ds,du) + \int_{\sigma_q}^{t}\int_{\mathfrak{U}} P(s,z_s,u)\tilde{K}(ds,du)\Bigg] I_{[\sigma_q,\sigma_{q+1})}(t) \end{aligned}$$

where $\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) = b_q(\delta_q)b_{q-1}(\delta_{q-1})\cdots b_\imath(\delta_\imath)$, and $I_{\mathcal{L}}(.)$ is the index function, i.e.,

$$I_{\mathcal{L}}(t) = \begin{cases} 1 & if \quad t \in \mathcal{L}, \\ 0 & if \quad t \notin \mathcal{L}. \end{cases}$$

Definition 2.2 (HU stability)**.** *Suppose that $w(t)$ is a $\mathbb{R}^d$-valued stochastic process. If there exists a real number $N > 0$, such that for arbitrary $\epsilon \ge 0$, satisfying*

$$\begin{aligned} \mathbb{E}\Bigg\| w(t) - \sum_{q=0}^{\infty}\Bigg[&\prod_{\imath=1}^{q} b_\imath(\delta_\imath)\sigma(0) - r(z) + h(0,\sigma) - \prod_{\imath=1}^{q} b_\imath(\delta_\imath)h(t,z_t) + \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath} f(s,z_s)\,ds \\ &+ \int_{\sigma_q}^{t} f(s,z_s)\,ds + \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_0^s k(s,\varsigma,z_\varsigma)\,d\varsigma\,ds + \int_{\sigma_q}^{\zeta}\int_0^s k(s,\varsigma,z_\varsigma)\,d\varsigma\,ds \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath} g(s,z_s)\,dW(s) + \int_{\sigma_q}^{t} g(s,z_s)\,dW(s) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_{\mathfrak{U}} P(s,z_s,u)\tilde{K}(ds,du) + \int_{\sigma_q}^{t}\int_{\mathfrak{U}} P(s,z_s,u)\tilde{K}(ds,du)\Bigg] I_{[\sigma_q,\sigma_{q+1})}(t)\Bigg\|^p \le \epsilon. \end{aligned}$$

For each solution $z(t)$ with the initial value $z_{t_0} = w_{t_0} = \sigma$, if there exists a solution $z(t)$ of equations (1.1)-(1.3) with

$$\mathbb{E}\|w(t) - z(t)\| \le N\epsilon, \quad \forall\; t\; (t_0-\tau,\mathcal{T}).$$

Then equation (1.1)-(1.3) has the HU stability.

Lemma 2.3 ([8]). *Let $\vartheta, \psi \in \mathfrak{C}([a,b], \mathbb{R}^d)$ be two functions. We suppose that $\vartheta(t)$ is nondecreasing. If $z(t) \in \mathfrak{C}([a,b], \mathbb{R}^d)$ is a solution of the following inequality*

$$z(t) \leq \vartheta(t) + \int_a^t \psi(s) z(s)\, ds, \quad t \in [a,b],$$

then $z(t) \leq \vartheta(t) \exp\left(\int_a^t \psi(s)\, ds\right)$.

Lemma 2.4 ([17]). *For any $p \geq 1$ and for any predictable process $z \in \pounds^p_{d\times m} = [0, \mathcal{T}]$ the inequality holds,*

$$\sup \mathbb{E}\|z(t)\, dW(t)\|^p \leq (p/2(p-1))^{p/2} \left(\int_0^t \mathbb{E}\|z(s)\|^p)^{2/p} ds\right)^{p/2}, \quad t \in [0, \mathcal{T}].$$

3 Main results

In order to derive the existence and uniqueness of the system (1.1)-(1.3), we shall impose the following assumptions:

(A1): The functions $h : [t_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, $f : [t_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, and $g : [t_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^{d\times m}$. There exist positive constant $L_h > 0$, $L_f > 0$ and $L_g > 0$ such that,

$$\begin{aligned}
\mathbb{E}\|h(t,\psi_1) - h(t,\psi_2)\|^p &\leq L_h \mathbb{E}\|\psi_1 - \psi_2\|^p_{\mathfrak{C}},\\
\mathbb{E}\|h(t,\psi)\|^p &\leq L_h \mathbb{E}\|\psi\|^p_{\mathfrak{C}}.\\
\mathbb{E}\|f(t,\psi_1) - f(t,\psi_2)\|^p &\leq L_f \mathbb{E}\|\psi_1 - \psi_2\|^p_{\mathfrak{C}},\\
\mathbb{E}\|f(t,\psi)\|^p &\leq L_f \mathbb{E}\|\psi\|^p_{\mathfrak{C}}.\\
\mathbb{E}\|g(t,\psi_1) - g(\zeta,\psi_2)\|^p &\leq L_g \mathbb{E}\|\psi_1 - \psi_2\|^p_{\mathfrak{C}},\\
\mathbb{E}\|g(t,\psi)\|^p &\leq L_g \mathbb{E}\|\psi\|^p_{\mathfrak{C}},
\end{aligned}$$

for all $t \in [t_0, \mathcal{T}]$ and ψ_1, ψ_2 and $\psi \in \mathfrak{C}$.

(A2): The function $k : [t_0, \mathcal{T}] \times [t_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, there exists a positive constant $L_k > 0$ such that,

$$\begin{aligned}
\int_0^t \mathbb{E}\|k(t,s,\psi_1) - k(t,s,\psi_2)\|^p &\leq L_k \mathbb{E}\|\psi_1 - \psi_2\|^p_{\mathfrak{C}},\\
\int_0^t \mathbb{E}\|k(t,s,\psi)\|^p &\leq L_k \mathbb{E}\|\psi\|^p_{\mathfrak{C}},
\end{aligned}$$

for all $t \in [t_0, \mathcal{T}]$ and ψ_1, ψ_2 and $\psi \in \mathfrak{C}$.

(A3): The condition $\max_{\imath,q}\left\{\prod_{\jmath=\imath}^{q}\|b_\jmath(\tau_\jmath)\|\right\}<\infty$. That is to say, there exists a constant $C>0$ such that

$$\mathbb{E}\left(\max_{\imath,q}\left\{\prod_{\jmath=\imath}^{q}\|b_\imath(\tau_\jmath)\|\right\}\right)^p\leq C.$$

(A4): The function $P:[t_0,\mathcal{T}]\times\mathfrak{C}\times\mathfrak{U}\to\mathbb{R}^d$, there exists a positive constant $L_P>0$ such that,

$$\int_{\mathfrak{U}}\mathbb{E}\|P(t,\psi_1,u)-P(t,\psi_2,u)\|^p\nu\,dz\leq L_P\mathbb{E}\|\psi_1-\psi_2\|^p_{\mathfrak{C}},$$
$$\int_{\mathfrak{U}}\mathbb{E}\|P(t,s,\psi)\|^p\nu\,dz\leq L_P\mathbb{E}\|\psi\|^p_{\mathfrak{C}},$$

for all $t\in[t_0,\mathcal{T}]$ and ψ_1,ψ_2 and $\psi\in\mathfrak{C}$.

(A5): The function $r:\mathfrak{C}\to\mathfrak{C}$ is continuous and there exists some constant $L_r>0$ such that,

$$\mathbb{E}\|r(t,\psi_1)-r(t,\psi_2)\|^p\leq L_r\mathbb{E}\|\psi_1-\psi_2\|^p_{\mathfrak{C}},$$
$$\mathbb{E}\|r(t,\psi)\|^p\leq L_r\mathbb{E}\|\psi\|^p_{\mathfrak{C}},$$

for all $t\in[t_0,\mathcal{T}]$ and ψ_1,ψ_2 and $\psi\in\mathfrak{C}$.

Theorem 3.1. *Assume that the assumptions (A1)–(A5) are satisfied. Then the system* (1.1)-(1.3) *has a unique solution in* $\mathcal{B}$.

Proof. Let $\mathcal{B}$ be the phase space $\mathcal{B}=\mathfrak{C}([t_0-\delta,\mathcal{T}],\mathcal{L}^p(\Omega,\mathbb{R}^d))$ endowed with the norm

$$\|z\|^p_{\mathcal{B}}=\sup_{t\in[t_0,\mathcal{T}]}\|z_t\|^p_{\mathfrak{C}},$$

where $\|z_t\|_{\mathfrak{C}}=\sup_{-\delta\leq s\leq t}\mathbb{E}\|z_t\|^p$. Denote $B_m=\{z\in\mathcal{B},\|z\|^p_{\mathcal{B}}\leq m\}$, which is the closed ball with center z and radius $m>0$. For any initial value $(t_0,z_0,)$ with $t_0\geq 0$ and $z_0\in B_m$, we define the operator $S:\mathcal{B}\to\mathcal{B}$ by

$$(Sz)(t)=\begin{cases}\sigma(t)-r(t), \qquad t\in(\infty,t_0]\\ \sum_{q=0}^{\infty}\Big[\prod_{\imath=1}^{q}b_\imath(\delta_\imath)\sigma(0)-r(t)+h(0,\sigma)-\prod_{\imath=1}^{q}b_\imath[(\delta_\imath)h(t,z_t)\\ +\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\tau_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}f(s,z_s)ds+\int_{\sigma_q}^{t}f(s,z_s)ds+\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_0^s k(s,\varsigma,z_\varsigma)d\varsigma ds\\ +\int_{\sigma_q}^{t}\int_0^s k(s,\varsigma,z_\varsigma)d\varsigma+\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}g(s,z_s)dW(s)+\int_{\sigma_q}^{t}g(s,z_s)dW(s)\\ +\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_{\mathfrak{U}}P(s,z_s,u)\tilde{K}(ds,du)\\ +\int_{\sigma_q}^{t}\int_{\mathfrak{U}}P(s,z_s,u)\tilde{K}(ds,du)\Big]I_{[\sigma_q,\sigma_{q+1})}(t), \qquad t\in[t_0,\mathcal{T}].\end{cases}$$

Now we have to prove that S maps $\mathcal{B}$ into itself.

$$\begin{aligned}\|(Sz)(t)\|^p = \Big\| \sum_{q=0}^{\infty} \Big[\prod_{\imath=1}^{q} b_\imath(\delta_\imath)[\sigma(0) - r(t) + h(0,\sigma)] - \prod_{\imath=1}^{q} b_\imath(\delta_\imath) h(t, z_t) \\ + \Big[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} f(s, z_s)\, ds + \int_{\sigma_q}^{\sigma} f(s, z_s)\, ds \Big] \\ + \Big[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \int_0^s k(s, \varsigma, z_\varsigma)\, d\varsigma\, ds + \int_{\sigma_q}^{\sigma} \int_0^s k(s, \varsigma, z_\varsigma)\, d\varsigma \Big] \\ + \Big[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} g(s, z_s)\, dW(s) + \int_{\sigma_q}^{\sigma} g(s, z_s)\, dW(s) \Big] \\ + \Big[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \int_{\mathfrak{U}} P(s, z_s, u) \tilde{K}(ds, du) + \int_{\sigma_q}^{t} \int_{\mathfrak{U}} P(s, z_s, u) \tilde{K}(ds, du) \Big] \Big] I_{[\sigma_q, \sigma_{q+1})}(t) \Big\|^p\end{aligned}$$

$$\begin{aligned}\mathbb{E}\|Sz(t)\|^p &\le 4^{p-1} \mathbb{E}\Big[\max_{\imath,q} \Big\{ \prod_{\imath=\jmath}^{q} \|b_\jmath(\delta_\jmath)\| \Big\} \Big]^p [\|\sigma(0) - r(z) + h(0,\sigma)\|^p] \\ &+ 4^{p-1} \mathbb{E}\Big[\max_{\imath,q} \Big\{ \prod_{\imath=\jmath}^{q} \|b_\jmath(\delta_\jmath)\| \Big\} \Big]^p \|h(t, z_t)\|^p \\ &+ 4^{p-1} \mathbb{E}\Big[\max_{\imath,q} \Big\{ 1, \prod_{\imath=\jmath}^{q} \|b_\jmath(\delta_\jmath)\| \Big\} \Big]^p \Big[\int_{t_0}^{t} \|f(s, z_s)\|\, ds\, I_{[\sigma_q, \sigma_{q+1})}(t) \Big]^p \\ &+ 4^{p-1} \mathbb{E}\Big[\max_{\imath,q} \Big\{ 1, \prod_{\imath=\jmath}^{q} \|b_\jmath(\delta_\jmath)\| \Big\} \Big]^p \Big[\int_{t_0}^{t} \int_0^s \|k(s, \varsigma, z_\varsigma)\, d\varsigma\, ds\, I_{[\sigma_q, \sigma_{q+1})}(t) \Big]^p \\ &+ 4^{p-1} \mathbb{E}\Big[\max_{\imath,q} \Big\{ 1, \prod_{\imath=\jmath}^{q} \|b_\jmath(\delta_\jmath)\| \Big\} \Big]^p \Big[\int_{t_0}^{t} \|g(s, z_s) dW(s)\|\, ds\, I_{[\sigma_q, \sigma_{q+1})}(t) \Big]^p \\ &+ 4^{p-1} \mathbb{E}\Big[\max_{\imath,q} \Big\{ 1, \prod_{\imath=\jmath}^{q} \|b_\jmath(\delta_\jmath)\| \Big\} \Big]^p \Big[\int_{t_0}^{t} \int_{\mathfrak{U}} \|P(s, z_s, u) \tilde{K}(ds, du)\|\, ds\, I_{[\sigma_q, \sigma_{q+1})}(t) \Big]^p \\ &\le 4^{p-1} C[\mathbb{E}\|\sigma(0)\|^p + L_r \mathbb{E}\|z\|^p] + 4^{p-1} C L_h \mathbb{E}\|\sigma\|^p + 4^{p-1} C L_h \mathbb{E}\|z_t\|^p_{\mathfrak{C}} \\ &+ 4^{p-1} \max\{1, C\}(t - t_0)^{p-1} L_f \int_{t_0}^{t} \mathbb{E}\|z_s\|^p_{\mathfrak{C}}\, ds + 4^{p-1} \max\{1, C\}(t - t_0)^{p-1} L_k \int_{t_0}^{t} \mathbb{E}\|z_s\|^p_{\mathfrak{C}}\, ds \\ &+ 4^{p-1} \max\{1, C\}(t - t_0)^{p/2-1} L_g L_p \int_{t_0}^{t} \mathbb{E}\|z_s\|^p_{\mathfrak{C}}\, ds + 4^{p-1} \max\{1, C\}(t - t_0)^{p/2} L_P c_p \int_{t_0}^{t} \mathbb{E}\|z_s\|^p_{\mathfrak{C}}\, ds.\end{aligned}$$

Thus

$$\begin{aligned}\sup_{s\in[t-\tau,t]} \mathbb{E}\|Sz(t)\|^p \le 4^{p-1} C[\mathbb{E}\|\sigma(0)\|^p + L_h \mathbb{E}\|\sigma\|^p] + \big\{ 4^{p-1} C(L_r + L_h)(t - t_0)^{-1} \\ + 4^{p-1} \max\{1, C\} \big[(t - t_0)^{p-1} L_f + (t - t_0)^{p-1} L_k + (t - t_0)^{p/2-1} L_g L_p \\ + (t - t_0)^{p-1} c_p (L_P + L_P^{P/2}) \big] \big\} (t - t_0) \sup_{s\in[t-\delta,t]} \mathbb{E}\|z_s\|^p_{\mathfrak{C}}.\end{aligned}$$

Therefore S maps $\mathcal{B}$ into itself.

Now, we have to prove that S is a contraction mapping.

$$\begin{aligned}
\mathbb{E}\|(Sz)(t)-(Sw)(t)\|^p \leq\ & 3^{p-1}\mathbb{E}\Big[\max_{\imath,q}\Big\{\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Big\}\Big]^p\Big[\|r(z)-r(w)\|I_{[\sigma_q,\sigma_{q+1})}(t)\Big]^p\\
&+3^{p-1}\mathbb{E}\Big[\max_{\imath,q}\Big\{\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Big\}\Big]^p\Big[\|h(t,z_t)-h(t,w_t)\|I_{[\sigma_q,\sigma_{q+1})}(t)\Big]^p\\
&+3^{p-1}\mathbb{E}\Big[\max_{\imath,q}\Big\{1,\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Big\}\Big]^p\Big[\int_{t_0}^{t}\|f(s,z_s)-f(s,w_s)\|dsI_{[\sigma_q,\sigma_{q+1})}(t)\Big]^p\\
&+3^{p-1}\mathbb{E}\Big[\max_{\imath,q}\Big\{1,\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Big\}\Big]^p\Big[\int_{t_0}^{t}\int_0^s\|k(s,\varsigma,z_\varsigma)-k(s,\varsigma,z_\varsigma)\|d\varsigma dsI_{[\sigma_q,\sigma_{q+1})}(t)\Big]^p\\
&+3^{p-1}\mathbb{E}\Big[\max_{\imath,q}\Big\{1,\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Big\}\Big]^p\Big[\int_{t_0}^{t}\|g(s,z_s)-g(s,w_s)\|dW(s)dsI_{[\sigma_q,\sigma_{q+1})}(t)\Big]^p\\
&+3^{p-1}\mathbb{E}\Big[\max_{\imath,q}\Big\{1,\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Big\}\Big]^p\Big[\int_{t_0}^{t}\int_{\mathfrak{U}}\|P(s,z_s,u)-P(s,z_s,u)\|\tilde{K}(ds,du)I_{[\sigma_q,\sigma_{q+1})}(t)\Big]^p\\
&\leq 3^{p-1}C\mathbb{E}\|r(z)-r(w)\|^p+3^{p-1}C\mathbb{E}\|h(t,z_t)-h(t,w_t)\|^p\\
&+3^{p-1}\max\{1,C\}(t-t_0)^pL_f\times\int_{t_0}^{t}\mathbb{E}\|f(s,z_s)-f(s,w_s)\|^pds\\
&+3^{p-1}\max\{1,C\}(t-t_0)^pL_k\int_{t_0}^{t}\int_0^s\mathbb{E}\|k(s,\varsigma,z_\varsigma)-k(s,\varsigma,w_\varsigma)\|^pd\varsigma ds\\
&+3^{p-1}\max\{1,C\}(t-t_0)^{p/2}L_pL_g\int_{t_0}^{t}\mathbb{E}\|g(s,z_s)-g(s,w_s)\|^pdW(s)\\
&+3^{p-1}\max\{1,C\}(t-t_0)^pc_pL_P\int_{t_0}^{t}\mathbb{E}\|P(s,z_s,u)-P(s,z_s,u)\|^pds\\
&\leq 3^{p-1}CL_r\mathbb{E}\|z-w\|_{\mathfrak{C}}^p+3^{p-1}CL_h\mathbb{E}\|z-w\|_{\mathfrak{C}}^p\\
&+3^{p-1}\max\{1,C\}(t-t_0)^pL_f\mathbb{E}\|z_s-w_s\|_{\mathfrak{C}}^pds\\
&+3^{p-1}\max\{1,C\}(t-t_0)^pL_k\mathbb{E}\|z_s-w_s\|_{\mathfrak{C}}^pds\\
&+3^{p-1}\max\{1,C\}(t-t_0)^{p/2}L_pL_g\mathbb{E}\|z_s-w_s\|_{\mathfrak{C}}^pds\\
&+3^{p-1}\max\{1,C\}(t-t_0)^pc_p(L_P+L_P^{p/2})\mathbb{E}\|z_s-w_s\|_{\mathfrak{C}}^pds\\
&\leq\big\{3^{p-1}C(L_r+L_h)+3^{p-1}\max\{1,C\}\big[\big(t-t_0)^pL_f+(t-t_0)^pL_k\\
&+(t-t_0)^{p/2}L_pL_g+(t-t_0)^pc_p(L_P+L_P^{p/2})\big]\big\}\sup_{\theta\in[-\delta,0]}\mathbb{E}\|z(t+\theta)-w(t+\theta)\|_{\mathfrak{C}}^p\\
&\leq\big\{3^{p-1}C(L_r+L_h)+3^{p-1}\max\{1,C\}\big[\big(t-t_0)^pL_f+(t-t_0)^pL_k\\
&+(t-t_0)^{p/2}L_pL_g+(t-t_0)^pc_p(L_P+L_P^{p/2})\big]\big\}\sup_{s\in[t-\delta,t]}\mathbb{E}\|z(s)-w(s)\|_{\mathfrak{C}}^p.
\end{aligned}$$

Taking the supremum over t, we get

$$\|(Sz)(t)-(Sw)(t)\|_{\mathcal{B}}^p\leq\mathfrak{A}(\mathcal{T})\mathbb{E}\|z-w\|_{\mathcal{B}}^p,$$

with

$$\mathfrak{A}(\mathcal{T})=3^{p-1}C(L_r+L_h)+3^{p-1}\max\{1,C\}\big[\big(t-t_0)^p(L_f+L_k+c_p(L_P+L_P^{p/2}))+(t-t_0)^{p/2}L_pL_g\big].$$

By taking a suitable $0 < \mathcal{T}_1 < \mathcal{T}$ sufficient small such that $\mathfrak{A}(\mathcal{T}) < 1$. Hence S is a contraction on $\mathcal{B}_{\mathcal{T}_1}$. $Sz = z$ is a unique solution of equation (1.1)-(1.3) by the Banach fixed point theorem. □

4 Stability

The stability through continuous dependence of solutions on the initial condition is investigated.

Definition 4.1 ([4]). *A mild solution $z(t)$ of the system* (1.1) *and* (1.2) *with initial condition σ satisfying* (2.1) *is said to be stable in the mean square if for all $\epsilon > 0$, there exist, $\beta > 0$ such that,*

$$\mathbb{E}\|z - w\|_t^p \leq \epsilon, \quad \textit{whenever},$$
$$\mathbb{E}\|\sigma_1 - \sigma_2\|^p < \beta, \quad \textit{for all} \quad t \in [t_0, T],$$

where $w(t)$ is another mild solution of the system (1.1) *and* (1.2) *with initial value σ defined in* (1.3)*.*

Theorem 4.2. *Let $z(t)$ and $w(t)$ be mild solutions of the system* (1.1)-(1.3) *with initial values σ_1 and σ_2 respectively. If the hypotheses of theorem 3.1 are fulfilled, the mean solution of the system* (1.1)- (1.3) *is stable in the mean square.*

Proof. Under assumptions, $z(t)$ and $w(t)$ be two mild solutions of the system (1.1)-(1.3) with initial values σ_1 and σ_2 respectively.

$$\begin{aligned} z(t) - w(t) = \sum_{q=0}^{\infty} \Bigg[&\prod_{\imath=1}^{q} b_\imath(\delta_\imath)[\sigma_1 - \sigma_2] + [r(z) - r(w)] + [h(0,\sigma_1) - h(0,\sigma_2)] + \prod_{\imath=1}^{q} b_\imath(\delta_\imath)[h(t,z_t) - h(t,w_t)] \\ &+ \Bigg[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} [f(s,z_s) - f(s,w_s)]\, ds + \int_{\sigma_q}^{t} [f(s,z_s) - f(s,w_s)]\, ds \Bigg] \\ &+ \Bigg[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \int_0^s [k(s,\varsigma,z_\varsigma) - k(s,\varsigma,w_\varsigma)]\, d\varsigma\, ds + \int_{\sigma_q}^{t} \int_0^s [k(s,\varsigma,z_\varsigma) - k(s,\varsigma,w_\varsigma)]\, d\varsigma \Bigg] \\ &+ \Bigg[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} [g(s,z_s) - g(s,w_s)]\, dW(s) + \int_{\sigma_q}^{t} [g(s,z_s) - g(s,w_s)]\, dW(s) \Bigg] \\ &+ \Bigg[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \int_{\mathfrak{U}} [P(s,z_s,u) - P(s,w_s,u)] \tilde{K}(ds,du) \\ &+ \int_{\sigma_q}^{t} \int_{\mathfrak{U}} [P(s,z_s,u) - P(s,w_s,u)] \tilde{K}(ds,du) \Bigg] \Bigg] I_{[\sigma_q,\sigma_{q+1})}(t). \end{aligned}$$

Then,

$$\begin{aligned}
\mathbb{E}\|z(t)-w(t)\|^p \leq{} & 4^{p-1}C(1+L_h)\mathbb{E}\|\sigma_1-\sigma_2\|^p + 4^{p-1}CL_r\mathbb{E}\|z-w\|^p + 4^{p-1}C^pL_h\mathbb{E}\|z(t)-w(t)\|^p \\
& + 4^{p-1}\max\{1,C\}(t-t_0)^{p-1}L_f\int_{t_0}^t \mathbb{E}\|z(s)-w(s)\|^p ds \\
& + 4^{p-1}\max\{1,C\}(t-t_0)^{p-1}L_k\int_{t_0}^t \mathbb{E}\|z(s)-w(s)\|^p ds \\
& + 4^{p-1}\max\{1,C\}(t-t_0)^{p/2-1}L_gL_p\int_{t_0}^t \mathbb{E}\|z(s)-w(s)\|^p ds \\
& + 4^{p-1}\max\{1,C\}(t-t_0)^{p/2}L_Pc_p\int_{t_0}^t \mathbb{E}\|z(s)-w(s)\|^p ds.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\sup_{s\in[t-\tau,t]}\mathbb{E}\|z(t)-w(t)\|^p \leq{} & 4^{p-1}C(1+L_h)\mathbb{E}\|\sigma_1-\sigma_2\|^p + 4^{p-1}C(L_r+L_h)\sup_{t\in[t-\tau,t]}\mathbb{E}\|z(t)-w(t)\|^p \\
& + 4^{p-1}\max\{1,C\}(t-t_0)^{p-1}L_f\int_{t_0}^t \sup_{s\in[t-\tau,t]}\mathbb{E}\|z(s)-w(s)\|^p ds \\
& + 4^{p-1}\max\{1,C\}(t-t_0)^{p-1}L_k\int_{t_0}^t \sup_{s\in[t-\tau,t]}\mathbb{E}\|z(s)-w(s)\|^p ds \\
& + 4^{p-1}\max\{1,C\}(t-t_0)^{p/2-1}L_gL_p\int_{t_0}^t \sup_{s\in[t-\tau,t]}\mathbb{E}\|z(s)-w(s)\|^p ds \\
& + 4^{p-1}\max\{1,C\}(t-t_0)^{p-1}L_Pc_p\int_{t_0}^t \sup_{s\in[t-\tau,t]}\mathbb{E}\|z(s)-w(s)\|^p ds.
\end{aligned}$$

Thus,

$$\sup_{s\in[t-\tau,t]}\mathbb{E}\|z(t)-w(t)\|^p \leq \gamma\mathbb{E}\|\sigma_1-\sigma_2\|^p,$$

where,

$$\gamma = \frac{4^{p-1}C(1+L_h)}{1-\left[4^{p-1}C(L_r+L_h)+4^{p-1}\max\{1,C\}(t-t_0)^p[(L_f+L_k)+(t-t_0)^{-p/2}L_gL_p+c_p(L_P+L_P^{p/2})]\right]}.$$

Given $\epsilon>0$, choose $\beta=\frac{\epsilon}{\gamma}$ such that $\mathbb{E}\|\sigma_1-\sigma_2\|^p<\beta$. Then,

$$\|z-w\|_{\mathcal{B}}^p \leq \epsilon.$$

This completes the proof. □

5 HU stability

In this section, we investigate the HU stability of equations (1.1)-(1.3) under the assumptions (A1)-(A5). We have the following HU stability theorem.

Theorem 5.1. *Under the assumptions (A1)-(A5). Then equations* (1.1)-(1.3) *has the HU stability.*

Proof.

$$\begin{aligned} z(t) = \sum_{q=0}^{\infty} \Big[&\prod_{\imath=1}^{q} b_\imath(\delta_\imath)\sigma(0) - r(z) + h(0,\sigma) - \prod_{\imath=1}^{q} b_\imath(\tau_\imath) h(t, z_t) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} f(s, z_s)\, ds + \int_{\sigma_q}^{t} f(s, z_s)\, ds \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_0^s k(s,\varsigma, z_\varsigma)\, d\varsigma\, ds + \int_{\sigma_q}^{\zeta}\int_0^s k(s,\varsigma, z_\varsigma)\, d\varsigma\, ds \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} g(s, z_s)\, dW(s) + \int_{\sigma_q}^{t} g(s, z_s)\, dW(s) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_{\mathfrak{U}} P(s, z_s, u)\tilde{K}(ds, du) + \int_{\sigma_q}^{t}\int_{\mathfrak{U}} P(s, z_s, u)\tilde{K}(ds, du)\Big] I_{[\sigma_q,\sigma_{q+1})}(\zeta). \end{aligned}$$

It follows from the condition that

$$\begin{aligned} \mathbb{E}\Big\| w(s) - \sum_{q=0}^{\infty} \Big[&\prod_{\imath=1}^{q} b_\imath(\delta_\imath)\sigma(0) - r(z) + h(0,\sigma) - \prod_{\imath=1}^{q} b_\imath(\tau_\imath) h(t, z_t) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} f(s, z_s)\, ds + \int_{\sigma_q}^{t} f(s, z_s)\, ds \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_0^s k(s,\varsigma, z_\varsigma)\, d\varsigma\, ds + \int_{\sigma_q}^{t}\int_0^s k(s,\varsigma, z_\varsigma)\, d\varsigma\, ds \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} g(s, z_s)\, dW(s) + \int_{\sigma_q}^{t} g(s, z_s)\, dW(s) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_{\mathfrak{U}} P(s, z_s, u)\tilde{K}(ds, du) + \int_{\sigma_q}^{t}\int_{\mathfrak{U}} P(s, z_s, u)\tilde{K}(ds, du)\Big] I_{[\sigma_q,\sigma_{q+1})}(t)\Big\|^p \leq \epsilon. \end{aligned}$$

When $t \in [t_0 - \delta, t_0]$, we get $\mathbb{E}\|w(t) - z(t)\|^p = 0$. And when $t \in [0, \mathcal{T}]$, we get

$$\begin{aligned} \mathbb{E}\|w(t) - z(t)\|^p \leq 2^{p-1}\mathbb{E}\Big\| w(s) - \sum_{q=0}^{\infty} \Big[&\prod_{\imath=1}^{q} b_\imath(\delta_\imath)\sigma(0) - r(z) + h(0,\sigma) - \prod_{\imath=1}^{q} b_\imath(\delta_\imath) h(t, z_t) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\tau_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} f(s, z_s)\, ds + \int_{\sigma_q}^{t} f(s, z_s)\, ds + \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_0^s k(s,\varsigma, z_\varsigma)\, d\varsigma\, ds \end{aligned}$$

$$
\begin{aligned}
&+\int_{\sigma_q}^{t}\int_0^s k(s,\varsigma,z_\varsigma)\,d\varsigma ds+\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath} g(s,z_s)\,dW(s)+\int_{\sigma_q}^{t} g(s,z_s)\,dW(s)\\
&+\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_{\mathfrak{U}} P(s,z_s,u)\tilde{K}(ds,du)+\int_{\sigma_q}^{t}\int_{\mathfrak{U}} P(s,z_s,u)\tilde{K}(ds,du)\Big]I_{[\sigma_q,\sigma_{q+1})}(t)\\
&+2^{p-1}\mathbb{E}\Big\|\sum_{q=0}^{\infty}\Big[\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}[f(s,z_s)-f(s,w_s)]\,ds+\int_{\sigma_q}^{t}[f(s,z_s)-f(s,w_s)]\,ds\Big]\\
&+\Big[\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_0^s[k(s,\varsigma,z_\varsigma)-k(s,\varsigma,w_\varsigma)]\,d\varsigma\,ds+\int_{\sigma_q}^{t}\int_0^s[k(s,\varsigma,z_\varsigma)-k(s,\varsigma,w_\varsigma)]\,d\varsigma\Big]\\
&+\Big[\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}[g(s,z_s)-g(s,w_s)]\,d\mathcal{W}(s)+\int_{\sigma_q}^{t}[g(s,z_s)-g(s,w_s)]\,d\mathcal{W}(s)\Big]\\
&+\Big[\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_{\mathfrak{U}}[P(s,z_s,u)-P(s,w_s,u)]\tilde{K}(ds,du)\\
&+\int_{\sigma_q}^{t}\int_{\mathfrak{U}}[P(s,z_s,u)-P(s,w_s,u)]\tilde{K}(ds,du)\Big]I_{[\sigma_q,\sigma_{q+1})}(t)\Big\|^p\\
&\le 2^{p-1}\epsilon+2^{p-1}N,
\end{aligned}
$$

where

$$
\begin{aligned}
N=4^{p-1}\mathbb{E}\Big\|&\sum_{q=0}^{\infty}\Big[\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}[f(s,z_s)-f(s,w_s)]ds+\int_{\sigma_q}^{t}[f(s,z_s)-f(s,w_s)]ds\Big]\\
&+\Big[\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_0^s[k(s,\varsigma,z_\varsigma)-k(s,\varsigma,w_\varsigma)]d\varsigma ds+\int_{\sigma_q}^{t}\int_0^s[k(s,\varsigma,z_\varsigma)-k(s,\varsigma,w_\varsigma)]d\varsigma\Big]\\
&+\Big[\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}[g(s,z_s)-g(s,w_s)]dW(s)+\int_{\sigma_q}^{t}[g(s,z_s)-g(s,w_s)]dW(s)\Big]\\
&+\Big[\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_{\mathfrak{U}}[P(s,z_s,u)-P(s,w_s,u)]\tilde{K}(ds,du)\\
&+\int_{\sigma_q}^{t}\int_{\mathfrak{U}}[P(s,z_s,u)-P(s,w_s,u)]\tilde{K}(ds,du)\Big]\Big]I_{[\sigma_q,\sigma_{q+1})}(t)\Big\|^p\\
&\le 4^{p-1}(\mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}).
\end{aligned}
$$

Take

$$
\begin{aligned}
\mathcal{A}&=\mathbb{E}\Big\|\sum_{q=0}^{\infty}\Big[\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\big[f(s,z_s)-f(s,w_s)\big]ds+\int_{\sigma_q}^{t}\big[f(s,z_s)-f(s,w_s)\big]ds\Big]I_{[\sigma_q,\sigma_{q+1})}(t)\Big\|^p\\
&\le(C^p+1)(\mathcal{T}-t_0)^{p-1}\int_{t_0}^{t}\mathbb{E}\big\|f(s,z_s)-f(s,w_s)\big\|^p ds\\
&\le(C^p+1)(\mathcal{T}-t_0)^{p-1}\int_{t_0}^{t}\mathbb{E}\big\|f(s,z_s)-f(s,w_s)\big\|^p ds\\
&\le(C^p+1)L_f(\mathcal{T}-t_0)^{p-1}\int_{t_0}^{t}\big\|z_s-w_s\big\|_{\mathfrak{C}}^p ds.
\end{aligned}
$$

By (A2), we have

$$\begin{aligned}\mathcal{B} &= \mathbb{E}\Big\| \sum_{q=0}^{\infty} \Big[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \big[k(s,\varsigma,z_\varsigma) - k(s,\varsigma,w_\varsigma)\big] d\varsigma\, ds \\ &+ \int_{\sigma_q}^{t} \int_0^s \big[k(s,\varsigma,z_\varsigma) - k(s,\varsigma,w_\varsigma)\big]\, d\varsigma\, ds \Big] I_{[\sigma_q,\sigma_{q+1})}(t) \Big\|^p \\ &\le (C^p+1)(\mathcal{T}-t_0)^{p-1} \int_{t_0}^{t} \mathbb{E}\big\| k(s,\varsigma,z_\varsigma) - k(s,\varsigma,w_\varsigma)\big] d\varsigma ds \\ &\le (C^p+1)L_k(\mathcal{T}-t_0)^{p-1} \int_{t_0}^{t} \big\| z_s - w_s \big\|_{\mathfrak{C}}^p ds.\end{aligned}$$

Using Lemma 2.4, we have

$$\begin{aligned}\mathcal{C} &= \mathbb{E}\Big\| \sum_{q=0}^{\infty} \Big[\sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \big[g(s,z_s) - g(s,w_s)\big] dW(s) \\ &+ \int_{\sigma_q}^{t} \big[g(s,z_s) - g(s,w_s)\big] dW(s) \Big] I_{[\sigma_q,\sigma_{q+1})}(t) \Big\|^p \\ &\le (C^p+1)(p(p-1)/2)(\mathcal{T}-t_0)^{p-2/2} \int_{t_0}^{t} \mathbb{E}\big\| g(s,z_s) - g(s,w_s)\big\|^p ds \\ &\le (C^p+1)L_g(p(p-1)/2)(\mathcal{T}-t_0)^{p-2/2} \int_{t_0}^{t} \big\| z_s - w_s \big\|_{\mathfrak{C}}^p ds.\end{aligned}$$

By (A4), we have

$$\begin{aligned}\mathcal{D} &= \mathbb{E}\Big\| \sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \int_{\mathfrak{U}} [P(s,z_s,u) - P(s,w_s,u)] \tilde{K}(ds,du) \\ &+ \int_{\sigma_q}^{t} \int_{\mathfrak{U}} [P(s,z_s,u) - P(s,w_s,u)] \tilde{K}(ds,du) \Big] I_{[\sigma_q,\sigma_{q+1})}(t) \Big\|^p \\ &\le (C^p+1)c_p(\mathcal{T}-t_0)^{p-1} \Big[\int_{t_0}^{t} \int_{\mathfrak{U}} \mathbb{E}\|P(s,z_s,u) - P(s,w_s,u)\|^p \nu(dz)\, ds \\ &+ \Big(\int_{t_0}^{t} \int_{\mathfrak{U}} \mathbb{E}\|P(s,z_s,u) - P(s,w_s,u)\|^{p/2} \nu(dz)\, ds \Big)^{1/2} \Big] \\ &\le (C^p+1)c_p(L_P + L_P^{p/2})(\mathcal{T}-t_0)^{p-1} \int_{t_0}^{t} \big\| z_s - w_s \big\|_{\mathfrak{C}}^p ds.\end{aligned}$$

Therefore,

$$F = H \int_{t_0}^{t} \big\| z(s) - w(s) \big\|_{\mathfrak{C}}^p ds, \quad \text{with}$$

$H = 4^{p-1}(C^p+1)(\mathcal{T}-\zeta_0)^{p/2-1}\big[L_f(\mathcal{T}-t_0)^{p/2} + L_k(\mathcal{T}-t_0)^{p/2} + L_g(p(p-1)/2)^{p/2} + c_p(L_P + L_P^{p/2})(\mathcal{T}-t_0)^{p/2}\big]$.

Then, we get that

$$\mathbb{E}\|z(t) - w(t)\|^p \le 2^{p-1}\epsilon + 2^{p-1} H \int_{t_0}^{t} \|w(s) - z(s)\|_{\mathfrak{C}}^p ds.$$

Considering,

$$\begin{aligned}\int_{t_0}^{t} \left\|w(s)-z(s)\right\|_{\mathfrak{C}}^{p} ds &= \int_{t_0}^{t} \sup_{\theta\in[-\tau,0]} \mathbb{E}\|w(s+\theta)-z(s+\theta)\|^p ds \\ &= \sup_{\theta\in[-\tau,0]} \int_{\zeta_0}^{t} \mathbb{E}\|w(s+\theta)-z(s+\theta)\|^p ds \\ &= \sup_{\theta\in[-\tau,0]} \int_{t_0+\theta}^{t+\theta} \mathbb{E}\|w(I)-z(I)\|^p dI.\end{aligned}$$

Notice that, when $t\in[t_0-\tau,t_0]$,

$$\mathbb{E}\|w(I)-z(I)\|^p dI = 0.$$

Therefore,

$$\int_{t_0}^{t} \left\|w_s-z_s\right\|_{\mathfrak{C}}^{p} ds = \sup_{\theta\in[-\tau,0]} \int_{t_0}^{t+\theta} \mathbb{E}\|w(I)-z(I)\|^p dI = \int_{t_0}^{t} \mathbb{E}\|w(I)-z(I)\|^p dI.$$

So, we get

$$\mathbb{E}\|w(t)-z(t)\|^p \leq 2^{p-1}\epsilon + 2^{p-1}H\int_{t_0}^{t} \mathbb{E}\|w(I)-z(I)\|^p dI.$$

By Lemma 2.3, we have

$$\mathbb{E}\|w(t)-z(t)\|^p \leq 2^{p-1}\epsilon + 2^{p-1}\exp(2^{p-1}H).$$

Therefore, there exists $N = 2^{p-1}\exp(2^{p-1}K)$ such that

$$\mathbb{E}\|w(t)-z(t)\|^p \leq N\epsilon.$$

Thus the proof gets completed. □

6 An application

The considered NRINSIDEs with Poisson jumps is of the form

$$\begin{aligned} d\left[(z(\zeta)+\int_{-\alpha}^{0} u_1(\theta)z(\zeta+\theta)\right] &= \left[\int_{-\alpha}^{0} u_2(\theta)z(\zeta+\theta) + \int_{-\alpha}^{0}\int_{0}^{\zeta} u_3(\theta)z(\zeta+\theta)\right]d\zeta \\ &+ \left[\int_{-\alpha}^{0} u_4(\theta)z(\zeta+\theta)\right]dW(\zeta) \\ &+ \left[\int_{-\alpha}^{0}\int_{\mathfrak{U}} u_5(\theta)z(\zeta+\theta)\right]\tilde{K}(ds,du), \quad t\geq t_0, \quad t\neq\zeta_q \end{aligned}$$

$$z(\sigma_q) = b_q(\delta_q) z(\sigma_q^-), \quad q = 1, 2, \dots,$$

$$z(0) + \sum_{\imath}^{m} c_\imath(r_{\imath,z}) = z_0, \quad 0 \le r_1 \le r_2 \le \cdots \le r_p \le \mathcal{T}.$$

Let $\alpha > 0$, z is $\mathbb{R}$-valued stochastic process, $\sigma \in \mathfrak{C}([-\delta, 0], \mathscr{L}^2(\Omega, \mathbb{R}))$. δ_q is defined from Ω to $\mathcal{D}_q \overset{\text{def}}{=} (0, d_q)$ for all $q = 1, 2, \dots$ Suppose that τ_q follow Erlang distribution and let $\delta_\imath$ and $\delta_\jmath$ are independent of every other as $\imath \neq \jmath$ for $\imath, \jmath = 1, 2, \dots$, $\zeta_0 = \sigma_0 < \sigma_1 < \sigma_2 < \cdots$ and $\sigma_q = \sigma_{q-1} + \tau_q$ for $q = 1, 2, \dots$ Let $W(t) \in \mathbb{R}$ be a one-dimensional Brownian motion, where b is a function of q. $u_1, u_2, u_3 : [-\delta, 0] \to \mathbb{R}$ are continuous functions. Define $h : [\zeta_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, $f : [\zeta_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, $g : [\zeta_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^{d \times m}$, $r : \mathfrak{C} \to \mathfrak{C}$, $k : [\zeta_0, \mathcal{T}] \times [\zeta_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, $P : [\zeta_0, \mathcal{T}] \times \mathfrak{C} \times \mathfrak{U} \to \mathbb{R}^d$, and $b_q : \mathcal{D}_q \to \mathbb{R}^{d \times d}$ by

$$h(\zeta, z(\zeta))(.) = \int_{-\alpha}^{0} u_1(\theta) z(\zeta + \theta)\, d\theta(.), \quad f(\zeta, z(\zeta))(.) = \int_{-\alpha}^{0} u_2(\theta) z(\zeta + \theta)\, d\theta(.),$$

$$k(\zeta, z(\zeta))(.) = \int_{-\alpha}^{0} u_3(\theta) z(\zeta + \theta)\, d\theta(.), \quad g(\zeta, z(\zeta))(.) = \int_{-\alpha}^{0} u_4(\theta) z(\zeta + \theta)\, d\theta(.),$$

$$P(\zeta, z(\zeta))(.) = \int_{-\alpha}^{0} u_3(\theta) z(\zeta + \theta)\, d\theta(.).$$

For $z(t + \theta) \in \mathfrak{C}$, we suppose that the following conditions hold:

(1) $\max_{\imath, q} \left\{ \prod_{\jmath = \imath}^{q} \mathbb{E} \| b_\imath(\delta_\imath) \|^2 \right\} < \infty,$

(2) $\int_{-\alpha}^{0} u_1(\theta)^2 d\theta, \int_{-\alpha}^{0} u_2(\theta)^2 d\theta, \int_{-\alpha}^{0} u_3(\theta)^2 d\theta < \int_{-\alpha}^{0} u_4(\theta)^2 d\theta < \int_{-\alpha}^{0} u_5(\theta)^2 d\theta < \infty.$

Suppose the conditions (1) and (2) are fulfilled. Then the assumptions (A1)-(A5) holds. The system (1.1)-(1.3) has a unique mild solution z and is HU stable.

Lemma 6.1. *If $P = 0$ in* (1.1)-(1.3)*, then the system behaves as NRINSIDEs of the form:*

$$d[z(t) + h(t, z_t)] = \big[f(t, z_t) + \int_0^t k(t, s, z_s)\, ds\big] dt + g(t, z_t)\, dW(t), \quad t \ge t_0, \quad t \neq t_q,$$

$$z(\sigma_q) = b_q(\delta_q) z(\sigma_q^-), \quad q = 1, 2, \dots,$$

$$z_{t_0} + r(z) = z_0 = \sigma = \{\sigma(\theta) : -\delta \le \theta \le 0\}$$

By applying Theorem 3.1 under the assumptions (A1)-(A5), then the above guarantees the existence of the mild solution.

7 Conclusion

This article is devoted to discuss the existence and HU stability. First, we used the Banach fixed point theorem to demonstrate the existence of mild solutions to the equations (1.1)-(1.3). Then, we examined the stability via the continuous dependence of solutions on the initial value. Next, we investigated the HU stability results under the Lipschitz condition on a bounded and closed interval. In addition, this result could be extended to investigate the controllability of random impulsive neutral stochastic differential equations finite/infinite state-dependent delay in the future. The fractional order of NRINSDEs with Poisson jumps would be quite interesting. This will be the focus of future research.

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References

[1] A. Anguraj and K. Ravikumar, "Existence and stability of impulsive stochastic partial neutral functional differential equations with infinite delays and Poisson jumps", *J. Appl. Nonlinear Dyn.*, vol. 9, no. 2, pp. 245–255, 2020.

[2] A. Anguraj and K. Ravikumar, "Existence, uniqueness and stability of impulsive stochastic partial neutral functional differential equations with infinite delays driven by a fractional Brownian motion", *J. Appl. Nonlinear Dyn.*, vol. 9, no. 2, pp. 327–337, 2020.

[3] A. Anguraj , K. Ravikumar and J. J. Nieto, "On stability of stochastic differential equations with random impulses driven by Poisson jumps", *Stochastics*, vol. 93, no. 5, pp. 682–696, 2021.

[4] A. Anguraj, K. Ramkumar and K. Ravikumar, "Existence and Hyers-Ulam stability of random impulsive stochastic functional integrodifferential equations with finite delays", *Comput. Methods Differ. Equ.*, vol. 10, no. 1, pp. 191–199, 2022.

[5] D. Baleanu, R. Kasinathan, R. Kasinathan and V. Sandrasekaran, "Existence, uniqueness and Hyers-Ulam stability of random impulsive stochastic integro-differential equations with nonlocal conditions", *AIMS Math.*, vol. 8, no. 2, pp. 2556–2575, 2023.

[6] D. Chalishajar, R. Kasinathan, R. Kasinathan and G. Cox, "Existence Uniqueness and Stability of Nonlocal Neutral Stochastic Differential Equations with Random Impulses and Poisson Jumps", *RNA*, vol. 5, no. 3, pp. 250–62, 2022.

[7] S. Deng, X. Shu and J. Mao, "Existence and exponential stability for impulsive neutral stochastic functional differential equations driven by fBm with non compact semigroup via Mönch fixed point", *J. Math. Anal. Appl.*, vol. 467, no. 1, pp. 398–420, 2018.

[8] S. Dragomir, *Some Gronwall type inequalities and applications*, Melbourne, Australia: RGMIA Monographs, 2002. Available: `https://rgmia.org/papers/monographs/standard.pdf`.

[9] M. Gowrisankar, P. Mohankumar and A. Vinodkumar, "Stability results of random impulsive semilinear differential equations", *Acta Math. Sci. (English Ed.)*, vol. 34, no. 4, pp. 1055–1071, 2014.

[10] A. Hamoud, "Existence and uniqueness of solutions for fractional neutral Volterra-Fredholm integro differential equations", *ATNAA*, vol. 4, no. 4, pp. 321–331, 2020.

[11] E. Hernández, M. Rabello and H. R. Henríquez, "Existence of solutions for impulsive partial neutral functional differential equations", *J. Math. Anal. Appl.*, vol. 311, no. 2, pp. 1135–1158, 2007.

[12] R. S. Jain and M. B. Dhakne, "On impulsive nonlocal integro-differential equations with finite delay", *Int. J. Math. Res.*, vol. 5, no. 4, pp. 361–373, 2013.

[13] R. S. Jain, B. S. Reddy, and S. D. Kadam, "Approximate solutions of impulsive integro-differential equations", *Arab. J. Math.*, vol. 7, no. 4, pp. 273–279, 2018.

[14] R. Kasinathan, R. Kasinathan, V. Sandrasekaran and J. J. Nieto, "Qualitative Behaviour of Stochastic Integro-differential Equations with Random Impulses", *Qual. Theory Dyn. Syst*, vol. 22, no. 2, Art. ID 61, 2023.

[15] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*. Singapore: World Scientific Publishing Company, 1989.

[16] W. Lang , S. Deng, X.B. Shu and F. Xu, "Existence and Ulam-Hyers-Rassias stability of stochastic differential equations with random impulses", *Filomat*, vol. 35, no. 2, pp. 399–407, 2021.

[17] X. Mao, *Stochastic Differential Equations and Applications*. Chichester, England: Horwood Publishing Limited, 1997.

[18] B. Radhakrishnan and M. Tanilarasi, "Existence of solutions for quasilinear random impulsive neutral differential evolution equation", *Arab J. Math. Sci.*, vol. 24, no. 2, pp. 235–246, 2018.

[19] A. Vinodkumar, M. Gowrisankar and P. Mohankumar, "Existence, uniqueness and stability of random impulsive neutral partial differential equations", *J. Egyptian Math. Soc.*, vol. 23, no. 1, pp. 31–36, 2015.

[20] A. Vinodkumar, K. Malar, M. Gowrisankar and P. Mohankumar, "Existence, uniqueness and stability of random impulsive fractional differential equations", *Acta Math. Sci. (English Ed.)*, vol. 36, no. 2, pp. 428–442, 2016.

[21] S. Wu and X. Meng, "Boundedness of nonlinear differential systems with impulsive effect on random moments", *Acta Math. Appl. Sin.*, vol. 20, no. 1, pp. 147–154, 2004.

[22] X. Yang, X. Li, Q. Xi and P. Duan, "Review of stability and stabilization for impulsive delayed systems", *Math. Biosci. Eng.*, vol. 15, no. 6, pp. 1495–1515, 2018.

[23] X. Yang and Q. Zhu, "pth moment exponential stability of stochastic partial differential equations with Poisson jumps", *Asian J. Control*, vol. 16, no. 5, pp. 1482–1491, 2014.

[24] S. Zhang and W. Jiang, "The existence and exponential stability of random impulsive fractional differential equations", *Adv. Difference Equ.*, vol. 2018, Art. ID 404, 2018.