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On generalized Hardy spaces associated with singular partial differential operators

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ABSTRACT

We define and study the Hardy spaces associated with singular partial differential operators. Also, a characterization by mean of atomic decomposition is investigated.

RESUMEN

Definimos y estudiamos los espacios de Hardy asociados a operadores diferenciales parciales singulares. También investigamos una caracterización por medio de la descomposición atómica.

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1 Introduction

The foundations of the real Hardy space $H^p(\mathbb{R}^n)$, $p \in [1,+\infty[$, were started with the works of C. Fefferman and E. M. Stein [10]. Hardy spaces were deeply developed later by R. Coifman and G. Weiss [8]. The theory of Hardy spaces $H^p(\mathbb{R}^n)$, plays a very important role in harmonic analysis and operator theory and it is shown that it has many interesting applications, for more details we refer the reader to [20]. In the euclidean case, there are many equivalent definitions of the Hardy spaces $H^p(\mathbb{R}^n)$ either by using the Poisson maximal function or by using the atomic decomposition. Uchiyama [19] characterized also the Hardy spaces $H^p(\mathbb{R}^n)$ by means of Littlewood-Paley g-function.

In [5], Baccar, Ben Hamadi and Rachdi have considered the following singular partial differential operators

$$\begin{cases} \Delta_1 = \dfrac{\partial}{\partial x}, \\ \Delta_2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{2\alpha+1}{r}\dfrac{\partial}{\partial r} - \dfrac{\partial^2}{\partial x^2}; \quad (r,x) \in]0,+\infty[\times\mathbb{R}; \quad \alpha \geqslant 0, \end{cases}$$

and they associated to Δ_1 and Δ_2 the so called Riemann-Liouville operator $\mathscr{R}_\alpha$ defined on $\mathscr{C}_e(\mathbb{R}^2)$ $\big($The space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable$\big)$, by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}\,dt\,ds, & \text{if } \alpha > 0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f(r\sqrt{1-t^2}, x+rt)\dfrac{dt}{\sqrt{(1-t^2)}}, & \text{if } \alpha = 0. \end{cases}$$

The Riemann-Liouville operator $\mathscr{R}_\alpha$ generalizes the spherical mean operator given by

$$\mathscr{R}_0(f)(r,x) = \frac{1}{2\pi}\int_0^{2\pi} f(r\sin\theta, x+r\cos\theta)\,d\theta,$$

which plays an important role in image processing of the so-called synthetic aperture radar (SAR) data, and in the linearized inverse scattering problem in acoustics, as well as in the interpretation of many physical phenomena in quantum mechanics, see [9, 11, 12].

According to [5], the Fourier transform $\mathscr{F}_\alpha$ associated with the Riemann-Liouville operator is defined for every $(s,y) \in \Upsilon$, by

$$\mathscr{F}_\alpha(f)(s,y) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\mathscr{R}_\alpha\left(\cos(s.)e^{-iy.}\right)(r,x)\frac{r^{2\alpha+1}dr\,dx}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}},$$

for a suitable integrable function, where Υ is a set that will be defined later.

Many harmonic analysis results have been already proved by Baccar, Ben Hamadi, Rachdi, Rouz and Omri for the Riemann-Liouville operator and its Fourier transform [3, 4, 5, 6, 7, 18]. Hleili, Mejjaoli, Omri and Rachdi have also established several uncertainty principles for the same Fourier transform $\mathscr{F}_\alpha$ [13, 15, 16, 17] .

Our purpose in this work is to define and study the Hardy's spaces $\mathcal{H}^p_\alpha$ related to the Riemann-Liouville operator and to characterize theses spaces for $p \in [1,+\infty[$ by using Poisson maximal operator associated to $\mathscr{R}_\alpha$ and by using atomic decomposition as well.

The paper is organized as follows. In the second section, we give some classical harmonic analysis results related to the Riemann-Liouville operator, the third section is devoted to the characterization of the Hardy spaces related to $\mathscr{R}_\alpha$ by using its Poisson maximal function. In the last section, we introduce the atomic decomposition which allows us to characterize $\mathcal{H}^1_\alpha$.

2 Riemann-Liouville operator

In this section we give and develop some harmonic analysis results related to the Riemann-Liouville operator that we will use later. For the proofs of these results we refer the reader to [5] and [7]. In [5] Baccar, Ben Hamadi and Rachdi considered the following system

$$\begin{cases} \triangle_1 u = -i\lambda u(r,x) \\ \triangle_2 u = -\mu^2 u(r,x) \\ u(0,0) = 1, \quad \frac{\partial u}{\partial x}(0,x), \quad \forall x \in \mathbb{R} \end{cases}$$

and showed that for all $(\mu,\lambda) \in \mathbb{C}^2$, this system admits a unique infinitely differentiable solution given by

$$\varphi_{\mu,\lambda}(r,x) = j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right) e^{-i\lambda x},$$

where j_α is the modified Bessel function of the first kind and index α, see [14, 21].

The function $\varphi_{\mu,\lambda}$ is bounded on $[0,+\infty[\times\mathbb{R}$ if and only if (μ,λ) belongs to the set

$$\Upsilon = \mathbb{R}^2 \cup \big\{(ir,x),\ (r,x) \in \times\mathbb{R}^2,\ |r| \leqslant |x|\big\}.$$

In this case, we have

$$\sup_{(r,x)\in\mathbb{R}^2} |\varphi_{\mu,\lambda}(r,x)| = 1.$$

In the following we denote by

- ν_α the measure defined on $[0,+\infty[\times\mathbb{R}$ by

$$d\nu_\alpha(r,x) = \frac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\,dr\,dx,$$

- $L^p(d\nu_\alpha)$, $p\in[1,+\infty]$, is the Lebesgue space of all measurable functions f on $[0,+\infty[\times\mathbb{R}$ such that $\|f\|_{p,\nu_\alpha} < +\infty$, where

$$\|f\|_{p,\nu_\alpha} = \begin{cases} \left(\int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|^p d\nu_\alpha(r,x)\right)^{\frac{1}{p}}, & \text{if } p\in[1,+\infty[\\ \underset{(r,x)\in[0,+\infty[\times\mathbb{R}}{\text{ess sup}} |f(r,x)|, & \text{if } p=+\infty. \end{cases}$$

- $L^1_{loc}(d\nu_\alpha)$ the space of measurable functions on $[0,+\infty[\times\mathbb{R}$ that are locally integrable on $[0,+\infty[\times\mathbb{R}$ with respect to the measure ν_α.

  According to [2], the eigenfunction $\varphi_{\mu,\lambda}$ satisfies the following product formula

$$\varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \left(\varphi_{\mu,\lambda}(\sqrt{r^2+s^2+2rs\cos\theta}, x+y)\right)\sin^{2\alpha}\theta\,d\theta.$$

- $\langle .|.\rangle_\alpha$is the inner product on the Hilbert space $L^2(d\nu_\alpha)$ defined by

$$\langle f|g\rangle_\alpha = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\overline{g(r,x)}\,d\nu_\alpha(r,x).$$

This allows us to define the translation operators as follows.

Definition 2.1. *For every $(r,x)\in[0,+\infty[\times\mathbb{R}$, the translation operator $\mathcal{T}_{(r,x)}$ associated with the operator $\mathscr{R}_\alpha$ is defined on $L^1(d\nu_\alpha)$ by, for every $(s,y)\in[0,+\infty[\times\mathbb{R}$*

$$\mathcal{T}_{(r,x)}(f)(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi f\left(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\right)(\sin\theta)^{2\alpha}d\theta,$$

whenever the integral on the right hand side is well defined.

Proposition 2.2. *Let f be in $L^1(d\nu_\alpha)$, then for every $(r,x)\in[0,+\infty[\times\mathbb{R}$, we have*

$$\int_0^{+\infty}\int_{\mathbb{R}} \mathcal{T}_{(r,x)}(f)(s,y)\,d\nu_\alpha(s,y) = \int_0^{+\infty}\int_{\mathbb{R}} f(s,y)\,d\nu_\alpha(s,y).$$

Proposition 2.3. *For every $f\in L^p(d\nu_\alpha)$, $1\leqslant p\leqslant+\infty$, and for every $(r,x)\in[0,+\infty[\times\mathbb{R}$, the function $\mathcal{T}_{(r,x)}(f)$ belongs to $L^p(d\nu_\alpha)$ and we have*

$$\|\mathcal{T}_{(r,x)}(f)\|_{p,\nu_\alpha} \leqslant \|f\|_{p,\nu_\alpha}. \tag{2.1}$$

Definition 2.4. *The convolution product of two measurable functions* f *and* g *on* $[0,+\infty[\times\mathbb{R}$ *is defined on* $[0,+\infty[\times\mathbb{R}$*, by*

$$(f*g)(r,x)=\int_0^{+\infty}\int_{\mathbb{R}}\mathcal{T}_{(r,-x)}(\check{f})(s,y)g(s,y)\,d\nu_\alpha(s,y),$$

where $\check{f}(s,y)=f(s,-y)$*, whenever the integral an the right hand side is well defined.*

Theorem 2.5. *If $p, q, r \in [1, +\infty]$ are such that $\frac{1}{r}+1 = \frac{1}{p}+\frac{1}{q}$ then, for every function $f \in L^p(d\nu_\alpha)$ and $g \in L^q(d\nu_\alpha)$, $f * g$ belongs to $L^r(d\nu_\alpha)$ and we have the Young's inequality*

$$\|f * g\|_{r,d\nu_\alpha} \leqslant \|f\|_{p,\nu_\alpha}\|g\|_{q,\nu_\alpha}.$$

Definition 2.6. *The Fourier transform $\mathscr{F}_\alpha$ associated with the operator $\mathscr{R}_\alpha$ is defined for every integrable function f on $[0, +\infty[\times\mathbb{R}$ with respect to the measure ν_α, by*

$$\forall(\mu, \lambda) \in \Upsilon, \quad \mathscr{F}_\alpha(f)(\mu, \lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r, x)\varphi_{\mu,\lambda}(r, x)\, d\nu_\alpha(r, x).$$

Proposition 2.7.

(i) Let $f \in L^1(d\nu_\alpha)$ and $(r, x) \in [0, +\infty[\times\mathbb{R}$ we have

$$\forall(\mu, \lambda) \in \Upsilon, \quad \mathscr{F}_\alpha(\mathcal{T}_{(r,-x)}(f))(\mu, \lambda) = \varphi_{\mu,\lambda}(r, x)\mathscr{F}_\alpha(f)(\mu, \lambda).$$

(ii) Let $f, g \in L^1(d\nu_\alpha)$, then we have

$$\forall(\mu, \lambda) \in \Upsilon, \quad \mathscr{F}_\alpha(f * g)(\mu, \lambda) = \mathscr{F}_\alpha(f)(\mu, \lambda)\mathscr{F}_\alpha(g)(\mu, \lambda).$$

In the following, we denote by

- Υ_+ the subspace of Υ given by

$$\Upsilon_+ = [0, +\infty[\times\mathbb{R} \cup \big\{(ir, x),\ (r, x) \in [0, +\infty[\times\mathbb{R},\ 0 \leqslant r \leqslant |x|\big\}.$$

- B_{Υ_+} the $\sigma-$algebra defined on Υ_+ by

$$B_{\Upsilon_+} = \{\theta^{-1}(B),\ B \in Bor([0, +\infty[\times\mathbb{R})\}$$

  where $Bor([0, +\infty[\times\mathbb{R})$ is the usual Borel $\sigma-$algebra on $[0, +\infty[\times\mathbb{R}$ and θ is bijective function defined by

$$\begin{array}{rccc} \theta: & \Upsilon_+ & \longrightarrow & [0, +\infty[\times\mathbb{R} \\ & (\mu, \lambda) & \longmapsto & \left(\sqrt{\mu^2 + \lambda^2}, \lambda\right). \end{array}$$

- γ_α the mesure defined on B_{Υ_+} by

$$\gamma_\alpha(A) = \nu_\alpha(\theta(A)), \quad \forall A \in B_{\Upsilon_+}.$$

- $L^p(d\gamma_\alpha)$, $p \in [1,+\infty]$ is the Lebesgue space of measurable functions f defined on Υ_+ satisfying $\|f\|_{p,\gamma_\alpha} < +\infty$, where

$$\|f\|_{p,\gamma_\alpha} = \begin{cases} \left(\int_{\Upsilon_+} |f(\mu,\lambda)|^p d\gamma_\alpha(\mu,\lambda)\right)^{\frac{1}{p}}, & \text{if } p \in [1,+\infty[\\ \underset{(\mu,\lambda)\in\Upsilon_+}{\text{ess sup}} |f(\mu,\lambda)|, & \text{if } p = +\infty. \end{cases}$$

- $\mathscr{S}_e(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, rapidly decreassing together with all their derivatives, even with respect the first variable.

  The space $\mathscr{S}_e(\mathbb{R}^2)$ is equipped with the topology associated to the countable family of norms

$$\forall m \in \mathbb{N}, \quad \rho_m(\varphi) = \sup_{\substack{(r,x)\in[0,+\infty[\times\mathbb{R} \\ k+|\beta|\leq m}} (1+r^2+x^2)^k |D^\beta(\varphi)(r,x)|.$$

- $\mathscr{D}_e(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$ with compact support, even with respect the first variable.

Proposition 2.8. *Let $f \in L^1(d\nu_\alpha)$. For every $(\mu,\lambda) \in \Upsilon$, we have*

$$\mathscr{F}_\alpha(f)(\mu,\lambda) = \widetilde{\mathscr{F}}_\alpha(f) \circ \theta(\mu,\lambda),$$

where

$$\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda) = \int_0^{+\infty} \int_{\mathbb{R}} f(r,x) j_\alpha(r\mu) e^{-i\lambda x} d\nu_\alpha(r,x).$$

Theorem 2.9. *$\widetilde{\mathscr{F}}_\alpha$ is an isomorphism from $\mathscr{S}_e(\mathbb{R}^2)$ onto itself.*

Proposition 2.10. *For every $f \in L^1(d\nu_\alpha)$ and for all $(r,x),(\mu,\lambda) \in [0,+\infty[\times\mathbb{R}$, we have*

$$\widetilde{\mathscr{F}}_\alpha(\mathcal{T}_{(r,x)}f)(\mu,\lambda) = j_\alpha(r\mu)e^{-ix\lambda}\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda).$$

Theorem 2.11 (Inversion formula for $\mathscr{F}_\alpha$)**.** *Let $f \in L^1(d\nu_\alpha)$ such that $\mathscr{F}_\alpha(f)$ belongs to $L^1(d\gamma_\alpha)$, then for almost every $(r,x) \in [0,+\infty[\times\mathbb{R}$,*

$$f(r,x) = \int_{\Upsilon_+} \mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}\, d\gamma_\alpha(\mu,\lambda).$$

Theorem 2.12 (Plancherel's theorem)**.** *The Fourier transform $\mathscr{F}_\alpha$ can be extended to an isometric isomorphism from $L^2(d\nu_\alpha)$ onto $L^2(d\gamma_\alpha)$.*

3 Hardy space associated with the Riemann-Liouville operator

Definition 3.1. *For every $t > 0$, the Poisson kernel p_t, associated with the Riemann-Liouville operator $\mathscr{R}_\alpha$ is defined on $\mathbb{R}^2$ by*

$$p_t(r,x) = \int_{\Upsilon_+} e^{-t\sqrt{s^2+2y^2}}\overline{\varphi_{s,y}(r,x)}\,d\gamma_\alpha(s,y) = \mathscr{F}_\alpha^{-1}\left(e^{-t\sqrt{.^2+2.^2}}\right)(r,x).$$

Lemma 3.2. *For every $t > 0$, the Poisson kernel p_t is given by*

$$\forall (r,x) \in \mathbb{R}^2, \quad p_t(r,x) = 2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)\frac{t}{(t^2+r^2+x^2)^{\alpha+2}}.$$

Proof. See [2]. □

Definition 3.3 (Bounded distribution)**.** *Let $v \in \mathscr{S}_e'(\mathbb{R}^2)$, we say that v is a bounded tempered distribution, if*

$$\forall \varphi \in \mathscr{S}_e(\mathbb{R}^2), \quad \varphi * v \in L^\infty(d\nu_\alpha)$$

and if the operator

$$\begin{array}{rccc} \phi_v: & \mathscr{S}_e(\mathbb{R}^2) & \longrightarrow & L^\infty(d\nu_\alpha) \\ & \varphi & \longmapsto & \varphi * v \end{array}$$

is bounded.

Proposition 3.4. *Let $v \in \mathscr{S}_e'(\mathbb{R}^2)$ be a bounded tempered distribution and $f \in L^1(d\nu_\alpha)$. Then, for every $\varphi \in \mathscr{S}_e(\mathbb{R}^2)$*

$$\langle f * v, \varphi\rangle = \int_0^{+\infty}\int_{\mathbb{R}} \breve{\varphi} * v(r,x)\check{f}(r,x)\,d\nu_\alpha(r,x).$$

where $\breve{\varphi}(r,x) = \varphi(-r,-x)$, is well defined. Moreover, $f * v$ is a tempered distribution.

Proof. Let v be a bounded tempered distribution. For all $f \in L^1(d\nu_\alpha)$ and $\varphi \in \mathscr{S}_e(\mathbb{R}^2)$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}} |\breve{\varphi} * v(r,x)||\check{f}(r,x)|\,d\nu_\alpha(r,x) \leq \|\breve{\varphi} * v\|_{\infty,\nu_\alpha}\|f\|_{1,\nu_\alpha} < +\infty$$

and consequently, the integral

$$\langle f * v, \varphi\rangle = \int_0^{+\infty}\int_{\mathbb{R}} \breve{\varphi} * v(r,x)\check{f}(r,x)\,d\nu_\alpha(r,x)$$

is well defined.

It is clear that $f * v$ is a linear operator. Now, for every $\varphi \in \mathscr{S}_e(\mathbb{R}^2)$, we have

$$\begin{aligned}|\langle f * v, \varphi\rangle| &\leqslant \int_0^{+\infty}\int_{\mathbb{R}} |\check{\varphi} * v(r,x)||\check{f}(r,x)|\, d\nu_\alpha(r,x)\\ &\leqslant \|\check{\varphi} * v\|_{\infty, d\nu_\alpha}\|\check{f}\|_{1,\nu_\alpha} \leqslant C\rho_m(\check{\varphi})\|f\|_{1,\nu_\alpha}\\ &\leqslant C\|f\|_{1,d\nu_\alpha}\rho_m(\varphi).\end{aligned}$$

Then, $f * v$ is a tempered distribution. □

Proposition 3.5. *For every bounded tempered distribution $v \in \mathscr{S}'_e(\mathbb{R}^2)$ and for every $t > 0$, $p_t * v \in L^\infty(d\nu_\alpha)$.*

Proof. By Urysohn's lemma, we know that there exits a function $f \in \mathcal{D}_e(\mathbb{R}^2)$ such that

$$\begin{cases} f \equiv 1, & \text{on} \quad \overline{B(0,1/2)}\\ f \equiv 0, & \text{on} \quad B^c(0,1)\\ 0 \leqslant f \leqslant 1. \end{cases}$$

Let $\varphi = \widetilde{\mathscr{F}}_\alpha^{-1}(f)$ then $\varphi \in \mathscr{S}_e(\mathbb{R}^2)$ and $\widetilde{\mathscr{F}}_\alpha(\varphi) = f = 1$ on $\overline{B(0,1/2)}$, hence for $\eta = 1 - \widetilde{\mathscr{F}}_\alpha(\varphi)$, we deduce that $\eta \in \mathscr{C}_e^\infty(\mathbb{R}^2)$ and $\eta = 0$ on $\overline{B(0,1/2)}$. Finally, let g the function defined by $g(r,x) = e^{-|(r,x)|}\eta(r,x)$ and $\psi = \widetilde{\mathscr{F}}_\alpha^{-1}(g)$, then for all $t > 0$ and for all $(r,x) \in \mathbb{R}^2$, we have

$$\begin{aligned}\widetilde{\mathscr{F}}_\alpha(p_t)(r,x) &= e^{-t\sqrt{r^2+x^2}}\\ &= e^{-t\sqrt{r^2+x^2}}\left(\widetilde{\mathscr{F}}_\alpha(\varphi)(tr,tx) + \eta(tr,tx)\right)\\ &= e^{-t\sqrt{r^2+x^2}}\widetilde{\mathscr{F}}_\alpha(\varphi_t)(r,x) + \widetilde{\mathscr{F}}_\alpha(\psi_t)(r,x)\\ &= \widetilde{\mathscr{F}}_\alpha(p_t)(r,x)\widetilde{\mathscr{F}}_\alpha(\varphi_t)(r,x) + \widetilde{\mathscr{F}}_\alpha(\psi_t)(r,x)\\ &= \widetilde{\mathscr{F}}_\alpha(p_t * \varphi_t + \psi_t)(r,x)\end{aligned}$$

Consequently, by the fact that $\widetilde{\mathscr{F}}_\alpha$ is injective, we get

$$p_t = p_t * \varphi_t + \psi_t$$

and therefore

$$p_t * v = p_t * \varphi_t * v + \psi_t * v.$$

Since φ_t and ψ_t belongs to $\mathscr{S}_e(\mathbb{R}^2)$, $\varphi_t * v$ and $\psi_t * v$ are bounded on $\mathbb{R}^2$ and $p_t \in L^1(\nu_\alpha)$, then $p_t * \varphi_t * v$ is a bounded function and the same holds for $p_t * v$. □

Definition 3.6. *Let $f \in \mathcal{S}'_e(\mathbb{R}^2)$ be a bounded tempered distribution. The Poisson maximal function $\mathcal{P}^\alpha_f$ associated with the Riemann-Liouville operator $\mathcal{R}_\alpha$ is defined on $\mathbb{R}^2$ by*

$$\mathcal{P}^\alpha_f(r,x) = \sup_{t>0} |p_t * f(r,x)|.$$

Definition 3.7 (Hardy space)**.** *For every $p \in [1, +\infty[$, the Hardy space $\mathcal{H}^p_\alpha$ associated with the Riemann-Liouville operator is the space of all the bounded tempered distributions f on $\mathbb{R}^2$ satisfying*

$$\mathcal{P}^\alpha_f \in L^p(d\nu_\alpha).$$

We set

$$\|f\|_{\mathcal{H}^p_\alpha} = \|\mathcal{P}^\alpha_f\|_{p,\nu_\alpha}. \tag{3.1}$$

Proposition 3.8. *Let $f \in \mathcal{S}'_e(\mathbb{R}^2)$ be a bounded tempered distribution. Then,*

$$\lim_{t\to 0} p_t * f = f \quad in \quad \mathcal{S}'_e(\mathbb{R}^2).$$

Proof. Let $\eta \in \mathcal{S}_e(\mathbb{R}^2)$. First, we will show that $\lim\limits_{t\to 0} p_t * \eta * f = \eta * f$ in $\mathcal{S}'_e(\mathbb{R}^2)$, thus by using Fubini's theorem, we deduce that for every $\psi \in \mathcal{S}_e(\mathbb{R}^2)$, we have

$$\begin{aligned}
\langle p_t * \eta * f, \psi\rangle_\alpha &= \int_0^{+\infty}\int_{\mathbb{R}} (p_t * \eta * f)(r,x)\psi(r,x)\, d\nu_\alpha(r,x) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} \left(\int_0^{+\infty}\int_{\mathbb{R}} \mathcal{T}_{(r,-x)}(\check{p}_t)(s,u)\eta * f(s,u) d\nu_\alpha(s,u)\right)\psi(r,x)\, d\nu_\alpha(r,x) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} \left(\int_0^{+\infty}\int_{\mathbb{R}} \mathcal{T}_{(r,-x)}(\check{p}_t)(s,u)\psi(r,x)\, d\nu_\alpha(r,x)\right)\eta * f(s,u)\, d\nu_\alpha(s,u) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} \left(\int_0^{+\infty}\int_{\mathbb{R}} \mathcal{T}_{(s,-u)}(\check{p}_t)(r,x))\psi(r,x)\, d\nu_\alpha(r,x)\right)\eta * f(s,u)\, d\nu_\alpha(s,u) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} p_t * \psi(s,t)\eta * f(s,t)\, d\nu_\alpha(s,t).
\end{aligned}$$

Using the dominated convergence theorem, we have

$$\begin{aligned}
\lim_{t\to 0}\langle p_t * \eta * f, \psi\rangle_\alpha &= \lim_{t\to 0}\int_0^{+\infty}\int_{\mathbb{R}} p_t * \psi(s,t)\eta * f(s,t)\, d\nu_\alpha(s,t) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} \lim_{t\to 0} p_t * \psi(s,t)\eta * f(s,t)\, d\nu_\alpha(s,t) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} \psi(s,t)\ \eta * f(s,t)\, d\nu_\alpha(s,t) \\
&= \langle \eta * f, \psi\rangle_\alpha.
\end{aligned}$$

Then,

$$\lim_{t\to 0} p_t * \eta * f = \eta * f \quad \text{in} \quad \mathscr{S}'_e(\mathbb{R}^2). \tag{3.2}$$

Now, we want to show that

$$\lim_{t\to 0} \mathscr{F}_\alpha(p_t)(1-\mathscr{F}_\alpha(\eta))\mathscr{F}_\alpha(f) = (1-\mathscr{F}_\alpha(\eta))\mathscr{F}_\alpha(f) \quad \text{in} \quad \mathscr{S}'_e(\mathbb{R}^2).$$

$\mathscr{F}_\alpha(p_t)(1-\mathscr{F}_\alpha(\eta))$ is a infinitely differentiable function on $\mathbb{R}^2$, then for any $\psi \in \mathscr{S}_e(\mathbb{R}^2)$, we have

$$\begin{aligned}
\lim_{t\to 0}\langle \mathscr{F}_\alpha(p_t)(1-\mathscr{F}_\alpha(\eta))\mathscr{F}_\alpha(f), \psi\rangle_\alpha &= \lim_{t\to 0}\langle \mathscr{F}_\alpha(f), \mathscr{F}_\alpha(p_t)(1-\mathscr{F}_\alpha(\eta))\psi\rangle_\alpha \\
&= \left\langle \mathscr{F}_\alpha(f), \lim_{t\to 0} \mathscr{F}_\alpha(p_t)(1-\mathscr{F}_\alpha(\eta))\psi \right\rangle_\alpha \\
&= \langle \mathscr{F}_\alpha(f), (1-\mathscr{F}_\alpha(\eta))\psi\rangle_\alpha \\
&= \langle (1-\mathscr{F}_\alpha(\eta))\mathscr{F}_\alpha(f), \psi\rangle_\alpha.
\end{aligned}$$

Hence,

$$\lim_{t\to 0} \mathscr{F}_\alpha(p_t)(1-\mathscr{F}_\alpha(\eta))\mathscr{F}_\alpha(f) = (1-\mathscr{F}_\alpha(\eta))\mathscr{F}_\alpha(f) \quad \text{in} \quad \mathscr{S}'_e(\mathbb{R}^2). \tag{3.3}$$

Consequently,

$$\lim_{t\to 0} \mathscr{F}_\alpha(p_t * f - p_t * \eta * f) = \mathscr{F}_\alpha(f - \eta * f) \quad \text{in} \quad \mathscr{S}'_e(\mathbb{R}^2),$$

which implies that

$$\lim_{t\to 0} p_t * f - p_t * \eta * f = f - \eta * f.$$

Then,

$$\lim_{t\to 0} p_t * f - \lim_{t\to 0} p_t * \eta * f = f - \eta * f.$$

From the Relation (3.2), we have

$$\lim_{t\to 0} p_t * f - \eta * f = f - \eta * f.$$

Then,

$$\lim_{t\to 0} p_t * f = f - \eta * f + \eta * f = f,$$

which achieves the proof. □

Definition 3.9 (Hardy-Littlewood maximal function)**.** *Let $f \in L^1_{loc}(d\nu_\alpha)$. The Hardy-Littlewood maximal function $M_\alpha(f)$ associated with the Riemann-Liouville operator $\mathcal{R}_\alpha$ is defined on $[0,+\infty[\times\mathbb{R}$, by*

$$M_\alpha(f)(r,x) = \sup_{\eta>0} \frac{1}{\nu_\alpha(B((0,0),\eta))} \int_{B((0,0),\eta)} \mathcal{T}_{(r,-x)}(|\check{f}|)(s,y)\, d\nu_\alpha(s,y).$$

Theorem 3.10 (The boundedness of M_α). *For every $p \in]1,+\infty]$, the maximal operator M_α is of strong type (p,p) from $L^p(d\nu_\alpha)$ into itself, that is for every $p \in]1,+\infty[$ there exists $C_p > 0$ such that for every $f \in L^p(d\nu_\alpha)$*

$$\|M_\alpha(f)\|_{p,\nu_\alpha} \leqslant C_p \|f\|_{p,\nu_\alpha}$$

Proof. See [1]. □

Proposition 3.11. *Let k be a nonnegative decreasing funtion on $[0,+\infty[$ which is continuous except possibly at finite number of points. We define the function K on $[0,+\infty[\times\mathbb{R}$ by*

$$K(r,x) = k\left(\sqrt{r^2+x^2}\right).$$

Then, for every locally integrable function f on $[0,+\infty[\times\mathbb{R}$ we have

$$\sup_{\epsilon>0}(K_\epsilon * |f|(r,x)) \leqslant \|K\|_{1,\nu_\alpha} M_\alpha(f)(r,x), \tag{3.4}$$

where $K_\epsilon(r,x) = \dfrac{1}{\epsilon^{2\alpha+3}} K\left(\dfrac{r}{\epsilon}, \dfrac{x}{\epsilon}\right)$.

Proof. First, we prove the relation (3.4), when K is continuous with compact support such that $supp(K) \subset B(0,R)$, where $R > 0$ and $f \in L^1_{loc}(d\nu_\alpha)$. We will prove that

$$\sup_{\epsilon>0}(K_\epsilon * |f|(0,0)) \leqslant \frac{1}{(2\pi)^{\frac{1}{2}} 2^\alpha \Gamma(\alpha+1)} M_\alpha(f)(0,0)\|K\|_{1,\nu_\alpha}. \tag{3.5}$$

$$\begin{aligned}
K_\epsilon * |f|(0,0) &= \int_0^{+\infty}\int_{\mathbb{R}} |f(s,x)| K_\epsilon(s,x)\, d\nu_\alpha(s,x) \\
&= \int_0^{+\infty}\int_{\mathbb{R}} |f(s,x)| K_\epsilon(s,x) \frac{s^{2\alpha+1}}{2^\alpha\sqrt{2\pi}\,\Gamma(\alpha+1)} ds\, dx \\
&= \int_0^{+\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f(r\cos\theta, r\sin\theta)| K_\epsilon(r,0) r \frac{(r\cos(\theta))^{2\alpha+1}}{2^\alpha\sqrt{2\pi}\,\Gamma(\alpha+1)} dr\, d\theta.
\end{aligned}$$

Let F and G be the functions defined on $[0,+\infty[$ by

$$\begin{aligned}
F(r) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f(r\cos\theta, r\sin\theta))| \cos^{2\alpha+1}\theta \frac{d\theta}{\sqrt{2\pi}} \\
G(r) &= \int_0^r F(y) y^{2\alpha+2} \frac{dy}{2^\alpha \Gamma(\alpha+1)}.
\end{aligned}$$

By integration by parts, we obtain

$$\begin{aligned}
K_\epsilon * |f|(0,0) &= \int_0^{+\infty} F(r)K_\epsilon(r,0)r^{2\alpha+2}\frac{dr}{2^\alpha\Gamma(\alpha+1)}\\
&= \int_0^{\epsilon R} F(r)K_\epsilon(r,0)r^{2\alpha+2}\frac{dr}{2^\alpha\Gamma(\alpha+1)}\\
&= G(\epsilon R)K_\epsilon(\epsilon R,0) - G(0)K_\epsilon(0,0) - \int_0^{\epsilon R} G(r)dK_\epsilon(r,0)\\
&= -\int_0^{\epsilon R} G(r)dK_\epsilon(r,0)\\
&= \int_0^{+\infty} G(r)d\left(-K_\epsilon\right)(r,0),
\end{aligned}$$

where the last integrals are understood in the Lebesgue-Stieltjes sense.

On the other hand,

$$\begin{aligned}
G(r) &= \int_0^r F(y)\frac{y^{2\alpha+2}}{2^\alpha\Gamma(\alpha+1)}dy\\
&= \int_0^r\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f(r\cos\theta, r\sin\theta))|\frac{y^{2\alpha+2}\cos^{2\alpha+1}\theta}{2^\alpha\sqrt{2\pi}\,\Gamma(\alpha+1)}d\theta\,dy\\
&= \int_{\left\{(s,x)\in[0,+\infty[\times\mathbb{R}:\ \sqrt{s^2+x^2}\leqslant r\right\}} |f(s,x)|\,d\nu_\alpha(s,x)\\
&\leqslant M_\alpha(f)(0,0)\nu_\alpha\left(\left\{(s,x)\in[0,+\infty[\times\mathbb{R}:\sqrt{s^2+x^2}\leqslant r\right\}\right)\\
&= M_\alpha(f)(0,0)\nu_\alpha\left(\left\{(s,x)\in[0,+\infty[\times\mathbb{R}:\sqrt{s^2+x^2}\leqslant 1\right\}\right)r^{2\alpha+3}.
\end{aligned}$$

Consequently, we use the integration by parts, we obtain

$$\begin{aligned}
&\int_0^{+\infty} G(r)d\left(-K_\epsilon\right)(r,0)\\
&\leqslant M_\alpha(f)(0,0)\nu_\alpha\left(\left\{(s,x)\in[0,+\infty[\times\mathbb{R}:\sqrt{s^2+x^2}\leqslant 1\right\}\right)\left(\int_0^{+\infty} r^{2\alpha+3}d\left(-K_\epsilon\right)(r,0)\right)\\
&= (2\alpha+3)M_\alpha(f)(0,0)\nu_\alpha\left(\left\{(s,x)\in[0,+\infty[\times\mathbb{R}:\sqrt{s^2+x^2}\leqslant 1\right\}\right)\int_0^{+\infty} r^{2\alpha+2}K_\epsilon(r,0)\,dr
\end{aligned}$$

Since,

$$\begin{aligned}
\nu_\alpha\left(\left\{(s,x)\in[0,+\infty[\times\mathbb{R}:\sqrt{s^2+x^2}\leqslant 1\right\}\right) &= \int_{\left\{(r,x)\in[0,+\infty[\times\mathbb{R}:\sqrt{s^2+x^2}\leqslant 1\right\}}\frac{s^{2\alpha+1}}{2^\alpha\sqrt{2\pi}\,\Gamma(\alpha+1)}ds\,dx\\
&= \frac{1}{2^\alpha\sqrt{2\pi}\,\Gamma(\alpha+1)}\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(r\cos\theta)^{2\alpha+1}r\,dr\,d\theta\\
&= \frac{1}{2^\alpha\sqrt{2\pi}\,\Gamma(\alpha+1)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2\alpha+3}\cos^{2\alpha+1}\theta\,d\theta.
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_0^{+\infty} G(r)d(-K_\epsilon)(r,0)\\
&\leqslant \frac{2\alpha+3}{(2\pi)^{\frac{1}{2}}2^\alpha\Gamma(\alpha+1)}M_\alpha(f)(0,0)\int_0^{+\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2\alpha+3}(r\cos\theta)^{2\alpha+1}rK_\epsilon(r,0)\,dr\,d\theta\\
&= M_\alpha(f)(0,0)\|K_\epsilon\|_{1,\nu_\alpha}\\
&= M_\alpha(f)(0,0)\|K\|_{1,\nu_\alpha}.
\end{aligned}$$

For the general case, let us consider an integrable function K on $[0,+\infty[\times\mathbb{R}$. We know that $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$. Then, for every $K\in L^1(d\nu_\alpha)$, there exists a sequence $(K_j)_{j\in\mathbb{N}}$ of radial, compactly supported, continuous functions increase to K such that

$$\lim_{j\to+\infty} K_j = K$$

From the Relation (3.5), we have

$$\lim_{j\to+\infty}\sup_{\epsilon>0}\left(K_{j,\epsilon}*|f|(0,0)\right)\leqslant \lim_{j\to+\infty} M_\alpha(f)(0,0)\|K_j\|_{1,\nu_\alpha}.$$

Then,

$$\sup_{\epsilon>0}\left(K_\epsilon*|f|(0,0)\right)\leqslant M_\alpha(f)(0,0)\|K\|_{1,\nu_\alpha}.$$

Let $f\in L^1_{loc}(d\nu_\alpha)$ and $(\mu,\lambda)\in[0,+\infty[\times\mathbb{R}$, we denote by

$$g(x,y)=\mathcal{T}_{(\mu,-\lambda)}(|\check{f}|)(x,y),\quad \forall (x,y)\in[0,+\infty[\times\mathbb{R}.$$

$$\begin{aligned}
M_\alpha(g)(0,0) &= \sup_{\eta>0}\frac{1}{\nu_\alpha(B((0,0),\eta))}\int_{B((0,0),\eta)}\mathcal{T}_{(0,0)}(|\check{g}|)(s,y)\,d\nu_\alpha(s,y)\\
&= \sup_{\eta>0}\frac{1}{\nu_\alpha(B((0,0),\eta))}\int_{B((0,0),\eta)}|\check{g}|(s,y)\,d\nu_\alpha(s,y)\\
&= \sup_{\eta>0}\frac{1}{\nu_\alpha(B((0,0),\eta))}\int_{B((0,0),\eta)}\mathcal{T}_{(\mu,-\lambda)}(|\check{f}|)(s,y)\,d\nu_\alpha(s,y)\\
&= M_\alpha(f)(\mu,\lambda).
\end{aligned}$$

Moreover, for all $\epsilon>0$ we have

$$\begin{aligned}
K_\epsilon*|g|(0,0) &= \int_0^{+\infty}\int_{\mathbb{R}}\mathcal{T}_{(0,0)}(|\check{g}|)(s,y)K_\epsilon(s,x)\,d\nu_\alpha(s,x) = \int_0^{+\infty}\int_{\mathbb{R}}|\check{g}|(s,y)K_\epsilon(s,x)\,d\nu_\alpha(s,x)\\
&= K_\epsilon*|f|(r,x).
\end{aligned}$$

Using the Relation (3.5)

$$\sup_{\epsilon>0}(K_\epsilon * |f|(r,x)) \leqslant M_\alpha(f)(r,x)\|K\|_{1,\nu_\alpha}. \qquad \square$$

Theorem 3.12. *For every $p \in]1,+\infty[$, $\mathcal{H}^p_\alpha$ coincides with $L^p(d\nu_\alpha)$. Moreover, there exists a constant $C_p > 0$ such that for every $f \in \mathcal{H}^p_\alpha$, we have*

$$\|f\|_{p,\nu_\alpha} \leqslant \|f\|_{\mathcal{H}^p_\alpha} \leqslant C_p\|f\|_{p,\nu_\alpha}.$$

Proof. Let $f \in \mathcal{H}^p_\alpha$. Using the Relation (3.1),

$$|p_t * f(r,x)| \leqslant \mathcal{P}^\alpha_f(r,x), \quad \forall (r,x) \in \mathbb{R}^2.$$

This implies that

$$\|p_t * f\|_{p,\nu_\alpha} \leqslant \|\mathcal{P}^\alpha_f\|_{p,\nu_\alpha} = \|f\|_{\mathcal{H}^p_\alpha} < +\infty.$$

We deduce that the set $\{p_t * f, t > 0\}$ lies in the closed ball $\overline{B(0,\|f\|_{\mathcal{H}^p_\alpha})}$ of $L^p(d\nu_\alpha)$. Moreover, $L^p(d\nu_\alpha)$ is the dual space of $L^q(d\nu_\alpha)$, where q is the conjugate exponent of p.

We define

$$\begin{array}{rccc} \Phi: & L^p(d\nu_\alpha) & \longrightarrow & (L^q(d\nu_\alpha))^* \\ & f & \longmapsto & \Phi_f \end{array}$$

where,

$$\begin{array}{rccc} \Phi_f: & L^q(d\nu_\alpha) & \longrightarrow & \mathbb{C} \\ & g & \longmapsto & \displaystyle\int_0^{+\infty}\int_{\mathbb{R}} f(r,x)g(r,x)\,d\nu_d(r,x). \end{array}$$

We know that for every $f \in L^p(d\nu_\alpha)$,

$$\|\Phi_f\|_{(L^q(d\nu_\alpha))^*} = \|f\|_{p,\nu_\alpha}.$$

Then,

$$\|\Phi_{p_t*f}\|_{(L^q(d\nu_\alpha))^*} = \|p_t * f\|_{p,\nu_\alpha} \leqslant \|f\|_{\mathcal{H}^p_\alpha} < +\infty.$$

We deduce that the set $\{\Phi_{p_t*f}, t > 0\}$ lies in the closed ball $\overline{B(0,\|f\|_{\mathcal{H}^p_\alpha})}$ of $L^p(d\nu_\alpha)$. Hence, by Banach-Alaoglu theorem, there exist a sequence $(t_j)_{j\in\mathbb{N}}$ and $f_0 \in L^p(d\nu_\alpha)$ such that

$$\lim_{t_j\to 0} \Phi_{p_{t_j}*f} = \Phi_{f_0}$$

in the the weak* topology of $L^p(d\nu_\alpha)$. Then,

$$\lim_{t_j\to 0} p_{t_j} * f = f_0 \quad \text{in} \quad L^p(d\nu_\alpha).$$

By Proposition 3.8, we obtain that for every $f \in S'_e(\mathbb{R}^2)$ bounded tempered distribution

$$\lim_{t\to 0} p_t * f = f \quad \text{in} \quad S'_e(\mathbb{R}^2).$$

Thus, f and f_0 coincides. We have (see [2])

$$\lim_{t\to 0} \|p_t * f - f\|_{p,\nu_\alpha} = 0.$$

Moreover, we have

$$\begin{aligned} \|f\|_{p,\nu_\alpha} &\leqslant \|p_t * f - f\|_{p,\nu_\alpha} + \|p_t * f\|_{p,\nu_\alpha} \\ &\leqslant \|p_t * f - f\|_{p,\nu_\alpha} + \|\mathcal{P}_f^\alpha\|_{p,\nu_\alpha}. \end{aligned}$$

Then,

$$\|f\|_{p,\nu_\alpha} \leqslant \|\mathcal{P}_f^\alpha\|_{p,\nu_\alpha} = \|f\|_{\mathcal{H}_\alpha^p}.$$

Using Proposition 3.11 for the function p_t, we have

$$\sup_{t>0} |p_t * f| \leqslant M_\alpha(f).$$

Then,

$$\|\mathcal{P}_f^\alpha\|_{p,\nu_\alpha} \leqslant \|M_\alpha(f)\|_{p,\alpha}.$$

Thus,

$$\|f\|_{\mathcal{H}_\alpha^p} \leqslant \|M_\alpha(f)\|_{p,\alpha}.$$

Now, from Theorem 3.10 we know that M_α is of strong type (p,p), $p \in]1,+\infty]$, we deduce that there exists a constant $C_p > 0$ such that

$$\|f\|_{\mathcal{H}_\alpha^p} \leqslant \|M_\alpha(f)\|_{p,\nu_\alpha} \leqslant C_p \|f\|_{p,\nu_\alpha},$$

which achieves the proof. □

Throughout this paper C denotes a positive constant that can change frome one line to next.

4 Atomic Decomposition of Hardy Spaces

Definition 4.1 (Cube)**.** *A cube of* $[0,+\infty[\times\mathbb{R}$ *is a subset of* $\mathbb{R}^2$ *such that*

$$Q = [a_0, b_0] \times [a_1, b_1],$$

where $b_0 - a_0 = b_1 - a_1 = L > 0$.

Definition 4.2 (Atomic Decomposition)**.** *A measurable function* f *on* $\mathbb{R}^2$ *even with respect to the first variable is called an* L^∞*-atom for* $\mathcal{H}^1_\alpha$*, if there exists a cube* Q *satisfying*

(i) $Supp(f) \subset Q$.

(ii) $\|f\|_{\infty,\nu_\alpha} \leqslant \dfrac{1}{\nu_\alpha(Q)}$.

(iii) $\displaystyle\int_Q f(r,x)d\nu_\alpha(r,x) = 0$.

In the next we define the atomic space $\mathcal{H}^\alpha_{atomic}$

Definition 4.3. *The space* $\mathcal{H}^\alpha_{atomic}$ *is defined as the vector space of all functions* $f \in L^1(d\nu_\alpha)$ *for which there exists a sequence* $\{f_i\}_{i\in\mathbb{N}}$ *of* L^∞*-atoms of* $\mathcal{H}^1_\alpha$ *and a sequence* $(\lambda_i)_{i\in\mathbb{N}} \in \ell^1(\mathbb{N})$*, such that*

$$f = \sum_{i=1}^{+\infty} \lambda_i f_i.$$

We set

$$\|f\|_{\mathcal{H}^\alpha_{atomic}} = \inf\left\{\sum_{i=1}^{+\infty} |\lambda_i| \mid f = \sum_{i=1}^{+\infty} \lambda_i f_i\right\}.$$

Now we introduce the following notations

- $Z^{[\alpha]}$ the set of functions $\varphi \in \mathcal{C}^1([0,+\infty[\times\mathbb{R},\mathbb{C})$ satisfying $\varphi(0,0) > 0$ and for every $(x,y) \in [0,+\infty[\times\mathbb{R}$.
  - $0 \leqslant \varphi(x,y) \leqslant \dfrac{C}{(1+x^2+y^2)^{\alpha+2}}$.
  - $0 \leqslant \dfrac{\partial\varphi}{\partial x}(x,y) \leqslant \dfrac{Cx}{(1+x^2+y^2)^{\alpha+3}}$.
  - $0 \leqslant \dfrac{\partial\varphi}{\partial y}(x,y) \leqslant \dfrac{Cy}{(1+x^2+y^2)^{\alpha+3}}$.

  Where C a positive constant depending on φ.
- We define the function h on $]0,+\infty[\times]0,+\infty[$ by

$$h(r,\gamma) = \begin{cases} \gamma r^{-2\alpha-1} & \text{if} \quad \gamma < r^{2\alpha+2}, \\ \gamma^{\frac{1}{2\alpha+2}} & \text{if} \quad \gamma \geqslant r^{2\alpha+2}. \end{cases}$$

- For every $\gamma > 0$ and for every $\varphi \in Z^{[\alpha]}$, $(r,x) \in]0,+\infty[\times\mathbb{R}$ and $(s,y) \in [0,+\infty[\times\mathbb{R}$, we set

$$\Phi^\gamma((r,x),(s,y)) = \gamma \mathcal{T}_{(r,x)}(\varphi_{h(r,\gamma)})(-s,-y),$$

where $\varphi_{h(r,\gamma)}(r,x) = \frac{1}{(h(r,\gamma))^{2\alpha+2}}\varphi\left(\frac{r}{h(r,\gamma)}, \frac{x}{h(r,\gamma)}\right)$.

- $d_\alpha((r,x),(s,y)) = \max\left(\left|\int_r^s t^{2\alpha+1}dt\right|, |x-y|\right)$, where $(r,x),(s,y) \in [0,+\infty[\times\mathbb{R}$.
- $p((r,x),(s,y)) = \max(|r-s|, |x-y|)$, where $(r,x),(s,y) \in [0,+\infty[\times\mathbb{R}$.
- $p'((r,x),(y,y')) = \max(|r-s|^{\frac{1}{2\alpha+2}}, |x-y|)$, where $(r,x),(s,y) \in [0,+\infty[\times\mathbb{R}$.

Our goal now, is to prove that $\|.\|_{\mathcal{H}^1_\alpha}$ and $\|.\|_{\mathcal{H}^\alpha_{atomic}}$ are equivalent. To do this, we need some preparation.

Proposition 4.4. *Let $f \in L^1(d\nu_\alpha)$. For every $\lambda > 0$ and $(r,x),(s,y) \in [0,+\infty[\times\mathbb{R}$, we have*

$$\mathcal{T}_{(\lambda r,\lambda x)}(f)(\lambda s,\lambda y) = \lambda^{-2\alpha-2}\mathcal{T}_{(r,x)}(f_{\lambda^{-1}})(s,y).$$

Proof.

$$\begin{aligned}
\mathcal{T}_{(\lambda r,\lambda x)}(f)(\lambda s,\lambda y) &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi f\left(\sqrt{(\lambda r)^2+(\lambda s)^2+2\lambda^2 rs\cos\theta}, \lambda x+\lambda y\right)\sin^{2\alpha}\theta\,d\theta \\
&= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi f\left(\lambda\sqrt{r^2+s^2+2rs\cos\theta}, \lambda(x+y)\right)\sin^{2\alpha}\theta\,d\theta \\
&= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \lambda^{-2\alpha-2}f_{\lambda^{-1}}\left(\lambda\sqrt{r^2+s^2+2rs\cos\theta}, \lambda(x+y)\right)\sin^{2\alpha}\theta\,d\theta \\
&= \lambda^{-2\alpha-2}\mathcal{T}_{(r,x)}(f_{\lambda^{-1}})(s,y).
\end{aligned}$$

□

Proposition 4.5. *For every $\gamma, \lambda > 0$ and $(r,x),(s,y) \in [0,+\infty[\times\mathbb{R}$, we have*

(i) $h(\lambda x, \lambda^{2\alpha+2}\gamma) = \lambda h(r,\gamma)$.

(ii) $d_\alpha((\lambda r, \lambda^{2\alpha+2}x),(\lambda s,\lambda^{2\alpha+2}y)) = \lambda^{2\alpha+2}d_\alpha((r,x),(s,y))$.

(iii) $\Phi^\gamma((\lambda r,\lambda x),(\lambda s,\lambda y)) = \Phi^{\lambda^{-2\alpha-2}\gamma}((r,x),(s,y))$.

Proof. (i)

$$\begin{aligned}
h(\lambda x, \lambda^{2\alpha+2}\gamma) &= \begin{cases} \lambda^{2\alpha+2}\gamma(\lambda x)^{-2\alpha-1}; & \text{if} \quad \lambda^{2\alpha+2}\gamma < (\lambda x)^{2\alpha+2}, \\ \lambda\gamma^{\frac{1}{2\alpha+2}}; & \text{if} \quad \gamma \geqslant x^{2\alpha+2}. \end{cases} \\
&= \begin{cases} \lambda\gamma x^{-2\alpha-1}; & \text{if} \quad \gamma < x^{2\alpha+2}, \\ \lambda\gamma^{\frac{1}{2\alpha+2}}; & \text{if} \quad \gamma \geqslant x^{2\alpha+2}. \end{cases} \\
&= \lambda h(r,\gamma).
\end{aligned}$$

(ii)

$$
\begin{aligned}
d_\alpha((\lambda r, \lambda^{2\alpha+2}x), (\lambda s, \lambda^{2\alpha+2}y)) &= \max\left(\left|\int_{\lambda s}^{\lambda r} t^{2\alpha+1}dt\right|, |\lambda^{2\alpha+2}x - \lambda^{2\alpha+2}y|\right) \\
&= \max\left(\lambda^{2\alpha+2}\left|\int_s^r t^{2\alpha+1}dt\right|, \lambda^{2\alpha+2}|x-y|\right) \\
&= \lambda^{2\alpha+2} d_\alpha((r,x),(s,y)).
\end{aligned}
$$

(iii) Using Proposition 4.4, we get

$$
\begin{aligned}
\Phi^\gamma((\lambda r, \lambda x), (\lambda s, \lambda y)) &= \gamma \mathcal{T}_{(\lambda r, \lambda x)}(\varphi_{h(\lambda r, \gamma)})(-\lambda s, -\lambda y) \\
&= \gamma \mathcal{T}_{(r,x)}(\varphi_{\lambda h(r, \lambda^{-2\alpha-2}\gamma)})(-\lambda s, -\lambda y) \\
&= \lambda^{-2\alpha-2}\gamma \mathcal{T}_{(r,x)}(\varphi_{h(r,\lambda^{-2\alpha-2}\gamma)})(-s,-y) \\
&= \Phi^{\lambda^{-2\alpha-2}\gamma}((r,x),(s,y)).
\end{aligned}
$$

□

Lemma 4.6. *There exist constants $C > 0$, $\beta > 0$ such that for every $(r,x),(s,y),(t,z) \in [0,+\infty[\times\mathbb{R}$ and $\gamma > 0$ we have*

$$
|\Phi^\gamma((r,x),(s,y)) - \Phi^\gamma((r,x),(t,z))| \leqslant C\left(\frac{d_\alpha((s,y),(t,z))}{\gamma}\right)^\beta. \tag{4.1}
$$

Proof. It is sufficient to prove the Relation (4.1) for $d_\alpha((s,y),(t,z)) < \frac{\gamma}{C}$, where C is a fixed constant large enough.

First, we will show that

$$
L = |\Phi^\gamma((1,x),(s,y)) - \Phi^\gamma((1,x),(t,z))| \leqslant C\left(\frac{d_\alpha((s,y),(t,z))}{r}\right)^\beta. \tag{4.2}
$$

$$
\begin{aligned}
L &= C\gamma\left|\int_0^\pi \varphi_{h(1,\gamma)}\left(\sqrt{1+s^2-2s\cos\theta}, x-y\right)\sin^{2\alpha}\theta\, d\theta\right. \\
&\quad \left. - \int_0^\pi \varphi_{h(1,\gamma)}\left(\sqrt{1+t^2-2t\cos\theta}, x-z\right)\sin^{2\alpha}\theta\, d\theta\right| \\
&= C\frac{\gamma}{(h(1,\gamma))^{2\alpha+2}}\left|\int_0^\pi \left(\varphi\left(\frac{\sqrt{1+s^2-2s\cos\theta}}{h(1,\gamma)}, \frac{x-y}{t(1,\gamma)}\right)\right.\right. \\
&\quad \left.\left. - \varphi\left(\frac{\sqrt{1+t^2-2t\cos\theta}}{h(1,\gamma)}, \frac{x-z}{h(1,\gamma)}\right)\right)\sin^{2\alpha}\theta\, d\theta\right|.
\end{aligned}
$$

Let f be a function defined by

$$
\begin{array}{rccc}
g: & [0,+\infty[\times\mathbb{R} & \longrightarrow & \mathbb{R} \\
 & (s,t) & \longmapsto & (g_1(s,t), g_2(s,t)) = \left(\dfrac{\sqrt{1+s^2-2s\cos\theta}}{h(1,\gamma)}, \dfrac{x-t}{h(1,\gamma)}\right)
\end{array}
$$

$$\begin{aligned}\frac{\partial(\varphi\circ g)}{\partial s}(s,t) &= \frac{\partial\varphi}{\partial s}(g(s,t))\frac{\partial g_1}{\partial s}(s,t)+\frac{\partial\varphi}{\partial t}(g(s,t))\frac{\partial g_2}{\partial s}(s,t)\\ &= \frac{1}{h(1,\gamma))}\left(\frac{s-\cos\theta}{\sqrt{1+s^2-2s\cos\theta}}\frac{\partial\varphi}{\partial s}(g(s,t))-\frac{\partial\varphi}{\partial t}(g(s,t))\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial(\varphi\circ g)}{\partial t}(s,t) &= \frac{\partial\varphi}{\partial s}(g(s,t))\frac{\partial g_1}{\partial t}(s,t)+\frac{\partial\varphi}{\partial t}(g(s,t))\frac{\partial g_2}{\partial t}(s,t)\\ &= -\frac{1}{h(1,\gamma)}\frac{\partial\varphi}{\partial t}(g(s,t)).\end{aligned}$$

Since $\varphi\in Z^{[\alpha]}$, we use the mean value theorem , there exist $(u,u')\in[(s,y),(t,z)]$ such that

$$\begin{aligned}|\varphi\circ g(s,y)-\varphi\circ g(t,z)| &\leqslant \|(s,y)-(t,z)\|_\infty \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}\|d(\varphi\circ g)(u,u')\|\\ &= p((s,y),(t,z))\sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}\|d(\varphi\circ g)(u,u')\|\end{aligned}$$

Then,

$$\begin{aligned}L &\leqslant C\frac{\gamma}{(h(1,\gamma))^{2\alpha+2}}p((s,y),(t,z))\left|\int_0^\pi \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}\|d(\varphi\circ g)(u,u')\|\sin^{2\alpha}\theta\,d\theta\right|\\ &\leqslant C\frac{\gamma}{(h(1,\gamma))^{2\alpha+3}}p((s,y),(t,z))\\ &\times \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}\int_0^\pi\left|\frac{u-\cos\theta}{\sqrt{1+u^2-2u\cos\theta}}\frac{\partial\varphi}{\partial s}\left(\frac{\sqrt{1+u^2-2u\cos\theta}}{h(1,\gamma)},\frac{x-u'}{h(1,\gamma)}\right)\right|\\ &+2\left|\frac{\partial\varphi}{\partial t}\left(\frac{\sqrt{1+u^2-2u\cos\theta}}{h(1,\gamma)},\frac{x-u'}{h(1,\gamma)}\right)\right|\sin^{2\alpha}\theta\,d\theta\\ &\leqslant C\gamma(h(1,\gamma))^2p((s,y),(t,z))\\ &\times \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}\int_0^\pi\frac{(|u-\cos\theta|+2|x-u'|)}{((h(1,\gamma))^2+1+u^2-2u\cos\theta+(x-u')^2)^{\alpha+3}}\sin^{2\alpha}\theta\,d\theta\\ &\leqslant C\gamma(h(1,\gamma))^2p((s,y),(t,z))\\ &\times \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}\int_0^\pi\frac{(1-\cos\theta)+|u-1|+2|x-u'|)}{((h(1,\gamma))^2+1+u^2-2u\cos\theta+(x-u')^2)^{\alpha+3}}\sin^{2\alpha}\theta\,d\theta\\ &\leqslant C\gamma(h(1,\gamma))^2p((s,y),(t,z))\\ &\times \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}\int_0^\pi\frac{(1-\cos\theta)+3p((1,x),(u,u'))}{((h(1,\gamma))^2+1+u^2-2u\cos\theta+(x-u')^2)^{\alpha+3}}\sin^{2\alpha}\theta\,d\theta\\ &\leqslant C\gamma(h(1,\gamma))^2p((s,y),(t,z))\\ &\sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}\int_0^\pi\frac{(1-\cos\theta)+3p((1,x),(u,u'))}{((h(1,\gamma))^2+(p((1,x),(u,u')))^2+2u(1-\cos\theta))^{\alpha+3}}\sin^{2\alpha}\theta\,d\theta\end{aligned}$$

Set

$$E(\gamma,u,u',\theta)=\frac{(1-\cos\theta)+3p((1,x),(u,u'))}{((h(1,\gamma))^2+(p((1,x),(u,u')))^2+2u(1-\cos\theta))^{\alpha+3}}.$$

Case 1 $\gamma\geqslant 1$**:** we have $h(1,\gamma)=\gamma^{\frac{1}{2\alpha+2}}$

- If $p((1,x),(u,u'))\leqslant\gamma^{\frac{1}{2\alpha+2}}$. Then,

$$E(\gamma,u,u',\theta)\leqslant(2+3\gamma^{\frac{1}{2\alpha+2}})\gamma^{-\frac{2\alpha+6}{2\alpha+2}}=3\gamma^{-\frac{2\alpha+5}{2\alpha+2}}(\gamma^{-\frac{1}{2\alpha+2}}+1)\leqslant 6\gamma^{-\frac{2\alpha+5}{2\alpha+2}}.$$

We have

$$p((s,y),(t,z))\leqslant(d_\alpha((s,y),(t,z)))^{\frac{1}{2\alpha+2}}.$$

Then,

$$L\leqslant C\gamma\gamma^{\frac{2}{2\alpha+2}}\gamma^{-\frac{2\alpha+5}{2\alpha+2}}p((y,y'),(z,z'))\leqslant C\frac{p((y,y'),(z,z'))}{\gamma^{\frac{1}{2\alpha+2}}}\leqslant C\left(\frac{d_\alpha((y,y'),(z,z'))}{\gamma}\right)^{\frac{1}{2\alpha+2}}$$

- If $p((1,x),(u,u'))\geqslant\gamma^{\frac{1}{2\alpha+2}}$. Then,

$$E(r,u,u',\theta)\leqslant 3\frac{1+p((1,x),(u,u'))}{(p((1,x),(u,u')))^{2\alpha+6}}\leqslant 6\frac{p((1,x),(u,u'))}{(p((1,x),(u,u')))^{2\alpha+6}}=6(p((1,x),(u,u')))^{-2\alpha-5}$$

Thus,

$$\begin{aligned}L&\leqslant C\gamma p((s,y),(t,z))\gamma^{\frac{2}{2\alpha+2}}\sup_{(u,u')\in[0,+\infty[\times\mathbb{R}}(p((1,x),(u,u')))^{-2\alpha-5}\\&\leqslant C\frac{p((s,y),(t,z))}{\gamma^{\frac{1}{2\alpha+2}}}\\&\leqslant C\left(\frac{d_\alpha((s,y),(t,z))}{\gamma}\right)^{\frac{1}{2\alpha+2}}.\end{aligned}$$

Case 2 $\gamma<1$**:** $h(1,\gamma)=\gamma$.

- $p((1,x),(u,u'))\geqslant\frac{1}{4}$

$$\begin{aligned}E(\gamma,u,u',\theta)&=\frac{(1-\cos\theta)+3p((1,x),(u,u'))}{(\gamma^2+(p((1,x),(u,u'))^2+2u(1-\cos\theta))^{\alpha+3}}\\&\leqslant\frac{2+3p((1,x),(u,u'))}{(p((1,x),(u,u')))^{2\alpha+6}}\\&\leqslant 11(p((1,x),(u,u')))^{-2\alpha-5}\\&\leqslant 4^{2\alpha+5}11.\end{aligned}$$

Then,

$$L\leqslant C\gamma^2p((s,y),(t,z))\leqslant C\left(\frac{d_\alpha((s,y),(t,z))}{\gamma}\right)^{\frac{1}{2\alpha+2}}.$$

- If $p((1,x),(u,u')) < \frac{1}{4}$ and $p((1,x),(u,u')) > \frac{\gamma}{4}$.

$$\begin{aligned}
L &\leqslant C\gamma^3 p((s,y),(t,z)) \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}} \int_0^{p((1,x),(u,u'))} \frac{\theta^{2\alpha}(\frac{\theta^2}{2}+3p((1,x),(u,u')))}{(p((1,x),(u,u')))^{2\alpha+6}}\,d\theta \\
&+ \int_{p((1,x),(u,u'))}^{\pi} \frac{\theta^{2\alpha}(\frac{\theta^2}{2}+3p((1,x),(u,u')))}{\theta^{2\alpha+6}} d\theta \\
&\leqslant C\gamma^3 p((s,y),(t,z)) \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}} \frac{p((1,x),(u,u'))+1}{(p((1,x),(u,u')))^4} \\
&+ \int_{p((1,x),(u,u'))}^{\pi} \frac{1}{2\theta^4} + \frac{3p((1,x),(u,u'))}{\theta^5}\,d\theta \\
&\leqslant C\gamma^3 p((s,y),(t,z)) \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}} \frac{5}{4(p((1,x),(u,u')))^4} + \frac{2}{(p((1,x),(u,u')))^3} \\
&\leqslant C\gamma^3 p((s,y),(t,z)) \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}} \frac{1}{(p((1,x),(u,u')))^4} \\
&\leqslant C\frac{p((s,y),(t,z))}{\gamma}.
\end{aligned}$$

- If $p((1,x),(u,u')) < \frac{\gamma}{4}$

$$\begin{aligned}
L &\leqslant C\gamma^3 p((s,y),(t,z)) \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}} \int_0^{\gamma} \frac{\theta^{2\alpha}(\frac{\theta^2}{2}+2p((1,x),(u,u')))}{\gamma^{2\alpha+6}}\,d\theta \\
&+ \int_{\gamma}^{\frac{\pi}{2}} \frac{\theta^{2\alpha}(\frac{\theta^2}{4}+2p((1,x),(u,u')))}{\theta^{2\alpha+6}} d\theta + \int_{\frac{\pi}{2}}^{\pi} \frac{\theta^{2\alpha}(\frac{\theta^2}{4}+2p((1,x),(u,u')))}{\theta^{2\alpha+6}} d\theta \\
&\leqslant C\gamma^3 p((s,y),(t,z)) \\
&\times \sup_{(u,u')\in[0,+\infty[\times\mathbb{R}} \left(\frac{1}{4\gamma^3} + \frac{2p((1,x),(u,u'))}{\gamma^5} + \frac{1}{8\gamma^3} + \frac{p((1,x),(u,u'))}{2\gamma^5} + 2p((1,x),(u,u'))\right) \\
&\leqslant C\gamma^3 p((s,y),(t,z)) \left(\frac{3}{8\gamma^3} + \frac{1}{\gamma^4} + \frac{\gamma}{2}\right) \\
&\leqslant Cp((s,y),(t,z)) \left(\frac{3}{8} + \frac{1}{\gamma} + \frac{\gamma^3}{2}\right) \\
&\leqslant C\frac{p((s,y),(t,z))}{\gamma}.
\end{aligned}$$

Using (ii) and (iii) of Proposition 4.5 and the Relation (4.2), we get

$$|\Phi^\gamma((r,x),(s,y)) - \Phi^\gamma((r,x),(t,z))| \leqslant C\left(\frac{d_\alpha((s,y),(t,z))}{\gamma}\right)^\beta. \qquad \square$$

Proposition 4.7. *There exist constants $A > 0$ and $\beta > 0$ such that*

(i) $\Phi^\gamma((r,x),(r,x)) > \frac{1}{A}$, $\gamma > 0$ *and* $(r,x) \in [0,+\infty[\times\mathbb{R}$.

(ii) $0 \leqslant \Phi^\gamma((r,x),(s,y)) \leqslant A\left(1 + \frac{d_\alpha((r,x),(s,y))}{\gamma}\right)^{-1-\beta}$, $\gamma > 0$ *and* $(r,x),(s,y) \in [0,+\infty[\times\mathbb{R}$.

(iii) For every $\gamma > 0$ *and* $(r,x),(s,y),(t,z) \in [0,+\infty[\times\mathbb{R}$*, such that*

$$d_\alpha((s,y),(t,z)) \leqslant \frac{1}{4A}(\gamma + d_\alpha((r,x),(s,y))),$$

we have

$$|\Phi^\gamma((r,x),(s,y)) - \Phi^\gamma((r,x),(t,z))| \leqslant A\left(\frac{d_\alpha((s,y),(t,z))}{\gamma}\right)^\beta \left(1+\frac{d_\alpha((r,x),(s,y))}{\gamma}\right)^{-1-2\beta}.$$

Proof. Let $\varphi \in Z^{[\alpha]}$

(i) First, we will show that there exists a constant $A > 0$ such that

$$\forall x \in \mathbb{R}, \quad \Phi^\gamma((1,x),(1,x)) > \frac{1}{A}.$$

We know that $\varphi(0,0) > 0$. Then, there exist constants $a > 0$ and $b > 0$ such that for every $0 < r < b$ we have

$$\varphi(r,0) > a. \tag{4.3}$$

- If $\gamma < 1$, then $h(1,\gamma) = \gamma$.

$$\begin{aligned}\Phi^\gamma((1,x),(1,x)) &= \gamma \mathcal{T}_{(1,x)}(\varphi_\gamma)(-1,-x)\\ &= \gamma\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \varphi_\gamma\left(\sqrt{2(1-\cos\theta)},0\right)\sin^{2\alpha}\theta\,d\theta.\\ &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \frac{1}{\gamma^{2\alpha+1}}\varphi\left(\frac{\sqrt{2(1-\cos\theta)}}{\gamma},0\right)\sin^{2\alpha}\theta\,d\theta.\end{aligned}$$

By the Relation (4.3), there exists b' such that for every $0 < \theta < \gamma b'$, we have

$$\varphi\left(\frac{\sqrt{2(1-\cos\theta)}}{\gamma},0\right) > a.$$

Then,

$$\begin{aligned}\Phi^\gamma((1,x),(1,x)) &\geqslant \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^{\gamma b'} \frac{a}{\gamma^{2\alpha+1}}\sin^{2\alpha}\theta\,d\theta.\\ &\geqslant \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^{\gamma b'} a\sin^{2\alpha}\theta\,d\theta.\end{aligned}$$

- If $\gamma \geqslant 1$, then $h(1,\gamma) = \gamma^{\frac{1}{2\alpha+2}}$.

$$\begin{aligned}\Phi^\gamma((1,x),(1,x)) &= \gamma \mathcal{T}_{(1,x)}\left(\varphi_{\gamma^{\frac{1}{2\alpha+2}}}\right)(-1,-x)\\ &= \gamma\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \varphi_{\gamma^{\frac{1}{2\alpha+2}}}\left(\sqrt{2(1-\cos\theta)},0\right)\sin^{2\alpha}\theta\,d\theta\\ &= \gamma\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \frac{1}{\gamma}\varphi\left(\frac{\sqrt{2(1-\cos\theta)}}{\gamma^{\frac{1}{2\alpha+2}}},0\right)\sin^{2\alpha}\theta\,d\theta\end{aligned}$$

$$= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^{\pi}\varphi\left(\frac{\sqrt{2(1-\cos\theta)}}{\gamma^{\frac{1}{2\alpha+2}}},0\right)\sin^{2\alpha}\theta\,d\theta$$
$$\geqslant \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^{b'} a\sin^{2\alpha}\theta\,d\theta.$$

Thus,

$$\Phi^{\gamma}((1,x),(1,x)) > \frac{1}{A}, \quad \forall x\in\mathbb{R}.$$

Let $(r,x)\in]0,+\infty[\times\mathbb{R}$. Using (iii) of Proposition 4.5 we have

$$\Phi^{\gamma}((r,x),(r,x)) = \Phi^{\gamma r^{-2\alpha-3}}\left(\left(1,\frac{x}{r}\right),\left(1,\frac{x}{r}\right)\right)\geqslant\frac{1}{A}.$$

For $r=0$ is obvious .

(ii) First, we have to show that

$$0\leqslant\Phi^{\gamma}((1,x),(s,y))\leqslant C\left(1+\frac{d_\alpha((1,x),(s,y))}{\gamma}\right)^{-1-\beta}. \tag{4.4}$$

Case 1 $\gamma<1$**:** $h(1,\gamma)=\gamma$.

$$\begin{aligned}\Phi^{\gamma}((1,x),(s,y)) &= \gamma\mathcal{T}_{(1,x)}(\varphi_\gamma)(-s,-y)\\
&= \gamma\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^{\pi}\varphi_\gamma\left(\sqrt{1+s^2-2s\cos\theta},\frac{x-y}{\gamma}\right)\sin^{2\alpha}\theta\,d\theta\\
&= \gamma\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^{\pi}\frac{1}{\gamma^{2\alpha+2}}\varphi\left(\frac{\sqrt{1+s^2-2s\cos\theta}}{\gamma},\frac{x-y}{\gamma}\right)\sin^{2\alpha}\theta\,d\theta\\
&\leqslant \gamma\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^{\pi}\frac{1}{\gamma^{2\alpha+2}}\frac{\gamma^{2\alpha+4}}{(\gamma^2+1+s^2-2s\cos\theta+(x-y)^2)^{\alpha+2}}\sin^{2\alpha}\theta\,d\theta\\
&\leqslant \gamma^3\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^{\pi}\frac{1}{(\gamma^2+(1-s)^2+2s-2s\cos\theta+(x-y)^2)^{\alpha+2}}\sin^{2\alpha}\theta\,d\theta.\end{aligned}$$

- If $\frac{\gamma}{2}\leqslant|1-s|$ and $\frac{1}{2}\leqslant s\leqslant 2$, then

$$d_\alpha((1,x),(s,y))\sim p\left((1,x),(s,y)\right).$$

In fact,

$$\frac{1}{2^{2\alpha+1}}|1-s|\leqslant\left|\int_1^s t^{2\alpha+1}dt\right|\leqslant 2^{2\alpha+1}|1-s|.$$

Then,

$$\frac{1}{2^{2\alpha+1}}p\left((1,x),(s,y)\right)\leqslant d_\alpha\left((1,x),(s,y)\right)\leqslant 2^{2\alpha+1}p\left((1,x),(s,y)\right).$$

$$\begin{aligned}
&\Phi^\gamma((1,x),(s,y))\\
&\leqslant \gamma^3 \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \frac{1}{\left(\gamma^2+\left(p\left((1,x),(s,y)\right)\right)^2+2s(1-\cos\theta)\right)^{\alpha+2}}\sin^{2\alpha}\theta\, d\theta\\
&\leqslant C\gamma^3\left(\int_0^{p((1,x),(s,y))}\frac{\sin^{2\alpha}(\theta)}{(p\left((1,x),(s,y)\right))^{2\alpha+4}}\,d\theta+\int_{p((1,x),(s,y))}^{\pi}\frac{\sin^{2\alpha}\theta}{\theta^{2\alpha+4}}\,d\theta\right)\\
&\leqslant C\gamma^3\left(\int_0^{p((1,x),(s,y))}\frac{\theta^{2\alpha}}{(p\left((1,x),(s,y)\right))^{2\alpha+4}}\,d\theta+\int_{p((1,x),(s,y))}^{\pi}\frac{\theta^{2\alpha}}{\theta^{2\alpha+4}}\,d\theta\right)\\
&\leqslant C\gamma^3\left(\frac{1}{(p\left((1,x),(s,y)\right))^3}-\frac{1}{3\pi^2}+\frac{1}{3(p\left((1,x),(s,y)\right))^3}\right)\\
&\leqslant C\gamma^3\frac{1}{(p\left((1,x),(s,y)\right))^3}.
\end{aligned}$$

Since, $\frac{\gamma}{2}\leqslant|1-s|\leqslant p\left((1,x),(s,y)\right)$ then

$$\gamma+p\left((1,x),(s,y)\right)\leqslant 3p\left((1,x),(s,y)\right).$$

Then,

$$\begin{aligned}
\Phi^\gamma((1,x),(s,y))&\leqslant C\gamma^3\frac{1}{(\gamma+p\left((1,x),(s,y)\right))^2}\\
&=C\gamma^3\left(1+\frac{p\left((1,x),(s,y)\right)}{\gamma}\right)^{-3}\\
&\leqslant C\left(1+\frac{d_\alpha\left((1,x),(s,y)\right)}{\gamma}\right)^{-3}.
\end{aligned}$$

- If $\frac{\gamma}{2}\leqslant|1-s|$, $|x-y|>1$ and $|1-s|>\frac{1}{2}$, then

$$d_\alpha((1,x),(s,y))\leqslant 2^{2\alpha+1}\left(p\left((1,x),(s,y)\right)\right)^{2\alpha+2}.$$

In fact, we have

$$\frac{1}{2^{2\alpha+1}}\left|\int_1^s t^{2\alpha+1}dt\right|\leqslant|1-s||1-s|^{2\alpha+1}\leqslant\left(p\left((1,x),(s,y)\right)\right)^{2\alpha+2},$$

and

$$|x-y|\leqslant|x-y|^{2\alpha+2}.$$

$$\begin{aligned}
\Phi^\gamma((1,x),(s,y))&\leqslant\gamma^3C\int_0^\pi\frac{\sin^{2\alpha}\theta}{(p\left((1,x),(s,y)\right))^{2\alpha+4}}\,d\theta\\
&\leqslant\gamma^3C\frac{1}{(p\left((1,x),(s,y)\right))^{2\alpha+4}}\\
&\leqslant\gamma^3C\frac{1}{(d_\alpha((1,x),(s,y)))^{\frac{2\alpha+4}{2\alpha+2}}}
\end{aligned}$$

$$
\begin{aligned}
&= \gamma^3 C \frac{1}{(d_\alpha((1,x),(s,y)))^{1+\frac{1}{\alpha+1}}}\\
&\leqslant \gamma^3 C \frac{1}{(r + d_\alpha((1,x),(s,y)))^{1+\frac{1}{\alpha+1}}}\\
&\leqslant \gamma^{1+\frac{1}{\alpha+1}} C \frac{1}{(\gamma + d_\alpha((1,x),(s,y)))^{1+\frac{1}{\alpha+1}}}\\
&= C\left(1 + \frac{d_\alpha((1,x),(s,y))}{\gamma}\right)^{-1-\frac{1}{\alpha+1}}
\end{aligned}
$$

- If $\frac{\gamma}{2} \leqslant |1-s|$, $|x-y| \leqslant 1$ and $|1-s| > \frac{1}{2}$, then

$$\frac{1}{2^{2\alpha+1}} d_\alpha((1,x),(s,y)) \leqslant \left(p'\left((1,x),(s,y)\right)\right)^{2\alpha+2}.$$

In fact, we have

$$\frac{1}{2^{2\alpha+1}} \left|\int_1^s t^{2\alpha+1} dt\right| \leqslant |1-s||1-s|^{2\alpha+1} \leqslant |1-s|^{2\alpha+2},$$

and

$$|x-y| \leqslant \left(|x-y|^{\frac{1}{2\alpha+2}}\right)^{2\alpha+2}.$$

$$
\begin{aligned}
\Phi^\gamma((1,x),(s,y)) &\leqslant \gamma^3 C \int_0^\pi \frac{\sin^{2\alpha}\theta}{(p'\left((1,x),(s,y)\right))^{2\alpha+4}} \, d\theta\\
&\leqslant \gamma^3 C \frac{1}{(p\left((1,x),(s,y)\right))^{2\alpha+4}}\\
&\leqslant \gamma^3 C \frac{1}{(d_\alpha((1,x),(s,y)))^{\frac{2\alpha+4}{2\alpha+2}}}\\
&= \gamma^3 C \frac{1}{(d_\alpha((1,x),(s,y)))^{1+\frac{1}{\alpha+1}}}\\
&\leqslant \gamma^3 C \frac{1}{(\gamma + d_\alpha((1,x),(s,y)))^{1+\frac{1}{\alpha+1}}}\\
&\leqslant \gamma^{1+\frac{1}{\alpha+1}} C \frac{1}{(\gamma + d_\alpha((1,x),(s,y)))^{1+\frac{1}{\alpha+1}}}\\
&= C\left(1 + \frac{d_\alpha((1,x),(s,y))}{\gamma}\right)^{-1-\frac{1}{\alpha+1}}
\end{aligned}
$$

- If $\frac{\gamma}{2} > |1-s|$ and $|x-y| < \frac{\gamma}{2}$ then $\frac{1}{2} < s < \frac{3}{2}$ and $p\left((1,x),(s,y)\right) \sim d_\alpha\left((1,x),(s,y)\right).$
In fact, we have

$$\left(\frac{1}{2}\right)^{2\alpha+1} |1-s| \leqslant \left|\int_1^s t^{2\alpha+1} dt\right| \leqslant \left(\frac{3}{2}\right)^{2\alpha+1} |1-s|.$$

Then,

$$\left(\frac{1}{2}\right)^{2\alpha+1} p\left((1,x),(s,y)\right) \leqslant d_\alpha\left((1,x),(s,y)\right) \leqslant \left(\frac{3}{2}\right)^{2\alpha+1} p\left((1,x),(s,y)\right).$$

$$\begin{aligned}
\Phi^\gamma((1,x),(y,y')) &\leqslant \gamma^3 C\left(\int_0^\gamma \frac{\sin^{2\alpha}\theta}{\gamma^{2\alpha+4}}\,d\theta + \int_\gamma^\pi \frac{\sin^{2\alpha}\theta}{\theta^{2\alpha+4}}\,d\theta\right)\\
&\leqslant \gamma^3 C\left(\int_0^\gamma \frac{\theta^{2\alpha}}{\gamma^{2\alpha+4}}\,d\theta + \int_\gamma^\pi \frac{\theta^{2\alpha}}{\theta^{2\alpha+4}}\,d\theta\right)\\
&\leqslant \gamma^3 C\left(\frac{\gamma^{2\alpha+1}}{\gamma^{2\alpha+4}} + \int_\gamma^\pi \frac{1}{\theta^4}\,d\theta\right)\\
&\leqslant C.
\end{aligned}$$

Since $\frac{\gamma}{2} > p\left((1,x),(s,y)\right)$ then

$$1 + \frac{p\left((1,x),(s,y)\right)}{\gamma} < \frac{3}{2}.$$

Thus,

$$\frac{8}{27}\left(1 + \frac{p\left((1,x),(s,y)\right)}{\gamma}\right)^{-3} > 1.$$

Then

$$\begin{aligned}
\Phi^\gamma((1,x),(s,y)) &\leqslant C\left(1 + \frac{p\left((1,x),(s,y)\right)}{r}\right)^{-3}\\
&\leqslant C\left(1 + \frac{d_\alpha\left((1,x),(s,y)\right)}{\gamma}\right)^{-3}
\end{aligned}$$

- If $\frac{\gamma}{2} > |1-s|$ and $|x-y| \geqslant \frac{\gamma}{2}$ then, we have

$$\frac{1}{2} < s < \frac{3}{2} \text{ and } p\left((1,x),(s,y)\right) \leqslant d_\alpha\left((1,x),(s,y)\right).$$

$$\Phi^\gamma((1,x),(s,y)) \leqslant \gamma^3 \frac{1}{(p\left((1,x),(s,y)\right))^3}.$$

Since, $\frac{\gamma}{2} \leqslant p\left((1,x),(s,y)\right)$ then, we have

$$\gamma + p\left((1,x),(s,y)\right) \leqslant 3p\left((1,x),(s,y)\right).$$

Then,

$$\begin{aligned}\Phi^{\gamma}((1,x),(s,y)) &\leqslant C\gamma^{2}\frac{1}{(\gamma+p\left((1,x),(s,y)\right))^{3}}\\ &= C\left(1+\frac{p\left((1,x),(s,y)\right)}{\gamma}\right)^{-3}\\ &\leqslant C\left(1+\frac{d_{\alpha}\left((1,x),(s,y)\right)}{\gamma}\right)^{-3}.\end{aligned}$$

Case 2 $\gamma \geqslant 1$**:** $h(1,\gamma)=\gamma^{\frac{1}{2\alpha+2}}$.

$$\begin{aligned}\Phi^{\gamma}((1,x),(s,y)) &= \gamma\mathcal{T}_{(1,x)}\left(\varphi_{\gamma^{\frac{1}{2\alpha+2}}}\right)(-s,-y)\\ &\leqslant \gamma^{\frac{2\alpha+4}{2\alpha+2}}C\int_{0}^{\pi}\frac{\sin^{2\alpha}\theta}{(\gamma^{\frac{2}{2\alpha+2}}+(p\left((1,x),(s,y)\right))^{2}+2s(1-\cos\theta))^{\alpha+2}}\,d\theta\end{aligned}$$

- If $\gamma^{\frac{1}{2\alpha+2}}\leqslant|1-s|$ and $|x-y|>\gamma^{\frac{1}{2\alpha+2}}$, then $s\geqslant 2$ and

  $$d_{\alpha}((1,x),(s,y))^{\frac{1}{2\alpha+2}}\sim p\left((1,x),(s,y)\right).$$

  In fact, we have

  $$|1-s|^{2\alpha+2}\leqslant\left|\int_{1}^{s}t^{2\alpha+1}dt\right|$$

  and

  $$|x-y|^{2\alpha+2}\leqslant|x-y|.$$

  Then,

  $$p\left((1,x),(s,y)\right)^{2\alpha+2}\leqslant d_{\alpha}((1,x),(s,y)).$$

  We use the fact that $f(y)=\dfrac{y^{2\alpha+2}-1}{(y-1)^{2\alpha+2}}$ is a bounded function in $[2,+\infty[$. Then,

  $$\left|\int_{1}^{s}t^{2\alpha+1}dt\right|\leqslant C|1-s|^{2\alpha+2}\tag{4.5}$$

  $$|1-x|\leqslant|1-x|^{2\alpha+2}.$$

  Then,

  $$d_{\alpha}((1,x),(s,y))\leqslant Cp\left((1,x),(s,y)\right)^{2\alpha+2}.$$

$$\begin{aligned}\Phi^{\gamma}((1,x),(s,y)) &\leqslant \gamma^{\frac{2\alpha+4}{2\alpha+2}}C\int_{0}^{\pi}\frac{\sin^{2\alpha}\theta}{(p\left((1,x),(s,y)\right))^{2\alpha+4}}\,d\theta\\ &\leqslant \gamma^{\frac{2\alpha+4}{2\alpha+2}}C\frac{1}{(p\left((1,x),(s,y)\right))^{2\alpha+4}}\end{aligned}$$

$$\leqslant C \frac{1}{\left(\frac{d_\alpha((1,x),(s,y))}{\gamma}\right)^{\frac{2\alpha+4}{2\alpha+2}}}$$

$$\leqslant C\left(1+\frac{d_\alpha((1,x),(s,y))}{\gamma}\right)^{-1-\frac{1}{\alpha+1}}.$$

- If $\gamma^{\frac{1}{2\alpha+2}} \leqslant |1-s|$ and $|x-y| \leqslant \gamma^{\frac{1}{2\alpha+2}}$, then $s \geqslant 2$ and $|x-y| \leqslant |1-s|$. Using the Relation (4.5), we get

$$\left|\int_1^s t^{2\alpha+1}dt\right| \leqslant C|1-s|^{2\alpha+2} \leqslant Cp\left((1,x),(s,y)\right)^{2\alpha+2}$$

$$|x-y| \leqslant |1-s| \leqslant |1-s|^{2\alpha+2} \leqslant p\left((1,x),(s,y)\right)^{2\alpha+2}$$

$$d_\alpha((1,x),(s,y))^{\frac{1}{2\alpha+2}} \leqslant p\left((1,x),(s,y)\right).$$

Thus,

$$\Phi^\gamma((1,x),(s,y)) \leqslant C\left(1+\frac{d_\alpha((1,x),(s,y))}{\gamma}\right)^{-1-\frac{1}{\alpha+1}}.$$

- If $\gamma^{\frac{1}{2\alpha+2}} > |1-s|$ and $\gamma^{\frac{1}{2\alpha+2}} > |x-y|$, then

$$d_\alpha((1,x),(s,y)) \leqslant 2^{2\alpha+2}\gamma.$$

In fact,

$$\left|\int_1^s t^{2\alpha+1}dt\right| \leqslant (\gamma^{\frac{1}{2\alpha+2}}+1)|1-s| \leqslant (\gamma^{\frac{1}{2\alpha+2}}+1)\gamma^{\frac{1}{2\alpha+2}} \leqslant (\gamma^{\frac{1}{2\alpha+2}}+1)\gamma \leqslant 2^{2\alpha+2}\gamma,$$

and

$$|x-y| \leqslant \gamma^{\frac{1}{2\alpha+2}} \leqslant \gamma$$

$$\Phi^\gamma((1,x),(s,y)) \leqslant \gamma^{\frac{2\alpha+4}{2\alpha+2}} C\int_0^\pi \frac{\sin^{2\alpha}\theta}{\gamma^{\frac{2\alpha+4}{2\alpha+2}}}\,d\theta \leqslant C \leqslant C\left(1+\frac{d_\alpha((1,x),(s,y))}{\gamma}\right)^{-3}.$$

- If $\gamma^{\frac{1}{2\alpha+2}} > |1-s|$ and $\gamma^{\frac{1}{2\alpha+2}} \leqslant |x-y|$, then

$$d_\alpha((1,x),(s,y)) \leqslant (p((1,x),(s,y)))^{2\alpha+2}.$$

In fact, we have

$$|1-s| < |x-y|.$$

This implies that

$$p((1,x),(s,y)) = |x-y|$$

$$\left|\int_1^s t^{2\alpha+1}dt\right| \leqslant (\gamma^{\frac{1}{2\alpha+2}}+1)^{2\alpha+1} \leqslant 2^{2\alpha}(\gamma+1) \leqslant 2^{2\alpha+1}\gamma \leqslant 2^{2\alpha+1}|x-y|^{2\alpha+2},$$

and

$$|x-y| < |x-y|^{2\alpha+2}$$

$$\tilde{\phi}_r((1,x),(s,y)) \leqslant C\left(1+\frac{d_\alpha((1,x),(s,y))}{\gamma}\right)^{-1-\frac{1}{\alpha+1}}.$$

From (4.4), we get

$$0 \leqslant \Phi^\gamma((r,x),(s,y)) \leqslant C\left(1+\frac{d_\alpha((r,x),(s,y))}{\gamma}\right)^{-1-\beta}.$$

Now, we will prove (iii) of the Proposition 4.7. Assume that for every $\gamma > 0$ and $(r,x),(s,y),(t,z) \in [0,+\infty[\times\mathbb{R}$, we have

$$d_\alpha((s,y),(t,z)) \leqslant \frac{r+d_\alpha((r,x),(s,y))}{4C}.$$

Then for every $\gamma' > 0$, we have

$$\left(1+\frac{d_\alpha((r,x),(t,z))}{\gamma}\right)^{-1-\gamma'} \leqslant \left(1+\frac{d_\alpha((r,x),(s,y))}{\gamma}\right)^{-1-\gamma'}. \tag{4.6}$$

Using (ii) and the Relation (4.6) , we have

$$|\Phi^\gamma((r,x),(s,y)) - \Phi^\gamma((r,x),(t,z))| \leqslant C\left(\frac{d_\alpha((s,y),(t,z))}{\gamma}\right)^{-1-\beta}. \tag{4.7}$$

Finally, using Lemma 4.1 and the Relation (4.8) we have

$$|\Phi^\gamma((r,x),(s,y)) - \Phi^\gamma((r,x),(t,z)))| \leqslant C\left(\frac{d_\alpha((s,y),(t,z))}{\gamma}\right)^{\beta}\left(1+\frac{d_\alpha((r,x),(s,y))}{\gamma}\right)^{-1-2\beta}. \quad \square$$

Proposition 4.8. *There exists a constant $C > 0$ such that for every $f \in L^1(d\nu_\alpha)$ we have*

$$\frac{1}{C}\|f\|_{\mathcal{H}^1_\alpha} \leqslant \|f\|_{\mathcal{H}^\alpha_{atomic}} \leqslant C\|f\|_{\mathcal{H}^1_\alpha}. \tag{4.8}$$

Proof. It is clear that $p_t \in Z^{[\alpha]}$. Using Proposition 4.7, Corollary 1 of [19]. Thus, we have show that (4.6) holds. $\square$

References

[1] B. Amri, “The Hardy-Littlewood operator associated with the Riemann-Liouville transform”, *Indag. Math. (N.S.)*, vol. 29, no. 5, pp. 1270–1289, 2018. doi: 10.1016/j.indag.2018.05.007

[2] B. Amri and L. Rachdi, “The Littlewood-Paley g-function associated with the Riemann-Liouville operator”, *Ann. Univ. Paedagog. Crac. Stud. Math.*, vol. 12, pp. 31–58, 2013.

[3] C. Baccar, N. Ben Hamadi, and L. Rachdi, “Best approximation for Weierstrass transform connected with Riemann-Liouville operator”, *Commun. Math. Anal.*, vol. 5, no. 1, pp. 65–83, 2008.

[4] C. Baccar, N. Ben Hamadi and S. Omri, “Fourier multipliers associated with singular partial differential operators”, *Oper. Matrices*, vol. 11, no. 1, pp. 37–53, 2017. doi: 10.7153/oam-11-03

[5] C. Baccar, N. Ben Hamadi and L. T. Rachdi, “Inversion formulas for Riemann-Liouville transform and its dual associated with singular partial differential operators”, *Int. J. Math. Math. Sci.*, vol. 2006, Art. ID 086238, 2006. doi: 10.1155/IJMMS/2006/86238

[6] C. Baccar and L. T. Rachdi, “Spaces of D_{L^p}-type and a convolution product associated with the Riemann-Liouville operators”, *Bull. Math. Anal. Appl.*, vol. 1, no. 3, pp. 16–41, 2009.

[7] N. Ben Hamadi and L. T. Rachdi, “Weyl transforms associated with the Riemann-Liouville operator”. *Int. J. Math. Math. Sci.*, vol. 2006, Art. ID 094768, 2006. doi: 10.1155/IJMMS/2006/94768

[8] R. Coifman and G. Weiss, “Extensions of Hardy spaces and their use in analysis”, *Bull. Amer. Math. Soc.*, vol. 83, no. 4, pp. 569–645, 1977. doi: 10.1090/S0002-9904-1977-14325-5

[9] J. A. Fawcett, “Inversion of n-dimensional spherical averages”, *SIAM J. Appl. Math.*, vol. 45, no. 2, pp. 336–341, 1985. doi: 10.1137/0145018

[10] C. Fefferman and E. N. Stein, “H^p spaces of several variables”, *Acta Math.*, vol. 129, no. 3-4, pp. 137–193, 1972. doi: 10.1007/BF02392215

[11] H. Helesten and L. E. Anderson, “An inverse method for the processing of synthetic aperture radar data”, *Inverse Problems*, vol. 3, no. 1, pp. 111–124, 1987. doi: 10.1088/0266-5611/3/1/013

[12] M. Herberthson, “A numerical implementation of an inverse formula for CARABAS raw Data”. Internal Report D 30430-3.2. National Defense Research Institute, FOA, Box 1165; S-581 11, Sweden, 1986.

[13] K. Hleili, S. Omri and L. T. Rachdi, “Uncertainty principle for the Riemann-Liouville operator”, *Cubo*, vol. 13, no. 3, pp. 91–115, 2011. doi: 10.4067/s0719-06462011000300006

[14] N. N. Lebedev, *Special Functions and Their Applications.* New York, USA: Dover Publications, Inc., 1972.

[15] H. Mejjaoli and S. Omri, “Boundedness and compactness of Reimann-Liouville two-wavelet multipliers”, *J. Pseudo-Differ. Oper. Appl.*, vol. 9, no. 2, pp. 189–213, 2018. doi: 10.1007/s11868-018-0235-2

[16] S. Omri and L. T. Rachdi, “An L^p-L^q version of Morgan’s theorem associated with Riemann-Liouville transform”, *Int. J. Math. Anal. (Ruse)*, vol. 1, no. 17-20, pp. 805–824, 2007.

[17] S. Omri and L. T. Rachdi, “Heisenberg-Pauli-Weyl uncertainty principle for the Riemann-Liouville Operator”, *JIPAM. J. Inequal. Pure Appl. Math.*, vol. 9, no. 3, Art. ID 88, 2008.

[18] L. T. Rachdi and A. Rouz, “On the range of the Fourier transform connected with Riemann-Liouville operator”, *Ann. Math. Blaise Pascal*, vol. 16, no. 2, pp. 355–397, 2009. doi: 10.5802/ambp.272

[19] A. Uchiyama, “A maximal function characterization of H^p on the space of homogeneous type”, *Trans. Amer. Math. Soc.*, vol. 262, no. 2, pp. 579–592, 1980. doi: 10.2307/1999848

[20] A. Uchiyama, *Hardy Spaces on the Euclidean Space*, Springer Monographs in Mathematics, Tokyo, Japan: Springer-Verlag, 2001. doi: 10.1007/978-4-431-67905-9

[21] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Mathematical Library, Cambridge, UK: Cambridge University Press, 1995.