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Diophantine Approximation, Continued Fractions, and Dynamical Spectrums

(디오판틴 근사, 연분수, 동역학적 스펙트럼)

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Diophantine Approximation, Continued Fractions, and Dynamical Spectrums

A dissertation
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Abstract

Diophantine Approximation, Continued Fractions, and Dynamical Spectrums

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Diophantine approximation is a rational approximation to an irrational number, which has been investigated using continued fractions. In the thesis, we deal with three topics related to Diophantine approximation and continued fractions.

The first topic is the Markoff and Lagrange spectrum associated with the Hecke group. The classical Markoff and Lagrange spectrum is associated with the modular group $\mathrm{PSL}(2,\mathbb{Z}) = \mathbf{H}_3$, which has been studied using the regular continued fraction. We consider the Markoff and Lagrange spectrum associated with $\mathbf{H}_4$ and $\mathbf{H}_6$. We use the Romik dynamical system to show that some results on the classical Markoff and Lagrange spectra appear in the Markoff and Lagrange spectra associated with the Hecke group.

The second topic is the exponents of repetition of Sturmian words. The exponent of repetition of a Sturmian word gives the irrationality exponent of the Sturmian number associated with the Sturmian word. For an irrational number θ, we determine the minimum of the exponents of repetition of Sturmian words of slope θ. We also investigate the spectrum of the exponents of repetition of Sturmian words of the golden ratio.

The last topic is quasi-Sturmian colorings on regular trees. We characterize quasi-Sturmian colorings of regular trees by its quotient graph and its recurrence function. We obtain an induction algorithm of quasi-Sturmian colorings which is analogous to the continued fraction algorithm of Sturmian words.

Keywords: Diophantine approximation, Continued fractions, Lagrange numbers, Markoff numbers, Sturmian words, Colorings of trees
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Chapter 1

Introduction

I. The Markoff and Lagrange spectra on the Hecke group

Dirichlet showed that for any $\theta \in \mathbb{R} \setminus \mathbb{Q}$, the inequality

$$\left|\theta - \frac{p}{q}\right| \leq \frac{1}{q^2}$$

has infinitely many integral solutions $p, q > 0$ [25]. Hurwitz improved Dirichlet's result by proving that the inequality

$$\left|\theta - \frac{p}{q}\right| \leq \frac{1}{\sqrt{5}q^2} \tag{1.1}$$

has infinitely many integral solutions $p, q > 0$. Moreover, he proved that $\sqrt{5}$ is the largest constant in the sense that for all irrational θ, the inequality (1.1) has infinitely many integral solutions [33]. For each irrational θ, we can consider improving the Hurwitz bound $\sqrt{5}$. This motivates the following definition:

Definition. For any $\theta \in \mathbb{R} \setminus \mathbb{Q}$, we define the Lagrange value $L(\theta)$ by

$$L(\theta) := \sup \left\{ c \geq 1 : \left|\theta - \frac{p}{q}\right| \leq \frac{1}{cq^2} \text{ for infinitely many } p \in \mathbb{Z}, q \in \mathbb{N} \right\}.$$

The Lagrange spectrum $\mathscr{L}$ is defined to be

$$\mathscr{L} = \{L(\theta) : \theta \in \mathbb{R} \setminus \mathbb{Q}\}.$$

The Markoff value and the Markoff spectrum are defined as follows:

Definition. For an indefinite binary quadratic form

$$f(x,y) = ax^2 + bxy + cy^2$$

with $a, b, c \in \mathbb{R}$ and $\delta(f) := b^2 - 4ac > 0$, we define the Markoff number $M(f)$ by

$$M(f) := \sup\left\{ \frac{\sqrt{\delta(f)}}{|f(x,y)|} : x, y \in \mathbb{Z}^2 \setminus \{(0,0)\} \right\}.$$

The Markoff spectrum $\mathscr{M}$ is

$$\mathscr{M} = \left\{ M(f) : \begin{array}{c} f(x,y) = ax^2 + bxy + cy^2 \text{ is indefinite with } a, b, c \in \mathbb{R}, \\ \delta(f) > 0 \end{array} \right\}.$$

The study on the geometric structure of $\mathscr{L}$ and $\mathscr{M}$ is a classical topic, which began with Markoff [46]:

$$\mathscr{L} \cap (0,3) = \mathscr{M} \cap (0,3) = \left\{ \sqrt{9 - \frac{4}{z^2}} : x^2 + y^2 + z^2 = 3xyz,\ x, y \le z,\ x, y, z \in \mathbb{Z} \right\}.$$

This means that $\mathscr{L}$ and $\mathscr{M}$ below 3 are discrete. After the Markoff's result, Tornheim proved that $\mathscr{L} \subset \mathscr{M}$ [65]. Perron showed that there exists maximal gaps in $\mathscr{M}$ [53]. Hall found a ray in $\mathscr{L}$, which is called Hall's ray [32]. Freiman determined the minimum of the Hall's ray in $\mathscr{L}$ [30].

On the other hand, Perron's formula induces that we can interpret the Markoff and Lagrange values as the supremum and limit supremum of heights of geodesics into the cusp of the modular surface [47]. We define the Markoff and Lagrange spectra associated with the Hecke group by the set of the Markoff and Lagrange values, respectively, which are the supremum and limit supremum of heights of geodesics into the cusp of the hyperbolic space which is the quotient space by the Hecke group. We prove that the Markoff and Lagrange values associated with $\mathbf{H}_4$ and $\mathbf{H}_6$ are expressed in terms of doubly-infinite Romik sequences.

Schmidt and Vulakh independently showed that the Markoff and Lagrange spectra associated with $\mathbf{H}_4$ below the first limit point $2\sqrt{2}$ are discrete as an analogy of Markoff theorem. We prove that the Markoff and Lagrange spectra associated with $\mathbf{H}_4$ above $2\sqrt{2}$ have positive Hausdorff dimension. We also show that the Markoff and Lagrange spectra associated with $\mathbf{H}_4$ have similar geometric structure with $\mathscr{L}$ and $\mathscr{M}$ as Figure 1.1.

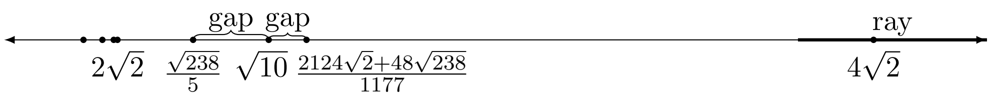


Figure 1.1 Gaps and a ray in the Markoff and Lagrange spectra associated with $\mathbf{H}_4$

Schmidt also showed that the Markoff and Lagrange spectra associated with $\mathbf{H}_6$ below the first limit point $4/\sqrt{3}$ are discrete as an analogy of Markoff theorem. We prove that the Markoff and Lagrange spectra associated with $\mathbf{H}_6$ above $4/\sqrt{3}$ have positive Hausdorff dimension. We also show that the Markoff and Lagrange spectra associated with $\mathbf{H}_6$ have similar geometric structure with $\mathscr{L}$ and $\mathscr{M}$ as Figure 1.2.

Figure 1.2 Gaps in the Markoff and Lagrange spectra associated with $\mathbf{H}_6$

II. The exponent of repetition

A word is a sequence of finite or infinite letters. For a word with finite letters, subword complexity (or factor complexity) is the function assigning n to the number of distinct subwords of length n appearing in the word. Morse and Hedlund showed that an infinite word is eventually periodic if and only if its subword complexity is bounded [50]. Thus, the smallest subword complexity of a non-eventually periodic word is $n+1$. We say a word is Sturmian if the subword complexity of the word is $n+1$. Sturmian words have some characterizations because a Sturmian word can be defined as a balanced non-periodic word or an irrational mechanical word.

Yann Bugeaud and Dong Han Kim suggested a new complexity function and characterized Sturmian words and eventually periodic words in terms of the complexity function. They introduced the exponent of repetition of a Sturmian word, which is defined as the limit infimum of the ratio of the new complexity function and the length of a subword: For an infinite word $\mathbf{x}$, the exponent of repetition of $\mathbf{x}$ is defined by

$$\operatorname{rep}(\mathbf{x}) := \liminf_{n\to\infty} \frac{r(n,\mathbf{x})}{n},$$

where $r(n, \mathbf{x})$ is the length of the smallest prefix in which some subword of length n occurs twice. The exponent of repetition of a Sturmian word gives the irrationality exponent of the Sturmian word.

Theorem ([15, Theorem 4.5]) *For a Sturmian word* $\mathbf{x} = x_1 x_2 \ldots$, *an integer* $b \geq 2$, *and a Sturmian number* $r_{\mathbf{x}} = \sum_{k \geq 1} \frac{x_k}{b^k}$, *the irrationality exponent of* $r_{\mathbf{x}}$ *is given by*

$$\mu(r_{\mathbf{x}}) = \frac{\operatorname{rep}(\mathbf{x})}{\operatorname{rep}(\mathbf{x}) - 1}.$$

In Chapter 5, we study the spectrum of the exponents of repetition. For $\theta \in (0,1) \setminus \mathbb{Q}$,

$$\mathscr{L}(\theta) := \{\operatorname{rep}(\mathbf{x}) : \mathbf{x} \text{ is a Sturmian word of slope } \theta\}.$$

Boris Adamczewski and Yann Bugeaud showed $\mathscr{L}(\theta) = \{1\}$ where θ has unbounded partial quotients. We determined the minimum of $\mathscr{L}(\theta)$ where θ has bounded partial quotients.

Theorem *Let* $\theta = [0; a_1, a_2, \ldots]$ *have bounded partial quotients. We have*

$$\min \mathscr{L}(\theta) = \varliminf_{k \to \infty} [1; 1 + a_k, a_{k-1}, a_{k-2}, \ldots, a_1].$$

We look into $\mathscr{L}(\varphi)$ for $\varphi = \frac{\sqrt{5}-1}{2} = [0; \overline{1}]$. Let us define

$$\mu_{\max} := 1 + \varphi = 1.618\ldots, \quad \mu_2 := 4\varphi - 1 = 1.472\ldots, \quad \mu_3 := \frac{5 - 5\varphi}{7\varphi - 3} = 1.440\ldots,$$
$$\mu_4 := \frac{73\varphi - 42}{65\varphi - 38} = 1.434\ldots, \quad \mu_{\min} := 2 - \varphi = 1.381\ldots.$$

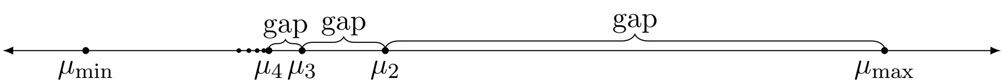


Figure 1.3 $\mu_{\max}, \mu_2, \mu_3, \mu_4, \mu_{\min}$ in $\mathscr{L}(\varphi)$

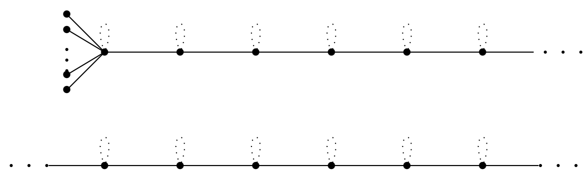
In Section 5.2, we prove that $\mu_{\max}$ is the maximum of $\mathscr{L}(\varphi)$. Next, we show that $\mu_{\max}, \mu_2, \mu_3, \mu_4$ are the four largest points in $\mathscr{L}(\varphi)$ and μ_4 is an accumulation point of $\mathscr{L}(\varphi)$. For $\mu \in \{\mu_{\max}, \mu_2, \mu_3, \mu_4, \mu_{\min}\}$, we give the necessary and sufficient condition for $\operatorname{rep}(\mathbf{x}) = \mu$ and the cardinality of the set of Sturmian words $\mathbf{x}$ satisfying $\operatorname{rep}(\mathbf{x}) = \mu$.

III. Quasi-Sturmian colorings on regular trees

Dong Han Kim and Seonhee Lim studied vertex colorings of regular trees which are maps from the vertex set of a tree to a finite set of letters. They generalized factor complexity of a word to define factor complexity $b_\phi(n)$ of a coloring ϕ where $b_\phi(n)$ is the number of colored balls of radius n up to isomorphisms preserving ϕ. They showed the analogy of Morse-Hedlund theorem and generalized Sturmian words to Sturmian colorings on regular trees [37]. They also found the induction algorithm of Sturmian colorings [38].

In Chapter 6, we define quasi-Sturmian colorings of regular trees. We characterize the quotient graph of a quasi-Sturmian coloring. The n-th factor graph is the graph whose vertices are the colored n-balls and its edges are pairs of colored n-balls whose centers are adjacent in the tree. The evolution of the factor graph tells us how the pattern of the coloring is.

Theorem *The quotient graph of a quasi-Sturmian coloring is either the union of a finite graph and a geodesic ray or a bi-infinite geodesic.*



For a quasi-Sturmian coloring with no cycle on its factor graph, the factor graph of the coloring belongs to one of (I), (II), or (III) in Figure 1.4. The factor graphs evolve as

$$\text{(I)} \to \text{(II)} \to \cdots \to \text{(II)} \to \text{(I)} \quad \text{or} \quad \text{(I)} \to \text{(II)} \to \cdots \to \text{(II)} \to \text{(III)} \to \text{(I)}.$$

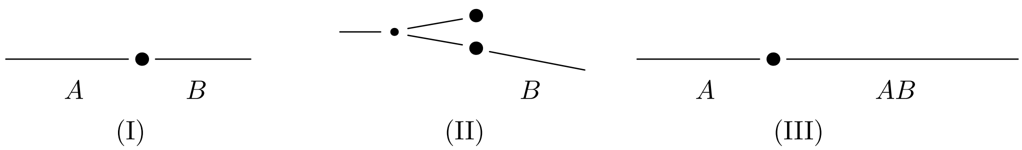


Figure 1.4 The evolution of the factor graph of a quasi-Sturmian coloring

This thesis is organized as follows. In Chapter 2, we review some definitions and properties of the regular continued fraction and recall some results on the Markoff and Lagrange spectra. In Chapter 3, we deal with the Markoff and Lagrange spectra on the Hecke group $\mathbf{H}_4$ and $\mathbf{H}_6$. We introduce the Romik dynamical system and use it to define the Markoff and Lagrange values, which are equal to the supremum and limit supremum of the heights of geodesic on the hyperbolic space which is the quotient space by the Hecke group. We prove that both spectra have positive Hausdorff dimension after the first limit point. After the first limit point, we show that there exist maximal gaps in both spectra. We also prove both spectra contains a ray which is called Hall's ray. We follow two papers [22, 39], which are the joint works with Byungchul Cha and Dong Han Kim.

In Chapter 4, we define Sturmian words and review some characterizations of Sturmian words. We also recall the definition of the exponent of repetition and its properties. In Chapter 5, we look into the spectrum of the exponents of repetition of Sturmian words. In Section 5.1, we determine the minimum of the spectrum of the exponents of repetition of Sturmian words. In Section 5.2, we investigate the spectrum of the exponents of repetition of Fibonacci words. We follow the paper [64].

In Chapter 6, we review some definitions of colorings of trees and recall the results on Sturmian colorings of trees. We characterize quasi-Sturmian colorings of a regular tree by its quotient graph and its recurrence function. We also look into an induction algorithm of a quasi-Sturmian coloring. We follow the paper [36], which is the joint work with Dong Han Kim, Seul Bee Lee, and Seonhee Lim.

Chapter 2

Diophantine approximation

Diophantine approximation is the study on the approximation of an irrational number by rational numbers. Dirichlet showed the following statement related Diophantine approximation.

Theorem 2.0.1 (Dirichlet) *Given $x \in \mathbb{R}$ and $t > 1$, there exist integers p, q such that*

$$|qx - p| \le \frac{1}{t}, \quad 1 \le q < t.$$

We obtain the following corollary from Dirichlet theorem.

Corollary 2.0.2. For any $\theta \in \mathbb{R} \setminus \mathbb{Q}$, there exist infinitely many integers $p, q > 0$ such that

$$\left|\theta - \frac{p}{q}\right| \le \frac{1}{q^2}.$$

Hurwitz determined the best bound for Corollary 2.0.2.

Theorem 2.0.3 *For any $\theta \in \mathbb{R} \setminus \mathbb{Q}$, there exist infinitely many integers $p, q > 0$ such that*

$$\left|\theta - \frac{p}{q}\right| \le \frac{1}{\sqrt{5}q^2}.$$

The equality holds if and only if $\theta = \dfrac{a\varphi + b}{c\varphi + d}$ *for some* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z})$, *where* $\varphi = \frac{1+\sqrt{5}}{2}$.

Similar to Theorem 2.0.3, we can consider the best bound for each irrational number. In this chapter, we study the best bound for all irrational numbers which are called Lagrange values. We also investigate Markoff value, which is interpreted similarly to Lagrange value. We look into the regular continued fraction, a method to obtain Markoff and Lagrange values, and its properties.

2.1 Continued fraction

In this section, we review some definitions and properties of the regular continued fractions, following [19, 26].

A *regular continued fraction* is a formal expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

where $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N}$ for all $n \in \mathbb{N}$. We denote a continued fraction as above by

$$[a_0; a_1, a_2, \dots].$$

We also write

$$[a_0; a_1, \dots, a_n]$$

for a finite fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}.$$

2.1.1 Basic properties

Let us start with the crucial lemma for many of the basic properties of the regular continued fraction.

Lemma 2.1.1. For a sequence $(a_n)_{n\geq 0}$ with $a_0 \in \mathbb{N}_0$ and $a_n \in \mathbb{N}$ for all $n \in \mathbb{N}$, the rational numbers

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, a_n]$$

for $n \geq 0$ with coprime numerator p_n and denominator q_n can be found recursively from the ralation

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \text{ for } n \geq 0, \tag{2.1}$$

where we set $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = a_0$, and $q_0 = 1$.

In the above lemma, each a_n is called *n-th partial quotient* of $[a_0; a_1, a_2, \dots]$. The finite fractions p_n/q_n is called *the principal convergent* of $[a_0; a_1, a_2, \dots]$. Lemma 2.1.1 immediately implies a pair of recursive formulas:

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, \quad q_{n+1} = a_{n+1}q_n + q_{n-1} \tag{2.2}$$

for all $n \geq 1$. Taking determinants in (2.1), we have

$$p_{n+1}q_n - p_n q_{n+1} = (-1)^n \tag{2.3}$$

and thus, we recursively have

$$\begin{aligned} \frac{p_n}{q_n} &= \frac{p_{n-1}}{q_{n-1}} + (-1)^{n+1}\frac{1}{q_{n-1}q_n} \\ &= a_0 + \frac{1}{q_0q_1} - \frac{1}{q_1q_2} + \cdots + (-1)^{n+1}\frac{1}{q_{n-1}q_n} \end{aligned}$$

for $n \geq 1$. Hence, $[a_0; a_1, a_2, \dots]$ is not just a formal expression and have a value as the limit of the principal convergents $\lim_{n\to\infty} p_n/q_n$. If $\theta = \lim_{n\to\infty} \frac{p_n}{q_n}$, then we say that $[a_0; a_1, a_2, \dots]$ is the *continued fraction expansion* for θ. When we want to emphasize θ, we denote a_n and p_n/q_n by $a_n(\theta)$ and $p_n(\theta)/q_n(\theta)$, respectively.

The principal convergent of an irrational number gives *the best approximants* in the following sense.

Proposition 2.1.2 ([26, Proposition 3.3])**.** Let $\theta = [a_0; a_1, a_2, \dots] \in \mathbb{R} \setminus \mathbb{Q}$. For any $n > 1$ and p, q with $0 < q \leq q_n$, if $\frac{p}{q} \neq \frac{p_n}{q_n}$, then

$$|q_n\theta - p_n| < |q\theta - p|.$$

Moreover, p_{n+1}, q_{n+1} are the solution with minimal $q \geq q_n$ such that $|q\theta - p| < |q_n\theta - p_n|$. See [19, 26].

2.1.2 Gauss map

In the section, we introduce a dynamical system related to the regular continued fraction.

We define a map $T : [0,1] \setminus \mathbb{Q} \to [0,1] \setminus \mathbb{Q}$ by

$$T(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. The map T is called *Gauss map or continued fraction map*. The Gauss map is piecewise invertible and has infinitely many branches as Figure 2.1.

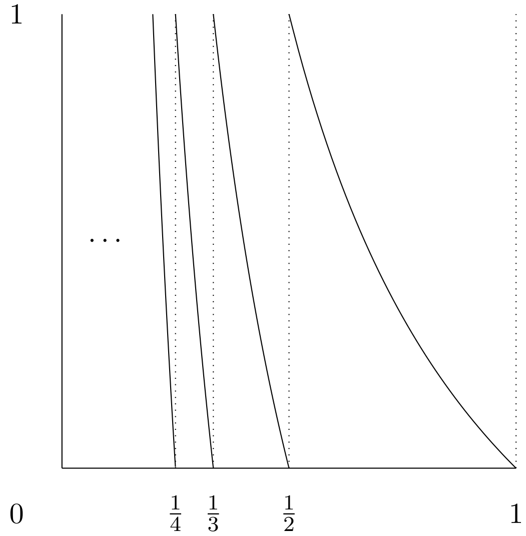


Figure 2.1 Gauss map

By definition, the Gauss map is the left shift map of continued fraction expansion:

$$T([0; a_1, a_2, \dots]) = [0; a_2, a_3, \dots]. \tag{2.4}$$

For $\theta \in (0,1) \setminus \mathbb{Q}$, the Gauss map gives the n-th partial quotient $a_n(\theta)$ as $a_n(\theta) = \lfloor (T^{n-1}(\theta))^{-1} \rfloor$ for $n \in \mathbb{N}$. Conversely, for a sequence $\{a_n(\theta)\}$ defined by $a_n(\theta) = \lfloor (T^{n-1}(\theta))^{-1} \rfloor$ for $n \in \mathbb{N}$,

$$\theta = [0; a_1(\theta), a_2(\theta), \dots].$$

Using (2.1) and (2.4), we obtain

$$\theta = \frac{p_n + p_{n-1}T^n(\theta)}{q_n + q_{n-1}T^n(\theta)} \tag{2.5}$$

for $\theta = [0; a_1, a_2, \dots] \in (0,1) \setminus \mathbb{Q}$.

Gauss showed that T preserves the probability measure defined by

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} dx.$$

Since

$$T^{-1}([0,s]) = \bigsqcup_{n=1}^{\infty} \left[\frac{1}{s+n}, \frac{1}{n} \right],$$

we obtain $\mu(T^{-1}([0,s])) = \mu([0,s])$ as follows:

$$\begin{aligned} &\frac{1}{\log 2} \sum_{n=1}^{\infty} \int_{\frac{1}{s+n}}^{\frac{1}{n}} \frac{1}{1+x} dx = \frac{1}{\log 2} \sum_{n=1}^{\infty} \log \frac{n+1}{n} \cdot \frac{s+n}{s+n+1} \\ =&\frac{1}{\log 2} \sum_{n=1}^{\infty} \log \left(\frac{1+\frac{s}{n}}{1+\frac{s}{n+1}} \right) = \frac{1}{\log 2} \sum_{n=1}^{\infty} \int_{\frac{s}{n+1}}^{\frac{s}{n}} \frac{1}{1+x} dx = \frac{1}{\log 2} \int_0^s \frac{1}{1+x} dx. \end{aligned}$$

2.2 The Markoff and Lagrange spectra

From Theorem 2.0.3, we can define the best bound of each irrational number for Corollary 2.0.2.

Definition 2.2.1. For any $\theta \in \mathbb{R} \setminus \mathbb{Q}$, we define *the Lagrange value* $L(\theta)$ by

$$L(\theta) := \sup \left\{ c \geq 1 : |q\theta - p| \leq \frac{1}{cq} \text{ for infinitely many } p \in \mathbb{Z}, q \in \mathbb{N} \right\}.$$

The Lagrange spectrum $\mathscr{L}$ is defined to be

$$\mathscr{L} = \{ L(\theta) : \theta \in \mathbb{R} \setminus \mathbb{Q} \}.$$

The following theorem, called Perron's formula, tells us that for each irrational number, its Lagrange value is obtained by the regular continued fraction of the number [53].

Theorem 2.2.2 *For any* $\theta = [a_1; a_2, a_3, \dots] \in \mathbb{R} \setminus \mathbb{Q}$,

$$L(\theta) = \limsup_{n\in\mathbb{N}} \left([0; a_n, a_{n-1}, \dots, a_1] + [a_{n+1}; a_{n+2}, \dots]\right).$$

Proof. Using (2.2), (2.5), and Proposition 2.1.2, we have

$$\begin{aligned}
L(\alpha) &= \limsup_{p\in\mathbb{Z}, q\in\mathbb{N}} |q(q\alpha - p)|^{-1} \\
&= \limsup_{n\in\mathbb{N}} |q_n(q_n\alpha - p_n)|^{-1} \\
&= \limsup_{n\in\mathbb{N}} \left(q_n^2 \left| \frac{p_n + p_{n-1}T^n(\alpha)}{q_n + q_{n-1}T^n(\alpha)} - \frac{p_n}{q_n} \right| \right)^{-1} \\
&= \limsup_{n\in\mathbb{N}} \left(q_n^2 \left| \frac{T^n(\alpha)}{q_n(q_n + q_{n-1}T^n(\alpha))} \right| \right)^{-1} \\
&= \limsup_{n\in\mathbb{N}} \left| (T^n(\alpha))^{-1} + \frac{q_{n-1}}{q_n} \right| \\
&= \limsup_{n\in\mathbb{N}} \left([a_{n+1}; a_{n+2}, \dots] + [0; a_n, \dots, a_1]\right).
\end{aligned}$$

□

Let us define the Markoff value and the Markoff spectrum.

Definition 2.2.3. For an indefinite binary quadratic form

$$f(x, y) = ax^2 + bxy + cy^2$$

with $a, b, c \in \mathbb{R}$ and $\delta(f) = b^2 - 4ac > 0$, we define *the Markoff number* $M(f)$ by

$$M(f) := \sup \left\{ \frac{\sqrt{\delta(f)}}{|f(x, y)|} : x, y \in \mathbb{Z}^2 \setminus \{(0, 0)\} \right\}.$$

The Markoff spectrum $\mathscr{M}$ is

$$\mathscr{M} = \left\{ M(f) : \begin{array}{c} f(x, y) = ax^2 + bxy + cy^2 \text{ is indefinite with } a, b, c \in \mathbb{R}, \\ \delta(f) > 0 \end{array} \right\}.$$

Let $f(x, y) = ax^2 + bxy + cy^2$ be an indefinite binary quadratic form and $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Compare

$$\limsup_{p\in\mathbb{Z}, q\in\mathbb{N}} \left(q^2 \left| \theta - \frac{p}{q} \right| \right)^{-1} \quad \text{and} \quad \sup_{(x,y)\in\mathbb{Z}^2 \setminus \{(0,0)\}} \frac{\sqrt{\delta(f)}}{|f(x, y)|}.$$

If $a = 0, b = 1, c = -\theta$, then

$$\frac{\sqrt{\delta(f)}}{|f(x,y)|} = \frac{1}{|xy - \theta y^2|} = \left(y^2 \left|\frac{x}{y} - \theta\right|\right)^{-1}.$$

Remark 2.2.4.

$$\mathscr{L} = \left\{ \limsup_{(x,y)\in\mathbb{Z}^2\setminus\mathbb{Z}\times\{0\}} \frac{\sqrt{\delta(f)}}{|f(x,y)|} : f(x,y) = xy - \theta y^2 \text{ with } \theta \in \mathbb{R}\setminus\mathbb{Q} \right\}.$$

Let us write

$$f(x,y) = ax^2 + bxy + cy^2 = (\alpha x - \beta y)(\gamma x - \delta y)$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Then,

$$\delta(f) = b^2 - 4ac = (\beta\gamma - \alpha\delta)^2 \text{ and } \frac{\sqrt{\delta(f)}}{|f(1,0)|} = \left|\frac{\beta}{\alpha} - \frac{\delta}{\gamma}\right|.$$

For coprime integers x, y, choose $M \in \mathrm{PSL}(2,\mathbb{Z})$ satisfying $M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Let

$$M\begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \tilde{\beta} \\ \tilde{\alpha} \end{pmatrix}, \quad M\begin{pmatrix} \delta \\ \gamma \end{pmatrix} = \begin{pmatrix} \tilde{\delta} \\ \tilde{\gamma} \end{pmatrix}.$$

We have

$$\begin{aligned}\frac{\sqrt{\delta(f)}}{|f(x,y)|} &= \left| \frac{\det\begin{pmatrix} \beta & \delta \\ \alpha & \gamma \end{pmatrix}}{\det\begin{pmatrix} x & \beta \\ y & \alpha \end{pmatrix} \cdot \det\begin{pmatrix} x & \delta \\ y & \gamma \end{pmatrix}} \right| = \left| \frac{\det\begin{pmatrix} \tilde{\beta} & \tilde{\delta} \\ \tilde{\alpha} & \tilde{\gamma} \end{pmatrix}}{\det\begin{pmatrix} 1 & \tilde{\beta} \\ 0 & \tilde{\alpha} \end{pmatrix} \cdot \det\begin{pmatrix} 1 & \tilde{\delta} \\ 0 & \tilde{\gamma} \end{pmatrix}} \right| \\ &= \left|\frac{\tilde{\beta}}{\tilde{\alpha}} - \frac{\tilde{\delta}}{\tilde{\gamma}}\right| = \left|M\cdot\frac{\beta}{\alpha} - M\cdot\frac{\delta}{\gamma}\right|.\end{aligned}$$

Hence,

$$\sup_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{\sqrt{\delta(f)}}{|f(x,y)|} = \sup_{M\in\mathrm{PSL}(2,\mathbb{Z})} \left|M\cdot\frac{\beta}{\alpha} - M\cdot\frac{\delta}{\gamma}\right|. \tag{2.6}$$

Figure 2.2 The discrete set, gaps, and a ray in $\mathscr{M}$ and $\mathscr{L}$

Note that $\dfrac{\beta}{\alpha}$ and $\dfrac{\delta}{\gamma}$ are roots of $at^2 + bt + c = (\alpha t - \beta)(\gamma t - \delta) = 0$. Therefore,

$$\begin{aligned}\mathscr{M} &= \left\{M(f) : f = ax^2 + bxy + cy^2 \text{ is indefinite with } a, b, c \in \mathbb{R} \text{ and } \delta(f) > 0\right\} \\ &= \left\{\sup_{M \in \mathrm{PSL}(2,\mathbb{Z})} |M \cdot x - M \cdot y| : x, y \in \mathbb{R}\right\}.\end{aligned}$$

Remark 2.2.5.

$$\begin{aligned}\mathscr{L} &= \left\{\limsup_{(x,y) \in \mathbb{Z}^2 \setminus \mathbb{Z} \times \{0\}} \frac{\sqrt{\delta(f)}}{|f(x,y)|} : f(x,y) = xy - \theta y^2 \text{ with } \theta \in \mathbb{R} \setminus \mathbb{Q}\right\} \\ &= \left\{\limsup_{M \in \mathrm{PSL}(2,\mathbb{Z}) \setminus S} |M \cdot 0 - M \cdot \theta^{-1}| : \theta \in \mathbb{R} \setminus \mathbb{Q}\right\}, \text{ where } S = \left\{\begin{pmatrix} n & -1 \\ 1 & 0 \end{pmatrix} : n \in \mathbb{Z}\right\}.\end{aligned}$$

Hence, the Markoff and Lagrange spectrum can be defined in terms of bi-infinite or infinite positive integer sequences.

Proposition 2.2.6. The Markoff spectrum $\mathscr{M}$ is the set of

$$M(A) := \sup_{n \in \mathbb{Z}} \left([0; a_{n-1}, a_{n-2}, \dots] + [a_n; a_{n+1}, \dots]\right)$$

as $A := \dots, a_{n-1}, a_n, a_{n+1}, \dots$ runs through all of bi-infinite sequences.

The Lagrange spectrum $\mathscr{L}$ is the set of

$$L(B) := \limsup_{n \in \mathbb{N}} \left([0; a_{n-1}, a_{n-2}, \dots, a_1] + [a_n; a_{n+1}, \dots]\right)$$

as $B := a_1, a_2, \dots$ runs through all of infinite sequences.

Let us recall some results on Markoff and Lagrange spectrum as Figure 2.2. Markoff showed that both $\mathscr{M}$ and $\mathscr{L}$ below 3 are discrete sets [46].

Theorem 2.2.7 *The Markoff (or Lagrange) spectrum below 3 consists of the number* $\sqrt{9m^2-4}/m$*, when* m *is a positive integer such that*

$$m^2 + {m_1}^2 + {m_2}^2 = 3mm_1m_2, \quad m \geq m_1, m_2$$

for some $m_1, m_2 \in \mathbb{N}$*. Given such a triple* m, m_1, m_2*, define* u *to be the least positive residue of* $\pm\, m_1/m_2 \pmod{m}$ *and define* v *by*

$$u^2 + 1 = vm.$$

If we define the quadratic form $f_m(x, y)$ *by*

$$f_m(x, y) = mx^2 + (3m - 2u)xy + (v - 3u)y^2,$$

then $f_m(x, 1) = 0$ *has a root* α *such that*

$$L(\alpha) = \sqrt{9m^2-4}/m.$$

Tornheim showed $\mathscr{L} \subset \mathscr{M}$ [65]. Perron proved that there exist gaps in $\mathscr{M}$ [53].

Theorem 2.2.8 *The intervals*

$$(\sqrt{12}, \sqrt{13}) \text{ and } (\sqrt{13}, (9\sqrt{3} + 65)/22)$$

are maximal gaps in $\mathscr{M}$*.*

Hall showed the existence of a ray in $\mathscr{L}$ [32].

Theorem 2.2.9 *Any real number can be written in the form*

$$a + [0; b_1, b_2, \ldots] + [0; c_1, c_2, \ldots],$$

where a *is an integer and the partial quotients* b_i *and* c_i *do not exceed 4 for all* $i \in \mathbb{N}$*.*

Chapter 3

The Markoff and Lagrange spectra associated with the Hecke group

We extend the classical Lagrange and Markoff spectra into the case when the quotient group is not the modular group $\mathbf{\Gamma}$. Let $\mathbf{G}$ be a subgroup of $\mathrm{SL}_2(\mathbb{R})$. Then we define the Markoff spectrum on group $\mathbf{G}$ as

$$\mathscr{M}(\mathbf{G}) := \left\{ \frac{\sqrt{\delta(f)}}{\inf_{M\in\mathbf{G}} |f(M)|} \;\middle|\; \delta(f) > 0 \right\}, \qquad f(M) := f\left(M \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right).$$

Then, by applying (2.6) to $\mathbf{G}$, we deduce that

$$\sup_{M\in\mathbf{G}} \frac{\sqrt{\delta(f)}}{|f(M)|} = \sup_{M\in\mathbf{G}} \left| M^{-1}\cdot\frac{u_1}{u_2} - M^{-1}\cdot\frac{v_1}{v_2} \right|. \tag{3.1}$$

The Markoff spectrum $\mathscr{M}(\mathbf{G})$ is the set of the maximum heights of geodesics in $\mathbb{H}/\mathbf{G}$. We define the Lagrange spectrum on group $\mathbf{G}$ as

$$\mathscr{L}(\mathbf{G}) := \left\{ \limsup_{M\in\mathbf{G}} \frac{\sqrt{\delta(f)}}{|f(M)|} \;\middle|\; \delta(f) > 0 \right\}.$$

The Hecke group $\mathbf{H}_q$ is the group generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}$, where $\lambda_q = 2\cos\frac{\pi}{q}$ and $q \geq 3$ is an integer. When $q = 3$, then $\lambda_3 = 1$ and the Hecke group $\mathbf{H}_3$ is the modular group $\mathbf{\Gamma} = \mathrm{SL}_2(\mathbb{Z})$. Thus,

$$\mathscr{M} = \mathscr{M}(\mathbf{H}_3), \qquad \mathscr{L} = \mathscr{L}(\mathbf{H}_3).$$

The minimum of Lagrange spectrum, which is called Hurwitz's constant, for the Hecke group $\mathbf{H}_q$ was studied in [31,42]. In particular, if q is even, then the minimum of the Lagrange spectrum $\mathscr{L}(\mathbf{H}_q)$ is always equal to 2. In this chapter, we deal with the Lagrange and Markoff spectra on the Hecke group $\mathbf{H}_4$ and $\mathbf{H}_6$.

3.1 The Markoff and Lagrange spectra on $\mathbf{H}_4$

3.1.1 The Markoff and Lagrange spectra of the index 2 sublattice

The Markoff spectrum of 2-minimal forms

Let Λ be an index 2 sublattice of $\mathbb{Z}^2$. For an indefinite quadratic form f, we set

$$m_\Lambda(f) := \min\left\{ \inf_{(x,y)\in\Lambda\setminus\{(0,0)\}} \frac{|f(x,y)|}{2}, \inf_{(x,y)\in\mathbb{Z}^2\setminus\Lambda} |f(x,y)| \right\}$$

and

$$\mathscr{M}_2 := \left\{ \frac{\sqrt{\delta(f)}}{m_\Lambda(f)} : \delta(f) > 0 \right\}.$$

Let $\mathrm{Ht}_\Lambda : \mathbb{Q} \to \mathbb{R}$ be the height function defined as

$$\mathrm{Ht}_\Lambda\left(\frac{p}{q}\right) := \begin{cases} q^2/2, & \text{if } (p,q) \in \Lambda, \\ q^2, & \text{if } (p,q) \notin \Lambda. \end{cases}$$

For an irrational α, let

$$L_\Lambda(\alpha) := \limsup_{p/q\in\mathbb{Q}} \left(\mathrm{Ht}_\Lambda\left(\frac{p}{q}\right) \left|\alpha - \frac{p}{q}\right| \right)^{-1}$$

and

$$\mathscr{L}_2 := \{L_\Lambda(\alpha) \in \mathbb{R} \mid \alpha \in \mathbb{R}\setminus\mathbb{Q}\}.$$

We claim that $\mathscr{M}_2$ and $\mathscr{L}_2$ do not depend on the choice of the index 2 sublattice Λ.

There are three index 2 sublattices of $\mathbb{Z}^2$. Let

$$\Lambda_0 = \{(n,m) \in \mathbb{Z}^2 \mid n+m \equiv 0 \pmod 2\}.$$

Then other sublattices are represented as

$$\Lambda_1 = 2\mathbb{Z} \times \mathbb{Z} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Lambda_0, \qquad \Lambda_2 = \mathbb{Z} \times 2\mathbb{Z} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \Lambda_0.$$

For an indefinite quadratic form f, let

$$f_1(x,y) = f(x+y,y), \qquad f_2(x,y) = f(x,x+y).$$

It is straightforward to check that $\delta(f) = \delta(f_1) = \delta(f_2)$ and

$$m_{\Lambda_0}(f_1) = m_{\Lambda_1}(f), \qquad m_{\Lambda_0}(f_2) = m_{\Lambda_2}(f).$$

Since the maps $f \mapsto f_1$ and $f \mapsto f_2$ are 1 to 1 correspondence from the set of indefinite quadratic forms to itself, the Markoff spectrum $\mathscr{M}_2$ does not depend on the choice of the sublattice Λ. We will show that the Lagrange spectrum $\mathscr{L}_2$ does not depend on the sublattice Λ.

Let

$$\varphi_1(x) := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot x = x+1, \qquad \varphi_2(x) := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot x = \frac{x}{x+1}.$$

Then

$$\mathrm{Ht}_{\Lambda_1}\left(\varphi_1\left(\frac{p}{q}\right)\right) = \mathrm{Ht}_{\Lambda_0}\left(\frac{p}{q}\right), \qquad \mathrm{Ht}_{\Lambda_2}\left(\varphi_2\left(\frac{p}{q}\right)\right) = \frac{(p+q)^2}{q^2}\,\mathrm{Ht}_{\Lambda_0}\left(\frac{p}{q}\right).$$

Since

$$\frac{d\varphi_1(x)}{dx} = 1, \qquad \frac{d\varphi_2(x)}{dx} = \frac{1}{(x+1)^2},$$

we have

$$L_{\Lambda_0}(\alpha) = L_{\Lambda_1}(\alpha) = L_{\Lambda_2}(\alpha).$$

The Markoff spectrum of the index 2 sublattices

We show $\mathscr{M}_2$ coincides with the Markoff spectrum on sublattice of index 2 studied by Vulakh.

Let Λ be an index 2 sublattice of $\mathbb{Z}^2$ and $\mathcal{F}_\Lambda$ be the set of real, indefinite quadratic forms

$$f(x,y) = ax^2 + bxy + cy^2, \quad a,b,c \in \mathbb{R}, \quad \delta(f) = b^2 - 4ac > 0$$

satisfying the condition

$$|f(x,y)| \geq 2m(f) \qquad \text{for } (x,y) \in \Lambda \setminus \{(0,0)\}.$$

We set

$$\mathscr{M}^{(2)} = \left\{ \frac{\sqrt{\delta(f)}}{m(f)} \,\middle|\, f \in \mathcal{F}_\Lambda \right\}.$$

Note that

$$\mathscr{M}_2 = \left\{ \frac{\sqrt{\delta(f)}}{m_\Lambda(f)} \,\middle|\, \delta(f) > 0 \right\}.$$

We will call a vector $\mathbf{v} \in \Lambda$ is primitive if there is no $\mathbf{w} \in \Lambda$ such that $\mathbf{v} = k\mathbf{w}$ for some integer $k \geq 2$. By a direct calculation we have the following lemma:

Lemma 3.1.1. The map

$$\varphi : (x,y) \mapsto \frac{1}{2}(x+y, x-y)$$

is a bijection from the set of primitive vectors of Λ_0 to the set of primitive vectors in $\mathbb{Z}^2 \setminus \Lambda_0$,

Proof. Any common factor of $x+y$, $x-y$ is 2 or a factor of x and y. Therefore, if x, y are coprime, then $x+y$, $x-y$ have no common factor except for 2.

If (x,y) is a primitive vector in Λ_0, then x, y are both odd and $\varphi(x,y) = \frac{1}{2}(x+y, x-y)$ is a primitive vector in $\mathbb{Z}^2$. Since $\frac{x+y}{2} + \frac{x-y}{2} = x \equiv 1 \pmod 2$, $\varphi(x,y)$ does not belong to Λ_0.

If (x,y) is a primitive vector in $\mathbb{Z}^2 \setminus \Lambda_0$, then one of x, y is odd and the other is even. Therefore, $x+y$, $x-y$ are both odd, thus $x+y$, $x-y$ are coprime. Hence, $\varphi^{-1}(x,y) = (x+y, x-y)$ is a primitive vector in Λ_0. □

Theorem 3.1.2 *We have*

$$\mathscr{M}^{(2)} = \mathscr{M}_2.$$

Proof. Let

$$m^p_\Lambda(f) = \inf_{\substack{(x,y)\in\Lambda \\ x,y \text{ coprime}}} |f(x,y)|, \qquad m^c_\Lambda(f) = \inf_{\substack{(x,y)\in\mathbb{Z}^2\setminus\Lambda \\ x,y \text{ coprime}}} |f(x,y)|.$$

Then we have

$$\mathcal{F}_\Lambda = \{f(x,y) \mid m^p_\Lambda(f) \geq 2m^c_\Lambda(f)\}$$

and

$$m_\Lambda(f) = \min\left\{\frac{m^p_\Lambda(f)}{2}, m^c_\Lambda(f)\right\}.$$

For an indefinite quadratic form f we set

$$\tilde{f}(x,y) = f\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right).$$

Then we have by Lemma 3.1.1

$$m^p_\Lambda(\tilde{f}) = \inf_{\substack{(x,y)\in\Lambda \\ x,y \text{ coprime}}} f\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right) = \inf_{\substack{(x',y')\in\mathbb{Z}^2\setminus\Lambda \\ x',y' \text{ coprime}}} f\left(\sqrt{2}(x',y')\right) = 2m^c_\Lambda(f),$$

$$m^c_\Lambda(\tilde{f}) = \inf_{\substack{(x,y)\in\mathbb{Z}^2\setminus\Lambda \\ x,y \text{ coprime}}} f\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right) = \inf_{\substack{(x',y')\in\Lambda \\ x',y' \text{ coprime}}} f\left(\frac{1}{\sqrt{2}}(x',y')\right) = \frac{m^p_\Lambda(f)}{2}.$$

Therefore, if $f \notin \mathcal{F}_\Lambda$, then $m^p_\Lambda(f) < 2m^c_\Lambda(f)$, thus $2m^c_\Lambda(\tilde{f}) < m^p_\Lambda(\tilde{f})$ and $\tilde{f} \in \mathcal{F}_\Lambda$.

Since

$$m_\Lambda(\tilde{f}) = \min\left\{\frac{m^p_\Lambda(\tilde{f})}{2}, m^c_\Lambda(\tilde{f})\right\} = \min\left\{m^c_\Lambda(f), \frac{m^p_\Lambda(f)}{2}\right\} = m_\Lambda(f)$$

and $\delta(\tilde{f}) = \delta(f)$, we conclude the theorem. □

The Markoff spectrum on the Hecke group $\mathbf{H}_4$

In [59, page 364], Schmidt considered the Markoff spectrum in the group

$$\mathbf{\Delta}_2 = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad - bc = 1,\ b \equiv 0 \pmod 2,\ a, c, d \in \mathbb{Z}\right\}$$

$$\cup \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad - bc = 2,\; a \equiv b \equiv d \equiv 0 \pmod 2,\; c \in \mathbb{Z} \right\}.$$

For any primitive integral vector $(x, y) \in \mathbb{Z}^2$, there exists

$$M = \begin{pmatrix} x & * \\ y & * \end{pmatrix} \in \mathbf{\Delta}_2 \text{ if } x \text{ is odd}, \qquad M = \frac{1}{\sqrt{2}} \begin{pmatrix} x & * \\ y & * \end{pmatrix} \in \mathbf{\Delta}_2 \text{ if } x \text{ is even}.$$

Therefore, we have for $\Lambda = 2\mathbb{Z} \times \mathbb{Z}$

$$\inf_{M \in \mathbf{\Delta}_2} |f(M)| = m_\Lambda(f)$$

and

$$\mathscr{M}(\mathbf{\Delta}_2) = \mathscr{M}_2.$$

The subgroup $\mathbf{\Delta}_2$ of $\mathrm{SL}_2(\mathbb{R})$ is conjugate to $\mathbf{H}_4$, i.e.,

$$\mathbf{\Delta}_2 = U \mathbf{H}_4 U^{-1} \qquad \text{where } U = \begin{pmatrix} \sqrt[4]{2} & 0 \\ 0 & 1/\sqrt[4]{2} \end{pmatrix}.$$

Indeed, we check that

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = U \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix} U^{-1}, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix} = U \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} U^{-1}.$$

They are generators of $\mathbf{\Delta}_2$. The fundamental domain of $\mathbf{\Delta}_2$ is given in Figure 3.1. For an $M \in \mathbf{\Delta}_2$, there exists $H \in \mathbf{H}_4$ such that $M = UHU^{-1}$. Therefore for any $\xi, \eta \in \mathbb{R}$,

$$\sup_{M \in \mathbf{\Delta}_2} |M \cdot \xi - M \cdot \eta| = \sup_{H \in \mathbf{H}_4} \left| UHU^{-1} \cdot \xi - UHU^{-1} \cdot \eta \right| = \sqrt{2} \sup_{H \in \mathbf{H}_4} \left| H \cdot \frac{\xi}{\sqrt{2}} - H \cdot \frac{\eta}{\sqrt{2}} \right|.$$

Hence, by (3.1), we have

$$\mathscr{M}(\mathbf{\Delta}_2) = \mathscr{M}_2 = \sqrt{2} \mathscr{M}(\mathbf{H}_4).$$

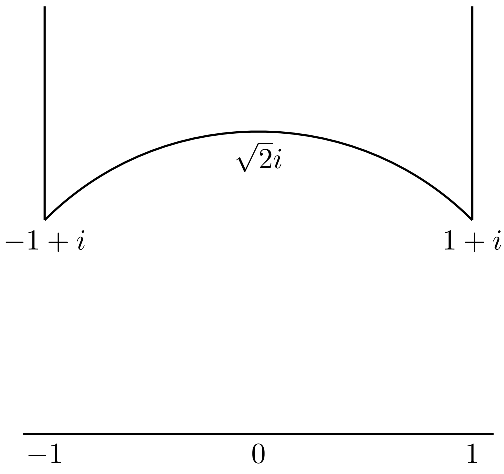


Figure 3.1 The fundamental domain of $\mathbf{\Delta}_2$.

The Markoff and Lagrange spectra of the unit circle

The Lagrange spectrum $\mathscr{L}_2$ coincides with the Lagrange spectrum for the intrinsic Diophantine approximation on the unit circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

A rational point $\mathbf{z} = (\frac{a}{c}, \frac{b}{c}) \in S^1$ is denoted by a primitive Pythagorean triple (a, b, c) satisfying $a^2 + b^2 = c^2$ where $a, b \in \mathbb{Z}$ and $c \in \mathbb{N}$. Define the height function for $\mathbf{z} = (\frac{a}{c}, \frac{b}{c})$ in S^1 as $\mathrm{Ht}_{S^1}(\mathbf{z}) = c$. We define the Lagrange number for a point $(\alpha, \beta) \in S^1$

$$L_{S^1}(\alpha, \beta) = \limsup_{\mathbf{z} \in S^1 \cap \mathbb{Q}^2} \frac{1}{\mathrm{Ht}_{S^1}(\mathbf{z}) \cdot \|(\alpha, \beta) - \mathbf{z}\|}$$

and the Lagrange spectrum as

$$\mathscr{L}(S^1) = \left\{ L_{S^1}(\alpha, \beta) \,|\, (\alpha, \beta) \in S^1 \setminus \mathbb{Q}^2 \right\}.$$

Let $\boldsymbol{\phi} : \mathbb{R} \to S^1 \setminus \{(0, 1)\}$ be the inverse of the stereographic projection given by

$$\boldsymbol{\phi}(t) = \left(\frac{2t}{t^2 + 1}, \frac{t^2 - 1}{t^2 + 1} \right).$$

Then, $\boldsymbol{\phi}$ gives a one-to-one correspondence between the rational number $\frac{p}{q}$ and the Pythagorean triple $\left(2pq, p^2-q^2, p^2+q^2\right)$ for $(p,q) \notin \Lambda$ or $\left(pq, \frac{p^2-q^2}{2}, \frac{p^2+q^2}{2}\right)$ for $(p,q) \in \Lambda$, where

$$\Lambda = \{(n,m) \in \mathbb{Z}^2 \,|\, n+m \equiv 0 \pmod 2\}.$$

Hence, we have

$$\begin{aligned} L_\Lambda(t) &= \limsup_{p/q \in \mathbb{Q}} \left(\mathrm{Ht}_\Lambda\left(\frac{p}{q}\right) \left| t - \frac{p}{q} \right| \right)^{-1} \\ &= 2 \limsup_{(a,b,c)} \left(\mathrm{Ht}_{S^1}\left(\frac{a}{c}, \frac{b}{c}\right) \left\| \boldsymbol{\phi}(t) - \left(\frac{a}{c}, \frac{b}{c}\right) \right\| \right)^{-1} = 2L_{S^1}(\boldsymbol{\phi}(t)). \end{aligned}$$

Therefore, we have

$$\mathscr{L}_2 = 2\mathscr{L}(S^1).$$

See [40] and [21] for the detail.

3.1.2 The Markoff spectrum and the Romik expansion

The Romik's dynamical system on S^1 and digit expansions

Let

$$\mathcal{Q} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1 \text{ and } \ x, y \geq 0\}$$

be the quarter circle of S^1. Recall from [55] that the Romik's dynamical system $(\mathcal{Q}, T)$ is defined by

$$T(x,y) = \left(\frac{|2-x-2y|}{3-2x-2y}, \frac{|2-2x-y|}{3-2x-2y} \right) \tag{3.2}$$

for $(x,y) \in \mathcal{Q}$.

To each $P = (x,y) \in \mathcal{Q}$, we assign a *Romik digit* $d(P)$ to be

$$d(P) = \begin{cases} 1 & \text{if } \frac{4}{5} \leq x \leq 1, \\ 2 & \text{if } \frac{3}{5} \leq x \leq \frac{4}{5}, \\ 3 & \text{if } 0 \leq x \leq \frac{3}{5}. \end{cases} \tag{3.3}$$

Then the *j-th Romik digit of* P is defined to be

$$d_j = d(T^{j-1}(P)) \text{ for } j = 1, 2, \dots.$$

The sequence $\{d_j\}_{j=1}^{\infty}$ will be called *the Romik digit expansion* of P and we write

$$P = (x, y) = [d_1, d_2, \dots]_{\mathcal{Q}}. \tag{3.4}$$

The map T shifts digits to the left, so that

$$T^k(P) = (\overbrace{T \circ \cdots \circ T}^{k \text{ times}})(P) = [d_{k+1}, d_{k+2}, \dots]_{\mathcal{Q}}.$$

For instance,

$$(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) = [2, 2, \dots]_{\mathcal{Q}} \text{ and } (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) = [3, 1, 3, 1, \dots]_{\mathcal{Q}}.$$

We denote the infinite successions of 1's and 3's by 1^∞ and 3^∞ respectively. Since the points $(1, 0)$ and $(0, 1)$ are fixed by T, we have

$$(1, 0) = [1, 1, 1, \dots]_{\mathcal{Q}} = [1^\infty]_{\mathcal{Q}} \text{ and } (0, 1) = [3, 3, 3, \dots]_{\mathcal{Q}} = [3^\infty]_{\mathcal{Q}}.$$

All irrational points on $\mathcal{Q}$ have a unique Romik digit expansions of the forms $[d_1, d_2, \dots]_{\mathcal{Q}}$. In what follows, we identify $P \in \mathcal{Q}$ with an element in $\{1, 2, 3\}^{\mathbb{N}}$ using Romik digit expansion of P. By the *infinite Romik sequence*, we mean an element of $\{1, 2, 3\}^{\mathbb{N}}$.

The map T originates from an old theorem on trees of *primitive Pythagorean triples*, that is, triples (a, b, c) of (pairwise) coprime positive integers a, b, c with $a^2 + b^2 = c^2$, which is often attributed to Berggren [10] and Barning [9]. We define U_1 and U_3 as the reflection by the x-axis and the y-axis respectively on S^1. Let H be the reflection by the line $x + y = 1$ and $U_2 = U_1 \circ U_3 = U_3 \circ U_1$. Then we have

$$U_1(x, y) = (x, -y), \qquad U_2(x, y) = (-x, -y), \qquad U_3(x, y) = (-x, y),$$

$$H(x, y) = \left(\frac{2 - x - 2y}{3 - 2x - 2y}, \frac{2 - 2x - y}{3 - 2x - 2y} \right).$$

Note that the Romik map T is defined as acting H first and applying U_d in order to $U_d(H(P)) \in \mathcal{Q}$, i.e.,

$$T(P) = (U_d \circ H)(P) \quad \text{for} \quad d_1(P) = d.$$

For an infinite Romik sequence $P = [d_1, d_2, d_3, \dots]_{\mathcal{Q}} \in \mathcal{Q}$, we define

$$P^* = [\dots, d_3, d_2, d_1]_{\mathcal{Q}} := H(P) \in S^1 \setminus \mathcal{Q}.$$

In what follows, we identify $P^* \in S^1 \setminus \mathcal{Q}$ with an element in $\{1,2,3\}^{\mathbb{Z}_{\leq 0}}$ using the Romik digit expansion of P^*. We check the idempotent maps U_i act on S^1 as follows

$$U_d\left([\dots, d_2, d_1, d]_{\mathcal{Q}}\right) = [d_1, d_2, \dots]_{\mathcal{Q}}, \qquad U_1([\dots, d_2, d_1, 2]_{\mathcal{Q}}) = [\dots, d_2, d_1, 3]_{\mathcal{Q}}, \tag{3.5}$$
$$U_2([\dots, d_2, d_1, 3]_{\mathcal{Q}}) = [\dots, d_2, d_1, 1]_{\mathcal{Q}}, \qquad U_3([\dots, d_2, d_1, 1]_{\mathcal{Q}}) = [\dots, d_2, d_1, 2]_{\mathcal{Q}}. \tag{3.6}$$

Let $\vee : S^1 \to S^1$ be the reflection given by $(\alpha, \beta)^\vee = (\beta, \alpha)$. Then for a given infinite Romik sequence $P = [d_1, d_2, \dots]_{\mathcal{Q}}$, $P^\vee = [d_1^\vee, d_2^\vee, \dots]_{\mathcal{Q}}$ where

$$d^\vee = \begin{cases} 3 & \text{if } d = 1, \\ 2 & \text{if } d = 2, \\ 1 & \text{if } d = 3. \end{cases}$$

Stereographic projection to the extended real line

For $P = (\alpha, \beta) \in S^1$, we define a modified stereographic projection following [21]

$$[P] := \frac{1}{\sqrt{2}}\left(\frac{\alpha}{1-\beta} - 1\right). \tag{3.7}$$

For an infinite Romik sequence $P = [a_1, a_2, a_3, \dots]_{\mathcal{Q}} \in \mathcal{Q}$, we denote

$$[P] = [a_1, a_2, a_3, \dots] \in [0, \infty] \subset \mathbb{R} \cup \{\infty\} =: \hat{\mathbb{R}}.$$

For $P^* = [\dots, b_3, b_2, b_1]_{\mathcal{Q}} \in S^1 \setminus \mathcal{Q}$, we write

$$[\dots, b_3, b_2, b_1] = [P^*] = [H(P)] = -[P] = -[b_1, b_2, b_3, \dots].$$

The reflection $\vee : S^1 \to S^1$ given by $(\alpha, \beta)^\vee = (\beta, \alpha)$ induces a map $\vee : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$ given by

$$\left(\frac{\alpha + \beta - 1}{\sqrt{2}(1-\beta)}\right)^\vee = \frac{\alpha + \beta - 1}{\sqrt{2}(1-\alpha)} = \frac{\sqrt{2}(1-\beta)}{\alpha + \beta - 1}.$$

Therefore, we have

$$[P^\vee] := \frac{1}{[P]} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot [P].$$

Since $T(P) = (U_d \circ H)(P)$ for $d_1(P) = d$, we have

$$[d, P] = (H \circ U_d \circ U_d \circ H)([d, P]) = (H \circ U_d)([P]) \quad \text{for } d = 1, 2, 3.$$

Thus, we deduce that

$$[1, P] = \frac{[P]}{\sqrt{2}[P] + 1}, \quad [2, P] = \frac{[P] + \sqrt{2}}{\sqrt{2}[P] + 1}, \quad [3, P] = \sqrt{2} + [P] \tag{3.8}$$

for $P \in \{1, 2, 3\}^{\mathbb{N}}$. Let $N_d = HU_d$. Then

$$N_1 = \begin{pmatrix} 1 & 0 \\ \sqrt{2} & 1 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}.$$

Then, using (3.5) and (3.6) we deduce

$$\begin{aligned} N_d \cdot [d_1, d_2, \dots] &= [d, d_1, d_2, \dots] \\ N_d \cdot [\dots, d_2, d_1] &= \begin{cases} [\dots, d_3, d_2] & \text{if } d = d_1, \\ [c, d_2, d_3, \dots] & \text{if } d \neq d_1, \end{cases} \end{aligned} \tag{3.9}$$

where $c \in \{1, 2, 3\}$ is the digit of $c \neq d$ and $c \neq d_1$. In particular we check

$$[1, P] = N_1 \cdot [P], \quad [2, P] = N_2 \cdot [P], \quad [3, P] = N_3 \cdot [P]$$

and deduce that

$$0 \le [1, P] \le \frac{1}{\sqrt{2}}, \qquad \frac{1}{\sqrt{2}} \le [2, P] \le \sqrt{2}, \qquad \sqrt{2} \le [3, P].$$

See [21] for the detail. Some cylinder sets of the Romik expansion are given in Figure 3.2.

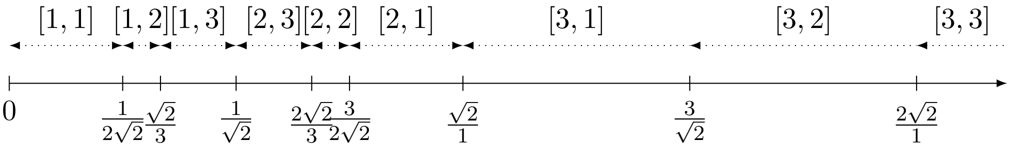


Figure 3.2 Cylinder sets on $\mathbb{R}$

By direct calculations using (3.9), we have the following lemma.

Lemma 3.1.3. Let $[P^*] = [\dots, a_2, a_1, a_0]$, $[Q] = [b_1, b_2, \dots]$ be two distinct points. Let $M = N_{d_1} \cdots N_{d_m}$. If $d_{m-j} = a_j$ for $0 \le j \le m-1$, then

$$M \cdot [P^*] = [\dots, a_{m+2}, a_{m+1}, a_m], \quad M \cdot [Q] = [a_{m-1}, \dots, a_0, b_1, b_2, \dots].$$

If there exists $0 \le k \le m-1$ such that $d_{m-j} = a_j$ for $0 \le j \le k-1$ and $d_{m-k} \ne a_k$, then

$$M \cdot [P^*] = [d_1, \dots, d_{m-k-1}, c, a_{k+1}, a_{k+2}, \dots], M \cdot [Q] = [d_1, \dots, d_{m-k}, a_{k-1}, \dots, a_0, b_1, b_2, \dots]$$

where $c \ne a_k$ and $c \ne d_{m-k}$.

The action of the Hecke group $\mathbf{H}_4$ and the Romik map

Let

$$H = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 & -\sqrt{2} \\ 0 & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\mathbf{G}_4$ be the group of 2×2 matrices generated by reflections H, U, J. We note that

$$HU = T = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}, \quad UH = T^{-1} = \begin{pmatrix} 1 & -\sqrt{2} \\ 0 & 1 \end{pmatrix}, \quad HJ = JH = S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the Hecke group $\mathbf{H}_4$ is generated by S and T. It is straightforward to check that $\mathbf{G}_4 = \mathbf{H}_4 \cup H\mathbf{H}_4$ and the Hecke group $\mathbf{H}_4$ is an index 2 subgroup of $\mathbf{G}_4$. The fundamental domain of $\mathbf{G}_4$ is given in Figure 3.3.

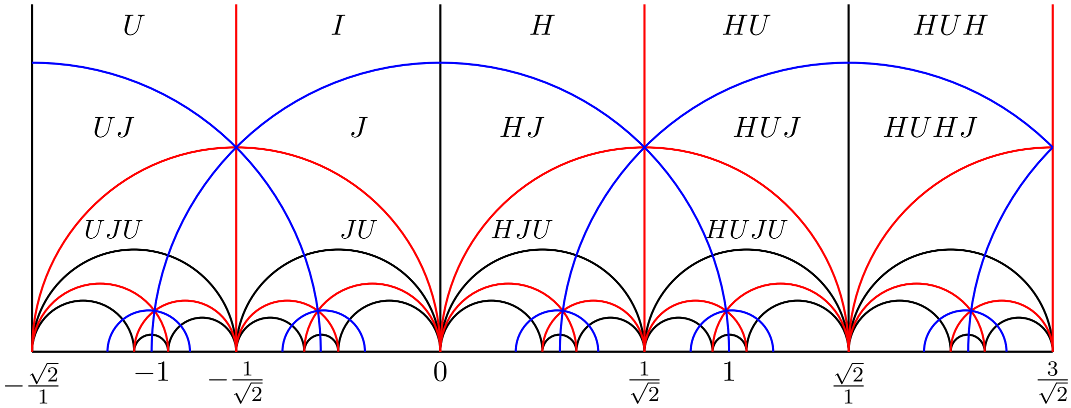


Figure 3.3 The fundamental domain of the group $\mathbf{G}_4$ on the upper half space.

Using the fact that $(JU)^4 = I$, we have $JUJU = UJUJ$, thus all elements in $\mathbf{G}_4$ generated by J and U are

$$I, \quad J, \quad U = HN_3, \quad JU = HN_1J, \quad UJ = HN_3J,$$
$$JUJ = HN_1, \quad UJU = HN_2J, \quad JUJU = UJUJ = HN_2.$$

Since

$$HJ = JH, \quad N_1J = JN_3, \quad N_2J = JN_2, \quad N_3J = JN_1,$$

we establish the following proposition:

Proposition 3.1.4. Any element in $\mathbf{G}_4$ is one of the following forms

$$I, \quad H, \quad N_{d_1} \cdots N_{d_m}, \quad HN_{d_1} \cdots N_{d_m}, \quad N_{d_1} \cdots N_{d_m}H, \quad HN_{d_1} \cdots N_{d_m}H,$$
$$J, \quad HJ, \quad N_{d_1} \cdots N_{d_m}J, \quad HN_{d_1} \cdots N_{d_m}J, \quad N_{d_1} \cdots N_{d_m}HJ, \quad HN_{d_1} \cdots N_{d_m}HJ.$$

We remark that the Romik expansion of a point P in S^1 is related with the even integer continued fraction ([60], [41]; see also [62]) when P is projected into the real line by the standard stereographic projection $(\alpha, \beta) \mapsto \frac{\alpha}{1-\beta}$ instead of the skewed projection in (3.7). Let $[d_1, d_2, \dots]$ be the Romik digit expansion of P and $(k_i)_{i \geq 0}$ be chosen as the subsequence of $d_{k_i} \neq 3$ with $k_0 = 0$. The even integer continued fraction expansion of

$$\frac{\alpha}{1-\beta} = 2a_0 + \cfrac{\varepsilon_1}{2a_1 + \cfrac{\varepsilon_2}{2a_2 + \ddots}} \qquad \text{with} \quad a_i \in \mathbb{N}, \quad \varepsilon_i \in \{1, -1\}$$

should satisfy

$$a_i = k_{i+1} - k_i, \quad \epsilon_i = \begin{cases} -1 & \text{if } d_{k_i} = 1, \\ +1 & \text{if } d_{k_i} = 2. \end{cases}$$

For the relation between Romik map and the even integer continued fraction, consult [35]. The expansion by the matrices N_1, N_2, N_3 in $\mathbf{H}_4$ enjoys more symmetry than the expansion by the Rosen continued fraction ([56], [63]; see also [48] for the dual map).

Expression of the Markoff number using the Romik sequence

For infinite Romik sequences $P = (a_n)_{n\geq 1}$ and $Q = (b_n)_{n\geq 1}$, we define a combined two-sided Romik sequence

$$P^*|Q := (c_n)_{n\in\mathbb{Z}}, \qquad c_n = \begin{cases} b_n, & \text{if } n \geq 1, \\ a_{-n+1}, & \text{if } n \geq 1, \end{cases}$$

which is an element of $\{1,2,3\}^{\mathbb{Z}}$. We give an equivalent relation $(a_n) \sim (b_n)$ in $\{1,2,3\}^{\mathbb{Z}}$ if and only if there exists some $k \in \mathbb{Z}$ such that $a_{n+k} = b_n$ for all $n \in \mathbb{Z}$. Then an equivalent class of $\{1,2,3\}^{\mathbb{Z}}$ under the equivalence relation is called a *doubly-infinite Romik sequence.* A *section* of a doubly-infinite Romik sequence is an element in the equivalent class. For a doubly-infinite Romik sequence T with a section $P^*|Q$, we define $T^\vee$ and T^* as the doubly-infinite Romik sequences with a section $(P^\vee)^*|Q^\vee$ and $Q^*|P$ respectively.

For distinct boundary points $\xi, \eta \in \partial\mathbb{H}$, we define a *reduced* two-sided Romik sequence $P^*|Q$ given by

$$\begin{cases} [P^*] = \xi = [\dots, a_2, a_1], \\ [Q] = \eta = [b_1, b_2, \dots], \end{cases} \qquad \text{for} \quad \begin{cases} \xi = [\dots, a_2, a_1], \\ \eta = [b_1, b_2, \dots], \end{cases}$$

$$\begin{cases} [P^*] = N_{a_k} \cdots N_{a_1} \cdot \xi = [\ldots, a_{k+2}, a_{k+1}], \\ [Q] = N_{a_k} \cdots N_{a_1} \cdot \eta = [c, b_{k+1}, b_{k+2}, \ldots], \end{cases} \quad \text{for} \quad \begin{cases} \xi = [\ldots, a_2, a_1], \\ \eta = [\ldots, b_2, b_1], \end{cases}$$

$$\begin{cases} [P^*] = N_{a_k} \cdots N_{a_1} H \cdot \xi = [\ldots, a_{k+2}, a_{k+1}], \\ [Q] = N_{a_k} \cdots N_{a_1} H \cdot \eta = [c, b_{k+1}, b_{k+2}, \ldots], \end{cases} \quad \text{for} \quad \begin{cases} \xi = [a_1, a_2, \ldots], \\ \eta = [b_1, b_2, \ldots], \end{cases}$$

where k is the largest integer satisfying that $a_j = b_j$ for $1 \le j \le k-1$, $a_k > b_k$ and c is the digit of $c \neq b_k$, $c \neq a_k$.

Proposition 3.1.5. Let $\xi, \eta \in \hat{\mathbb{R}}$ be two distinct points on the boundary of $\mathbb{H}$ and $P^*|Q$ be the reduced two-sided Romik sequence of ξ, η. Then we have

$$[Q] + [P] = [Q] - [P^*] \ge |\eta - \xi| \,.$$

Proof. First, we assume

$$\xi = [c_1, \ldots, c_{k-1}, a_k, a_{k+1}, \ldots], \;\; \eta = [c_1, \ldots, c_{k-1}, b_k, b_{k+1}, \ldots]$$

with $a_k > b_k, c \neq a_k, c \neq b_k$. By (3.8), we have

$$\big|[P] - [Q]\big| \ge \left| \frac{[P] - [Q]}{(\sqrt{2}[P] + 1)(\sqrt{2}[Q] + 1)} \right| = \big|[1, P] - [1, Q]\big| = \big|[2, Q] - [2, P]\big|$$

and

$$[P] - [Q] = [3, P] - [3, Q].$$

Therefore

$$|[b_k, b_{k+1}, \ldots] - [a_k, a_{k+1}, \ldots]| \ge |\eta - \xi|.$$

By (3.8) again, we have

$$\begin{aligned} [Q] + [1, P] &= [Q] + \frac{[P]}{\sqrt{2}[P] + 1} = [Q] + \sqrt{2} - \frac{[P] + \sqrt{2}}{\sqrt{2}[P] + 1} = [3, Q] - [2, P], \\ [Q] + [2, P] &= [Q] + \frac{[P] + \sqrt{2}}{\sqrt{2}[P] + 1} = [Q] + \sqrt{2} - \frac{[P]}{\sqrt{2}[P] + 1} = [3, Q] - [1, P], \\ [Q] + [3, P] &= [Q] - [P^*, 3] \ge \sqrt{2} \ge [2, Q] - [1, P]. \end{aligned}$$

Therefore, we have

$$[c, b_{k+1}, b_{k+2}, \ldots] + [a_{k+1}, a_{k+2}, \ldots] \ge |[b_k, b_{k+1}, \ldots] - [a_k, a_{k+1}, \ldots]| \ge |\eta - \xi|. \quad \square$$

A geodesic γ in $\mathbb{H}$ is determined by two end points ξ, η in $\partial\mathbb{H}$. Therefore, for each geodesic γ, we define the reduced two-sided Romik sequence $P^*|Q$ and also associate a doubly-infinite Romik sequence T with a section $P^*|Q$.

Proposition 3.1.6. Let $\gamma, \tilde{\gamma}$ be geodesics of $\mathbb{H}$ with associated doubly-infinite Romik sequences T, $\tilde{T}$ respectively. There exists $M \in \mathbf{G}_4$ such that $\gamma = M \cdot \tilde{\gamma}$ if and only if $T \in \{\tilde{T}, \tilde{T}^*, \tilde{T}^\vee, (\tilde{T}^\vee)^*\}$.

Proof. Let $P^*|Q$, $R^*|S$ be the reduced two-sided Romik sequences of geodesics γ, $\tilde{\gamma}$. Then there exist $M_1, M_2 \in \mathbf{G}_4$ such that $[P^*] = M_1 \cdot \xi$, $[Q] = M_1 \cdot \eta$ and $[R^*] = M_2 \cdot \tilde{\xi}$, $[S] = M_2 \cdot \tilde{\eta}$ for the endpoints ξ, η of γ and $\tilde{\xi}, \tilde{\eta}$ of $\tilde{\gamma}$.

If $T = \tilde{T}$, then $P^*|Q \sim R^*|S$. By Lemma 3.1.3, there exists M such that

$$[P^*] = M \cdot [R^*], \qquad [Q] = M \cdot [S].$$

Then we have

$$\xi = M_1^{-1} M M_2 \cdot \tilde{\xi} \quad \text{and} \quad \eta = M_1^{-1} M M_2 \cdot \tilde{\eta}.$$

For the case $T = \tilde{T}^*$, $T = \tilde{T}^\vee$, $T = (\tilde{T}^\vee)^*$, by the same way, we can find M such that

$$\xi = M_1^{-1} M M_2 \cdot \tilde{\eta}, \qquad \eta = M_1^{-1} M M_2 \cdot \tilde{\xi}.$$

$$\xi = M_1^{-1} M J M_2 \cdot \tilde{\xi} \qquad \eta = M_1^{-1} M J M_2 \cdot \tilde{\eta}.$$

$$\xi = M_1^{-1} M J M_2 \cdot \tilde{\eta} \qquad \eta = M_1^{-1} M J M_2 \cdot \tilde{\xi},$$

respectively.

On the other hand, if there exists $M \in \mathbf{G}_4$ such that $\xi = M \cdot \tilde{\xi}$ and, $\eta = M \cdot \tilde{\eta}$, then

$$[P^*] = M_1 M M_2^{-1} \cdot [R^*], \quad [Q] = M_1 M M_2^{-1} \cdot [S].$$

By Proposition 3.1.4 and Lemma 3.1.3, $M_1 M M_2^{-1}$ is I or $N_{d_1} \cdots N_{d_m}$ or $H N_{d_1} \cdots N_{d_m} H$ or J or $N_{d_1} \cdots N_{d_m} J$ or $H N_{d_1} \cdots N_{d_m} H J$. If $M_1 M M_2^{-1}$ is one of I, $N_{d_1} \cdots N_{d_m}$, $H N_{d_1} \cdots N_{d_m} H$, then $P^*|Q \sim R^*|S$. Otherwise, $P^*|Q \sim (R^*|S)^\vee$. If there exists $M \in \mathbf{G}_4$ such that $\xi = M \cdot \tilde{\eta}$, $\eta = M \cdot \tilde{\xi}$, then by the same way, we deduce that $P^*|Q \sim S^*|R$ or $P^*|Q \sim (S^*|R)^\vee$. □

Let $\xi, \eta \in \hat{\mathbb{R}}$ be two distinct points on the boundary of $\mathbb{H}$ and T be the associated doubly-infinite Romik sequence of ξ, η. Then Propositions 3.1.5 and 3.1.6 imply that

$$\sup_{M \in \mathbf{G}_4} |M \cdot \xi - M \cdot \eta| = \max \left\{ \sup_{P^*|Q} \left| [Q] - [P^*] \right|, \sup_{P^*|Q} \left| [Q^\vee] - [(P^\vee)^*] \right| \right\}$$

where $P^*|Q$ runs over all sections of T. Let

$$L(P^*|Q) := [Q] - [P^*] = [P] + [Q].$$

Using (3.1), Proposition 3.1.5 implies the following proposition.

Theorem 3.1.7 *Let T be a doubly-infinite Romik sequence. We define $\mathcal{M}(T)$ by the maximum of two supremum values as follows:*

$$\mathcal{M}(T) := \sup_{P^*|Q} \max \left\{ L(P^*|Q), L((P^\vee)^*|Q^\vee) \right\},$$

where $P^*|Q$ runs over all sections of T. The Markoff spectrum is the set of the Markoff numbers taken by $\mathcal{M}(T)$ as T runs through all of doubly-infinite Romik sequences.

$$\mathscr{M}(\mathbf{H}_4) = \{\mathcal{M}(T) \in \mathbb{R} \,|\, T \textit{ is a doubly-infinite Romik sequence}\}$$

Theorem 3.1.8 ([21, Corollary 2.17]) *Let T be a doubly-infinite Romik sequence. We define $\mathcal{L}(T)$ by the maximum of two limit superior values as follows:*

$$\mathcal{L}(T) := \limsup_{P^*|Q} \max \left\{ L(P^*|Q), L((P^\vee)^*|Q^\vee) \right\},$$

where $P^*|Q$ runs over all sections of T. For an infinite Romik sequence P, we define

$$\mathcal{L}(P) := \mathcal{L}({}^\infty 3P).$$

The Lagrange spectrum is the set of the Lagrange numbers taken by $\mathcal{L}(T)$ as T runs through all of doubly-infinite Romik sequences.

$$\begin{aligned} \mathscr{L}(\mathbf{H}_4) &= \{\mathcal{L}(T) \in \mathbb{R} \,|\, T \textit{ is a doubly-infinite Romik sequence}\} \\ &= \{\mathcal{L}(P) \in \mathbb{R} \,|\, P \textit{ is an infinite Romik sequence}\}. \end{aligned}$$

3.1.3 Closedness of the Markoff spectrum

We follow the argument of Bombieri in [12, page 191]. Given the discrete topology on $\{1,2,3\}$, the product space $\{1,2,3\}^{\mathbb{Z}}$ is compact due to Tychonoff's theorem.

Lemma 3.1.9. Let T be a doubly-infinite Romik sequence. If $\mathcal{M}(T)$ is finite, then there exists a doubly-infinite Romik sequence $\tilde{T}$ with a section $P^*|Q$ such that $\mathcal{M}(T)=\mathcal{M}(\tilde{T})=L(P^*|Q)$.

Proof. There exists a sequence of sections $\{P_n^*|Q_n\}_{n\in\mathbb{N}}$ of T or $T^\vee$, say T, satisfying that $\lim_{n\to\infty} L(P_n^*|Q_n)=\mathcal{M}(T)$. Since the product space $\{1,2,3\}^{\mathbb{Z}}$ is compact, there exists a subsequence $\{P_{n_k}^*|Q_{n_k}\}_{k\in\mathbb{N}}$ which converges to a section $P^*|Q$ of a doubly-infinite Romik sequence $\tilde{T}$. By the continuity of L, we have $L(P^*|Q)=\mathcal{M}(T)\leq\mathcal{M}(\tilde{T})$.

If $R^*|S$ is another section of $\tilde{T}$, then $R^*|S$ is a limit of $\{R_{n_k}^*|S_{n_k}\}_{k\in\mathbb{N}}$, which is a shifted subsequence of $\{P_{n_k}^*|Q_{n_k}\}$. Thus $L(R^*|S)\leq\mathcal{M}(T)$, which implies that $\mathcal{M}(\tilde{T})\leq\mathcal{M}(T)$. □

Theorem 3.1.10 *The Markoff spectrum $\mathscr{M}(\mathbf{H}_4)$ is closed.*

Proof. Choose a convergent sequence $\{m_n\}_{n\in\mathbb{N}}$ in $\mathscr{M}(\mathbf{H}_4)$. By Lemma 3.1.9, there exist a sequence of doubly-infinite Romik sequences $\{T_n\}_{n\in\mathbb{N}}$ with a sequence of sections of $\{P_n^*|Q_n\}_{n\in\mathbb{N}}$ such that $m_n=L(P_n^*|Q_n)$ for all $n\in\mathbb{N}$. By the compactness of $\{1,2,3\}^{\mathbb{Z}}$, we have a converging subsequence $\{P_{n_k}^*|Q_{n_k}\}_{k\in\mathbb{N}}$ to the limit $P^*|Q$ which is a section of a doubly-infinite Romik sequence T. By the continuity of L, we have $\lim m_n=L(P^*|Q)$, thus $\lim m_n\leq\mathcal{M}(T)$.

Let $R^*|S$ be another section of T. Then $R^*|S$ is a limit of finite shifts of subsequence of $\{P_{n_k}^*|Q_{n_k}\}_{k\in\mathbb{N}}$. Therefore $L(R^*|S)\leq\mathcal{M}(T_n)$ and $\mathcal{M}(T)\leq\lim m_n$. Hence, $\mathcal{M}(T)=\lim m_n$ and we conclude that the Markoff spectrum is closed. □

Theorem 3.1.11 *The Lagrange spectrum $\mathscr{L}(\mathbf{H}_4)$ is contained in the Markoff spectrum $\mathscr{M}(\mathbf{H}_4)$, i.e., $\mathscr{L}(\mathbf{H}_4)\subset\mathscr{M}(\mathbf{H}_4)$.*

Proof. For a doubly-infinite Romik sequence T, there exists a sequence of sections $\{P_n^*|Q_n\}_{n\in\mathbb{N}}$ of T or $T^\vee$, say T, such that $\mathcal{L}(T)=\lim_{n\to\infty}L(P_n^*|Q_n)$. Since the product space $\{1,2,3\}^{\mathbb{Z}}$ is compact, there exists a subsequence $\{P_{n_k}^*|Q_{n_k}\}_{k\in\mathbb{N}}$ which converges to an element $P^*|Q\in\{1,2,3\}^{\mathbb{Z}}$, which is a section of a doubly-infinite sequence $\tilde{T}$. By the continuity of L, we deduce that $\mathcal{L}(T)\leq\mathcal{M}(\tilde{T})$.

For another section $R^*|S$ of $\tilde{T}$, we have $L(R^*|S) \leq \mathcal{L}(T)$ since $R^*|S$ is a limit of a sequence of sections of T. Therefore, $\mathcal{M}(\tilde{T}) \leq \mathcal{L}(T)$. Hence, $\mathcal{L}(T) = \mathcal{M}(\tilde{T}) \in \mathscr{M}(\mathbf{H}_4)$ □

3.1.4 Hausdorff dimension of the Lagrange spectrum

In this section, we show that the Lagrange spectrum has positive Hausdorff dimension after the first accumulation point.

Assume that $\varepsilon > 0$ is given. Since

$$[32^\infty] = \sqrt{2}+1, \qquad [12^\infty] = \sqrt{2}-1,$$

there exists $m \geq 0$ such that

$$[(32^{2m+2}1)^\infty] + [(12^{2m}3)^\infty] < [32^\infty] + [12^\infty] + \varepsilon = 2\sqrt{2} + \varepsilon. \tag{3.10}$$

Let $A = 32^{2m+2}1$, $B = 32^{2m}1$. Define

$$\begin{aligned}\tilde{E} &= \{P \in \{1,2,3\}^{\mathbb{N}} \,|\, P = B^{m_1}A^{n_1}B^{m_2}A^{n_2}\cdots \quad \text{for all } i,\ n_i, m_i \in \mathbb{N}\},\\ E &= \{P \in \{1,2,3\}^{\mathbb{N}} \,|\, P = B^{m_1}A^{n_1}B^{m_2}A^{n_2}\cdots \quad \text{for all } i,\ n_i, m_i \in \{1,2\}\}.\end{aligned}$$

Lemma 3.1.12. We have

$$\dim_H(\{[P] \,|\, P \in E\}) > 0.$$

Proof. Let

$$\alpha := [(B^2A)^\infty], \quad \beta := [(BA^2)^\infty].$$

Then for each $P \in E$, we have

$$\alpha \leq [P] \leq \beta.$$

Let

$$\begin{aligned} N_A :&= N_3 N_2^{2m+2} N_1 \\ &= \frac{1}{2}\begin{pmatrix} (1+\sqrt{2})^{2m+4} + (1-\sqrt{2})^{2m+4} & (1+\sqrt{2})^{2m+3} - (1-\sqrt{2})^{2m+3} \\ (1+\sqrt{2})^{2m+3} - (1-\sqrt{2})^{2m+3} & (1+\sqrt{2})^{2m+2} + (1-\sqrt{2})^{2m+2} \end{pmatrix}, \\ N_B :&= N_3 N_2^{2m} N_1 \\ &= \frac{1}{2}\begin{pmatrix} (1+\sqrt{2})^{2m+2} + (1-\sqrt{2})^{2m+2} & (1+\sqrt{2})^{2m+1} - (1-\sqrt{2})^{2m+1} \\ (1+\sqrt{2})^{2m+1} - (1-\sqrt{2})^{2m+1} & (1+\sqrt{2})^{2m} + (1-\sqrt{2})^{2m} \end{pmatrix}. \end{aligned}$$

Then, we have

$$N_B^2 N_A \cdot \alpha \le [B^2AP] \le N_B^2 N_A \cdot \beta, \quad N_B^2 N_A^2 \cdot \alpha \le [B^2A^2P] \le N_B^2 N_A^2 \cdot \beta,$$
$$N_B N_A \cdot \alpha \le [BAP] \le N_B N_A \cdot \beta, \quad N_B N_A^2 \cdot \alpha \le [BA^2P] \le N_B N_A^2 \cdot \beta.$$

Let $D = [\alpha, \beta]$ be the closed interval in $\mathbb{R}$ and define $f_i : D \to D$ as

$$f_1(x) = N_B^2 N_A \cdot x, \quad f_2(x) = N_B^2 N_A^2 \cdot x, \quad f_3(x) = N_B N_A \cdot x, \quad f_4(x) = N_B N_A^2 \cdot x.$$

Then $\{f_1, f_2, f_3, f_4\}$ is a family of contracting functions, which is called an iterated function system (see e.g. [27]). We check that there are $c_i > 0$ for $i = 1, 2, 3, 4$ such that $|f_i(x) - f_i(y)| \ge c_i|x - y|$ for $x, y \in D$. The set

$$F = \{[P] \mid P \in E\}$$

satisfies

$$F = f_1(F) \cup f_2(F) \cup f_3(F) \cup f_4(F).$$

By [27, Proposition 9.7], we conclude that

$$\dim_H(F) \ge s,$$

where $s > 0$ is the constant satisfying

$$c_1^s + c_2^s + c_3^s + c_4^s = 1. \qquad \square$$

Choose

$$P = B^{m_1} A^{n_1} B^{m_2} A^{n_2} \cdots \in E,$$

where $n_i, m_i \in \{1, 2\}$. Let

$$W_k = B^{m_1} A^{n_1} B^{m_2} \cdots A^{n_k}$$

and

$$T_P = {}^{\infty}BA^3W_1B^2A^3W_2B^3A^3W_3B^4A^3W_4 \cdots B^kA^3W_kB^{k+1}A^3W_{k+1} \cdots .$$

Lemma 3.1.13. We have

$$\mathcal{L}(T_P) = \frac{1}{[B^{\infty}]} + [A^3P].$$

Proof. Let $(R^\vee)^*32^k|2^\ell 1S$ be a section of T_P. Then we have for $k \geq 1, \ell \geq 0$

$$L((R^\vee)^*32^k|2^\ell 1S) = [2^k 3R^\vee] + [2^\ell 1S] \leq [223^\infty] + [21^\infty] = \frac{3}{2\sqrt{2}} + \sqrt{2} < 2\sqrt{2}$$

and for $k = 0$

$$L((R^\vee)^*32^k|2^\ell 1S) = L((R^\vee)^*|32^\ell 1S).$$

Therefore, we have

$$\begin{aligned}\mathcal{L}(T_P) &= \limsup_{(R^\vee)^*|S} \max\left(L((R^\vee)^*|S), L((S^\vee)^*|R)\right) \\ &= \max\left\{\limsup_{(R^\vee)^*|S}\left(\frac{1}{[R]} + [S]\right), \limsup_{(R^\vee)^*|S}\left(\frac{1}{[S]} + [R]\right)\right\},\end{aligned}$$

where $(R^\vee)^*|S$ runs over all sections of T_P such that S and R are infinite Romik sequences of concatenations of A, B. Using the fact that for $n > m \geq 0$ and $Q, R \in \tilde{E}$,

$$[A^n Q] > [A^m R],$$

we conclude that

$$\begin{aligned}\mathcal{L}(T_P) &= \limsup_{k\to\infty} L(\cdots B_{k-1}A^3W_{k-1}B^k|A^3W_kB^{k+1}A^3W_{k+1}\cdots) \\ &= L({}^\infty B|A^3P) = [(B^\vee)^\infty] + [A^3P] = \frac{1}{[B^\infty]} + [A^3P].\end{aligned}$$ □

Let

$$K = \left\{\frac{1}{[B^\infty]} + [A^3P] \mid P \in E\right\}.$$

Then, Lemma 3.1.13 and (3.10) yield that

$$K \subset \mathscr{L}(\mathbf{H}_4) \cap (0, 2\sqrt{2} + \varepsilon). \tag{3.11}$$

Since $[P] \mapsto [A^3P] = N_A^3 \cdot [P]$ is a bi-Lipschitz function on the closed interval $D = [\alpha, \beta]$, Lemma 3.1.12 implies that $\dim_H(K) > 0$ and we obtain the following statement.

Theorem 3.1.14 *For any $\varepsilon > 0$, we have*

$$\dim_H\left(\mathscr{M}(\mathbf{H}_4) \cap \left[0, 2\sqrt{2} + \epsilon\right)\right) \geq \dim_H\left(\mathscr{L}(\mathbf{H}_4) \cap \left[0, 2\sqrt{2} + \epsilon\right)\right) > 0.$$

3.1.5 Gaps of the Markoff spectrum

We investigate the existence of gaps in $\mathscr{M}(\mathbf{H}_4)$ above the first limit point $2\sqrt{2}$ in this section. In what follows, we say an interval (a,b) is a *maximal gap* in $\mathscr{M}(\mathbf{H}_4)$ if $(a,b)\cap\mathscr{M}(\mathbf{H}_4)=\emptyset$ and $a,b\in\mathscr{M}(\mathbf{H}_4)$. We denote k consecutive $W\cdots W$ by W^k. We denote an infinite sequence with period W and a doubly infinite sequence with period W by W^∞ and ${}^\infty W^\infty$. For example, $(122)^3=122122122$, $132(13)^\infty=132131313\ldots$, and ${}^\infty(23)^\infty=\ldots 232323\ldots$.

Theorem 3.1.15 *The interval*

$$\left(\sqrt{10},\frac{2124\sqrt{2}+48\sqrt{238}}{1177}\right)=(3.162\ldots,3.181\ldots)$$

is a maximal gap in $\mathscr{M}(\mathbf{H}_4)$. Moreover, $\mathcal{M}(T)=\sqrt{10}$ for $T={}^\infty(32)^\infty$ and $\mathcal{M}(U)=\frac{2124\sqrt{2}+48\sqrt{238}}{1177}$ for $U=S^*23232S$ where $S=(31321312)^\infty$. Moreover, $\mathcal{M}(U)$ is a limit point of $\mathscr{M}(\mathbf{H}_4)$.

Proof. Let $m_0=\frac{2124\sqrt{2}+48\sqrt{238}}{1177}$ and $I=(\sqrt{10},m_0)$. We check that $\mathcal{M}({}^\infty(32)^\infty)=\sqrt{10}$ and $\mathcal{M}(S^*23232S)=m_0$ for $S=(31321312)^\infty$. Let us prove that any infinite Romik sequence does not have its Markoff number in I. Let T be a doubly infinite Romik sequence. Suppose that $\mathcal{M}(T)\in I$.

First, if T or $T^\vee$, say T, contains 333, then

$$\mathcal{M}(T)\geq L(P^*|333Q)=[P]+[Q]+3\sqrt{2}\geq 3\sqrt{2}>m_0$$

for some infinite Romik sequences P,Q with $T=P^*333Q$. Therefore, T and $T^\vee$ do not contain 333.

Next, assume that T or $T^\vee$, say T, contains 33. If T contains 233, then

$$\mathcal{M}(T)\geq L(P^*2|33Q)=[2,P]+[Q]+2\sqrt{2}\geq\frac{1}{\sqrt{2}}+2\sqrt{2}=\frac{5}{\sqrt{2}}>m_0$$

for some infinite Romik sequences P,Q with $T=P^*233Q$. If T contains 1331, then

$$\mathcal{M}(T)\geq L(P^*1|331Q)=[1,Q]+[1,P]+2\sqrt{2}$$

for some infinite Romik sequences P,Q with $T=P^*1331Q$. Since

$$[1,P]\geq[\overline{1,1,2,3,3,2}]=\frac{\sqrt{7}-\sqrt{2}}{5}>0.2463\ldots$$

for any infinite Romik sequence $1P$ contained in T, we have

$$\mathcal{M}(T) \geq 2 \cdot \frac{\sqrt{7}-\sqrt{2}}{5} + 2\sqrt{2} > m_0.$$

Hence, T and $T^\vee$ do not contain 33. Since

$$\begin{aligned} L(P^*2|2Q) &= [2,P] + [2,Q] \leq \sqrt{2} + \sqrt{2} < \sqrt{10}, \\ L(P^*1|2Q) &= [1,P] + [2,Q] \leq \frac{1}{\sqrt{2}} + \sqrt{2} < \sqrt{10}, \\ L(P^*1|1Q) &= [1,P] + [1,Q] \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} < \sqrt{10}, \end{aligned}$$

T or $T^\vee$, say T, contains 3. We note that

$$L(P^*1|31Q) = [1,P] + [1,Q] + \sqrt{2} \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \sqrt{2} < \sqrt{10}$$

and

$$\begin{aligned} L(P^*1|32Q) &= [1,P] + [2,Q] + \sqrt{2} \\ &\leq [1,\overline{3,2,1,2}] + [2,\overline{1,2,3,2}] + \sqrt{2} = 2\sqrt{2} + \frac{\sqrt{2}}{2+\sqrt{7}} < \sqrt{10} \end{aligned}$$

for any infinite Romik sequences P, Q contained in T. Hence, T or $T^\vee$, say T, contains 232. Clearly, $T \neq {}^\infty(32)^\infty$. If T contains $(23)^\infty$, then there exists an infinite Romik sequence P such that P does not start with 32 and $T = P^*23(23)^\infty$. Thus,

$$\begin{aligned} \mathcal{M}(T) &\geq L(P^*2|3(23)^\infty) = [3,\overline{2,3}] + [2,P] \geq \frac{1+\sqrt{5}}{\sqrt{2}} + [2,3,1,\overline{3,2,1,2}] \\ &= \frac{1+\sqrt{5}}{\sqrt{2}} + \frac{\sqrt{7}+77}{63\sqrt{2}} > m_0. \end{aligned}$$

Thus, each block $232\ldots232$ appearing in T has a finite length. If $T = P^*1232Q$ for some infinite Romik sequences P,Q, then

$$\mathcal{M}(T) \geq L(P^*12|32Q) = [2,1,P] + [2,Q] + \sqrt{2} \geq \frac{3}{2\sqrt{2}} + \frac{1}{\sqrt{2}} + \sqrt{2} > m_0.$$

Hence, 1232 and 3212 do not appear in T. Thus, for P appearing in T, $[P] \leq [S]$ if P does not start with 32. If T contains 2323232, then there exists an infinite Romik sequence P such that P does not start with 32 and $T = P^*2323232Q$ for some infinite

Romik sequence Q. Hence,

$$\begin{aligned}\mathcal{M}(T) &\geq L(P^*2|323232Q) = [2,P] + [2,3,2,3,2,Q] + \sqrt{2} \\ &> [2,S] + [2,3,2,S] + \sqrt{2} = \mathcal{M}(S^*23232S) = m_0.\end{aligned}$$

On the other hand, if T does not contain 23232, then there exist infinite Romik sequences P, Q such that both P and Q do not start with 32 and $T = P^*232Q$. Thus,

$$\begin{aligned}\mathcal{M}(T) &\geq L(P^*2|32Q) = [2,Q] + [2,P] + \sqrt{2} \geq [2,S] + [2,S] + \sqrt{2} \\ &> [2,S] + [2,3,2,S] + \sqrt{2} = \mathcal{M}(S^*23232S) = m_0.\end{aligned}$$

Hence, we obtain $T = P^*23232Q$ for some infinite Romik sequences P, Q which do not begin with 32. Since $[2,P] + [2,3,2,P]$ is decreasing on $[P]$ and $[P], [Q] \leq [S]$,

$$\begin{aligned}\mathcal{M}(T) &\geq \frac{1}{2}(L(P^*2|3232Q) + L(P^*232|32Q)) \\ &= \frac{1}{2}\left([2,P] + [2,3,2,P]\right) + \frac{1}{2}\left([2,Q] + [2,3,2,Q]\right) + \sqrt{2} \\ &\geq \frac{1}{2}\left([2,S] + [2,3,2,S]\right) + \frac{1}{2}\left([2,S] + [2,3,2,S]\right) + \sqrt{2} \\ &= [2,S] + [2,3,2,S] + \sqrt{2} = \mathcal{M}(S^*23232S).\end{aligned}$$

Hence, any doubly infinite Romik sequence does not have its Markoff number in I. In other words, I is a maximal gap in $\mathscr{M}(\mathbf{H}_4)$.

Finally, let us show that m_0 is a limit point of $\mathscr{M}(\mathbf{H}_4)$. For $k \geq 1$, let $U_k := S^*23232S_k3232S$ where $S_k := (31321312)^k3132$. Since $U_k = {U_k}^*$, we have $\mathcal{M}(U_k) = [3,2,S_k,3,2,3,2,S] + [2,3,2,S]$. Thus, $\lim_{k\to\infty} \mathcal{M}(U_k) = m_0$. Since $[P] \leq [S]$ for any infinite Romik sequence P starting with S_k, $\mathcal{M}(U_k) \geq m_0$ for all k. Hence, m_0 is a limit point of $\mathscr{M}(\mathbf{H}_4)$. □

Theorem 3.1.16 *The interval*

$$\left(\frac{\sqrt{238}}{5}, \sqrt{10}\right) = (3.085\ldots, 3.162\ldots)$$

is a maximal gap in $\mathscr{M}(\mathbf{H}_4)$. Moreover, $\mathcal{M}(T) = \frac{\sqrt{238}}{5}$ for $T = {}^\infty(31321312)^\infty$.

Proof. Let $I = (\frac{\sqrt{238}}{5}, \sqrt{10})$ and T be a doubly infinite Romik sequence. From Theorem 3.1.15, $\sqrt{10} \in \mathscr{M}(\mathbf{H}_4)$. Suppose that $\mathcal{M}(T) \in I$. From the proof of Theorem

3.1.15, both T and $T^\vee$ contain neither 33 nor 232. We assume that T does not contain 33,11,212,232.

If T does not contain 2, then $T = {}^\infty(31)^\infty$ and $\mathcal{M}(T) = \sqrt{6} \notin I$. Thus, 2 appears in T. Let $S = (31321312)^\infty$. For any infinite Romik sequence R appearing in T, $[S^\vee] \le [R] \le [S]$ if R does not start with 32 or 12. Thus, for infinite Romik sequences P, Q with $T = P^*2Q$,

$$L(P^*|2Q) = [P] + [2, Q] \le [S] + [2, S^\vee] = \mathcal{M}((S^\vee)^*2S).$$

For infinite Romik sequences P, Q with $T = P^*13Q$, $[Q] \le [2, S^\vee]$ and $[3, 1, P] \le [S]$. Thus,

$$L(P^*1|3Q) = [1, P] + [Q] + \sqrt{2} \le [3, 1, P] + [2, S^\vee] < [S] + [2, S^\vee] = \mathcal{M}((S^\vee)^*2S).$$

Since each section of T is in the form of $P^*|2Q, P^*2|Q, P^*1|3Q$, or $P^*3|1Q$, we deduce that $\mathcal{M}(T) \le \mathcal{M}((S^\vee)^*2S)$. Hence, any doubly infinite Romik sequence does not have its Markoff number in I. It is obvious that $\mathcal{M}(T) = \frac{\sqrt{238}}{5}$ for $T = {}^\infty(31321312)^\infty$. Thus, I is a maximal gap in $\mathscr{M}(\mathbf{H}_4)$. $\square$

3.1.6 Hall's Ray

In this section, we prove the existence of Hall's ray. Let

$$F = \{\, [P] \mid P \in \{1, 2\} \times \{1, 2, 3\}^{\mathbb{N}} \text{ contains neither 111 nor 333}\}.$$

Let $S = (332112)^\infty$. Then, the minimum of F is $\frac{\sqrt{7}-\sqrt{2}}{5} = [S^\vee] = [1, 1, 2, S]$ and the maximum of F is $\sqrt{7} - \sqrt{2} = [2, S^\vee] = [2, 1, 1, 2, S]$. For $a \ne b$, denote

$$\langle a, b\rangle = \{t \in \mathbb{R} \mid \min\{a, b\} \le t \le \max\{a, b\}\}.$$

First, let us verify that F can be obtained by applying the Cantor dissection process to the interval

$$F_0 := \langle [S^\vee], [2, S^\vee]\rangle = \left[\frac{\sqrt{7}-\sqrt{2}}{5}, \sqrt{7}-\sqrt{2}\right].$$

Now, let us define six types of intervals as follows. In a dissection process, each type of interval is divided by the following rules:

(I) For $a_{n-1} \neq 1$,

$$\langle\, [a_1, \ldots, a_{n-1}, S^\vee],\, [a_1, \ldots, a_{n-1}, 1, S]\,\rangle$$

is divided into the union of $\langle\, [a_1, \ldots, a_{n-1}, S^\vee],\, [a_1, \ldots, a_{n-1}, 1, 2, S^\vee]\,\rangle$ and $\langle\, [a_1, \ldots, a_{n-1}, 1, 3, S^\vee],\, [a_1, \ldots, a_{n-1}, 1, S]\,\rangle$. Thus, each interval of type (I) is divided into one interval of type (III) and one interval of type (VI).

(II) For any a_{n-1},

$$\langle\, [a_1, \ldots, a_{n-1}, 2, S^\vee],\, [a_1, \ldots, a_{n-1}, 2, S]\,\rangle$$

is divided into the union of $\langle\, [a_1, \ldots, a_{n-1}, 2, S^\vee],\, [a_1, \ldots, a_{n-1}, 2, 2, S^\vee]\,\rangle$ and $\langle\, [a_1, \ldots, a_{n-1}, 2, 3, S^\vee],\, [a_1, \ldots, a_{n-1}, 2, S]\,\rangle$. Thus, each interval of type (II) is divided into one interval of type (III) and one interval of type (V).

(III) For $a_{n-1} \neq 3$,

$$\begin{aligned} &\langle\, [a_1, \ldots, a_{n-1}, 3, S^\vee],\, [a_1, \ldots, a_{n-1}, S]\,\rangle \\ =&\langle\, [a_1, \ldots, a_{n-1}, 3, S^\vee],\, [a_1, \ldots, a_{n-1}, 3, 3, 2, S^\vee]\,\rangle \end{aligned}$$

is divided into the union of $\langle\, [a_1, \ldots, a_{n-1}, 3, S^\vee],\, [a_1, \ldots, a_{n-1}, 3, 2, S^\vee]\,\rangle$ and $\langle\, [a_1, \ldots, a_{n-1}, 3, 3, S^\vee],\, [a_1, \ldots, a_{n-1}, 3, 3, 2, S^\vee]\,\rangle$. Thus, each interval of type (III) is divided into two intervals of type (V).

(IV) For $a_{n-2} \neq 1$,

$$\begin{aligned} &\langle\, [a_1, \ldots, a_{n-2}, S^\vee],\, [a_1, \ldots, a_{n-2}, 1, 1, S]\,\rangle \\ =&\langle\, [a_1, \ldots, a_{n-2}, 1, 1, 2, S],\, [a_1, \ldots, a_{n-2}, 1, 1, S]\,\rangle \end{aligned}$$

is divided into the union of $\langle\, [a_1, \ldots, a_{n-2}, 1, 1, 2, S],\, [a_1, \ldots, a_{n-2}, 1, 1, 2, S^\vee]\,\rangle$ and $\langle\, [a_1, \ldots, a_{n-2}, 1, 1, 3, S^\vee],\, [a_1, \ldots, a_{n-2}, 1, 1, S]\,\rangle$. Thus, each interval of type (IV) is divided into one interval of type (II) and one interval of type (III).

(V) For $a_n \neq 1$,

$$\langle\, [a_1, \ldots, a_n, S^\vee],\, [a_1, \ldots, a_n, 2, S^\vee]\,\rangle$$

is divided into the union of

$$\langle\, [a_1, \ldots, a_n, S^\vee],\, [a_1, \ldots, a_n, 1, S]\,\rangle \cup \langle\, [a_1, \ldots, a_{n-1}, 2, S],\, [a_1, \ldots, a_{n-1}, 2, S^\vee]\,\rangle.$$

Thus, each interval of type (V) is divided into one interval of type (I) and one interval of type (II).

(VI) For $a_{n-1} \neq 1$,

$$\langle\, [a_1, \ldots, a_{n-1}, S^\vee],\, [a_1, \ldots, a_{n-1}, 1, 2, S^\vee]\,\rangle$$

is divided into the union of $\langle\, [a_1, \ldots, a_{n-1}, S^\vee],\, [a_1, \ldots, a_{n-1}, 1, 1, S]\,\rangle$ and $\langle\, [a_1, \ldots, a_{n-1}, 1, 2, S],\, [a_1, \ldots, a_{n-1}, 1, 2, S^\vee]\,\rangle$. Thus, each interval of type (VI) is divided into one interval of type (II) and one interval of type (IV).

We note that F_0 is of type (V) and each type of interval is dissected into two intervals contained in 6 types of intervals. Hence, starting from F_0, the dissection process can be continued by the above 6 rules. Consequently, we obtain the Cantor set $F = \cap_{k=0}^{\infty} F_k$.

Lemma 3.1.17. Let I_0 be a closed interval of type (I) to type (VI). In the Cantor dissection process, we have closed intervals I_1, I_2 in I_0 satisfying $I_0 \setminus J = I_1 \cup I_2$ for an open interval J. Then

$$|I_i| \geq |J| \quad \text{for} \quad i = 1, 2.$$

Proof. For $\alpha, \beta \in F$, let $\alpha := [d_1, \ldots, d_n, P]$, $\beta := [d_1, \ldots, d_n, Q]$ for $P, Q \in \{1, 2, 3\}^{\mathbb{N}}$. Let

$$M = N_{d_1} N_{d_2} \cdots N_{d_n} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$|M \cdot [P] - M \cdot [Q]| = \frac{|[P] - [Q]|}{(c[P] + d)(c[Q] + d)}.$$

Note that

$$\begin{aligned}
[S] &= \sqrt{7} + \sqrt{2}, & [S^\vee] &= \frac{1}{[S]} = \frac{\sqrt{7} - \sqrt{2}}{5}, \\
[1, S] &= \frac{4\sqrt{2} - \sqrt{7}}{5}, & [3, S^\vee] &= \frac{1}{[1, S]} = \frac{4\sqrt{2} + \sqrt{7}}{5}, \\
[2, S] &= \frac{\sqrt{7} + \sqrt{2}}{5}, & [2, S^\vee] &= \frac{1}{[2, S]} = \sqrt{7} - \sqrt{2}, \\
[1, 2, S] &= \frac{1}{\sqrt{7}}, & [3, 2, S^\vee] &= \frac{1}{[1, 2, S]} = \sqrt{7}.
\end{aligned}$$

For an interval I_0 of type (I), we have

$$|J| = \big|[d_1, \dots, d_{n-1}, 1, 2, S^\vee] - [d_1, \dots, d_{n-1}, 1, 3, S^\vee]\big| = \frac{[3, S^\vee] - [2, S^\vee]}{(c[3, S^\vee] + d)(c[2, S^\vee] + d)},$$

$$|I_1| = \big|[d_1, \dots, d_{n-1}, 1, 2, S^\vee] - [d_1, \dots, d_{n-1}, 1, 1, 2, S]\big| = \frac{[2, S^\vee] - [1, 2, S]}{(c[2, S^\vee] + d)(c[1, 2, S] + d)},$$

$$|I_2| = \big|[d_1, \dots, d_{n-1}, 1, S] - [d_1, \dots, d_{n-1}, 1, 3, S^\vee]\big| = \frac{[S] - [3, S^\vee]}{(c[S] + d)(c[3, S^\vee] + d)}.$$

Therefore, we obtain

$$\frac{|J|}{|I_1|} = \frac{(c[1, 2, S] + d)([3, S^\vee] - [2, S^\vee])}{(c[3, S^\vee] + d)([2, S^\vee] - [1, 2, S])} < \frac{[3, S^\vee] - [2, S^\vee]}{[2, S^\vee] - [1, 2, S]} = 0.5025 \cdots < 1,$$

$$\frac{|J|}{|I_2|} = \frac{(c[S] + d)([3, S^\vee] - [2, S^\vee])}{(c[2, S^\vee] + d)([S] - [3, S^\vee])} < \frac{[S]([3, S^\vee] - [2, S^\vee])}{[2, S^\vee]([S] - [3, S^\vee])} = 0.5893 \cdots < 1.$$

For an interval I_0 of type (II), we have

$$|J| = \big|[d_1, \dots, d_{n-1}, 2, 3, S^\vee] - [d_1, \dots, d_{n-1}, 2, 2, S^\vee]\big| = \frac{[3, S^\vee] - [2, S^\vee]}{(c[3, S^\vee] + d)(c[2, S^\vee] + d)},$$

$$|I_1| = \big|[d_1, \dots, d_{n-1}, 2, S] - [d_1, \dots, d_{n-1}, 2, 3, S^\vee]\big| = \frac{[S] - [3, S^\vee]}{(c[S] + d)(c[3, S^\vee] + d)},$$

$$|I_2| = \big|[d_1, \dots, d_{n-1}, 2, 2, S^\vee] - [d_1, \dots, d_{n-1}, 2, S^\vee]\big| = \frac{[2, S^\vee] - [S^\vee]}{(c[2, S^\vee] + d)(c[S^\vee] + d)}.$$

Therefore, we obtain

$$\frac{|J|}{|I_1|} = \frac{(c[S] + d)([3, S^\vee] - [2, S^\vee])}{(c[2, S^\vee] + d)([S] - [3, S^\vee])} < \frac{[S]([3, S^\vee] - [2, S^\vee])}{[2, S^\vee]([S] - [3, S^\vee])} = 0.5893 \cdots < 1,$$

$$\frac{|J|}{|I_2|} = \frac{(c[S^\vee] + d)([3, S^\vee] - [2, S^\vee])}{(c[3, S^\vee] + d)([2, S^\vee] - [S^\vee])} < \frac{[3, S^\vee] - [2, S^\vee]}{[2, S^\vee] - [S^\vee]} = 0.4354 \cdots < 1.$$

For an interval I_0 of type (III), we have

$$|J| = \big|[d_1, \dots, d_{n-1}, 3, 3, S^\vee] - [d_1, \dots, d_{n-1}, 3, 2, S^\vee]\big| = \frac{[3, S^\vee] - [2, S^\vee]}{(c[3, S^\vee] + d)(c[2, S^\vee] + d)},$$

$$|I_1| = \big|[d_1, \dots, d_{n-1}, 3, 3, 2, S^\vee] - [d_1, \dots, d_{n-1}, 3, 3, S^\vee]\big| = \frac{[3, 2, S^\vee] - [3, S^\vee]}{(c[3, 2, S^\vee] + d)(c[3, S^\vee] + d)},$$

$$|I_2| = \big|[d_1, \dots, d_{n-1}, 3, 2, S^\vee] - [d_1, \dots, d_{n-1}, 3, S^\vee]\big| = \frac{[2, S^\vee] - [S^\vee]}{(c[2, S^\vee] + d)(c[S^\vee] + d)}.$$

Therefore, we obtain

$$\frac{|J|}{|I_1|} = \frac{(c[3,2,S^\vee]+d)([3,S^\vee]-[2,S^\vee])}{(c[2,S^\vee]+d)([3,2,S^\vee]-[3,S^\vee])} < \frac{[3,2,S^\vee]([3,S^\vee]-[2,S^\vee])}{[2,S^\vee]([3,2,S^\vee]-[3,S^\vee])} = 0.9354\cdots < 1,$$
$$\frac{|J|}{|I_2|} = \frac{(c[S^\vee]+d)([3,S^\vee]-[2,S^\vee])}{(c[3,S^\vee]+d)([2,S^\vee]-[S^\vee])} < \frac{[3,S^\vee]-[2,S^\vee]}{[2,S^\vee]-[S^\vee]} = 0.4354\cdots < 1.$$

For an interval I_0 of type (IV), we have

$$|J| = \big|[d_1,\dots,d_{n-2},1,1,3,S^\vee]-[d_1,\dots,d_{n-2},1,1,2,S^\vee]\big| = \frac{[3,S^\vee]-[2,S^\vee]}{(c[3,S^\vee]+d)(c[2,S^\vee]+d)},$$
$$|I_1| = \big|[d_1,\dots,d_{n-2},1,1,S]-[d_1,\dots,d_{n-2},1,1,3,S^\vee]\big| = \frac{[S]-[3,S^\vee]}{(c[S]+d)(c[3,S^\vee]+d)},$$
$$|I_2| = \big|[d_1,\dots,d_{n-2},1,1,2,S^\vee]-[d_1,\dots,d_{n-2},1,1,2,S]\big| = \frac{[2,S^\vee]-[2,S]}{(c[2,S^\vee]+d)(c[2,S]+d)}.$$

Using the condition that $d_{n-2} \neq 1$ and $d_{n-1} = d_n = 1$, the matrix $N_{d_1}\dots N_{d_n} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies $\frac{d}{c} \le \frac{\sqrt{2}}{5}$. Therefore,

$$\frac{|J|}{|I_1|} = \frac{(c[S]+d)([3,S^\vee]-[2,S^\vee])}{(c[2,S^\vee]+d)([S]-[3,S^\vee])} < \frac{[S]([3,S^\vee]-[2,S^\vee])}{[2,S^\vee]([S]-[3,S^\vee])} = 0.5893\cdots < 1,$$
$$\frac{|J|}{|I_2|} = \frac{(c[2,S]+d)([3,S^\vee]-[2,S^\vee])}{(c[3,S^\vee]+d)([2,S^\vee]-[2,S])} \le \frac{[2,S]+\sqrt{2}/5}{[3,S^\vee]+\sqrt{2}/5}\,\frac{[3,S^\vee]-[2,S^\vee]}{[2,S^\vee]-[2,S]} = 0.5760\cdots < 1.$$

For an interval I_0 of type (V), we have

$$|J| = \big|[d_1,\dots,d_n,2,S]-[d_1,\dots,d_n,1,S]\big| = \frac{[2,S]-[1,S]}{(c[2,S]+d)(c[1,S]+d)},$$
$$|I_1| = \big|[d_1,\dots,d_n,2,S]-[d_1,\dots,d_n,2,S^\vee]\big| = \frac{[2,S^\vee]-[2,S]}{(c[2,S^\vee]+d)(c[2,S]+d)},$$
$$|I_2| = \big|[d_1,\dots,d_n,1,1,2,S]-[d_1,\dots,d_n,1,S]\big| = \frac{[1,S]-[S^\vee]}{(c[1,S]+d)(c[S^\vee]+d)}.$$

Using the condition that $d_n \neq 1$, the matrix $N_{d_1}\dots N_{d_n} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies $\frac{c}{d} \le \sqrt{2}$.

Therefore,

$$\frac{|J|}{|I_1|} = \frac{(c[2,S^\vee]+d)([2,S]-[1,S])}{(c[1,S]+d)([2,S^\vee]-[2,S])} \le \frac{\sqrt{2}[2,S^\vee]+1}{\sqrt{2}[1,S]+1}\,\frac{[2,S]-[1,S]}{[2,S^\vee]-[2,S]} = 0.7403\cdots < 1,$$

$$\frac{|J|}{|I_2|} = \frac{(c[S^\vee]+d)([2,S]-[1,S])}{(c[2,S]+d)([1,S]-[S^\vee])} < \frac{[2,S]-[1,S]}{[1,S]-[S^\vee]} = 0.5893\cdots < 1.$$

For an interval I_0 of type (VI), we have

$$|J| = \big|[d_1,\dots,d_{n-1},1,2,S]-[d_1,\dots,d_{n-1},1,1,S]\big| = \frac{[2,S]-[1,S]}{(c[2,S]+d)(c[1,S]+d)},$$
$$|I_1| = \big|[d_1,\dots,d_{n-1},1,2,S^\vee]-[d_1,\dots,d_{n-1},1,2,S]\big| = \frac{[2,S^\vee]-[2,S]}{(c[2,S^\vee]+d)(c[2,S]+d)},$$
$$|I_2| = \big|[d_1,\dots,d_{n-1},1,1,S]-[d_1,\dots,d_{n-1},1,1,2,S]\big| = \frac{[1,S]-[1,2,S]}{(c[1,S]+d)(c[1,2,S]+d)}.$$

Using the condition that $d_{n-1}\neq 1$ and $d_n = 1$, the matrix $N_{d_1}\dots N_{d_n} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ satisfies $\frac{c}{d}\leq 2\sqrt{2}$. Therefore,

$$\frac{|J|}{|I_1|} = \frac{(c[2,S^\vee]+d)([2,S]-[1,S])}{(c[1,S]+d)([2,S^\vee]-[2,S])} < \frac{2\sqrt{2}[2,S^\vee]+1}{2\sqrt{2}[1,S]+1}\,\frac{[2,S]-[1,S]}{[2,S^\vee]-[2,S]} = 0.8292\cdots < 1,$$
$$\frac{|J|}{|I_2|} = \frac{(c[1,2,S]+d)([2,S]-[1,S])}{(c[2,S]+d)([1,S]-[1,2,S])} < \frac{[2,S]-[1,S]}{[1,S]-[1,2,S]} = 0.9354\cdots < 1. \qquad \square$$

Lemma 3.1.18. ([24, Chapter 4, Lemma 3]) Let B be the union of disjoint closed intervals $A_1, A_2, \dots, A_r$. Given an open interval I in A_1, let A_{r+1}, A_{r+2} be the disjoint closed intervals such that $A_1\setminus I = A_{r+1}\cup A_{r+2}$. Let B^* be the union of $A_2, A_3, \dots, A_{r+1}, A_{r+2}$. If $|A_i|\geq |I|$ for $i = 2,\dots,r+2$, then

$$B + B = B^* + B^*.$$

Lemma 3.1.19. ([24, Chapter 4, Lemma 4]) If $C_1, C_2, \dots$ is a sequence of the bounded closed sets such that C_i contains C_{i+1} for all $i\geq 1$, then

$$\cap_{i=1}^\infty C_i + \cap_{i=1}^\infty C_i = \cap_{i=1}^\infty (C_i + C_i).$$

Now, using Lemmas 3.1.17, 3.1.18 and 3.1.19, let us prove $F_0 + F_0 = F + F$.

Theorem 3.1.20 *We have* $F + F = \left[\frac{2\sqrt{7}-2\sqrt{2}}{5}, 2\sqrt{7}-2\sqrt{2}\right]$.

Proof. Recall $F_0 = \left[\frac{\sqrt{7}-\sqrt{2}}{5}, \sqrt{7}-\sqrt{2}\right]$. Now, let us prove that $F+F = F_0+F_0$. Let us construct a sequence $\{F_n\}_{n=0}^\infty$ satisfying the following four properties:

1. Each F_n is closed and bounded.

2. $F_n \supset F_{n+1}$ for all $n \geq 0$.

3. $\bigcap_{n=0}^{\infty} F_n = F$.

4. $F_n + F_n = F_{n+1} + F_{n+1}$ for $n \geq 0$.

We already verified that F is obtained from F_0 by removing an infinite number of disjoint open intervals which belong to 6 types intervals from (I) to (VI). Now, let us arrange the set of an infinite number of the open intervals in decreasing order of length. Let us denote the arranged open intervals by $D_0, D_1, \ldots$. For $n \geq 0$, we set $F_{n+1} = F_n \setminus D_n$. By the definition of F_n, three properties (1), (2), (3) are satisfied. Thus, it is enough to show that $F_n + F_n = F_{n+1} + F_{n+1}$.

Let us use an induction on n. Let A_1, A_2 be the disjoint closed intervals such that $F_0 \setminus D_0 = A_1 \cup A_2$. By Lemma 3.1.17, $|A_1|, |A_2| \geq |D_0|$. Thus, $F_0 + F_0 = F_1 + F_1$ by Lemma 3.1.18. Assume that $F_{n-1} + F_{n-1} = F_n + F_n$ for some n. Let I be the closed interval from which D_n is removed and I_1, I_2 be the disjoint closed intervals such that $I \setminus D_n = I_1 \cup I_2$. By Lemma 3.1.17, $|I_1|, |I_2| \geq |D_n|$. By the definition of D_{n-1} and Lemma 3.1.17, each closed interval in F_n has length equal to or greater than $|D_{n-1}|$. Hence, each closed interval in F_{n+1} has length equal to or greater than $|D_n|$. By Lemma 3.1.18, $F_n + F_n = F_{n+1} + F_{n+1}$. Therefore, by Lemma 3.1.19, $F + F = (\cap_{i=1}^{\infty} F_i) + (\cap_{i=1}^{\infty} F_i) = \cap_{i=1}^{\infty}(F_i + F_i) = F_0 + F_0$. □

Since the length of $F_0 + F_0 = \left[\frac{2\sqrt{7}-2\sqrt{2}}{5}, 2\sqrt{7} - 2\sqrt{2}\right]$ is greater than $\sqrt{2}$, Theorem 3.1.20 implies the following corollary.

Corollary 3.1.21. Any real number is expressed as $\sqrt{2}n + [P] + [Q]$ for $n \in \mathbb{Z}$, $P, Q \in F$.

Hence, we obtain the existence of Hall's ray.

Theorem 3.1.22 *The Lagrange spectrum $\mathscr{L}(\mathbf{H}_4)$ contains every real number greater than $4\sqrt{2}$, i.e. $(4\sqrt{2}, \infty) \subset \mathscr{L}(\mathbf{H}_4) \subset \mathscr{M}(\mathbf{H}_4)$.*

Proof. Let $\alpha > 4\sqrt{2}$. By Corollary 3.1.21, there exist two Romik sequences $P_1, P_2 \in F$ and $n \in \mathbb{Z}$ such that $\alpha = \sqrt{2}n + [P_1] + [P_2]$. Since $[P_1], [P_2] \leq \sqrt{2}$, $n \geq 2$. We set P_1, P_2 as $d_{-1}, d_{-2}, \ldots$ and $d_0, d_1, d_2, \ldots$ respectively. We define a doubly-infinite

Romik sequence $T = (t_i)_{i\in\mathbb{Z}}$ with a section ${P_1}^*|WP_2 = \ldots t_{-1}|t_0, t_1 \ldots$ where W is a subsequence $33\ldots3$ of length n. By definition, T contains neither 111 nor 333 except for W. Thus, $L(\ldots t_m|t_{m+1}\ldots) \le 4\sqrt{2}$ for $m \le -2$ or $m \ge n$ and $L(\ldots t_m|t_{m+1}\ldots) = L({P_1}^*|WP_2) = [P_1] + [P_2] + \sqrt{2}n = \alpha > 4\sqrt{2}$ for $-1 \le m < n$. Since $T^\vee$ contains no 333, $L(\ldots {t_m}^\vee|{t_{m+1}}^\vee \ldots) \le 4\sqrt{2}$ for any $m \in \mathbb{Z}$. Hence,

$$\mathcal{M}(T) = \max\left\{\sup_{m\in\mathbb{Z}} L(\ldots t_m|t_{m+1}\ldots), \sup_{m\in\mathbb{Z}} L(\ldots {t_m}^\vee|{t_{m+1}}^\vee\ldots)\right\} = \alpha.$$

Hence, $\mathscr{M}(\mathbf{H}_4)$ contains every real number greater than $4\sqrt{2}$.

Let us prove that $\mathscr{L}(\mathbf{H}_4)$ contains every real number greater than $4\sqrt{2}$. By the definition of t_i, $t_0 = t_1 = 3$ and $t_{-1} = d_{-1} \in \{1, 2\}$. We define a doubly-infinite Romik sequence $A' = (t'_m)_{m\in\mathbb{Z}}$ with a section

$$\ldots, t_{-k_2}, \ldots, t_{l_2}, t_{-k_1}, \ldots, t_{l_1}|t_{-k_1}, \ldots, t_{l_1}, t_{-k_2}, \ldots, t_{l_2}, \ldots$$

where $k_1 < k_2 < \ldots$ and $l_1 < l_2 < \ldots$ are increasing sequences,

$$\begin{cases} d_{-j} = 2 \text{ for all } j \in \{k_i\}_{i=1}^\infty \text{ if 2 appears infinitely many in } P_1, \\ d_{-j} = 3 \text{ for all } j \in \{k_i\}_{i=1}^\infty \text{ if 2 appears finitely many in } P_1, \end{cases}$$

and

$$\begin{cases} d_j = 2 \text{ for all } j \in \{l_i\}_{i=1}^\infty \text{ if 2 appears infinitely many in } P_2, \\ d_j = 1 \text{ for all } j \in \{l_i\}_{i=1}^\infty \text{ if 2 appears finitely many in } P_2. \end{cases}$$

Hence,

$$\begin{aligned} &\limsup_{m\in\mathbb{Z}} L(\ldots t'_m|t'_{m+1}\ldots) \\ =&\limsup_{j\to\infty} \left([t_0, t_1, \ldots, t_{l_j}, t_{-k_{j+1}}, \ldots] + [t_{-1}, t_{-2}, \ldots, t_{-k_j}, t_{l_{j-1}}, \ldots]\right) \\ =&[P_1] + [P_2] + \sqrt{2}n = \alpha. \end{aligned}$$

Since $(P')^\vee$ contains no 333, we have

$$L(\ldots {t'_m}^\vee|{t'_{m+1}}^\vee\ldots) = [{t'_m}^\vee, {t'_{m-1}}^\vee, \ldots] + [{t'_{m+1}}^\vee, {t'_{m+2}}^\vee, \ldots] \le 4\sqrt{2}$$

for any $m \in \mathbb{Z}$. Thus,

$$\limsup_{m \in \mathbb{Z}} L(\dots {t'_m}^{\vee} | {t'_{m+1}}^{\vee} \dots) < \alpha.$$

Therefore, $\mathcal{L}(A') = \alpha$ and $\mathscr{L}(\mathbf{H}_4)$ contains every real number greater than $4\sqrt{2}$. $\square$

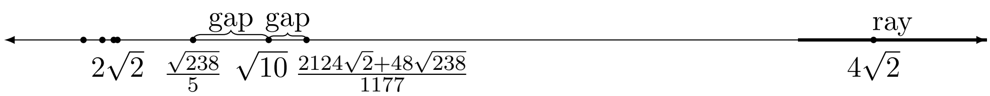


Figure 3.4 Gaps and a ray in $\mathscr{M}(\mathbf{H}_4)$

3.2 The Markoff and Lagrange spectra on $\mathbf{H}_6$

3.2.1 The Markoff spectrum and the Romik expansion

The Hecke group $\mathbf{H}_q$ is defined by the subgroup of $\mathrm{SL}_2(\mathbb{R})$ generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix},$$

where

$$\lambda_q = 2\cos\left(\frac{\pi}{q}\right).$$

When $q = 3$, the Hecke group $\mathbf{H}_3$ is the modular group $\mathrm{SL}_2(\mathbb{Z})$. In this paper we consider the Hecke group $\mathbf{H}_6$ for the case of $q = 6$. In this case, $\lambda_6 = \sqrt{3}$.

We follow the notations in [22]:

$$N_1 = \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix},$$
$$N_4 = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}, \quad N_5 = \begin{pmatrix} 1 & 0 \\ \sqrt{3} & 1 \end{pmatrix}.$$

For $P \in \{1, 2, 3, 4, 5\}^{\mathbb{N}}$, we have

$$[d, P] = N_d \cdot [P]. \tag{3.12}$$

Let

$$H = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 & -\sqrt{3} \\ 0 & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\mathbf{G}_6$ be the group of 2×2 matrices generated by reflections H, U, J. Note that $(HJ)^2 = I$ and $(UJ)^6 = I$ in $\mathrm{PSL}_2(\mathbb{R})$. See Figure 3.5 for the fundamental domain of $\mathbf{G}_6$.

Since $S = HJ$ and $T = HU$, the Hecke group $\mathbf{H}_6$ is a subgroup of $\mathbf{G}$. Indeed, using $HJ = JH = S$, $HU = T$, $UH = T^{-1}$, we have

$$\mathbf{G}_6 = \mathbf{H}_6 \cup \mathbf{H}_6 H$$

and $\mathbf{H}_6$ is an index 2 subgroup of $\mathbf{G}_6$.

Note that

$$\begin{aligned} N_1 &= HU, & N_5 &= JN_1J = JHUJ = HJUJ, \\ N_2 &= HUJUJ, & N_4 &= JN_2J = JHUJU = HJUJU, \\ N_3 &= HUJUJU, & N_3 &= JN_3J = HJUJUJUJ. \end{aligned}$$

Since $(UJ)^6 = I$, or $UJUJUJ = JUJUJU$, elements generated by U and J in $\mathbf{G}_6$ are

$$I,\ U,\ J,\ UJ,\ JU,\ UJU,\ JUJ,\ UJUJ,\ JUJU,$$
$$UJUJU,\ JUJUJ,\ UJUJUJ = JUJUJU,$$

which are represented as

$$\begin{aligned} U &= U_1, & UJ &= U_1J, \\ JU &= U_5J, & UJU &= U_2J, \\ JUJ &= U_5, & UJUJ &= U_2, \\ JUJU &= U_4, & UJUJU &= U_3, \\ JUJUJ &= U_4J, & UJUJUJ &= U_3J. \end{aligned}$$

Then,

$$U_d^{-1} = U_{\hat{d}},\ N_d = HU_d.$$

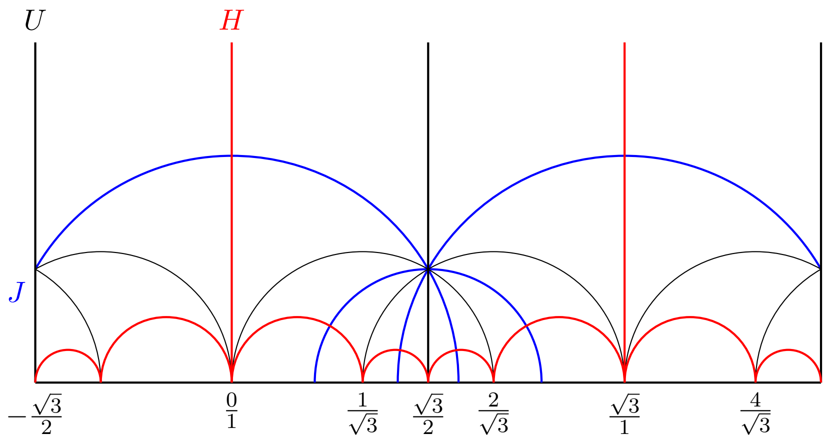


Figure 3.5 The fundamental domain of the group $\mathbf{G}_6$ on the upper half space.

We have $HJ = JH$ and $N_d J = J N_{d^\vee}$. Therefore, any element M of $\mathbf{G}_6$ is one of the forms

$$N_{d_1}\cdots N_{d_m},\ N_{d_1}\cdots N_{d_m}J,\ HN_{d_1}\cdots N_{d_m},\ HN_{d_1}\cdots N_{d_m}J,$$
$$N_{d_1}\cdots N_{d_m}H,\ HN_{d_1}\cdots N_{d_m}HJ,\ HN_{d_1}\cdots N_{d_m}H,\ HN_{d_1}\cdots N_{d_m}HJ.$$

Since $U_1^2 = U_3^2 = U_5^2 = I$ and $U_2^3 = U_4^3 = I$, the group $\{I, U_1, U_2, U_3, U_4, U_5\}$ is isomorphic to the symmetry group S_3.

We consider a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2^{\pm}(\mathbb{Z})$. If $\det(M) = 1$, then $M \cdot z = \dfrac{az+b}{cz+d}$. If $\det(M) = -1$, then $M \cdot z = \dfrac{a\bar{z}+b}{c\bar{z}+d}$.

Define

$$\hat{d} = \begin{cases} 1 & \text{if } d = 1, \\ 4 & \text{if } d = 2, \\ 3 & \text{if } d = 3, \\ 2 & \text{if } d = 4, \\ 5 & \text{if } d = 5. \end{cases} \qquad \text{and} \qquad d^\vee = \begin{cases} 5 & \text{if } d = 1, \\ 4 & \text{if } d = 2, \\ 3 & \text{if } d = 3, \\ 2 & \text{if } d = 4, \\ 1 & \text{if } d = 5. \end{cases}$$

For an infinite Romik sequence $P = (a_1, a_2, a_3, \dots) \in \{1, 2, 3, 4, 5\}^{\mathbb{N}}$, we denote

$$[P] = [a_1, a_2, a_3, \dots] \in [0, \infty].$$

For $Q = (b_1, b_2, b_3, \dots) \in \{1,2,3,4,5\}^{\mathbb{N}}$, we define

$$[\dots, \hat{b}_3, \hat{b}_2, \hat{b}_1] := [\hat{Q}^*] = H([b_1, b_2, b_3, \dots]) = -[b_1, b_2, b_3, \dots].$$

In what follows, we identify $[-\infty, 0]$ with an element in $\{1,2,3,4,5\}^{\mathbb{N}_{\leq 0}}$ using Romik digit expansion.

Since

$$[d, P] = N_d \cdot [P] = (HU_d) \cdot [P],$$

we have

$$U_d([P]) = H([d, P]) = [\hat{P}^*, \hat{d}] \quad \text{and} \quad N_d \cdot [P^*, d] = (HU_d) \cdot [P^*, d] = [P^*].$$

Thus

$$U_d \cdot [P^*, d] = H([P^*]) = [\hat{P}] \quad \text{and} \quad U_d^{-1}([\hat{P}]) = U_{\hat{d}}([\hat{P}]) = [P^*, d].$$

We check the idempotent maps U_1, U_3, U_5 acts on $\mathbb{R}^\infty$ as follows

$$\begin{aligned}
&U_1([P^*, 1]) = [\hat{P}],\\
&U_1([P^*, 2]) = U_3U_2([P^*, 2]) = U_3([\hat{P}]) = [P^*, 3],\\
&U_1([P^*, 4]) = U_5U_4([P^*, 4]) = U_5([\hat{P}]) = [P^*, 5],\\
&U_3([P^*, 3]) = [\hat{P}],\\
&U_3([P^*, 1]) = U_2U_1([P^*, 1]) = U_2([\hat{P}]) = [P^*, 4],\\
&U_3([P^*, 2]) = U_5U_2([P^*, 2]) = U_5([\hat{P}]) = [P^*, 5],\\
&U_5([P^*, 5]) = [\hat{P}],\\
&U_5([P^*, 1]) = U_4U_1([P^*, 1]) = U_4([\hat{P}]) = [P^*, 2],\\
&U_5([P^*, 3]) = U_2U_3([P^*, 3]) = U_2([\hat{P}]) = [P^*, 4].
\end{aligned}$$

For the maps U_2, U_4, we have

$$\begin{aligned}
&U_2([P^*, 2]) = [\hat{P}],\\
&U_2([P^*, 1]) = U_3U_1([P^*, 1]) = U_3([\hat{P}]) = [P^*, 3],\\
&U_2([P^*, 3]) = U_5U_3([P^*, 3]) = U_5([\hat{P}]) = [P^*, 5],\\
&U_2([P^*, 5]) = U_1U_5([P^*, 5]) = U_1([\hat{P}]) = [P^*, 1],\\
&U_2([P^*, 4]) = U_4U_4([P^*, 4]) = U_4([\hat{P}]) = [P^*, 2],
\end{aligned}$$

$$U_4([P^*,4]) = [\hat{P}],$$
$$U_4([P^*,1]) = U_5U_1([P^*,1]) = U_5([\hat{P}]) = [P^*,5],$$
$$U_4([P^*,3]) = U_1U_3([P^*,3]) = U_1([\hat{P}]) = [P^*,1],$$
$$U_4([P^*,5]) = U_3U_5([P^*,5]) = U_3([\hat{P}]) = [P^*,3],$$
$$U_4([P^*,2]) = U_2U_2([P^*,2]) = U_2([\hat{P}]) = [P^*,4].$$

Let $P^*\,|Q = [\dots,a_{-2},a_{-1},a_0\,|\,a_1,a_2,\dots]$ be the geodesic on $\mathbb{H}^2$ whose two endpoint are $P^* = [\dots,a_{-2},a_{-1},a_0]$ and $Q = [a_1,a_2,\dots]$. Then N_{a_0}acts as a right shift of the two-sided sequence

$$N_{a_0}[\dots,a_{-2},a_{-1},a_0\,|\,a_1,a_2,\dots] = [\dots,a_{-2},a_{-1}\,|\,a_0,a_1,a_2,\dots].$$

Definition 3.2.1. For two bi-infinite sequence $(a_n)_{n\in\mathbb{Z}},(b_n)_{n\in\mathbb{Z}}$ in $\{1,2,3,4,5\}^{\mathbb{Z}}$, we give an equivalence relation $(a_n)_{n\in\mathbb{Z}} \sim (b_n)_{n\in\mathbb{Z}}$ if there exists an integer $k\in\mathbb{Z}$ such that $a_{n+k} = b_n$ for all $n\in\mathbb{N}$. We call an equivalence class a *bi-infinite Romik sequence* and an element in the equivalence class is called a *section* of the bi-infinite Romik sequence. Let A be a bi-infinite Romik sequence. For a section $P^*|Q$ of A, we define

$$L(P^*|Q) := [Q] - [P^*] = [Q] + H([P^*]) = [Q] + [\hat{P}].$$

Definition 3.2.2. We define $\mathcal{M}(A)$ by the maximum of two supremum values as follows:

$$\mathcal{M}(A) := \sup_{P^*|Q} \max\{L(P^*|Q), L((P^\vee)^*|Q^\vee)\},$$

where $P^*|Q$ runs over all sections of A. The Markoff spectrum $\mathscr{M}(\mathbf{H}_6)$ is defined by the set of the Markoff numbers taken by $\mathcal{M}(A)$ as A runs through all of bi-infinite Romik sequences.

Definition 3.2.3. We define $\mathcal{L}(A)$ by the maximum of two limit superior values as follows:

$$\mathcal{L}(A) := \limsup_{P^*|Q} \max\{L(P^*|Q), L((P^\vee)^*|Q^\vee)\},$$

where $P^*|Q$ runs over all sections of A. The Lagrange spectrum $\mathscr{L}(\mathbf{H}_6)$ is defined by the set of the Lagrange numbers taken by $\mathcal{L}(A)$ as A runs through all of bi-infinite Romik sequences.

3.2.2 Closedness of the Markoff spectrum

Given the discrete topology on $\{1,2,3,4,5\}$, the product space $\{1,2,3,4,5\}^{\mathbb{Z}}$ is compact due to Tychonoff's theorem.

Lemma 3.2.4. Let A be a bi-infinite Romik sequence A with a section $(a_k)_{k\in\mathbb{Z}}$. If $\mathcal{M}(A)$ is finite, then there exists a bi-infinite Romik sequence B with a section $P^*|Q$ such that $\mathcal{M}(A) = \mathcal{M}(B) = L(P^*|Q)$.

Proof. By considering A or $A^\vee$, we may assume that there exists a subsequence $\{k_n\}_{n\geq 1}$ such that $\lim_{n\to\infty} L(\ldots a_{k_n-1}|a_{k_n}\ldots) = \mathcal{M}(A)$. Let $A_n = \ldots a_{k_n-1}|a_{k_n}\ldots$ be a section of A. By the compactness of the space $\{1,2,3,4,5\}^{\mathbb{Z}}$, there exists a subsequence $\{A_{n_i}\}$ converging to $P^*|Q$ which is a section of a bi-infinite Romik sequence B. By the continuity of M, we have $L(P^*|Q) = \mathcal{M}(A)$. □

Theorem 3.2.5 *The Markoff spectrum $\mathscr{M}(\mathbf{H}_6)$ is closed.*

Proof. Choose a convergent sequence $\{M_n\}_{n\geq 1}$ in $\mathscr{M}(\mathbf{H}_6)$. By Lemma 3.2.4, there exist bi-infinite Romik sequences $\{A_n\}$ with a section $P_n^*|Q_n$ such that $M_n = L(P_n^*|Q_n)$ for all $n \in \mathbb{N}$. By the compactness of the space $\{1,2,3,4,5\}^{\mathbb{Z}}$, we have a subsequence $\{n_i\}$ such that $P_{n_i}^*|Q_{n_i}$ converges to $P^*|Q$. By the continuity of L, M_{n_i} converges to $L(P^*|Q) \leq \mathcal{M}(B)$ where B is a bi-infinite Romik sequence with a section $P^*|Q$. Hence, $\lim_{i\to\infty} M_{n_i} \leq \mathcal{M}(B)$ For any section $R^*|S$ of B, $R^*|S$ is a limit of finite shifts of $P_{n_i}^*|Q_{n_i}$. Thus, $L(R^*|S) \leq \lim_{i\to\infty} M_{n_i}$, which implies $\mathcal{M}(B) \leq \lim_{i\to\infty} M_{n_i}$. Hence, the Markoff spectrum is closed. □

Theorem 3.2.6 *The Lagrange spectrum is contained in the Markoff spectrum: $\mathscr{L}(\mathbf{H}_6) \subset \mathscr{M}(\mathbf{H}_6)$.*

Proof. For any bi-infinite Romik sequence A, there exists a sequence of sections $\{P_n^*|Q_n\}$ such that $\mathcal{L}(A) = \lim_{n\to\infty} L(P_n^*|Q_n)$. By the compactness of the space $\{1,2,3,4,5\}^{\mathbb{Z}}$, there exists a subsequence $\{n_i\}$ such that $P_{n_i}^*|Q_{n_i}$ converges to $P^*|Q$ where B is a bi-infinite Romik sequence with a section $P^*|Q$. By the continuity of L, we deduce that $\mathcal{L}(A) = L(P^*|Q) \leq \mathcal{M}(B)$. For any section $R^*|S$ of B, $R^*|S$ is a limit of finite shifts of $P_{n_i}^*|Q_{n_i}$. Thus, $L(R^*|S) \leq \mathcal{L}(A)$, which concludes $\mathcal{L}(A) = \mathcal{M}(B)$. □

3.2.3 Hausdorff dimension of the Lagrange spectrum

In this section, we show that the Lagrange spectrum has positive Hausdorff dimension after the first accumulation point.

Let $\varepsilon > 0$. Choose $m \in \mathbb{N}$ such that

$$[(2^{2m+2}3)\infty] + [(34^{2m})\infty] < [2^\infty] + [34^\infty] + \varepsilon = \frac{4}{\sqrt{3}} + \varepsilon. \tag{3.13}$$

Let

$$A = 2^{2m+2}3, \qquad B = 2^{2m}3.$$

Consider

$$\begin{aligned} E &= \{P \in \{2,3\}^{\mathbb{N}} \mid P = B^{m_1}A^{n_1}B^{m_2}A^{n_2}\cdots, n_i, m_i \in \{1,2\}\}, \\ \tilde{E} &= \{P \in \{2,3\}^{\mathbb{N}} \mid P = B^{m_1}A^{n_1}B^{m_2}A^{n_2}\cdots, n_i, m_i \in \mathbb{N}\}. \end{aligned}$$

Lemma 3.2.7. We have

$$\dim_H(\{[P] \mid P \in E\}) > 0.$$

Proof. Let

$$\alpha := [(B^2A)^\infty], \quad \beta := [(BA^2)^\infty].$$

Then for each $P \in E$, we have

$$\alpha \le [P] \le \beta.$$

From [20, Proposition 38],

$$\begin{aligned} N_A :&= N_2^{2m+2}N_3 \\ &= \frac{1}{\sqrt{13}}\begin{pmatrix} \frac{1+\sqrt{13}}{2}\lambda^{2m+3} - \frac{1-\sqrt{13}}{2}\overline{\lambda}^{2m+3} & \frac{\sqrt{39}-\sqrt{3}}{2}\lambda^{2m+3} + \frac{\sqrt{39}+\sqrt{3}}{2}\overline{\lambda}^{2m+3} \\ \sqrt{3}(\lambda^{2m+3} - \overline{\lambda}^{2m+3}) & \frac{7-\sqrt{13}}{2}\lambda^{2m+3} - \frac{7+\sqrt{13}}{2}\overline{\lambda}^{2m+3} \end{pmatrix} \\ N_B :&= N_2^{2m}N_3 \\ &= \frac{1}{\sqrt{13}}\begin{pmatrix} \frac{1+\sqrt{13}}{2}\lambda^{2m+1} - \frac{1-\sqrt{13}}{2}\overline{\lambda}^{2m+1} & \frac{\sqrt{39}-\sqrt{3}}{2}\lambda^{2m+1} + \frac{\sqrt{39}+\sqrt{3}}{2}\overline{\lambda}^{2m+1} \\ \sqrt{3}(\lambda^{2m+1} - \overline{\lambda}^{2m+1}) & \frac{7-\sqrt{13}}{2}\lambda^{2m+1} - \frac{7+\sqrt{13}}{2}\overline{\lambda}^{2m+1} \end{pmatrix} \end{aligned}$$

where $\lambda = \frac{3+\sqrt{13}}{2}$, $\overline{\lambda} = \frac{3-\sqrt{13}}{2}$. Then, we have

$$N_B^2N_A \cdot \alpha \le [B^2AP] \le N_B^2N_A \cdot \beta, \quad N_B^2N_A^2 \cdot \alpha \le [B^2A^2P] \le N_B^2N_A^2 \cdot \beta,$$

$$N_BN_A \cdot \alpha \leq [BAP] \leq N_BN_A \cdot \beta, \quad N_BN_A^2 \cdot \alpha \leq [BA^2P] \leq N_BN_A^2 \cdot \beta.$$

Let $D = [\alpha, \beta]$ be the closed interval in $\mathbb{R}$ and define $f_i : D \to D$ as

$$f_1(x) = N_B^2N_A \cdot x, \quad f_2(x) = N_B^2N_A^2 \cdot x, \quad f_3(x) = N_BN_A \cdot x, \quad f_4(x) = N_BN_A^2 \cdot x.$$

Let $f_i(x) = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \cdot x$ for $i = 1, 2, 3, 4$. Since

$$|f_i(x) - f_i(y)| = \frac{|x - y|}{(C_ix + D_i)(C_iy + D_i)} \text{ and } C_i, D_i > 1$$

for $i = 1, 2, 3, 4$, $\{f_1, f_2, f_3, f_4\}$ is a family of contracting functions, which is called an iterated function system (see e.g. [27]). We check that there are $c_i > 0$ for $i = 1, 2, 3, 4$ such that $|f_i(x) - f_i(y)| \geq c_i|x - y|$ for $x, y \in D$. By the definition of E and f_i's, the set

$$F = \{[P] \mid P \in E\}$$

satisfies

$$F = f_1(F) \cup f_2(F) \cup f_3(F) \cup f_4(F).$$

By [27, Proposition 9.7], we conclude that $\dim_H(F) \geq s$, where $s > 0$ is the constant satisfying $c_1^s + c_2^s + c_3^s + c_4^s = 1$. □

Choose

$$P = B^{m_1}A^{n_1}B^{m_2}A^{n_2} \cdots \in E,$$

where $n_i, m_i \in \{1, 2\}$. Let

$$W_k = B^{m_1}A^{n_1}B^{m_2} \cdots A^{n_k}$$

and

$$T_P = {}^\infty BA^3W_1B^2A^3W_2B^3A^3W_3B^4A^3W_4 \cdots B^kA^3W_kB^{k+1}A^3W_{k+1} \cdots .$$

Lemma 3.2.8. We have

$$\mathcal{L}(T_P) = [(\hat{B}^*)^\infty] + [A^3P].$$

Proof. Let $R^*32^k|2^\ell 3S$ be a section of T_P. Then we have

$$L(R^*32^k|2^\ell 3S) = [4^k3\hat{R}] + [2^\ell 3S]$$

$$\begin{aligned}
&\leq [4432^\infty] + [23^\infty] = \begin{pmatrix} 17 & 10\sqrt{3} \\ 13\sqrt{3} & 23 \end{pmatrix} \cdot [2^\infty] + \begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \cdot [3^\infty] \\
&= \frac{17}{13\sqrt{3}} - \frac{1}{507[2^\infty] + 200\sqrt{3}} + \left(\frac{2}{\sqrt{3}} + \frac{1}{3[3^\infty] + \sqrt{3}} \right) \\
&< \frac{17}{13\sqrt{3}} + \left(\frac{2}{\sqrt{3}} + \frac{1}{2\sqrt{3}} \right) < \frac{4}{\sqrt{3}}
\end{aligned}$$

for $k \geq 1, \ell \geq 0$. Since R begins with 2,

$$L(R^* 32^k | 2^\ell 3S) = [3\hat{R}] + [2^\ell 3S] > [\hat{R}] + [32^\ell 3S] = L(R^* | 32^\ell 3S)$$

for $k = 0$. Therefore, we have

$$\begin{aligned}
\mathcal{L}(T_P) &= \limsup_{R^*|S} \max \left(L(R^*|S), L((R^\vee)^*|S^\vee) \right) \\
&= \max \left\{ \limsup_{R^*|S} \left([\hat{R}] + [S] \right), \limsup_{R^*|S} \left([\hat{R}^\vee] + [S^\vee] \right) \right\},
\end{aligned}$$

where $R^*|S$ runs over all sections of T_P such that both S and R are concatenations of A, B. Using the fact that for $n > m \geq 0$ and $Q, R \in \tilde{E}$,

$$[A^n Q] > [A^m R],$$

we conclude that

$$\begin{aligned}
\mathcal{L}(T_P) &= \limsup_{k \to \infty} L(\cdots B_{k-1} A^3 W_{k-1} B^k | A^3 W_k B^{k+1} A^3 W_{k+1} \cdots) \\
&= L({}^\infty B | A^3 P) = [(\hat{B}^*)^\infty] + [A^3 P].
\end{aligned}$$

□

Let

$$K = \left\{ [(\hat{B}^*)^\infty] + [A^3 P] \mid P \in E \right\}.$$

Then, Lemma 3.2.8 and (3.13) yield that

$$K \subset \mathscr{L}(\mathbf{H}_6) \cap \left(0, \frac{4}{\sqrt{3}} + \varepsilon \right). \tag{3.14}$$

Since $[P] \mapsto [A^3 P] = N_A^3 \cdot [P]$ is a bi-Lipschitz function on D, Lemma 3.2.7 implies $\dim_H(K) > 0$ and we obtain the following statement.

Theorem 3.2.9 *For any $\varepsilon > 0$, we have*

$$\dim_H \left(\mathscr{M}(\mathbf{H}_6) \cap \left[0, \frac{4}{\sqrt{3}} + \epsilon\right)\right) \geq \dim_H \left(\mathscr{L}(\mathbf{H}_6) \cap \left[0, \frac{4}{\sqrt{3}} + \epsilon\right)\right) > 0.$$

3.2.4 Gaps of the Markoff spectrum

We investigate the existence of gaps in $\mathscr{M}(\mathbf{H}_6) \cap [\frac{4}{\sqrt{3}}, \infty)$ in the section. We say an interval (a, b) is a *maximal gap* in $\mathscr{M}(\mathbf{H}_6)$ if $(a, b) \cap \mathscr{M}(\mathbf{H}_6) = \emptyset$ and $a, b \in \mathscr{M}(\mathbf{H}_6)$. We denote k consecutive $W \cdots W$ by W^k. We denote an infinite sequence with period W and a bi-infinite sequence with period W by W^∞ and ${}^\infty W^\infty$. For example, $(234)^3 = 234234234$, $153(13)^\infty = 153131313\ldots$, and ${}^\infty(23)^\infty = \ldots 232323\ldots$.

Lemma 3.2.10. Let A be a bi-infinite Romik sequence.

(1) If A or $A^\vee$ contains 1, then $\mathcal{M}(A) \geq \frac{7-\sqrt{5}}{\sqrt{3}} = 2.750\ldots$ or $A = {}^\infty(15)^\infty$.

(2) The sequence $A \in \{2,3,4\}^{\mathbb{Z}}$ does not contain 24 and 42 if and only if $\mathcal{M}(A) \leq \frac{\sqrt{143}}{5}$.

(3) If $A \in \{2,3,4\}^{\mathbb{Z}}$ contains 24 or 42 and does not contain 243, 423, 424, 242, 342, 324, then $\mathcal{M}(A) \geq \sqrt{7}$. The equality holds for $A = {}^\infty(2244)^\infty$.

Proof. (1) Assume that $\mathcal{M}(A) < \frac{7-\sqrt{5}}{\sqrt{3}}$. Since

$$L(P^*1|1Q) = [1\hat{P}] + [1Q] \geq 2\sqrt{3} = 3.464\ldots,$$
$$L(P^*1|2Q) = [1\hat{P}] + [2Q] \geq \frac{5}{3}\sqrt{3} = 2.886\ldots$$

for any infinite Romik sequences P, Q, A and $A^\vee$ do not contain $11, 12$. Since $[(53)^\infty] \leq [P] \leq [(13)^\infty]$ for an infinite Romik sequence P in A,

$$L(P^*1|3Q) = [1\hat{P}] + [3Q] \geq [1(53)^\infty] + [(35)^\infty] = \sqrt{3} + \frac{\sqrt{15}}{3} = 3.023\ldots,$$
$$L(P^*1|4Q) = [1\hat{P}] + [4Q] \geq [1(53)^\infty] + [4(13)^\infty] = \frac{7-\sqrt{5}}{\sqrt{3}} = 2.750\ldots.$$

Thus, A and $A^\vee$ do not contain 11, 12, 13, and 14. Hence, if A or $A^\vee$ contains 1, then A is ${}^\infty(15)^\infty$.

(2) Let A and $A^\vee$ do not contain 24. We have

$$[P] \leq [(3234)^\infty],\ [Q] \leq [(2343)^\infty]$$

where P, Q are infinite Romik sequences starting with 3,2 in A, respectively. Since $\hat{P}$ or Q does not start with 2 for any section $P^*|Q$ of A or $A^\vee$,

$$L(P^*|Q) = [\hat{P}] + [Q] \leq [(3234)^\infty] + [(2343)^\infty]$$

for any section $P^*|Q$ of A or $A^\vee$. Hence,

$$\mathcal{M}(A) \leq [(3234)^\infty] + [(2343)^\infty] = \frac{\sqrt{143}}{5}.$$

Conversely, if A or $A^\vee$ contains 24, then

$$L(P^*4|2Q) = [2\hat{P}] + [2Q] \geq [2(24)^\infty] + [2(24)^\infty] = 2.644\ldots$$

for any section $P^*4|2Q$ of A or $A^\vee$. Hence, $\mathcal{M}(A) > \frac{\sqrt{143}}{5}$.

(3) Let A or $A^\vee$ contain 24. Assume that A and $A^\vee$ do not contain 243, 424, 342. Hence, $[P] \geq [22(444222)^\infty]$ for an infinite Romik sequence P starting with 2 in A and $A^\vee$. Since $[P] \geq [2(2343)^\infty]$ for an infinite Romik sequence P starting with 223 in A and $A^\vee$,

$$L(P^*4|223Q) = [2\hat{P}] + [223Q] \geq [22(444222)^\infty] + [2(2343)^\infty] = 2.648\ldots,$$
$$L(P^*4|222Q) = [2\hat{P}] + [222Q] \geq [22(444222)^\infty] + [2222(444222)^\infty] = 2.652\ldots.$$

Thus, if 4223 and 4222 do not appear in A and $A^\vee$, then 42 is extended to 4224 in A and $A^\vee$. Hence, $A = {}^\infty(4422)^\infty$ and $\mathcal{M}(A) = [(2244)^\infty] + [(2244)^\infty] = \sqrt{7}$. Therefore, $\mathcal{M}(A) \geq \sqrt{7}$. □

Theorem 3.2.11 *The interval*

$$\left(\frac{\sqrt{143}}{5}, \sqrt{7}\right) = (2.391\ldots, 2.645\ldots)$$

is a maximal gap in $\mathscr{M}(\mathbf{H}_6)$. Moreover, $\mathcal{M}(A) = \frac{\sqrt{143}}{5}$ for $A = {}^\infty(2343)^\infty$, $\mathcal{M}(B) = \sqrt{7}$ for $B = {}^\infty(2244)^\infty$ and $\mathcal{M}(B)$ is a limit point of $\mathscr{M}(\mathbf{H}_6)$.

Proof. Let A be a bi-infinite Romik sequence. Assume that $\mathcal{M}(A) \in \left(\frac{\sqrt{143}}{5}, \sqrt{7}\right)$. If A or $A^\vee$ contains 1, then by Lemma 3.2.10 (1), $A = {}^\infty(15)^\infty$. Since A is periodic, $\mathcal{M}(A) = [(51)^\infty] + [(15)^\infty] = \sqrt{7}$. Hence, A and $A^\vee$ do not contain 1 and A only consists of $2, 3, 4$. From Lemma 3.2.10 (2), A or $A^\vee$ contains 42. Since $[P] \leq [(24)^\infty] = \sqrt{2}$ for an infinite Romik sequence P in A,

$$\begin{aligned} L(P^*4|23Q) &= [2\hat{P}] + [23Q] \geq [2(24)^\infty] + [23(24)^\infty] = 2.684\ldots, \\ L(P^*4|24Q) &= [2\hat{P}] + [24Q] \geq [2(24)^\infty] + [24(42)^\infty] = 2.726\ldots. \end{aligned}$$

Thus, A and $A^\vee$ do not contain 423, 424, 342. From Lemma 3.2.10 (3), $\mathcal{M}(A) \geq \sqrt{7}$. Therefore, $\left(\frac{\sqrt{143}}{5}, \sqrt{7}\right)$ does not contain any Markoff numbers and is a maximal gap in $\mathscr{M}(\mathbf{H}_6)$.

On the other hand, we have

$$[(3234)^k3^\infty] + [(2343)^m3^\infty] \xrightarrow[k,m\to\infty]{} [(3234)^\infty] + [(2343)^\infty] = \mathcal{M}({}^\infty(3432)^\infty),$$

$$\begin{aligned} \mathcal{M}({}^\infty(3)(2343)^{k+m}3^\infty) &\geq L({}^\infty(3)(2343)^k|(2343)^m3^\infty) \\ &= [(3234)^k3^\infty] + [(2343)^m3^\infty] \end{aligned}$$

for all $k, m \geq 1$. Since $\mathcal{M}({}^\infty(3)(2343)^{k+m}3^\infty) \leq \frac{\sqrt{143}}{5} = \mathcal{M}({}^\infty(3432)^\infty)$ by Lemma 3.2.10 (2), $\mathcal{M}({}^\infty(3432)^\infty)$ is a limit point of $\mathscr{M}(\mathbf{H}_6)$. □

Theorem 3.2.12 *The interval*

$$\left(\sqrt{7}, \frac{13\sqrt{3} + 13\sqrt{7} + \sqrt{143}}{26}\right) = (2.645\ldots, 2.648\ldots)$$

is a maximal gap in $\mathscr{M}(\mathbf{H}_6)$. Moreover, $\mathcal{M}({}^\infty(4422)(3432)^\infty) = \frac{13\sqrt{3}+13\sqrt{7}+\sqrt{143}}{26}$ is a limit point of $\mathscr{M}(\mathbf{H}_6)$.

Proof. Let A be a bi-infinite Romik sequence. Suppose that $\mathcal{M}(A)$ belongs to $\left(\sqrt{7}, \frac{1}{26}(13\sqrt{3} + 13\sqrt{7} + \sqrt{143})\right)$. By Lemma 3.2.10 (1), A consists of 2, 3, 4. Moreover, by Lemma 3.2.10 (2), A or $A^\vee$ contains 42, and from the proof of Lemma 3.2.10 (3), A and $A^\vee$ do not contain 342, 424, 423, 4222. Thus, if A or $A^\vee$ contains 4223, say A, then 4223 is extended to 44223 in A. Then, we have

$$\mathcal{M}(A) \geq L(P^*44|223Q) = [22\hat{P}] + [223Q]$$

$$\geq [(2244)^\infty] + [2(2343)^\infty] = \frac{13\sqrt{3}+13\sqrt{7}+\sqrt{143}}{26}$$

for a section $P^*44|223Q$ of A or $A^\vee$. Hence, $\left(\sqrt{7}, \frac{1}{26}(13\sqrt{3}+13\sqrt{7}+\sqrt{143})\right)$ does not contain any Markoff numbers and is a maximal gap in $\mathscr{M}(\mathbf{H}_6)$.

On the other hand, we have

$$\begin{aligned}\mathcal{M}({}^\infty(4422)(3432)^k3^\infty) &\geq L({}^\infty(2244)|22(3432)^k3^\infty)\\ &= [(2244)^\infty] + [22(3432)^k3^\infty],\end{aligned}$$

$$[(2244)^\infty] + [22(3432)^k3^\infty] \xrightarrow[k\to\infty]{} [(2244)^\infty] + [2(2343)^\infty] = \mathcal{M}({}^\infty(4422)(3432)^\infty)$$

for all $k \geq 1$. Since

$$[23\dots] + [32\dots] \leq N_2N_3\cdot 0 + N_3N_2\cdot 0 = \frac{7\sqrt{3}}{5} < \sqrt{7},$$

we have

$$\mathcal{M}({}^\infty(4422)(3432)^k3^\infty) = [(2244)^\infty] + [22(3432)^k3^\infty].$$

Thus, $\mathcal{M}({}^\infty(4422)(3432)^\infty)$ is a limit point of $\mathscr{M}(\mathbf{H}_6)$. □

Theorem 3.2.13 *The interval*

$$\left(\frac{2\sqrt{506}}{19}, \frac{2\sqrt{2803333}}{1405}\right) = (2.3678361\dots, 2.3833675\dots)$$

is a maximal gap in $\mathscr{M}(\mathbf{H}_6)$. *Moreover,* $\mathcal{M}({}^\infty(433233)^\infty) = \frac{2\sqrt{506}}{19}$ *and* $\mathcal{M}({}^\infty(4343223)^\infty) = \frac{2\sqrt{2803333}}{1405}$.

Proof. Let A be a bi-infinite Romik sequence. Suppose that $\mathcal{M}(A) \in \left(\frac{2\sqrt{506}}{19}, \frac{2\sqrt{2803333}}{1405}\right)$. By Lemma 3.2.10 (1), A consists of 2, 3, 4. Moreover, from Lemma 3.2.10 (2), A and $A^\vee$ do not contain 42. Since $[(2343)^\infty] + [4(4323)^\infty] = 2.1232\dots$ and $2[(3234)^\infty] = 2.0452\dots$, we have

$$\begin{aligned}L(P^*4|3Q) &= [2\hat{P}] + [3Q]\\ &\geq [2(2343)^\infty] + [(3432)^\infty] = 2.3038\dots\\ &> \max\left\{[(2343)^\infty] + [4(4323)^\infty], 2[(3234)^\infty]\right\}\\ &\geq \max\left\{L(P'^*4|4Q'), L(P''^*3|3Q'')\right\}\end{aligned}$$

where $P^*4|3Q$ is a section of A or $A^\vee$, and $P'^*44Q', P''^*33Q'' \in \{2,3,4\}^{\mathbb{Z}}$ are bi-infinite Romik sequences which do not contain 24, 42. Then, it is enough to consider $L(\dots4|3\dots)$, $L(\dots3|2\dots)$ for $\mathcal{M}(A)$. Assume that A and $A^\vee$ do not contain 234. For a section $P^*|Q$ of A, we have

$$[P] \leq [(332334)^\infty], [Q] \leq [(233433)^\infty]$$

where infinite Romik sequences P, Q start with 3,2, respectively. Thus, we have

$$\mathcal{M}(A) \leq [(233433)^\infty] + [(332334)^\infty] = \mathcal{M}({}^\infty(433233)^\infty) = \frac{2\sqrt{506}}{19}.$$

Hence, A or $A^\vee$ contains 234. Suppose that both A and $A^\vee$ do not contain 3234 and 2343. Since

$$\begin{aligned}
L(P^*4|32Q) &= [2\hat{P}] + [32Q] \leq [(22234443)^\infty] + [(32223444)^\infty] = 2.3503\dots,\\
L(P^*4|33Q) &= [2\hat{P}] + [33Q] \leq [(233433)^\infty] + [(332334)^\infty] \leq \mathcal{M}({}^\infty(433233)^\infty),\\
L(P^*4|34Q) &= [2\hat{P}] + [34Q] \leq [(233433)^\infty] + [344(32223444)^\infty] = 2.3463\dots,
\end{aligned}$$

we have $\mathcal{M}(A) \leq \mathcal{M}({}^\infty(433233)^\infty)$. Hence, A or $A^\vee$ contains 3432 or 4323. On the other hand, we can check the following 5 cases.

(1) If 234323 or 343234 appears in A or $A^\vee$, then

$$\mathcal{M}(A) \geq L(P^*343|234Q) \geq [323(2343)^\infty] + [234(4323)^\infty] = 2.3910352\dots.$$

(2) If 234322 or 443234 appears in A or $A^\vee$, then

$$\mathcal{M}(A) \geq L(P^*443|234Q) \geq [322(3432)^\infty] + [234(4323)^\infty] = 2.3890563\dots.$$

(3) If 334322 or 443233 appears in A or $A^\vee$, then

$$\mathcal{M}(A) \geq L(P^*443|233Q) \geq [322(3432)^\infty] + [233(2343)^\infty] = 2.3861379\dots.$$

(4) If 434323 or 343232 appears in A or $A^\vee$, then

$$\mathcal{M}(A) \geq L(P^*343|232Q) \geq [323(2343)^\infty] + [232(3432)^\infty] = 2.3853441\dots.$$

(5) If 334323 or 343233 appears in A or $A^\vee$, then

$$\mathcal{M}(A) \geq L(P^*343|233Q) \geq [323(2343)^\infty] + [233(2343)^\infty] = 2.3881168\dots.$$

Thus, 3432 and 4323 are extended to $4\underline{3432}2$ and $4\underline{4323}2$ in A and $A^\vee$, respectively, where $a\underline{w}b$ is an extension with a to the left and b to the right of w. If 4434322 or 4443232 occurs in A or $A^\vee$, then we have

$$\mathcal{M}(A) \geq L(P^*4434|322Q) \geq [2322(2343)^\infty] + [322(3432)^\infty] = 2.3836369\ldots,$$
$$\mathcal{M}(A) \geq L(P^*4443|232Q) \geq [3222(2343)^\infty] + [232(3432)^\infty] = 2.3835556\ldots.$$

Hence, 3432 and 4323 are extended to $34\underline{3432}2$ and $34\underline{4323}2$ in A and $A^\vee$, respectively. Since 4323 and 3432 appears in $4432323 = (3434322)^{\vee *}$ and $4343223 = (3443232)^{\vee *}$, 3432 and 4323 are extended to $34\underline{3432}23$ and $34\underline{4323}23$ in A and $A^\vee$, respectively. Since $34343223 = (34432323)^{\vee *}$, we assume 34432323 appears in A or $A^\vee$. If 334432323 or 434432323 or 344323233 or 344323232 appears in A or $A^\vee$, then we have

$$\mathcal{M}(A) \geq L(P^*33443|2323Q) \geq [3223(3432)^\infty] + [2323(4323)^\infty] = 2.3833785\ldots,$$
$$\mathcal{M}(A) \geq L(P^*43443|2323Q) \geq [32232P] + [2323(4323)^\infty] > 2.3833785\ldots,$$
$$\mathcal{M}(A) \geq L(P^*3443|23233Q) \geq [3223(4323)^\infty] + [2323(3432)^\infty] = 2.3833843\ldots,$$
$$\mathcal{M}(A) \geq L(P^*3443|23232Q) \geq [3223(4323)^\infty] + [23232Q] > 2.3833843\ldots.$$

Thus, 2344323234 occurs in A or $A^\vee$. Since 3234 is extended to $432\underline{3234}432$ in A and $A^\vee$, 2344323234 is extended to $\underline{2344323234}432$ in A and $A^\vee$. Hence, 2344323234432 occurs in A or $A^\vee$. If 22344323234432 occurs in A or $A^\vee$, then we have

$$\begin{aligned}\mathcal{M}(A) &\geq L(P^*223443|23234432Q) \\ &\geq [322344(4323)^\infty] + [23234432(2343)^\infty] = 2.3833686\ldots.\end{aligned}$$

Hence, $(3234432)^2$ occurs in A or $A^\vee$. Since 3234 is extended to $432\underline{3234}432$ in A and $A^\vee$, $432(3234432)^2$ occurs in A or $A^\vee$. Applying the same argument to A and $A^\vee$, ${}^\infty(3443232)^\infty$ occurs in A or $A^\vee$, and $\mathcal{M}(A) = \frac{2\sqrt{2803333}}{1405}$. Hence, $\left(\frac{2\sqrt{506}}{19}, \frac{2\sqrt{2803333}}{1405}\right)$ does not contain any Markoff numbers in $\mathscr{M}(\mathbf{H}_6)$. □

Figure 3.6 Gaps in $\mathscr{M}(\mathbf{H}_6)$

Proposition 3.2.14. The gap $\left(\frac{\sqrt{143}}{5}, \sqrt{7}\right)$ is the longest gap in $\mathscr{M}(\mathbf{H}_6)$.

Proof. Let us check some Markoff numbers for $k \geq 0$:

$$\mathcal{M}({}^{\infty}(2)1^k(2)^{\infty}) = \frac{\sqrt{13}}{\sqrt{3}} + \sqrt{3}k,\ \mathcal{M}({}^{\infty}(15142)1^k(15142)^{\infty}) = \frac{2\sqrt{104}}{7} + \sqrt{3}k,$$
$$\mathcal{M}({}^{\infty}(2)31^k(2)^{\infty}) = \frac{4}{\sqrt{3}} + \sqrt{3}k,\ \mathcal{M}({}^{\infty}(142)1^k(142)^{\infty}) = \sqrt{10} + \sqrt{3}k,$$
$$\mathcal{M}({}^{\infty}(2343)1^k(2343)^{\infty}) = \frac{\sqrt{143}}{5} + \sqrt{3}k,\ \mathcal{M}({}^{\infty}(1452)1^k(1452)^{\infty}) = \sqrt{11} + \sqrt{3}k,$$
$$\mathcal{M}({}^{\infty}(2244)1^k(2244)^{\infty}) = \sqrt{7} + \sqrt{3}k,\ \mathcal{M}({}^{\infty}(132)1^k(132)^{\infty}) = \frac{\sqrt{435}}{6} + \sqrt{3}k,$$
$$\mathcal{M}({}^{\infty}(24)1^k(24)^{\infty}) = \sqrt{8} + \sqrt{3}k,\ \mathcal{M}({}^{\infty}(14)1^k(14)^{\infty}) = \frac{2\sqrt{10}}{\sqrt{3}} + \sqrt{3}k.$$

First of all, since $[2\cdots] \leq \sqrt{3}$, we have

$$L(P^*|Q) \leq 2\sqrt{3} \tag{3.15}$$

for infinite Romik sequences $\hat{P}, Q$ not starting with 1.

1) $\mathcal{M}({}^{\infty}(2)1^k(2)^{\infty}) = \frac{\sqrt{13}}{\sqrt{3}} + \sqrt{3}k$.

For $k = 0$, $\mathcal{M}({}^{\infty}(2)^{\infty}) = [4^{\infty}] + [2^{\infty}] = \frac{\sqrt{13}-1}{2\sqrt{3}} + \frac{\sqrt{13}+1}{2\sqrt{3}} = \sqrt{\frac{13}{3}}$. For $k \geq 1$, since $[5\cdots], [4\cdots] \leq \frac{\sqrt{3}}{2}$ and $[2\cdots] \leq \sqrt{3}$, we have

$$L(P^*2|2Q), L(P^*4|4Q), L(P^*5|4Q), L(P^*4|5Q) \leq \frac{3\sqrt{3}}{2} < \sqrt{\frac{13}{3}} + \sqrt{3}.$$

Thus,

$$\mathcal{M}({}^{\infty}(2)1^k(2)^{\infty}) = [4^{\infty}] + [1^k 2^{\infty}] = \sqrt{3}k + [4^{\infty}] + [2^{\infty}] = \sqrt{3}k + \sqrt{\frac{13}{3}}.$$

2) $\mathcal{M}({}^{\infty}(2)31^k(2)^{\infty}) = \frac{4}{\sqrt{3}} + \sqrt{3}k$.

For $k = 0$, note that

$$L({}^{\infty}2|2Q) = [4^{\infty}] + [2Q] \leq [4^{\infty}] + [232^{\infty}] \leq [4^{\infty}] + [235^{\infty}] \leq \frac{\sqrt{13}-1}{2\sqrt{3}} + \frac{4\sqrt{3}}{5},$$
$$L(P^*2|2^{\infty}) = [4\hat{P}] + [2^{\infty}] \leq [4434^{\infty}] + [2^{\infty}] \leq [441^{\infty}] + [2^{\infty}] \leq \frac{4}{3\sqrt{3}} + \frac{\sqrt{13}+1}{2\sqrt{3}}$$

for infinite Romik sequences P, Q starting with 2 or 3. Since the above two values are less than $\frac{4}{\sqrt{3}}$,

$$\mathcal{M}({}^\infty(2)3(2)^\infty) = [34^\infty] + [2^\infty] = \frac{7-\sqrt{13}}{2\sqrt{3}} + \frac{1+\sqrt{13}}{2\sqrt{3}} = \frac{4}{\sqrt{3}}.$$

For $k \geq 1$, by (3.15) and $2\sqrt{3} < \sqrt{3} + \frac{4}{\sqrt{3}}$,

$$\mathcal{M}({}^\infty(2)31^k(2)^\infty) = [34^\infty] + [1^k2^\infty] = \sqrt{3}k + [34^\infty] + [2^\infty] = \sqrt{3}k + \frac{4}{\sqrt{3}}.$$

3) $\mathcal{M}({}^\infty(2343)1^k(2343)^\infty) = \frac{\sqrt{143}}{5} + \sqrt{3}k$.

Theorem 3.2.11 implies the case for $k = 0$. For $k \geq 1$, by (3.15) and $2\sqrt{3} < \sqrt{3} + \frac{\sqrt{143}}{5}$,

$$\begin{aligned}&\mathcal{M}({}^\infty(2343)1^k(2343)^\infty) = [(3234)^\infty] + [1^k(2343)^\infty]\\ =&\sqrt{3}k + [(3234)^\infty] + [(2343)^\infty] = \sqrt{3}k + \frac{\sqrt{143}}{5}.\end{aligned}$$

4) $\mathcal{M}({}^\infty(2244)1^k(2244)^\infty) = \sqrt{7} + \sqrt{3}k$.

Theorem 3.2.11 implies the case for $k = 0$. For $k \geq 1$, by (3.15) and $2\sqrt{3} < \sqrt{3} + \sqrt{7}$,

$$\begin{aligned}&\mathcal{M}({}^\infty(2244)1^k(2244)^\infty) = [(2244)^\infty] + [1^k(2244)^\infty]\\ =&\sqrt{3}k + [(2244)^\infty] + [(2244)^\infty] = \sqrt{3}k + \sqrt{7}.\end{aligned}$$

5) $\mathcal{M}({}^\infty(24)1^k(24)^\infty) = \sqrt{8} + \sqrt{3}k$.

For $k = 0$, $\mathcal{M}({}^\infty(42)^\infty) = [(24)^\infty] + [(24)^\infty] = 2\sqrt{2}$. For $k \geq 1$, by (3.15) and $2\sqrt{3} < \sqrt{3} + \sqrt{8}$,

$$\begin{aligned}&\mathcal{M}({}^\infty(24)1^k(24)^\infty) = [(24)^\infty] + [1^k(24)^\infty]\\ =&\sqrt{3}k + [(24)^\infty] + [(24)^\infty] = \sqrt{3}k + 2\sqrt{2}.\end{aligned}$$

6) $\mathcal{M}({}^\infty(15142)1^k(15142)^\infty) = \frac{2\sqrt{104}}{7} + \sqrt{3}k$.

For $k = 0$, we check

$$\begin{aligned}&L({}^\infty(15245)|(15245)^\infty) = L({}^\infty(52451)|(52451)^\infty) = [(15245)^\infty] + [(52451)^\infty],\\ &L({}^\infty(14215)|(14215)^\infty) = L({}^\infty(51421)|(51421)^\infty)\end{aligned}$$

$$= L({}^{\infty}(15142)|(15142)^{\infty}) = L({}^{\infty}(42151)|(42151)^{\infty}) = [(14215)^{\infty}] + [(51421)^{\infty}]$$

and $[5\cdots]+[2\cdots], [4\cdots]+[4\cdots] \le \frac{4}{\sqrt{3}}$. Thus,

$$\mathcal{M}({}^{\infty}(15142)^{\infty}) = [(14215)^{\infty}] + [(51421)^{\infty}] = \frac{2\sqrt{104}}{7}.$$

For $k \ge 1$, since $[5\cdots], [4\cdots] \le \frac{\sqrt{3}}{2}$, we have

$$L(P^{*}2|14Q), L(P^{*}2|15Q), L(P^{*}5|14Q), L(P^{*}5|15Q) \le 2\sqrt{3}. \tag{3.16}$$

Combining with (3.15) and (3.16),

$$\begin{aligned}\mathcal{M}({}^{\infty}(15142)1^{k}(15142)^{\infty}) &= [(42151)^{\infty}] + [1^{k}(15142)^{\infty}]\\ =&\sqrt{3}k + [(42151)^{\infty}] + [(15142)^{\infty}] = \frac{2\sqrt{104}}{7} + \sqrt{3}k.\end{aligned}$$

7) $\mathcal{M}({}^{\infty}(142)1^{k}(142)^{\infty}) = \sqrt{10} + \sqrt{3}k$.

For $k = 0$, since

$$[5\cdots]+[2\cdots], [4\cdots]+[4\cdots] \le \frac{4}{\sqrt{3}}, \quad 2[22\cdots] \le \frac{14}{3\sqrt{3}},$$

we have

$$L({}^{\infty}(452)|(452)^{\infty}), L({}^{\infty}(245)|(245)^{\infty}), L({}^{\infty}(214)|(214)^{\infty}) < \sqrt{10}.$$

Hence, $\mathcal{M}({}^{\infty}(142)^{\infty}) = [(142)^{\infty}] + [(421)^{\infty}] = \sqrt{10}$. For $k \ge 1$, combining with (3.15) and (3.16),

$$\begin{aligned}\mathcal{M}({}^{\infty}(142)1^{k}(142)^{\infty}) &= [(421)^{\infty}] + [1^{k}(142)^{\infty}]\\ =&\sqrt{3}k + [(421)^{\infty}] + [(142)^{\infty}] = \sqrt{10} + \sqrt{3}k.\end{aligned}$$

8) $\mathcal{M}({}^{\infty}(1452)1^{k}(1452)^{\infty}) = \sqrt{11} + \sqrt{3}k$.

For $k = 0$, note that $({}^{\infty}(1452)^{\infty})^{\vee} = {}^{\infty}(5214)^{\infty}$. Thus,

$$\begin{aligned}\mathcal{M}({}^{\infty}(1452)^{\infty}) =& \max\left\{L({}^{\infty}(1452)|(1452)^{\infty}), L({}^{\infty}(5214)|(5214)^{\infty})\right\}\\ =& \max\left\{[(4521)^{\infty}] + [(1452)^{\infty}], [(2145)^{\infty}] + [(5214)^{\infty}]\right\}\\ =&[(4521)^{\infty}] + [(1452)^{\infty}] = \sqrt{11}.\end{aligned}$$

For $k \geq 1$, by (3.15) and (3.16),

$$\mathcal{M}({}^{\infty}(1452)1^k(1452)^{\infty}) = [(4521)^{\infty}] + [1^k(1452)^{\infty}]$$

$$= \sqrt{3}k + [(4521)^{\infty}] + [(1452)^{\infty}] = \sqrt{11} + \sqrt{3}k.$$

9) $\mathcal{M}({}^{\infty}(132)1^k(132)^{\infty}) = \frac{\sqrt{435}}{6} + \sqrt{3}k$.

For $k = 0$, by (3.15), $\mathcal{M}({}^{\infty}(132)^{\infty}) = [(132)^{\infty}] + [(431)^{\infty}] = \frac{\sqrt{435}}{6}$. For $k \geq 1$, since $[4\cdots], [3\cdots] \leq \frac{2}{\sqrt{3}}$ and $[14\cdots], [13\cdots] \leq \frac{5}{\sqrt{3}}$,

$$L(P^*2|13Q), L(P^*21|3Q) \leq \frac{7}{\sqrt{3}} < \frac{\sqrt{435}}{6} + \sqrt{3}.$$

Hence, by (3.15),

$$\mathcal{M}({}^{\infty}(132)1^k(132)^{\infty}) = [(431)^{\infty}] + [1^k(132)^{\infty}]$$

$$= \sqrt{3}k + [(431)^{\infty}] + [(132)^{\infty}] = \frac{\sqrt{435}}{6} + \sqrt{3}k.$$

10) $\mathcal{M}({}^{\infty}(14)1^k(14)^{\infty}) = \frac{2\sqrt{10}}{\sqrt{3}} + \sqrt{3}k$.

For $k = 0$, $\mathcal{M}({}^{\infty}(14)^{\infty}) = [(21)^{\infty}] + [(14)^{\infty}] = \frac{2\sqrt{10}}{\sqrt{3}}$. For $k \geq 1$, by (3.15), $L(P^*14|14Q), L(P^*41|41Q) \leq 3\sqrt{3} < \frac{2\sqrt{10}}{\sqrt{3}} + \sqrt{3}$. Hence, by (3.15),

$$\mathcal{M}({}^{\infty}(14)1^k(14)^{\infty}) = [(21)^{\infty}] + [1^k(14)^{\infty}]$$

$$= \sqrt{3}k + [(21)^{\infty}] + [(14)^{\infty}] = \frac{2\sqrt{10}}{\sqrt{3}} + \sqrt{3}k.$$

Therefore, any interval longer than the length of $\left(\frac{\sqrt{143}}{5}, \sqrt{7}\right)$ contains at least one point of 10 types for some k, which implies $\left(\frac{\sqrt{143}}{5}, \sqrt{7}\right)$ is the longest gap in $\mathscr{M}(\mathbf{H}_6)$. $\square$

Romik Sequences	Markoff numbers
${}^{\infty}(2)1^k(2)^{\infty}$	$\frac{\sqrt{13}}{\sqrt{3}} + \sqrt{3}k = 2.081\cdots + \sqrt{3}k$
${}^{\infty}(2)1^k3(2)^{\infty}$	$\frac{4}{\sqrt{3}} + \sqrt{3}k = 2.309\cdots + \sqrt{3}k$
${}^{\infty}(2343)1^k(2343)^{\infty}$	$\frac{\sqrt{143}}{5} + \sqrt{3}k = 2.391\cdots + \sqrt{3}k$
${}^{\infty}(2244)1^k(2244)^{\infty}$	$\sqrt{7} + \sqrt{3}k = 2.645\cdots + \sqrt{3}k$
${}^{\infty}(24)1^k(24)^{\infty}$	$\sqrt{8} + \sqrt{3}k = 2.828\cdots + \sqrt{3}k$
${}^{\infty}(15142)1^k(15142)^{\infty}$	$\frac{2\sqrt{104}}{7} + \sqrt{3}k = 2.913\cdots + \sqrt{3}k$
${}^{\infty}(142)1^k(142)^{\infty}$	$\sqrt{10} + \sqrt{3}k = 3.162\cdots + \sqrt{3}k$
${}^{\infty}(1452)1^k(1452)^{\infty}$	$\sqrt{11} + \sqrt{3}k = 3.316\cdots + \sqrt{3}k$
${}^{\infty}(132)1^k(132)^{\infty}$	$\frac{\sqrt{435}}{6} + \sqrt{3}k = 3.476\cdots + \sqrt{3}k$
${}^{\infty}(14)1^k(14)^{\infty}$	$\frac{2\sqrt{10}}{\sqrt{3}} + \sqrt{3}k = 3.651\cdots + \sqrt{3}k$

Table 3.1 Markoff numbers in $\mathscr{M}(\mathbf{H}_6) \cap [\frac{\sqrt{13}}{\sqrt{3}} + \sqrt{3}k, \frac{2\sqrt{10}}{\sqrt{3}} + \sqrt{3}k]$ for $k \geq 0$

Chapter 4

Combinatorics on words

In this chapter, we introduce a Sturmian word and investigate its characterizations. For the next chapter, we define the exponent of repetition and we look into the exponent of repetition of Sturmian words.

4.1 Sturmian words

Let us consider a finite or countably infinite set $\mathcal{A}$. We call $\mathcal{A}$ an *alphabet* and an element of $\mathcal{A}$ a *letter*. A *word* $\mathbf{x}$ over $\mathcal{A}$ is a finite or infinite sequence of finite letters. For each integer $n \geq 1$, $p(n, \mathbf{x})$ is defined by the number of distinct subwords of length n appearing in the word $\mathbf{x}$ and is called by the *subword complexity* of $\mathbf{x}$. Morse and Hedlund showed that an infinite word is eventually periodic if and only if its subword complexity is bounded [50]. Thus, a non-eventually periodic word $\mathbf{x}$ with the smallest subword complexity satisfies $p(n, \mathbf{x}) = n + 1$ for all $n \geq 1$.

Definition 4.1.1. A *Sturmian word* is an infinite word $\mathbf{x}$ over $\mathcal{A} = \{0, 1\}$ satisfying $p(n, \mathbf{x}) = n + 1$ for all $n \geq 1$.

Example 4.1.2. Let $\mathbf{f}_n$ be a sequence of finite words such that $\mathbf{f}_0 = 0$, $\mathbf{f}_1 = 1$, and $\mathbf{f}_{n+2} = \mathbf{f}_{n+1}\mathbf{f}_n$ for $n \geq 0$. Let $\mathbf{f} = \lim_{k\to\infty} \mathbf{f}_k$. Then,

$$\mathbf{f} = 1011010110110\ldots.$$

The sequence $\mathbf{f}$ is Sturmian. We call $\mathbf{f}$ *Fibonacci word*.

First, we can characterize a Sturmian word by the coding of an irrational rotation.

Definition 4.1.3. For $\theta \in (0,1)$ and a real number ρ, let

$$s_n = \begin{cases} 0 & \text{if} \quad \lfloor (n+1)\theta + \rho \rfloor - \lfloor n\theta + \rho \rfloor = 0, \\ 1 & \text{if} \quad \lfloor (n+1)\theta + \rho \rfloor - \lfloor n\theta + \rho \rfloor = 1, \end{cases}$$

and

$$s'_n = \begin{cases} 0 & \text{if} \quad \lceil (n+1)\theta + \rho \rceil - \lceil n\theta + \rho \rceil = 0, \\ 1 & \text{if} \quad \lceil (n+1)\theta + \rho \rceil - \lceil n\theta + \rho \rceil = 1. \end{cases}$$

We call $\mathbf{s}_{\theta,\rho} := (s_n)$ a lower mechanical word and $\mathbf{s}'_{\theta,\rho} := (s'_n)$ a upper mechanical word of *slope* θ and *intercept* ρ, respectively. We say a mechanical word $s_{\theta,\rho}$ (or $s'_{\theta,\rho}$) is *irrational* if θ is irrational.

The next characterization of a Sturmian word is that subwords of the same length include nearly same number of 1. We define *the height* $h(u)$ of a finite word u by the number of 1 in u.

Definition 4.1.4. A finite or infinite word $\mathbf{u}$ is a *balanced* word if

$$|h(u) - h(v)| \leq 1$$

for all subword u, v in $\mathbf{u}$ with the same length.

Morse and Hedlund characterized Sturmian words [50].

Proposition 4.1.5. For an infinite word $\mathbf{x}$, the following statements are equivalent.

(1) $\mathbf{x}$ is Sturmian.

(2) $\mathbf{x}$ is balanced and non-periodic.

(3) $\mathbf{x}$ is irrational mechanical.

Let $\theta \in (0,1) \setminus \mathbb{Q}$. The *rotation* of angle θ is a map R_θ from $[0,1)$ into itself defined by

$$R_\theta(x) = x + \theta \ (\mathrm{mod}\ 1).$$

The following proposition means that a Sturmian word is a coding of an orbit of a point on $[0,1)$ under R_θ.

Proposition 4.1.6. If $\mathbf{s}_{\theta,\rho} = (s_n)$ and $\mathbf{s}'_{\theta,\rho} = (s'_n)$, then

$$s_n = \begin{cases} 0, & \text{if } R_\theta^n(\rho) \in [0, 1-\theta), \\ 1, & \text{if } R_\theta^n(\rho) \in [1-\theta, 1), \end{cases} \quad \text{and } s'_n = \begin{cases} 0, & \text{if } R_\theta^n(\rho) \in (0, 1-\theta], \\ 1, & \text{if } R_\theta^n(\rho) \in (1-\theta, 1]. \end{cases}$$

Definition 4.1.7. A *characteristic* word $\mathbf{c}_\theta$ of slope θ is defined by

$$\mathbf{c}_\theta := \mathbf{s}_{\theta,\theta} = \mathbf{s}'_{\theta,\theta}.$$

Let $\theta := [0; a_1, a_2, \ldots]$. We define a sequence $\{M_k\}_{k\geq 0}$ in the following way: Let us define $M_0 := 0$, $M_1 := 0^{a_1-1}1$, and $M_{k+1} := M_k^{a_{k+1}} M_{k-1}$ for $k \geq 1$. Then, the characteristic Sturmian word of slope θ is obtained by

$$\mathbf{c}_\theta := \lim_{k\to\infty} M_k. \tag{4.1}$$

For a non-empty finite word V, let us denote by V^- the word V with the last letter removed. Let $k \geq 1$. Note that $M_k M_{k-1}$ and $M_{k-1}M_k$ are identical, except for the last two letters [43, Proposition 2.2.2]. Let $\widetilde{M_k} := M_k M_{k-1}^{--} = M_{k-1} M_k^{--}$. The last two letters of $M_k M_{k-1}$ is 01 (resp., 10) if and only if the last two letters of $M_{k-1}M_k$ is 10 (resp., 01). We denote by D_k, D'_k the last two letters such that $M_k M_{k-1} = \widetilde{M_k} D_k$, $M_{k-1} M_k = \widetilde{M_k} D'_k$, respectively.

From now on, let $\mathbf{x} = x_1 x_2 \ldots$ be a Sturmian word of slope θ. By Lemma 7.2 in [15], for any $k \geq 1$, there exists a unique word W_k satisfying one of the following cases

(i) $\mathbf{x} = W_k M_k \widetilde{M_k} \ldots$, where W_k is a non-empty suffix of M_k,

(ii) $\mathbf{x} = W_k M_{k-1} M_k \widetilde{M_k} \ldots$, where W_k is a non-empty suffix of M_k,

(iii) $\mathbf{x} = W_k M_k \widetilde{M_k} \ldots$, where W_k is a non-empty suffix of M_{k-1}.

For case (i) and case (ii), there exist q_k non-empty suffices of M_k. For case (iii), there exist q_{k-1} non-empty suffices of M_{k-1}. Lemma 7.2 in [15] also gives that all the $(2q_k + q_{k-1})$ cases are mutually exclusive. For each $k \geq 1$, we say that $\mathbf{x}$ *belongs to case (i), (ii), (iii) at level* k if W_k satisfies case (i), (ii), (iii), respectively. We denote by $\mathcal{C}_k^{(\mathrm{i})}$, $\mathcal{C}_k^{(\mathrm{ii})}$, $\mathcal{C}_k^{(\mathrm{iii})}$ the set of Sturmian words which belong to case (i), (ii), (iii) at level k, respectively. For each $\mathbf{x}$, we have an infinite sequence of (i), (ii) and (iii) for which case $\mathbf{x}$ belongs to at level $1, 2, \ldots$, called the *locating chain* of $\mathbf{x}$. In the locating chain of $\mathbf{x}$, let $u^d := \underbrace{uu\ldots u}_{d}$ where u is a finite word of (i),(ii),(iii).

Example 4.1.8. Let $\mathbf{x} = \mathbf{c}_\varphi$, i.e. the characteristic Sturmian word of slope $\varphi = [0; \overline{1}]$. Since $\mathbf{x}$ starts with $M_{k+1}M_kM_{k+1} = M_kM_{k-1}M_k\widetilde{M}_kD_k$ for any $k \geq 1$, $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{ii})}$ and $W_k = M_k$ for all $k \geq 1$. Hence, the locating chain of $\mathbf{x}$ is $\overline{\text{(ii)}}$.

Example 4.1.9. Let $\mathbf{x} = 1\mathbf{c}_\varphi$. Since $\mathbf{x}$ starts with $1M_{k+2} = 1M_kM_{k-1}M_k = 1M_k\widetilde{M}_kD'_k$ for any $k \geq 1$, $W_k = 1$ for $k \geq 1$. Moreover, $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{i})}$ if k is odd, and $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{iii})}$ if k is even. Hence, the locating chain of $\mathbf{x}$ is $\overline{\text{(i)(iii)}}$.

Example 4.1.10. Let $\mathbf{x} = 10101M_4M_5\ldots$. Since $\mathbf{x}$ starts with $W_1M_0M_1 = 101$, $\mathbf{x} \in \mathcal{C}_1^{(\mathrm{ii})}$ and $W_2 = W_1M_0 = 10$. Since $\mathbf{x}$ starts with $W_2M_2\widetilde{M}_2 = 10101$, $\mathbf{x} \in \mathcal{C}_2^{(\mathrm{i})}$ and $W_3 = W_2 = 10$. Since $\mathbf{x}$ starts with $W_3M_3M_4 = W_3M_3\widetilde{M}_3D_3$, $\mathbf{x} \in \mathcal{C}_3^{(\mathrm{iii})}$ and $W_4 = W_3 = 10$. Since $\mathbf{x}$ starts with $W_4M_3M_4\widetilde{M}_4$, $\mathbf{x} \in \mathcal{C}_4^{(\mathrm{ii})}$ and $W_5 = W_4M_3 = 10101$. Moreover, for $k \geq 5$, $\mathbf{x}$ starts with $W_kM_{k-1}M_k\widetilde{M}_k$ and $W_{k+1} = W_kM_{k-1}$ where W_k is a non-empty suffix of M_k. Hence, $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{ii})}$ for $k \geq 5$. Therefore, the locating chain of $\mathbf{x} = \text{(ii)(i)(iii)}\overline{\text{(ii)}}$.

4.2 The exponent of repetition

Sturmian words have been studied in many different areas [6, 16, 34, 43]. Various complexities have been looked into characterize Sturmian words such as Cassaigne's versions of the recurrence function [18] or rectangle complexity [11]. In this section, we focus on a new complexity function $r(n, \mathbf{x})$ suggested by Bugeaud and Kim.

Definition 4.2.1. Given an infinite word $\mathbf{x} = x_1x_2\ldots$, let $r(n, \mathbf{x})$ denote the length of the smallest prefix in which some subword of length n occurs twice. More precisely,

$$r(n, \mathbf{x}) := \min\{m : x_jx_{j+1}\ldots x_{j+n-1} = x_{m-n+1}x_{m-n+2}\ldots x_m \text{ for some } 1 \leq j \leq m-n\}.$$

The *exponent of repetition* of $\mathbf{x}$ is defined by

$$\mathrm{rep}(\mathbf{x}) := \liminf_{n\to\infty} \frac{r(n, \mathbf{x})}{n}.$$

Example 4.2.2. In Example 4.1.2, $\mathbf{f} = 1011010110110\ldots$. By the definition of the Fibonacci word and (4.1), the slope of $\mathbf{f}$ is $\frac{\sqrt{5}+1}{2}$. Then,

$$r(1,\mathbf{f}) = 3,\ r(2,\mathbf{f}) = 5,\ r(3,\mathbf{f}) = 6, \text{ and } \operatorname{rep}(\mathbf{f}) = \frac{\sqrt{5}+1}{2}.$$

Remark. By definition, we obtain the following statements.

(1) $r(n+1,\mathbf{x}) \geq r(n,\mathbf{x}) + 1$ for $n \geq 1$.

(2) $\operatorname{rep}(c\mathbf{x}) = \operatorname{rep}(\mathbf{x})$ for any finite word c.

The exponent of repetition gives another characterization of Sturmian words and eventually periodic words.

Theorem 4.2.3 ([15, Theorem 2.3 and 2.4]) *The following statements hold.*

(1) $\mathbf{x}$ *is eventually periodic if and only if* $r(n,\mathbf{x}) \leq 2n$ *for all sufficiently large integers* n.

(2) $\mathbf{x}$ *is a Sturmian word if and only if* $r(n,\mathbf{x}) \leq 2n+1$ *for all* $n \geq 1$ *and equality holds for infinitely many* n.

We say a real number is a *Sturmian number* if there exists an integer $b \geq 2$ such that the b-ary expansion of the real number is a Sturmian word over $\{0, 1, \ldots, b-1\}$. For a Sturmian word $\mathbf{x}$ over $\{0, 1, \ldots, b-1\}$, we say $r_{\mathbf{x}} := \sum_{k\geq 1} \frac{x_k}{b^k}$ is a Sturmian number *associated with* $\mathbf{x}$. Recall that the irrationality exponent of a real number α is defined by

$$\mu(\alpha) := \sup\left\{ w \in \mathbb{R} : \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^w} \text{ for infinitely many } p, q \right\}.$$

Then, the exponent of repetition of $\mathbf{x}$ gives the irrationality exponent of the Sturmian number associated with $\mathbf{x}$.

Theorem 4.2.4 ([15, Theorem 4.5]) *For a Sturmian word* $\mathbf{x} = x_1x_2\ldots$, *an integer* $b \geq 2$, *and a Sturmian number* $r_{\mathbf{x}} = \sum_{k\geq 1} \frac{x_k}{b^k}$, *the irrationality exponent of* $r_{\mathbf{x}}$ *is given by*

$$\mu(r_{\mathbf{x}}) = \frac{\operatorname{rep}(\mathbf{x})}{\operatorname{rep}(\mathbf{x}) - 1}.$$

Thus, we look into the spectrum of the exponents of repetition of Sturmian words at the next chapter. From now on, we call $r_{\mathbf{x}}$ a *Sturmian number of slope* θ if $\mathbf{x}$ is a Sturmian word of slope θ.

Theorem 2.4 in [15] says that $r(n, \mathbf{x}) \leq 2n + 1$ for all $n \geq 1$ and equality holds for infinitely many n. Let

$$\Lambda(\mathbf{x}) := \{n \in \mathbb{N} : r(n, \mathbf{x}) = 2n + 1\}.$$

We have $\Lambda(\mathbf{x}) = \{n_1, n_2, \dots\}$ for an increasing sequence $\{n_i\}_{i \geq 1}$. From Lemma 5.3 in [15], $r(n, \mathbf{x}) \leq 2n$ implies $r(n, \mathbf{x}) \leq r(n-1, \mathbf{x}) + 1$. Thus,

$$r(n, \mathbf{x}) = r(n-1, \mathbf{x}) + 1 \quad \text{if } n \notin \Lambda(\mathbf{x}).$$

Hence, the sequence $\left\{\dfrac{r(n, \mathbf{x})}{n}\right\}_{n \geq 1}$ is decreasing on each interval $[n_i, n_{i+1} - 1]$. It gives

$$\operatorname{rep}(\mathbf{x}) = \liminf_{i \to \infty} \left(1 + \frac{n_i}{n_{i+1}}\right).$$

Chapter 5

The spectrum of the exponents of repetition

For any irrational number $\theta \in (0,1)$, let $\mathscr{L}(\theta)$ be the set of the exponents of repetition of Sturmian words of slope θ, i.e.

$$\mathscr{L}(\theta) = \{\text{rep}(\mathbf{x}) \,:\, \mathbf{x} \text{ is a Sturmian word of slope } \theta\}.$$

In this chapter, we mainly investigate $\mathscr{L}(\theta)$. We review the historical results of $\mathscr{L}(\theta)$ in Section . We determine the minimum of $\mathscr{L}(\theta)$ where θ has bounded partial quotients. In the last section, we look into $\mathscr{L}(\varphi)$ for $\varphi := \frac{\sqrt{5}-1}{2} = [0; \overline{1}]$.

5.1 The exponents of repetition of Sturmian words

Theorem 3.3 in [15] gives $\mathscr{L}(\theta) = \{1\}$ where θ has unbounded partial quotients. We find the minimum of $\mathscr{L}(\theta)$ where θ has bounded partial quotients. We keep the notations in Section 4.2.

The following lemma shows that for $k \geq 1$, there is a relation between cases which $\mathbf{x}$ belongs to at level k and $k+1$.

Lemma 5.1.1. Let $k \geq 1$. The following statements hold.

(1) If $\mathbf{x} \in \mathcal{C}_k^{(\text{i})}$ and $\mathbf{x} \in \mathcal{C}_{k+1}^{(\text{i})} \cup \mathcal{C}_{k+1}^{(\text{ii})}$, then $W_{k+1} = W_k M_k^t M_{k-1}$ for some $1 \leq t \leq a_{k+1} - 1$.

(2) If $\mathbf{x} \in \mathcal{C}_k^{(\text{i})}$ and $\mathbf{x} \in \mathcal{C}_{k+1}^{(\text{iii})}$, then $W_{k+1} = W_k$.

(3) If $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{ii})}$, then $\mathbf{x} \in \mathcal{C}_{k+1}^{(\mathrm{i})} \cup \mathcal{C}_{k+1}^{(\mathrm{ii})}$ and $W_{k+1} = W_k M_{k-1}$.

(4) If $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{iii})}$, then $\mathbf{x} \in \mathcal{C}_{k+1}^{(\mathrm{i})} \cup \mathcal{C}_{k+1}^{(\mathrm{ii})}$ and $W_{k+1} = W_k$.

Proof. In this proof, for all $k \geq 1$, let W_k be the unique prefix of $\mathbf{x}$ defined in which case $\mathbf{x}$ belongs to at level k. Note that $\widetilde{M}_k$ is a prefix of M_{k+1}^{--} by definition.

(1) Let $\mathbf{x} \in \mathcal{C}_{k+1}^{(\mathrm{i})} \cup \mathcal{C}_{k+1}^{(\mathrm{ii})}$. Note that $\mathbf{x}$ starts with $W_{k+1} M_k \widetilde{M}_k$ for the suffix W_{k+1} of M_{k+1}. If W_{k+1} is a non-empty suffix of M_{k-1}, then $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{iii})}$. It is a contradiction. If $W_{k+1} = W_k' M_{k-1}$ for some non-empty suffix W_k' of M_k, then $\mathbf{x}$ starts with $W_k' M_{k-1} M_k \widetilde{M}_k$. Thus, $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{ii})}$. It is a contradiction. Hence, $a_{k+1} > 1$ and $W_{k+1} = W_k'' M_k^t M_{k-1}$ for some $1 \leq t \leq a_{k+1} - 1$ and some non-empty suffix W_k'' of M_k. Consequently, $\mathbf{x}$ starts with $W_k'' M_k \widetilde{M}_k$. By the uniqueness of W_k, $W_k = W_k''$. It gives $W_{k+1} = W_k M_k^t M_{k-1}$.

(2) Let $\mathbf{x} \in \mathcal{C}_{k+1}^{(\mathrm{iii})}$. We have $\mathbf{x} = W_{k+1} M_{k+1} \widetilde{M}_{k+1}$ for a non-empty suffix W_{k+1} of M_k. Since $M_{k+1} \widetilde{M}_{k+1} = M_k M_{k+1}^{--} D_{k+1} M_{k+1}^{--}$, $\mathbf{x}$ starts with $W_{k+1} M_k \widetilde{M}_k$. By the uniqueness of W_k, $W_{k+1} = W_k$.

(3) Let $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{ii})}$. Note that $\mathbf{x}$ starts with $W_k M_{k-1} M_k \widetilde{M}_k$ for a non-empty suffix W_k of M_k. Assume that $\mathbf{x} \in \mathcal{C}_{k+1}^{(\mathrm{iii})}$. Since $\mathbf{x}$ starts with $W_{k+1} M_{k+1} \widetilde{M}_{k+1}$ for the suffix W_{k+1} of M_k, $\mathbf{x}$ starts with $W_{k+1} M_k \widetilde{M}_k$. Hence, $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{i})}$. It is a contradiction. Hence, $\mathbf{x} \in \mathcal{C}_{k+1}^{(\mathrm{i})} \cup \mathcal{C}_{k+1}^{(\mathrm{ii})}$. Thus, $\mathbf{x}$ starts with $W_{k+1} M_k \widetilde{M}_k$ for the suffix W_{k+1} of M_{k+1}. If W_{k+1} is a non-empty suffix of M_{k-1}, then $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{iii})}$. It is a contradiction. If $W_{k+1} = W_k' M_k^t M_{k-1}$ for some $1 \leq t \leq a_{k+1} - 1$ and some non-empty suffix W_k' of M_k, then $\mathbf{x}$ starts with $W_k' M_k \widetilde{M}_k$. Thus, $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{i})}$. It is a contradiction. Hence, $W_{k+1} = W_k' M_{k-1}$ for some non-empty suffix W_k' of M_k. By the uniqueness of W_k, $W_{k+1} = W_k M_{k-1}$.

(4) Let $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{iii})}$. Note that $\mathbf{x}$ starts with $W_k M_k \widetilde{M}_k$ for the suffix W_k of M_{k-1}. Assume that $\mathbf{x} \in \mathcal{C}_{k+1}^{(\mathrm{iii})}$. Since $\mathbf{x}$ starts with $W_{k+1} M_{k+1} \widetilde{M}_{k+1}$ for the suffix W_{k+1} of M_k, $\mathbf{x}$ starts with $W_{k+1} M_k \widetilde{M}_k$. Hence, $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{i})}$. It is a contradiction. Hence, $\mathbf{x} \in \mathcal{C}_{k+1}^{(\mathrm{i})} \cup \mathcal{C}_{k+1}^{(\mathrm{ii})}$. Thus, $\mathbf{x}$ starts with $W_{k+1} M_k \widetilde{M}_k$ for the suffix W_{k+1} of M_{k+1}. If $W_{k+1} = W_k' M_k^t M_{k-1}$ for $0 \leq t \leq a_{k+1} - 1$ and some non-empty suffix W_k' of M_k, then $\mathbf{x}$ starts with $W_k' M_{k-1} M_k \widetilde{M}_k$ or $W_k' M_k \widetilde{M}_k$. Thus, $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{i})} \cup \mathcal{C}_k^{(\mathrm{ii})}$. It is a contradiction. Hence, $W_{k+1} = W_{k-1}'$ for some non-empty suffix W_{k-1}' of M_{k-1}. By the uniqueness of W_k, $W_{k+1} = W_k$. □

k	$k+1$	The relation between W_{k+1} and W_k
case (i)	case (i) case (ii)	$W_{k+1} = W_k {M_k}^t M_{k-1}$ $(1 \leq t \leq a_{k+1}-1)$
	case (iii)	$W_{k+1} = W_k$
case (ii)	case (i) case (ii)	$W_{k+1} = W_k M_{k-1}$
case (iii)	case (i) case (ii)	$W_{k+1} = W_k$

Table 5.1 The relation between W_{k+1} and W_k in Sturmian words

Let

$$u_{t,k} = tq_k + q_{k-1} - 1, \qquad v_{t,k} = |W_k| + tq_k + q_{k-1} - 1,$$
$$u'_k = q_{k+1} - 1, \qquad v'_k = |W_k| + q_{k+1} - 1.$$

The following lemma shows that all of elements in $\Lambda(\mathbf{x}) \cap [u_{1,1}, \infty)$ are expressed in terms of q_k's and $|W_k|$'s.

Lemma 5.1.2. Let $k \geq 1$.

(1) If $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{i})} \cap \mathcal{C}_{k+1}^{(\mathrm{i})}$, then

$$\Lambda(\mathbf{x}) \cap [u_{1,k}, u_{1,k+1} - 1] = \begin{cases} \{v_{t,k}, u'_k\} & \text{for } t = a_{k+1} - 1, \\ \{v_{t,k}, u_{t+1,k}, v_{t+1,k}, u'_k\} & \text{for } t < a_{k+1} - 1 \end{cases}$$

where t satisfies $W_{k+1} = W_k {M_k}^t M_{k-1}$.

(2) If $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{i})} \cap \mathcal{C}_{k+1}^{(\mathrm{ii})}$, then

$$\Lambda(\mathbf{x}) \cap [u_{1,k}, u_{1,k+1} - 1] = \{v_{t,k}, u_{t+1,k}, v_{t+1,k}\}$$

where t satisfies $W_{k+1} = W_k {M_k}^t M_{k-1}$.

(3) If $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{i})} \cap \mathcal{C}_{k+1}^{(\mathrm{iii})} \cap \mathcal{C}_{k+2}^{(\mathrm{i})}$, then

$$\Lambda(\mathbf{x}) \cap [u_{1,k}, u_{1,k+2} - 1] = \{v'_k, u'_{k+1}\}.$$

(4) If $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{i})} \cap \mathcal{C}_{k+1}^{(\mathrm{iii})} \cap \mathcal{C}_{k+2}^{(\mathrm{ii})}$, then

$$\Lambda(\mathbf{x}) \cap [u_{1,k}, u_{1,k+2} - 1] = \{v'_k\}.$$

(5) If $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{ii})} \cap \mathcal{C}_{k+1}^{(\mathrm{i})}$, then

$$\Lambda(\mathbf{x}) \cap [u_{1,k}, u_{1,k+1} - 1] = \begin{cases} \{u_{1,k}\} & \text{for } a_{k+1} = 1, \\ \{u_{1,k}, v_{1,k}, u'_k\} & \text{for } a_{k+1} > 1. \end{cases}$$

(6) If $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{ii})} \cap \mathcal{C}_{k+1}^{(\mathrm{ii})}$, then

$$\Lambda(\mathbf{x}) \cap [u_{1,k}, u_{1,k+1} - 1] = \{u_{1,k}, v_{1,k}\}.$$

Proof. For $\mathbf{x} = x_1 x_2 \ldots$, let $x_i^j := x_i x_{i+1} \ldots x_j$.

(1) Suppose that $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{i})} \cap \mathcal{C}_{k+1}^{(\mathrm{i})}$. Since $M_k M_{k-1}$ is primitive, Lemma 7.1 in [15] implies that for

$$\mathbf{x} = W_k M_k \widetilde{M}_k \cdots = W_k M_k M_{k-1} M_k{}^{--} = W_k M_{k-1} M_k{}^{--} D_k M_{k-1}{}^{--} \ldots,$$

the first q_k subwords of length $u_{1,k}$ are mutually distinct. From $x_1^{u_{1,k}} = x_{q_k+1}^{u_{2,k}}$, $r(u_{1,k}, \mathbf{x}) = u_{2,k}$. Note that

$$\begin{aligned} \mathbf{x} &= W_{k+1} M_{k+1} \widetilde{M}_{k+1} \cdots = W_k M_k{}^t M_{k-1} M_k{}^{a_{k+1}} M_{k-1} \widetilde{M}_{k+1} \ldots \\ &= W_k M_k{}^{t+1} M_{k-1}{}^{--} D'_k M_k{}^{a_{k+1}-1} M_{k-1} \widetilde{M}_{k+1} \ldots . \end{aligned}$$

Since $x_1^{v_{t,k}-1} = x_{q_k+1}^{v_{t+1,k}-1}$, $r(v_{t,k}-1, \mathbf{x}) \le v_{t,k}-1$. The fact that $r(n+1, \mathbf{x}) \ge r(n, \mathbf{x})+1$ for any $n \ge 1$ gives

$$r(n, \mathbf{x}) = n + q_k$$

for $u_{1,k} \le n \le v_{t,k} - 1$. Moreover, we have $r(v_{t,k}, \mathbf{x}) \ge r(v_{t,k} - 1, \mathbf{x}) + 2$. Hence, $r(v_{t,k}, \mathbf{x}) = 2v_{t,k} + 1$ by Theorem 2.4 and Lemma 5.3 in [15]. Since $x_1^{u_{t+1,k}-1} = x_{v_{t,k}+2}^{v_{2t+1,k}+q_{k-1}-1}$, we have $r(u_{t+1,k}-1, \mathbf{x}) \le v_{2t+1,k}+q_{k-1}-1$. The fact that $r(n+1, \mathbf{x}) \ge r(n, \mathbf{x}) + 1$ for any $n \ge 1$ gives

$$r(n, \mathbf{x}) = n + v_{t,k} + 1$$

for $v_{t,k} \le n \le u_{t+1,k} - 1$. Note that

$$\begin{aligned} \mathbf{x} &= W_{k+1} M_{k+1} \widetilde{M}_{k+1} \cdots = W_k M_k{}^t M_{k-1} M_k{}^{a_{k+1}} M_{k-1} \widetilde{M}_{k+1} \ldots \\ &= W_k M_k{}^{t+1} M_{k-1}{}^{--} D'_k M_k{}^{a_{k+1}-1} M_{k-1} \widetilde{M}_{k+1} \cdots = W_k M_k{}^t M_{k-1} M_k{}^{t+1} M_{k-1} \ldots . \end{aligned}$$

It gives $r(u_{t+1,k}, \mathbf{x}) \geq r(u_{t+1,k} - 1, \mathbf{x}) + 2$. Thus, we have $r(u_{t+1,k}, \mathbf{x}) = 2u_{t+1,k} + 1$ by Theorem 2.4 and Lemma 5.3 in [15]. If $t = a_{k+1} - 1$, then $(t+1)q_k + q_{k-1} = q_{k+1}$. By the argument used at level k, $r(u_{1,k+1} - 1, \mathbf{x}) = u_{2,k+1} - 1$. The fact that $r(n+1, \mathbf{x}) \geq r(n, \mathbf{x}) + 1$ for any $n \geq 1$ gives

$$r(n, \mathbf{x}) = n + u'_k + 1$$

for $u'_k \leq n \leq u_{1,k+1} - 1$. It follows that

$$\Lambda(\mathbf{x}) \cap [u_{1,k}, u_{1,k+1} - 1] = \{v_{t,k}, u'_k\}.$$

Now, let $t < a_{k+1} - 1$. Note that

$$\begin{aligned}
\mathbf{x} = W_{k+1} M_{k+1} \widetilde{M}_{k+1} \cdots &= W_k {M_k}^t M_{k-1} {M_k}^{a_{k+1}} M_{k-1} M_k {M_{k+1}}^{--} \ldots \\
&= W_k {M_k}^t M_{k-1} M_k {M_k}^{a_{k+1}-2} M_k M_{k-1} M_k {M_{k+1}}^{--} \ldots \\
&= W_k {M_k}^{t+1} {M_{k-1}}^{--} D_k {M_k}^{a_{k+1}-1} M_k {M_{k-1}}^{--} D'_k {M_{k+1}}^{--} \ldots .
\end{aligned}$$

Since $x_1^{v_{t+1,k}-1} = x_{u_{t+1,k}+2}^{|W_k|+tq_k+q_{k-1}+((t+2)q_k+q_{k-1}-2)}$, we have $r(v_{t+1,k} - 1, \mathbf{x}) \leq v_{2t+2,k} + q_{k-1} - 1$. The fact that $r(n+1, \mathbf{x}) \geq r(n, \mathbf{x}) + 1$ for any $n \geq 1$ gives

$$r(n, \mathbf{x}) = n + u_{t+1,k} + 1$$

for $u_{t+1,k} \leq n \leq v_{t+1,k} - 1$. Note that

$$\begin{aligned}
\mathbf{x} = W_{k+1} M_{k+1} \widetilde{M}_{k+1} \cdots &= W_k {M_k}^t M_{k-1} {M_k}^{a_{k+1}} M_{k-1} M_k {M_{k+1}}^{--} \ldots \\
&= W_k {M_k}^{t+1} {M_{k-1}}^{--} D'_k {M_k}^{a_{k+1}-1} M_k {M_{k-1}}^{--} D'_k {M_{k+1}}^{--} \ldots \\
&= W_k {M_k}^t M_{k-1} {M_k}^{t+2} M_{k-1} \ldots .
\end{aligned}$$

It gives $r(v_{t+1,k}, \mathbf{x}) \geq r(v_{t+1,k} - 1, \mathbf{x}) + 2$. Hence, we have $r(v_{t+1,k}, \mathbf{x}) = 2v_{t+1,k} + 1$ from Theorem 2.4 and Lemma 5.3 in [15]. On the other hand, from

$$\begin{aligned}
\mathbf{x} = W_{k+1} M_{k+1} \widetilde{M}_{k+1} \cdots &= W_k {M_k}^t M_{k-1} M_k {M_k}^{a_{k+1}-1} M_{k-1} M_k {M_{k+1}}^{--} \ldots \\
&= W_k {M_k}^t M_{k-1} M_k {M_k}^{a_{k+1}} {M_{k-1}}^{--} D'_k {M_{k+1}}^{--} \ldots ,
\end{aligned}$$

we have $x_{|W_k|+tq_k+q_{k-1}+1}^{|W_k|+tq_k+q_{k-1}+(a_{k+1}q_k+q_{k-1}-2)} = x_{|W_k|+(t+1)q_k+q_{k-1}+1}^{|W_k|+(t+1)q_k+q_{k-1}+(a_{k+1}q_k+q_{k-1}-2)}$. It gives $r(u'_k - 1, \mathbf{x}) \leq (v_{t+1,k} + 1) + (u'_k - 1)$. The fact that $r(n+1, \mathbf{x}) \geq r(n, \mathbf{x}) + 1$ for any $n \geq 1$ gives

$$r(n, \mathbf{x}) = n + v_{t+1,k} + 1$$

for $v_{t+1,k} \le n \le u'_k - 1$. Moreover, $r(u'_k, \mathbf{x}) \ge r(u'_k - 1, \mathbf{x}) + 2$. From Theorem 2.4 and Lemma 5.3 in [15], $r(u'_k, \mathbf{x}) = 2u'_k + 1$. Note that $\mathbf{x} \in \mathcal{C}^{(\mathrm{i})}_{k+1}$. By the argument used at level k, $r(u_{1,k+1} - 1, \mathbf{x}) = u_{2,k+1} - 1$. The fact that $r(n+1, \mathbf{x}) \ge r(n, \mathbf{x}) + 1$ for any $n \ge 1$ gives

$$r(n, \mathbf{x}) = n + u'_k + 1$$

for $u'_k \le n \le u_{1,k+1} - 1$. It follows that

$$\Lambda(\mathbf{x}) \cap [u_{1,k}, u_{1,k+1} - 1] = \left\{v_{t,k}, u_{t+1,k}, v_{t+1,k}, u'_k\right\}.$$

(2) Suppose that $\mathbf{x} \in \mathcal{C}^{(\mathrm{i})}_k \cap \mathcal{C}^{(\mathrm{ii})}_{k+1}$. Note that

$$\mathbf{x} = W_{k+1} M_k \cdots = W_k {M_k}^t M_{k-1} M_k \cdots = W_k {M_k}^{t+1} {M_{k-1}}^{--} D'_k \ldots.$$

Use the argument used at level k in (1). Since $x_1^{v_{t,k}-1} = x_{q_k+1}^{v_{t+1,k}-1}$,

$$r(n, \mathbf{x}) = n + q_k$$

for $u_{1,k} \le n \le v_{t,k} - 1$. Note that

$$\begin{aligned}\mathbf{x} &= W_{k+1} M_k M_{k+1} \widetilde{M}_{k+1} \cdots = W_k {M_k}^t M_{k-1} M_k {M_k}^{a_{k+1}} M_{k-1} \widetilde{M}_{k+1} \ldots \\ &= W_k {M_k}^{t+1} {M_{k-1}}^{--} D'_k {M_k}^{a_{k+1}} M_{k-1} \widetilde{M}_{k+1} \cdots = W_k {M_k}^t M_{k-1} {M_k}^{t+1} M_{k-1} \ldots.\end{aligned}$$

Since $x_{|W_k|+1}^{|W_k|+u_{t+1,k}-1} = x_{v_{t,k}+2}^{v_{t,k}+1+u_{t+1,k}-1}$, $r(u_{t+1,k}-1, \mathbf{x}) \le v_{2t+1,k} + q_{k-1} - 1$. Moreover, $r(v_{t,k}, \mathbf{x}) \ge r(v_{t,k} - 1, \mathbf{x}) + 2$. From Theorem 2.4 and Lemma 5.3 in [15], $r(v_{t,k}, \mathbf{x}) = 2v_{t,k} + 1$. The fact that $r(n+1, \mathbf{x}) \ge r(n, \mathbf{x}) + 1$ for any $n \ge 1$ gives

$$r(n, \mathbf{x}) = n + v_{t,k} + 1$$

for $v_{t,k} \le n \le u_{t+1,k} - 1$. Moreover, $r(u_{t+1,k}, \mathbf{x}) \ge r(u_{t+1,k} - 1, \mathbf{x}) + 2$. From Theorem 2.4 and Lemma 5.3 in [15], $r(u_{t+1,k}, \mathbf{x}) = 2u_{t+1,k} + 1$. On the other hand, note that

$$\begin{aligned}\mathbf{x} &= W_{k+1} M_k M_{k+1} \widetilde{M}_{k+1} \ldots \\ &= W_k {M_k}^t M_{k-1} M_k {M_k}^{a_{k+1}} M_{k-1} M_k \ldots \\ &= W_k {M_k}^t M_{k-1} M_k {M_k}^{a_{k+1}+1} {M_{k-1}}^{--} \ldots \\ &= W_k {M_k}^{t+1} {M_{k-1}}^{--} D'_k {M_k}^{a_{k+1}+1} {M_{k-1}}^{--} \ldots.\end{aligned}$$

Since $x_1^{v_{t+1,k}-1} = x_{u_{t+1,k}+2}^{u_{t+1,k}+1+v_{t+1,k}-1}$, $r(v_{t+1,k}-1,\mathbf{x}) \le v_{2t+2,k}+q_{k-1}-1$. The fact that $r(n+1,\mathbf{x}) \ge r(n,\mathbf{x})+1$ for any $n \ge 1$ gives

$$r(n,\mathbf{x}) = n + u_{t+1,k} + 1$$

for $u_{t+1,k} \le n \le v_{t+1,k}-1$. Moreover, $r(v_{t+1,k},\mathbf{x}) \ge r(v_{t+1,k}-1,\mathbf{x})+2$. From Theorem 2.4 and Lemma 5.3 in [15], $r(v_{t+1,k},\mathbf{x}) = 2v_{t+1,k}+1$. On the other hand, note that

$$\begin{aligned}\mathbf{x} &= W_{k+1}M_kM_{k+1}\widetilde{M}_{k+1}\cdots = W_kM_k{}^tM_{k-1}M_kM_k{}^{a_{k+1}}M_{k-1}M_k\ldots\\ &= W_kM_k{}^tM_{k-1}M_kM_k{}^{a_{k+1}+1}M_{k-1}{}^{--}D'_k\ldots.\end{aligned}$$

Since $x_{v_{t,k}+2}^{v_{t,k}+1+u_{a_{k+1}+1,k}-1} = x_{v_{t+1,k}+2}^{v_{t+1,k}+1+u_{a_{k+1}+1,k}-1}$, $r(u_{1,k+1}-1,\mathbf{x}) \le u_{1,k+1}-1+v_{t+1,k}+1$. The fact that $r(n+1,\mathbf{x}) \ge r(n,\mathbf{x})+1$ for any $n \ge 1$ gives

$$r(n,\mathbf{x}) = n + v_{t+1,k} + 1$$

for $v_{t+1,k} \le n \le u_{1,k+1}-1$. Hence,

$$\Lambda(\mathbf{x}) \cap [u_{1,k}, u_{1,k+1}-1] = \{v_{t,k}, u_{t+1,k}, v_{t+1,k}\}.$$

(3) Suppose that $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{i})} \cap \mathcal{C}_{k+1}^{(\mathrm{iii})} \cap \mathcal{C}_{k+2}^{(\mathrm{i})}$. Since $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{i})}$, the argument used at level k in (1) yields $r(u_{1,k},\mathbf{x}) = u_{2,k}$. Note that

$$\begin{aligned}\mathbf{x} &= W_{k+1}M_{k+1}\widetilde{M}_{k+1}\cdots = W_kM_k{}^{a_{k+1}}M_{k-1}M_kM_{k+1}{}^{--}\ldots\\ &= W_kM_k{}^{a_{k+1}}\widetilde{M}_kD'_kM_{k+1}{}^{--}\ldots.\end{aligned}$$

Since $x_1^{v_{a_{k+1},k}-1} = x_{q_k+1}^{v_{a_{k+1},k}+q_k-1}$, $r(v'_k-1,\mathbf{x}) \le v'_k+q_k-1$. The fact that $r(n+1,\mathbf{x}) \ge r(n,\mathbf{x})+1$ for any $n \ge 1$ gives

$$r(n,\mathbf{x}) = n + q_k$$

for $u_{1,k} \le n \le v'_k-1$. Moreover, $r(v'_k,\mathbf{x}) \ge r(v'_k-1,\mathbf{x})+2$. From Theorem 2.4 and Lemma 5.3 in [15], $r(v'_k,\mathbf{x}) = 2v'_k+1$. Note that

$$\begin{aligned}\mathbf{x} &= W_{k+2}M_{k+2}\widetilde{M}_{k+2}\cdots = W_kM_{k+1}M_{k+1}{}^{a_{k+2}-1}M_k\widetilde{M}_{k+2}\ldots\\ &= W_kM_{k+1}M_{k+1}{}^{a_{k+2}}M_k{}^{--}D'_{k+1}M_{k+2}{}^{--}\ldots.\end{aligned}$$

Since $x_{|W_k|+1}^{|W_k|+u_{a_{k+2},k+1}-1} = x_{|W_k|+q_{k+1}+1}^{|W_k|+u_{a_{k+2}+1,k+1}-1}$, $r(u'_{k+1}-1,\mathbf{x}) \le |W_k|+q_{k+1}+u'_{k+1}-1$. The fact that $r(n+1,\mathbf{x}) \ge r(n,\mathbf{x})+1$ for any $n \ge 1$ gives

$$r(n,\mathbf{x}) = n + v'_k + 1$$

for $v'_k \le n \le u'_{k+1}-1$. Moreover, $r(u'_{k+1},\mathbf{x}) \ge r(u'_{k+1}-1,\mathbf{x})+2$. From Theorem 2.4 and Lemma 5.3 in [15], $r(u'_{k+1},\mathbf{x}) = 2u'_{k+1}+1$. Since $\mathbf{x} \in \mathcal{C}_{k+2}^{(\mathrm{i})}$, the argument used at level k in (1) implies $r(u_{1,k+2},\mathbf{x}) = u_{2,k+2}$. The fact that $r(n+1,\mathbf{x}) \ge r(n,\mathbf{x})+1$ for any $n \ge 1$ gives

$$r(n,\mathbf{x}) = n + u'_{k+1} + 1$$

for $u'_{k+1} \le n \le u_{1,k+2}$. Hence,

$$\Lambda(\mathbf{x}) \cap [u_{1,k}, u_{1,k+2}-1] = \{v'_k, u'_{k+1}\}.$$

(4) Suppose that $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{i})} \cap \mathcal{C}_{k+1}^{(\mathrm{iii})} \cap \mathcal{C}_{k+2}^{(\mathrm{ii})}$. Use the same argument with (3). Since $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{i})} \cap \mathcal{C}_{k+1}^{(\mathrm{iii})}$, we have

$$r(n,\mathbf{x}) = n + q_k$$

for $u_{1,k} \le n \le v'_k - 1$ and $r(v'_k,\mathbf{x}) = 2v'_k+1$. On the other hand, note that

$$\begin{aligned}\mathbf{x} &= W_{k+2}M_{k+1}M_{k+2}\widetilde{M}_{k+2}\cdots = W_kM_{k+1}{M_{k+1}}^{a_{k+2}}M_k\widetilde{M}_{k+2}\ldots\\ &= W_kM_{k+1}{M_{k+1}}^{a_{k+2}+1}{M_k}^{--}D'_{k+1}{M_{k+2}}^{--}\ldots.\end{aligned}$$

Since $x_{|W_k|+1}^{|W_k|+u_{a_{k+2}+1,k+1}-1} = x_{|W_k|+q_{k+1}+1}^{|W_k|+q_{k+1}+u_{a_{k+2}+1,k+1}-1}$, $r(u_{1,k+2}-1,\mathbf{x}) \le |W_k|+q_{k+1}+u_{1,k+2}-1$. The fact that $r(n+1,\mathbf{x}) \ge r(n,\mathbf{x})+1$ for any $n \ge 1$ gives

$$r(n,\mathbf{x}) = n + v'_k + 1$$

for $v'_k \le n \le u_{1,k+2}-1$. Hence,

$$\Lambda(\mathbf{x}) \cap [u_{1,k}, u_{1,k+2}-1] = \{v'_k\}.$$

(5) Suppose that $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{ii})} \cap \mathcal{C}_{k+1}^{(\mathrm{i})}$. Note that

$$\mathbf{x} = W_kM_{k-1}M_k\widetilde{M}_k\cdots = W_kM_{k-1}M_kM_{k-1}{M_k}^{--}\ldots.$$

Since M_kM_{k-1} is primitive, Lemma 7.1 in [15] implies that the first $(u_{1,k}+1)$ subwords of length $u_{1,k}$ are mutually distinct. Thus, $r(u_{1,k},\mathbf{x}) = 2u_{1,k}+1$. Since $x_1^{v_{1,k}-1} =$

$x_{q_k+q_{k-1}+1}^{q_k+q_{k-1}+v_{1,k}-1}$, $r(v_{1,k}-1,\mathbf{x}) \le q_k + q_{k-1} + v_{1,k} - 1$. The fact that $r(n+1,\mathbf{x}) \ge r(n,\mathbf{x})+1$ for any $n \ge 1$ gives

$$r(n,\mathbf{x}) = n + u_{1,k} + 1$$

for $u_{1,k} \le n \le v_{1,k} - 1$. Note that

$$\mathbf{x} = W_{k+1}M_{k+1}\widetilde{M}_{k+1}\cdots = W_kM_{k-1}{M_k}^{a_{k+1}}M_{k-1}M_k{M_{k+1}}^{--}\ldots.$$

If $a_{k+1} = 1$, then $x_1^{v_{2,k}+q_{k-1}-1} = x_{q_k+q_{k-1}+1}^{q_k+q_{k-1}+v_{2,k}+q_{k-1}-1}$. It implies $r(v_{2,k}+q_{k-1}-1,\mathbf{x}) \le v_{3,k} + 2q_{k-1} - 1$. Since $r(n+1,\mathbf{x}) \ge r(n,\mathbf{x})+1$ for any $n \ge 1$,

$$r(n,\mathbf{x}) = n + u_{1,k} + 1$$

for $u_{1,k} \le n \le v_k' + q_{k+1} - 1$. Hence,

$$\Lambda(\mathbf{x}) \cap [u_{1,k}, u_{1,k+1} - 1] = \{u_{1,k}\}.$$

Now, let $a_{k+1} > 1$. Since $\mathbf{x} = W_kM_{k-1}M_kM_kM_{k-1}\ldots$, $r(v_{1,k}\mathbf{x}) \ge r(v_{1,k}-1,\mathbf{x}) + 2$. From Theorem 2.4 and Lemma 5.3 in [15], $r(v_{1,k},\mathbf{x}) = 2v_{1,k}+1$. On the other hand, note that

$$\begin{aligned}\mathbf{x} &= W_{k+1}M_{k+1}\widetilde{M}_{k+1}\cdots = W_kM_{k-1}{M_k}^{a_{k+1}}M_{k-1}\widetilde{M}_{k+1}^{--}\cdots \\ &= W_kM_{k-1}M_k{M_k}^{a_{k+1}-1}M_{k-1}\widetilde{M}_{k+1}^{--}\cdots = W_kM_{k-1}M_k{M_k}^{a_{k+1}}M_{k-1}^{--}D_k'M_{k+1}^{--}\ldots.\end{aligned}$$

Since $x_{|W_k|+q_{k-1}+1}^{|W_k|+q_{k-1}+u_{a_{k+1},k}-1} = x_{|W_k|+q_k+q_{k-1}+1}^{|W_k|+q_k+q_{k-1}+u_{a_{k+1},k}-1}$, $r(u_k'-1,\mathbf{x}) \le |W_k|+q_k+q_{k-1}+u_k'-1$. The fact that $r(n+1,\mathbf{x}) \ge r(n,\mathbf{x})+1$ for any $n \ge 1$ gives

$$r(n,\mathbf{x}) = n + v_{1,k} + 1$$

for $v_{1,k} \le n \le u_k' - 1$. Moreover, $r(u_k',\mathbf{x}) \ge r(u_k'-1,\mathbf{x}) + 2$. From Theorem 2.4 and Lemma 5.3 in [15], $r(u_k',\mathbf{x}) = 2u_k' + 1$. Since $\mathbf{x} \in \mathcal{C}_{k+1}^{(\mathrm{i})}$, the argument used at level k in (1) implies $r(u_{1,k+1},\mathbf{x}) = u_{2,k+1}$. The fact that $r(n+1,\mathbf{x}) \ge r(n,\mathbf{x})+1$ for any $n \ge 1$ gives

$$r(n,\mathbf{x}) = n + u_k' + 1$$

for $u_k' \le n \le u_{1,k+1}$. Hence,

$$\Lambda(\mathbf{x}) \cap [u_{1,k}, u_{1,k+1} - 1] = \{u_{1,k}, v_{1,k}, u_k'\}.$$

(6) Suppose that $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{ii})} \cap \mathcal{C}_{k+1}^{(\mathrm{ii})}$. Use the same argument with (5). Since $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{ii})}$, we have

$$r(n, \mathbf{x}) = n + u_{1,k} + 1$$

for $u_{1,k} \leq n \leq v_{1,k} - 1$. Note that

$$\begin{aligned}\mathbf{x} &= W_{k+1} M_k M_{k+1} \widetilde{M}_{k+1} \cdots = W_k M_{k-1} M_k M_k{}^{a_{k+1}} M_{k-1} M_k M_{k+1}{}^{--} \dots \\ &= W_k M_{k-1} M_k M_k{}^{a_{k+1}+1} M_{k-1}{}^{--} D'_k M_{k+1}{}^{--} \dots .\end{aligned}$$

Since $\mathbf{x} = W_k M_{k-1} M_k M_k M_{k-1} \dots$, $r(v_{1,k}, \mathbf{x}) \geq r(v_{1,k} - 1, \mathbf{x}) + 2$. From Theorem 2.4 and Lemma 5.3 in [15], $r(v_{1,k}, \mathbf{x}) = 2v_{1,k} + 1$. Moreover, since $x_{|W_k|+q_{k-1}+1}^{|W_k|+q_{k-1}+u_{a_{k+1}+1,k}-1} = x_{v_{1,k}+2}^{v_{1,k}+1+u_{a_{k+1}+1,k}-1}$, $r(u_{1,k+1} - 1, \mathbf{x}) \leq |W_k| + q_k + q_{k-1} + u_{1,k+1} - 1$. The fact that $r(n+1, \mathbf{x}) \geq r(n, \mathbf{x}) + 1$ for any $n \geq 1$ gives

$$r(n, \mathbf{x}) = n + v_{1,k} + 1$$

for $v_{1,k} \leq n \leq u_{1,k+1} - 1$. Hence,

$$\Lambda(\mathbf{x}) \cap [u_{1,k}, u_{1,k+1} - 1] = \{u_{1,k}, v_{1,k}\}. \qquad \square$$

Remark. When $\mathbf{x} \in \mathcal{C}_1^{(\mathrm{iii})}$, Lemma 5.1.2 does not determine the elements in $\Lambda(\mathbf{x}) \cap [u_{1,1}, u_{1,2} - 1]$. Thus, we should check how the elements of $\Lambda(\mathbf{x}) \cap [u_{1,1}, u_{1,2} - 1]$ are expressed in terms of q_k's and $|W_k|$'s. If $\mathbf{x} \in \mathcal{C}_1^{(\mathrm{iii})}$, then $\mathbf{x}$ starts with $0^{a_1}10^{a_1-1}$. Thus, $r(n, \mathbf{x}) = n + 1$ for $1 \leq n \leq v'_0 - 1$. Moreover, since $\mathbf{x} \in \mathcal{C}_2^{(\mathrm{i})} \cup \mathcal{C}_2^{(\mathrm{ii})}$, $\mathbf{x}$ starts with $0^{a_1}10^{a_1}$ or $0^{a_1}10^{a_1-1}1$. Thus, $r(v'_0, \mathbf{x}) = 2v'_0 + 1$. Hence, we can follow the proof of Lemma 5.1.2 (3) and (4). We have $\Lambda(\mathbf{x}) \cap [u_{1,1}, u_{1,2} - 1] = \{v'_0, u'_1\}$ for $\mathbf{x} \in \mathcal{C}_2^{(\mathrm{i})}$ and $\Lambda(\mathbf{x}) \cap [u_{1,1}, u_{1,2} - 1] = \{v'_0\}$ for $\mathbf{x} \in \mathcal{C}_2^{(\mathrm{ii})}$.

For $l = 1, 2, \dots, 6$, define

$$\Lambda_l(\mathbf{x}) := \{n \in \Lambda(\mathbf{x}) : n \text{ appears in } (l) \text{ of Lemma 5.1.2}\}.$$

It is obvious that $\Lambda_l(\mathbf{x})$'s are mutually distinct and $\sqcup_{l=1}^{6} \Lambda_l(\mathbf{x}) = \Lambda(\mathbf{x}) \cap [u_{1,1}, \infty)$. Now, we find the minimum of $\mathscr{L}(\theta)$ where θ has bounded partial quotients.

Theorem 5.1.3 *Let $\theta = [0; a_1, a_2, \dots]$ have bounded partial quotients. We have*

$$\min \mathscr{L}(\theta) = \varliminf_{k \to \infty} [1; 1 + a_k, a_{k-1}, a_{k-2}, \dots, a_1].$$

Proof. Let $\mathbf{x}$ be a Sturmian word of slope θ. For $k \geq 1$, set $\eta_k := \dfrac{q_{k-1}}{q_k}, t_k := \frac{|W_k|}{q_k}, \epsilon_k := \frac{1}{q_k}$. Note that $t_k \leq 1$, $\lim\limits_{k\to\infty} \epsilon_k = 0$, and $\eta_k \geq \epsilon_k$ for $k \geq 1$. Set $m_\theta := \varliminf\limits_{i\to\infty} a_i, M_\theta := \varlimsup\limits_{i\to\infty} a_i$. Let $\liminf\limits_{k\to\infty} \eta_k = [0; b_1, b_2, \dots]$.

First, assume that $a_i > 1$ for infinitely many i. Since $\eta_k = [0; a_k, a_{k-1}, \dots, a_1]$ and $M_\theta \geq 2$, we have $b_1 = M_\theta$, $\liminf\limits_{k\to\infty} \eta_k < \frac{1}{2}$. Using Lemma 5.1.2, let us prove that $\liminf\limits_{n_i \in \Lambda(\mathbf{x})} \frac{n_i}{n_{i+1}} \geq \liminf\limits_{k\to\infty} [0; 1 + a_{k+1}, a_k, \dots]$ through the 7 cases below. Note that $\liminf\limits_{k\to\infty} \eta_k > \dfrac{\liminf\limits_{k\to\infty} \eta_k}{1 + \liminf\limits_{k\to\infty} \eta_k} = \liminf\limits_{k\to\infty} \frac{\eta_k}{1+\eta_k} = \liminf\limits_{k\to\infty} [0; 1 + a_k, a_{k-1}, \dots]$.

(1) For any $n_i \in \Lambda_1(\mathbf{x})$ with $n_{i+1} \in \Lambda_1(\mathbf{x}) \cup \Lambda_2(\mathbf{x})$, $\frac{n_i}{n_{i+1}}$ is

$$\frac{t_k + t + \eta_k - \epsilon_k}{t + 1 + \eta_k - \epsilon_k}, \frac{t + 1 + \eta_k - \epsilon_k}{t + 1 + t_k + \eta_k - \epsilon_k},$$
$$\frac{\eta_{k+1}(t + 1 + t_k + \eta_k - \epsilon_k)}{1 - \epsilon_{k+1}}, \text{ or } \frac{1 - \epsilon_{k+1}}{t' + t_{k+1} + \eta_{k+1} - \epsilon_{k+1}}$$

for some $k = k(i)$ and t, t' satisfying $W_{k+1} = W_k M_k^t M_{k-1}, W_{k+2} = W_{k+1} M_{k+1}^{t'} M_k$. We have

$$\liminf_{k\to\infty} \frac{t_k + t + \eta_k - \epsilon_k}{t + 1 + \eta_k - \epsilon_k},\ \liminf_{k\to\infty} \frac{t + 1 + \eta_k - \epsilon_k}{t + 1 + t_k + \eta_k - \epsilon_k} \geq \frac{t}{t+1} \geq \frac{1}{2} > \liminf_{k\to\infty} \eta_k,$$

$$\liminf_{k\to\infty} \frac{\eta_{k+1}}{1 - \epsilon_{k+1}}(t + 1 + t_k + \eta_k - \epsilon_k) \geq \liminf_{k\to\infty} 2\eta_{k+1} \geq \liminf_{k\to\infty} \eta_k,$$

and

$$\begin{aligned}\liminf_{k\to\infty} \frac{1 - \epsilon_{k+1}}{t_{k+1} + t' + \eta_{k+1} - \epsilon_{k+1}} &= \liminf_{k\to\infty} \frac{1}{t_{k+1} + t' + \eta_{k+1}} \\ &\geq \liminf_{k\to\infty} \frac{1}{a_{k+2} + \eta_{k+1}} = \liminf_{k\to\infty} \eta_{k+2}.\end{aligned}$$

(2) For any $n_i \in \Lambda_2(\mathbf{x})$, $\frac{n_i}{n_{i+1}}$ is

$$\frac{t_k + t + \eta_k - \epsilon_k}{t + 1 + \eta_k - \epsilon_k}, \frac{t + 1 + \eta_k - \epsilon_k}{t + 1 + t_k + \eta_k - \epsilon_k}, \text{ or } \frac{\eta_{k+1}(t + 1 + t_k + \eta_k - \epsilon_k)}{1 + \eta_{k+1} - \epsilon_{k+1}}$$

for some $k = k(i)$ and t satisfying $W_{k+1} = W_k M_k^t M_{k-1}$. From the previous case,

$$\liminf_{k\to\infty} \frac{t_k + t + \eta_k - \epsilon_k}{t + 1 + \eta_k - \epsilon_k},\ \liminf_{k\to\infty} \frac{t + 1 + \eta_k - \epsilon_k}{t + 1 + t_k + \eta_k - \epsilon_k} \geq \frac{1}{2} > \liminf_{k\to\infty} \eta_k.$$

We also have

$$\liminf_{k\to\infty} \frac{\eta_{k+1}}{1+\eta_{k+1}-\epsilon_{k+1}}(t+1+t_k+\eta_k-\epsilon_k) \geq \liminf_{k\to\infty} \frac{2\eta_{k+1}}{1+\eta_{k+1}} \geq \liminf_{k\to\infty} \eta_k.$$

(3) For any $n_i \in \Lambda_5(\mathbf{x})$ with $n_{i+1} \in \Lambda_1(\mathbf{x}) \cup \Lambda_2(\mathbf{x}) \cup \Lambda_5(\mathbf{x})$, $\frac{n_i}{n_{i+1}}$ is

$$\frac{1+\eta_k-\epsilon_k}{1+t_k+\eta_k-\epsilon_k},\ \frac{\eta_{k+1}}{1-\epsilon_{k+1}}(1+t_k+\eta_k-\epsilon_k), \text{ or } \frac{1-\epsilon_{k+1}}{t+t_{k+1}+\eta_{k+1}-\epsilon_{k+1}}$$

for some $k = k(i)$ and t satisfying $W_{k+2} = W_{k+1}M_{k+1}^t M_k$. We have

$$\liminf_{k\to\infty} \frac{1+\eta_k-\epsilon_k}{1+t_k+\eta_k-\epsilon_k} \geq \liminf_{k\to\infty} \frac{1}{1+t_k} \geq \frac{1}{2} > \liminf_{k\to\infty} \eta_k,$$

$$\liminf_{k\to\infty} \frac{\eta_{k+1}}{1-\epsilon_{k+1}}(1+t_k+\eta_k-\epsilon_k) \geq \liminf_{k\to\infty} \eta_{k+1},$$

and

$$\begin{aligned}\liminf_{k\to\infty} \frac{1-\epsilon_{k+1}}{t_{k+1}+t+\eta_{k+1}-\epsilon_{k+1}} &= \liminf_{k\to\infty} \frac{1}{t_{k+1}+t+\eta_{k+1}} \\ &\geq \liminf_{k\to\infty} \frac{1}{a_{k+2}+\eta_{k+1}} = \liminf_{k\to\infty} \eta_{k+2}.\end{aligned}$$

(4) For any $n_i \in \Lambda_6(\mathbf{x})$, $\frac{n_i}{n_{i+1}}$ is

$$\frac{1+\eta_k-\epsilon_k}{1+t_k+\eta_k-\epsilon_k} \text{ or } \frac{\eta_{k+1}}{1+\eta_{k+1}-\epsilon_{k+1}}(1+t_k+\eta_k-\epsilon_k)$$

for some $k = k(i)$. From the previous case,

$$\liminf_{k\to\infty} \frac{1+\eta_k-\epsilon_k}{1+t_k+\eta_k-\epsilon_k} > \liminf_{k\to\infty} \eta_k.$$

We also have

$$\begin{aligned}&\liminf_{k\to\infty} \frac{\eta_{k+1}}{1+\eta_{k+1}-\epsilon_{k+1}}(1+t_k+\eta_k-\epsilon_k) \geq \liminf_{k\to\infty} \frac{\eta_{k+1}}{1+\eta_{k+1}} \liminf_{k\to\infty}(1+t_k+\eta_k) \\ &\geq \frac{\liminf_{k\to\infty} \eta_{k+1}}{1+\liminf_{k\to\infty} \eta_{k+1}}(1+\liminf_{k\to\infty} \eta_k) = \liminf_{k\to\infty} \eta_{k+1}.\end{aligned}$$

(5) For any $n_i \in \Lambda_3(\mathbf{x})$ with $n_{i+1} \in \Lambda_1(\mathbf{x}) \cup \Lambda_2(\mathbf{x}) \cup \Lambda_3(\mathbf{x})$, $\frac{n_i}{n_{i+1}}$ is

$$(1 + t_k \eta_{k+1} - \epsilon_{k+1}) \frac{\eta_{k+2}}{1 - \epsilon_{k+2}} \text{ or } \frac{1 - \epsilon_{k+2}}{t + t_{k+2} + \eta_{k+2} - \epsilon_{k+2}}$$

for some $k = k(i)$ and t satisfying $W_{k+3} = W_{k+2} M_{k+2}^t M_{k+1}$. We have

$$\liminf_{k\to\infty} \frac{(1 + t_k \eta_{k+1} - \epsilon_{k+1})\eta_{k+2}}{1 - \epsilon_{k+2}} \geq \liminf_{k\to\infty} \frac{(1 - \epsilon_{k+1})\eta_{k+2}}{1 - \epsilon_{k+2}} = \liminf_{k\to\infty} \eta_{k+2}$$

and

$$\begin{aligned}\liminf_{k\to\infty} \frac{1 - \epsilon_{k+2}}{t_{k+2} + t + \eta_{k+2} - \epsilon_{k+2}} &= \liminf_{k\to\infty} \frac{1}{t_{k+2} + t + \eta_{k+2}} \\ &\geq \liminf_{k\to\infty} \frac{1}{a_{k+3} + \eta_{k+2}} = \liminf_{k\to\infty} \eta_{k+3}.\end{aligned}$$

(6) For any $n_i \in \Lambda_1(\mathbf{x}) \cup \Lambda_3(\mathbf{x}) \cup \Lambda_5(\mathbf{x})$ with $n_{i+1} \in \Lambda_3(\mathbf{x}) \cup \Lambda_4(\mathbf{x})$, $\frac{n_i}{n_{i+1}}$ is

$$\frac{\eta_{k+1} - \epsilon_{k+1}}{1 + t_k \eta_{k+1} - \epsilon_{k+1}}$$

for some $k = k(i)$. We have

$$\liminf_{k\to\infty} \frac{\eta_{k+1} - \epsilon_{k+1}}{1 + t_k \eta_{k+1} - \epsilon_{k+1}} \geq \liminf_{k\to\infty} \frac{\eta_{k+1}}{1 + \eta_{k+1}} = \liminf_{k\to\infty} [0; 1 + a_{k+1}, a_k, a_{k-1}, \dots].$$

(7) For any $n_i \in \Lambda_4(\mathbf{x})$, $\frac{n_i}{n_{i+1}}$ is

$$\frac{\eta_{k+2}}{1 + \eta_{k+2} - \epsilon_{k+2}} (1 + t_k \eta_{k+1} - \epsilon_{k+1})$$

for some $k = k(i)$. We have

$$\liminf_{k\to\infty} \frac{\eta_{k+2}(1 + t_k \eta_{k+1} - \epsilon_{k+1})}{1 + \eta_{k+2} - \epsilon_{k+2}} \geq \liminf_{k\to\infty} \frac{\eta_{k+2}}{1 + \eta_{k+2}} = \liminf_{k\to\infty} [0; 1 + a_{k+2}, a_{k+1}, a_k, \dots].$$

Hence, from (1)-(7),

$$\operatorname{rep}(\mathbf{x}) \geq \liminf_{k\to\infty} [1; 1 + a_k, a_{k-1}, \dots]$$

where $a_i > 1$ for infinitely many i.

Now, assume that there exists an integer $I > 0$ such that $a_i = 1$ for $i \geq I$. The assumptions of Lemma 5.1.1 (1) and (2) are not satisfied for any level $k \geq I$. In other words, $\Lambda_1(\mathbf{x}) \cup \Lambda_2(\mathbf{x})$ is finite. Thus, it is sufficient to consider (3)-(7). Note that $\liminf_{k\to\infty} \eta_k = [0; \overline{1}] = \varphi$. Using $1 \geq t_k$, $\lim_{k\to\infty} \epsilon_k = 0$,

$$\liminf_{k\to\infty} \frac{1+\eta_k-\epsilon_k}{1+t_k+\eta_k-\epsilon_k} \geq \liminf_{k\to\infty} \frac{1+\eta_k}{2+\eta_k} = \frac{1+\liminf_{k\to\infty}\eta_k}{2+\liminf_{k\to\infty}\eta_k} = \varphi = \liminf_{k\to\infty}\eta_k,$$

(3)-(7) are similarly proved. Hence, $\mathrm{rep}(\mathbf{x}) \geq [1; 2, \overline{1}]$ for a Sturmian word $\mathbf{x}$ of slope φ. Therefore,

$$\mathrm{rep}(\mathbf{x}) \geq \liminf_{k\to\infty} [1; 1+a_k, a_{k-1}, \dots]$$

for a Sturmian word $\mathbf{x}$ of slope θ.

The equality holds in the following setting. Choose the sequence $\{k_j\}$ such that $\eta_{k_j} \to \liminf_{k\to\infty} \eta_k$, $\lim_{j\to\infty} (k_{j+1} - k_j) = \infty$, and $k_{j+1} - k_j$ is odd for all j. Let $\mathbf{x} \in \mathcal{C}_{k_j}^{(\mathrm{ii})}$ for all k_j and $\mathbf{x} \in \mathcal{C}_{k_j+2l-1}^{(\mathrm{i})} \cap \mathcal{C}_{k_j+2l}^{(\mathrm{iii})}$ for all $0 < l \leq \frac{k_{j+1}-k_j-1}{2}$. Thus, $W_{k_j+1} = W_{k_j} M_{k_j-1}$ for all k_j and $W_{k+1} = W_k$ for all $k \neq k_j$. In the proof of (7), we have $\lim_{j\to\infty} t_{k_j} = 0$ and $\mathrm{rep}(\mathbf{x}) = \varliminf_{k\to\infty} [1; 1+a_k, a_{k-1}, a_{k-2}, \dots]$. In conclusion, $\min \mathscr{L}(\theta) = \varliminf_{k\to\infty} [1; 1+a_k, a_{k-1}, a_{k-2}, \dots]$. □

5.2 The spectrum of the exponents of repetition of Fibonacci words

We keep the notations in Section 4.2 with the slope $\varphi = [0; \bar{1}]$. In this section, we investigate $\mathscr{L}(\varphi)$. Let us define

$$\mu_{\max} := 1+\varphi = 1.618\dots, \quad \mu_2 := 4\varphi - 1 = 1.472\dots, \quad \mu_3 := \frac{5-5\varphi}{7\varphi-3} = 1.440\dots,$$
$$\mu_4 := \frac{73\varphi-42}{65\varphi-38} = 1.434\dots, \quad \mu_{\min} := 2-\varphi = 1.381\dots.$$

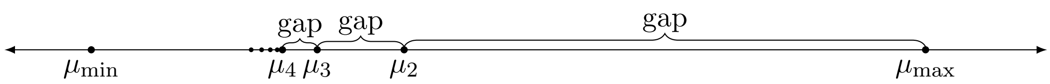


Figure 5.1 $\mu_{\max}, \mu_2, \mu_3, \mu_4, \mu_{\min}$ in $\mathscr{L}(\varphi)$

Our first goal is to prove that $\mu_{\max}$ is the maximum of $\mathscr{L}(\varphi)$. Next, we show that $\mu_{\max}, \mu_2, \mu_3, \mu_4$ are the four largest points in $\mathscr{L}(\varphi)$ and μ_4 is an accumulation

k	$k+1$	The relation between W_{k+1} and W_k
case (i)	case (iii)	$W_{k+1} = W_k$
case (ii)	case (i) case (ii)	$W_{k+1} = W_k M_{k-1}$
case (iii)	case (i) case (ii)	$W_{k+1} = W_k$

Table 5.2 The relation between W_{k+1} and W_k in Fibonacci words

point of $\mathscr{L}(\varphi)$. For $\mu \in \{\mu_{\max}, \mu_2, \mu_3, \mu_4, \mu_{\min}\}$, we give the necessary and sufficient condition for $\operatorname{rep}(\mathbf{x}) = \mu$ and the cardinality of the set of Sturmian words $\mathbf{x}$ satisfying $\operatorname{rep}(\mathbf{x}) = \mu$.

Note that $M_{k+1} = {M_k}^{a_{k+1}} M_{k-1} = M_k M_{k-1}$ for all $k \geq 1$. The following lemma is a special case of Lemma 5.1.1.

Lemma 5.2.1. Let $k \geq 1$. The following statements hold.

(1) If $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{i})}$, then $\mathbf{x} \in \mathcal{C}_{k+1}^{(\mathrm{iii})}$ and $W_{k+1} = W_k$.

(2) If $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{ii})}$, then $\mathbf{x} \in \mathcal{C}_{k+1}^{(\mathrm{i})} \cup \mathcal{C}_{k+1}^{(\mathrm{ii})}$ and $W_{k+1} = W_k M_{k-1}$.

(3) If $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{iii})}$, then $\mathbf{x} \in \mathcal{C}_{k+1}^{(\mathrm{i})} \cup \mathcal{C}_{k+1}^{(\mathrm{ii})}$ and $W_{k+1} = W_k$.

Proof. For $k \geq 1$, let W_k be the unique non-empty prefix of $\mathbf{x}$ defined in which case $\mathbf{x}$ belongs to at level k.

(1) Since $a_k = 1$ for all $k \geq 1$, the assumption of (1) in Lemma 5.1.1 cannot be satisfied. Hence, $\mathbf{x} \in \mathcal{C}_{k+1}^{(\mathrm{iii})}$. Since $\mathbf{x}$ starts with $W_{k+1} M_{k+1} \widetilde{M}_{k+1} = W_{k+1} M_k \widetilde{M}_k D'_k M_{k+1}^{--}$ for the suffix W_{k+1} of M_k, $W_{k+1} = W_k$ by the uniqueness of W_k.

(2) and (3) are equivalent to (3) and (4) in Lemma 5.1.1 respectively. □

By Lemma 5.2.1, only (iii) should follow (i) in the locating chain of $\mathbf{x}$: $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{i})}$ implies $\mathbf{x} \in \mathcal{C}_{k+1}^{(\mathrm{iii})}$. Hence, if $\mathbf{x} \notin \mathcal{C}_1^{(\mathrm{iii})}$, then the locating chain of $\mathbf{x}$ can be expressed as an infinite sequence of (i)(iii) and (ii). If $\mathbf{x} \in \mathcal{C}_1^{(\mathrm{iii})}$, then the locating chain of $\mathbf{x}$ is an infinite sequence of (i)(iii) and (ii), except for the first letter (iii). Let us denote (i)(iii) and (ii) by a and b, respectively.

Since only (iii) should follow (i) in the locating chain of $\mathbf{x}$, the assumptions of Lemma 5.1.2 (1) and (2) cannot be satisfied. We have the following lemma corresponding to Lemma 5.1.2. Using $q_{k+1} = q_k + q_{k-1}$ for all $k \geq 1$, Lemma 5.1.2 (3)-(6) are equivalent to (1)-(4) of the following lemma, respectively.

Lemma 5.2.2. Let $k \geq 1$.

(1) If $\mathbf{x} \in \mathcal{C}_k^{(i)} \cap \mathcal{C}_{k+2}^{(i)}$, then $\Lambda(\mathbf{x}) \cap [q_{k+1}-1, q_{k+3}-2] = \{q_{k+1}+|W_k|-1, q_{k+2}-1\}$.

(2) If $\mathbf{x} \in \mathcal{C}_k^{(i)} \cap \mathcal{C}_{k+2}^{(ii)}$, then $\Lambda(\mathbf{x}) \cap [q_{k+1}-1, q_{k+3}-2] = \{q_{k+1}+|W_k|-1\}$.

(3) If $\mathbf{x} \in \mathcal{C}_k^{(ii)} \cap \mathcal{C}_{k+1}^{(i)}$, then $\Lambda(\mathbf{x}) \cap [q_{k+1}-1, q_{k+2}-2] = \{q_{k+1}-1\}$.

(4) If $\mathbf{x} \in \mathcal{C}_k^{(ii)} \cap \mathcal{C}_{k+1}^{(ii)}$, then $\Lambda(\mathbf{x}) \cap [q_{k+1}-1, q_{k+2}-2] = \{q_{k+1}-1, q_{k+1}+|W_k|-1\}$.

For $l = 1, 2, 3, 4$, define

$$\Lambda_l'(\mathbf{x}) := \{n \in \Lambda(\mathbf{x}) : n \text{ appears in } (l) \text{ of Lemma 5.2.2}\}.$$

It is obvious that $\Lambda_l'(\mathbf{x})$'s are mutually distinct and $\cup_{l=1}^4 \Lambda_l'(\mathbf{x}) = \Lambda(\mathbf{x}) \cap [q_2-1, \infty) = \Lambda(\mathbf{x})$. From the definition of $\Lambda_l(\mathbf{x})$, $\Lambda_l'(\mathbf{x}) = \Lambda_{l+2}(\mathbf{x})$ for $l = 1, 2, 3, 4$ where the slope of $\mathbf{x}$ is φ.

Note that $\operatorname{rep}(\mathbf{x})$ is the limit infimum of $(1 + \frac{n_i}{n_{i+1}})$'s for $n_i \in \cup_{l=1}^4 \Lambda_l'(\mathbf{x})$. The following lemma says that it is enough to consider the elements of $\Lambda_2'(\mathbf{x})$ and $\Lambda_3'(\mathbf{x})$ to obtain $\operatorname{rep}(\mathbf{x})$.

Lemma 5.2.3. Suppose that both a and b appear infinitely often in the locating chain of $\mathbf{x}$. Then,

$$\operatorname{rep}(\mathbf{x}) = \liminf_{n_i \in \Lambda_2'(\mathbf{x}) \cup \Lambda_3'(\mathbf{x})} \left(1 + \frac{n_i}{n_{i+1}}\right).$$

Proof. First, for each k satisfying $\mathbf{x} \in \mathcal{C}_k^{(ii)} \cap \mathcal{C}_{k+1}^{(i)}$, there exists $d(k) > 0$ such that $\mathbf{x} \in \mathcal{C}_k^{(ii)} \cap \mathcal{C}_{k+2d(k)+1}^{(ii)}$ and $\mathbf{x} \in \mathcal{C}_{k+2d-1}^{(i)} \cap \mathcal{C}_{k+2d}^{(iii)}$ for $1 \leq d \leq d(k)$. By Lemma 5.2.2, $\mathcal{J}_k := \Lambda(\mathbf{x}) \cap [q_{k+1}-1, q_{k+2d(k)+2}-2] \subset \Lambda_1'(\mathbf{x}) \cup \Lambda_2'(\mathbf{x}) \cup \Lambda_3'(\mathbf{x})$. Note that $W_{k+1} = \cdots = W_{k+2d(k)+1}$ and

$$\frac{q_{j+1}+|W_j|-1}{q_{j+2}-1} \geq \frac{q_{j+1}-1}{q_{j+2}+|W_{j+1}|-1}$$

for $j = k+1, k+3, \ldots, k+2d(k)-1$. Since

$$\frac{q_{j+1}-1}{q_{j+2}+|W_{j+1}|-1}$$

is increasing for $j = k, k+1, \ldots, k+2d(k)-2$,

$$\min_{n_i \in \mathcal{J}_k} \frac{n_i}{n_{i+1}} = \min\left\{\frac{q_{k+1}-1}{q_{k+2}+|W_{k+1}|-1}, \frac{q_{k+2d(k)}+|W_{k+2d(k)-1}|-1}{q_{k+2d(k)+2}-1}\right\}$$

$$= \min_{n_i \in (\Lambda_2'(\mathbf{x}) \cup \Lambda_3'(\mathbf{x})) \cap \mathcal{J}_k} \frac{n_i}{n_{i+1}}.$$

Second, for each l satisfying $\mathbf{x} \in \mathcal{C}_l^{(\text{iii})} \cap \mathcal{C}_{l+1}^{(\text{ii})}$, there exists $d'(l) > 0$ such that $\mathbf{x} \in \mathcal{C}_l^{(\text{iii})} \cap \mathcal{C}_{l+d'(l)+1}^{(\text{i})}$ and $\mathbf{x} \in \mathcal{C}_j^{(\text{ii})}$ for $l+1 \leq j \leq l+d'(l)$. By Lemma 5.2.2, $\mathcal{J}_l' := \Lambda(\mathbf{x}) \cap [q_l - 1, q_{l+d'(l)+2} - 2] \subset \Lambda_2'(\mathbf{x}) \cup \Lambda_3'(\mathbf{x}) \cup \Lambda_4'(\mathbf{x})$. Note that

$$\frac{q_l + |W_{l-1}| - 1}{q_{l+2} - 1} \leq \frac{q_{l+1} - 1}{q_{l+2} - 1} \text{ and } \frac{q_{j+1} - 1}{q_{j+2} - 1} \leq \frac{q_{j+1} - 1}{q_{j+1} + |W_j| - 1}, \frac{q_{j+1} + |W_j| - 1}{q_{j+2} - 1}$$

for $j = l+1, \ldots, l+d'(l)-1$. Since

$$\frac{q_j - 1}{q_{j+1} - 1}$$

is increasing for $j = l+1, \ldots, l+d'(l)$,

$$\min_{n_i \in \mathcal{J}_l'} \frac{n_i}{n_{i+1}} = \min\left\{\frac{q_l + |W_{l-1}| - 1}{q_{l+2} - 1}, \frac{q_{l+d'(l)+1} - 1}{q_{l+d'(l)+2} + |W_{l+d'(l)+1}| - 1}\right\}$$
$$= \min_{n_i \in (\Lambda_2'(\mathbf{x}) \cup \Lambda_3'(\mathbf{x})) \cap \mathcal{J}_l'} \frac{n_i}{n_{i+1}}.$$

Since $\Lambda(\mathbf{x})$ is the union of $\mathcal{J}_k$'s and $\mathcal{J}_l'$'s, $\text{rep}(\mathbf{x}) = \liminf_{n_i \in \Lambda_2'(\mathbf{x}) \cup \Lambda_3'(\mathbf{x})} \left(1 + \frac{n_i}{n_{i+1}}\right)$. □

Let d be a positive integer. We define an *a-chain* to be a subword $aa \cdots a$ in the locating chain of $\mathbf{x}$ before and after which b appears. For example, if the locating chain of $\mathbf{x} = abbaaabaab\ldots$, then a-chains are $a, aaa, aa, \ldots$. Similarly, a *b-chain* is defined as a subword $bb \cdots b$ in the locating chain of $\mathbf{x}$ before and after which a appears. We say that an a-chain or a b-chain is a *chain*. From the definition of a chain, a-chains and b-chains alternatively appear in the locating chain of $\mathbf{x}$. We can choose two sequences $\{m_i(\mathbf{x})\}_{i \geq 1}$ and $\{l_j(\mathbf{x})\}_{j \geq 1}$ defined as follows: Let $m_i(\mathbf{x})$ (resp., $l_j(\mathbf{x})$) be the length of the ith a-chain (resp., the jth b-chain) in the locating chain of $\mathbf{x}$. Let $b^{l_1(\mathbf{x})}$ follow $a^{m_1(\mathbf{x})}$. In other words, the locating chain of $\mathbf{x}$ is $c(\mathbf{x}) a^{m_1(\mathbf{x})} b^{l_1(\mathbf{x})} a^{m_2(\mathbf{x})} b^{l_2(\mathbf{x})} \ldots$. for the unique finite word $c(\mathbf{x})$. For example, if the locating chain of $\mathbf{x}$ is (iii)$bbabbabaabbba\ldots$, then $c(\mathbf{x}) = $ (iii)bb, $m_1(\mathbf{x}) = 1$, $m_2(\mathbf{x}) = 1$, $m_3(\mathbf{x}) = 2$, $l_1(\mathbf{x}) = 2$, $l_2(\mathbf{x}) = 1$, $l_3(\mathbf{x}) = 3$. Let

$$S_d := \left\{ \mathbf{x} : \begin{array}{c} aa \ldots a \text{ or } bb \ldots b \text{ of length greater than or equal to } d \\ \text{appears infinitely often in the locating chain of } \mathbf{x} \end{array} \right\}.$$

In other words, S_d is the set of Sturmian words $\mathbf{x}$ such that $m_i(\mathbf{x})$ or $l_i(\mathbf{x})$ is larger than or equal to d for infinitely many i. By definition, $S_{d+1} \subset S_d$. In what follows, we will write $m_i(\mathbf{x})$ and $l_j(\mathbf{x})$ simply m_i and l_j, when no confusion can arise. Now, let us show that $\mu_{\max}$ is the maximum of $\mathscr{L}(\varphi)$. We give the necessary and sufficient condition for $\mathrm{rep}(\mathbf{x}) = \mu_{\min}$ or $\mu_{\max}$.

Theorem 5.2.4 *Let $\mathbf{x}$ be a Sturmian word of slope φ. Then, $\mu_{\min} \le \mathrm{rep}(\mathbf{x}) \le \mu_{\max}$. Moreover, the locating chain of $\mathbf{x}$ is $u\bar{a}$ or $v\bar{b}$ for some finite words u, v if and only if $\mathrm{rep}(\mathbf{x}) = \mu_{\max}$. We have $\mathbf{x} \in S_d$ for any $d \ge 1$ if and only if $\mathrm{rep}(\mathbf{x}) = \mu_{\min}$.*

Proof. First, assume that there exists a constant K such that $\mathbf{x} \in \mathcal{C}_k^{(\mathrm{ii})}$ for all $k \ge K$. By Lemma 5.2.2, $\Lambda(\mathbf{x}) \cap [q_{K+1} - 1, \infty) = \{q_k - 1, q_k + |W_{k-1}| - 1 : k \ge K+1\}$. Since $W_{k+1} = W_k M_{k-1}$ for any $k \ge K$,

$$\begin{aligned}\liminf_{n_i \in \Lambda(\mathbf{x})} \frac{n_i}{n_{i+1}} &= \liminf_{k \ge K+1} \left\{ \frac{q_k - 1}{q_k + |W_{k-1}| - 1}, \frac{q_k + |W_{k-1}| - 1}{q_{k+1} - 1} \right\} \\ &= \min\left\{ \liminf_{k \ge K+1} \left(\frac{q_k - 1}{q_k + |W_{k-1}| - 1} \right), \liminf_{k \ge K+1} \left(\frac{q_k + |W_{k-1}| - 1}{q_{k+1} - 1} \right) \right\} = \varphi.\end{aligned}$$

Hence, $\mathrm{rep}(\mathbf{x}) = \liminf_{n_i \in \Lambda(\mathbf{x})} \left(1 + \frac{n_i}{n_{i+1}}\right) = \mu_{\max}$.

Second, assume that there exists a constant K such that $\mathbf{x} \in \mathcal{C}_{K+2l}^{(\mathrm{i})} \cap \mathcal{C}_{K+2l+1}^{(\mathrm{iii})}$ for all $l \ge 0$. By Lemma 5.2.2, $\Lambda(\mathbf{x}) \cap [q_{K+1} - 1, \infty) = \{q_{K+2l+1} + |W_{K+2l}| - 1, q_{K+2l+2} - 1 : l \ge 0\}$. Since $|W_k|$ is constant for $k \ge K$,

$$\begin{aligned}&\liminf_{n_i \in \Lambda(\mathbf{x})} \frac{n_i}{n_{i+1}} \\ &= \liminf_{l \ge 0} \left\{ \frac{q_{k+2l+1} + |W_{k+2l}| - 1}{q_{K+2l+2} - 1}, \frac{q_{K+2l+2} - 1}{q_{k+2l+3} + |W_{k+2l+2}| - 1} \right\} \\ &= \min\left\{ \liminf_{l \ge 0} \left(\frac{q_{k+2l+1} + |W_{k+2l}| - 1}{q_{K+2l+2} - 1} \right), \liminf_{l \ge 1} \left(\frac{q_{K+2l} - 1}{q_{k+2l+1} + |W_{k+2l}| - 1} \right) \right\} = \varphi.\end{aligned}$$

Hence, $\mathrm{rep}(\mathbf{x}) = \liminf_{n_i \in \Lambda(\mathbf{x})} \left(1 + \frac{n_i}{n_{i+1}}\right) = \mu_{\max}$.

Now, let both a and b occur infinitely often in the locating chain of $\mathbf{x}$. Since ba appears infinitely often in the locating chain of $\mathbf{x}$, we can choose an infinite sequence $\{n_{i(j)}\}_{j \ge 1} \subset \Lambda_3'(\mathbf{x})$. For each $j \ge 1$, Lemma 5.2.2 gives $n_{i(j)} = q_{k+1} - 1$,$n_{i(j)+1} =$

$q_{k+2}+|W_{k+1}|-1$ for some $k=k(j)$. Note that $W_{k(j)+1}=W_{k(j)}M_{k(j)-1}$ By definition,

$$\begin{aligned}\operatorname{rep}(\mathbf{x}) &\leq \liminf_{j\to\infty}\left(1+\frac{q_{k(j)+1}-1}{q_{k(j)+2}+|W_{k(j)+1}|-1}\right)\\ &\leq \liminf_{j\to\infty}\left(1+\frac{q_{k(j)+1}-1}{q_{k(j)+2}+q_{k(j)-1}-1}\right)=1+\frac{\varphi}{1+\varphi^3}<\mu_{\max}.\end{aligned}$$

Hence, $\operatorname{rep}(\mathbf{x})<\mu_{\max}$. In other words, $\operatorname{rep}(\mathbf{x})=\mu_{\max}$ implies that the locating chain of $\mathbf{x}$ is $u\bar{a}$ or $v\bar{b}$ for some finite words u,v.

From Theorem 5.1.3, the minimum of $\mathscr{L}(\varphi)$ is $\mu_{\min}$. Let us use Lemma 5.2.3 to determine $\mathbf{x}$ satisfying $\operatorname{rep}(\mathbf{x})=\mu_{\min}$. For $n_i\in\Lambda_2'(\mathbf{x})$, let $n_i=q_k+|W_{k-1}|-1$, $n_{i+1}=q_{k+2}-1$ for some $k=k(i)$. For $n_i\in\Lambda_3'(\mathbf{x})$, let $n_i=q_{k'+1}-1$, $n_{i+1}=q_{k'+2}+|W_{k'+1}|-1$ for some $k'=k'(i)$. Thus,

$$\begin{aligned}&\operatorname{rep}(\mathbf{x})\\ &=\min\left\{\liminf_{n_i\in\Lambda_2'(\mathbf{x})}\left(1+\frac{n_i}{n_{i+1}}\right),\liminf_{n_i\in\Lambda_3'(\mathbf{x})}\left(1+\frac{n_i}{n_{i+1}}\right)\right\}\\ &=\min\left\{\liminf_{i\to\infty}\left(1+\frac{q_{k(i)}+|W_{k(i)-1}|-1}{q_{k(i)+2}-1}\right),\liminf_{i\to\infty}\left(1+\frac{q_{k'(i)+1}-1}{q_{k'(i)+2}+|W_{k'(i)+1}|-1}\right)\right\}\\ &\geq\min\left\{\liminf_{i\to\infty}\left(1+\frac{q_{k(i)}-1}{q_{k(i)+2}-1}\right),\liminf_{i\to\infty}\left(1+\frac{q_{k'(i)+1}-1}{q_{k'(i)+2}+q_{k'(i)+1}-1}\right)\right\}\\ &=\min\left\{1+\varphi^2,1+\frac{\varphi}{\varphi+1}\right\}=\mu_{\min}.\end{aligned}$$

The necessary and sufficient condition for $\operatorname{rep}(\mathbf{x})=\mu_{\min}$ is

$$\liminf_{i\to\infty}\frac{|W_{k(i)-1}|}{q_{k(i)}}=0 \text{ or } \limsup_{i\to\infty}\frac{|W_{k'(i)+1}|}{q_{k'(i)+1}}=1.$$

Hence, arbitrarily long sequence $aa\dots a$ or $bb\dots b$ should occur in the locating chain of $\mathbf{x}$, i.e. $\mathbf{x}\in S_d$ for any $d\geq 1$. □

The following result states that μ_2,μ_3 are the second and third largest points in $\mathscr{L}(\varphi)$. The necessary and sufficient condition for $\operatorname{rep}(\mathbf{x})=\mu_2$ or μ_3 is determined.

Theorem 5.2.5 *The intervals $(\mu_2,\mu_{\max}),(\mu_3,\mu_2)$ are maximal gaps in $\mathscr{L}(\varphi)$. Moreover, the locating chain of $\mathbf{x}$ is $u\overline{ab}$ for some finite word u if and only if* rep($\mathbf{x}$) $= \mu_2$. *The locating chain of* $\mathbf{x}$ *is* $v\overline{b^2a^2}$ *for some finite word* v *if and only if* rep($\mathbf{x}$) $= \mu_3$.

Proof. If rep($\mathbf{x}$) $< \mu_{\max}$, then $\mathbf{x} \in S_1$ by Theorem 5.2.4.

First, let $\mathbf{x} \in S_1 \cap S_2^c$. Since any chains of length greater than 1 occur at most finitely often in the locating chain of $\mathbf{x}$, there exists an integer $I > 0$ satisfying $m_j = l_j = 1$ for $j \geq I$. Thus, the locating chain of $\mathbf{x}$ is $u\overline{ab}$ for some finite word u. Using Lemma 5.2.2 and 5.2.3, we obtain rep($\mathbf{x}$) $= \mu_2$.

Second, let $\mathbf{x} \in S_2$. Using Lemma 5.2.2 and 5.2.3, rep($\mathbf{x}$) $= \mu_3$ where the locating chain of $\mathbf{x}$ is $v\overline{a^2b^2}$ for some finite word v. If $m_j \geq 3$ for infinitely many j, then there exists an infinite sequence $\{k(j)\}$ satisfying $\mathbf{x} \in \mathcal{C}^{(\mathrm{i})}_{k(j)-6} \cap \mathcal{C}^{(\mathrm{i})}_{k(j)-4} \cap \mathcal{C}^{(\mathrm{i})}_{k(j)-2} \cap \mathcal{C}^{(\mathrm{ii})}_{k(j)}$ for all j. By Lemma 5.2.3,

$$\mathrm{rep}(\mathbf{x}) \leq \liminf_{j\to\infty} \left(1 + \frac{q_{k(j)-1} + |W_{k(j)-6}| - 1}{q_{k(j)+1} - 1}\right) \leq 1 + \varphi^2 + \varphi^7$$

where $W_{k(j)} = W_{k(j)-6}$ for all j. If $l_j \geq 3$ for infinitely many j, then there exists an infinite sequence $\{k(j)\}$ satisfying $\mathbf{x} \in \mathcal{C}^{(\mathrm{ii})}_{k(j)-2} \cap \mathcal{C}^{(\mathrm{ii})}_{k(j)-1} \cap \mathcal{C}^{(\mathrm{ii})}_{k(j)} \cap \mathcal{C}^{(\mathrm{i})}_{k(j)+1}$ for all j. By Lemma 5.2.3,

$$\begin{aligned}\mathrm{rep}(\mathbf{x}) &\leq \liminf_{j\to\infty} \left(1 + \frac{q_{k(j)+1} - 1}{q_{k(j)+2} + |W_{k(j)+1}| - 1}\right)\\ &= \liminf_{j\to\infty} \left(1 + \frac{q_{k(j)+1} - 1}{q_{k(j)+2} + q_{k(j)-1} + q_{k(j)-2} + q_{k(j)-3} + |W_{k(j)-2}| - 1}\right)\\ &\leq \liminf_{j\to\infty} \left(1 + \frac{q_{k(j)+1} - 1}{q_{k(j)+2} + 2q_{k(j)-1} - 1}\right) = 1 + \frac{\varphi}{1 + 2\varphi^3}.\end{aligned}$$

Since $1 + \varphi^2 + \varphi^7, 1 + \dfrac{\varphi}{1+2\varphi^3} < \mu_3$, rep($\mathbf{x}$) $< \mu_3$ for $\mathbf{x} \in S_3$. Now, let $\mathbf{x} \in S_2 \cap S_3^c$. By definition, there exists an integer $I > 0$ such that $m_j, l_j \leq 2$ for $j \geq I$. If $l_j = 1$, $m_{j+1} = 2$ for infinitely many j, then there exists an infinite sequence $\{k(j)\}$ satisfying $\mathbf{x} \in \mathcal{C}^{(\mathrm{i})}_{k(j)-7} \cap \mathcal{C}^{(\mathrm{ii})}_{k(j)-5} \cap \mathcal{C}^{(\mathrm{i})}_{k(j)-4} \cap \mathcal{C}^{(\mathrm{i})}_{k(j)-2} \cap \mathcal{C}^{(\mathrm{ii})}_{k(j)}$ for all j. By Lemma 5.2.3,

$$\mathrm{rep}(\mathbf{x}) \leq \liminf_{j\to\infty} \left(1 + \frac{q_{k(j)-1} + q_{k(j)-6} + |W_{k(j)-7}| - 1}{q_{k(j)+1} - 1}\right) \leq 1 + \varphi^2 + \varphi^7 + \varphi^8$$

where $W_{k(j)} = W_{k(j)-4}$, $W_{k(j)-5} = W_{k(j)-7}$, and $W_{k(j)-4} = W_{k(j)-5}M_{k(j)-6}$ for all j. If $m_j = 1$, $l_j = 2$ for infinitely many j, then there exists an infinite sequence $\{k(j)\}$

satisfying $\mathbf{x} \in \mathcal{C}^{(\mathrm{ii})}_{k(j)-4} \cap \mathcal{C}^{(\mathrm{i})}_{k(j)-3} \cap \mathcal{C}^{(\mathrm{ii})}_{k(j)-1} \cap \mathcal{C}^{(\mathrm{ii})}_{k(j)} \cap \mathcal{C}^{(\mathrm{i})}_{k(j)+1}$ for all j. By Lemma 5.2.3,

$$\begin{aligned}\operatorname{rep}(\mathbf{x}) &\le \liminf_{j\to\infty}\left(1+\frac{q_{k(j)+1}-1}{q_{k(j)+2}+|W_{k(j)+1}|-1}\right)\\ &= \liminf_{j\to\infty}\left(1+\frac{q_{k(j)+1}-1}{q_{k(j)+2}+q_{k(j)-1}+q_{k(j)-2}+|W_{k(j)-3}|-1}\right)\\ &\le \liminf_{j\to\infty}\left(1+\frac{q_{k(j)+1}-1}{q_{k(j)+2}+q_{k(j)}+q_{k(j)-5}-1}\right) = 1+\frac{\varphi}{\varphi^7+\varphi^2+1}\end{aligned}$$

where $W_{k(j)-1} = W_{k(j)-3}$ and $W_{k(j)-3} = W_{k(j)-4}M_{k(j)-5}$ for all j. Since $1+\varphi^2+\varphi^7+\varphi^8, 1+\dfrac{\varphi}{\varphi^7+\varphi^2+1} < \mu_3$, $\operatorname{rep}(\mathbf{x}) < \mu_3$ where the locating chain of $\mathbf{x}$ is not $v\overline{a^2b^2}$ for some finite word v. Hence, $\max\{\operatorname{rep}(\mathbf{x}) : \mathbf{x} \in S_2\} = \mu_3$. Moreover, $\operatorname{rep}(\mathbf{x}) = \mu_3$ if and only if the locating chain of $\mathbf{x}$ is $v\overline{a^2b^2}$ for some finite word v. Therefore, two intervals $(\mu_2, \mu_{\max})$, (μ_3, μ_2) are maximal gaps in $\mathscr{L}(\varphi)$. On the other hand, by Theorem 5.2.4, $\operatorname{rep}(\mathbf{x}) = \mu_{\max}$ if and only if $\mathbf{x} \in S_1^c$. In the proof above, $\operatorname{rep}(\mathbf{x}) = \mu_2$ for $\mathbf{x} \in S_1 \cap S_2^c$, and $\operatorname{rep}(\mathbf{x}) \le \mu_3$ for $\mathbf{x} \in S_2$. Hence, $\operatorname{rep}(\mathbf{x}) = \mu_2$ if and only if $\mathbf{x} \in S_1 \cap S_2^c$. □

In the next theorem, we assert that μ_4 is the fourth largest point in $\mathscr{L}(\varphi)$ and a limit point of $\mathscr{L}(\varphi)$. We give the necessary and sufficient condition for $\operatorname{rep}(\mathbf{x}) = \mu_4$.

Theorem 5.2.6 *The interval (μ_4, μ_3) is a maximal gap in $\mathscr{L}(\varphi)$. Moreover, $\operatorname{rep}(\mathbf{x}) = \mu_4$ if and only if $\mathbf{x} \in S_2 \cap S_3^c$ satisfies the following two conditions:*

1) *The locating chain of $\mathbf{x}$ is $u(b^2a^2)^{e_1}ba(b^2a^2)^{e_2}ba\ldots$ for some finite word u and integers $e_i \ge 1$.*

2) $\limsup_{i\ge 1}\{e_i\} = \infty$.

Furthermore, μ_4 is a limit point of $\mathscr{L}(\varphi)$.

Proof. From Lemma 5.2.2 and 5.2.3, $\operatorname{rep}(\mathbf{x}) = \mu_4$ if $x \in S_2 \cap S_3^c$ satisfies the above two conditions 1) and 2). Assume that a Sturmian word $\mathbf{x}$ satisfies $\operatorname{rep}(\mathbf{x}) \in (\mu_4, \mu_3)$. In the proof of Theorem 5.2.5, $\operatorname{rep}(\mathbf{x}) \ge \mu_2$ for $\mathbf{x} \in S_2^c$, and $\operatorname{rep}(\mathbf{x}) \le \min\{1+\varphi^2+\varphi^7, 1+\dfrac{\varphi}{2\varphi^3+1}\} < \mu_4$ for $\mathbf{x} \in S_3$. Thus, $\mathbf{x} \in S_2 \cap S_3^c$. By definition, there exists an integer $I > 0$ such that $m_j, l_j \le 2$ for all $j \ge I$. Moreover, $\operatorname{rep}(\mathbf{x}) < \mu_3$ implies that $\{j : m_j = 1\} \cup \{j : l_j = 1\}$ is infinite. Hence, $\{j : m_j = 1, l_j = 2\} \cup \{j : l_j = 1, m_{j+1} = 2\}$ is infinite.

First, if $\{j : l_j = 1, m_{j+1} = 2\}$ is infinite, then $\mathrm{rep}(\mathbf{x}) < 1.432 < \mu_4$. Thus, $\{j : l_j = 1, m_{j+1} = 2\}$ is finite. In other words, there exists an integer $I' > 0$ such that $m_j = 2$ implies $l_{j-1} = 2$ for all $j > I'$. Since $\{j : m_j = 1, l_j = 2\} \cup \{j : l_j = 1, m_{j+1} = 2\}$ is infinite, $\{j : m_j = 1, l_j = 2\}$ is infinite. Now, let us show that both $\{j : m_j = 1, l_{j-1} = l_j = 2\}$ and $\{j : m_{j-1} = m_j = 1, l_{j-1} = 1, l_j = 2\}$ are finite. If $\{j : m_j = 1, l_{j-1} = l_j = 2\}$ is infinite, then there exists an infinite sequence $\{k(j)\}$ such that $\mathbf{x} \in \mathcal{C}^{(\mathrm{ii})}_{k(j)-5} \cap \mathcal{C}^{(\mathrm{ii})}_{k(j)-4} \cap \mathcal{C}^{(\mathrm{i})}_{k(j)-3} \cap \mathcal{C}^{(\mathrm{ii})}_{k(j)-1} \cap \mathcal{C}^{(\mathrm{ii})}_{k(j)} \cap \mathcal{C}^{(\mathrm{i})}_{k(j)+1}$ for all j. Hence,

$$\begin{aligned}
\mathrm{rep}(\mathbf{x}) &\le \liminf_{j\to\infty} \left(1 + \frac{q_{k(j)+1} - 1}{q_{k(j)+2} + |W_{k(j)+1}| - 1}\right) \\
&= \liminf_{j\to\infty} \left(1 + \frac{q_{k(j)+1} - 1}{q_{k(j)+2} + q_{k(j)-1} + q_{k(j)-2} + q_{k(j)-5} + q_{k(j)-6} + |W_{k(j)-5}| - 1}\right) \\
&\le \liminf_{j\to\infty} \left(1 + \frac{q_{k(j)+1} - 1}{q_{k(j)+2} + q_{k(j)-1} + q_{k(j)-2} + q_{k(j)-5} + q_{k(j)-6} - 1}\right) \\
&= 1 + \frac{\varphi}{1 + \varphi^3 + \varphi^4 + \varphi^7 + \varphi^8} < \mu_4.
\end{aligned}$$

It follows that $\{j : m_j = 1, l_{j-1} = l_j = 2\}$ is finite.

If $\{j : m_{j-1} = m_j = 1, l_{j-1} = 1, l_j = 2\}$ is infinite, then there exists an infinite sequence $\{k(j)\}$ such that $\mathbf{x} \in \mathcal{C}^{(\mathrm{ii})}_{k(j)-7} \cap \mathcal{C}^{(\mathrm{i})}_{k(j)-6} \cap \mathcal{C}^{(\mathrm{ii})}_{k(j)-4} \cap \mathcal{C}^{(\mathrm{i})}_{k(j)-3} \cap \mathcal{C}^{(\mathrm{ii})}_{k(j)-1} \cap \mathcal{C}^{(\mathrm{ii})}_{k(j)} \cap \mathcal{C}^{(\mathrm{i})}_{k(j)+1}$ for all j. Hence,

$$\begin{aligned}
\mathrm{rep}(\mathbf{x}) &\le \liminf_{j\to\infty} \left(1 + \frac{q_{k(j)+1} - 1}{q_{k(j)+2} + |W_{k(j)+1}| - 1}\right) \\
&= \liminf_{j\to\infty} \left(1 + \frac{q_{k(j)+1} - 1}{q_{k(j)+2} + q_{k(j)-1} + q_{k(j)-2} + q_{k(j)-5} + q_{k(j)-8} + |W_{k(j)-7}| - 1}\right) \\
&\le \liminf_{j\to\infty} \left(1 + \frac{q_{k(j)+1} - 1}{q_{k(j)+2} + q_{k(j)-1} + q_{k(j)-2} + q_{k(j)-5} + q_{k(j)-8} - 1}\right) \\
&= 1 + \frac{\varphi}{1 + \varphi^3 + \varphi^4 + \varphi^7 + \varphi^{10}} < \mu_4.
\end{aligned}$$

It follows that $\{j : m_{j-1} = m_j = 1, l_{j-1} = 1, l_j = 2\}$ is finite. Therefore, both $\{j : m_j = 1, l_{j-1} = l_j = 2\}$ and $\{j : m_{j-1} = m_j = 1, l_{j-1} = 1, l_j = 2\}$ are finite.

Next, let us prove that $\{j : m_j = m_{j+1} = 1\}$ is finite. Suppose that $\{j : m_j = m_{j+1} = 1\}$ is infinite. Then, $\{j : m_j = m_{j+1} = 1, l_j = 1\} \cup \{j : m_j = m_{j+1} =$

$1, l_j = 2\}$ is infinite. Note that $\{j : l_j = 1, m_{j+1} = 2\}$ is finite. In other words, there exists an integer $I'' > 0$ such that $l_j = 1$ implies $m_{j+1} = 1$ for all $j > I''$. Thus, if $\{j : m_j = m_{j+1} = 1, l_j = 1\}$ is infinite, then $\{j : m_j = m_{j+1} = 1, l_j = 1, l_{j+1} = 2\} \cup \{j : m_j = m_{j+1} = m_{j+2} = 1, l_j = l_{j+1} = 1\}$ is infinite. Since $\{j : m_j = m_{j+1} = 1, l_j = 1, l_{j+1} = 2\}$ is finite, $\{j : m_j = m_{j+1} = m_{j+2} = 1, l_j = l_{j+1} = 1\}$ is infinite. Thus, $\{j : m_j = m_{j+1} = m_{j+2} = 1, l_j = l_{j+1} = 1, l_{j+2} = 2\} \cup \{j : m_j = m_{j+1} = m_{j+2} = m_{j+3} = 1, l_j = l_{j+1} = l_{j+2} = 1\}$ is infinite. Since $\{j : m_j = m_{j+1} = 1, l_j = 1, l_{j+1} = 2\}$ is finite, $\{j : m_j = m_{j+1} = m_{j+2} = m_{j+3} = 1, l_j = l_{j+1} = l_{j+2} = 1\}$ is infinite. By the same argument, it follows that the locating chain of $\mathbf{x}$ is $u\overline{ab}$ for some finite word u, which leads a contradiction with $x \in S_2$. Hence, $\{j : m_j = m_{j+1} = 1, l_j = 1\}$ is finite. Similarly, we use the same argument to induce that $\{j : m_j = m_{j+1} = 1, l_j = 2\}$ is finite. Therefore, $\{j : m_j = m_{j+1} = 1\}$ is finite.

From the above arguments, we have the locating chain of $\mathbf{x}$ is

$$u(b^2a^2)^{e_1}b^{f_1}a(b^2a^2)^{e_2}b^{f_2}a\dots$$

for integers $e_i \geq 1$, $f_j = 1$ or 2, and some finite word u. Since $\{j : m_j = 1, l_{j-1} = l_j = 2\}$ is finite, we can assume that $f_j = 1$ for all j. The locating chain of $\mathbf{x}$ is $u(b^2a^2)^{e_1}ba(b^2a^2)^{e_2}ba\dots$. Moreover, if $d = \limsup_{i\geq 1}\{e_i\} < \infty$, then

$$\text{rep}(\mathbf{x}) \leq 1 + \left(\frac{1}{\varphi} + \varphi + \frac{\varphi^6}{1-\varphi^{6d+3}} + \frac{\varphi^{10}}{1-\varphi^6}\frac{1-\varphi^{6d}}{1-\varphi^{6d+3}}\right)^{-1} < \mu_4.$$

It follows $\limsup_{i\geq 1}\{e_i\} = \infty$. However, for a Sturmian word $\mathbf{x}$ such that the locating chain of $\mathbf{x}$ is $u(b^2a^2)^{e_1}ba(b^2a^2)^{e_2}ba\dots$ and $\limsup_{i\geq 1}\{e_i\} = \infty$, $\text{rep}(\mathbf{x}) = \mu_4$. It implies that there does not exist a Sturmian word $\mathbf{x}$ satisfying $\text{rep}(\mathbf{x}) \in (\mu_4, \mu_3)$. Hence, (μ_4, μ_3) is a maximal gap in $\mathscr{L}(\varphi)$. Furthermore,

$$\begin{aligned}\text{rep}(\dots\overline{(b^2a^2)^d ba}) &= 1 + \left(\frac{1}{\varphi} + \varphi + \frac{\varphi^6}{1-\varphi^{6d+3}} + \frac{\varphi^{10}}{1-\varphi^6}\frac{1-\varphi^{6d}}{1-\varphi^{6d+3}}\right)^{-1} \\ &\to \mu_4 \text{ as } d \to \infty.\end{aligned}$$

Hence, μ_4 is a limit point of $\mathscr{L}(\varphi)$. □

Proposition 5.2.7. For $\alpha \in \{\mu_{\max}, \mu_2, \mu_3\}$, there are only countably many Sturmian words $\mathbf{x}$ of slope φ satisfying $\mathrm{rep}(\mathbf{x}) = \alpha$. For $\beta \in \{\mu_4, \mu_{\min}\}$, there are uncountably many Sturmian words $\mathbf{x}$ of slope φ satisfying $\mathrm{rep}(\mathbf{x}) = \beta$.

Proof. From Theorem 5.2.4, $\mathrm{rep}(\mathbf{x}) = \mu_{\max}$ if and only if the locating chain of $\mathbf{x}$ is $u\overline{a}$ or $v\overline{b}$ for some finite words u, v. Thus, $\mathbf{x}$ satisfying $\mathrm{rep}(\mathbf{x}) = \mu_{\max}$ is completely determined by the choice of u or v. Hence, there exist only countably many Sturmian words $\mathbf{x}$ of slope φ with $\mathrm{rep}(\mathbf{x}) = \mu_{\max}$. Theorem 5.2.4 also implies that $\mathrm{rep}(\mathbf{x}) = \mu_{\min}$ if and only if $\mathbf{x} \in S_d$ for any $d \geq 1$. Hence, it is possible to choose $2^{\aleph_0}$ sequences $\{m_i\} \cup \{l_j\}$. Namely, there exist uncountably many Sturmian words $\mathbf{x}$ of slope φ with $\mathrm{rep}(\mathbf{x}) = \mu_{\min}$.

On the other hand, Theorem 5.2.5 implies that $\mathrm{rep}(\mathbf{x}) = \mu_2$ if and only if the locating chain of $\mathbf{x}$ is $u\overline{ba}$ for some finite word u. Thus, $\mathbf{x}$ satisfying $\mathrm{rep}(\mathbf{x}) = \mu_2$ is completely determined by the choice of u. Hence, there exist only countably many Sturmian words $\mathbf{x}$ of slope φ with $\mathrm{rep}(\mathbf{x}) = \mu_2$. Theorem 5.2.5 also implies that $\mathrm{rep}(\mathbf{x}) = \mu_3$ if and only if the locating chain of $\mathbf{x}$ is $u\overline{b^2a^2}$ for some finite word u. Thus, $\mathbf{x}$ satisfying $\mathrm{rep}(\mathbf{x}) = \mu_3$ is completely determined by the choice of u. Hence, there exist only countably many Sturmian words $\mathbf{x}$ of slope φ with $\mathrm{rep}(\mathbf{x}) = \mu_3$.

Finally, Theorem 5.2.6 implies that $\mathrm{rep}(\mathbf{x}) = \mu_4$ if and only if the locating chain of $\mathbf{x}$ is $u(b^2a^2)^{e_1}ba(b^2a^2)^{e_2}ba\ldots$ for some finite word u and integers $e_i \geq 1$ satisfying $\limsup_{i\geq 1}\{e_i\} = \infty$. Hence, it is possible to choose $2^{\aleph_0}$ sequences $\{e_i\}$. Namely, there exist uncountably many Sturmian words $\mathbf{x}$ of slope φ with $\mathrm{rep}(\mathbf{x}) = \mu_4$. □

Chapter 6

Colorings of regular trees

6.1 Sturmian colorings of trees

In this section, we study Sturmian colorings on regular trees. Let us begin by some notations in graphs and trees, following [61].

A *graph* $\mathcal{G}$ consists of a set of vertices $V\mathcal{G}$ and a set of edges $E\mathcal{G}$. The vertex set $V\mathcal{G}$ is defined to be a finite or countably infinite set. The edge set $E\mathcal{G}$ is defined by a subset of ordered pairs of two distinct vertices. The *degree* of a vertex v is defined to be the number of edges starting from v. We say that a graph is *k-regular* if all vertices of the graph have the same degree k. A *tree* is a graph with no cycle.

Let $\mathcal{T}$ be a k-regular tree. By a *coloring* of a regular tree $\mathcal{T}$, we mean a vertex coloring with finite alphabet, i.e. a surjective map $\phi : V\mathcal{T} \to \mathcal{A}$ from the vertex set $V\mathcal{T}$ to the set $\mathcal{A}$ such that $|\mathcal{A}| < \infty$. For subtrees $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\mathcal{T}$, we define a *color-preserving homomorphism* $f : \mathcal{T}_1 \to \mathcal{T}_2$ of a coloring ϕ as a graph homomorphism such that $\phi(v) = \phi(f(v))$ for all $v \in V\mathcal{T}_1$. We say that two vertices u, v are *in the same class* if there is a color-preserving isometry of $\mathcal{T}$ such that $f(u) = v$. For a given coloring ϕ, let Γ_ϕ be the group of color-preserving isometries of $\mathcal{T}$. Then, Γ_ϕ is a subgroup of $\mathrm{Aut}(\mathcal{T})$.

For a given coloring ϕ, let $\Gamma := \Gamma_\phi$ be the group of color-preserving isometries of $\mathcal{T}$. The quotient $X = \Gamma \backslash \mathcal{T}$ has a structure of an *edge-indexed graph*, which is a graph equipped with an index map $i : EX \to \mathbb{N}$ defined as follows: Let $e \in EX$ be an oriented edge with the initial vertex $x \in VX$ and the terminal vertex $y \in VX$. Let $\tilde{x}$ be a lift of x in $\mathcal{T}$. The index $i(e)$ is the number of lifts of y among the neighboring vertices of $\tilde{x}$. We sometimes denote e by $[x, y]$ and denote $i(e)$ by $i(x, y)$. We call $\mathcal{X} = (X, i)$ the *quotient (edge-indexed) graph of* $(\mathcal{T}, \phi)$. Let $\pi : \mathcal{T} \to X$ be the covering map. There is a coloring ϕ_0 of X such that $\phi = \phi_0 \circ \pi$.

Dong Han Kim and Seonhee Lim generalized a Sturmian word and its factor complexity to a Sturmian coloring of a tree [37].

The *n-ball $\mathcal{B}_n(u)$ of center u* is defined by the closed ball of radius n and center u. We say two n-balls $\mathcal{B}_n(u)$ and $\mathcal{B}_n(v)$ are *equivalent* if there is a color-preserving isometry $f : \mathcal{B}_n(u) \to \mathcal{B}_n(v)$. We denote by $[\mathcal{B}_n(u)]$ the equivalence class of $\mathcal{B}_n(u)$ and call it a *colored n-ball.* The set of colored n-balls of ϕ is denoted by $\mathbb{B}_\phi(n)$.

For $n \geq 0$, the *factor complexity* $b_\phi(n)$ of a coloring ϕ is defined to be the number of colored n-balls in $(\mathcal{T}, \phi)$. In other words,

$$b_\phi(n) = |\mathbb{B}_\phi(n)|.$$

Clearly, $b_\phi(0) = |\mathcal{A}|$.

If $b_\phi(n+1) > b_\phi(n)$, there are at least two distinct n-balls $\mathcal{B}_n(u)$ and $\mathcal{B}_n(v)$ such that $[\mathcal{B}_n(u)] = [\mathcal{B}_n(v)]$ but $[\mathcal{B}_{n+1}(u)] \neq [\mathcal{B}_{n+1}(v)]$. We call such a colored n-ball $[\mathcal{B}_n(u)]$ *special.* Then, we say that $[\mathcal{B}_{n+1}(u)]$ and $[\mathcal{B}_{n+1}(v)]$ are extensions of $[\mathcal{B}_n(u)]$.

The *type set* Λ_u of a vertex $u \in V\mathcal{T}$ is the set of integers n for which $[\mathcal{B}_n(u)]$ is special. A vertex u is said to be of *bounded type* if Λ_u is a finite set. For a vertex u of bounded type, the *maximal type* $\tau(u)$ of u is the maximum of elements in Λ_u. We say that a coloring ϕ is of *bounded type* if each vertex (or equivalently a vertex) of $(\mathcal{T}, \phi)$ is of bounded type. Otherwise, we say that a coloring ϕ is of *unbounded type.*

We say that a coloring is *periodic* if its quotient graph $\Gamma\backslash\mathcal{T}$ is a finite graph. Dong Han Kim and Seonhee Lim showed the analogous theorem of Morse-Hedlund theorem which is to characterize periodic words and Sturmian words by subword complexity. They generalized Sturmian words to Sturmian colorings of regular trees [37].

Theorem 6.1.1 ([37, Theorem 2.7]) *The followings are equivalent.*

(1) The coloring ϕ is periodic.

(2) The factor complexity satisfies $b_\phi(n) = b_\phi(n+1)$ for some n.

(3) The factor complexity $b_\phi(n)$ is bounded.

Suppose that $b_\phi(n)$ is not bounded. From Theorem 6.1.1, $b_\phi(n)$ is strictly increasing for all n. Since $b_\phi(0) \geq 2$, $b_\phi(n) \geq n+2$ for all n. In other words, non-periodic coloring has at least $n+2$ factor complexity.

Definition 6.1.2. A *Sturmian coloring* $(\mathcal{T}, \phi)$ is a coloring with factor complexity $b_\phi(n) = n + 2$.

They also characterized a Sturmian coloring of a regular tree by its quotient graph.

Theorem 6.1.3 ([37, Theorem 3.4 and 3.9]) *If ϕ is a Sturmian coloring, then its quotient graph is one of a geodesic ray or a bi-infinite geodesic with possibly attached loops at each vertex.*

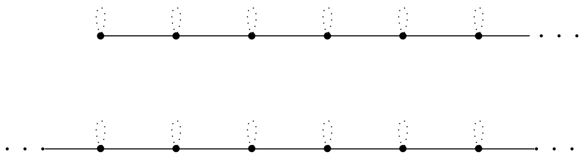


Figure 6.1 The quotient graph of a Sturmian coloring

The factor graph $\mathcal{G}_n$ is defined as the graph whose vertices are the colored n-balls. Its edges are pairs of colored n-balls appearing in $(\mathcal{T}, \phi)$ of distance 1, i.e. (D_n, E_n) such that $[\mathcal{B}_n(v)] = D_n$, $[\mathcal{B}_n(w)] = E_n$ for some vertices $v, w \in V\mathcal{T}$ with $\mathbf{d}(v, w) = 1$. From the definition of a Sturmian coloring, there exists a unique special n-ball for each n. We denote by S_n, C_n the special n-ball, the centered colored n-ball of S_{n+1}, respectively. The special n-ball S_n has two extension to colored $(n+1)$-balls. We denote by A_{n+1}, B_{n+1} the colored $(n+1)$-ball extensions of S_n. Let us choose $\{A_n\}, \{B_n\}$ such that A_{n+1} contains more A_n than B_n.

Now, We define the subgraphs $\mathcal{G}_n^A, \mathcal{G}_n^B$ of $\mathcal{G}_n$. The subgraphs $\mathcal{G}_n^A, \mathcal{G}_n^B$ consist of the colored n-balls adjacent to S_n in A_{n+1}, B_{n+1}, respectively. We denote by $\bowtie$ the concatenation of $\mathcal{G}_n^A$ and $\mathcal{G}_n^B$ as follows: If $C_n \neq S_n$, then we define

$$V(\mathcal{G}_n^A \bowtie \mathcal{G}_n^B) = V\mathcal{G}_n^A \cup V\mathcal{G}_n^B \text{ and } E(\mathcal{G}_n^A \bowtie \mathcal{G}_n^B) = E\mathcal{G}_n^A \cup E\mathcal{G}_n^B$$

where the end vertices C_n in $\mathcal{G}_n^A$ and C_n in $\mathcal{G}_n^B$ are identified and the loops at C_n in $\mathcal{G}_n^A$ and $\mathcal{G}_n^B$ are identified in $\mathcal{G}_n^A \bowtie \mathcal{G}_n^B$. If $C_n = S_n$, then we define

$$V(\mathcal{G}_n^A \bowtie \mathcal{G}_n^B) = V\mathcal{G}_n^A \sqcup V\mathcal{G}_n^B \text{ and } E(\mathcal{G}_n^A \bowtie \mathcal{G}_n^B) = E\mathcal{G}_n^A \sqcup E\mathcal{G}_n^B \sqcup \{e\}$$

where C_n in $\mathcal{G}_n^A$ and $\mathcal{G}_n^B$ remain two distinct vertices in $\mathcal{G}_n^A \bowtie \mathcal{G}_n^B$ and e is the edge connecting with two end vertices C_n in $\mathcal{G}_n^A$ and $\mathcal{G}_n^B$.

Theorem 6.1.4 ([38, Theorem 1.2]) *Let ϕ be a Sturmian coloring.*

(1) If ϕ is such that $\mathcal{G}_n$ does not have any cycle for all n, then there exists $K \in [0, \infty]$ and a sequence $(n_k)_{k\geq 0}$ such that $n_k = k$ for $0 \leq k \leq K$ and

$$\begin{aligned}
&\mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A, \quad \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^A \bowtie \mathcal{G}_{n-1}^B, && \text{if } 0 \leq n < K,\\
&\left.\begin{array}{l}\mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A \bowtie \mathcal{G}_{n-1}^B, \quad \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^A \bowtie \mathcal{G}_{n-1}^B, \text{ or}\\ \mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A \bowtie \mathcal{G}_{n-1}^B, \quad \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^B,\end{array}\right\} && \text{if } 0 \leq n = K,\\
&\mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A, \quad \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^B, && \text{if } n \neq n_k, n > K,\\
&\left.\begin{array}{l}\mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A \bowtie \mathcal{G}_{n-1}^B, \quad \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^B, \text{ or}\\ \mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A, \quad \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^B,\end{array}\right\} && \text{if } n = n_k, n > K.
\end{aligned}$$

(2) If ϕ is such that $\mathcal{G}_n$ has a cycle for some n, then ϕ is of bounded type. The coloring ϕ is of bounded type if and only if either $\mathcal{G}_n^A$ or $\mathcal{G}_n^B$ eventually stabilizes.

6.2 Quasi-Sturmian colorings

In this section, we look into quasi-Sturmian colorings of regular trees. Quasi-Sturmian coloring is similarly defined with the definition of quasi-Sturmian words.

Definition 6.2.1. We say that a coloring is *quasi-Sturmian* if there exists a pair of integers c and N_0 such that $b(n) = n + c$ for $n \geq N_0$, i.e.

$$b(n+1) - b(n) = 1 \text{ for each } n \geq N_0. \tag{6.1}$$

We assume that N_0 is the minimal integer satisfying (6.1). From the definition, a quasi-Sturmian coloring has a unique special n-ball for all $n \geq N_0$ which we denote by S_n.

6.2.1 Quotient graphs of quasi-Sturmian colorings

In this section, we characterize the quotient graphs of quasi-Sturmian colorings. The quotient graph of a quasi-Sturmian coloring of bounded type is a union of a finite graph and a geodesic ray. For a quasi-Sturmian coloring of unbounded type, the quotient graph is a geodesic ray or an infinite geodesic.

For $u \in V\mathcal{T}$, $\tau(u) \leq m$ if and only if $[\mathcal{B}_{m+1}(u)] = [\mathcal{B}_{m+1}(v)]$ implies that u and v are in the same class. If two vertices u and v are in the same class, then u and v have the same maximal type. Kim and Lim proved that the converse is also true in the case of a Sturmian coloring (see Proposition 3.2 in [37]). We observe that the same proof holds in quasi-Sturmian colorings as long as $b(n+1) - b(n) = 1$. We provide the proof for completeness.

Lemma 6.2.2. Suppose that $b(n)$ is a strictly increasing function. If $b(n+1)-b(n) = 1$ and two vertices u and v have maximal type n, then u and v are in the same class.

Proof. Suppose that $b(n+1) - b(n) = 1$ and there exist two vertices u and v not in the same class such that $\tau(u) = \tau(v) = n$. Since the alphabet $\mathcal{A}$ is finite, there is a number N such that $\mathcal{B}_N(w)$ contains a special n-ball for each $w \in V\mathcal{T}$ (see Lemma 2.16 in [37]).

Fix a vertex w and let z be the center of a special n-ball contained in $\mathcal{B}_N(w)$. Since the special n-ball is unique and it has only two extensions of radius $n+1$, either $[\mathcal{B}_{n+1}(z)] = [\mathcal{B}_{n+1}(u)]$ or $[\mathcal{B}_{n+1}(z)] = [\mathcal{B}_{n+1}(v)]$, thus z is in the same class of u or v. Since $w \in \mathcal{B}_N(z)$, the tree $\mathcal{T}$ is covered by N-balls whose centers are in the same class of u or v. Thus, the maximal types of vertices of $\mathcal{T}$ is bounded by $M = \max\{\tau(p) : p \in \mathcal{B}_N(u) \cup \mathcal{B}_N(v)\}$. It contradicts that $b(n)$ is strictly increasing. □

Corollary 6.2.3. Let $(\mathcal{T}, \phi)$ be a quasi-Sturmian coloring of bounded type with factor complexity $b(n) = n + c$ for $n \geq N_0$. If two vertices u and v of $(\mathcal{T}, \phi)$ have the same maximal type greater than or equal to N_0, then u and v are in the same class.

Lemma 6.2.4. If a vertex u of a quasi-Sturmian coloring $(\mathcal{T}, \phi)$ is of maximal type m, then the following hold.

(1) If $m \geq N_0$, its neighboring vertices are of maximal type $m-1$, m, $m+1$.
If $m = N_0 - 1$, its neighboring vertices are of maximal type at most N_0.
If $m \leq N_0 - 2$, its neighboring vertices are of maximal type at most $N_0 - 1$.

(2) If $m \geq N_0$, one of its neighboring vertices is of maximal type $m+1$.

(3) If $m \geq N_0$ is not minimum among maximal types of vertices, one of its neighboring vertices is of maximal type $m-1$.

Proof. Let $\{u_i\}_{i=1,\cdots,d}$ be the neighboring vertices of u, where d is the degree of T.

(1) Let $\tau = \max\{\tau(u_i)\}_{i=1,\cdots,d}$. Choose u_k such that $\tau(u_k) = \tau$. There is a vertex v such that $[\mathcal{B}_\tau(u_k)] = [\mathcal{B}_\tau(v)]$ but $[\mathcal{B}_{\tau+1}(u_k)] \neq [\mathcal{B}_{\tau+1}(v)]$. Let $f : \mathcal{B}_\tau(u_k) \to \mathcal{B}_\tau(v)$ be a color-preserving isometry. Let $w = f(u)$. Suppose that $\tau > m+1$. Since $\mathcal{B}_{m+1}(u) \subset \mathcal{B}_\tau(u_k)$, $[\mathcal{B}_{m+1}(u)] = [\mathcal{B}_{m+1}(w)]$. Thus, u and w are in the same class. Since $\mathbf{d}(w, v) = 1$, u_j and v are in the same class for some j. We have

$$[\mathcal{B}_\tau(u_j)] = [\mathcal{B}_\tau(v)] = [\mathcal{B}_\tau(u_k)] \text{ and } [\mathcal{B}_{\tau+1}(u_j)] = [\mathcal{B}_{\tau+1}(v)] \neq [\mathcal{B}_{\tau+1}(u_k)],$$

thus $\tau(u_j) \geq \tau$. By the maximality of τ, $\tau(u_j) = \tau$. By Corollary 6.2.3, if $\tau \geq N_0$, then u_k and u_j are in the same class. It contradicts $[\mathcal{B}_{\tau+1}(u_k)] \neq [\mathcal{B}_{\tau+1}(u_j)]$. Hence, $\tau < N_0$.

We conclude that $\tau > m+1$ implies $\tau < N_0$. If $m \geq N_0 - 1$, then $\tau \leq m+1$. If $m < N_0 - 1$, then $\tau \leq N_0 - 1$. In other words, for u, v such that $\mathbf{d}(u, v)$=1, if $|\tau(u) - \tau(v)| \geq 2$, then $\tau(u), \tau(v) \leq N_0 - 1$. Thus if $m \geq N_0$, then $\tau(u_i) \geq m-1$.

(2) Let $m \geq N_0$. Suppose that there is no u_i such that $\tau(u_i) = m+1$. By (1), $m-1 \leq \tau(u_i) \leq m$ for each i. If $\tau(u_i) = m-1$, then there is no vertices on $\mathcal{B}_1(u_j)$ of maximal type greater than m. Even if $\tau(u_i) = m$, since u and u_i are in the same class by Corollary 6.2.3, we have the same conclusion. Thus, there is no vertex on $\mathcal{B}_2(u)$ of maximal type greater than m. Inductively, every vertex is of maximal type less than $m+1$. It contradicts the fact that $b(n)$ is strictly increasing.

(3) We can show it by the similar argument of the proof of (2). □

For a quasi-Sturmian coloring of bounded type, we define

$$N_1 = \max\{N_0, \ \min\{\tau(x) : x \in V\mathcal{T}\}\}. \tag{6.2}$$

For a coloring of bounded type, we define the subgraph G of X as the graph consisting of the vertices of maximal type less than or equal to N_1. The next proposition follows from Corollary 6.2.3 and Lemma 6.2.4.

Proposition 6.2.5. For the quotient graph $\mathcal{X} = (X, i)$ of a quasi-Sturmian coloring ϕ of bounded type, the quotient graph X is a union of G and a geodesic ray (see the following figure).

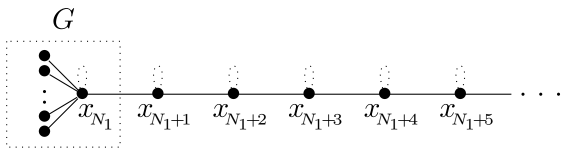


Figure 6.2 Quotient graphs of quasi-Sturmian colorings of bounded type

The quotient graph X is linear from the vertex of maximal type $N_1 + 1$. In the figure, the vertex labeled by x_k is of maximal type k.

The quotient graph of a Sturmian coloring of unbounded type is a geodesic ray or an infinite geodesic (see Theorem 3.8 in [37]). Now, we show that the same property holds for quasi-Sturmian colorings of unbounded type.

Proposition 6.2.6. For a quasi-Sturmian coloring of unbounded type, the vertices of a 1-ball have at most three distinct type sets.

Proof. Let us assume that there are three neighboring vertices u_1, u_2, u_3 of u such that the type sets of u, u_1, u_2, u_3 are all distinct. Since each special n-ball is unique for $n \geq N_0$, if there is $n \in \Lambda_u \cap \Lambda_v$ such that $n \geq N_0$, then $[\mathcal{B}_n(u)] = [\mathcal{B}_n(v)]$. Thus, if $\Lambda_u \cap \Lambda_v$ is infinite, then $\Lambda_u = \Lambda_v$. Let $N = \max \Lambda_u \cap \Lambda_v$. Note that $\Lambda_u \cap \Lambda_v$ is non-empty since every type set contains -1. Choose such N for each pair of vertices from different classes in $\mathcal{B}_2(u)$ and let M be the maximum of such N's. Then, the type sets of two non-equivalent vertices in $\mathcal{B}_2(u)$ intersected with $\{M+1, M+2, \cdots\}$ are all mutually disjoint.

Now let $l > M + 1$ be in the type set Λ_u. Such l exists since the coloring is of unbounded type. At least one of u_1, u_2, u_3 has a type set disjoint from $\{l-1, l, l+1\}$, say u_i. Since $l \in \Lambda_u$, there is v such that $[\mathcal{B}_l(u)] = [\mathcal{B}_l(v)]$ but $[\mathcal{B}_{l+1}(u)] \neq [\mathcal{B}_{l+1}(v)]$. Let $f : \mathcal{B}_l(u) \to \mathcal{B}_l(v)$ be a color-preserving isometry. Then $[\mathcal{B}_{l-1}(u_i)] = [\mathcal{B}_{l-1}(f(u_i))]$.

Let $p = \min\{k \geq l-1 : k \in \Lambda_{u_i}\}$. Since $p > l+1$, $[\mathcal{B}_{l-1}(u_i)]$ has a unique extension to $[\mathcal{B}_p(u_i)]$. Thus, $[\mathcal{B}_p(u_i)]$ and $[\mathcal{B}_p(f(u_i))]$ are equivalent by a color-preserving isometry g. Since $[\mathcal{B}_{p-1}(g^{-1}(v))] = [\mathcal{B}_{p-1}(v)]$ and $p-1 > l$, $[\mathcal{B}_l(g^{-1}(v))] = [\mathcal{B}_l(v)] = [\mathcal{B}_l(u)]$ and $[\mathcal{B}_{l+1}(g^{-1}(v))] = [\mathcal{B}_{l+1}(v)] \neq [\mathcal{B}_{l+1}(u)]$. Thus, $g^{-1}(v) \neq u$ and $\Lambda_{g^{-1}(v)} \cap \Lambda_u$

contains $l > M+1$. However, since $\mathbf{d}(g^{-1}(v), u) \leq 2$, it contradicts that $\Lambda_{g^{-1}(v)} \cap \Lambda_u \cap \{M+1, M+2, \cdots\}$ is empty. □

Let $(\mathcal{T}, \phi)$ be a quasi-Sturmian coloring of a tree and $\mathcal{X} = (X, i)$ be its quotient graph. If two vertices u, v have the same type set, they have the same colored n-balls for every n, i.e. u, v are equivalent (see Lemma 2.4 in [37]). By Proposition 6.2.6, there are at most 2 adjacent vertices of each vertex $x \in VX$.

For a quasi-Sturmian coloring of unbounded type, we define G as the set of vertices which have only one adjacent vertex in X. Since factor complexity of ϕ is unbounded, X is an infinite graph. Since X is connected, G is empty or G has a single element. Thus, we obtain the following theorem characterizing the quotient graphs of quasi-Sturmian colorings of trees.

Theorem 6.2.7 *If ϕ is a quasi-Sturmian coloring, then its quotient graph is one of the following graphs.*

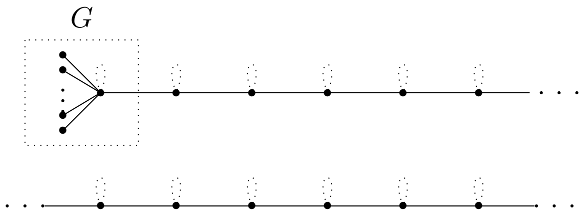


Figure 6.3 Quotient graphs of quasi-Sturmian colorings

More precisely, the quotient graph of a coloring of bounded type is the first graph, where as the quotient graph of a coloring of unbounded type is a geodesic ray or a bi-infinite geodesic.

6.2.2 Evolution of factor graphs

In this section, we look into quasi-Sturmian colorings of unbounded type in details. Let us begin by explaining an induction algorithm for quasi-Sturmian colorings of bounded type. As in [38], for $n \geq N_0$, S_n denotes a unique special n-ball, C_n denotes a centered n-ball of S_{n+1}, and A_{n+1}, B_{n+1} denote two types of extensions of S_n. For a class of n-balls $B = [\mathcal{B}_n(x)]$, denote the class of $[\mathcal{B}_{n+1}(x)]$ by $\overline{B}$ and the class of $[\mathcal{B}_{n-1}(x)]$ by $\underline{B}$. Note that if B is not special, then $\overline{B}$ is well-defined.

Recall from the introduction that for a given quasi-Sturmian coloring ϕ, for $n \geq N_0 + 1$, the *factor graph* $\mathcal{G}_n$ has $\mathbb{B}_\phi(n)$ as its vertex set. There is an edge between two colored n-balls D, E if there exist n-balls centered at x, y in the classes D, E, respectively, such that $\mathbf{d}(x, y)$=1.

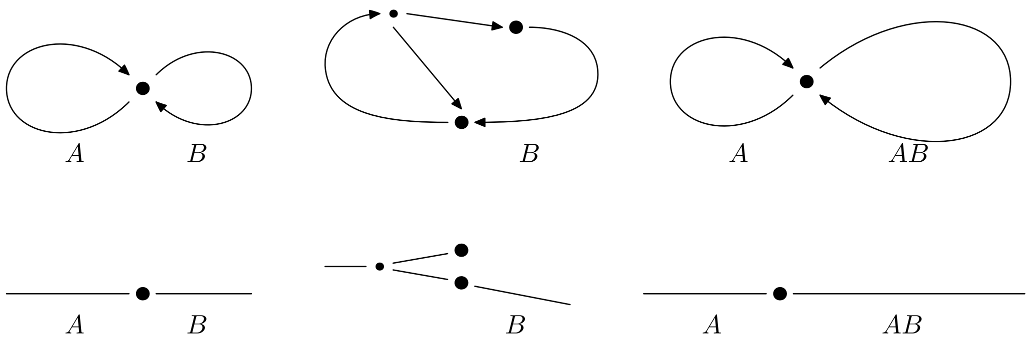


Figure 6.4 The evolution of Rauzy graphs of a quasi-Sturmian word (above) and the evolution of $\mathcal{G}_n$ of a quasi-Sturmian coloring on a tree (below)

Cyclic quasi-Sturmian colorings

Now, we gather preliminaries of cyclic quasi-Sturmian colorings.

Definition 6.2.8. We say that D is *weakly adjacent* to E if there exist $v, w \in V\mathcal{T}$ such that $\mathbf{d}(v, w) = 1$ and $[\mathcal{B}_n(v)] = D$ and $[\mathcal{B}_m(w)] = E$ for some n, m.

We also say that D is *strongly adjacent* to E if for any $\mathcal{B}_n(x)$ in the class D, there exists a vertex y such that $\mathcal{B}_m(y) \in E$ and $\mathbf{d}(x, y) = 1$. If D is strongly adjacent to E and vice versa, then we say that D and E are strongly adjacent.

We remark the following fact. If $[\mathcal{B}_{n+1}(u)] = [\mathcal{B}_{n+1}(v)]$ and $[\mathcal{B}_{n+2}(u)] \neq [\mathcal{B}_{n+2}(v)]$, then there exist neighboring vertices u' and v' of u and v, respectively, such that $[\mathcal{B}_n(u')] = [\mathcal{B}_n(v')]$ and $[\mathcal{B}_{n+1}(u')] \neq [\mathcal{B}_{n+1}(v')]$ (see Lemma 2.11 in [37] for details). Thus, S_{n+1} is strongly adjacent to S_n for $n \geq N_0$.

Lemma 6.2.9. Let $(\mathcal{T}, \phi)$ be a quasi-Sturmian coloring and $n \geq N_0$.

(1) We can choose $\{A_n\}_{n \geq N_0+1}$, $\{B_n\}_{n \geq N_0+1}$ so that A_{n+1}, B_{n+1} are strongly adjacent to A_n, B_n, respectively. Moreover, A_{n+1}, B_{n+1} are uniquely determined if we give the condition that A_{n+1} contains more balls of the class A_n than B_{n+1} does.

(2) For each vertex x in $\mathcal{T} - \widetilde{G}$ and $n \geq N_0 + 1$, the n-balls with centers adjacent to x belong to at most two classes of n-balls apart from $[\mathcal{B}_n(x)]$. Thus, for any class $D \neq S_n$ of n-balls with centers in $\mathcal{T} - \widetilde{G}$, each vertex of $\mathcal{G}_n$ has degree at most 2.

(3) If $A_n \neq S_n$ (respectively $B_n \neq S_n$), then A_n (respectively B_n) is strongly adjacent to S_n.

(4) The two classes S_n, C_n are strongly adjacent.

We will specify the choice of A_{N_0+1} from the two extensions of S_{N_0} for acyclic quasi-Sturmian colorings later.

Lemma 6.2.10. Let ϕ be a quasi-Sturmian coloring and n be greater than N_0. Let D be a colored n-ball other than A_n, B_n and S_n. Assume that S_n and D are weakly adjacent. Then, we have that

(1) the special ball S_n and D are strongly adjacent, and

(2) if $D \neq C_n$, then $S_n \neq C_n$.

Proposition 6.2.11. If there are two vertices of degree at least three in $\mathcal{G}_n$ for some $n > N_0$, then the quasi-Sturmian coloring $(\mathcal{T}, \phi)$ is of bounded type.

Proof. If ϕ is of unbounded type, S_n is the unique vertex adjacent to distinct three classes of n-balls in $\mathcal{G}_n$ by Lemma 6.2.9 (2). Thus, there is at most one vertex of degree at least three in $\mathcal{G}_n$. □

Definition 6.2.12. A quasi-Sturmian coloring is *cyclic* if there is a cycle containing S_n in $\mathcal{G}_n$ for some $n > N_0$. If not, we say that a quasi-Sturmian coloring is *acyclic*.

Lemma 6.2.13. Suppose that $\mathcal{G}_n$ has a cycle whose lift in X is not contained in G for some $n \geq N_0 + 1$. The following statements hold.

(1) The special ball S_n is in the cycle.

(2) If $D \neq A_n, B_n, C_n, S_n$, then D is not weakly adjacent to S_n.

Lemma 6.2.14. For $n > N_0$, suppose that $\mathcal{G}_n$ has a cycle whose lift in X is not contained in G.

(1) If C_n is not contained in the cycle, then $\mathcal{G}_{n+l}$ has a cycle containing C_{n+l} for some $l \geq 1$.

(2) If $C_n = S_n$, then $\mathcal{G}_{n+1}$ has a cycle containing C_{n+1} and $C_{n+1} \neq S_{n+1}$.

Proposition 6.2.15. (1) Let $n \geq N_0 + 1$. If there is a ball D which is weakly adjacent to S_n and different from $A_n, B_n, C_n,$ and S_n, then $\mathcal{G}_{n+1}$ has a cycle containing $\overline{D}$.

(2) Any cyclic quasi-Sturmian coloring is of bounded type.

Acyclic quasi-Sturmian colorings

Lemma 6.2.16. Let ϕ be an acyclic quasi-Sturmian coloring. If $A_N = S_N = C_N$ for some $N > N_0 + 1$, then $A_n = S_n = C_n$ for all $N_0 + 1 \leq n < N$.

We choose A_n as $S_n = C_n = A_n$ if there exists $n > N_0$ such that $S_n = C_n$ is identical to A_n or B_n. Define

$$K = \min\{n > N_0 : A_n, S_n, C_n \text{ are not all identical}\}$$

as in [38]. Note that K may be infinity.

For an acyclic quasi-Sturmian coloring, for each $n \geq K$, neither A_n, S_n, C_n nor B_n, S_n, C_n are identical. Therefore, the colored n-balls S_n, A_n, B_n, C_n satisfy one of the following conditions.

(I) S_n, C_n are distinct, but one of S_n, C_n is identical to A_n or B_n.

(II) S_n, A_n, B_n, C_n are all distinct.

(III) S_n, A_n, B_n are distinct, but $S_n = C_n$.

Case (I) is divided into three subcases:

(I-a) A_n, B_n, S_n are distinct and $C_n = A_n$ or B_n,

(I-b) A_n, B_n, C_n are distinct and $S_n = A_n$ or B_n,

(I-c) $A_n = S_n$, $B_n = C_n$ are distinct, By Lemma 6.2.13 and Lemma 6.2.15, we deduce that S_n is a vertex of degree 3 in $\mathcal{G}_n$ for Case (II), but for Case (I) and (III), $\mathcal{G}_n$ is a linear graph and S_n is of degree 1 or 2.

Proposition 6.2.17. Suppose that $\mathcal{G}_n$ corresponds to Case (I). Then S_n is a vertex of degree 2 or 1 in $\mathcal{G}_n$. Thus $\mathcal{G}_n$ is a linear graph. Let m be the number of vertices connected to S_n through C_n. Note that $m \geq 1$ since C_n is not identical to S_n. Then we have $\mathcal{G}_{n+k}$ belongs to Case (II) for all $0 < k < m$ and either $\mathcal{G}_{n+m}$ belongs to Case (I) or $\mathcal{G}_{n+m}$ belongs to Case (III) and $\mathcal{G}_{n+m+1}$ belongs to Case (I).

Proof. If S_n and C_n are distinct, then $\mathcal{G}_n$ belongs to Case (I) or (II). We deduce that S_{n+1}, A_{n+1}, B_{n+1} are distinct. If C_n is of degree 2, then there exists D neighboring C_n which is not S_n. Thus $\overline{D}$ is weakly adjacent to S_{n+1} but different from $S_{n+1}, A_{n+1}, B_{n+1}$, which implies that $\overline{D} = C_{n+1}$, which corresponds Case (II). In this case, the number of vertices connected to S_{n+1} through C_{n+1} decreases by 1.

If C_n is of degree 1, then $m = 1$. In this case, S_{n+1} is connected to only two extensions A_{n+1}, B_{n+1} of S_n in $\mathcal{G}_{n+1}$, which implies that $C_{n+1} = S_{n+1}$, i.e. Case (III) or $C_{n+1} = A_{n+1}$ or B_{n+1}, i.e. Case (I-a).

If $\mathcal{G}_n$ belongs to Case (III), then $S_n = C_n$, thus we have either $S_{n+1} = A_{n+1}$ or $S_{n+1} = B_{n+1}$, say $S_{n+1} = A_{n+1}$. Since $\overline{A_n}$ is weakly adjacent to $A_{n+1} = S_{n+1}$ and $\overline{A_n}$ cannot be A_{n+1} nor B_{n+1}, we deduce that $C_{n+1} = \overline{A_n}$. Therefore, $\mathcal{G}_{n+1}$ belongs to the Case (I-b).

We remark that Case (I-c) can happen only for $n = K$. □

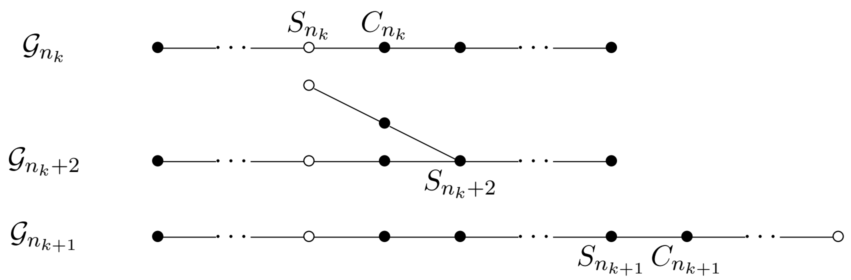


Figure 6.5 The evolution of $\mathcal{G}_{n_k}$ along the path (I) $\to$ (II) $\to \cdots \to$ (II) $\to$ (I) where the vertex $\circ$ represents either S_{n_k} or the extensions of S_{n_k}

We denote by (n_k) the subsequence for which $\mathcal{G}_{n_k}$ is of Case (I). The evolution of $\mathcal{G}_n$ from $n = n_k$ to $n = n_{k+1}$ is shown in Figure 6.5. Compare with Sturmian words (see Figure 6.4): there are infinitely many n's such that the Rauzy graph has disjoint two cycles starting from a common bi-special word (see e.g. [2]). It corresponds to the factor graph $\mathcal{G}_n$ belongs to Case (I).

6.2.3 Quasi-Sturmian colorings of bounded type

In this section, we investigate a necessary and sufficient condition for a quotient graph to be a quotient graph of a quasi-Sturmian coloring of bounded type.

Let x be a vertex of the quotient graph X. For the two lifts $\tilde{x}$ and $\tilde{x}'$ of x, $[\mathcal{B}_n(\tilde{x})] = [\mathcal{B}_n(\tilde{x}')]$ for all n. Then, $\tau(\tilde{x}) = \tau(\tilde{x}')$. By abuse of notation, define $[\mathcal{B}_n(x)]$ as a class $[\mathcal{B}_n(\tilde{x})]$. Define the maximal type $\tau(x)$ of x as $\tau(\tilde{x})$.

Recall the examples in Section 6.2.1. Let $\mathcal{X} = (X, i)$ be the quotient graph for each of them. We obtain a periodic edge-indexed subgraph X' of X by removing a finite subgraph G in Proposition 6.2.5. Then, a lift of $(X', i|_{EX'})$ can be extended to a periodic coloring of a tree. It is natural to guess that the property holds for every quasi-Sturmian coloring.

From now on, let $(\mathcal{T}, \phi)$ be a quasi-Sturmian coloring of bounded type. By Proposition 6.2.5, the quotient graph X of $(\mathcal{T}, \phi)$ is the graph in Figure 6.2. Let $\widetilde{G}$ be the union of lifts of G. A connected component of $\mathcal{T} - \widetilde{G}$ is a lift of $(X - G, i|_{E(X-G)})$. Thus, all connected components of $\mathcal{T} - \widetilde{G}$ are equivalent to each other. Let Y be a connected component of $\mathcal{T} - \widetilde{G}$.

Lemma 6.2.18. If u, v are vertices of Y with $[\mathcal{B}_{N_1}(u)] = [\mathcal{B}_{N_1}(v)]$, where N_1 is as in (6.2), then we have $[\mathcal{B}_{N_1+1}(u)] = [\mathcal{B}_{N_1+1}(v)]$.

Proof. It suffices to consider the case of $[\mathcal{B}_{N_1}(u)] = S_{N_1}$. Every vertex of maximal type N_1 is the center of either A_{N_1+1} or B_{N_1+1}, say A_{N_1+1}. Since vertices of $X - G$ are of maximal type bigger than N_1, if u is a vertex of Y and $[\mathcal{B}_{N_1}(u)] = S_{N_1}$, then $[\mathcal{B}_{N_1+1}(u)] = B_{N_1+1}$. □

We define an edge-indexed graph $\mathcal{Z} = (Z, i_Z)$ as follows : the vertices of Z are of the form $[\mathcal{B}_{N_1}(u)]$ for a vertex u in Y or $X - G$, and any two vertices D, E of Z are adjacent if D and E are weakly adjacent. The index $i_Z(D, E)$ is the number of E which are adjacent to D. The indices are well-defined by Lemma 6.2.18. Since any vertex in $X - G$ is adjacent to at most two vertices besides itself, the graph Z is a line segment or a cycle.

Lemma 6.2.19. A restriction of ϕ on any connected component of $\mathcal{T} - \widetilde{G}$ has a periodic extension to $\mathcal{T}$.

Proof. Let u be the vertex of Y. Define a coloring ψ_k on $\mathcal{B}_k(u)$ with the alphabet $VZ = \{[\mathcal{B}_{N_1}(v)] \,|\, v \in Y\}$ recursively: Put $\psi_0(u) = [\mathcal{B}_{N_1}(u)] \in VZ$. Define $\psi_{k+1}(v) = \psi_k(v)$ for $v \in \mathcal{B}_k(u)$. Choose $w \in V\mathcal{T}$ with $\mathbf{d}(u, w) = k$ and let w_α $(\alpha = 0, \dots d-1)$ be the neighboring vertices of w with $\mathbf{d}(u, w_\alpha) = k+1$ for $\alpha \geq 1$ and $\mathbf{d}(u, w_0) = k-1$. We define $\psi_{k+1}(w_\alpha)$ for $\alpha \geq 1$ in the following ways.

If $w \notin Y$, then $w_\alpha \notin Y$ for all $\alpha \geq 1$. Let $D_0 = \psi_k(w_0)$ and D_j be a colored N_1-ball satisfying $i_Z(\psi_k(w), D_j) > 0$ with $j = 0, 1, 2$ or $j = 0, 1$. We assign $\psi_{k+1}(w_\alpha)$ as D_0 for $1 \leq \alpha < i_Z(\psi_k(w), D_0)$ and, for $\ell \neq 0$,

$$\psi_{k+1}(w_\alpha) = D_\ell \quad \text{for} \quad \sum_{j=0}^{\ell-1} i_Z(\psi_k(w), D_j) \leq \alpha \leq \sum_{j=0}^{\ell} i_Z(\psi_k(w), D_j) - 1.$$

Then we have

$$i_Z(\psi_{k+1}(w), D) = \#\{0 \leq \alpha \leq d \,|\, \psi_{k+1}(w_\alpha) = D\} \tag{6.3}$$

for each $D \in VZ$.

If $w \in Y$, then we put $\psi_{k+1}(w_\alpha) = [\mathcal{B}_{N_1}(w_\alpha)]$ for all $\alpha \geq 1$. Using the fact that Y is an infinite subgraph of T, Lemma 6.2.18 implies that there exists a vertex v such that $\mathcal{B}_{N_1+1}(v) \subset Y$ and $[\mathcal{B}_{N_1+1}(v)] = [\mathcal{B}_{N_1+1}(w)]$, thus $\psi_{k+1}(w_\alpha) = [\mathcal{B}_{N_1}(w_\alpha)] \in VZ$ and (6.3) is satisfied.

Since $\psi_{k+\ell}|_{\mathcal{B}_k(u)} = \psi_k$ for $\ell \geq 1$, the coloring $\psi = \lim_{k \to \infty} \psi_k$ on $\mathcal{T}$ with alphabet VZ exists. By (6.3), we deduce that $\mathcal{Z}$ is the quotient graph of ψ. Since $\psi(u) = [\mathcal{B}_{N_1}(u)]$ on Y, by the coloring which gives the color of the center of $\psi(u)$, we complete the proof. $\square$

Theorem 6.2.20 (Quotient graphs of colorings of bounded type) *Let $\mathcal{X} = (X, i)$ be the quotient graph of a coloring $(\mathcal{T}, \phi)$. The following statements are equivalent.*

(1) The coloring ϕ is a quasi-Sturmian coloring of bounded type.

(2) *There is a finite connected subgraph G of the quotient graph X such that $X - G$ is a connected infinite ray and any connected component of $\mathcal{T} - \widetilde{G}$ has a periodic extension to $\mathcal{T}$ where $\widetilde{G}$ is the union of lifts of G.*

Proof. By Lemma 6.2.18 and Lemma 6.2.19, (1) implies (2). Now we assume (2) holds. Let $\mathcal{A}$ be the alphabet of ϕ. Let $\tilde{x}$ be a lift of $x \in VX$. Define a new coloring ψ with an alphabet $\mathcal{A} \sqcup VG$ as

$$\psi(v) = \begin{cases} x & \text{if } v = \tilde{x} \text{ for some } x \in VG, \\ \phi(v) & \text{otherwise.} \end{cases}$$

Denote by $[\mathcal{B}_n(u)]_\psi$ a ψ-colored n-ball. As ever $[\mathcal{B}_n(u)]$ means a ϕ-colored n-ball. A map $\mathbb{B}_\psi(n) \to \mathbb{B}_\phi(n)$ which defined by $[\mathcal{B}_n(x)]_\psi \mapsto [\mathcal{B}_n(x)]$ is surjective. It implies $b_\phi(n) \leq b_\psi(n)$. Since X is not a finite graph, $b_\phi(n)$ is strictly increasing. Thus, it is enough to show that b_ψ is linear.

Let us denote by $\mathbf{d}(x, G) = \min\{\mathbf{d}(x, g) : g \in VG\}$ for $x \in VX$. Fix a positive integer n. If $\mathbf{d}(x, G) \leq n$, then $[\mathcal{B}_n(x)]_\psi \neq [\mathcal{B}_n(y)]_\psi$ for any other $y \in VX$. If x is a vertex such that $\mathbf{d}(x, G) > n + 1$, then $[\mathcal{B}_{n+1}(x)]_\psi = [\mathcal{B}_{n+1}(x)]$. Thus, $[\mathcal{B}_n(x)]$ has a unique extension to a colored $(n + 1)$-ball. Since X is not finite, ψ has at least one special n-ball for each n. Thus, for x such that $\mathbf{d}(x, G) = n+1$, $[\mathcal{B}_n(x)]$ is the unique special n-ball and it has exactly two extensions to colored $(n + 1)$-balls. It means that $b_\psi(n) = n + |\mathcal{A}| + |VG|$ for all n. □

6.2.4 Recurrence functions of colorings of trees

In this section, we will extend the notion of recurrence functions $R(n)$, $R''(n)$ for words to colorings of trees. We will show that the quasi-Sturmian colorings of trees satisfy a certain inequality between $R''(n)$ and $b(n)$. We also explain that the existence of $R(n)$ is related to unboundedness of the quasi-Sturmian colorings of trees.

Let us briefly recall recurrence functions of words (see Section 10.9 in [7] for definitions and details). Recurrence functions are important objects related to symbolic dynamics. Let Σ be a finite alphabet. Let Σ^* be the set of finite words over Σ and $\Sigma^{\mathbb{N}}$ be the set of infinite words over Σ. For $\mathbf{u} \in \Sigma^* \cup \Sigma^{\mathbb{N}}$, we denote by $F_n(\mathbf{u})$ the set of factors of length n of $\mathbf{u}$.

A recurrence function $R_{\mathbf{u}}(n)$ is defined as the smallest integer $m \geq 1$ such that every factor of length m contains all factors of length n. It is known that such an integer $R_{\mathbf{u}}(n)$ exists for all n if and only if the word is *uniformly recurrent*, i.e. any subword of the word infinitely occurs with bounded gaps. Another recurrence function $R''_{\mathbf{u}}(n)$ is defined by

$$R''_{\mathbf{u}}(n) = \min\{m \in \mathbb{N} \mid F_n(\mathbf{u}) = F_n(\omega) \text{ for some } \omega \in F_m(\mathbf{u})\},$$

i.e. it is the length of the smallest factor of $\mathbf{u}$ that contains all factors of length n of $\mathbf{u}$. From the definition, the following fact immediately holds.

Remark. For all $n \geq 0$, $R''_{\mathbf{u}}(n) \geq p_{\mathbf{u}}(n) + n - 1$ for any word $\mathbf{u}$.

Recall that a word $\mathbf{u}$ is said to have *grouped factors* if, for all $n \geq 0$, it satisfies $R''_{\mathbf{u}}(n) = p_{\mathbf{u}}(n) + n - 1$. If there is n_0 such that the equality holds for all $n \geq n_0$, we say that $\mathbf{u}$ has *ultimately grouped factors*. Cassaigne suggested some conditions that guarantee the equality.

Theorem 6.2.21 ([18]) *A word* $\mathbf{u}$ *is Sturmian if and only if* $R''_{\mathbf{u}}(n) = 2n$ *for every* $n \geq 0$*. A uniformly recurrent word on a binary alphabet has ultimately grouped factors if and only if it is periodic or quasi-Sturmian.*

We want analogous results for quasi-Sturmian colorings of trees. Let $(\mathcal{T}, \phi)$ be a quasi-Sturmian coloring of a tree and $\mathcal{X} = (X, i)$ be the quotient graph of $(\mathcal{T}, \phi)$. We define $R_\phi(n)$ as the smallest radius m such that every colored n-ball of ϕ occurs in $[\mathcal{B}_m(x)]$ for all $x \in V\mathcal{T}$. We define $R''_\phi(n)$ as the smallest radius m such that every colored n-ball of ϕ occurs in $[\mathcal{B}_m(x)]$ for some $x \in V\mathcal{T}$.

Definition 6.2.22. A coloring of a tree $(\mathcal{T}, \phi)$ is said to be *recurrent* if, for any compact subtree $\mathcal{T}'$, every colored ball appears in $\mathcal{T} - \mathcal{T}'$. A coloring of a tree is said to be *uniformly recurrent* if $R_\phi(n) < \infty$ for all n.

Proposition 6.2.23. Let $(\mathcal{T}, \phi)$ be a quasi-Sturmian coloring of a tree. The following conditions are equivalent.

(1) $(\mathcal{T}, \phi)$ is of unbounded type.

(2) $(\mathcal{T}, \phi)$ is uniformly recurrent.

(3) For any colored ball, it appears in $\mathcal{T} - \pi^{-1}(S)$ for any finite set $S \subset X$.

Proof. (1) implies (2) : Suppose $(\mathcal{T}, \phi)$ is of unbounded type. Let $n \geq N_0$. For each colored n-ball $E = [\mathcal{B}_n(w)]$, we define m_E to be the smallest element of $\Lambda_w \cap \{n, n+1, \dots\}$ which is not empty since Λ_w is infinite. Note that m_E depends only on E and not on w.

Choose a vertex $v \in V\mathcal{T}$ and a colored n-ball E which is distinct from $[\mathcal{B}_n(v)]$. Let $m = m_E$. Denote $F^1 = [\mathcal{B}_m(v)]$ which is not S_m. Let $[F^1 - F^2 - \cdots - F^l - S_m]$ be the shortest path from F^1 to S_m in $\mathcal{G}_m$. For arbitrary colored m-balls F and F', if $F \neq S_m$, then F has the unique extension. Thus, if F is weakly adjacent to F', then F is strongly adjacent to F'. Therefore, there is a path $[v - v_2 - v_3 - \cdots - v_l - w']$ in $\mathcal{T}$ such that $[\mathcal{B}_m(v_i)] = F^i$, $i = 2, \cdots, l$, and $[\mathcal{B}_m(w')] = S_m$.

Since S_m occurs in $[\mathcal{B}_{m+l}(v)]$, E occurs in $[\mathcal{B}_{n+l}(v)]$. Since $l \leq |V\mathcal{G}_m| = m + c$, E occurs in $[\mathcal{B}_{n+m+c}(v)]$. Every colored n-ball occurs in $[\mathcal{B}_{n+M+c}(v)]$ where $M = \max\{m_E : E \in \mathbb{B}_\phi(n)\}$. Thus, $R_\phi(n) \leq n + M + c$.

(2) implies (3) : Suppose that $R_\phi(n)$ exists for all n. Since the quotient graph X is infinite, for any finite $S \subset X$, there is x such that $\mathcal{B}_{R_\phi(n)}(x) \subset \mathcal{T} - \pi^{-1}(S)$.

(3) implies (1) : Assume that $(\mathcal{T},\phi)$ is of bounded type. Let v be a vertex of maximal type N_1. By Proposition 6.2.5, all vertices in $X - G$ are of maximal type larger than N_1. Therefore, $[\mathcal{B}_{N_1+1}(v)]$ does not appear in $\mathcal{T} - \pi^{-1}(G)$. □

Recall that we denote by $\mathcal{Z}$ the quotient graph of $\mathcal{T} - \widetilde{G}$ with respect to the coloring ϕ. By abuse of notation, let $\mathbf{d}$ be the metric on X or $\mathcal{G}_n$ induced by $\mathbf{d}$ on T. Let us denote by

$$\mathbf{r}(x, G) = \max\{\mathbf{d}(x, y) : y \in VG\}.$$

Proposition 6.2.24. Let $(\mathcal{T}, \phi)$ be a quasi-Sturmian coloring.

(1) Let ϕ be of unbounded type. As in Proposition 6.2.17, the factor graph $\mathcal{G}_n$ is of Case (I) on $n = n_k$. Then, we have

$$R''_\phi(n) = n + \left\lfloor \frac{b_\phi(n_k)}{2} \right\rfloor \quad \text{for } n_{k-1} < n \leq n_k.$$

(2) Let ϕ be of bounded type. Let x_{N_1} be the vertex of X which is of maximal type N_1.

(a) If Z is acyclic, then we have

$$R''_\phi(n) = n + \left\lfloor \frac{1}{2}(b_\phi(n_k) - |VG| + \mathbf{r}(x_{N_1}, G) + 1) \right\rfloor \quad \text{for } n_{k-1} < n \leq n_k.$$

(b) If Z is cyclic, then we have

$$R''_\phi(n) = n + \left\lfloor \frac{1}{2}(b_\phi(n) - |VG| + \mathbf{r}(x_{N_1}, G) + 1) \right\rfloor \quad \text{for all } n \geq N_1.$$

Proof. (1) In the case of a quasi-Sturmian coloring of unbounded type, the evolution of the factor graph follows Proposition 6.2.17. Let D and E be n_k-balls which are weakly adjacent. If $D \neq S_{n_k}$ or if $D = S_{n_k}$, $E = C_{n_k}$, then D and E are strongly adjacent by Lemma 6.2.9 (3), (4). If $D = S_{n_k}$ and $E \neq C_{n_k}$, then there exist vertices v, u and w in T with $\mathbf{d}(v,u) = \mathbf{d}(v,w) = 1$ such that $D = [\mathcal{B}_{n_k}(v)]$, $E = [\mathcal{B}_{n_k}(u)]$ and $C_{n_k} = [\mathcal{B}_{n_k}(w)]$. Therefore, we can take a path with length $b_\phi(n_k) - 1$ consisting of centers of all the colored n_k-balls in $\mathcal{T}$. Thus, we have

$$R''_\phi(n_k) \leq n_k + \left\lfloor \frac{b_\phi(n_k)}{2} \right\rfloor.$$

Let D_{n_k}, E_{n_k} be the colored n_k-balls which are the end points of the graph $\mathcal{G}_{n_k}$. The distance between D_{n_k} and E_{n_k} in $\mathcal{G}_{n_k}$ is $b_\phi(n_k) - 1$, thus for any vertices z, z' in T such that $[\mathcal{B}_{n_k}(z)] = D_{n_k}$ and $[\mathcal{B}_{n_k}(z')] = E_{n_k}$, we have $\mathbf{d}(z,z') \geq b_\phi(n_k) - 1$. Therefore, it follows that

$$R''_\phi(n_k) = n_k + \left\lfloor \frac{b_\phi(n_k)}{2} \right\rfloor.$$

Now, let us consider the case $n_{k-1} < n < n_k$, then $\mathcal{G}_n$ is of Case (II) or Case (III). We define D_n, E_n and F_n as the colored n-balls which are the vertices of degree 1 and connected to S_n through A_n, B_n, C_n in $\mathcal{G}_n$, respectively. Note that if $S_n = C_n$, then we define $F_n = C_n$. Any vertex of the center of special ball S_n in T is adjacent to either centers of A_n and C_n or centers of B_n and C_n. Thus, the distance between the centers of D_n and E_n in T is at least $\mathbf{d}(D_n, F_n) + \mathbf{d}(E_n, F_n)$.

If $\mathcal{G}_n$ is of Case (II) for all $n_{k-1} < n < n_k$, then $\mathbf{d}(D_n, F_n) + \mathbf{d}(E_n, F_n) = b_\phi(n_k) - 1$. Otherwise, $\mathcal{G}_n$ is of Case (III) for $n = n_k - 1$ and $\mathcal{G}_n$ is of Case (II) for $n_{k-1} < n < n_k - 1$. Then, $\mathbf{d}(D_n, F_n) + \mathbf{d}(E_n, F_n) = b_\phi(n_k - 1) - 1$. However, on T, a path from a center of D_n to a center of E_n has at least two vertices which are centers of F_n where they are extended to two distinct colored n_k-balls C_{n_k} and S_{n_k}. It means that the length of the path is at least $b_\phi(n_k - 1) - 1 + 1 = b_\phi(n_k) - 1$. Thus,

$$R''_\phi(n) \geq n + \left\lfloor \frac{b_\phi(n_k)}{2} \right\rfloor \quad \text{for } n_{k-1} < n < n_k.$$

On the other hand, since each n-ball is the restriction of an n_k-ball, there exists a path with length $b_\phi(n_k) - 1$ consisting of centers of all the colored n_k-balls in $\mathcal{T}$. Thus we have the conclusion.

(2)-(a) If Z is acyclic and $n \geq N_1$, then the evolution of the factor graph $\mathcal{G}_n$ also follows Proposition 6.2.17. Hence, we apply the argument similar to the argument in (1). The difference between (1) and (2)-(a) is the existence of the compact part G of the quotient graph X. Take a finite graph G' in $\mathcal{G}_{n_k}$ isomorphic to G. Since every vertex in $\mathcal{G}_{n_k} - G'$ has at most degree 2, the maximal distance between any two vertices in $\mathcal{G}_{n_k}$ is $b_\phi(n_k) - |VG| + \mathbf{r}(x_{N_1}, G)$. Thus, by the similar argument with (1), we have for $n_{k-1} < n \leq n_k$

$$R''_\phi(n) = n + \left\lfloor \frac{1}{2}(b_\phi(n_k) - |VG| + \mathbf{r}(x_{N_1}, G) + 1) \right\rfloor.$$

(2)-(b) Let Z be cyclic and assume that $n \geq N_1$. Let G' be the subgraph of $\mathcal{G}_n$, which is isomorphic to G. Then $\mathcal{G}_n - G'$ consists of a cyclic graph isomorphic to Z and a finite linear graph with a common vertex S_n which is the unique vertex of degree 3 in $\mathcal{G}_n - G'$. We may assume that A_n belong to the cycle in $\mathcal{G}_n - G'$. Consider the path $P = [A_n - \cdots - C_n - S_n - B_n - \cdots - [\mathcal{B}_n(\tilde{x}_{N_1})]]$ in $\mathcal{G}_n$, where a vertex $\tilde{x}_{N_1}$ is a lifting of x_{N_1} in $\mathcal{T}$. Since a vertex in T which is the center of B_{n+1} is a center of S_n and adjacent to centers of B_n, C_n (Lemma 6.2.9), there exists a lifting of a path P in T. Since the length of the path P is $b_\phi(n) - |VG|$, the maximal distance between any two vertices in $\mathcal{G}_n$ is also $b_\phi(n) - |VG| + \mathbf{r}(x_{N_1}, G)$. By the similar argument before, we have the third assertion. □

We note that the converse of the proposition does not hold. Consider a sequence of words

$$X_k = \begin{cases} aL_kaL_kbL_ka, & \text{if } k \text{ is odd,} \\ bL_kaL_kbL_kb, & \text{if } k \text{ is even,} \end{cases}$$

where L_k is given by $L_1 = \varepsilon$, the empty word and $L_{k+1} = L_kaL_k$ for odd k, $L_{k+1} = L_kbL_k$ for even k recursively. Then L_k is a palindrome and we get

$$X_1 = aaba, \qquad X_2 = baaabab, \qquad X_3 = aabaaabababaa, \qquad \ldots$$

Since X_k is a factor of X_{k+1}, we have a coloring ϕ of a 2-regular tree by the limit of X_k. Let $n_k = |L_ka_kL_k| = 2^k - 1$. Then we can check that for $n_{k-1} < n \leq n_k$, we have

$$R''_\phi(n) - n = \left\lfloor \frac{|X_k|}{2} \right\rfloor$$

and

$$b_\phi(n_k) = |X_k|.$$

Thus, we have

$$R''_\phi(n) = n + \left\lfloor \frac{b_\phi(n_k)}{2} \right\rfloor \quad \text{for } \ n_{k-1} < n \leq n_k.$$

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국문초록

디오판틴 근사는 무리수의 유리수 근사를 뜻하는데 연분수를 사용하여 연구되어 왔습니다. 이 논문에서는 디오판틴 근사와 연분수에 관련된 세 가지 주제를 다루고 있습니다.

첫 번째 주제는 헤케군에 관련된 마르코프와 라그랑지 스펙트럼입니다. 고전적인 마르코프와 라그랑지 스펙트럼은 모듈러군 $\mathrm{PSL}(2,\mathbb{Z}) = \mathbf{H}_3$와 관련이 있는데, 단순연분수를 사용하여 연구되어 왔습니다. 우리는 $\mathbf{H}_4$와 $\mathbf{H}_6$에 관련된 마르코프와 라그랑지 스펙트럼을 다룹니다. 우리는 로믹 동역학을 이용하여 고전적인 마르코프와 라그랑지 스펙트럼에서 발견된 결과가 헤케군에 관련된 마르코프와 라그랑지 스펙트럼에서도 나타남을 보입니다.

두 번째 주제는 스터미안 단어의 반복지수입니다. 스터미안 단어의 반복지수는 그 스터미안 단어와 연관된 스터미안 수의 비합리성 지수를 줍니다. 주어진 무리수 θ에 대해, 우리는 기울기가 θ인 스터미안 단어의 반복지수 중 최소값을 밝힙니다. 또한 우리는 황금비를 기울기로 갖는 스터미안 단어의 반복지수들의 스펙트럼을 연구합니다.

마지막 주제는 정규나무 위에서의 준-스터미안 채색입니다. 우리는 정규나무의 준-스터미안 채색을 이것의 몫 그래프와 회귀함수로 구분짓습니다. 우리는 스터미안 단어의 연분수 알고리즘과 유사한 준-스터미안 채색의 귀납적 알고리즘을 밝힙니다.

주요어: 디오판틴 근사, 연분수, 라그랑지 수, 마르코프 수, 스터미안 단어, 나무의 채색

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감사의 글

졸업 논문을 쓰면서 지난 학위 과정 동안 많은 분들의 도움과 수고가 있었음을 느낍니다. 먼저 저의 지도교수님이신 임선희 교수님께 감사의 마음을 전합니다. 군대 문제로 학위 과정이 순탄하지 않았음에도 교수님 덕분에 학위 과정으로 다시 돌아올 수 있었고 무사히 마칠 수 있었습니다. 그리고 저에게 재밌는 문제를 주시고 함께 지도해주신 김동한 교수님께도 감사의 마음을 전합니다. 두 교수님 덕분에 연구 방향을 잡을 수 있었고, 부족한 저에게 연구자로서의 태도와 마음가짐을 알게 해주셔서 감사합니다. 학위 과정 동안 함께 연구하며 부족한 저를 도와주신 차병철 교수님과 슬비누나에게도 감사의 마음을 전합니다. 논문 피드백을 주시고 리뷰어로 기회를 주셨던 Yann Bugeaud 교수님께도 감사의 마음을 전합니다. 그리고 바쁘신 와중에도 제 구술고사와 학위 논문을 심사해주신 국웅 교수님, Otto van Koert 교수님, 김도형 교수님께도 감사드립니다.

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