CUBO, A Mathematical Journal
Vol. 25, no. 2, pp. 231–250, August 2023
DOI: 10.56754/0719-0646.2502.231

Existence and controllability of integrodifferential equations with non-instantaneous impulses in Fréchet spaces

Abdelhamid Bensalem[1]
Abdelkrim Salim[1,2]
Bashir Ahmad[3,] ✉
Mouffak Benchohra[1,3]

[1] *Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria.*
bensalem.abdelhamid@yahoo.com
benchohra@yahoo.com

[2] *Faculty of Technology, Hassiba Benbouali University of Chlef, P.O. Box 151 Chlef 02000, Algeria.*
salim.abdelkrim@yahoo.com

[3] *Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia.*
bashirahmad_qau@yahoo.com ✉

ABSTRACT

In this paper, we investigate existence of mild solutions to a non-instantaneous integrodifferential equation via resolvent operators in the sense of Grimmer in Fréchet spaces. Utilizing the technique of measures of noncompactness in conjunction with the Darbo's fixed point theorem, we present sufficient criteria ensuring the controllability of the given problem. An illustrative example is also discussed.

RESUMEN

En este artículo, investigamos la existencia de soluciones mild de una ecuación integrodiferencial no-instantánea vía operadores resolventes en el sentido de Grimmer en espacios de Fréchet. Usando la técnica de medidas de nocompacidad junto con el teorema de punto fijo de Darbo, presentamos criterios suficientes para asegurar la controlabilidad del problema dado. Se discute, además, un ejemplo ilustrativo.

Keywords and Phrases: Integrodifferential equation, mild solution, measures of noncompactness, resolvent operator controllability, fixed point theorem, Fréchet space.

2020 AMS Mathematics Subject Classification: 93B05, 34D23, 47H10, 46A04, 45J05, 47H08, 35D30, 47B40.

Accepted: 31 May, 2023
Received: 20 November, 2022

1 Introduction

In recent years, the theory of fractional differential equations has been extensively developed by many authors. For a detailed account of the subject, we refer the reader to [1, 3, 4, 33]. Hernández and O'Regan initiated the theory of abstract impulsive differential equations with non-instantaneous impulses in [21]. Later, the authors studied instantaneous and non-instantaneous impulsive integrodifferential equations in Banach spaces in [2].

The controllability of linear and nonlinear differential systems in finite dimensional spaces received considerable attention, for example, see [8, 9, 10], while some interesting results on the controllability of such systems in infinite-dimensional Banach spaces with unbounded operators can be found in the monographs [10, 12, 23, 31]. For more details on the subject, see the papers [5, 13, 19, 20, 32] and the references cited therein. Lasiecka and Triggiani [22] discussed the exact controllability of semilinear abstract systems with application to waves and plates boundary control problems. For some results on evolution equations, for instance, see [1, 11, 28, 29].

Recently, in [15], the authors used Schauder's fixed point theorem to study the existence of mild solutions by considering two cases of the resolvent operators for the following integrodifferential problem:

$$\begin{cases} \xi'(t) = \Psi_1\xi(t) + \int_0^t \Psi_2(t-\theta)\xi(\theta)\,d\theta + \wp\left(t,\xi(t),(H\xi)(t)\right); & \text{if } t\in[0,a], \\ \xi(0) = g(\xi)+\xi_0. \end{cases}$$

Motivated by the works [2, 15], we will investigate the existence and controllability of mild solutions to the following impulsive integrodifferential equations via resolvent operators:

$$\begin{cases} \xi'(t) = \Psi_1\xi(t) + \wp\left(t,\xi(t),(H\xi)(t)\right) + \int_0^t \Psi_2(t-\theta)\xi(\theta)\,d\theta; & \text{if } t\in\Theta_j; \quad j=0,1,\ldots, \\ \xi(t) = \varpi_j\left(t,\xi\left(t_j^-\right)\right); \quad \text{if } t\in\tilde{\Theta}_j, \quad j=1,2,\ldots, \\ \xi(0)=\xi_0, \end{cases} \tag{1.1}$$

where $\Theta_0 = [0,\ t_1], \Theta_j := (\theta_j, t_{j+1}]$ and $\tilde{\Theta}_j = (t_j, \theta_j]$ with $0=\theta_0 < t_1 \le \theta_1 \le t_2 < \cdots < \theta_{\ell-1} \le t_\ell \le \theta_\ell \le t_{\ell+1} \le \cdots \le +\infty$, $\Psi_1 : D(\Psi_1)\subset \Xi \to \Xi$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\ge 0}$, $\Psi_2(t)$ is a closed linear operator with domain $D(\Psi_1)\subset D(\Psi_2(t))$, the operator H is defined by

$$(H\xi)(t) = \int_0^a \hbar(t,\theta,\xi(\theta))\,d\theta,$$

for $a>0$, $D_\hbar = \{(t,\theta)\in\mathbb{R}^2\ ;\ 0\le\theta\le t\le a\}$ and $\hbar : D_\hbar\times\Xi\to\Xi$. The nonlinear term

$\wp : \Theta_j \times \Xi \times \Xi \to \Xi$; $j = 0, \dots,$ $\varpi_j : \tilde{\Theta}_j \times \Xi \to \Xi$; $j = 1, 2, \dots,$ are given functions, where $\Theta = [0, +\infty)$, and $(\Xi, \|\cdot\|)$ is a Banach space, $\xi_0 \in \Xi$.

We emphasize that the novelty of our work includes the investigation of problem (1.1) under a diverse set of conditions. Specifically, we incorporated non-instantaneous impulses in the integrodifferential system on an unbounded domain to broaden its scope, in contrast to previous research efforts. The controllability of the given integrodifferential problem with non-instantaneous impulses is also studied. Our results generalize the ones presented in the articles [2, 15].

The rest of this paper is organized as follows. In Section 2, we recall some preliminary results and definitions related to our study. In Section 3, we will present the existence result by using the technique of measures of noncompactness in conjunction with the Darbo's fixed point theorem. We will also study the controllability for the given problem. An example is given to illustrate the applicability of the abstract results.

2 Preliminaries

Let us begin this section with some preliminary concepts related to the study of the problem at hand. Let $C(\Theta, \Xi)$ be the space of continuous functions from $\Theta := [0; +\infty)$ into Ξ and $B(\Xi)$ denotes the space of all bounded linear operators from Ξ into Ξ equipped with the norm

$$\|T\|_{B(\Xi)} = \sup\{\|T(\xi)\| \ : \ \|\xi\| = 1\}.$$

A measurable function $\xi : [0; +\infty) \to \Xi$ is Bochner integrable if and only if $\|\xi\|$ is Lebesgue integrable. For the properties of the Bochner integral, for instance, see [30].

Let $L^1([0; +\infty), \Xi)$ denote the Banach space of measurable functions $\xi : [0; +\infty) \to \Xi$ which are Bochner integrable, with the norm

$$\|\xi\|_{L^1} = \int_0^{+\infty} \|\xi(t)\| \, dt.$$

We consider the following Cauchy problem

$$\begin{cases} \xi'(t) = \Psi_1 \xi(t) + \displaystyle\int_0^t \Psi_2(t - \theta)\xi(\theta) \, d\theta, \quad \text{for } t \geq 0, \\ \xi(0) = \xi_0 \in \Xi. \end{cases} \tag{2.1}$$

The existence and properties of the resolvent operator have been discussed in [18]. In what follows, we suppose the following assumptions:

$(R1)$ Ψ_1 is the infinitesimal generator of a uniformly continuous semigroup $\{T(t)\}_{t>0}$;

$(R2)$ For all $t \geq 0, \Psi_2(t)$ is a closed linear operator from $\mathcal{D}(\Psi_1)$ to Ξ and $\Psi_2(t) \in \Psi_2(D(\Psi_1), \Xi)$. For any $\xi \in D(\Psi_1)$, the map $t \to \Psi_2(t)\xi$ is bounded, differentiable and the derivative $t \to \Psi_2{}'(t)\xi$ is bounded and uniformly continuous on $\mathbb{R}^+$.

Theorem 2.1 ([18]). *If the assumptions $(R1)$ and $(R2)$ are satisfied, then the problem (2.1) has a unique resolvent operator.*

Let $\{t_i\}_{i=0}^{\infty}$ be the sequence of real numbers such that

$$0 = t_0 < t_1 < t_2 < \cdots, \text{ and } \lim_{i \to +\infty} t_i = +\infty.$$

Let $PC(\mathbb{R}^+, \Xi)$ be the Banach space defined by

$$PC(\mathbb{R}^+, \Xi) = \Big\{ \xi : \mathbb{R}^+ \to \Xi : \xi|_{\tilde{\Theta}_j} = \varpi_j; j = 1, \dots, \ell, \xi|_{\Theta_j}; j = 0, \dots, \ell, \text{ are continuous } \xi\left(\theta_j^-\right), \ \xi\left(\theta_j^+\right), \ \xi\left(t_j^-\right) \text{ and } \xi\left(t_j^+\right) \text{ exist with } \xi\left(t_j^-\right) = \xi\left(t_j\right) \Big\},$$

endowed with the family of seminorms:

$$\|x\|_n = \sup\{\|x(t)\| \ : \ t \in [0, t_n]\}, \ n = 1, 2, \dots$$

Define by $\mathfrak{F} = C\left(\Theta, \Xi\right)$ the Fréchet space of continuous functions $\Im$ from $\mathbb{R}_+$ into Ξ, with the norm

$$\|\Im\|_n = \sup_{t \in \tilde{\Theta}_n} \|\Im(t)\|, \quad \tilde{\Theta}_n := [0, n], \ n \in \mathbb{N},$$

and the distance

$$d(\xi, \Im) = \sum_{n=1}^{\infty} \frac{2^{-n}\|\xi - \Im\|_n}{1 + \|\xi - \Im\|_n}; \quad \xi, \Im \in C\left(\mathbb{R}_+, \Xi\right).$$

Let χ represent the Kuratowski measure of noncompactness in Ξ. The properties of χ can be found in [6].

Definition 2.2 ([16]). *Let $\beth_{\mathfrak{F}}$ be the family of all nonempty and bounded subsets of a Fréchet space $\mathfrak{F}$. A family of functions $\{\chi_n\}_{n \in \mathbb{N}}$, where $\chi_n : \beth_{\mathfrak{F}} \to [0, \infty)$ is a family of measures of noncompactness in the real Fréchet space $\mathfrak{F}$, if for all Ω, Ω_1 and $\Omega_2 \in \beth_{\mathfrak{F}}$, the following conditions are satisfied:*

(C_1) *$\{\chi_n\}_{n \in \mathbb{N}}$ is full, that is $\chi_n(\Omega) = 0$ for $n \in \mathbb{N}$ if and only if Ω is precompact;*

(C_2) *$\chi_n\left(\Omega_1\right) < \chi_n\left(\Omega_2\right)$, for $\Omega_1 \subset \Omega_2$ and $n \in \mathbb{N}$;*

(C_3) *$\chi_n(\operatorname{Conv}\Omega) = \chi_n(\Omega)$, for $n \in \mathbb{N}$;*

(C_4) *If* $\{\Omega_i\}$ *is a sequence of closed sets from* $\beth_{\mathfrak{F}}$, *such that* $\Omega_{i+1} \subset \Omega_i, i = 1, \ldots,$ *and* $\lim_{i\to\infty} \chi_n(\Omega_i) = 0$, *for each* $n \in \mathbb{N}$, *then the intersection set* $\Omega_\infty = \bigcap_{i=1}^{\infty} \Omega_i$ *is nonempty.*

Example 2.3. *For* $\Omega \in \beth_{\mathfrak{F}},\ x \in \Omega,\ n \in \mathbb{N}$ *and* $\epsilon > 0$, *let us denote by* $\beta^n(x, \epsilon)$ *for* $n \in \mathbb{N}$, *the modulus of continuity of the function* x *on the interval* $\tilde{\Theta}_n$ *defined by*

$$\beta^n(x, \epsilon) = \sup\{|x(t) - x(\theta)|\ ;\ t, \theta \in \tilde{\Theta}_n \quad |t - \theta| < \epsilon\}.$$

Further, let us set

$$\beta^n(\Omega, \epsilon) = \sup\{\beta^n(x, \epsilon)\ ;\ x \in \Omega\}, \quad \beta_0^n(\Omega) = \lim_{\epsilon\to 0^+} \beta^n(\Omega, \epsilon)$$

and

$$\alpha_n(\Omega) = \beta_0^n(\Omega) + \sup_{t\in\tilde{\Theta}_n} \chi\big(\Omega(t)\big).$$

If the family of mappings $\{\alpha_n\}_{n\in\mathbb{N}}$, *where* $\alpha_n : \beth_{\mathfrak{F}} \to \Theta$, *satisfies the conditions* (C_1)*–*(C_4), *then the family of maps* $\{\alpha_n\}_{n\in\mathbb{N}}$ *defined above is a family of measures of noncompactness in the Fréchet space* $\mathfrak{F}$.

Definition 2.4 ([27])**.** *A nonempty subset* $\Omega \subset \mathfrak{F}$ *is bounded if, for* $n \in \mathbb{N}$, *there exists* $\beth_n > 0$, *such that*

$$\|\xi\|_n \leq \beth_n,\ \textit{for each}\ \xi \in \Omega.$$

Lemma 2.5 ([16])**.** *If* $\mathfrak{M}$ *is a bounded subset of a Banach space* Ξ, *then for each* $\epsilon > 0$, *there is a sequence* $\{\xi_j\}_{j=1}^{\infty} \subset \mathfrak{M}$ *such that*

$$\chi(\mathfrak{M}) \leq 2\chi\left(\{\xi_j\}_{j=1}^{\infty}\right) + \epsilon.$$

Lemma 2.6 ([24])**.** *If* $\{\xi_j\}_{j=0}^{\infty} \subset L^1$ *is uniformly integrable, then the function* $t \to \alpha(\{\xi_j(t)\}_{j=0}^{\infty})$ *is measurable and*

$$\chi\left(\left\{\int_0^t \xi_j(\theta)d\theta\right\}_{j=0}^{\infty}\right) \leq 2\int_0^t \chi\left(\{\xi_j(\theta)\}_{j=0}^{\infty}\right)d\theta,\ \textit{for}\ t \in \tilde{\Theta}_n,\ n \in \mathbb{N}.$$

For more details about measures of noncompactness, see [7, 16, 17].

3 Main results

In this subsection, we discuss the existence of mild solutions for the problem (1.1).

3.1 Existence of mild solutions

Definition 3.1. *A function $\xi \in PC(\mathbb{R}^+, \Xi)$ is called a mild solution to the problem (1.1) if it satisfies*

$$\xi(t) = \begin{cases} R(t)\xi_0 + \displaystyle\int_0^t R(t-\theta)\wp(\theta, \xi(\theta), (H\xi)(\theta))\, d\theta; \quad \textit{if } t \in \Theta_0, \\ R(t-\theta_j)\left[\varpi_j(\theta, \xi(\theta_j^-))\right] + \displaystyle\int_{\theta_j}^t R(t-\theta)\wp(\theta, \xi(\theta), (H\xi)(\theta))\, d\theta; \quad \textit{if } t \in \Theta_j, \\ \varpi_j(t, \xi(t_j^-)); \quad \textit{if } t \in \tilde{\Theta}_j, \end{cases}$$

where $j = 1, 2, \ldots$

In the sequel, we need the following hypotheses.

$(A1)$ (i) $\wp : \Theta \times \Xi \times \Xi \to \Xi$ is a Carathéodory function and there exist a function $p \in L^1(\Theta, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : \Theta \to (0, +\infty)$, such that

$$||\wp(t, \xi, \bar{\xi})|| \leq p(t)\psi(\|\xi\| + \|\bar{\xi}\|), \quad \text{for } \xi,\ \bar{\xi} \in \Xi.$$

(ii) There exists a function $l_\wp \in L^1(\Theta, \mathbb{R}^+)$ such that for any bounded set $B \subset \Xi$ and $t \in \Theta$, we have

$$\chi(\wp(t, B, H(B))) \leq l_\wp(t)\chi(B).$$

$(A2)$ The function $\hbar : D_\hbar \times \Xi \times \Xi \to \Xi$ is continuous and there exists $c_1 > 0$ such that,

$$\|\hbar(t, \theta, \xi) - \hbar(t, \theta, \bar{\xi})\| \leq c_1\|\xi - \bar{\xi}\|, \quad \text{for each } (t, \theta) \in D_\hbar \text{ and } \xi,\ \bar{\xi} \in \Xi,$$

with

$$\hbar^* = \sup\{\|\hbar(t, \theta, 0)\|\ ,\ (t, \theta) \in D_\hbar\} < \infty.$$

$(A3)$ $\varpi_j : \tilde{\Theta}_j \times \Xi \to \Xi$ are continuous and there exist positive constants $L_{\varpi_j}, j \in \mathbb{N}$ and $\tau > 1$

such that

$$\|\varpi_j(.,\xi) - \varpi_j(.,\Im)\| \leq \frac{L_{\varpi_j}}{\tau}\|\xi - \Im\|, \quad \text{for all } \xi, \Im \in \Xi,\ j = 1, 2, \ldots$$

$(A4)$ Assume that $(R1) - (R2)$ hold, and there exist $\gimel_R \geq 1$ and $b \geq 0$, such that

$$\|R(t)\|_{B(\Xi)} \leq \gimel_R e^{-bt}.$$

Using the methods employed in [25, 26], we can verify that the following example contains a family of measures of noncompactness in $PC(\mathbb{R}^+, \Xi)$:

$$\chi_n(\Pi) = \max_{i=0,\ldots,\ell} \beta_0(\gamma_i^p, \Pi) + \sup_{t\in\tilde{\Theta}_n} \left\{e^{-\tau\widetilde{\zeta}(t)}\chi(\Pi(t))\right\}; \quad p = 0, 1, 2 \text{ and } \ell = 0, 1, \ldots,$$

with γ_i^p a partition of $\mathbb{R}^+$. In particular,

$$\gamma_i^p = \begin{cases} \Theta_0; & \text{if } p = 0,\ \ell = 0, \\ \Theta_\ell; & \text{if } p = 1,\ \ell = 1, 2, \ldots, \\ \tilde{\Theta}_\ell; & \text{if } p = 2,\ \ell = 1, 2, \ldots, \end{cases}$$

and $\widetilde{\zeta}(t) = \int_0^t \zeta(\theta)\, d\theta,\ \zeta(t) = 4\gimel_R l(t),\ \tau > 1$, where $\Pi(t) = \{\pi(t) \in \mathfrak{F}\ ;\ \pi \in \Pi\}$, $t \in \tilde{\Theta}_n$. Moreover, if the set Π is equicontinuous, then $\beta_0(\gamma_i^p, \Pi) = 0$.

Theorem 3.2. *If the conditions $(A1) - (A4)$ are satisfied and*

$$\gimel_R L_{\varpi_j} < \tau,$$

then the system (1.1) has at least one mild solution.

Proof. Transform the problem (1.1) into a fixed point problem by introducing an operator $\aleph : PC(\mathbb{R}^+, \Xi) \to PC(\mathbb{R}^+, \Xi)$ as

$$\aleph\xi(t) = \begin{cases} R(t)\xi_0 + \int_0^t R(t-\theta)\wp(\theta, \xi(\theta), (H\xi)(\theta))\, d\theta; & \text{if } t \in \Theta_0, \\ R(t-\theta_j)\left[\varpi_j(\theta, \xi(\theta_j^-))\right] + \int_{\theta_j}^t R(t-\theta)\wp(\theta, \xi(\theta), (H\xi)(\theta))\, d\theta; & \text{if } t \in \Theta_j, \\ \varpi_j(t, \xi(t_j^-)); & \text{if } t \in \tilde{\Theta}_j, \end{cases}$$

where $j = 1, 2, \ldots$ Clearly, the fixed points of the operator $\aleph$ are mild solutions to the problem (1.1). Next, we verify that the operator $\aleph$ satisfies the hypothesis of Darbo's fixed point theorem [16].

Let $\delta_n > 0$ and $D_{\delta_n} = \{\xi \in PC(\mathbb{R}^+, \Xi);\ \|\xi\|_n \leq \delta_n\}$, where

$$\max\left\{\beth_R(\|\xi_0\| + \psi(K^*_{\delta_n})), \frac{\beth_R(\varpi_0 + \psi(K^*_{\delta_n}))}{1 - \frac{\beth_R L_{\varpi_j}}{\tau}}\right\} \leq \delta_n,$$

and

$$K^*_{\delta_n} = \big((c_1+1)\delta_n + a\hbar^*\big)\|p\|_{L^1}.$$

Notice that the set D_{δ_n} is bounded, closed and convex.

Step 1: $\aleph(D_{\delta_n}) \subset D_{\delta_n}$.

- **Case 1:** For any $n \in \mathbb{N}$, $\xi \in D_{\delta_n}$, $t \in \Theta_0 \cap \tilde{\Theta}_n$, it follows by $(A1)$ that

$$\begin{aligned}\|\aleph\xi(t)\| &\leq \beth_R\|\xi_0\| + \beth_R\int_0^t \psi(\|\xi(\theta)\| + \|H\xi(\theta)\|)p(\theta)\,d\theta \\ &\leq \beth_R\|\xi_0\| + \beth_R\psi((c_1+1)\delta_n + a\hbar^*)\|p\|_{L^1}.\end{aligned}$$

  Then we have

$$\|\aleph\xi\|_n \leq \beth_R\Big[\|\xi_0\| + \psi((c_1+1)\delta_n + a\hbar^*)\|p\|_{L^1}\Big].$$

- **Case 2:** For $t \in \Theta_j \cap \tilde{\Theta}_n$ and for each $\xi \in D_{\delta_n}$, by $(A1), (A2)$ and $(A3)$, we have

$$\|\varpi_j(\cdot, \xi(\cdot))\| \leq \frac{L_{\varpi_j}}{\tau}\|\xi(\cdot)\| + \varpi_0,$$

  and

$$\|\aleph\xi\|_n \leq \beth_R\left[\frac{L_{\varpi_j}}{\tau}\delta_n + \varpi_0 + \psi((c_1+1)\delta_n + a\hbar^*)\|p\|_{L^1}\right].$$

- **Case 3:** For $t \in \tilde{\Theta}_j \cap \tilde{\Theta}_n$, and for each $\xi \in D_{\delta_n}$, by $(A3)$, we have

$$\|\aleph\xi\|_n \leq \frac{L_{\varpi_j}}{\tau}\delta_n + \varpi_0.$$

Thus,

$$\|\aleph\xi\|_n \leq \delta_n.$$

Step 2: $\aleph$ is continuous.

Let ξ_ℓ be a sequence such that $\xi_\ell \to \xi_*$ in Ξ. We complete the proof in several steps.

- **Case 1:** For $t \in \Theta_0 \cap \tilde{\Theta}_n$, we have

$$\|(\aleph\xi_\ell)(t) - (\aleph\xi_*)(t)\| \leq \daleth_R \int_{\theta_j}^{t} \|\wp(\theta, \xi_\ell(\theta), H\xi_\ell(\theta)) - \wp(\theta, \xi_*(\theta), H\xi_*(\theta))\| \, d\theta.$$

It follows by continuity of $\hbar$ and $\wp$ that

$$\hbar(t, \theta, \xi_\ell(\theta)) \to \hbar(t, \theta, \xi_*(\theta)) \quad \text{as } \ell \to +\infty,$$

and

$$\|\hbar(t, \theta, \xi_\ell(\theta)) - \hbar(t, \theta, \xi_*(\theta))\| \leq c_1 \|\xi_\ell - \xi_*\|.$$

By the Lebesgue dominated convergence theorem, we obtain

$$\int_0^t \hbar(t, \theta, \xi_\ell(\theta)) \, d\theta \to \int_0^t \hbar(t, \theta, \xi_*(\theta)) \, d\theta, \quad \text{as } \ell \to +\infty.$$

Then, by $(A1)$, we get

$$\wp(\theta, \xi_\ell(\theta), H\xi_\ell(\theta)) \to \wp(\theta, \xi_*(\theta), H\xi_*(\theta)), \quad \text{as } \ell \to +\infty,$$

which implies that

$$\|(\aleph\xi_\ell) - (\aleph\xi_*)\|_n \to 0, \quad \text{as } \ell \to +\infty.$$

- **Case 2:** Let $t \in \Theta_j \cap \tilde{\Theta}_n$. Then we have

$$\begin{aligned} \|(\aleph\xi_\ell)(t) - (\aleph\xi_*)(t)\| \leq{}& \daleth_R \|\varpi_j(\theta_j, \xi_\ell(\theta_j^-)) - \varpi_j((\theta_j, \xi_*(\theta_j^-)))\| \\ &+ \daleth_R \int_0^t \|\wp(\theta, \xi_\ell(\theta), H\xi_\ell(\theta)) - \wp(\theta, \xi_*(\theta), H\xi_*(\theta))\| \, d\theta. \end{aligned}$$

As argued in case 1, by the continuity of $\hbar, \wp$ and ϖ_j, we get

$$\|(\aleph\xi_\ell) - (\aleph\xi_*)\|_n \to 0, \quad \text{as } \ell \to +\infty.$$

- **Case 3:** For $t \in \tilde{\Theta}_j \cap \tilde{\Theta}_n$, we obtain

$$\|(\aleph\xi_\ell)(t) - (\aleph\xi_*)(t)\| \leq \|\varpi_j(t_j, \xi_\ell(t_j^-)) - \varpi_j((t_j, \xi_*(t_j^-)))\|,$$

which, in view of the continuity of ϖ_j, implies that

$$\|(\aleph\xi_\ell) - (\aleph\xi_*)\|_n \to 0, \quad \text{as } \ell \to +\infty.$$

Thus, $\aleph$ is continuous.

Step 3 Since we have $\aleph(D_{\delta_n}) \subset D_{\delta_n}$, therefore, $\aleph(D_{\delta_n})$ is bounded.

Step 4 Let Π be a bounded equicontinuous subset of D_{δ_n}, then $\{\aleph(\Pi)\}$ is equicontinuous, which implies that $\beta_0\left(\gamma_i^p, \aleph(\Pi)\right) = 0$. Now, for any $\varrho > 0$, there exists a sequence $\{\xi_\ell\}_{j=0}^{\infty} \subset \Pi$ and we complete the proof of this part in certain steps.

- **Case 1:** Let $t \in \Theta_0 \cap \tilde{\Theta}_n$. Setting $O_{fv(\theta)} = \wp(\theta, \xi(\theta), H\xi(\theta))$, we have

$$\begin{aligned}
\chi\left\{\int_0^t R(t-\theta)O_{fv(\theta)}d\theta\ ;\ \xi \in \Pi\right\} &\leq 2\chi\left\{\int_0^t R(t-\theta)O_{f\xi_\ell(\theta)}d\theta\ ; \xi \in \Pi\right\} + \varrho \\
&\leq 4\int_0^t \daleth_R l_{\wp}(\theta)\chi(\{\Pi(\theta)\})d\theta + \varrho \\
&\leq \int_0^t \zeta(\theta)\chi(\Pi(\theta))d\theta + \varrho \\
&\leq \int_0^t e^{\tau\widetilde{\zeta}(\theta)}e^{-\tau\widetilde{\zeta}(\theta)}\zeta(\theta)\chi(\Pi(\theta))d\theta + \varrho \\
&\leq \int_0^t \zeta(\theta)e^{\tau\widetilde{\zeta}(\theta)} \sup_{\theta\in[0,t]} e^{-\tau\widetilde{\zeta}(\theta)}\chi(\Pi(\theta))d\theta + \varrho \\
&\leq \chi_n(\Pi)\int_0^t \left(\frac{e^{\tau\widetilde{\zeta}(\theta)}}{\tau}\right)' d\theta + \varrho \\
&\leq \frac{e^{\tau\widetilde{\zeta}(t)}}{\tau}\chi_n(\Pi) + \varrho,
\end{aligned}$$

  which implies that

$$\chi(\aleph(\Pi)(t)) \leq \frac{e^{\tau\widetilde{\zeta}(t)}}{\tau}\chi_n(\Pi) + \varrho.$$

  Since ϱ is arbitrary, so

$$\chi(\aleph(\Pi)(t)) \leq \frac{e^{\tau\widetilde{\zeta}(t)}}{\tau}\chi_n(\Pi),$$

  and hence

$$\chi_n(\aleph(\Pi)) \leq \frac{1}{\tau}\chi_n(\Pi).$$

- **Case 2:** For $t \in \Theta_j \cap \widetilde{\Theta}_n$, we proceed as in the proof of Case 1 to obtain

$$\begin{aligned}
\chi(\aleph(\Pi)(t)) &\leq \daleth_R\ \chi\left(\left\{\varpi_j(\theta, \xi_\ell(\theta_j^-)); \xi \in \Pi\right\}\right) + \frac{e^{\tau\widetilde{\zeta}(t)}}{\tau}\chi_n(\Pi) + \varrho \\
&\leq \frac{e^{\tau\widetilde{\zeta}(t)}(\daleth_R L_{\varpi_j} + 1)}{\tau}\chi_n(\Pi) + \varrho.
\end{aligned}$$

Therefore, we get

$$\chi_n(\aleph(\Pi)) \quad \leq \quad \frac{(\daleth_R L_{\varpi_j}+1)}{\tau}\chi_n(\Pi).$$

- **Case 3:** Let $t \in \tilde{\Theta}_j \cap \tilde{\Theta}_n$. By $(A3)$, the set $\left\{\varpi_j(t,\xi_j^-)\right\}_{j=1}^n$ is equicontinuous, and that $\beta_0\left(\gamma_i^p, G(\Pi)\right)=0$, with $\{Gv(t)\}=\left\{\varpi_j(t,\xi_j^-)\right\}$.
On the other hand, we have

$$\|\varpi_j(t,\xi(.))-\varpi_j(t,\overline{\xi}(.))\| \leq \frac{L_{\varpi_j}}{\tau}\|\xi(.)-\overline{\xi}(.)\|,$$

which yields

$$e^{-\tau\widetilde{\zeta}(t)}\|\varpi_j(t,\xi(t_j^-))-\varpi_j(t,\overline{\xi}(t_j^-))\| \leq \frac{L_{\varpi_j}}{\tau}e^{-\tau\widetilde{\zeta}(t)}\|\xi(t_j^-)-\overline{\xi}(t_j^-)\|.$$

Hence, we get

$$\chi_n(\aleph(\Pi)) \quad \leq \quad \frac{L_{\varpi_j}}{\tau}\chi_n(\Pi),$$

which shows that $\aleph$ is contraction (in terms of a measure of noncompactness), since $\daleth_R L_{\varpi_j}+1<\tau$. Therefore, by Darbo's fixed point theorem [16], we deduce that $\aleph$ has at least one fixed point which is a mild solution to the problem (1.1). □

3.2 Controllability of the system

In this subsection, we discuss the controllability for the system:

$$\begin{cases} \xi'(t) &=& \Psi_1(t)\xi(t)+\wp(t,\xi(t),(H\xi)(t)) \\ & & +\displaystyle\int_0^t \Psi_2(t-\theta)\xi(\theta)\,d\theta + Cu(t); \quad \text{if } t\in\Theta_j, \quad j=0,1,\ldots, \\ \xi(t) &=& \varpi_j\left(t,\xi\left(t_j^-\right)\right); \quad \text{if } t\in\tilde{\Theta}_j, \quad j=1,2,\ldots, \\ \xi(0) &=& \xi_0, \end{cases} \tag{3.1}$$

where $u\in L^2(\Theta,\mathfrak{S})$ is the control function, $\mathfrak{S}$ is the Banach space of admissible control functions and C is a bounded linear operator from $\mathfrak{S}$ into Ξ. Before proceeding further, we define the solution for the problem (3.1).

Definition 3.3. *The system (3.1) is said to be controllable on the interval Θ, if for every initial function $\xi_0=\xi(0)\in\Xi$ and $\hat{\xi}\in\Xi$, there is some control $u\in L^2([0;n];\Xi)$ for some $n>0$, such that the mild solution $\xi(\cdot)$ of the system (3.1) satisfies the terminal condition $\xi(n)=\hat{\xi}$.*

To obtain the controllability of mild solutions of (3.1), we assume the following conditions.

$(A5)$ There exists a positive constant ρ_n, such that

$$\max\left\{\varphi_1^\rho; \varphi_2^\rho; \frac{\varpi_0}{1-\frac{L_{\varpi_j}}{\tau}}\right\} \leq \rho_n,$$

where

$$\varphi_1^\rho = \left\{\beth_R\left[\|\xi_0\| + \psi(K^*_{\rho_n})\|p\|_{L^1} + c_5c_6\left(\frac{\rho_n}{\beth_R} + \|\xi_0\| + \psi(K^*_{\rho_n})\|p\|_{L^1}\right)\right]\right\},$$

$$\varphi_2^\rho = \left\{\beth_R\left[\frac{L_{\varpi_j}}{\tau}\rho_n + \varpi_0 + \psi(K^*_{\rho_n})\|p\|_{L^1} + c_5c_6\left(\frac{\rho_n}{\beth_R} + \|\xi_0\| + \psi(K^*_{\rho_n})\|p\|_{L^1}\right)\right]\right\},$$

and

$$K^*_{\rho_n} = \big((c_1+1)\rho_n + a\hbar^*\big)\|p\|_{L^1}.$$

$(A6)$ (i) For each n, the linear operator $W : L^2(\tilde{\Theta}_n, \mathfrak{S}) \to \mathfrak{F}$, defined by

$$Wu = \int_0^n R(n-\theta)Cu(\theta)\,d\theta,$$

has a pseudo inverse operator W^{-1}, which takes values in $L^2(\tilde{\Theta}_n, \mathfrak{S})\diagdown \ker(W)$.

(ii) There exist positive constants c_5, c_6, such that

$$\|C\| \leq c_5 \quad \text{and} \quad \|W^{-1}\| \leq c_6.$$

(iii) There exist $p_w \in L^1(\Theta, \mathbb{R}^+)$, $k_C \geq 0$, and for any bounded sets $V_1 \subset \Xi$, $V_2 \subset \mathfrak{S}$,

$$\chi((W^{-1}V_1)(t)) \leq p_w(t)\chi(V_1), \qquad \chi((CV_2)(t)) \leq k_C\chi(V_2(t)).$$

Theorem 3.4. *Suppose that the hypotheses $(A1)-(A5)$ hold. Then the problem (3.1) is controllable.*

Proof. For $n \in \mathbb{N}$, we define a family of measures of non compactness in $PC(\Theta, \mathfrak{F})$ as

$$\widetilde{\chi}_n(\Pi) = \max_{i=0,\dots,\ell}\beta_0\left(\gamma_i^p, \Pi\right) + \sup_{t\in\tilde{\Theta}_n}\left\{e^{-\tau\widetilde{\varkappa}(t)}\chi(\Pi(t))\right\}, \quad p = 0,1,2 \quad \text{and } \ell = 0,1,\dots,$$

where $\widetilde{\varkappa}(t) = \int_0^t \varkappa(\theta)\,d\theta$, $\varkappa(t) = 4\beth_R(l_\wp(t) + k_C(\beth_R\|l_\wp\|^1)p_w(t))$, $\tau > 1$. Using $(A5)$, we define the control:

$$u_\xi(t)=\begin{cases} W^{-1}\Big[\xi(n)-R(n)\xi_0-\displaystyle\int_0^n R(n-\theta)\wp(\theta,\xi(\theta),(H\xi)(\theta))\,d\theta\Big]; \quad \text{if } t\in\Theta_0,\\ W^{-1}\Big[\xi(n)-R(n-\theta_j)\left[\varpi_j(\theta,\xi(\theta_j^-))\right]\\ -\displaystyle\int_{\theta_j}^n R(t-\theta)\wp(\theta,\xi(\theta),(H\xi)(\theta))\,d\theta\Big]; \quad \text{if } t\in\Theta_j, \quad j=1,2,\dots \end{cases}$$

Using the above control, it will be shown that the operator defined by

$$\Upsilon\xi(t)=\begin{cases} R(t)\xi_0+\displaystyle\int_0^t R(t-\theta)\wp(\theta,\xi(\theta),(H\xi)(\theta))\,d\theta+\int_0^t R(t-\theta)Cu_\xi(\theta)\,d\theta; \quad \text{if } t\in\Theta_0,\\ R(t-\theta_j)\left[\varpi_j(\theta,\xi(\theta_j^-))\right]+\displaystyle\int_{\theta_j}^t R(t-\theta)\wp(\theta,\xi(\theta),(H\xi)(\theta))\,d\theta\\ +\displaystyle\int_{\theta_j}^t R(t-\theta)Cu_\xi(\theta)\,d\theta; \quad \text{if } t\in\Theta_j, \quad j=1,2,\dots,\\ \varpi_j(t,\xi(t_j^-)); \quad \text{if } t\in\tilde{\Theta}_j, \quad j=1,2,\dots, \end{cases}$$

has a fixed point which is a mild solution to the system (3.1), and hence the system is controllable. By $(A4)$, we define a closed, bounded and convex subset B_{ρ_n} for any $n\in\mathbb{N}$ as follows: $B_{\rho_n}=B(0,\rho_n)=\{x\in PC\ :\ \|x\|_n\le\rho_n\}$. We establish the proof in several steps.

Step 1: $\aleph(B_{\rho_n})\subset B_{\rho_n}$. For any $\xi\in B_{\rho_n}$, we accomplish the following cases by using the assumptions $(A1),(A4)$ and $(A5)$.

- **Case 1:** Let $t\in\Theta_0\cap\tilde{\Theta}_n$. For any $n\in\mathbb{N}$, $\xi\in B_{\rho_n}$, $t\in\Theta_0\cap\tilde{\Theta}_n$, it follows by $(A1)$ that

$$\begin{aligned}\|\Upsilon\xi(t)\| &\le \gimel_R\Big(\|\xi_0\|+\psi((c_1+1)\rho_n+a\hbar^*)\|p\|_{L^1}+c_5c_6\big(\frac{\rho_n}{\gimel_R}+\|\xi_0\|+\psi(K^*_{\rho_n})\|p\|_{L^1}\big)\Big)\\ &\le \rho_n.\end{aligned}$$

- **Case 2:** For $t\in\Theta_j\cap\tilde{\Theta}_n$, and for each $\xi\in B_{\rho_n}$, by $(A1),(A2)$ and $(A3)$, we obtain

$$\begin{aligned}\|\Upsilon\xi(t)\| &\le \gimel_R\Big[\frac{L_{\varpi_j}}{\tau}\rho_n+\varpi_0+\psi((c_1+1)\delta_n+a\hbar^*)\|p\|_{L^1}\\ &\quad +c_5c_6\left(\frac{\rho_n}{\gimel_R}+\|\xi_0\|+\psi(K^*_{\rho_n})\|p\|_{L^1}\right)\Big]\\ &\le \rho_n.\end{aligned}$$

- **Case 3:** Let $t\in\tilde{\Theta}_j\cap\tilde{\Theta}_n$. Then, for each $\xi\in B_{\rho_n}$, it follows by $(A3)$ that

$$\|\Upsilon\xi(t)\| \le \frac{L_{\varpi_j}}{\tau}\rho_n+\varpi_0\le\rho_n.$$

Thus, we get

$$\|\Upsilon\xi\|_n \leq \rho_n,$$

which implies that $\Upsilon(B_{\rho_n}) \subset B_{\rho_n}$ and $\Upsilon(B_{\rho_n})$ is bounded.

Step 2: Υ is continuous on B_{ρ_n}. Let ξ_n be a sequence such that $\xi_n \to \xi_*$ in B_{ρ_n}. Since $\wp, \hbar, \varpi_j, C$ are continuous, therefore, it follows by the Lebesgue dominated convergence theorem that

$$\int_0^t R(t-\theta)Cu_{\xi_n}(\theta)\, d\theta \to \int_0^t R(t-\theta)Cu_{\xi_*}(\theta)\, d\theta,$$

which yields

$$\|(\Upsilon\xi_n) - (\Upsilon\xi_*)\|_n \to 0, \quad \text{as } n \to +\infty.$$

Thus, we deduce that Υ is continuous.

Step 3: Let Π be a bounded equicontinuous subset of B_{ρ_n}, then $\{\Upsilon(\Pi)\}$ is equicontinuous, and that $\beta_0(\gamma_i^p, \Upsilon(\Pi)) = 0$. Now, for any $\varrho > 0$, there exists a sequence $\{\xi_j\}_{j=0}^{\infty} \subset \Pi$. Then we complete the proof for several cases.

- **Case 1:** For $t \in \Theta_0 \cap \tilde{\Theta}_n$, we have

$$\begin{aligned}
\chi(\Upsilon(\Pi)(t)) &\leq 2\chi\left(\left\{\int_0^t R(t-\theta)(\wp(\theta,\xi_j(\theta),H\xi_j(\theta)) + u_{\xi_j}(\theta))\, d\theta \;;\; \xi \in \Pi\right\}\right) + \varrho \\
&\leq 4\int_0^t \beth_R(l_\wp(\theta) + k_C(\beth_R\|l_\wp\|_L^1)p_w(\theta))\chi(\{\Pi(\theta)\})\, d\theta + \varrho \\
&\leq \frac{e^{\tau\widetilde{\varkappa}(t)}}{\tau}\widetilde{\chi}_n(\Pi) + \varrho.
\end{aligned}$$

Since ϱ is arbitrary, we have

$$\chi(\Upsilon(\Pi)(t)) \leq \frac{e^{\tau L(t)}}{\tau}\widetilde{\chi}_n(\Pi),$$

and hence

$$\widetilde{\chi}_n(\Upsilon(\Pi)) \leq \frac{1}{\tau}\widetilde{\chi}_n(\Pi).$$

- **Case 2:** Let $t \in \Theta_j \cap \tilde{\Theta}_n$. Then, as in the proof of Case 1, we get

$$\begin{aligned}
\chi(\Upsilon(\Pi)(t)) &\leq 4\int_0^t \beth_R(l_\wp(\theta) + k_C(\beth_R\|l_\wp\|_L^1)p_w(\theta))\chi(\{\Pi(\theta)\})d\theta + \varrho + \frac{\beth_R L_{\varpi_j}}{\tau}\chi(\{\Pi(t)\}) \\
&\leq \frac{e^{\tau\widetilde{\varkappa}(t)}(\beth_R L_{\varpi_j} + 1)}{\tau}\widetilde{\chi}_n(\Pi) + \varrho.
\end{aligned}$$

Since ϱ is arbitrary, we obtain

$$\widetilde{\chi}_n(\Upsilon(\Pi)) \leq \frac{\beth_R L_{\varpi_j} + 1}{\tau}\widetilde{\chi}_n(\Pi).$$

- **Case 3:** Let $t \in \tilde{\Theta}_j \cap \tilde{\Theta}_n$. Then, by $(A3)$, the set $\left\{\varpi_j(t, z_j^-)\right\}_{j=1}^n$ is equicontinuous, and that $\beta_0\left(\gamma_i^p, G(\Pi)\right) = 0$, with $\{Gz(t)\} = \left\{\varpi_j(t, z_j^-)\right\}$. On the other hand, we have

$$\|\varpi_j(t, z(.)) - \varpi_j(t, \overline{z}(.))\| \leq \frac{L_{\varpi_j}}{\tau}\|z(.) - \overline{z}(.)\|,$$

which implies that

$$e^{-\tau\widetilde{\zeta}(t)}\|\varpi_j(t, z(t_j^-)) - \varpi_j(t, \overline{z}(t_j^-))\| \leq \frac{L_{\varpi_j}}{\tau}e^{-\tau\widetilde{\zeta}(t)}\|z(t_j^-) - \overline{z}(t_j^-)\|.$$

Therefore, we have

$$\widetilde{\chi}_n(\Upsilon(\Pi)) \quad \leq \quad \frac{L_{\varpi_j}}{\tau}\widetilde{\chi}_n(\Pi),$$

which shows that Υ is contraction in view of the assumption

$$\daleth_R L_{\varpi_j} + 1 < \tau.$$

Hence, by Darbo's fixed point theorem [16], the operator Υ has a fixed point, which implies that the given system is controllable. □

4 An example

Consider the following impulsive integro-differential equations:

$$\begin{cases} \dfrac{\partial}{\partial t}\gamma(t,x) = -\dfrac{\partial}{\partial x}\gamma(t,x) - \pi\gamma(t,x) - \displaystyle\int_0^t \Gamma(t-\theta)\left(\dfrac{\partial}{\partial x}\gamma(\theta,x) + \pi\gamma(\theta,x)\right)d\theta \\ \qquad\qquad + \dfrac{\|\gamma(t,x)\|_{L^2}}{1+t^3\sin^2(t)} + \left(1+t^3\sin^2(t)\right)^{-1}\sin\left[\displaystyle\int_0^a \cos^2(\theta t)|\gamma(\theta,x)|\,d\theta\right] \\ \qquad\qquad + Cu(t,x), \quad \text{if } t \in \Theta_j,\ x \in (0,1), \\ \gamma(t,x) = \dfrac{\|\gamma(2j^- - 1, x)\|_{L^2}}{1 + 17(\|\gamma(2j^- - 1, x)\|_{L^2} + 1)}, \quad \text{if } t \in \tilde{\Theta}_j, \quad x \in (0,1), \\ \gamma(t,0) = \gamma(t,1) = 0, \quad t \in \mathbb{R}^+, \\ \gamma(0,x) = e^x, \quad x \in (0,1), \end{cases} \tag{4.1}$$

where $\Theta_0 = (0,1],\ \Theta_j = (2j; 2j+1], j = 0, 1, \ldots,\ \tilde{\Theta}_j = (2j-1; 2j], j = 1, 2, \ldots$ Set $\mathfrak{F} = L^2(0,1)$ and let Ψ_1 be defined by

$$(\Psi_1\varphi)(x) = -\left(\frac{d}{dx}\varphi(x) + \pi\varphi(x)\right),$$

and

$$D(\Psi_1) = \{\varphi \in L^2(0,1)\ /\ \varphi, \Psi_1\varphi \in L^2(0,1)\ \ ;\ \ \varphi(0) = \varphi(1) = 0\}.$$

The operator Ψ_1 is the infinitesimal generator of a C_0-semigroup on $\mathfrak{F}$ with domain $D(\Psi_1)$. Now, we define the operator $\Psi_2(t) : \mathfrak{F} \mapsto \mathfrak{F}$ as follows:

$$\Psi_2(t)z = \Gamma(t)\Psi_1 z, \quad \text{for } t \geq 0, \quad z \in D(\Psi_1).$$

As argued in [14], for some $r_2 > r_1 > 0$, it follows that $\|\Gamma(t)\| \leq \dfrac{e^{-r_2 t}}{r_1}$, and $\|\Gamma'(t)\| \leq \dfrac{e^{-r_2 t}}{r_1^2}$. From [18], we have that the resolvent operator $(R(t))_{t\geq 0}$ exists on $\mathfrak{F}$ which is norm continuous and $\|R(t)\| \leq e^{(r_1^{-1}-1)t}$. Therefore, the assumption $(A4)$ holds with $\beth_R = 1$ and $b = 1 - r_1^{-1}$. Now, we define

$$\gamma(t)(x) = \gamma(t,x),$$

$$\wp(t,\gamma(t),H\gamma(t))(x) = \frac{\|\gamma(t,x)\|_{L^2}}{1+t^3\sin^2(t)} + \left(1+t^3\sin^2(t)\right)^{-1}\sin\left[\int_0^a \cos^2(\theta t)|\gamma(\theta,x)|\,d\theta\right],$$

$$H\gamma(t)(x) = \int_0^a \cos^2(\theta t)|\gamma(\theta,x)|\,d\theta,$$

and

$$\varpi_j(t,\gamma(t_{j^-},x)) = \frac{\|\gamma(2j^- - 1,x)\|_{L^2}}{1+17(\|\gamma(2j^- - 1,x)\|_{L^2}+1)}.$$

Case 01: $Cu = 0$. With the above setting, the system (4.1) can be expressed in the following abstract form:

$$\begin{cases} \gamma'(t) = \Psi_1\gamma(t) + \wp\left(t,\gamma(t),(H\gamma)(t)\right) + \displaystyle\int_0^t \Psi_2(t-\theta)\gamma(\theta)\,d\theta, & \text{if } t \in \Theta_j, \\ \gamma(t) = \varpi_j\left(t,\gamma\left(t_j^-\right)\right), \quad \text{if } t \in \tilde{\Theta}_j, \\ \gamma(0) = \gamma_0. \end{cases} \tag{4.2}$$

On the other hand, we have

$$|\wp(t,\gamma_1(t),\gamma_2(t))| \leq (1+t^3\sin^2(t))^{-1}\Big(|\gamma_1(t)| + |\gamma_2(t)| + 1\Big).$$

Also, for any bounded set $\Sigma \subset \mathfrak{F}$, we have

$$\chi(\wp(t,\Sigma,H(\Sigma))) \leq (1+t^3\sin^2(t))^{-1}\chi(\Sigma).$$

So

$$p(t) = (1+t^3\sin^2(t))^{-1}, \text{ which certainly belongs to } L^1(\Theta,\mathbb{R}^+),$$

and $\psi(t) = 1+t$ is a continuous nondecreasing function from Θ to $[1,+\infty)$. Moreover, we have the estimates:

$$\|\hbar(t,\theta,\gamma_1) - \hbar(t,\theta,\gamma_2)\|_{\mathfrak{F}} \leq a\|\gamma_1 - \gamma_2\|_{\mathfrak{F}},$$

and

$$\|\varpi_j(\gamma_1) - \varpi_j(\gamma_2)\|_{\mathfrak{F}} \leq \frac{1}{18}\|\gamma_1 - \gamma_2\|_{\mathfrak{F}}.$$

For $\daleth_R < 3$, all the assumptions of Theorem 3.2 are satisfied. Hence, the problem (4.1) has at least one mild solution defined on $\mathbb{R}^+$.

Case 02 : $Cu = \varkappa u(t,\gamma)$ for $\varkappa > 0$. Let the operator $C : L^2(0,1) \to L^2(0,1)$ be defined by $Cu = \varkappa u(t,\gamma)$. Then, the system (4.1) takes the form:

$$\begin{cases} \gamma'(t) = \Psi_1\gamma(t) + \wp\left(t, \gamma(t), (H\gamma)(t)\right) + \displaystyle\int_0^t \Psi_2(t-\theta)\gamma(\theta)\,d\theta + Cu(t), & \text{if } t \in \Theta_j, \\ \gamma(t) = \varpi_j\left(t, \gamma\left(t_j^-\right)\right), & \text{if } t \in \tilde{\Theta}_j, \\ \gamma(0) = \gamma_0. \end{cases} \tag{4.3}$$

As argued in Case 01, we can easily verify the assumptions $(A1) - (A5)$. If we assume that the operator W given by $Wu = \displaystyle\int_0^n R(n-\theta)\varkappa u(\theta)\,d\theta$, satisfies $(A6)$, then all the assumptions given in Theorem 3.4 are verified. Therefore, the problem (4.1) is controllable.

5 Conclusions

In this research, we investigated existence of mild solutions for a non-instantaneous integrodifferential equation via resolvent operators in the sense of Grimmer in a Fréchet space. We applied Darbo's fixed point theorem in conjunction with the technique of measures of noncompactness to establish the desired results. The controllability of the given problem is also discussed. An example is presented for illustrating the application of our key findings. Our results are novel in the given configuration and contribute significantly to the literature on the topic. We believe that the present study can lead to new avenues for research, such as coupled systems, problems with infinite delays, and their fractional counterparts. Thus, this article will serve as a starting point for future endeavors in aforementioned areas.

References

[1] S. Abbas and M. Benchohra, *Advanced Functional Evolution Equations and Inclusions*, Cham, Switzerland: Springer, 2015. doi: 10.1007/978-3-319-17768-7

[2] S. Abbas, M. Benchohra and G. M. N'Guérékata, "Instantaneous and noninstantaneous impulsive integrodifferential equations in Banach spaces", *Abstr. Appl. Anal.*, vol. 2020, Art. ID 2690125, 2020. doi: 10.1155/2020/2690125

[3] B. Ahmad, A. Alsaedi, S. K. Ntouyas and J. Tariboon, *Hadamard-type Fractional Differential Equations, Inclusions and Inequalities*, Cham, Switzerland: Springer, 2017. doi: 10.1007/978-3-319-52141-1

[4] B. Ahmad, J. Henderson, and R. Luca, *Boundary Value Problems for Fractional Differential Equations and Systems.* NJ, USA: World Scientific, 2021. doi: 10.1142/11942

[5] A. Baliki and M. Benchohra, "Global existence and stability for neutral functional evolution equations", *Rev. Roumaine Math. Pures Appl.*, vol. 60, no. 1, pp. 71–82, 2015.

[6] J. Banaś and K. Goebel, *Measure of Noncompactness in Banach Spaces.* Lecture Notes in Pure and Applied Math 60. New York, USA: Marcel Dekker, Inc., 1980.

[7] J. Banaś, "Measures of noncompactness in the space of continuous tempered functions", *Demonstratio Math.*, vol. 14, no. 1, pp. 127–133, 1981.

[8] M. Benchohra, F. Bouazzaoui, E. Karapinar and A. Salim, "Controllability of second order functional random differential equations with delay", *Mathematics*, vol. 10, no. 7, Art. ID 1120, 2022. doi: 10.3390/math10071120

[9] N. Benkhettou, K. Aissani, A. Salim, M. Benchohra and C. Tunc, "Controllability of fractional integro-differential equations with infinite delay and non-instantaneous impulses", *Appl. Anal. Optim.*, vol. 6, no. 1, pp. 79–94, 2022.

[10] A. Bensoussan, G. Da Prato, M. C. Delfour and S. K. Mitter, *Representation and Control of Infinite Dimension Systems, Vol. II*, Systems & Control: Foundations & Applications. Boston, MA, USA: Birkhäuser, Inc., 1993.

[11] P. Chen, Y. Li and X. Zhang, "Cauchy problem for stochastic non-autonomous evolution equations governed by noncompact evolution families", *Discrete Contin. Dyn. Syst. Ser. B.*, vol. 26, no. 3, pp. 1531–1547, 2021. doi: 10.3934/dcdsb.2020171

[12] R. Curtain and H. J. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory.* New-York, USA: Springer-Verlag, 1995. doi: 10.1007/978-1-4612-4224-6

[13] W. Desch, R. C. Grimmer and W. Schappacher, “Some considerations for linear integrodifffer-ential equations”, *J. Math. Anal. Appl.*, vol. 104, no. 1, pp. 219–234, 1984. doi: 10.1016/0022-247X(84)90044-1

[14] A. Diop, M. A. Diop, O. Diallo and M. B. Traoré, “Local attractivity for integro-differential equations with noncompact semigroups”, *Nonauton. Dyn. Syst.*, vol. 7, no. 1, pp. 102–117, 2020. doi: 10.1515/msds-2020-0113

[15] M. A. Diop, K. Ezzinbi and M. P. Ly, “Nonlocal problems for integrodifferential equation via resolvent operators and optimal control”, *Differ. Incl. Control Optim.*, vol. 42, no. 1, pp. 5–25, 2022. doi: 10.7151/dmdico.1231

[16] S. Dudek, “Fixed point theorems in Fréchet algebras and Fréchet spaces and applications to nonlinear integral equations”, *Appl. Anal. Disc. Math.*, vol. 11, no. 2, pp. 340–357, 2017. doi: 10.2298/AADM1702340D

[17] S. Dudek and L. Olszowy, “Continuous dependence of the solutions of nonlinear integral quadratic Volterra equation on the parameter”, *J. Funct. Spaces*, Art. ID 471-235, 2015. doi: 10.1155/2015/471235

[18] R. C. Grimmer, “Resolvent opeators for integral equations in a Banach space”, *Trans. Amer. Math. Soc.*, vol. 273, no. 1, pp. 333–349, 1982.

[19] X. Hao and L. Liu, “Mild solution of semilinear impulsive integro-differential evolution equation in Banach spaces”, *Math. Methods Appl. Sci.*, vol. 40, no. 13, pp. 4832–4841, 2017. doi: 10.1002/mma.4350

[20] A. Heris, A. Salim, M. Benchohra and E. Karapınar, “Fractional partial random differential equations with infinite delay”, *Results in Physics*, vol. 37, Art. ID 105557, 2022. doi: 10.1016/j.rinp.2022.105557

[21] E. Hernández and D. O’Regan, “On a new class of abstract impulsive differential equations”, *Proc. Amer. Math. Soc.*, vol. 141, no. 5, pp. 1641–1649, 2013. doi: 10.1090/S0002-9939-2012-11613-2

[22] I. Lasiecka and R. Triggiani, “Exact controllability of semilinear abstract systems with application to waves and plates boundary control problems”, *Appl. Math. Optim.*, vol. 23, no. 2, pp. 109–154, 1991. doi: 10.1007/BF01442394

[23] X. Li and J. Yong, *Optimal Control Theory for Infinite Dimensional Systems.* Systems & Control: Foundations & Applications. Boston, MA, USA: Birkhäuser Boston, Inc., 1995. doi: 10.1007/978-1-4612-4260-4

[24] H. Mönch, "Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces", *Nonlinear Anal.*, vol. 4, no. 5, pp. 985–999, 1980. doi: 10.1016/0362-546X(80)90010-3

[25] L. Olszowy and S. Wedrychowicz, "Mild solutions of semilinear evolution equation on an unbounded interval and their applications", *Nonlinear Anal.*, vol. 72, no. 3-4, pp. 2119–2126, 2010. doi: 10.1016/j.na.2009.10.012

[26] L. Olszowy, "Fixed point theorems in the Fréchet space $C(\mathbb{R}_+)$ and functional integral equations on an unbounded interval", *Appl. Math. Comput.*, vol. 218, no. 18, pp. 9066–9074, 2012. doi: 10.1016/j.amc.2012.03.044

[27] J. Simon, *Banach, Fréchet, Hilbert and Neumann spaces.* Mathematics and Statistics Series. Analysis for PDEs set, Vol. 1. Hoboken, NJ, USA: Wiley, 2017.

[28] H. Waheed, A. Zada and J. Xu, "Well-posedness and Hyers-Ulam results for a class of impulsive fractional evolution equations", *Math. Meth. Appl. Sci.*, vol. 44, no. 1, pp. 749–771, 2021. doi: 10.1002/mma.6784

[29] J. Xu, B. Pervaiz, A. Zada and S. O. Shah, "Stability analysis of causal integral evolution impulsive systems on time scales", *Acta Math. Sci. Ser. B (Engl. Ed.)*, vol. 41, no. 3, pp. 781–800, 2021. doi: 10.1007/s10473-021-0310-2

[30] K. Yosida, *Functional Analysis.* New York-Berlin, USA-Germany: Springer-Verlag, 1980.

[31] J. Zabczyk, *Mathematical Control Theory.* Berlin, Germany: Birkhäuser, 1992. doi: 10.1007/978-0-8176-4733-9

[32] B. Zhu and B. Han, "Approximate controllability for mixed type non-autonomous fractional differential equations", *Qual. Theory Dyn. Syst.*, vol. 21, no. 4, Art. ID 111, 2022. doi: 10.1007/s12346-022-00641-7

[33] B. Zhu and B. Han, "Existence and uniqueness of mild solutions for fractional partial integro-differential equations", *Mediterr. J. Math.*, vol. 17, no. 4, Art. ID 113, 2020. doi: 10.1007/s00009-020-01550-2