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RESEARCH

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Stability regions of discrete linear periodic systems with delayed feedback controls

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Abstract

We propose a geometric method to determine the stability region of the zero solution of a linear periodic difference equation via the delayed feedback control (briefly, DFC) with the commuting feedback gain. For the equation, our method is more effective than the Jury criterion. First, we give a relationship, named the *C-map theorem*, between the characteristic multipliers of an original equation and those of the equation via DFC. Next, we show the existence and *m*-starlike property, defined in this paper, of an *m-closed curve* induced from the C-map. Using this result, we prove that the region enclosed by the *m*-closed curve is the stability region of the zero solution of the equation via DFC.

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1 Introduction and preliminaries

1.1 Introduction

The delayed feedback control (DFC) is an important method for stabilizing the unstable periodic orbit $\phi(t)$ with period $\omega > 0$ to a differential equation

$$x'(t) = f(x), \quad x \in \Omega \subset \mathbb{R}^d, \tag{E}$$

embedded within a chaotic attractor. As DFCs, Pyragas [11] has firstly used a perturbation $u(t) = K(x(t-\omega) - x(t))$ to Equation (E), that is,

$$x'(t) = f\big(x(t)\big) + K\big(x(t-\omega) - x(t)\big), \tag{DF}$$

where a $d \times d$ real constant matrix K is the so-called feedback gain, and he numerically determined the feedback gain K so that the periodic solution of Equation (DF) is stable.

To stabilize theoretically the unstable periodic orbit, Miyazaki, Naito, and Shin [8] have used the method of linearization under the commuting condition for the gain K. Then, the linear variational equations around the orbit $\phi(t)$ for Equation (DF) becomes

$$y'(t) = A(t)y(t) + K\big(y(t-\omega) - y(t)\big), \tag{LDF}$$

where $A(t) = Df(\phi(t))$ is the Jacobian of $f(x)$.

A discrete version of Equation (DF) is given by the form

$$x(n+1) = f\big(x(n)\big) + u(n), \quad u(n) = K\big(x(n-\omega) - x(n)\big), n \in \mathbb{Z} := \{0, \pm1, \pm2, \ldots\},$$

where $\omega \in \mathbb{Z}_1^\infty := \{1, 2, \ldots\}$.

This type of feedback scheme has certain inherent limitations [12]. On the other hand, Buchner and Żebrowski [1] considered a perturbation of the echo-type formulated as $u(n) = K(x(n-\omega+1) - A(n)x(n))$ to study the stability and the bifurcation for the logistic map. This method is considered as a prediction-based feedback control [13] or nonlinear feedback control [14]. For other types of $u(n)$ see [15–17]. Furthermore, Ohta, Takahashi, and Miyazaki [9] made a remark that DFC of the echo type is more effective than Pyragas type for one-dimensional case.

As the first step of the study, we are interested in the problem of stabilizing the unstable zero solution to linear periodic difference equations of the form

$$x(n+1) = A(n)x(n), \quad n \in \mathbb{Z}, \tag{L}$$

apart from nonlinear difference equations $x(n+1) = f(x(n))$.

Here we assume that $A(n)$ is a $d \times d$ complex matrix with period ω and $x(n)$ belongs to the d dimensional complex Euclidean space $\mathbb{C}^d$.

In this paper, we adopt the perturbation of the echo type and consider the following equation with DFC

$$y(n+1) = A(n)y(n) + K\big(y(n-\omega+1) - A(n)y(n)\big). \tag{LF}$$

The goal of the paper is to describe the stability region, containing all the characteristic multipliers of Equation (L), of the zero solution to Equation (LF) for general period $\omega \geq 3$ (refer to [7] for $\omega = 2$). We develop a *geometric method* to characterize the stability region. As a next step, for periodic solutions with period ω, we will investigate the stability region in the forthcoming paper [5], whose main results rely strongly on this paper.

In general, the stability of the zero (or periodic) solution of Equation (LF) is determined by the absolute values of these characteristic multipliers, i.e., the roots of its characteristic polynomial. However, for the characteristic polynomial of Equation (LF), it is very difficult to apply the classical criteria of Schur–Cohn or Jury in [4] as well as to determine the stability region, since they are based on algebraic methods. Indeed, the order of the inner matrix becomes very large as the dimension d and the period ω increase. For example, according to our experimental calculation, the criteria of Schur–Cohn or Jury are very complicated even for the case when $\omega = 4$ and $d = 1$. This is a motivation for this paper.

To solve such a difficulty, we introduce a new geometric method. As a main result, we can theoretically determine the stability region in general when $K = kE$. In particular, when $\omega = 4$ and all the characteristic multipliers of Equation (L) are real, our method can give a more concrete and precise stability region (Fig. 2).

Our geometric method is developed as follows.

First, we establish a relationship, named the C-map theorem (Theorem 2.5 and Corollary 2.6), between characteristic multipliers of Equation (L) and Equation (LF). To carry out this, we introduce a C-map under the commuting condition: $KA(n) = A(n)K$ for all $n \in \{0, 1, 2, \dots, \omega - 1\}$, which is motivated by the paper [8] for a continuous system. For example, for a characteristic multiplier μ of Equation (L) and for a characteristic multiplier ν of Equation (LF) with $K = kE$, k a real number and E the identity matrix, the C-map is given by $\mu = C_{\omega,k}(\nu) = \nu(\frac{\nu-k}{(1-k)\nu})^{\omega}$.

Next, we give geometric properties of the image $B_{\omega,k}(\theta) := C_{\omega,k}(e^{i\theta}), \theta \in (-\pi, \pi]$ by the C-map of the unit circle in the complex plane. In general, the above image is not geometrically simple. We show the existence and the m-starlike property (Theorem 6.4) of an m-closed curve (Definition 6.3) as a part of the image $B_{\omega,k}(\theta)$.

Finally, using this result, we prove that if all the characteristic multipliers of Equation (L) are in the interior of the (stability) region enclosed by an m-closed curve, then the zero solution to Equation (LF) is asymptotically stable (Theorem 7.2). Furthermore, we give necessary and sufficient conditions for all the characteristic multipliers of Equation (L) to be in the interior of the region. Our method is illustrated for the cases when $\omega = 3, 4$ and all the characteristic multipliers of Equation (L) are real.

The paper is organized as follows.

Section 1. Introduction and preliminaries

Section 2. Characteristic multipliers for Equation (LF)

Section 3. Properties of the function $B_{\omega,k}(\theta)$

Section 4. Existence of solutions of Equation $\Im B_{\omega,k}(\theta) = 0$

Section 5. Equation $\Im B_{\omega,k}(\theta) = 0$

Section 6. Geometric properties of the function $B_{\omega,k}(\theta)$

Section 7. Stability regions

1.2 Preliminaries

In this subsection, we give some basic properties of the characteristic multipliers for Equation (L) and Equation (LF). Let X be a Banach space with $\dim X < \infty$ and $L : X \to X$ a bounded linear operator. We denote by $\mathcal{N}(L)$ the null space of L, and by $W_\eta(L)$ and $G_\eta(L)$ the eigenspace and the generalized eigenspace for $\eta \in \sigma(L)$, respectively, where $\sigma(L)$ stands for the set of all eigenvalues of L. Let $\mathbb{Z}_p^\infty = \{p, p + 1, \dots\}$ for $p \in \mathbb{Z}$. For any $m, n \in \mathbb{Z}$ with $m < n$ we set $\mathbb{Z}_m^n = \{m, m + 1, \dots, n - 1, n\}$.

First, we consider Equation (L), which has the matrix coefficient $A(n)$ with period ω. Throughout this paper we assume that

(A): $A(n)$ is nonsingular for all $n \in \mathbb{Z}_0^{\omega-1}$.

Then the unique solution $x(n; m, x^0)$ of Equation (L) through the initial point $(m, x^0) \in \mathbb{Z} \times \mathbb{C}^d$ is given by $x(n; m, x^0) = T(n, m)x^0$, where $T(n, m)$, $n, m \in \mathbb{Z}$ stands for the solution operator of Equation (L). Set $T(n) = T(n + \omega, n)$, $n \in \mathbb{Z}$. Then $T(0)$ is called the periodic operator of Equation (L). Then $T(n, m)$ $(m, n \in \mathbb{Z})$ and $T(0)$ are given by

$$T(n, m) = \prod_{i=m}^{n-1} A(i) (n \geq m) \quad \text{and} \quad T(0) = \prod_{i=0}^{\omega-1} A(i),$$

respectively, where

$$\prod_{i=m}^{n-1} A(i) = \begin{cases} A(n-1)A(n-2)\cdots A(m) & (n > m), \\ E & (n = m). \end{cases}$$

Thus $T(n,m)$, $n,m \in \mathbb{Z}$ has following properties (refer to [2, 10]):

(T1) $T(n,n) = E$, $n \in \mathbb{Z}$.

(T2) $T(n,m)T(m,r) = T(n,r)$, $m \in \mathbb{Z}_r^n$.

(T3) $T(n+\omega, m+\omega) = T(n,m)$, $m \leq n$.

Note that using ω–periodicity of $A(n)$,

$$T(1) = A(0)T(0)A(0)^{-1}. \tag{1}$$

A complete study of (L) is carried out by the so-called *Floquet theory* (see, for example, C. Pötzsche [10]) Note that $\sigma(T(n)) = \sigma(T(0))$ and $T(0)$ is nonsingular by Condition (A). Thus $0 \notin \sigma(T(0))$. From now on, $\mu \in \sigma(T(0))$ is called the *Floquet's multiplier* or *characteristic multiplier* of Equation (L) (refer to [2, 10]). We recall that the location of eigenvalues of $T(0)$ determines the stability properties of Equation (L).

Next, we consider Equation (LF). Let $\mathcal{C}_{\omega-1}$ be the set of all maps from $\mathbb{Z}_{-\omega+1}^0$ into $\mathbb{C}^d$, which is the Banach space equipped with the norm $|\varphi|_{\mathcal{C}_{\omega-1}} = \sup_{s\in\mathbb{Z}_{-\omega+1}^0} |\varphi(s)|$. It is obvious that $\dim \mathcal{C}_{\omega-1} = \omega d$. Let $m \in \mathbb{Z}$ be fixed. For any function $y : \mathbb{Z}_{m-\omega+1}^\infty \to \mathbb{C}^d$ and any $n \in \mathbb{Z}_m^\infty$, we define a function $y_n : \mathbb{Z}_{-\omega+1}^0 \to \mathbb{C}^d$ by $y_n(s) = y(n+s)$, $s \in \mathbb{Z}_{-\omega+1}^0$. For any $n \in \mathbb{Z}_m^\infty$ the unique solution $y_n(m,\varphi) \in \mathcal{C}_{\omega-1}$ of Equation (LF) through the initial point $(m,\varphi) \in \mathbb{Z} \times \mathcal{C}_{\omega-1}$ is given by $y_n(m,\varphi) = U_K(n,m)\varphi$, where $U_K(n,m) : \mathcal{C}_{\omega-1} \to \mathcal{C}_{\omega-1}$ stands for the solution operator of Equation (LF). Set $U_K(n) = U_K(n+\omega, n)$, $n \in \mathbb{Z}$. Then $U_K(0)$ is called the periodic operator of Equation (LF). Hereafter, if $K = kE$, then we denote by $U_k(n,m)$ and $U_k(0)$ the operators $U_K(n,m)$ and $U_K(0)$, respectively.

The following result can be proved by a similar argument as in the proof of [3, p. 237, Lemma 1.1].

Proposition 1.1 *ν is a characteristic multiplier of Equation* (LF) *if and only if there is a nontrivial solution y_n, $n \in \mathbb{Z}_0^\infty$ of Equation* (LF) *of the form*

$$y(n+\omega) = \nu y(n), \quad n \in \mathbb{Z}_{-\omega+1}^\infty. \tag{2}$$

Hereafter, we assume the following condition (K) for the feedback gain K:

(K-1) $\sigma(K) \subset \mathbb{R}$,

(K-2) $0 < |\kappa| < 1$ for all $\kappa \in \sigma(K)$,

(K-3) $\sigma(U_K(0)) \cap \sigma(K) = \emptyset$.

If $k \in \mathbb{R}$ with $0 < |k| < 1$, then $K = kE$ satisfies the condition (K) (see Lemma 1.3 for a proof). However, the condition (K-3) does not hold for a general matrix K, while it can be replaced by other conditions (see [6]).

Now, we introduce the commuting condition (C).

(C) $KA(n) = A(n)K$, $(n \in \mathbb{Z})$.

The proof of the following lemma is easy.

Lemma 1.2 *For Equation* (L) *the following statements are equivalent*:
(1) $A(n)K = KA(n)$, $n \in \mathbb{Z}$.
(2) $T(n,m)K = KT(n,m)$, $n,m \in \mathbb{Z}$.
(3) $T(n,0)K = KT(n,0)$, $n \in \mathbb{Z}$.

For the case $K = kE$, the following result holds.

Lemma 1.3 *Let* $K = kE$, $0 < |k| < 1$, *and* $k \in \mathbb{R}$. *If Condition* (A) *is satisfied, then Conditions* (C) *and* (K) *are satisfied.*

Proof Since $K = kE$, we obtain that the condition (C) is clearly satisfied. Moreover, $\sigma(K) = \{k\}$ and $W_k(K) = \mathbb{C}^d$. Now, we show by contradiction that the condition (K-3) is satisfied. Suppose $k \in \sigma(U_k(0))$. Then there exists a nontrivial solution $y(n)$ of Equation (LF) such that $y(n+\omega) = ky(n)$, $n \in \mathbb{Z}_{-\omega+1}$ by Proposition 1.1. Hence $y(n+1-\omega) = k^{-1}y(n+1)$, $n \in \mathbb{Z}$. Substituting this relation into Equation (LF), we have $y(n+1) = A(n)y(n) + k[k^{-1}y(n+1) - A(n)y(n)]$, which implies that $(1-k)A(n)y(n) = 0$. Since $k \neq 1$ and $A(n)$ is nonsingular, we have $y(n) = 0$. This leads to a contradiction, since $y(n)$ is a nontrivial solution. Hence, the condition (K-3) is satisfied. □

Hereafter, we always assume Conditions (A), (K), and (C) in this paper.

We note that under the condition (2), Equation (LF) becomes

$$y(n+1) = K(\nu)^{-1}A(n)y(n),$$

where

$$K(\nu) = \nu^{-1}(\nu E - K)(E - K)^{-1}. \tag{3}$$

Finally, we will transform Equation (LF) to the extended linear periodic difference equation. By transforming

$$y\bigl(n-(\omega-1)\bigr) = z(1;n), \qquad y\bigl(n-(\omega-2)\bigr) = z(2;n), \qquad \ldots, \qquad y(n) = z(\omega;n)$$

in Equation (LF) and setting $z(n) = {}^t({}^tz(1;n), {}^tz(2;n), \ldots, {}^tz(\omega;n))$, Equation (LF) becomes

$$z(n+1) = B_K(n)z(n), \tag{BE}$$

where

$$B_K(n) = \begin{pmatrix} 0 & E & 0 & \cdots & 0 & 0 \\ 0 & 0 & E & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & E \\ K & 0 & 0 & \cdots & 0 & (E-K)A(n) \end{pmatrix}, \tag{4}$$

which is called the *extended feedback equation* of Equation (LF).

Then the following result is easy to prove.

Lemma 1.4 $\det B_K(n) = (-1)^{(\omega-1)d} \det K$ *for all* $n \in \mathbb{Z}_0^\infty$.

Proof An easy calculation yields

$$\begin{aligned}\det B_K(n) &= (-1)^{(\omega-1)d} \det \begin{pmatrix} E & 0 & \cdots & 0 & 0 & 0 \\ 0 & E & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & E & 0 & 0 \\ 0 & 0 & \cdots & 0 & E & 0 \\ 0 & 0 & \cdots & 0 & (E-K)A(n) & K \end{pmatrix} \\ &= (-1)^{(\omega-1)d} \det \begin{pmatrix} E & 0 \\ (E-K)A(n) & K \end{pmatrix} \\ &= (-1)^{(\omega-1)d} \det K.\end{aligned}$$

This completes the proof. □

It follows from Lemma 1.4 that if $0 \notin \sigma(K)$, then the existence and uniqueness of solutions to Equation (BE) is guaranteed. We denote by $T_B(n,m)$ and $T_B(0)$ the solution operator and the periodic operator of Equation (BE), respectively. Let $\mathbb{C} := \mathbb{C}^1$ and $\mathbb{R}$ stand for the set of all the real numbers.

Now, we give a relationship between the operators $U_K(0)$ and $T_B(0)$.

Define a mapping $S_{\omega-1}$ from $\mathcal{C}_{\omega-1}$ into $\mathbb{C}^{\omega d} := \overbrace{\mathbb{C}^d \times \mathbb{C}^d \times \cdots \times \mathbb{C}^d}^{\omega}$ by

$$\varphi \in \mathcal{C}_{\omega-1} \mapsto {}^t\big({}^t\varphi(-\omega+1), \quad {}^t\varphi(-\omega+2), \dots, {}^t\varphi(-1), \quad {}^t\varphi(0)\big) \in \mathbb{C}^{\omega d}.$$

Then $S_{\omega-1}$ is bijective. Hence, we have $S_{\omega-1}U_K(n,m)\varphi = T_B(n,m)S_{\omega-1}\varphi$.

Indeed, we have

$$\begin{aligned}S_{\omega-1}U_K(n,m)\varphi &= S_{\omega-1}y_n(m,\varphi) \\ &= \begin{pmatrix} y(n-(\omega-1);m,\varphi) \\ y(n-(\omega-2);m,\varphi) \\ \vdots \\ y(n-1;m,\varphi) \\ y(n;m,\varphi) \end{pmatrix} \\ &= \begin{pmatrix} z(1;n) \\ z(2;n) \\ \vdots \\ z(\omega-1;n) \\ z(\omega;n) \end{pmatrix}\end{aligned}$$

$$= T_B(n,m)\begin{pmatrix}\varphi(-\omega+1)\\ \varphi(-\omega+2)\\ \vdots\\ \varphi(-1)\\ \varphi(0)\end{pmatrix}$$

$$= T_B(n,m)S_{\omega-1}\varphi.$$

So $U_K(n,m)$ is uniquely extended to $n < m$ as follows:

$$U_K(n,m) = S_{\omega-1}^{-1}T_B(n,m)S_{\omega-1}, \quad n < m.$$

From this we have

$$S_{\omega-1}U_K(n,m) = T_B(n,m)S_{\omega-1} \quad (m,n \in \mathbb{Z}), \qquad S_{\omega-1}U_K(0) = T_B(0)S_{\omega-1}.$$

Since $U_K(0)$ and $T_B(0)$ are similar, the following relations hold:

$$U_K(0)\varphi = \nu\varphi \quad \Longleftrightarrow \quad S_{\omega-1}U_K(0)\varphi = \nu S_{\omega-1}\varphi \quad \Longleftrightarrow \quad T_B(0)S_{\omega-1}\varphi = \nu S_{\omega-1}\varphi.$$

Therefore, we obtain the following result.

Lemma 1.5 *$\sigma(U_K(0)) = \sigma(T_B(0))$ and $0 \notin \sigma(U_K(0))$.*

Proof Combining Lemma 1.4 and the condition (K-2), we have $\det B_K(n) \neq 0$. Since $U_K(0)$ and $T_B(0)$ are similar, $\sigma(U_K(0)) = \sigma(T_B(0))$ and hence $0 \notin \sigma(U_K(0))$. □

2 Characteristic multipliers for Equation (LF)

In this section, we determine the spectrum $\sigma(U_K(0))$ of the periodic operator of Equation (LF) and establish the C-map theorem.

2.1 Spectrum of the periodic operator $U_K(0)$

Set

$$H_m^n = (E-K)^{n-m}T(n,m), \quad n \geq m.$$

Then H_m^n has the following properties:

$$H_k^k = E, \qquad H_k^n H_m^k = H_m^n, \qquad (E-K)A(n)H_m^n = H_m^{n+1}. \tag{5}$$

Indeed, using the commuting condition (C) and Lemma 1.2, we have

$$\begin{aligned}(E-K)A(n)H_m^n &= (E-K)^{n+1-m}A(n)T(n,m)\\ &= (E-K)^{n+1-m}T(n+1,m) = H_m^{n+1}.\end{aligned}$$

Inductively, we can obtain a representation of $T_B(0)$ as follows:

$$T_B(0) = \begin{pmatrix} K & 0 & 0 & \cdots & 0 & H_0^1 \\ KH_1^2 & K & 0 & \cdots & 0 & H_0^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ KH_1^{\omega-1} & KH_2^{\omega-1} & KH_3^{\omega-1} & \cdots & K & H_0^{\omega-1} \\ KH_1^{\omega} & KH_2^{\omega} & KH_3^{\omega} & \cdots & KH_{\omega-1}^{\omega} & H_0^{\omega}+K \end{pmatrix}.$$

Now, we will calculate $\det(T_B(0) - \nu E)$.

Proposition 2.1 *The characteristic polynomial of* $T_B(0)$ *is given as follows*:

$$\det\bigl(T_B(0) - \nu E\bigr) = \det\bigl[(-1)^{\omega}\nu^{\omega-1}(E-K)^{\omega}\bigr]\det\bigl[\nu K(\nu)^{\omega} - T(0)\bigr].$$

In particular, $\det(T_B(0) - \nu E) = 0$ *if and only if* $\det(\nu K(\nu)^{\omega} - T(0)) = 0$.

Proof Set

$$M = \begin{pmatrix} E & & & & & & \\ -H_1^2 & E & & & & \mathbf{0} & \\ & -H_2^3 & E & & & & \\ & & \ddots & \ddots & & & \\ & & & \ddots & \ddots & & \\ & \mathbf{0} & & & -H_{\omega-2}^{\omega-1} & E & \\ & & & & & -H_{\omega-1}^{\omega} & E \end{pmatrix}.$$

Then $\det M = 1$. Under the condition (C), by Schur's formula, we have

$$\begin{aligned} \det\bigl(T_B(0) - \nu E\bigr) &= \det\bigl[M\bigl(T_B(0) - \nu E\bigr)\bigr] \\ &= \det\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \\ &= \det M_{22} \det\bigl(M_{11} - M_{12}M_{22}^{-1}M_{21}\bigr), \end{aligned}$$

where

$$M_{11} = \begin{pmatrix} K-\nu E & & & & & \\ \nu H_1^2 & K-\nu E & & & & \\ & \nu H_2^3 & K-\nu E & & \mathbf{0} & \\ & & \ddots & \ddots & & \\ & \mathbf{0} & & \ddots & \ddots & \\ & & & & \nu H_{\omega-2}^{\omega-1} & K-\nu E \end{pmatrix}, \qquad M_{12} = \begin{pmatrix} H_0^1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

$$M_{21} = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & \nu H_{\omega-1}^{\omega} \end{pmatrix}, \qquad M_{22} = K - \nu E.$$

Here, we have used the condition (K-3) and the formula for the determinant of a block matrix with four submatrices. Thus we have

$$\begin{aligned}
&\det\bigl(T_B(0) - \nu E\bigr)\\
&\quad = \det(K - \nu E)\\
&\qquad \times \det\begin{pmatrix}
K-\nu E & & & & & -\nu(K-\nu E)^{-1}H_0^1 H_{\omega-1}^{\omega}\\
\nu H_1^2 & K-\nu E & & & & 0\\
& \nu H_2^3 & K-\nu E & & \mathbf{0} & 0\\
& & \ddots & \ddots & & \vdots\\
& \mathbf{0} & & \ddots & \ddots & \\
& & & & K-\nu E & 0\\
& & & & \nu H_{\omega-2}^{\omega-1} & K-\nu E
\end{pmatrix}\\
&\quad \vdots\\
&\quad = \det(K-\nu E)^{\omega-2}\det\begin{pmatrix} K-\nu E & (-\nu)^{\omega-2}\{(K-\nu E)^{-1}\}^{\omega-2}H_0^1 H_2^{\omega}\\ \nu H_1^2 & K-\nu E\end{pmatrix}.
\end{aligned} \tag{6}$$

It follows from (6) that

$$\begin{aligned}
&\det\bigl(T_B(0) - \nu E\bigr)\\
&\quad = \det(K-\nu E)^{\omega-1}\det\bigl[K-\nu E + (-\nu)^{\omega-1}\bigl\{(K-\nu E)^{-1}\bigr\}^{\omega-1}H_0^1 H_1^{\omega}\bigr]\\
&\quad = \det\bigl[(K-\nu E)^{\omega} + (-\nu)^{\omega-1}H_0^1 H_1^{\omega}\bigr]\\
&\quad = \det\bigl[(K-\nu E)^{\omega} + (-\nu)^{\omega-1}(E-K)^{\omega}T(1,0)T(\omega,1)\bigr]\\
&\quad = \det\bigl[(K-\nu E)^{\omega} + (-\nu)^{\omega-1}(E-K)^{\omega}T(1)\bigr] \quad \bigl(\text{by the property } (T3)\bigr)
\end{aligned}$$

Since $T(1) = A(0)T(0)A^{-1}(0)$ and $A(0)K = KA(0)$ hold, we have

$$\begin{aligned}
&\det\bigl(T_B(0) - \nu E\bigr)\\
&\quad = \det\bigl[(K-\nu E)^{\omega} + (-\nu)^{\omega-1}(E-K)^{\omega}T(1)\bigr]\\
&\quad = \det\bigl[(K-\nu E)^{\omega}A(0)A(0)^{-1} + (-\nu)^{\omega-1}(E-K)^{\omega}A(0)T(0)A(0)^{-1}\bigr](\text{by}(1))\\
&\quad = \det\bigl\{A(0)\bigl[(K-\nu E)^{\omega} + (-\nu)^{\omega-1}(E-K)^{\omega}T(0)\bigr]A(0)^{-1}\bigr\}\\
&\quad = \det\bigl[(-1)^{\omega}\nu^{\omega-1}(E-K)^{\omega}\bigl\{\nu\bigl(\nu^{-1}(E-K)^{-1}\bigr)^{\omega}(\nu E-K)^{\omega} - T(0)\bigr\}\bigr]\\
&\quad = \det\bigl[(-1)^{\omega}\nu^{\omega-1}(E-K)^{\omega}\bigr]\det\bigl[\nu K(\nu)^{\omega} - T(0)\bigr].
\end{aligned}$$

Since $\nu \neq 0$ by Lemma 1.5 and $1 \notin \sigma(K)$, we obtain that if $\det(T_B(0) - \nu E) = 0$, then $\det(\nu K(\nu)^{\omega} - T(0)) = 0$, and vice versa. □

Combining Proposition 2.1 and Lemma 1.5, we obtain the following equivalence.

Proposition 2.2 *The following statements are equivalent*:
(1) $\nu \in \sigma(U_K(0))$.
(2) $\det(T_B(0) - \nu E) = 0$.
(3) $\det(\nu K(\nu)^\omega - T(0)) = 0$.

Theorem 2.3 *The following statements hold.*
(1) *Let* $\nu \in \sigma(U_K(0))$. *Then* $\psi \in W_\nu(U_K(0)) \Longleftrightarrow S_{\omega-1}\psi \in W_\nu(T_B(0))$.
(2) *The characteristic equation* $\det(T_B(0) - \nu E) = 0$ *has* ωd *roots*.

Proof We prove only the assertion (1). Assume $\psi \in W_\nu(U_K(0))$. Since $U_K(0)\psi = \nu\psi$, we have $S_{\omega-1}U_K(0)\psi = S_{\omega-1}\nu\psi$. Since $S_{\omega-1}U_K(0)\psi = T_B(0)S_{\omega-1}\psi$, we obtain $T_B(0)S_{\omega-1}\psi = S_{\omega-1}\nu\psi$, that is, $S_{\omega-1}\psi \in W_\nu(T_B(0))$, and vice versa. □

Combining Theorem 2.3 with Lemma 1.4, we obtain the following result.

Proposition 2.4 *Let* $\nu_1, \dots, \nu_{\omega d}$, *counted with multiplicity, be all the characteristic multipliers of Equation* (LF). *Then* $\nu_1 \cdots \nu_{\omega d} = (\det K)^\omega$.

Proof Combining Theorem 2.3 with Lemma 1.4, we obtain

$$\begin{aligned} \nu_1 \cdots \nu_{\omega d} = \det T_B(0) &= \prod_{n=0}^{\omega-1} \det B_K(n) \\ &= \left((-1)^{(\omega-1)d} \det K\right)^\omega \\ &= (-1)^{\omega(\omega-1)d} (\det K)^\omega. \end{aligned}$$

Since $\omega(\omega - 1)d$ is an even number, the proof is complete. □

It follows from Proposition 2.4 that
(1) if $K = kE$, then

$$\nu_1\nu_2 \cdots \nu_{\omega d} = (k)^{\omega d};$$

(2) if $k_1, k_2, \dots, k_d$, counted with multiplicity, are eigenvalues of the matrix K, then $\det K = k_1k_2 \cdots k_d$ and

$$\nu_1 \cdots \nu_{\omega d} = (\det K)^\omega = (k_1k_2 \cdots k_d)^\omega.$$

This implies that if $\det K > 1$, then there exists a $\nu_i \in \sigma(U_K(0))$ such that $|\nu_i| > 1$. In other words, the zero solution of Equation (LF) is unstable if $\det K > 1$. Note that $\det K > 1$ if $K = kE$ and $|k| > 1$.

2.2 C-map theorems

In this subsection, we introduce the C-map Theorems, which give the relationship between the characteristic multipliers of Equations (L) and (LF) and play the crucial role throughout this paper. For commuting matrices A and B we set

$$\sigma[AB] = \left\{(\alpha, \beta) \in \sigma(A) \times \sigma(B) | \alpha\beta \in \sigma(AB)\right\},$$

where $\sigma(AB) = \{\alpha\beta | \alpha \in \sigma(A), \beta \in \sigma(B), W_\alpha(A) \cap W_\beta(B) \neq \emptyset\}$.

For a function $f(x,y)$, we denote by $f_x(y)$ and $f_y(x)$ the function $f(x,y)$ of y for each fixed x, and the function $f(x,y)$ of x for each fixed y, respectively.

In view of $K(\nu)^\omega$ in Proposition 2.1, we introduce

$$g(k,z)=\left(\frac{z-k}{(1-k)z}\right)^\omega : I\times D\to\mathbb{C}\setminus\{0\}\quad\text{and}\quad C_{\omega,k}(z)=zg(k,z),$$

where $I=\mathbb{R}\setminus\{1\}$ and $D=\mathbb{C}\setminus\mathbb{R}$. The function $C_{\omega,k}(z)$ is called the characteristic multiplier map (briefly, *C-map*) for Equation (LF). Note that $g(K,z)$ is well defined and $zg(K,z)$ is nonsingular for all $z\in D$, since $g(k,z)$ is analytic in k for all $z\in D$.

We are now in a position to state and prove the C-map theorem for Equation (LF).

Theorem 2.5 (C-map Theorem) *$\nu\in\sigma(U_K(0))$ if and only if there exists a $(k,\mu)\in\sigma[KT(0)]$ such that $\mu=C_{\omega,k}(\nu)$.*

Proof It follows from Proposition 2.2 that $\nu\in\sigma(U_K(0))$ if and only if $\det(\nu K(\nu)^\omega - T(0))=0$, that is, $0\in\sigma(\nu g(K,\nu)-T(0))$. According to the spectral mapping theorem, we have $\sigma(\nu g(K,\nu))=\{\nu g(k,\nu)|k\in\sigma(K)\}$. Moreover, it follows from Condition (C) and [8, Lemma 4.1] that $\nu g(K,\nu)$ and $T(0)$ commute. Therefore, by [8, Lemma A.1], the condition $0\in\sigma(\nu g(K,\nu)-T(0))$ implies that $\nu\in\sigma(U_K(0))$ if and only if there exist $k_0\in\sigma(K)$ and $\mu\in\sigma(T(0))$ such that

$$\mu=\nu g(k_0,\nu),\quad G_{\nu g(K,\nu)}\big(\nu g(k_0,\nu)\big)\cap G_{T(0)}(\mu)\neq\{0\}. \tag{7}$$

For such a $k_0\in\sigma(K)$, we denote by $\{k_0,k_1,\dots,k_p\}$, $p\le d-1$ the set of $k\in\sigma(K)$ such that $\nu g(k,\nu)=\nu g(k_0,\nu)$. Using the spectral mapping theorem again, we have $G_{\nu g(K,\nu)}(\nu g(k_0,\nu))=\bigoplus_{i=0}^p G_K(k_i)$. Therefore, we see that $G_{\nu g(K,\nu)}(\nu g(k_0,\nu))\cap G_{T(0)}(\mu)\neq\{0\}$ if and only if $G_{T(0)}(\mu)\cap\bigoplus_{i=0}^p G_K(k_i)\neq\{0\}$. Then $x\in G_{T(0)}(\mu)\cap\bigoplus_{i=1}^p G_K(k_i)$, $x\neq 0$ can be expressed as $x=\sum_{i=0}^p P_i x$, $P_i x\in G_K(k_i)$, where $P_i:\mathbb{C}^d\to G_K(k_i)$ is the projection. Since $T(0)$ and K commute, we have $T(0)P_i x=P_iT(0)x=P_i\mu x=\mu P_i x$, $i=0,\dots,p$. Since there is at least one i such that $P_i x\neq 0$, we have

$$G_K(k_i)\cap G_{T(0)}(\mu)\neq\{0\}. \tag{8}$$

It follows from [8, Lemma A.2] that the condition (8) is reduced to the condition $W_K(k_i)\cap W_{T(0)}(\mu)\neq\{0\}$. Hence the condition (7) is replaced by the condition

$$\mu=C_{\omega,k_i}(\nu),\quad W_K(k_i)\cap W_{T(0)}(\mu)\neq\{0\}.$$

Thus we have $(k_i,\mu)\in\sigma[KT(0)]$. This proves the theorem. □

Corollary 2.6 *Let $K=kE$. Then $\nu\in\sigma(U_k(0))$ if and only if $C_{\omega,k}(\nu)\in\sigma(T(0))$.*

The C-map $\mu=C_{\omega,k}(z)$ can be reformulated as

$$P_{\omega,k}(z;\mu)=(z-k)^\omega-\mu(1-k)^\omega z^{\omega-1}=0. \tag{9}$$

Using (9) and Corollary 2.6, we obtain the following result.

Corollary 2.7 *Let $K = kE$. Then for every $\mu \in \sigma(T(0))$ the equation $\mu = C_{\omega,k}(\nu)$ has ω solutions, counted with multiplicity, which belong to $\sigma(U_k(0))$.*

3 Properties of the function $B_{\omega,k}(\theta)$

In this section we consider several properties of the image

$$B_{\omega,k}(\theta) := C_{\omega,k}\left(e^{i\theta}\right) = \left(\frac{1 - ke^{-i\theta}}{1-k}\right)^{\omega} e^{i\theta}, \quad -\pi < \theta \le \pi \tag{10}$$

by the C-map $C_{\omega,k}(z)$ of the unit circle. Clearly, we have:

(1) $B_{\omega,k}(0) = 1 \in \mathbb{R}$.

(2) $B_{\omega,k}(\pi) = -(\frac{1+k}{1-k})^{\omega} \in \mathbb{R}$.

(3) $B_{\omega,k}(\theta)$ is differentiable on $[0,\pi]$.

Note that $\lim_{k\to 1} |B_{\omega,k}(\pi)| = \infty$.

Hereafter, we assume that $\omega \in \mathbb{Z}_3^{\infty}$

We denote by C the closed unit disc, i.e., $C = \{z||z| \le 1\}$, and denote by $n(\partial C, C_{\omega,k})$ the winding number of $C_{\omega,k}(\nu)$ when ν rotates along the unit circle ∂C centered at the origin in the positive direction.

Lemma 3.1 *The following statements hold.*

(1) $B_{\omega,k}(\theta) \neq 0$ *for all* $\theta \in (-\pi,\pi]$.

(2) $B_{\omega,k}(\theta + 2n\pi) = B_{\omega,k}(\theta)$*and*$B_{\omega,k}(\theta) = \overline{B_{\omega,k}(-\theta)}$, $(n \in \mathbb{Z}, -\pi < \theta \le \pi)$.

(3) $B_{\omega,k}(-\pi) := \lim_{\theta\to-\pi} B_{\omega,k}(\theta) = -(\frac{1+k}{1-k})^{\omega} \in \mathbb{R}$.

(4) $C_{\omega,k}(1) = 1$ *and* $C_{\omega,k}(\nu) = \overline{C_{\omega,k}(\overline{\nu})}$.

(5) $n(\partial C, C_{\omega,k}) = 1$.

Proof (1), (2), (3), and (4) are obvious. (5) By the argument principle, we have

$$n(\partial C, C_{\omega,k}) = \frac{1}{2\pi i}\int_{\partial C} \frac{C'_{\omega,k}(\nu)}{C_{\omega,k}(\nu)}\,d\nu = \frac{1}{2\pi i}\left[\int_{\partial C} \frac{\omega}{\nu - k}\,d\nu + \int_{\partial C} \frac{1-\omega}{\nu}\,d\nu\right] = 1$$

as required. □

To obtain a representation of $B_{\omega,k}(\theta)$, for any k, $0 < |k| < 1$ and any $\theta \in (-\pi,\pi]$, we define $\beta(k,\theta)$ as

$$\tan\beta(k,\theta) = \frac{k\sin\theta}{1 - k\cos\theta}, \quad \left|\beta(k,\theta)\right| < \frac{\pi}{2}. \tag{11}$$

Now, we give elementary properties of $\beta(k,\theta)$.

Lemma 3.2 *For* $\theta \in (0,\pi)$ *the following statements hold*:

(1) $0 < k < 1$ *if and only if* $0 < \beta(k,\theta) < \frac{\pi}{2}$ *for all* $\theta \in (0,\pi)$.

(2) $-1 < k < 0$ *if and only if* $-\frac{\pi}{2} < \beta(k,\theta) < 0$ *for all* $\theta \in (0,\pi)$.

(3) *For any* $\theta \in (0,\pi)$ $\beta(k,\theta)$ *is increasing in* k, $0 < |k| < 1$.

The assertions (1) and (2) in Lemma 3.2 imply that $\beta(k,\theta) \neq 0$ for all $\theta \in (0,\pi)$. Since $\frac{\sin\beta(k,\theta)}{\cos\beta(k,\theta)} = \frac{k\sin\theta}{1-k\cos\theta}$, (11) can be replaced by

$$k\sin\left(\beta(k,\theta) + \theta\right) = \sin\beta(k,\theta). \tag{12}$$

Next, we give two representations of $B_{\omega,k}(\theta)$. We need the following identity often

$$1 - ke^{-i\theta} = \sqrt{1 - 2k\cos\theta + k^2}e^{i\beta(k,\theta)}. \tag{13}$$

Now, the following representation of $B_{\omega,k}(\theta)$ is given by (13).

Proposition 3.3 *$B_{\omega,k}(\theta)$ can be reformulated as*

$$B_{\omega,k}(\theta) = \frac{(1 - 2k\cos\theta + k^2)^{\frac{\omega}{2}}}{(1-k)^{\omega}}e^{i\varphi_k(\theta)}, \quad \theta \in (-\pi,\pi],$$

where

$$\varphi_k(\theta) = \omega\beta(k,\theta) + \theta, \quad -\pi < \theta \leq \pi. \tag{14}$$

Corollary 3.4 *The following results hold.*

(1) $\beta(k,0) = 0$ *and* $\varphi_k(0) = 0$ *for all* k $(0 < |k| < 1)$.

(2) $\beta(k,\pi) = 0$ *and* $\varphi_k(\pi) = \pi$ *for all* k $(0 < |k| < 1)$.

(3) $\beta(k,\theta) \neq 0$ *for all* k $(0 < |k| < 1)$ *and* θ $(0 < |\theta| < \pi)$.

Using Proposition 3.3, we have

$$\left|B_{\omega,k}(\theta)\right| = \frac{(1 - 2k\cos\theta + k^2)^{\frac{\omega}{2}}}{(1-k)^{\omega}} \tag{15}$$

and $|\varphi_k(\theta)| < (\frac{\omega}{2} + 1)\pi$. Thus the following result holds.

Lemma 3.5 *Let* $\theta \in [0,\pi]$. *Then the following statements hold.*

(1) *If* $0 < k < 1$, *then* $|B_{\omega,k}(\theta)| \geq 1$ *and* $|B_{\omega,k}(\theta)|$ *is strictly increasing in* θ.

(2) *If* $-1 < k < 0$, *then* $|B_{\omega,k}(\theta)| \leq 1$ *and* $|B_{\omega,k}(\theta)|$ *is strictly decreasing in* θ.

Corollary 3.6 *The following statements hold.*

(1) *If* $0 < k < 1$, *then* $\min_{0\leq\theta\leq\pi} |B_{\omega,k}(\theta)| = B_{\omega,k}(0) = 1$ *and* $\max_{0\leq\theta\leq\pi} |B_{\omega,k}(\theta)| = |B_{\omega,k}(\pi)| = (\frac{1+k}{1-k})^{\omega}$.

(2) *If* $-1 < k < 0$, *then* $\min_{0\leq\theta\leq\pi} |B_{\omega,k}(\theta)| = |B_{\omega,k}(\pi)| = (\frac{1+k}{1-k})^{\omega}$ *and* $\max_{0\leq\theta\leq\pi} |B_{\omega,k}(\theta)| = B_{\omega,k}(0) = 1$.

Since

$$\left|B_{\omega,k}(\theta)\right|^{\frac{2}{\omega}} = 1 + \frac{2k}{(1-k)^2}(1 - \cos\theta), \tag{16}$$

we have the following lemma.

Lemma 3.7 *Let* $0 < |k| < 1$ *and* $\theta \in (0,\pi]$. *Then* $|B_{\omega,\theta}(k)|$ *is strictly increasing in* k.

Proof Set $b(k) := |B_{\omega,\theta}(k)|^{\frac{2}{\omega}}$. In view of (16), we have $b'(k) = \frac{2(1+k)}{(1-k)^3}(1 - \cos\theta) > 0$, and hence $b(k)$ is strictly increasing in k. □

4 Existence of solutions of equation $\Im B_{\omega,k}(\theta) = 0$

In this section, we give the criteria for the existence of solutions of Equation $\Im B_{\omega,k}(\theta) = 0$, i.e., $B_{\omega,k}(\theta) \in \mathbb{R}$ on $[0,\pi]$. In other words, $\sin\varphi_k(\theta) = 0$. Since $\beta(k,0) = \beta(k,\pi) = 0$ for any k, $0 < |k| < 1$ by Corollary 3.4, $\theta = 0, \pi$ are the solutions of Equation $\Im B_{\omega,k}(\theta) = 0$. Thus we consider the case $\theta \in (0,\pi)$. To discuss this problem, we investigate separately two cases $0 < |k| \leq \frac{1}{\omega-1}$ and $\frac{1}{\omega-1} < |k| < 1$.

First, we need the following properties of $\varphi_k(\theta)$. Since $\varphi_k(\theta) = \omega\beta(k,\theta) + \theta$, we have $\frac{d}{d\theta}\varphi_k(\theta) = \frac{\zeta(k,\theta)}{1-2k\cos\theta+k^2}$, where $\zeta(k,\theta) = k(\omega-2)\cos\theta - (\omega-1)k^2 + 1$.

Proposition 4.1 *The following statements hold for* $\theta \in [0,\pi]$.

(1) $\varphi_k'(\theta) = 0 \Longleftrightarrow \zeta_k(\theta) = 0$, i.e., $\cos\theta = \frac{(\omega-1)k^2-1}{k(\omega-2)}$.

(2) $\varphi_k'(\theta) > 0 \Longleftrightarrow \zeta_k(\theta) > 0$; $\varphi_k'(\theta) < 0 \Longleftrightarrow \zeta_k(\theta) < 0$.

(3) $\varphi_k'(\theta)$ *is continuous on* $[0,\pi]$.

(4) $\frac{1}{\omega-1} < |k| < 1 \Longleftrightarrow |\frac{(\omega-1)k^2-1}{k(\omega-2)}| < 1$.

Corollary 4.2 *The following statements hold.*

(1) $k = -\frac{1}{\omega-1}$ *if and only if* $\varphi_k'(0) = 0$.

(2) $k = \frac{1}{\omega-1}$ *if and only if* $\varphi_k'(\pi) = 0$.

The following result is easily derived from the above argument and Proposition 4.1.

Corollary 4.3 *For* $\theta = 0, \pi$ *the following statements hold.*

(1) *The case* $\theta = 0$.

(1-1) *If* $-\frac{1}{\omega-1} < k < 1$, *then* $\varphi_k'(0) > 0$.

(1-2) *If* $-1 < k < -\frac{1}{\omega-1}$, *then* $\varphi_k'(0) < 0$.

(2) *The case* $\theta = \pi$.

(2-1) *If* $-1 < k < \frac{1}{\omega-1}$, *then* $\varphi_k'(\pi) > 0$.

(2-2) *If* $\frac{1}{\omega-1} < k < 1$, *then* $\varphi_k'(\pi) < 0$.

Next, we show that solutions of Equation $\Im B_{\omega,k}(\theta) = 0$ on $[0,\pi]$ for the case $0 < |k| \leq \frac{1}{\omega-1}$ are $\theta = 0$ and π only.

We are now in a position to state and prove the first main theorem in this section.

Theorem 4.4 *Let* $\theta \in [0,\pi]$. *Suppose* $0 < |k| \leq \frac{1}{\omega-1}$. *Then* $\Im B_{\omega,k}(\theta) = 0$ *if and only if* $\theta = 0, \pi$.

Proof The proof is based on Proposition 4.1 and Corollary 4.3.

(1) If $0 < k \leq \frac{1}{\omega-1}$, then $\varphi_k'(\pi) \geq 0$ by Corollary 4.2 and Corollary 4.3. Moreover, Proposition 4.1 implies the inequality $\cos\pi = -1 \geq \frac{(\omega-1)k^2-1}{k(\omega-2)}$. On the other hand, we have the inequality $\cos\theta > \cos\pi \geq \frac{(\omega-1)k^2-1}{k(\omega-2)}$ on $[0,\pi)$. Thus $\varphi_k'(\theta) > 0$ on $[0,\pi)$ and $\varphi_k'(\pi) \geq 0$.

(2) If $-\frac{1}{\omega-1} \leq k < 0$, then it follows that $\varphi_k'(0) \geq 0$ by Corollary 4.3. Thus Proposition 4.1 implies the inequality $\cos 0 = 1 \leq \frac{(\omega-1)k^2-1}{k(\omega-2)}$. On the other hand, we have the inequality $\cos\theta < \cos 0 \leq \frac{(\omega-1)k^2-1}{k(\omega-2)}$ on $(0,\pi]$. Thus $\varphi_k'(\theta) > 0$ on $(0,\pi]$ and $\varphi_k'(0) \geq 0$.

Summing up these cases, we obtain that if $0 < |k| \leq \frac{1}{\omega-1}$, then $\varphi_k'(\theta) > 0$ on $(0,\pi)$. Thus, in view of Corollary 3.4, we see that $\varphi_k : [0,\pi] \to [0,\pi]$ is bijective. Therefore, $\sin\varphi_k(\theta) = 0$ if and only if $\theta = 0, \pi$. □

Remark 4.5 If $0 < |k| \le \frac{1}{\omega-1}$ in Theorem 4.4, then $\varphi_k'(\theta) \ge 0$ on $[0,\pi]$ and $\varphi_k(\theta) > 0$ $(0 < \theta < \pi)$, $\varphi_k(0) = 0$, $\varphi_k(\pi) = \pi$.

Finally, we discuss the existence of solutions of Equation $\Im B_{\omega,k}(\theta) = 0$ on $(0,\pi)$ for the case $\frac{1}{\omega-1} < |k| < 1$.

(1) *Properties of the sets* $\mathbb{Z}_+(\theta)$ *and* $\mathbb{Z}_-(\theta)$.

We turn to the existence of solutions in $(0,\pi)$ of Equation $\Im B_{\omega,k}(\theta) = 0$. Clearly, $\sin\varphi_k(\theta) = 0$ is reduced to

$$m\pi = \omega\beta(k,\theta) + \theta, \quad m \in \mathbb{Z}. \tag{17}$$

Thus $m = m(\theta,k,\omega)$ on $[0,\pi]$. To obtain such an m, we introduce the function

$$\beta_m(\theta) = \frac{m\pi - \theta}{\omega}, \quad 0 \le \theta \le \pi.$$

Then (17) is equivalent to $\beta(k,\theta) = \beta_m(\theta)$. Clearly, since $|\beta_m(\theta)| < \frac{\pi}{2}$, we define the set of all $m = m(\theta,\omega) \in \mathbb{Z}$ satisfying $|\beta_m(\theta)| < \frac{\pi}{2}$.

The following statements are obvious.

Lemma 4.6 *Let* $\frac{1}{\omega-1} < |k| < 1$. *Then the following statements are equivalent.*

(1) *For any* k, *equation* $\Im B_{\omega,k}(\theta) = 0$ *has a solution in* $(0,\pi)$.

(2) *For any* k, *equation* $\sin\varphi_k(\theta) = 0$ *has a solution in* $(0,\pi)$.

(3) *For any* k, *there exists an* $m \in \mathbb{Z}$ *and a* $\theta \in (0,\pi)$ *satisfying* $\varphi_k(\theta) = m\pi$.

(4) *For any* k *there exists an* $m \in \mathbb{Z}$ *and a* $\theta \in (0,\pi)$ *satisfying* $\beta(k,\theta) = \beta_m(\theta)$.

For $a \in \mathbb{R}$, the symbol $[a]$ stands for the maximum integer not greater than a. We set $\omega_0 = [\frac{\omega}{2}]$, $\mathbb{O} = \{2n+1 | n \in \mathbb{Z}\}$ and $\mathbb{E} = \{2n | n \in \mathbb{Z}\}$. Then we note that if $\omega \in \mathbb{O}$, then $\omega_0 = \frac{\omega}{2} - \frac{1}{2}$; if $\omega \in \mathbb{E}$, then $\omega_0 = \frac{\omega}{2}$. Since $|\beta_m(\theta)| < \frac{\pi}{2}$, it follows that $-\frac{\omega}{2} + \frac{\theta}{\pi} < m < \frac{\omega}{2} + \frac{\theta}{\pi}$. For $\theta \in (0,\pi)$ and $\omega \in \mathbb{Z}_3^\infty$, we define

$$\mathbb{Z}_+(\theta) = \left\{ m \in \mathbb{Z} | 1 \le m < \frac{\omega}{2} + \frac{\theta}{\pi} \right\}, \qquad \mathbb{Z}_-(\theta) = \left\{ m \in \mathbb{Z} | -\frac{\omega}{2} + \frac{\theta}{\pi} < m \le 0 \right\},$$

and $\mathbb{Z}(\theta) = \mathbb{Z}_+(\theta) \cup \mathbb{Z}_-(\theta)$. For $\theta = 0, \pi$, we define

$$\mathbb{Z}(0) = \mathbb{Z}_-(0) = \{0\} \quad \text{and} \quad \mathbb{Z}(\pi) = \mathbb{Z}_+(\pi) = \{1\}. \tag{18}$$

Next, by easy calculation, we can determine the sets $\mathbb{Z}_+(\theta)$ and $\mathbb{Z}_-(\theta)$ as follows.

Lemma 4.7 *Let* $\theta \in (0,\pi)$.

(1) *If* $\omega \in \mathbb{O}$, *then*

$$\mathbb{Z}_+(\theta) = \left\{ m \in \mathbb{Z} \middle| 1 \le m \le \begin{cases} \omega_0 & (0 < \theta \le \frac{\pi}{2}) \\ \omega_0 + 1 & (\frac{\pi}{2} < \theta < \pi) \end{cases} \right\},$$

$$\mathbb{Z}_-(\theta) = \left\{ m \in \mathbb{Z} \middle| \begin{array}{c} -\omega_0 (0 < \theta < \frac{\pi}{2}) \\ 1 - \omega_0 (\frac{\pi}{2} \le \theta < \pi) \end{array} \right\} \le m \le 0 \right\}.$$

(2) *If* $\omega \in \mathbb{E}$, *then*

$$\mathbb{Z}_+(\theta) = \{m \in \mathbb{Z} | 1 \le m \le \omega_0\}, \qquad \mathbb{Z}_-(\theta) = \{m \in \mathbb{Z} | 1 - \omega_0 \le m \le 0\}.$$

Now, we give a relationship between the set $\mathbb{Z}_+(\theta)$ (or $\mathbb{Z}_-(\theta)$) and $\beta_m(\theta)$.

Corollary 4.8 *The following statements hold*:
(1) $m \in \mathbb{Z}_+(\theta)$ *if and only if* $0 < \beta_m(\theta) < \frac{\pi}{2}$.
(2) $m \in \mathbb{Z}_-(\theta)$ *if and only if* $-\frac{\pi}{2} < \beta_m(\theta) < 0$.

The following lemma easily follows from Lemma 3.2 and Lemma 4.7.

Lemma 4.9 *The following statements hold.*
(1) *If* $0 < k < 1$, *then there exists an* $m \in \mathbb{Z}_+(\theta)$ *such that* $0 < \beta_m(\theta) < \frac{\pi}{2}$ *for all* $\theta \in (0, \pi)$.
(2) *If* $-1 < k < 0$, *then there exists an* $m \in \mathbb{Z}_-(\theta)$ *such that* $-\frac{\pi}{2} < \beta_m(\theta) < 0$ *for all* $\theta \in (0, \pi)$.

(2) *The function* $g_m(\theta)$.

In general, if there exist a k_*, $0 < |k_*| < 1$, a $\theta_* \in [0, \pi]$ and an $m_* \in \mathbb{Z}$ such that $\beta(k_*, \theta_*) = \beta_{m_*}(\theta_*)$, then $\sin \varphi_{k_*}(\theta_*) = \sin m_* \pi = 0$. It follows from Corollary 4.8 that $m_* \in \mathbb{Z}$ can be replaced by $m_* \in \mathbb{Z}(\theta_*)$. So, we have $m = 0$ and $m = 1$ in (17) for the solutions $\theta = 0$ and $\theta = \pi$ of the equation $\sin \varphi_k(\theta) = 0$.

First, we discuss the existence of solutions of Equation $\Im B_{\omega,k}(\theta) = 0$, i.e., $\sin \varphi_k(\theta) = 0$. We define

$$g_m(\theta) := g_{m,k}(\theta) = \sin \beta_m(\theta) - k \sin\bigl(\beta_m(\theta) + \theta\bigr), \quad \theta \in (0, \pi), \tag{19}$$

where $m \in \mathbb{Z}(\theta)$. It follows from Corollary 4.8 and Lemma 4.9 that if $0 < k < 1$, then $m \in \mathbb{Z}_+(\theta)$; if $-1 < k < 0$, then $m \in \mathbb{Z}_-(\theta)$. We define

$$g_{m,k}(0) := \lim_{\theta \to 0^+} g_{m,k}(\theta) = (1 - k) \sin \frac{m\pi}{\omega},$$

$$g_{m,k}(\pi) := \lim_{\theta \to \pi^-} g_{m,k}(\theta) = (1 + k) \sin \frac{(m-1)\pi}{\omega}.$$

Then $g_{m,k}(\theta)$ is well defined on $[0, \pi]$.

Lemma 4.10 *If for any* k, $\frac{1}{\omega - 1} < |k| < 1$, *there exist a* $\theta_* \in (0, \pi)$ *and an* $m_* \in \mathbb{Z}(\theta_*)$ *such that* $g_{m_*,k}(\theta_*) = 0$, *then* $\sin \varphi_k(\theta_*) = 0$, *and vice versa*.

Proof Since $g_{m_*,k}(\theta_*) = 0$, i.e., $\sin \beta_{m_*}(\theta_*) = k \sin(\beta_{m_*}(\theta_*) + \theta_*)$, (12) yields $\tan \beta_{m_*}(\theta_*) = \frac{k \sin \theta_*}{1 - k \cos \theta_*}$. This means $\beta(k, \theta_*) = \beta_{m_*}(\theta_*)$. Thus $\sin \varphi_k(\theta_*) = 0$ by Lemma 4.6.

Conversely, let for any k, $\frac{1}{\omega - 1} < |k| < 1$, there exist a $\theta_* \in (0, \pi)$ such that $\sin \varphi_k(\theta_*) = 0$. Then there exists an $m_* \in \mathbb{Z}(\theta)$ such that $\varphi_k(\theta_*) = m_* \pi$. Therefore, $\beta(k, \theta_*) = \beta_{m_*}(\theta_*)$ by Lemma 4.6. Thus $\beta_{m_*}(\theta_*)$ satisfies the equation $g_{m_*,k}(\theta_*) = 0$. □

The derivative of $g_m(\theta)$ becomes

$$g_m'(\theta) = -\frac{1}{\omega}\bigl[\cos \beta_m(\theta) + k(\omega - 1) \cos\bigl(\beta_m(\theta) + \theta\bigr)\bigr], \quad 0 \le \theta \le \pi.$$

We define

$$h_m(\theta) = -\frac{\cos(\beta_m(\theta)+\theta)}{\cos\beta_m(\theta)}, \quad 0 \le \theta \le \pi, m \in \mathbb{Z}(\theta).$$

Note that $|\beta_m(\theta)| < \frac{\pi}{2}$ for all $m \in \mathbb{Z}(\theta)$ by Corollary 4.8 and (18). Thus we obtain

(1) $h_m(0) = h_0(0) = -1$, $h_m(\pi) = h_1(\pi) = 1$ and

(2) $\cos\beta_m(\theta) > 0$, $m \in \mathbb{Z}(\theta)$ on $[0,\pi]$.

Therefore, the function $h_m(\theta)$ is well defined on $[0,\pi]$ and $g'_m(\theta)$ on $[0,\pi]$ is expressed as

$$g'_m(\theta) = -\frac{1}{\omega}\big[1 - k(\omega-1)h_m(\theta)\big]\cos\beta_m(\theta). \tag{20}$$

The proof of the following result easily follows from (20).

Proposition 4.11 *If $\theta \in [0,\pi]$ and $m \in \mathbb{Z}(\theta)$, then*

$$\begin{aligned}
&(1) \quad g'_m(\theta) < 0 \quad \Longleftrightarrow \quad k(\omega-1)h_m(\theta) < 1,\\
&(2) \quad g'_m(\theta) = 0 \quad \Longleftrightarrow \quad k(\omega-1)h_m(\theta) = 1,\\
&(3) \quad g'_m(\theta) > 0 \quad \Longleftrightarrow \quad k(\omega-1)h_m(\theta) > 1.
\end{aligned}$$

(3) *The existence of solutions of Equation $\Im B_{\omega,k}(\theta) = 0$ on $(0,\pi)$ for the case $\frac{1}{\omega-1} < |k| < 1$.*

We are now in a position to state and prove the second main theorem in this section.

Theorem 4.12 *Let $\theta \in (0,\pi)$. Then the following statements hold.*

(1) *If $\frac{1}{\omega-1} < k < 1$, then the equation $g_{1,k}(\theta) = 0$ has a solution in $(0,\pi)$.*

(2) *If $-1 < k < -\frac{1}{\omega-1}$, then the equation $g_{0,k}(\theta) = 0$ has a solution in $(0,\pi)$.*

Proof We consider the equations $g_{1,k}(\theta) = 0$ and $g_{0,k}(\theta) = 0$ using Lemma 4.10.

Now, we note that $\beta_m(\theta) \neq 0$ on $(0,\pi)$ by Corollary 4.8 for $m = 0$ or $m = 1$. Clearly, it is easy to see that $g_{m,k}(\theta)$ is well defined on $[0,\pi]$ and

$$g_{m,k}(0) = (1-k)\sin\frac{m\pi}{\omega}, \qquad g_{m,k}(\pi) = (1+k)\sin\frac{(m-1)\pi}{\omega}.$$

(1) First, we claim that for any k satisfying $\frac{1}{\omega-1} < k < 1$ the equation $g_{1,k}(\theta) = 0$ has a solution $\theta_* \in (0,\pi)$. Clearly, we have

$$g_{1,k}(0) = (1-k)\sin\frac{\pi}{\omega} > 0, \qquad g_{1,k}(\pi) = 0.$$

By Proposition 4.11 we obtain that if $\frac{1}{\omega-1} < k < 1$, then $g'_{1,k}(\pi) > 0$. Thus there exists a $\delta > 0$ such that $g'_{1,k}(\theta) > 0$ for all $\theta \in [\delta,\pi]$. Hence there exists an $\eta \in (\delta,\pi)$ such that

$$g_{1,k}(\pi) - g_{1,k}(\delta) = -g_{1,k}(\delta) = g'_{1,k}(\eta)(\pi-\delta) > 0,$$

which implies that $g_{1,k}(\delta) < 0$. Therefore, by the intermediate value theorem, the equation $g_{1,k}(\theta) = 0$ has a solution in $(0,\delta)$.

(2) Secondly, we claim that for any k satisfying $-1 < k < -\frac{1}{\omega-1}$, the equation $g_{0,k}(\theta) = 0$ has a solution $\theta_* \in (0,\pi)$. Clearly, we obtain

$$g_{0,k}(0) = 0, \qquad g_{0,k}(\pi) = -(1+k)\sin\frac{\pi}{\omega} < 0.$$

It follows from Proposition 4.11 that if $-1 < k < -\frac{1}{\omega-1}$, then $g'_{0,k}(0) > 0$. Thus there exists a $\delta > 0$ such that $g'_{0,k}(\theta) > 0$ for all $\theta \in [0,\delta]$. Hence there exists an $\eta \in (0,\delta)$ such that $g_{0,k}(\delta) - g_{0,k}(0) = g_{0,k}(\delta) = g'_{0,k}(\eta)\delta > 0$, which implies that $g_{0,k}(\delta) > 0$. Also, the equation $g_{0,k}(\theta) = 0$ has a solution on (δ,π). □

The following theorem is an immediate result of Theorem 4.12 and Lemma 4.10

Theorem 4.13 *Suppose* $\frac{1}{\omega-1} < |k| < 1$. *Then equation* $\Im B_{\omega,k}(\theta) = 0$ *has at least one solution in* $(0,\pi)$.

5 Equation $\Im B_{\omega,k}(\theta) = 0$

In this section, we solve equation $\Im B_{\omega,k}(\theta) = 0$ for the case $\frac{1}{\omega-1} < |k| < 1$ and consider the number of solutions in $(0,\pi)$. Now, we transform this equation to an algebraic equation of order $\omega - 2$. Since, in general,

$$\sin n\theta = \sin\theta \sum_{p=0}^{[(n-1)/2]} (-1)^p \binom{n-1-p}{p} (2\cos\theta)^{n-1-2p}, \tag{21}$$

we have the following result.

Proposition 5.1 *Let* $\theta \in (0,\pi)$ *and* $\frac{1}{\omega-1} < |k| < 1$. *Then equation* $\Im B_{\omega,k}(\theta) = 0$ *is equivalent to the equation*

$$1 + k\sum_{j=1}^{\omega-1} \binom{\omega}{j+1} (-k)^j \sum_{p=0}^{[(j-1)/2]} (-1)^p \binom{j-1-p}{p} X^{j-1-2p} = 0 \tag{22}$$

of order $\omega - 2$, *where* $X = 2\cos\theta$.

Proof Since $\frac{1}{\omega-1} < |k| < 1$, equation $\Im B_{\omega,k}(\theta) = 0$ has a solution in $(0,\pi)$ by Theorem 4.13. Using the definition of $B_{\omega,k}(\theta)$ and the binomial theorem, we have

$$\begin{aligned}
(1-k)^\omega B_{\omega,k}(\theta) &= \left(e^{i\theta} - k\right)^\omega e^{-i(\omega-1)\theta} \\
&= \sum_{j=0}^{\omega} \binom{\omega}{j} (-k)^j e^{i(1-j)\theta} \\
&= -\omega k + e^{i\theta} - k\sum_{j=1}^{\omega-1} \binom{\omega}{j+1} (-k)^j e^{-ij\theta}.
\end{aligned}$$

Therefore, Euler's formula yields that

$$\Im B_{\omega,k}(\theta) = 0 \iff \sin\theta + k\sum_{j=1}^{\omega-1} \binom{\omega}{j+1} (-k)^j \sin j\theta = 0.$$

Therefore, the proof follows from (21). □

The following result is an immediate consequence of Proposition 5.1.

Corollary 5.2 *The number of solutions in* $(0,\pi)$ *of equation* $\Im B_{\omega,k}(\theta)=0$ *is at most* $\omega-2$.

We can solve equation $\Im B_{\omega,k}(\theta)=0$ by the equation $g_{m,k}(\theta)=0$ in (19). This equation is transformed as follows.

Lemma 5.3 *For* $\theta\in(0,\pi)$ *the following statements hold.*

(1) *If* $m\in\mathbb{O}\cap\mathbb{Z}(\theta)$, *then the equation* $g_{m,k}(\theta)=0$ *is equivalent to the equation*

$$\sin\beta_m(\theta)=k\sin\big((\omega-1)\beta_m(\theta)\big).$$

(2) *If* $m\in\mathbb{E}\cap\mathbb{Z}(\theta)$, *then the equation* $g_{m,k}(\theta)=0$ *is equivalent to the equation*

$$\sin\beta_m(\theta)=-k\sin\big((\omega-1)\beta_m(\theta)\big).$$

Proof The assertions are easily obtained from

$$(\omega-1)\beta_m(\theta)=m\pi-\frac{m\pi+(\omega-1)\theta}{\omega},$$

which means $\beta_m(\theta)+\theta=m\pi-(\omega-1)\beta_m(\theta)$. Therefore, the proof easily follows. □

Based on this fact, we obtain the following result.

Proposition 5.4 *Suppose that* $\frac{1}{\omega-1}<|k|<1$, $\theta\in(0,\pi)$ *and set* $X=2\cos\beta_m(\theta)$.

(1) *Let* $m\in\mathbb{O}\cap\mathbb{Z}(\theta)$. *Then* $g_{m,k}(\theta)=0$ *if and only if*

$$1-k\sum_{p=0}^{\omega_0-1}(-1)^p\binom{\omega-2-p}{p}X^{\omega-2-2p}=0. \tag{23}$$

(2) *Let* $m\in\mathbb{E}\cap\mathbb{Z}(\theta)$. *Then* $g_{m,k}(\theta)=0$ *if and only if*

$$1+k\sum_{p=0}^{\omega_0-1}(-1)^p\binom{\omega-2-p}{p}X^{\omega-2-2p}=0. \tag{24}$$

Proof (1) Let $m\in\mathbb{O}$. Then it follows from Lemma 5.3 that $g_{m,k}(\theta)=0$ is equivalent to $\sin\beta_m(\theta)=k\sin[(\omega-1)\beta_m(\theta)]$. By (21) we have

$$\sin\big((\omega-1)\beta_m(\theta)\big)=\sin\beta_m(\theta)\sum_{p=0}^{\omega_0-1}(-1)^p\binom{\omega-2-p}{p}\big(2\cos\beta_m(\theta)\big)^{\omega-2-2p}.$$

Therefore, we obtain (23), since $\sin\beta_m(\theta)\neq0$.

(2) Let $m\in\mathbb{E}$. Then, by the same argument as above, we obtain (24). □

Applying the above two methods in Proposition 5.1 and Proposition 5.4, we can obtain the solutions of Equation $\Im B_{\omega,k}(\theta)=0$ for the period $\omega=4$.

Example 5.5 Let $\omega = 4$. Then the solutions $\gamma_\pm \in (0,\pi)$ of equation $\Im B_{4,k}(\theta) = 0$ are given as follows:

(1) If $\frac{1}{3} < k < 1$, then

$$\gamma_+ = \arccos\left(\frac{k^2 + 2k - 1}{2k^2}\right). \tag{25}$$

(2) If $-1 < k < -\frac{1}{3}$, then

$$\gamma_- = \arccos\left(\frac{-k^2 + 2k + 1}{2k^2}\right). \tag{26}$$

First, we verify this result by Proposition 5.1. Indeed, it follows from (22) that $1 - 6k^2 + k^4 + 4k^3X - k^4X^2 = 0$. Thus the solutions of Equation (22) are given by $X = 2\cos\gamma = \frac{2k\mp(1-k^2)}{k^2}$. If $\frac{1}{3} < |k| < 1$, then $2|\cos\gamma| < 2$, i.e., $|\cos\gamma| = |\frac{2k\mp(1-k^2)}{2k^2}| < 1$. Then the solutions $\gamma \in (0,\pi)$ of Equation $\Im B_{4,k}(\theta) = 0$ are given by (25) and (26).

Next, we verify the above result applying Theorem 5.4. Since $\omega = 4$, $\omega_0 = 2$, we have $\cup_{0<\theta<\pi}\mathbb{Z}^4_+(\theta) = \{1,2\}$ and $\cup_{0<\theta<\pi}\mathbb{Z}^4_-(\theta) = \{-1,0\}$.

(1) Let $\frac{1}{3} < k < 1$ and $m = 1$. Then (23) becomes $1 - k(X^2 - 1) = 0$, i.e., $\cos\beta_1(\theta) = \sqrt{\frac{k+1}{4k}}$. Since $\cos\frac{\pi-\theta}{4} = \sqrt{\frac{k+1}{4k}}$, we have $\cos\theta = \frac{k^2+2k-1}{2k^2}$.

If $m = 2 \in \mathbb{E}$, then (23) becomes $1 + k(X^2 - 1) = 0$ and $2\cos\beta_2(\theta) = X$. Thus $X^2 = \frac{k-1}{k} < 0$, which means that no solution exists.

(2) Let $-1 < k < -\frac{1}{3}$ and $m = 0$. Then (24) becomes $1 + k(X^2 - 1) = 0$, i.e., $\cos\beta_0(\theta) = \sqrt{\frac{k-1}{4k}}$. Since $\cos\frac{-\theta}{4} = \sqrt{\frac{k-1}{4k}}$, we have $\cos\theta = \frac{-k^2+2k+1}{2k^2}$.

If $m = -1 \in \mathbb{O}$, then (23) becomes $1 - k(X^2 - 1) = 0$ and $2\cos\beta_{-1}(\theta) = X$. Thus $X^2 = \frac{k+1}{k} < 0$, which means that no solution exists.

Therefore, we obtain Example 5.5.

Using the same argument as above, we can obtain the following result for the case $\omega = 3$.

Example 5.6 Let $\omega = 3$ and $\frac{1}{\omega-1} = \frac{1}{2} < |k| < 1$. Then the unique solution $\gamma \in (0,\pi)$ of equation $\Im B_{3,k}(\theta) = 0$ is given by $\gamma = \arccos(\frac{3k^2-1}{2k^3})$. In particular, we have:

(1) If $\frac{1}{\sqrt{3}} < k < 1$, then $0 < \gamma < \frac{\pi}{2}$.

(2) If $\frac{1}{2} < k \le \frac{1}{\sqrt{3}}$, then $\frac{\pi}{2} \le \gamma < \pi$.

(3) If $-\frac{1}{\sqrt{3}} \le k < -\frac{1}{2}$, then $0 < \gamma \le \frac{\pi}{2}$.

(4) If $-1 < k < -\frac{1}{\sqrt{3}}$, then $\frac{\pi}{2} < \gamma < \pi$.

6 Geometric properties of the function $B_{\omega,k}(\theta)$

In this section, we deal with geometric properties of the function $B_{\omega,k}(\theta)$, $\theta \in (-\pi,\pi]$. We denote by $\partial\Omega$ the boundary of a bounded domain Ω. Moreover, if $\partial\Omega$ is a simply closed curve, then Ω means the domain enclosed by $\partial\Omega$. We denote by $\operatorname{int}\Omega$ and $\operatorname{ext}\Omega$ the interior and the exterior of Ω, respectively. Define a positive number $\gamma \in (0,\pi]$ as

$$\gamma = \begin{cases} \pi, & 0 < |k| \le \frac{1}{\omega-1}, \\ \min\{\theta \in (0,\pi) | \Im B_{\omega,k}(\theta) = 0\}, & \frac{1}{\omega-1} < k < 1, \\ \max\{\theta \in (0,\pi) | \Im B_{\omega,k}(\theta) = 0\}, & -1 < k < -\frac{1}{\omega-1}. \end{cases}$$

and

$$I(\gamma)=\begin{cases}[0,\pi],\gamma=\pi, & 0<|k|\le\frac{1}{\omega-1},\\ [0,\gamma],\gamma\neq\pi, & \frac{1}{\omega-1}<k<1,\\ [\gamma,\pi],\gamma\neq\pi, & -1<k<-\frac{1}{\omega-1}.\end{cases}\tag{27}$$

Clearly, $B_{\omega,k}(\gamma)\in\mathbb{R}$. We denote by $B^{\gamma}_{\omega,k}(0)$ the domain enclosed by the line $\mathbb{R}$ and the restriction of the curve $B_{\omega,k}(\theta)$ to $I(\gamma)$. Moreover, We denote by $D^{\gamma}_{\omega,k}(0)$ the union of the domain $B^{\gamma}_{\omega,k}(0)$ and its symmetric domain on the line $\mathbb{R}$. The curve $\partial D^{\gamma}_{\omega,k}(0)$ is called the minimal and closed curve around the origin (briefly, *m-closed curve*). Then it has the following properties:

(1) $0\in\operatorname{int}D^{\gamma}_{\omega,k}(0)$.

(2) $\partial D^{\gamma}_{\omega,k}(0)\not\subset\mathbb{R}$ on $\operatorname{int}I(\gamma)$.

(3) $D^{\gamma}_{\omega,k}(0)$ is a simply connected domain.

Note that if $0<|k|\le\frac{1}{\omega-1}$, then $\partial D^{\gamma}_{\omega,k}(0)=\partial D^{\pi}_{\omega,k}(0)$. We denote by $\mathbb{C}_+$ and $\mathbb{C}_-$ the upper half plane $\{z\in\mathbb{C}|\Im z\ge 0\}$ and the lower half plane $\{z\in\mathbb{C}|\Im z\le 0\}$, respectively. Then $\partial B^{\gamma}_{\omega,k}(0)$ lies inside either $\mathbb{C}_+$ or $\mathbb{C}_-$. Each shaded region in Fig. 1 below for $\omega=3$ shows $D^{\gamma}_{3,k}(0)$.

Lemma 6.1 *$|B_{\omega,k}(\theta)|$ is bijective, continuous, and strictly monotone on $I(\gamma)$.*

Proof Set $\mu_\theta=|B_{\omega,k}(\theta)|$. Since $|B_{\omega,k}(\theta)|$ is strictly monotone on $I(\gamma)$ by Lemma 3.5, we see that if $0<k<1$, then the function $|B_{\omega,k}(\theta)|:[0,\gamma]\to[\mu_0,\mu_\gamma]$ is bijective. Similarly, if $-1<k<0$, then $|B_{\omega,k}(\theta)|:[\gamma,\pi]\to[\mu_\pi,\mu_\gamma]$ is also bijective. Thus the function $|B_{\omega,k}(\theta)|$ is also bijective on $I(\gamma)$. □

Lemma 6.2 *$\partial B^{\gamma}_{\omega,k}(0)\subset\mathbb{C}_+$ and $\varphi'_k(\gamma)\ge 0$. Moreover, $\varphi_k(\gamma)=\pi$ if $-\frac{1}{\omega-1}\le k<1$ $(k\neq 0)$; $\varphi_k(\gamma)=0$ if $-1<k<-\frac{1}{\omega-1}$.*

Proof (i) Let $\frac{1}{\omega-1}<k<1$. Then $I(\gamma)=[0,\gamma]$. Corollaries 3.4 and 4.3 imply $\varphi_k(0)=0$ and $\varphi'_k(0)>0$. Now we claim $\varphi_k(\gamma)=\pi$. Indeed, $\varphi_k(\gamma)=0$ or π. If $\varphi_k(\gamma)=0$, then $\beta(k,\gamma)=-\frac{\gamma}{\omega}<0$, which contradicts the assertion of Lemma 3.2. Therefore, $\partial B^{\gamma}_{\omega,k}(0)$ lies on $\mathbb{C}_+$.

Next, we claim $\varphi'_k(\gamma)\ge 0$. Indeed, for a contradiction, we assume $\varphi'_k(\gamma)<0$. Since $\varphi'_k(\theta)$ is continuous on $[0,\gamma]$, there exists a $\delta>0$ such that $\varphi'_k(\theta)<0$ on $[\gamma-\delta,\gamma]$ and $\varphi_k(\gamma-\delta)<2\pi$.

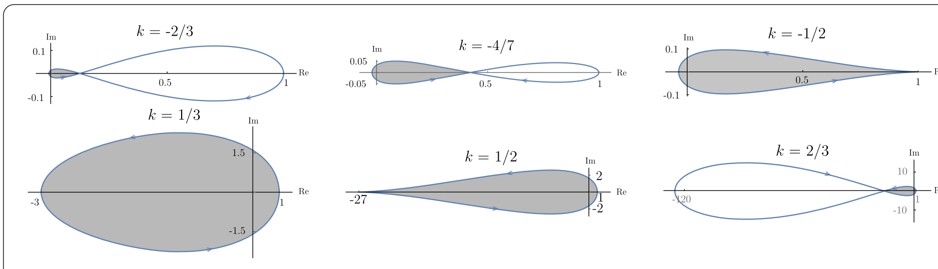


Figure 1 The graphs of $B_{3,k}(\theta)$ and $D^{\gamma}_{3,k}(0)$ on the complex plane

Thus there exists an $\eta \in (\gamma - \delta, \gamma)$ such that $\varphi_k(\gamma) - \varphi_k(\gamma - \delta) = \varphi_k'(\eta)\delta < 0$, and hence, $\varphi_k(\gamma) = \pi < \varphi_k(\gamma - \delta) < 2\pi$. This means $B_{\omega,k}(\gamma - \delta) \notin \mathbb{C}_+$, which yields a contradiction.

(ii) Let $-1 < k < -\frac{1}{\omega-1}$. Then $I(\gamma) = [\gamma, \pi]$. Corollaries 3.4 and 4.3 implies $\varphi_k'(\pi) > 0$. Now we claim $\varphi_k(\gamma) = 0$. Indeed, if $\varphi_k(\gamma) = \pi$, then $\beta(k,\gamma) = \frac{\pi-\gamma}{\omega} > 0$, which contradicts the assertion of Lemma 3.2. Therefore, $\partial B^{\gamma}_{\omega,k}(0)$ lies on $\mathbb{C}_+$. Then $\varphi_k'(\gamma) \geq 0$ is obtained by the same argument as above.

(iii) Let $0 < |k| \leq \frac{1}{\omega-1}$. Then $I(\gamma) = [0, \pi]$. Corollary 3.4 implies $\varphi_k(0) = 0$ and $\varphi_k(\pi) = \pi$. Moreover, it follows from Theorem 4.4 and Remark 4.5 that $\varphi_k(\theta) > 0$ on $(0, \pi)$ and $\varphi_k'(\gamma) \geq 0$ $(\gamma = 0, \pi)$.

Therefore, the proof is complete. □

Note that Lemma 6.1 and Lemma 6.2 imply that $|B_{\omega,k}(\theta)|$ and $\varphi_k(\theta)$ are monotone on $I(\gamma)$. We denote by L_μ or ℓ_φ the half line connecting a point $\mu = |\mu|e^{i\varphi} \in \mathbb{C}$, $\mu \neq 0$ from the origin.

Definition 6.3 Let $\partial\Omega$ be a closed curve around the origin. If $\partial\Omega \cap L_\mu$ has a unique element for every $\mu \in \mathbb{C} \setminus \{0\}$, then $\partial\Omega$ is called the monotone starlike curve (briefly, *m-starlike curve*).

For example, circles and ellipses whose center is the origin are m-starlike curves. Also the boundary of a convex domain containing the origin is an m-starlike curve.

Theorem 6.4 *The m-closed curve $\partial D^{\gamma}_{\omega,k}(0)$ is an m-starlike curve.*

Proof It suffices to prove the uniqueness of elements in $\partial B^{\gamma}_{\omega,k}(0) \cap L_\mu$ for any $\mu \in \mathbb{C}$ such that $0 \leq \mathrm{Arg}\mu \leq \pi$. It follows from Lemma 6.2 that $\partial B^{\gamma}_{\omega,k}(0) \cap L_\mu$ is contained in $\mathbb{C}_+$. For a contradiction, we assume that there exist $\mu := |\mu|e^{i\varphi}$ and δ_1, δ_2 $(|\delta_1| < |\delta_2|)$ satisfying $\partial B^{\gamma}_{\omega,k}(0) \cap L_\mu = \{\delta_1, \delta_2\}$. Then there are $\theta_1, \theta_2 \in I(\gamma)$ such that $\delta_i = B_{\omega,k}(\theta_i)$ and $\varphi = \varphi_k(\theta_i)$, $i = 1, 2$ by using Lemma 6.1 and Proposition 3.3. Since $|\delta_1| < |\delta_2|$, it follows from Lemma 3.5 that $\theta_1 < \theta_2$ if $0 < k < 1$; $\theta_2 < \theta_1$ if $-1 < k < 0$. Define

$$\psi = \sup\{\phi > \varphi \mid \partial D^{\gamma}_{\omega,k}(0) \cap \ell_\phi \text{ is not unique}\}.$$

Since $\psi \leq \pi$, there is a unique $\theta_0 \in I(\gamma)$ such that $\psi = \varphi_k(\theta_0)$ holds.

Note that the tangent line of $B_{\omega,k}(\theta)$ at θ_0 coincides with that of L_{δ_0}, $\delta_0 = B_{\omega,k}(\theta_0)$. Since $\frac{d}{d\theta}B_{\omega,k}(\theta) = \frac{i(1-ke^{-i\theta})^{\omega-1}}{(1-k)^\omega}[k\omega - k + e^{i\theta}]$, and since the slope of the half line L_{δ_0} is expressed as $B_{\omega,k}(\theta_0)$, we have

$$B_{\omega,k}(\theta_0)i\left[e^{i\theta_0} + k(\omega - 1)\right] = B_{\omega,k}(\theta_0)\left(e^{i\theta_0} - k\right),$$

that is, $i[e^{i\theta_0} + k(\omega - 1)] = (e^{i\theta_0} - k)$. This means

$$\begin{cases} lk\omega - k + \cos\theta_0 - \sin\theta_0 = 0, \\ k - \cos\theta_0 + \sin\theta_0 = 0, \end{cases}$$

$$\sin\theta_0 = \frac{k\omega}{2}, \quad \text{and} \quad \cos\theta_0 = -\frac{k(\omega - 2)}{2},$$

and hence

$$\tan\theta_0 = -\frac{\omega}{\omega-2} < 0.$$

Thus $-\frac{\pi}{2} < \theta_0 < 0$. This is a contradiction.

Therefore, $\partial B^{\gamma}_{\omega,k}(0) \cap L_\mu$ is unique. □

7 Stability regions: general case

In this section, we consider the criteria on the stabilization via DFC for the case $K = kE$, which are main results in this paper. For $K = kE$ and $\mu \in \sigma(T(0))$ we denote by $\sigma_\mu(U_k(0))$ the set of all $\nu \in \sigma(U_k(0))$ such that $\mu = C_{\omega,k}(\nu)$.

Now, we are in a position to state and prove two main theorems of this paper.

Theorem 7.1 *Suppose $K = kE$ and $\mu \in \sigma(T(0))$.*

(1) *If $\mu \in \operatorname{int} D^{\gamma}_{\omega,k}(0)$, then $|\nu| < 1$ for all $\nu \in \sigma_\mu(U_k(0))$.*

(2) *If $\mu \in \operatorname{ext} D^{\gamma}_{\omega,k}(0)$, then $|\nu| > 1$ for all $\nu \in \sigma_\mu(U_k(0))$.*

Proof Let $\mu = |\mu|e^{i\varphi} \in \sigma(T(0))$. Then Corollary 2.7 implies that all the solutions of the equation $\mu = C_{\omega,k}(\nu)$ belong to $\sigma_\mu(U_k(0))$.

(1) Let $\mu \in \operatorname{int} D^{\gamma}_{\omega,k}(0)$. Then we prove that the inequality $|\nu| < 1$ holds for all $\nu \in \sigma_\mu(U_k(0))$. For a contradiction, we assume that $|\nu| \geq 1$ holds for some $\nu \in \sigma_\mu(U_k(0))$. If $|\nu| = 1$, then $\mu \in \partial D^{\gamma}_{\omega,k}(0)$ or $\mu \in \operatorname{ext} D^{\gamma}_{\omega,k}(0)$. This is a contradiction.

Now we consider the case $|\nu| > 1$. We denote by C the closed unit disc centered at the origin. Then we can take a $\nu_0 \in \mathbb{R}$ such that $\nu_0 \in \operatorname{int} C$ and $\mu_0 := C_{\omega,k}(\nu_0) \in \operatorname{int} D^{\gamma}_{\omega,k}(0)$. Moreover, let L be the line segment connecting with ν_0 and ν. Then there exists a unique $\eta \in L$ such that $\eta \in \partial C$. In other words, η is the intersection of the line segment L and the unit circle ∂C. Hence $|\eta| = 1$. Since the mapping $C_{\omega,k}(\cdot)$ is an analytic function on a neighborhood of the point η, we have

$$\frac{d}{d\nu} C_{\omega,k}(\nu)|_{\nu=\eta} = \left(\frac{\nu-k}{(1-k)\nu}\right)^{\omega-1} \frac{\nu+(\omega-1)k}{(1-k)\nu}\bigg|_{\nu=\eta}.$$

Note that $\frac{d}{d\nu} C_{\omega,k}(\eta) = 0$ if and only if $\eta = -(\omega-1)k$. Thus $|\eta| = 1 = (\omega-1)|k|$.

(1-1) The case $|k| \neq \frac{1}{\omega-1}$. Since $\frac{d}{d\nu} C_{\omega,k}(\eta) \neq 0$, it is a conformal mapping at η, that is, the angle between two curves L and ∂C coincides with the angle between two curves $C_{\omega,k}(L)$ and $\partial D^{\gamma}_{\omega,k}(0)$. Thus, there exists a point in $\operatorname{ext} D^{\gamma}_{\omega,k}(0)$, which belongs to $C_{\omega,k}(L)$. On the other hand, since μ and μ_0 are connected via $C_{\omega,k}(L)$ and since μ and μ_0 belong to $\operatorname{int} D^{\gamma}_{\omega,k}(0)$, there exists another point $\xi \in L$ such that $C_{\omega,k}(\xi) \in \partial D^{\gamma}_{\omega,k}(0) \cap C_{\omega,k}(L)$. Since $\xi \in \partial C$, a contradiction follows from the uniqueness of η.

(1-2) The case $|k| = \frac{1}{\omega-1}$. It follows that $D^{\gamma}_{\omega,k}(0) = D^{\pi}_{\omega,k}(0)$ by Theorem 4.4 and $\frac{d}{d\nu} C_{\omega,k}(\eta) = 0$. Thus $\eta = \pm 1$. Since $\mu \in \mathbb{R}$ if $\nu \in \mathbb{R}$, we have $L \subset \mathbb{R}$ and $C_{\omega,k}(L) \subset \mathbb{R}$. In particular, $C_{\omega,k}(1) = 1$ and $C_{\omega,k}(-1) = -(\frac{1+k}{1-k})^{\omega}$.

Let $k = -\frac{1}{\omega-1}$. Then $\eta = 1$. Since $C_{\omega,-\frac{1}{\omega-1}}(x) = x(\frac{x(\omega-1)+1}{\omega x})^{\omega}$, we have $\frac{d}{dx} C_{\omega,-\frac{1}{\omega-1}}(1) = 0$ and $\frac{d^2}{dx^2} C_{\omega,-\frac{1}{\omega-1}}(1) > 0$, so that $C_{\omega,-\frac{1}{\omega-1}}(1) = 1$ is the minimal value on L. This means that $\mu = C_{\omega,-\frac{1}{\omega-1}}(\nu) \in (1,\infty)$, i.e., $\mu \in \operatorname{ext} D^{\pi}_{\omega,-\frac{1}{\omega-1}}(0)$, which leads to a contradiction.

Let $k = \frac{1}{\omega-1}$. Then we can apply the similar method to get the result.

(2) Let $\mu \in \operatorname{ext} D^{\gamma}_{\omega,k}(0)$. For a contradiction, we assume that there exists a $\nu \in \sigma_{\mu}(U_k(0))$ such that $|\nu| < 1$ and $\mu = C_{\omega,k}(\nu)$. Let L be the line segment connecting ν and k ($k \neq \nu$). Then $L \subset \operatorname{int} C$. Since $C_{\omega,k}(k) = 0$, there exists an $\eta \in C_{\omega,k}(L) \cap \partial D^{\gamma}_{\omega,k}(0) \neq \emptyset$. Thus there exists a $\xi \in L$ such that $\eta = C_{\omega,k}(\xi)$, which is a contradiction. □

The following result is an immediate consequence of Theorem 7.1.

Theorem 7.2 *Let $K = kE$.*

(1) *If $\mu \in \operatorname{int} D^{\gamma}_{\omega,k}(0)$ for all $\mu \in \sigma(T(0))$, then $|\nu| < 1$ for all $\nu \in \sigma(U_k(0))$.*

(2) *If there exists a $\mu \in \sigma(T(0))$ such that $\mu \in \operatorname{ext} D^{\gamma}_{\omega,k}(0)$, then $|\nu| > 1$ for all $\nu \in \sigma_{\mu}(U_k(0))$, and hence there exists a $\nu \in \sigma(U_k(0))$ such that $|\nu| > 1$.*

Remark 7.3 (1) Theorem 7.2 can be extended to more general commuting matrix K (see [6]).

(2) Combining Theorem 7.2 with nondegenerate properties, we can obtain a stability region for a periodic solution (see [5]).

Next, we give necessary and sufficient conditions for $\mu \in \operatorname{int} \mathcal{D}^{\gamma}_{\omega,k}(0)$. In relation to (15), we define a function of $k \in (-1, 1)$ as follows:

$$
\begin{aligned}
f_{\omega}\big(k;\theta,|\mu|\big) &= |\mu|^{\frac{2}{\omega}}(1-k)^2 - 1 + 2k\cos\theta - k^2 \\
&= \big(|\mu|^{\frac{2}{\omega}} - 1\big)k^2 - 2\big(|\mu|^{\frac{2}{\omega}} - \cos\theta\big)k + \big(|\mu|^{\frac{2}{\omega}} - 1\big),
\end{aligned}
\tag{28}
$$

where $-\pi < \theta \leq \pi$. Then $f_{\omega}(k;\theta,|\mu|) < 0$ if and only if $|\mu| < |B_{\omega,k}(\theta)|$. Since $\partial D^{\gamma}_{\omega,k}(0) \cap L_{\mu} = \{\delta_{\mu}\}$ for every $\mu \in \mathbb{C}$ is unique by Theorem 6.4, there exists a unique θ_{μ} such that $\delta_{\mu} = B_{\omega,k}(\theta_{\mu})$. Hereafter, such an argument θ_{μ} is called *the argument associated with* $(\mu, \partial D^{\gamma}_{\omega,k}(0))$.

Now, we give necessary and sufficient conditions for $\mu \in \operatorname{int} D^{\gamma}_{\omega,k}(0)$. The proof is easy.

Theorem 7.4 *Suppose $K = kE$ and $\mu \in \mathbb{C}$. If θ_{μ} is the argument associated with $(\mu, \partial D^{\gamma}_{\omega,k}(0))$, then the following statements are equivalent*:

(1) $\mu \in \operatorname{int} D^{\gamma}_{\omega,k}(0)$.

(2) $|\mu| < |B_{\omega,k}(\theta_{\mu})|$.

(3) $f_{\omega}(k;\theta_{\mu},|\mu|) < 0$

Using Theorem 7.4, we can easily obtain the following result.

Corollary 7.5 *Suppose $K = kE$ and $\mu \in \mathbb{C}$. If θ_{μ} is the argument associated with $(\mu, \partial D^{\gamma}_{\omega,k}(0))$, then the following statements are equivalent*:

(1) $\mu \in \operatorname{ext} D^{\gamma}_{\omega,k}(0)$.

(2) $|\mu| > |B_{\omega,k}(\theta_{\mu})|$.

(3) $f_{\omega}(k;\theta_{\mu},|\mu|) > 0$.

Finally, we illustrate our method (Theorem 7.2) for the case $\omega = 4$ to compare with the Jury criterion, provided that all $\mu \in \sigma(T(0))$ are real. Set $\sigma_{\mathbb{R}}(T(0)) = \sigma(T(0)) \cap \mathbb{R}$. Then the following lemmas are obtained by using Example 5.5.

Lemma 7.6 *If $\gamma_\pm \in (0,\pi)$ are given by (25) and (26) in Example 5.5, then $B_{4,k}(\gamma_-)$ and $B_{4,k}(\gamma_+)$ are given as follows*:

(1) *If $-1 < k \le -\frac{1}{3}$, then $B_{4,k}(\gamma_-) = \frac{(1+k)^4}{k^2(1-k)^2}$.*

(2) *If $\frac{1}{3} \le k < 1$, then $B_{4,k}(\gamma_+) = -\frac{(1+k)^2}{k^2}$.*

Proof Since $(1-k)^4 B_{4,k}(\theta) = e^{i\theta} - 4k + 6k^2 e^{-i\theta} - 4k^3 e^{-2i\theta} + k^4 e^{-3i\theta}$, we obtain

$$\begin{aligned}(1-k)^4 B_{4,k}(\gamma) \\ = \bigl(4\cos^3\gamma - 3\cos\gamma\bigr)k^4 - 4\bigl(2\cos^2\gamma - 1\bigr)k^3 + (6\cos\gamma)k^2 - 4k + \cos\gamma.\end{aligned} \quad (29)$$

Now, we substitute γ_+ and γ_- in Example 5.5 into (29).

(1) Let $-1 < k \le -\frac{1}{3}$. Then we obtain $(1-k)^4 B_{4,k}(\gamma) = \frac{(k-1)^2(k+1)^4}{k^2}$.

(2) Let $\frac{1}{3} \le k < 1$. Then we obtain $(1-k)^4 B_{4,k}(\gamma) = -\frac{(k+1)^2(k-1)^4}{k^2}$. □

The following lemma gives properties of $B_{4,k}(\gamma_-)$ and $B_{4,k}(\gamma_+)$.

Lemma 7.7 *The following statements hold*:

(1) *$B_{4,k}(\gamma_-) = \frac{(1+k)^4}{k^2(1-k)^2}$, $-1 < k < -\frac{1}{3}$ has the following properties*:

(1-1) $B_{4,-\frac{1}{3}}(\gamma_-) = 1$.

(1-2) $\lim_{k\to -1} B_{4,k}(\gamma_-) = 0$.

(1-3) *$B_{4,k}(\gamma_-)$ is increasing in $k \in (-1,-\frac{1}{3})$.*

(2) *$B_{4,k}(\gamma_+) = -\frac{(1+k)^2}{k^2}$, $(\frac{1}{3} < k < 1)$ has the following properties*:

(2-1) $B_{4,\frac{1}{3}}(\gamma_+) = -4^2$.

(2-2) $\lim_{k\to 1} B_{4,k}(\gamma_+) = -2^2$.

(2-3) *$B_{4,k}(\gamma_+)$ is decreasing in $k \in (\frac{1}{3},1)$.*

The following result (see Fig. 2) illustrates Theorem 7.2.

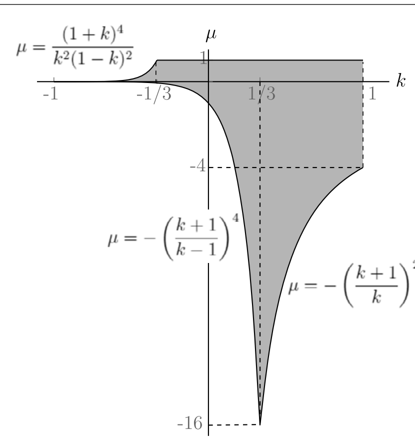
Proposition 7.8 *Suppose $\omega = 4$ and $\mu \in \sigma_{\mathbb{R}}(T(0))$.*

(1) *If μ is in the following regions, then $|\nu| < 1$ for all $\nu \in \sigma_\mu(U_k(0))$.*

(1-1) $0 < \mu < \frac{(1+k)^4}{k^2(1-k)^2}$ *and* $-1 < k < -\frac{1}{3}$.

(1-2) $0 < \mu < 1$ *and* $-\frac{1}{3} \le k < 1$, $k \ne 0$

Figure 2 The region of (k,μ) where $|\nu| < 1$ for all $\nu \in \sigma(U_k(0))$



(1-3) $-(\frac{k+1}{k-1})^4 < \mu < 0$ *and* $-1 < k \leq \frac{1}{3}$, $k \neq 0$.
(1-4) $-\frac{(1+k)^2}{k^2} < \mu < 0$ *and* $\frac{1}{3} < k < 1$.
(1-5) $\mu = -1$ *and* $0 < k < 1$.
(2) *If* μ *is in the following regions, then* $|\nu| > 1$ *for all* $\nu \in \sigma_\mu(U_k(0))$.
(2-1) $\mu > 1$ *and* $-\frac{1}{3} < k < 1$, $k \neq 0$.
(2-2) $\frac{(1+k)^4}{k^2(1-k)^2} < \mu$ *and* $-1 < k < -\frac{1}{3}$.
(2-3) $\mu < -(\frac{k+1}{k-1})^4$ *and* $-1 < k < \frac{1}{3}$, $k \neq 0$.
(2-4) $\mu < -\frac{(1+k)^2}{k^2}$ *and* $\frac{1}{3} < k < 1$.
(2-5) $\mu = -1$ *and* $-1 < k < 0$.

Proof We will verify the conditions in Proposition 7.4 and Corollary 7.5 to apply Theorem 7.1. Let θ_μ be the argument associated with $(\mu, \partial D^{\gamma}_{4,k}(0))$.

(A) The case $\mu > 0$.

(A-1) Let $1 < \mu$ and $-1 < k < -\frac{1}{3}$. Then $I(\gamma) = I(\gamma_-) = [\gamma_-, \pi]$ by using the definition of $I(\gamma)$ and Example 5.5. Thus $\theta_\mu = \gamma_-$. By Lemma 7.6 and Lemma 3.5 we have $0 < |B_{4,k}(\theta_\mu)| = |B_{4,k}(\gamma_-)| = \frac{(1+k)^4}{k^2(1-k)^2} \leq 1 < \mu$. By Corollary 7.5, we obtain $\mu \in \operatorname{ext} D^{\gamma}_{4,k}(0)$.

(A-2) Let $1 < \mu$ and $-\frac{1}{3} \leq k < 1$, $k \neq 0$. Then $I(\gamma) = [0, \gamma]$ and $\theta_\mu = 0$. Hence we see that $f_4(k; 0, |\mu|) = (|\mu|^{\frac{1}{2}} - 1)(k-1)^2 > 0$ if and only if $0 < |k| < 1$. Thus $f_4(k; 0, |\mu|) > 0$ for all $k \in [-\frac{1}{3}, 1) \setminus \{0\}$, and hence $\mu \in \operatorname{ext} D^{\gamma_-}_{4,k}(0)$.

(A-3) Let $0 < \mu < 1$ and $-1 < k < -\frac{1}{3}$. Then $I(\gamma) = [\gamma_-, \pi]$ and $\theta_\mu = \gamma_-$. Thus it follows from (A-1) that if $\mu < B_{4,k}(\gamma_-)$ for all $k \in (-1, -\frac{1}{3})$, then $\mu \in \operatorname{int} D^{\gamma_-}_{4,k}(0)$; if $1 > \mu > B_{4,k}(\gamma_-)$ for all $k \in (-1, -\frac{1}{3})$, then $\mu \in \operatorname{ext} D^{\gamma_-}_{4,k}(0)$.

(A-4) Let $0 < \mu < 1$ and $-\frac{1}{3} \leq k < 1$, $k \neq 0$. Then $\theta_\mu = 0$. Hence we see that $f_4(k; 0, |\mu|) < 0$ for all $k \in [-\frac{1}{3}, 1) \setminus \{0\}$. Thus $\mu \in \operatorname{int} D^{\gamma_-}_{4,k}(0)$.

(A-5) Let $\mu = 1$. If $-1 < k < -\frac{1}{3}$, then $\theta_\mu = \gamma_-$. Since $B_{4,k}(\gamma_-)$ is increasing in $k \in (-1, -\frac{1}{3})$ by Lemma 7.7 and $B_{4,k}(\gamma_-) < B_{4,-\frac{1}{3}}(\gamma_-) = 1$, we have $|B_{4,k}(\gamma_-)| < 1 = |\mu|$, and hence $\mu \in \operatorname{ext} D^{\gamma_-}_{4,k}(0)$. If $-\frac{1}{3} \leq k < 1$, $k \neq 0$, then $\theta_\mu = 0$, so that $f_4(k; 0, |\mu|) = 0$.

(B) The case $\mu < 0$. Set $b = |\mu|^{\frac{1}{4}}$. Then (28) with $\theta_\mu = \pi$ becomes

$$f_4(k; \pi, |\mu|) = (b^2 - 1)k^2 - 2(b^2 + 1)k + (b^2 - 1).$$

Thus we have $k_\pm(\pi) := \frac{b^2+1 \pm \sqrt{4b^2}}{b^2-1}$.

If $b \neq 1$, then two solutions $k_-(\pi)$ and $k_+(\pi)$ of the equation $f_4(k; \pi, |\mu|) = 0$ are given by $k_-(\pi) = \frac{b-1}{b+1}$, $k_+(\pi) = \frac{b+1}{b-1}$. If $0 < b < 1$, then $k_+(\pi) < -1 < k_-(\pi) < 0$; if $b > 1$, then $0 < k_-(\pi) < 1 < k_+(\pi)$. Moreover, we have $k = \frac{b-1}{b+1} \iff b = -\frac{k+1}{k-1} \text{or} |\mu| = (\frac{k+1}{k-1})^4$ and $k = \frac{b+1}{b-1} \iff b = \frac{k+1}{k-1} \text{or} |\mu| = (\frac{k+1}{k-1})^4$. Then the following statements hold:

(B-1) Let $-1 < k \leq \frac{1}{3}$. Then $\theta_\mu = \pi$. Let $-1 < \mu < 0$. Since $0 < b < 1$, we obtain that $f_4(k; \pi, |\mu|) < 0$ if and only if $k_-(\pi) < k \leq \frac{1}{3}$, i.e., $\frac{b-1}{b+1} < k \leq \frac{1}{3}$. So, it follows that if $-(\frac{k+1}{k-1})^4 < \mu < 0$, then $f_4(k; \pi, |\mu|) < 0$.

Let $\mu < -1$. Since $b > 1$, we obtain that $f_4(k; \pi, |\mu|) < 0$ if and only if $k_-(\pi) < k < 1$, i.e., $\frac{b-1}{b+1} < k < 1$. So it follows that if $-(\frac{k+1}{k-1})^4 < \mu < 0$, then $f_4(k; \pi, |\mu|) < 0$. Thus $\mu \in \operatorname{int} D^{\gamma_-}_{4,k}(0)$.

(B-2) Let $\frac{1}{3} < k < 1$. Then $\theta_\mu = \gamma_+$. Since $B_{4,k}(\gamma_+) = -\frac{(1+k)^2}{k^2} = -(1 + \frac{1}{k})^2 < 0$, we have $0 < |B_{4,k}(\gamma_+)| = (1 + \frac{1}{k})^2 \leq 16$. If $-\frac{(1+k)^2}{k^2} < \mu < 0$, then $|\mu| < B_{4,k}(\gamma_+)$. Thus $\mu \in \operatorname{int} D^{\gamma_-}_{4,k}(0)$.

(B-3) The case $\mu = -1$. If $-1 < k \leq \frac{1}{3}$, then $\theta_\mu = \pi$. Since $f_4(k; \pi, |\mu|) = -4k$, we see that if $-1 < k < 0$, then $f_4(k; \pi, |\mu|) > 0$; if $0 < k \leq \frac{1}{3}$, then $f_4(k; \pi, |\mu|) < 0$. If $\frac{1}{3} < k < 1$, then $\theta_\mu = \gamma_+$.

Thus we have $|B_{4,k}(\theta_\mu)| = |B_{4,k}(\gamma_+)| = \frac{(k+1)^2}{k^2} > 1 = |\mu|$. Thus $\mu \in \operatorname{int} D^{\gamma_-}_{4,k}(0)$. Summing up these cases, we obtain the proposition. □

Our new method works fine to determine the stability region for this case, but it is very complicated to check the Jury criterion.

The following result illustrates Theorem 7.2 for $\omega = 3$ and $\sigma_{\mathbb{R}}(T(0))$, which is proved by the same argument as above.

Proposition 7.9 *Suppose $\omega = 3$ and $\mu \in \sigma_{\mathbb{R}}(T(0))$.*

(1) *If μ is in the following regions, then $|\nu| < 1$ for all $\nu \in \sigma_\mu(U_k(0))$.*

(1-1) $0 < \mu < -(\frac{1+k}{k})^3$ *and* $-1 < k < -\frac{1}{2}$.

(1-2) $0 < \mu < 1$ *and* $-\frac{1}{2} \le k < 1, k \neq 0$.

(1-3) $-(\frac{k+1}{k-1})^3 < \mu < 0$ *and* $-1 < k \le \frac{1}{2}, k \neq 0$.

(1-4) $-(\frac{1+k}{k})^3 < \mu < 0$ *and* $\frac{1}{2} < k < 1$.

(1-5) $\mu = -1$ *and* $0 < k < 1$.

(2) *If μ is in the following regions, then $|\nu| > 1$ for all $\nu \in \sigma_\mu(U_k(0))$.*

(2-1) $\mu > 1$ *and* $-\frac{1}{2} < k < 1, k \neq 0$.

(2-2) $-(\frac{1+k}{k})^3 < \mu$ *and* $-1 < k < -\frac{1}{2}$.

(2-3) $\mu < -(\frac{k+1}{k-1})^3$ *and* $-1 < k < \frac{1}{2}, k \neq 0$.

(2-4) $\mu < -(\frac{1+k}{k})^3$ *and* $\frac{1}{2} < k < 1$.

(2-5) $\mu = 1$ *and* $-1 < k < -\frac{1}{2}$; $\mu = -1$ *and* $-1 < k < 0$.

The result of Proposition 7.9 just coincides with the one obtained from the Jury criterion.

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Declarations

Competing interests

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Author contributions

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