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Abduction as Deductive Saturation: a Proof-Theoretic Inquiry

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Abstract
Abductive reasoning involves finding the missing premise of an "unsaturated" deductive inference, thereby selecting a possible *explanans* for a conclusion based on a set of previously accepted premises. In this paper, we explore abductive reasoning from a structural proof-theory perspective. We present a hybrid sequent calculus for classical propositional logic that uses sequents and antisequents to define a procedure for identifying the set of analytic hypotheses that a rational agent would be expected to select as *explanans* when presented with an abductive problem. Specifically, we show that this set may not include the deductively minimal hypothesis due to the presence of redundant information. We also establish that the set of all analytic hypotheses exhausts all possible solutions to the given problem. Finally, we propose a deductive criterion for differentiating between the best *explanans* candidates and other hypotheses.

Keywords Abduction · Refutation · Analyticity · Proof theory

1 Introduction

Abductive processes are ubiquitous in scientific theorizing and everyday life. They involve inherent cognitive risk for a rational agent who must select possible *explanantes* for an *explanandum* based on incomplete or uncertain information.

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Although these processes are not deductive in nature, the ultimate goal of a rational agent in abductive reasoning can be described as the search for the missing premise of an "unsaturated" deductive inference. Charles Sanders Peirce presents this situation as an abductive scenario:

> *I once landed at a seaport in a Turkish province; and, as I was walking up to the house which I was to visit, I met a man upon horseback, surrounded by four horsemen holding a canopy over his head. As the governor of the province was the only personage I could think of who would be so greatly honored, I inferred that this was he. This was a hypothesis.* [23]

The treatment of abduction as an enthymematic deductive argument in reverse is guided by certain directives. Given a non empty set of premises Γ and a formula G such that $\Gamma \nvdash G$, we need to find a formula H satisfying three logical conditions:

$$\mathsf{A1}\colon \Gamma, H \vdash G \qquad \mathsf{A2}\colon H \nvdash G \qquad \mathsf{A3}\colon \Gamma, H \nvdash \bot$$

A1 – A3 are part of the tradition of twentieth-century philosophy of science as they can be traced back to Hempel's essential requirements for H to be considered an *explanans* of G given Γ ([15], pp. 277-78). Of course, the Hempelian account is no longer the prevailing approach to explanation among most contemporary philosophers of science. Over time, the Hempelian model has faced criticism and has been challenged by alternative accounts of explanation. Many philosophers now advocate for a more nuanced understanding of explanation that incorporates additional factors beyond simple deductive subsumption. Some of these alternative accounts include causal models, pragmatic approaches, and various forms of contextual explanations (for a survey see [35]). However, the Hempelian approach, with its focus on logical coherence and systematic analysis, aligns with a structured framework for understanding the problem of abduction from an abstract deductive perspective. A1 states that the formula G needs to be 'deductively reachable' from the set of premises $\Gamma \cup \{H\}$, that is H must bridge the deductive gap between Γ and G. A2 and A3 require that the formula H provides useful and non-trivial information. Specifically, A2 ensures that Γ is not a superfluous context by demanding that H alone does not imply G, while A3 requires that adding H to Γ should not make $\Gamma \cup \{H\}$ inconsistent.

To the extent that there exist infinitely many abductive formulas obeying A1 – A3 for any invalid sequent $\Gamma \nvdash G$, a natural question immediately arises: what strategy should be employed by a rational agent to select just one of these formulas? The following two-step strategy seems to be a reasonable one:

(1) restrict the search space to the (finite) set of abductive hypotheses that convey information already contained in Γ and G;
(2) investigate the search space enlarged with abductive hypotheses that satisfy conditions A1 – A3 and provide information not in Γ or G.

Several efforts have been made to address Step (1), which aims to define an *effective* procedure for generating and justifying hypotheses that satisfy A1 – A3. One traditional approach relies on the use of tableaux. Essentially, it consists in writing the refutation tree associated with the set $\Gamma, \neg G$, examining the open branches, and then identifying any cluster of formulas Δ which allow for the systematic closure of each one of the open branches in the tableau under consideration [3, 7, 18]. The formula H resulting from the *maximal* cluster of such formulas satisfies *deductive minimality* (DM, henceforth):

$$\text{DM: for any } H', \text{ if } \Gamma, H' \vdash G, \text{ then } H' \vdash H$$

H is regarded as the optimal hypothesis under the name of *least compromising hypothesis*.

In this paper, we prove *inter alia* that the condition of DM is not necessary for the optimality of H. In effect, DM fails to capture something fundamental to abductive reasoning: its purpose of finding the simplest and most relevant *explanans* from among many. To illustrate this failure from the perspective of a rational agent, let's consider two simple examples.

Example 1.1 *Consider the invalid sequent* $p \vee q \nvdash q$*. The resulting least compromising hypothesis is* $\neg p \vee q$*. However, it seems reasonable to assume that a rational agent would consider* $\neg p \vee q$ *too weak to properly saturate* $p \vee q \nvdash q$*. In fact,* $\neg p$ *seems to provide a better explanation for* $p \vee q \nvdash q$*, as it appeals to an instance of the disjunctive syllogism* $p \vee q, \neg p \vdash q$*.*

Example 1.2 *Consider the invalid sequent* $p \to q \nvdash r \to q$*. Inserting among the premises the least compromising hypothesis* $r \to (p \vee q)$ *is a detour for a rational agent seeking an optimal explanation for* $p \to q \nvdash r \to q$*. Instead,* $r \to p$ *fits the bill by referring to an instance of the hypothetical syllogism* $r \to p, p \to q \vdash r \to q$ (cf. [23], p. 472).

To overcome these difficulties, we design a sequent-based procedure that always *approximates* an abductive hypothesis providing a better explanation in our refined sense. Although our machinery hinges on the well-known duality between tableaux *à la* Smullyan and Kleene's sequent system [25, 32], we believe that explicitly handling sequents instead of tableaux results in a simpler formal approach, since sequents allow for a *local* control of information flow.

Furthermore, our approach can be usefully applied to Step (2), which concerns the search space expanded with abductive hypotheses that satisfy conditions A1 – A3 while providing additional information. We show how a generalized version of our procedure can track any abductive hypothesis with new information. Specifically, we establish that any formula in the expanded search space that satisfies conditions A1 – A3 must also imply one of these hypotheses that satisfy the same conditions. This result enables us to shift our attention to the (infinite) subspace of abductive hypotheses that respect conditions A1 to A3 and imply hypotheses that offer a better explanation. We hypothesize that this subspace includes the set of candidates for selection as the best explanans.

The paper is is organized as follows. Section 2 introduces the formal machinery that we use to develop our proposal, namely a hybrid sequent system (with both sequents and antisequents) that possesses crucial proof-theoretic properties. In Section 3, we describe a sequent-based procedure for generating the least compromising hypothesis, and we provide sufficient conditions for enforcing its satisfaction of conditions A2 and A3. Section 4, presents another sequent-based procedure for generating optimal approximations of the hypotheses, which are analytically obtained from the abductive problem and are expected to be selected as optimal by a rational agent. We also spell out sufficient conditions for ensuring that this procedure satisfies conditions A2 and A3. In Section 5, we generalize the procedure in Section 4 to obtain any possible strengthening of the least compromising hypothesis. This generalization lays the groundwork for a logical treatment of abduction in the presence of new information. Finally, in Section 6 we draw conclusions about our proposal and sketch some directions for future research. At the end of the paper, we include a legend of the terminology we employ, in order to improve readability.

2 Preliminary Notions and Results

We use capital Greek letters $\Gamma, \Delta, \ldots$ to denote finite *sets* of formulas, in particular $\Theta, \Lambda, \ldots$ are taken to stand for sets of *atomic* formulas. For any context Γ we shall be adopting the following conventions: If $\Gamma = \{A_1, A_2, \ldots, A_n\}$, then

$$\Gamma^{\perp} = \{\neg A_1, \neg A_2, \ldots, \neg A_n\} \quad \bigwedge \Gamma = A_1 \wedge A_2 \wedge \cdots \wedge A_n \quad \bigvee \Gamma = A_1 \vee A_2 \vee \cdots \vee A_n.$$

For $\Gamma = \varnothing$, we set $\Gamma^{\perp} = \Gamma$, $\bigwedge \Gamma = \top$, and $\bigvee \Gamma = \bot$, where $\top$ and $\bot$ stand for an arbitrarily chosen tautology and contradiction, respectively. For any formula A, $sub(A)$ denotes the set of its subformulas. In this way, $sub(\Gamma) = sub(A_1) \cup \cdots \cup sub(A_n)$.

In what follows, we shall be dealing with ordinary Gentzen-style sequents $\Gamma \vdash \Delta$ as well as *antisequents* $\Gamma \dashv \Delta$. Antisequents have been introduced in the literature on refutation calculi to indicate sequents asserting their own invalidity [14, 29]. In other words, the antisequent $\Gamma \dashv \Delta$ is valid if, and only if, the sequent $\Gamma \vdash \Delta$ is *in*valid, namely when there is some Boolean valuation verifying all the formulas in Γ and falsifying all those in Δ. Henceforth we use $\mathcal{S}, \mathcal{R}, \ldots$ as metavariables ranging over the sets of sequents and antisequents without distinction.

The system $\overline{\overline{\mathsf{G4}}}$ is imported from [25, 29] with the slight modification that logical contexts are considered as *sets* of formulas instead of ordinary multisets. In particular, $\overline{\overline{\mathsf{G4}}}$ is obtained by adding to Kleene's G4 rules the complementary axiom $\overline{\Gamma \dashv \Delta}\,^{ax}$, where Γ and Δ are two sets of atomic sentences such that $\Gamma \cap \Delta = \varnothing$. In Fig. 1, the $\overline{\overline{\mathsf{G4}}}$ sequent calculus is expressed in a compact way by writing ordinary sequents $\Gamma \vdash \Delta$ as $\Gamma \models^{1} \Delta$ and antisequents $\Gamma \dashv \Delta$ as $\Gamma \models^{0} \Delta$. Whenever we need to generalize over the union of sequents and antisequents, we shall be writing $\Gamma \models^{*} \Delta$.

Due to the hybrid nature of the calculus, a $\overline{\overline{\mathsf{G4}}}$ derivation δ may end either in a sequent $\Gamma \vdash \Delta$ or in an antisequent $\Gamma \dashv \Delta$. In the first case, we say that δ is a *proof* for $\Gamma \vdash \Delta$; in the second, δ qualifies as a *refutation* for $\Gamma \vdash \Delta$. That is, any $\overline{\overline{\mathsf{G4}}}$ derivation of $\Gamma \dashv \Delta$ counts as a refutation for $\Gamma \vdash \Delta$.

The rules of $\overline{\overline{\mathsf{G4}}}$ can be understood as a two-step procedure for decomposing any (anti)sequent $\Gamma \models^{*} \Delta$ into a set of atomic (anti)sequents, as follows:

(1) (bottom-up) Discount the indices and continue decomposing the (anti)sequent $\Gamma \models^{*} \Delta$ using the rules in Fig. 1 until each leaf of the resulting tree ends with a clause;

AXIOMS

$$\frac{}{\Theta, p \overset{1}{\vdash} p, \Lambda}\ ax \qquad\qquad \frac{}{\Theta \overset{0}{\vdash} \Lambda}\ \overline{ax}, \text{ where } \Theta \cap \Lambda = \varnothing$$

LOGICAL RULES

$$\frac{\Gamma \overset{i}{\vdash} \Delta, A}{\Gamma, \neg A \overset{i}{\vdash} \Delta}\ \neg_{\mathcal{L}} \qquad\qquad \frac{\Gamma, A \overset{i}{\vdash} \Delta}{\Gamma \overset{i}{\vdash} \Delta, \neg A}\ \neg_{\mathcal{R}}$$

$$\frac{\Gamma, A, B \overset{i}{\vdash} \Delta}{\Gamma, A \wedge B \overset{i}{\vdash} \Delta}\ \wedge_{\mathcal{L}} \qquad\qquad \frac{\Gamma \overset{i}{\vdash} \Delta, A \qquad \Gamma \overset{j}{\vdash} \Delta, B}{\Gamma \overset{i \cdot j}{\vdash} \Delta, A \wedge B}\ \wedge_{\mathcal{R}}$$

$$\frac{\Gamma, A \overset{i}{\vdash} \Delta \qquad \Gamma, B \overset{j}{\vdash} \Delta}{\Gamma, A \vee B \overset{i \cdot j}{\vdash} \Delta}\ \vee_{\mathcal{L}} \qquad\qquad \frac{\Gamma \overset{i}{\vdash} \Delta, A, B}{\Gamma \overset{i}{\vdash} \Delta, A \vee B}\ \vee_{\mathcal{R}}$$

$$\frac{\Gamma \overset{i}{\vdash} \Delta, A \qquad \Gamma, B \overset{j}{\vdash} \Delta}{\Gamma, A \to B \overset{i \cdot j}{\vdash} \Delta}\ \to_{\mathcal{L}} \qquad\qquad \frac{\Gamma, A \overset{i}{\vdash} \Delta, B}{\Gamma \overset{i}{\vdash} \Delta, A \to B}\ \to_{\mathcal{R}}$$

Fig. 1 G4 and $\overline{\overline{\mathsf{G4}}}$ sequent calculi

(2) (top-down) Decorate each sequent in the resulting tree with the correct index, starting from the leaves and following the rules of $\overline{\overline{\mathsf{G4}}}$.

We can now recall three features of the $\overline{\overline{\mathsf{G4}}}$ proof system [4, 25, 28].

Fact 2.1 $\overline{\overline{\mathsf{G4}}}$ *proves (refutes)* $\Gamma \vdash \Delta$ *if and only if the sequent* $\Gamma \vdash \Delta$ *is classically valid (* in*valid).*

Fact 2.2 (Stability). *Any two* $\overline{\overline{\mathsf{G4}}}$ *derivations ending with the same (anti)sequent display the same set of top clauses.*

We first observe that Fact 2.2 allows us to directly refer to the set of top-clauses associated with a certain (anti)sequent $\Gamma \overset{*}{\vdash} \Delta$, being such a decomposition independent of the specific derivation delivering it. In particular, we write $\mathsf{top}(\Gamma \overset{*}{\vdash} \Delta)$ to indicate the set of top-sequents associated with $\Gamma \overset{*}{\vdash} \Delta$. The two sets $\mathsf{top}_{\vdash}(\Gamma \overset{*}{\vdash} \Delta)$ and $\mathsf{top}_{\dashv}(\Gamma \overset{*}{\vdash} \Delta)$ partition $\mathsf{top}(\Gamma \overset{*}{\vdash} \Delta)$ collecting exactly those $\Theta \overset{i}{\vdash} \Lambda \in \mathsf{top}(\Gamma \overset{*}{\vdash} \Delta)$ such that $i = 1$ and $i = 0$, respectively. The third fact worth mentioning is a byproduct of the invertibility of G4 logical rules.

Fact 2.3 *Let* $\mathsf{top}_{\dashv}(\Gamma \overset{*}{\vdash} \Delta) = \left\{\Theta_1 \dashv \Lambda_1, \ldots, \Theta_n \dashv \Lambda_n\right\}$*, the two formulas* $\bigwedge \Gamma \to \bigvee \Delta$ *and* $\bigwedge_{i=1}^{n} (\bigwedge \Theta_i \to \bigvee \Lambda_i)$ *turn out to be logically equivalent.*

Proof By Fact 2.1, the two formulas $\bigwedge \Gamma \rightarrow \bigvee \Delta$ and $\bigwedge_{i=1}^{n} (\bigwedge \Theta_i \rightarrow \bigvee \Lambda_i)$ are logically equivalent if and only if $\overline{\overline{\mathsf{G4}}}$ proves the two sequents

$$\bigwedge \Gamma \rightarrow \bigvee \Delta \vdash \bigwedge_{i=1}^{n} (\bigwedge \Theta_i \rightarrow \bigvee \Lambda_i) \tag{1}$$

$$\bigwedge_{i=1}^{n} (\bigwedge \Theta_i \rightarrow \bigvee \Lambda_i) \vdash \bigwedge \Gamma \rightarrow \bigvee \Delta \tag{2}$$

We consider the two cases separately, reasoning by contradiction.

(i) If $\overline{\overline{\mathsf{G4}}}$ refutes Eq. 1, then $\overline{\overline{\mathsf{G4}}}$ refutes at least one sequent of the form $\vdash \bigwedge_{i=1}^{n} (\bigwedge \Theta_i \rightarrow \bigvee \Lambda_i), \bigwedge \Theta'_j \rightarrow \bigvee \Lambda'_j$, with $\Theta'_j \dashv \Lambda'_j \in \mathsf{top}_{\dashv}(\bigwedge \Gamma \rightarrow \bigvee \Delta \models^{*})$, by full invertibility of $\mathsf{G4}$: by the same token, $\overline{\overline{\mathsf{G4}}}$ refutes at least one sequent of the form $\Theta'_j, \Theta_i \vdash \Lambda_i, \Lambda'_j$. On the other hand, each sequent $\Theta'_j \vdash \Lambda'_j, \bigwedge \Gamma \rightarrow \bigvee \Delta$ is clearly provable: by full invertibility of $\mathsf{G4}$, this means that $\overline{\overline{\mathsf{G4}}}$ proves each sequent of the form $\Theta'_j, \Theta_i \vdash \Lambda_i, \Lambda'_j$ – a contradiction.

(ii) If $\overline{\overline{\mathsf{G4}}}$ refutes Eq. 2, then $\overline{\overline{\mathsf{G4}}}$ refutes at least one sequent of the form $\bigwedge \Theta_1 \rightarrow \bigvee \Lambda_1, \ldots, \bigwedge \Theta_i \rightarrow \bigvee \Lambda_i, \ldots, \bigwedge \Theta_n \rightarrow \bigvee \Lambda_n \vdash \bigwedge \Theta_i \rightarrow \bigvee \Lambda_i$ by full invertibility of $\mathsf{G4}$: this implies that $\overline{\overline{\mathsf{G4}}}$ refutes $\bigwedge \Theta_1 \rightarrow \bigvee \Lambda_1, \ldots, p, \ldots, \bigwedge \Theta_n \rightarrow \bigvee \Lambda_n, \Theta_i \vdash \Lambda_i$ for any $p \in \Lambda_i$ and $\bigwedge \Theta_1 \rightarrow \bigvee \Lambda_1, \ldots, \bigwedge \Theta_n \rightarrow \bigvee \Lambda_n, \Theta_i \vdash \Lambda_i, q$ for any $q \in \Theta_i$ – by the *ax* rule of $\overline{\overline{\mathsf{G4}}}$, a contradiction.

□

Remark 2.1 *As a limiting case of the previous fact, the formula* $\bigwedge \Gamma \rightarrow \bigvee \Delta$ *is equivalent to* $\top$ *just in case* $\mathsf{top}_{\dashv}(\Gamma \models^{*} \Delta) = \varnothing$, *i.e., exactly when* $\Gamma \vdash \Delta$ *is a classically valid sequent.*

Example 2.1 *This is a* $\overline{\overline{\mathsf{G4}}}$*-derivation ending with the antisequent* $p \rightarrow q, p \vee q \dashv r$ *and so qualifying as a refutation for* $p \rightarrow q, p \vee q \vdash r$

$$\dfrac{\dfrac{\dfrac{}{p \vdash p, r}\,ax. \quad \dfrac{}{q, p \dashv r}\,\overline{ax}}{p \vee q \dashv p, r}\,\vee_{\mathcal{L}} \quad \dfrac{\dfrac{}{p \dashv q, r}\,\overline{ax} \quad \dfrac{}{q \dashv r}\,\overline{ax}}{p \vee q, q \dashv r}\,\vee_{\mathcal{L}}}{p \rightarrow q, p \vee q \dashv r}\,\rightarrow_{\mathcal{L}}$$

In this case we have $\mathsf{top}(\Gamma \models^{*} \Delta) = \{p \vdash p, r\,;\ q \dashv p, r\,;\ p, q \dashv r\,;\ q \dashv r\}$ *with* $\mathsf{top}_{\vdash}(\Gamma \models^{*} \Delta) = \{p \vdash p, r\}$ *and* $\mathsf{top}_{\dashv}(\Gamma \models^{*} \Delta) = \{q \dashv p, r\,;\ p, q \dashv r\,;\ q \dashv r\}$. *According to what established by Fact 2.3, the two following formulas turn out to be logically equivalent:*

$$((p \rightarrow q) \wedge (p \vee q)) \rightarrow r \qquad (q \rightarrow (p \vee r)) \wedge ((p \wedge q) \rightarrow r) \wedge (q \rightarrow r)$$

3 Producing the Least Compromising Hypothesis

In what follows, by *abductive problem* we mean any expression of the form $\Gamma, ⓘ \vdash G$, with $\Gamma \neq \varnothing$ and such that $\overline{\overline{\mathsf{G4}}}$ refutes $\Gamma \vdash G$. Accordingly, by *abductive algorithm* we refer to any effective procedure that, given in input an abductive problem $\Gamma, ⓘ \vdash G$, returns an *abductive hypothesis* H such that $\Gamma, H \vdash G$ is provable in $\overline{\overline{\mathsf{G4}}}$.

In [7], the tableaux method is employed to design an elegant and effective abductive algorithm for producing what they call the *least compromising hypothesis*. We begin this section by proposing a sequent-based reading of the very same procedure. The switching from tableaux to sequents is here technically justified by the fact that sequent calculi facilitate the study of the structural properties of the algorithm. Due to the well-known duality between semantic tableaux *à la* Smullyan and Kleene's system G4 [32], any result obtained for one system can be nonetheless imported in the other.

Procedure 3.1 (Least Compromising Hypothesis). *For any abductive problem* $\Gamma, ⓘ \vdash G$*, the* least compromising hypothesis $\mathsf{LCH}\big(\Gamma, ⓘ \vdash G\big)$ *is the formula resulting from the following steps:*

(1) Decompose the antisequent $\Gamma \dashv G$ *till the set of clauses* $\mathsf{top}_{\dashv}(\Gamma \dashv G) = \big\{\Theta_1 \dashv \Lambda_1, \ldots, \Theta_n \dashv \Lambda_n\big\}$ *is fully accomplished.*
(2) For each clause $\Theta_i \dashv \Lambda_i \in \mathsf{top}_{\dashv}(\Gamma \dashv G)$ *consider the formula* $C_i \equiv \bigwedge \Theta_i \rightarrow \bigvee \Lambda_i$.
(3) Finally set $\mathsf{LCH}\big(\Gamma, ⓘ \vdash G\big) = C_1 \wedge \cdots \wedge C_n$.

Example 3.1 *We apply Procedure 3.1 to compute the formula* $\mathsf{LCH}(p \rightarrow q, p \vee q, ⓘ \vdash r)$*:*

(1) By looking at the $\overline{\overline{\mathsf{G4}}}$*-proof reported in Example 2.1, we immediately get*

$$\mathsf{top}_{\dashv}\big(p \rightarrow q, p \vee q, ⓘ \vdash r\big) = \big\{q \dashv p, r\ ;\ \ p, q \dashv r\ ;\ \ q \dashv r\big\}$$

(2) Then we turn each clause into its corresponding formula:

$$\begin{array}{rcl} q \dashv p, r & \Rightarrow & q \rightarrow (p \vee r) \\ p, q \dashv r & \Rightarrow & (p \wedge q) \rightarrow r \\ q \dashv r & \Rightarrow & q \rightarrow r \end{array}$$

(3) We finally lead up to the compound formula:

$$\mathsf{LCH}(p \rightarrow q, p \vee q, ⓘ \vdash r) = (q \rightarrow (p \vee r)) \wedge ((p \wedge q) \rightarrow r) \wedge (q \rightarrow r).$$

It is possible for the decomposition of the antisequent $\Gamma \dashv G$ to produce a set of complementary top-clauses $\Theta_1 \dashv \Lambda_1, \ldots, \Theta_n \dashv \Lambda_n$ such that there exists one $\Theta_i \dashv \Lambda_i$ which is a classical consequence of $\Theta_{j_1} \dashv \Lambda_{j_1}, \ldots, \Theta_{j_k} \dashv \Lambda_{j_k}$, with $1 \leq i \neq j_1 \neq \cdots \neq j_k \leq n$. For example, consider the LCH-hypothesis of Example 3.1, and note that $q \rightarrow (p \vee r)$ and $(p \wedge q) \rightarrow r$ are both classical consequences of $q \rightarrow r$, whereas $q \rightarrow r$ is a classical consequence of $q \rightarrow (p \vee r)$ and $(p \wedge q) \rightarrow r$. In general, it is reasonable to consider such a $\Theta_i \dashv \Lambda_i$ as redundant. Dropping $\bigwedge \Theta_i \rightarrow \bigvee \Lambda_i$

from the set of conjuncts of the LCH-hypothesis yields a logically equivalent formula, which is an optimized version of the former.

In [7], the authors demonstrate that one can generate an optimized version of the LCH-hypothesis by replacing Smullyan-style tableaux with *KE*-tableaux [6, 8], which are a dual presentation of a sequent calculus that does not enjoy admissibility of Cut [12]. In our sequent-based approach via G4, redundant clauses can be eliminated by utilizing the following rewriting rules:

$$\{\Theta \dashv \Lambda\} \cup \{\Theta', \Theta \dashv \Lambda, \Lambda'\} \rightarrow_w \{\Theta \dashv \Lambda\} \tag{3}$$

$$\bigcup_{i=1}^{n}\{\Theta_i \dashv \Lambda_i\} \cup \{\Theta'',\ \Theta'_1, \dots, \Theta_j, \dots, \Theta'_n \dashv \Lambda'_1, \dots, \Lambda_k, \dots, \Lambda'_n, \Lambda''\} \rightarrow_c \bigcup_{i=1}^{n}\{\Theta_i \dashv \Lambda_i\} \tag{4}$$

with $\Theta' \cup \Lambda' \neq \varnothing$, $n \geq 2$ and $1 \leq j \neq k \leq n$, provided that for any $1 \leq j' \neq j, k' \neq k, j' \neq k' \leq n$ and some Φ, Ψ we have that $\Theta'_{j'} = (\Phi \cup \Theta_{j'}) \setminus \{p\}$ and $\Lambda'_{k'} = (\Lambda_{k'} \cup \Psi) \setminus \{p\}$ for any $p \in (\Lambda_{k'} \cup \Psi) \cap (\Phi \cup \Theta_{j'})$.

The rationale for adopting these rewriting rules is that of avoiding cases in which $\overline{\overline{\mathsf{G4}}}$ derives at least one clause in $\mathcal{S}$ from other clauses in $\mathcal{S}$ either by applying (an invalidity-preserving version of) Weakening – as with the derivation of $\Theta', \Theta \dashv \Lambda, \Lambda'$ from $\Theta \dashv \Lambda$ –, or by applying in some order (invalidity-preserving versions of) Weakening and Cut – as with the derivation of $\Theta'', \Theta'_1, \dots, \Theta_j, \dots, \Theta'_n \dashv \Lambda'_1, \dots, \Lambda_k, \dots, \Lambda'_n, \Lambda''$ from $\bigcup_{i=1}^{n}\{\Theta_i \dashv \Lambda_i\}$.

For any set $\mathcal{S}$ of clauses, maximal application of the rewriting rules Eqs. 3 – 4 to $\mathcal{S}$ yields a (not necessarily unique) subset $\mathcal{T}$ of clauses where all redundant clauses from $\mathcal{S}$ have been dropped *modulo* logical equivalence: we refer to $\mathcal{T}$ as a *reduct under Weakening and Cut* of $\mathcal{S}$ after [24].

We can thus refine step (1) of Procedure 3.1 by taking a reduct under Weakening and Cut of the set of top-clauses which results from the decomposition of the abductive problem. If we consider once more Example 3.1, this refinement forces us to consider two possible optimizations of the LCH-hypothesis $(q \rightarrow (p \vee r)) \wedge ((p \wedge q) \rightarrow r) \wedge (q \rightarrow r)$: in one case, we first apply rule Eq. 3, thus dropping $q \dashv p, r$ and $p, q \dashv r$ from $\mathsf{top}_{\dashv}(p \rightarrow q, p \vee q, ? \vdash r)$ and getting $q \rightarrow r$ as optimized LCH-hypothesis; in the other case, we first apply rule Eq. 4, thus dropping $q \dashv r$ from $\mathsf{top}_{\dashv}(p \rightarrow q, p \vee q, ? \vdash r)$ and getting $(q \rightarrow (p \vee r)) \wedge ((p \wedge q) \rightarrow r)$ as a distinct (but logically equivalent) optimized version of the LCH-hypothesis.

We can now turn to the proof of the first basic result about the LCH-hypothesis:

Theorem 3.1 *For any (abductive) problem* $\Gamma, ? \vdash G$, $\overline{\overline{\mathsf{G4}}}$ *proves both* $\mathsf{LCH}(\Gamma \dashv G) \vdash \bigwedge \Gamma \rightarrow G$ *and* $\bigwedge \Gamma \rightarrow G \vdash \mathsf{LCH}(\Gamma \dashv G)$.

Proof Let $\mathsf{top}_{\vdash}(\Gamma \dashv G) = \{\Theta_1 \vdash \Lambda_1, \dots, \Theta_m \vdash \Lambda_m\}$ and $\mathsf{top}_{\dashv}(\Gamma \dashv G) = \{\Theta_{m+1} \dashv \Lambda_{m+1}, \dots, \Theta_{m+n} \dashv \Lambda_{m+n}\}$. It is a routine matter to verify that $\overline{\overline{\mathsf{G4}}}$ proves each of the following sequents:

$$(\bigwedge \Theta_1 \rightarrow \bigvee \Lambda_1) \wedge \cdots \wedge (\bigwedge \Theta_{m+n} \rightarrow \bigvee \Lambda_{m+n}) \vdash \bigwedge \Gamma \rightarrow G \tag{5}$$

$$\bigwedge \Gamma \rightarrow G \vdash (\bigwedge \Theta_1 \rightarrow \bigvee \Lambda_1) \wedge \cdots \wedge (\bigwedge \Theta_{m+n} \rightarrow \bigvee \Lambda_{m+n}) \tag{6}$$

$$\vdash (\bigwedge \Theta_1 \rightarrow \bigvee \Lambda_1) \wedge \cdots \wedge (\bigwedge \Theta_m \rightarrow \bigvee \Lambda_m) \tag{7}$$

The provability of sequents Eqs. 5 and 6 is an immediate consequence of Fact 2.3, whereas the provability of Eq. 7 straightforwardly follows from the fact that each clause $\Theta_i \vdash \Lambda_i$, with $1 \leq i \leq m$, is tautological. By $\wedge$-invertibility of G4, provability of sequents Eqs. 5 – 7 implies that the following sequents are provable:

$$(\bigwedge \Theta_1 \rightarrow \bigvee \Lambda_1), \ldots, (\bigwedge \Theta_{m+n} \rightarrow \bigvee \Lambda_{m+n}) \vdash \bigwedge \Gamma \rightarrow G \tag{8}$$

$$\bigwedge \Gamma \rightarrow G \vdash \bigwedge \Theta_j \rightarrow \bigvee \Lambda_j \tag{9}$$

$$\vdash \bigwedge \Theta_1 \rightarrow \bigvee \Lambda_1, \ldots, \bigwedge \Theta_m \rightarrow \bigvee \Lambda_m \tag{10}$$

with $1 \leq j \leq m+n$. By closure of G4 under Cut, provability of Eqs. 8 and 10 implies that the following sequent is provable:

$$(\bigwedge \Theta_{m+1} \rightarrow \bigvee \Lambda_{m+1}), \ldots, (\bigwedge \Theta_{m+n} \rightarrow \bigvee \Lambda_{m+n}) \vdash \bigwedge \Gamma \rightarrow G \tag{11}$$

Provability of Eqs. 9 and 11, together with the fact that

$$\mathsf{LCH}(\Gamma \dashv G) = (\bigwedge \Theta_{m+1} \rightarrow \bigvee \Lambda_{m+1}) \wedge \cdots \wedge (\bigwedge \Theta_{m+n} \rightarrow \bigvee \Lambda_{m+n})$$

yields the conclusion. □

We can now show that the LCH-hypothesis enjoys condition A1 (cf. Lemma 3.1 and Theorem 3.1 in [7]):

Corollary 3.1 *For any problem* $\Gamma, ⓘ \vdash G$, *the sequent* $\Gamma, \mathsf{LCH}(\Gamma \dashv G) \vdash G$ *is always provable in* $\overline{\overline{\mathsf{G4}}}$.

Proof The claim is an immediate consequence of Theorem 3.1 and full invertibility of G4. □

The previous result can be strengthened by showing that the LCH-abductive hypothesis turns out to be deductively minimal (*modulo* logical equivalence) with respect to the whole set of formulas obeying the condition A1. It should now be clear why in the literature the resulting abductive hypothesis is classified as the "least compromising" one:

Theorem 3.2 *For any problem* $\Gamma, ? \vdash G$, *if* $\overline{\overline{\mathsf{G4}}}$ *proves* $\Gamma, A \vdash G$, *then it also proves* $A \vdash \mathsf{LCH}(\Gamma \dashv G)$.

Proof If $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma, A \vdash G$, then it proves $A \vdash \bigwedge \Gamma \rightarrow G$ as well: by Theorem 3.1 and closure of G4 under Cut we get the desired conclusion. □

Minimality guarantees that if $\mathsf{LCH}(\Gamma \dashv G)$ does not satisfy A2 and A3, then no abductive hypothesis A can satisfy A2 and A3 at the same time. Since we are interested in abductive hypotheses that comply with the complete set of *desiderata* A1, A2, and A3, a natural question arises as to whether $\mathsf{LCH}(\Gamma \dashv G)$ always satisfies them simultaneously. Unfortunately, the answer is negative. For example, consider the problem $\neg p \vee \neg q, ⓘ \vdash p \wedge q$. According to Procedure 3.1, we have that $\mathsf{LCH}(\neg p \vee \neg q, ⓘ \vdash p \wedge q) = p \wedge q \wedge (p \vee q)$. The sequents $p \wedge q \wedge (p \vee q) \vdash p \wedge q$ and $\neg p \vee \neg q, p \wedge q \wedge (p \vee q) \vdash$ are both provable in $\overline{\overline{\mathsf{G4}}}$.

Upon closer examination, we can observe that the formula $\mathsf{LCH}(\Gamma \dashv G)$ satisfies conditions A2 and A3 in a limited number of cases characterized by the following result:

Theorem 3.3 *For any problem* $\Gamma, ⓘ \vdash G$*,* $\overline{\overline{\mathsf{G4}}}$ *refutes* $\mathsf{LCH}(\Gamma \dashv G) \vdash G$ *and* $\Gamma, \mathsf{LCH}(\Gamma \dashv G) \vdash$ *just in case* $\overline{\overline{\mathsf{G4}}}$ *refutes* $\neg G \vdash \bigwedge \Gamma$ *and* $\Gamma \vdash \neg G$*, respectively.*

Proof By Theorem 3.1 and G4 being closed under Cut, $\mathsf{LCH}(\Gamma \dashv G)$ does not satisfy condition A2 if and only if $\overline{\overline{\mathsf{G4}}}$ proves $\bigwedge \Gamma \rightarrow G \vdash G$, and $\mathsf{LCH}(\Gamma \dashv G)$ does not satisfy A3 if and only if $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma, \bigwedge \Gamma \rightarrow G \vdash$. We consider the two cases separately.

(i) If $\overline{\overline{\mathsf{G4}}}$ proves $\bigwedge \Gamma \rightarrow G \vdash G$, then $\overline{\overline{\mathsf{G4}}}$ proves $\vdash G, \bigwedge \Gamma$ by $\rightarrow$-invertibility of G4, and then $\neg G \vdash \bigwedge \Gamma$ by one application of $\neg_{\mathcal{L}}$. On the other hand, if $\overline{\overline{\mathsf{G4}}}$ proves $\neg G \vdash \bigwedge \Gamma$ then $\overline{\overline{\mathsf{G4}}}$ proves $\vdash G, \bigwedge \Gamma$ by $\neg$-invertibility of G4, and then derives $\bigwedge \Gamma \rightarrow G \vdash G$ from $G \vdash G$ by one application of $\rightarrow_{\mathcal{L}}$.

(ii) If $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma, \bigwedge \Gamma \rightarrow G \vdash$, then $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma, G \vdash$ by $\rightarrow$-invertibility of G4, and then $\Gamma \vdash \neg G$ by one application of $\neg_{\mathcal{R}}$. On the other hand, if $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma \vdash \neg G$ then $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma, G \vdash$ by $\neg$-invertibility of G4, and then derives $\Gamma, \bigwedge \Gamma \rightarrow G \vdash$ from $\Gamma \vdash \bigwedge \Gamma$ by one application of $\rightarrow_{\mathcal{L}}$.

By contraposition and using Fact 2.1, we then obtain the desired conclusion. □

From now on, we will call *explanans* any abductive hypothesis respecting conditions A2 and A3. Bearing in mind that B is deductively independent of A when $\overline{\overline{\mathsf{G4}}}$ refutes both the sequents $A \vdash B$ and $A, B \vdash$, we collect the following facts about any LCH-*explanans*:

Proposition 3.1 *For any problem* $\Gamma, ⓘ \vdash G$*, if* $\mathsf{LCH}(\Gamma \dashv G)$ *is an* explanans*, then:*

(i) $\bigwedge \Gamma$*,* G *and* $\mathsf{LCH}(\Gamma \dashv G)$ *are all truth-functionally contingent;*
(ii) $\mathsf{LCH}(\Gamma \dashv G)$ *and* $\bigwedge \Gamma$ *turn out to be deductively independent of each other;*
(iii) G *is deductively independent of* $\mathsf{LCH}(\Gamma \dashv G)$*, but not* vice versa*.*

Proof We prove each statement separately.

(i) If $\Gamma, ⓘ \vdash G$ is an abductive problem, then $\bigwedge \Gamma$ is not contradictory and G not tautological. On the other hand, if $\mathsf{LCH}(\Gamma \dashv G)$ respects conditions A2 and A3 then $\mathsf{LCH}(\Gamma \dashv G)$ is not contradictory, and by Theorem 3.3 neither $\bigwedge \Gamma$ nor $\neg G$ can be tautological.

(ii) $\mathsf{LCH}(\Gamma \dashv G)$ is deductively independent of $\bigwedge \Gamma$: by contradiction, if $\overline{\overline{\mathsf{G4}}}$ proved $\bigwedge \Gamma \vdash \mathsf{LCH}(\Gamma \dashv G)$, then it would prove $\bigwedge \Gamma \vdash G$ by Corollary 3.1 and closure of G4 under Cut; on the other hand, if $\overline{\overline{\mathsf{G4}}}$ proved $\bigwedge \Gamma \vdash \neg \mathsf{LCH}(\Gamma \dashv G)$, $\neg$-invertibility of $\overline{\overline{\mathsf{G4}}}$ would guarantee that $\bigwedge \Gamma, \mathsf{LCH}(\Gamma \dashv G)$ is provable – against condition A3. The fact that $\bigwedge \Gamma$ is deductively independent of $\mathsf{LCH}(\Gamma \dashv G)$ can be proved by an analogous argument.

(iii) G is deductively independent of $\mathsf{LCH}(\Gamma \dashv G)$: if $\overline{\overline{\mathsf{G4}}}$ proved $\mathsf{LCH}(\Gamma \dashv G), G \vdash$, then it would prove $\Gamma, \mathsf{LCH}(\Gamma \dashv G) \vdash$ by Corollary 3.1 and closure of G4 under Cut – against condition A3; if $\overline{\overline{\mathsf{G4}}}$ proved $\mathsf{LCH}(\Gamma \dashv G) \vdash G$, then condition A2 would be violated. On the other hand, it suffices to notice that $\overline{\overline{\mathsf{G4}}}$ proves $G \vdash \bigwedge \Gamma \rightarrow G$ to conclude, by Theorem 3.1 and closure of G4 under Cut, that $\mathsf{LCH}(\Gamma \dashv G)$ is not deductively independent of G. □

As a result, (i) and (ii) of Proposition 3.1 jointly state that a rational agent uses an LCH-*explanans* only if she uses LCH to lower the number of (contingent) facts independent of a (contingent) theoretical background: according to the terminology of [1], a rational agent uses LCH as an *explanans* only if she uses it to reduce the number of *novelties* w.r.t. the theoretical background. On the other hand, point (iii) of Proposition 3.1 shows that the minimal *explanans* LCH enjoys maximal *evidential support*, meaning that if the *explanandum* is true, then the LCH-*explanans* cannot fail to be true (cf. [10], p. 45).

We conclude this section by noticing that the LCH-abductive hypothesis is context-sensitive, that is to say the addition of premises in the theoretical background may alter the deductive strength of the LCH-abductive hypothesis:

Proposition 3.2 *For any two distinct problems* $\Gamma, ⓘ \vdash G$ *and* $\Gamma', \Gamma, ⓘ \vdash G$,

(i) $\overline{\overline{\mathsf{G4}}}$ *proves* $\mathsf{LCH}(\Gamma \dashv G) \vdash \mathsf{LCH}(\Gamma', \Gamma \dashv G)$;

(ii) $\overline{\overline{\mathsf{G4}}}$ *refutes* $\mathsf{LCH}(\Gamma', \Gamma \dashv G) \vdash \mathsf{LCH}(\Gamma \dashv G)$ *if and only if* $\overline{\overline{\mathsf{G4}}}$ *refutes* $\Gamma, \neg A \vdash G$ *for at least one formula* $A \in \Gamma'$.

Proof For Eq. 1 it suffices to consider that $\overline{\overline{\mathsf{G4}}}$ proves $\bigwedge \Gamma \rightarrow G \vdash \left(\bigwedge \Gamma' \wedge \bigwedge \Gamma\right) \rightarrow G$ and exploit Theorem 3.1. As to Eq. 2, we consider the two directions separately.

(i) If $\overline{\overline{\mathsf{G4}}}$ proves $\mathsf{LCH}(\Gamma', \Gamma \dashv G) \vdash \mathsf{LCH}(\Gamma \dashv G)$, then it proves $(\bigwedge \Gamma' \wedge \bigwedge \Gamma) \rightarrow G \vdash \bigwedge \Gamma \rightarrow G$ by Theorem 3.1 and closure of G4 under Cut. As a result, $\overline{\overline{\mathsf{G4}}}$ proves $\bigwedge \Gamma \vdash G, A$ for any $A \in \Gamma'$ by full invertibility of G4: by one application of $\neg_{\mathcal{L}}$ we get the result.

(ii) If $\overline{\overline{\mathsf{G4}}}$ proves $\bigwedge \Gamma, \neg A \vdash G$ for any $A \in \Gamma'$, then it proves $\bigwedge \Gamma \vdash G, A$ by $\neg$-invertibility of G4 and thus $(\bigwedge \Gamma' \wedge \bigwedge \Gamma) \rightarrow G \vdash \bigwedge \Gamma \rightarrow G$ by applications of $\wedge_{\mathcal{R}}$, $\rightarrow_{\mathcal{L}}$ and $\rightarrow_{\mathcal{R}}$: as a result, $\overline{\overline{\mathsf{G4}}}$ proves $\mathsf{LCH}(\Gamma', \Gamma \dashv G) \vdash \mathsf{LCH}(\Gamma \dashv G)$ by Theorem 3.1 and closure of G4 under Cut.

By contraposition, we exploit Fact 2.1 to get the conclusion. □

4 Deductive Minimality and Expected Explanation

It is easy to find problems in which a rational agent's preferred abductive hypothesis does not match the minimum deductive hypothesis. Some of these problems are illustrated in Fig. 2, in addition to those presented in the Introduction. In all these cases, the expected hypothesis satisfies conditions A1 – A3 and is obtained by dropping some atomic pieces of information from the least compromising hypothesis.

For the sake of optimality, it is plausible to assume that deleted atoms correspond to *redundant* information – information in the abductive problem that the rational agent treats as irrelevant against deductive saturation. Specifically, the rational agent seems to implicitly treat as irrelevant some atomic pieces of information, whether in the theoretical background or in the goal formula, that perform partial deductive saturation even before making an abductive inference. The results presented in this section explore this intuition.

Let us begin with some terminology. For any (anti)sequent $\Gamma \overset{*}{\vdash} \Delta$, if $\mathsf{top}(\Gamma \overset{*}{\vdash} \Delta) = \left\{\Theta_1 \overset{*}{\vdash} \Lambda_1, \ldots, \Theta_n \overset{*}{\vdash} \Lambda_n\right\}$ then we have that

	ABDUCTIVE PROBLEM	MINIMAL HYPOTHESIS	EXPECTED HYPOTHESIS
1	$p \to q, ⓘ \vdash q$	$q \vee p$	p
2	$p \to q, ? \vdash p \to r$	$(q \wedge p) \to r$	$q \to r$
3	$p \to q, t \to r, p, ? \vdash r$	$(p \wedge q) \to (t \vee r)$	$q \to t$
4	$(p \vee q) \to (r \wedge s) \vdash s$	$s \vee p \vee q$	$p \vee q$
5	$\neg q \to \neg p, ? \vdash q$	$p \vee q$	p
6	$p, ? \vdash p \wedge q$	$p \to q$	q
7	$p, t, ? \vdash (p \wedge q) \wedge (t \wedge r)$	$((p \wedge t) \to q) \wedge ((p \wedge t) \to r)$	$q \wedge r$
8	$p \vee q, p \to r, ? \vdash r$	$q \to (p \vee r)$	$\neg q$
9	$p \vee q, \neg p \vdash r$	$q \to (p \vee r)$	$q \to r$
10	$p \to (q \wedge t), ? \vdash q$	$p \vee q$	p
11	$(p \wedge q) \to (r \vee s), p, ? \vdash s$	$((r \wedge p) \to s) \wedge (p \to (s \vee q))$	$q \wedge \neg r$
12	$p \vee q, ? \vdash p \vee t$	$q \to (p \vee t)$	$q \to t$
13	$p \to q, r \to s, ? \vdash q \vee s$	$q \vee s \vee p \vee r$	$p \vee r$
14	$p \to q, p \vee r, ? \vdash q \vee s$	$r \to (p \vee q \vee \vee s)$	$r \to s$
15	$p \to q, r \to s, ? \vdash \neg p \vee \neg r$	$\neg p \vee \neg r \vee \neg q \vee \neg s$	$\neg q \vee \neg s$
16	$p \to q, \neg q \vee \neg s, ? \vdash \neg p \vee \neg r$	$(p \wedge q \wedge r) \to s$	$r \to s$
17	$(p \to r) \wedge (q \to r), ? \vdash r$	$r \vee p \vee q$	$p \vee q$
18	$(p \vee q) \to (p \vee t), ? \vdash t$	$(p \vee t \vee q) \wedge (t \vee \neg p)$	$q \wedge \neg p$
19	$p \to q, ? \vdash p \to (q \wedge r)$	$(p \wedge q) \to r$	$p \to r$
20	$p \to q, ? \vdash q \wedge r$	$(q \vee p) \wedge (q \to r) \vee (r \vee p)$	$p \wedge r$

Fig. 2 Examples of minimal and expected solutions

(a) $\mathsf{AT}(\Gamma \models^{*} \Delta) = \bigcup_{i=1}^{n} (\Theta_i \cup \Lambda_i)$;
(b) $\mathsf{ID}(\Gamma \models^{*} \Delta) = \bigcup_{i=1}^{n} (\Theta_i \cap \Lambda_i)$;
(c) $\mathsf{CUT}(\Gamma \models^{*} \Delta) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} (\Lambda_i \cap \Theta_j)$;
(d) $\mathsf{AT}(A) = \mathsf{AT}(\models^{*} A) = \mathsf{AT}(A \models^{*})$.

Example 4.1 *Take the* $\overline{\overline{\mathsf{G4}}}$ *derivation of* $p \to q, p \vee q \dashv r$ *in Example 2.1. We have that*

(i) $\mathsf{AT}(p \to q, p \vee q \dashv r) = \{p, q, r\}$;
(ii) $\mathsf{ID}(p \to q, p \vee q \dashv r) = \{p\}$;
(iii) $\mathsf{CUT}(p \to q, p \vee q \dashv r) = \{p\}$.

For any problem $\Gamma, ⓘ \vdash G$, we say that an atom $p \in \mathsf{AT}(\Gamma \dashv G)$ is *abductively redundant* if $p \in \mathsf{ID}(\Gamma \dashv G)$. In other words, an atom is abductively redundant when 'trivializes' a clause in the decomposition of the abductive problem. Intuitively, atomic sentences of this kind correspond to pieces of information which are trivially contained in the theoretical background, or trivially contained in the goal formula, or shared between theoretical background and goal formula.

If we revise $\mathsf{LCH}(\Gamma \dashv G)$ by erasing atoms in $S \subseteq \mathsf{ID}(\Gamma \dashv G)$, we can partially eliminate redundant information. According to the following proposition, a rational agent who eliminates all abductive redundant information also drops *all* the information contained in intermediate steps possibly used to "saturate" the abductive problem *via* the deductively minimal *explanans*:

Proposition 4.1 *For any problem* $\Gamma, ⓘ \vdash G$ *such that* $\mathsf{LCH}(\Gamma \dashv G)$ *is an* explanans, *if* $p \in \mathsf{ID}(\Gamma \dashv G)$, *then* $p \in \mathsf{CUT}(\bigwedge \Gamma \to G \dashv)$.

Proof Notice that $\mathsf{top}(\Gamma \dashv G) = \mathsf{top}(\dashv \bigwedge \Gamma \to G)$ due to $\to$-invertibility in $\mathsf{G4}$. Moreover, if $\mathsf{LCH}(\Gamma \dashv G)$ is an *explanans*, then $\overline{\overline{\mathsf{G4}}}$ refutes $\bigwedge \Gamma \to G \vdash$ by using Proposition 3.1, Fact 2.1, and Theorem 3.1. Based on this, we can prove a statement stronger than the one above. Namely, for any formula A, $p \in \mathsf{ID}(\dashv A)$ only if $p \in \mathsf{CUT}(A \dashv)$.

To prove this, we must perform an intermediate step. For any $\Theta \models^{*} \Lambda \in \mathsf{top}(\dashv A)$, suppose $\Theta = p_1, \ldots, p_m$ and $\Lambda = p_{m+1}, \ldots, p_{m+n}$ with $m, n \geq 0$ and $m + n > 0$. For any clause $\Phi \models^{*} \Psi \in \mathsf{top}(A \dashv)$, there is precisely one $p_h \in (\Phi \cup \Psi)$ such that $p_h \in \Psi$ if $1 \leq h \leq m$ and $p_h \in \Phi$ if $m + 1 \leq h \leq m + n$. Furthermore, for any two distinct $\Phi \models^{*} \Psi$ and $\Phi' \models^{*} \Psi' \in \mathsf{top}(A \dashv)$, there are at least two atoms p_h and $p_{h'}$ such that $p_h \in (\Phi \cup \Psi)$, $p_{h'} \in (\Phi' \cup \Psi')$, $p_{h'} \in \Psi'$ if $1 \leq h' \leq m$ and $p_{h'} \in \Phi'$ if $m + 1 \leq h' \leq m + n$, and $p_h \neq p_{h'}$.

We reason by (course-of-value) induction over the number $k \geq 0$ of connectives in A. If $k = 0$, the result is trivial. If $k = j + 1$ with $j \geq 0$, then it suffices to consider two cases.

(i) A is of the form $\neg B$: since $\mathsf{top}(\neg B \dashv) = \mathsf{top}(\dashv B)$ and $\mathsf{top}(\dashv \neg B) = \mathsf{top}(B \dashv)$ by $\neg$-invertibility of $\mathsf{G4}$, it suffices to apply the inductive hypothesis for $k < j$.

(ii) A is of the form $B \wedge C$: since $\mathsf{top}(B \wedge C \dashv) = \mathsf{top}(B, C \dashv)$ and $\mathsf{top}(\dashv B \wedge C) = \mathsf{top}(\dashv B) \cup \mathsf{top}(\dashv C)$ by $\wedge$-invertibility of G4, it suffices to apply twice the inductive hypothesis – with j being $j_1 + j_2$, j_1 being the number of connectives in B and j_2 the number of connectives in C.

It can now be proved that $p \in \mathsf{ID}(\dashv A)$ only if $p \in \mathsf{CUT}(A \dashv)$ (we omit the details). Notice that $\mathsf{CUT}(\mathsf{LCH}(\Gamma \dashv G) \dashv) \subseteq \mathsf{CUT}(\bigwedge \Gamma \to G \dashv)$ by Theorem 3.1. □

For any problem $\Gamma, ⓘ \vdash G$, the elimination of redundant information generates formulas according to the following procedure:

Procedure 4.1 (Approximation to an expected hypothesis). *For any problem $\Gamma, ⓘ \vdash G$ and any subset S of $\mathsf{ID}(\Gamma \dashv G)$, the S*-approximation to an expected hypothesis *$\mathsf{EH}_S\big(\Gamma, ⓘ \vdash G\big)$ is the formula obtained according to the following steps:*

(1) Decompose the antisequent $\Gamma \dashv G$ till the set of clauses $\mathsf{top}_{\dashv}(\Gamma \dashv G) = \big\{\Theta_1 \dashv \Lambda_1, \dots, \Theta_n \dashv \Lambda_n\big\}$ is fully accomplished.
(2) For each clause $\Theta_i \dashv \Lambda_i \in \mathsf{top}_{\dashv}(\Gamma \dashv G)$ take the largest clause $\Theta'_i \dashv \Lambda'_i$ such that $\Theta'_i \subseteq \Theta_i$, $\Lambda'_i \subseteq \Lambda_i$ and $\Theta'_i \cap S = \Lambda'_i \cap S = \varnothing$.
(3) For each clause $\Theta'_i \dashv \Lambda'_i$ thus obtained consider the formula $C_i \equiv \bigwedge \Theta'_i \to \bigvee \Lambda'_i$.
(4) Finally set $\mathsf{EH}_S\big(\Gamma, ⓘ \vdash G\big) = C_1 \wedge \dots \wedge C_n$ (avoiding repetition of conjuncts).

Notice that, for any problem $\Gamma, ⓘ \vdash G$, if $|\mathsf{ID}(\Gamma \dashv G)| = k$, then there are (at most) 2^k EH_S-hypotheses.

Remark that an $\mathsf{EH}_S(\Gamma \dashv G)$-hypothesis is just the $\mathsf{LCH}(\Gamma \dashv G)$-hypothesis whenever either $S = \varnothing$ or $\mathsf{AT}(\mathsf{LCH}(\Gamma \dashv G)) \cap S = \varnothing$.

Example 4.2 *We apply Procedure 4.1 to compute the formula $\mathsf{EH}_S\big((p \wedge q) \vee (r \wedge s), ⓘ \vdash p \wedge r\big)$, for any $S \subseteq \mathsf{ID}\big((p \wedge q) \vee (r \wedge s), ⓘ \vdash p \wedge r\big)$:*

(1) By performing decomposition we get

$$\mathsf{top}_{\dashv}\big((p \wedge q) \vee (r \wedge s), ⓘ \vdash p \wedge r\big) = \big\{r, s \dashv p \ ; \ \ p, q \dashv r\big\}$$

and

$$\mathsf{ID}\big((p \wedge q) \vee (r \wedge s), ⓘ \vdash p \wedge r\big) = \{p, r\}$$

(2) For any $S \subseteq \{p, r\}$ delete all occurrences of atoms in S from clauses $r, s \dashv p$ and $p, q \dashv r$, and take the formula translations of the resulting clauses.
(3) Finally, we obtain the following set of EH_S-abductive hypotheses:

$$((r \wedge s) \to p) \wedge ((p \wedge q) \to r)$$

$$(\neg r \vee \neg s) \wedge (q \to r)$$

$$(s \to p) \wedge (\neg p \vee \neg q)$$

$$\neg s \wedge \neg q$$

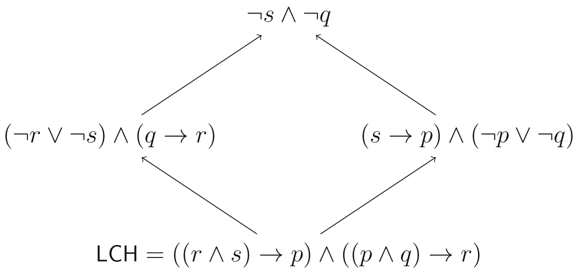

Fig. 3 Poset of EH_S-abductive hypotheses for $(p \wedge q) \vee (r \wedge s) \dashv p \wedge r \neg s \wedge \neg q$

We can refine Procedure 4.1 by taking a reduct under Weakening and Cut of the set of clauses resulting from step (3).

Let us define a partial order $\leq$ over the set of EH_S-hypotheses such that, for any $S, T \subseteq \mathsf{ID}(\Gamma \dashv G)$, $\mathsf{EH}_S(\Gamma \dashv G) \leq \mathsf{EH}_T(\Gamma \dashv G)$ if and only if $S \subseteq T$ (see Fig. 3). It is easy to prove that $\leq$ is monotonic w.r.t. deductive strength:

Theorem 4.1 *For any problem* $\Gamma, ? \vdash G$ *and any* $S, T \subseteq \mathsf{ID}(\Gamma \dashv G)$*, if* $\mathsf{EH}_S(\Gamma \dashv G) \leq \mathsf{EH}_T(\Gamma \dashv G)$ *then* $\overline{\overline{\mathsf{G4}}}$ *proves* $\mathsf{EH}_T(\Gamma \dashv G) \vdash \mathsf{EH}_S(\Gamma \dashv G)$.

Proof By construction, $\mathsf{EH}_T(\Gamma \dashv G)$ is of the following form

$$\left(\bigwedge \Theta_1 \rightarrow \bigvee \Lambda_1\right) \wedge \cdots \wedge \left(\bigwedge \Theta_n \rightarrow \bigvee \Lambda_n\right) \tag{12}$$

On the other hand, $\mathsf{EH}_S(\Gamma \dashv G)$ has by construction the following form

$$\left(\left(\bigwedge \Theta_1 \wedge \bigwedge \Theta'_1\right) \rightarrow \left(\bigvee \Lambda_1 \vee \bigvee \Lambda'_1\right) \wedge \cdots \wedge \left(\bigwedge \Theta_n \wedge \bigwedge \Theta'_n\right) \rightarrow \left(\bigvee \Lambda_n \vee \bigvee \Lambda'_n\right)\right) \tag{13}$$

with $\Theta'_i, \Lambda'_i \subseteq \left(T \setminus S\right)$ for any $1 \leq i \leq n$. By full invertibility in G4 we have that $\overline{\overline{\mathsf{G4}}}$ proves

$$\left(\bigwedge \Theta_i \rightarrow \bigvee \Lambda_i\right) \vdash \left(\bigwedge \Theta_i \wedge \bigwedge \Theta'_i\right) \rightarrow \left(\bigvee \Lambda_i \vee \bigvee \Lambda'_i\right) \tag{14}$$

for any $1 \leq i \leq n$. Provability of Eq. 14, together with the fact that $\mathsf{EH}_T(\Gamma \dashv G)$ and $\mathsf{EH}_S(\Gamma \dashv G)$ have the form displayed by Eqs. 12 and 13, respectively, implies that $\overline{\overline{\mathsf{G4}}}$ proves $\mathsf{EH}_T(\Gamma \dashv G) \vdash \mathsf{EH}_S(\Gamma \dashv G)$ by $n(n-1)$ applications of Left Weakening, $n(n-1)$ applications of $\wedge_{\mathcal{L}}$ and $n-1$ applications of $\wedge_{\mathcal{R}}$. □

Corollary 4.1 *For any problem* $\Gamma, ? \vdash G$ *and any* $S \subseteq \mathsf{ID}(\Gamma \dashv G)$*,* $\overline{\overline{\mathsf{G4}}}$ *proves* $\mathsf{EH}_S(\Gamma \dashv G) \vdash \mathsf{LCH}(\Gamma \dashv G)$.

We can now show that any formula obtained according to Procedure 4.1 satisfies condition A1, and is thus an abductive hypothesis:

Corollary 4.2 *For any problem* $\Gamma, ? \vdash G$ *and any* $S \subseteq \mathsf{ID}(\Gamma \dashv G)$*,* $\mathsf{EH}_S(\Gamma \dashv G)$ *satisfies condition* A1.

Proof Since $\mathsf{LCH}(\Gamma \dashv G)$ is always such that $\Gamma, \mathsf{LCH}(\Gamma \dashv G) \vdash G$ by Corollary 3.1, it suffices to exploit Corollary 4.1 and closure under Cut of G4 to get the result. □

If the elimination of redundant information from $\mathsf{top}_{\dashv}(\mathsf{LCH}(\Gamma \dashv G)) = \{\Theta_1 \dashv \Lambda_1, \dots, \Theta_n \dashv \Lambda_n\}$ is non-vacuous, then an EH_S-abductive hypothesis may not be logically equivalent to the deductively minimal hypothesis:

Proposition 4.2 *For any problem* $\Gamma, \circledcirc \vdash G$ *and any* $S \subseteq \mathsf{ID}(\Gamma \dashv G)$*, if* $\mathsf{LCH}(\Gamma \dashv G)$ *is an* explanans *then* $\overline{\overline{\mathsf{G4}}}$ *refutes* $\mathsf{LCH}(\Gamma \dashv G) \vdash \mathsf{EH}_S(\Gamma \dashv G)$ *if and only if one of the following holds:*

(i) $\mathsf{EH}_S(\Gamma \dashv G)$ *is contradictory;*

(ii) *if* $\Theta_i \dashv \Lambda_i, \Theta_j \dashv \Lambda_j$ *and* $\Theta_k \dashv \Lambda_k \in \mathsf{top}_{\dashv}(\dashv \mathsf{LCH}(\Gamma \dashv G))$*, with* $1 \leq i, j, k \leq n$*, then*

(a) *for any* i *such that either* $\Theta'_i = \Theta_i \setminus S$ *or* $\Lambda'_i = \Lambda_i \setminus S$ *is non empty, and any* $j \neq i$*, either there is one non empty* $\Theta'_j \subseteq \Theta_j$ *such that* $\Theta'_j \cap \Theta'_i = \varnothing$*, or there is one non empty* $\Lambda'_j \subseteq \Lambda_j$ *such that* $\Lambda'_j \cap \Lambda'_i = \varnothing$*;*

(b) *for any* j *there is (at least) an atom* p *such that either* $p \in \Lambda'_j$ *and* $p \notin \Theta'_k$*, or* $p \in \Theta'_j$ *and* $p \notin \Lambda'_k$ *– for any* $k \neq j$*.*

Proof Notice that $\mathsf{LCH}(\Gamma \dashv G)$ being an *explanans* implies that $\overline{\overline{\mathsf{G4}}}$ refutes $\mathsf{LCH}(\Gamma \dashv G) \vdash$, by Proposition 3.1 and Fact 2.1. We separately prove the two directions of the biconditional.

(i) If $\overline{\overline{\mathsf{G4}}}$ refutes $\mathsf{LCH}(\Gamma \dashv G) \vdash \mathsf{EH}_S(\Gamma \dashv G)$, then there must be (at least) one $\Theta_i \dashv \Lambda_i \in \mathsf{top}_{\dashv}(\dashv \mathsf{LCH}(\Gamma \dashv G))$ such that $(\Theta_i \cup \Lambda_i) \cap S \neq \varnothing$. Suppose by contradiction that $\mathsf{EH}_S(\Gamma \dashv G)$ is not contradictory and one of the following two holds:

(a) there is (at least) one distinct $\Theta_j \dashv \Lambda_j \in \mathsf{top}_{\dashv}(\dashv \mathsf{LCH}(\Gamma \dashv G))$ such that, if $\Theta'_i \neq \varnothing$, then for any non empty $\Theta'_j \subseteq \Theta_j$ we have that $\Theta'_j \cap \Theta'_i \neq \varnothing$, and, if $\Lambda'_i \neq \varnothing$, then for any non empty $\Lambda'_j \subseteq \Lambda_j$ we have that $\Lambda'_j \cap \Lambda'_i$;

(b) there is (at least) one $\Theta_j \dashv \Lambda_j \in \mathsf{top}_{\dashv}(\dashv \mathsf{LCH}(\Gamma \dashv G))$ such that, for any atom p, if $p \in \Lambda'_j$, then $p \in \Theta'_k$ for (at least) one $k \neq j$ – and, if $p \in \Theta'_j$, then $p \in \Lambda'_k$ for (at least) one $k \neq j$.

Any $\Phi \mathrel{|\!\!\!\overset{*}{=}} \Psi \in \mathsf{top}(\mathsf{LCH}(\Gamma \dashv G) \dashv)$ results from the selection of one (not necessarily distinct) atom for any $\Theta \mathrel{|\!\!\!\overset{*}{=}} \Lambda \in \mathsf{top}(\dashv \mathsf{LCH}(\Gamma \dashv G))$ taking care of placing on the left (resp. right) side of the sequent symbol the atoms selected on the right (resp. left) (cf. the proof of Proposition 4.1). As a consequence, if (a) is the case then for any $\Phi \mathrel{|\!\!\!\overset{*}{=}} \Psi \in \mathsf{top}(\mathsf{LCH}(\Gamma \dashv G) \dashv)$ we have that either $\Phi \cap \Lambda'_i \neq \varnothing$ or $\Psi \cap \Theta'_i \neq \varnothing$ and thus that $\mathsf{LCH}(\Gamma \dashv G) \vdash (\bigwedge \Theta'_i \to \bigvee \Lambda'_i)$ is provable – a contradiction. On the other hand, if (b) is the case then for any $\Phi \mathrel{|\!\!\!\overset{*}{=}} \Psi \in \mathsf{top}(\mathsf{LCH}(\Gamma \dashv G) \dashv)$ such that $\Phi \cap \Lambda'_i = \varnothing$ and $\Psi \cap \Theta'_i = \varnothing$ we have that $\overline{\overline{\mathsf{G4}}}$ proves $\Phi \vdash \Psi$ – again, a contradiction.

(ii) If $\mathsf{EH}_S(\Gamma \dashv G)$ is contradictory, then $\overline{\overline{\mathsf{G4}}}$ proves $\mathsf{LCH}(\Gamma \dashv G) \vdash \mathsf{EH}_S(\Gamma \dashv G)$ only if the sequent $\mathsf{LCH}(\Gamma \dashv G) \vdash$ is provable – a contradiction. On the other hand, suppose that there is (at least) one $\Theta_i \dashv \Lambda_i \in \mathsf{top}_{\dashv}(\dashv \mathsf{LCH}(\Gamma \dashv G))$ such that $(\Theta_i \cup \Lambda_i) \cap S \neq \varnothing$ and, for any distinct $\Theta_j \dashv \Lambda_j \in \mathsf{top}_{\dashv}(\dashv \mathsf{LCH}(\Gamma \dashv G))$, there is either a non empty $\Theta'_j \subseteq \Theta_j$ such that $\Theta'_j \cap \Theta'_i = \varnothing$ or a non empty $\Lambda'_j \subseteq \Lambda_j$ such that $\Lambda'_j \cap \Lambda'_i = \varnothing$, with $\Theta'_i = (\Theta_i \setminus S)$ and $\Lambda'_i = (\Lambda_i \setminus S)$. This means that there is (at least) one $\Phi \models^* \Psi \in \mathsf{top}(\mathsf{LCH}(\Gamma \dashv G) \dashv)$ such that $\Phi \cap \Lambda'_i = \varnothing$ and $\Psi \cap \Theta'_i = \varnothing$. If for any j there is (at least) an atom p such that either $p \in \Lambda'_j$ and $p \notin \Theta'_k$ for any $k \neq j$, or $p \in \Theta'_j$ and $p \notin \Lambda'_k$ for any $k \neq j$, then one can always pick a $\Phi \models^* \Psi \in \mathsf{top}(\mathsf{LCH}(\Gamma \dashv G) \dashv)$ such that $\Phi \cap \Psi = \varnothing$: as a result, $\overline{\overline{\mathsf{G4}}}$ refutes $\mathsf{LCH}(\Gamma \dashv G) \vdash (\bigwedge \Theta'_i \rightarrow \bigvee \Lambda'_i)$ – as desired.

□

Given the set $\mathcal{F}$ of all formulas, for any set S of atomic sentences we use $\mathcal{F} \ominus S$ to denote the largest set of formulas in which no atom from S occurs – more formally, $\mathcal{F} \ominus S = \{A \in \mathcal{F} \mid \mathsf{AT}(A) \cap S = \varnothing\}$. We can show that any non-contradictory EH_S-hypothesis is deductively minimal w.r.t. abductive hypotheses in $\mathcal{F} \ominus S$:

Proposition 4.3 *For any problem* $\Gamma, ⓘ \vdash G$ *and any* $S \subseteq \mathsf{ID}(\Gamma \dashv G)$*, if* A *is any abductive hypothesis such that* $A \in \mathcal{F} \ominus S$ *and* $\mathsf{EH}_S(\Gamma \dashv G)$ *is not contradictory, then* $\overline{\overline{\mathsf{G4}}}$ *proves* $A \vdash \mathsf{EH}_S(\Gamma \dashv G)$.

Proof Since $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma, A \vdash G$, if $\mathsf{top}_{\dashv}(\Gamma \dashv G) = \{\Theta_1 \dashv \Lambda_1, \dots, \Theta_m \dashv \Lambda_m\}$ then each sequent $\Theta_i, A \vdash \Lambda_i$, with $1 \leq i \leq m$, is provable. On the other hand, if $\mathsf{top}(A \dashv) = \{\Theta'_1 \models^* \Lambda'_1, \dots, \Theta'_n \models^* \Lambda'_n\}$, then each sequent $\Theta'_j, \Theta_i \vdash \Lambda_i, \Lambda'_j$, with $1 \leq j \leq n$, is provable. If $\mathsf{EH}_S(\Gamma \dashv G) = \left(\bigwedge \Phi_1 \rightarrow \bigvee \Psi_1\right) \wedge \dots \wedge \left(\bigwedge \Phi_m \rightarrow \bigvee \Psi_m\right)$, with $\Phi_i = \left(\Theta_i \setminus S\right)$ and $\Psi_i = \left(\Lambda_i \setminus S\right)$, we can prove that $\overline{\overline{\mathsf{G4}}}$ proves $\Phi_i, \Theta'_j \vdash \Lambda'_j, \Psi_i$: we just reason by cases over $\Theta'_j \models^* \Lambda'_j$.

(*i*) If $\overline{\overline{\mathsf{G4}}}$ proves $\Theta'_j \vdash \Lambda'_j$, then it proves $\Phi_i, \Theta'_j \vdash \Lambda'_j, \Psi_i$ since $\mathsf{G4}$ is closed under Weakening.

(*ii*) If $\overline{\overline{\mathsf{G4}}}$ refutes $\Theta'_j \vdash \Lambda'_j$, then $\Theta'_j \cap \Lambda'_j = \varnothing$: since $\Theta'_j, \Theta_i \vdash \Lambda_i, \Lambda'_j$ is provable, we have that either $\Theta'_j \cap \Lambda_i \neq \varnothing$ or $\Theta_i \cap \Lambda'_j \neq \varnothing$. The fact that $A \in \mathcal{F} \ominus S$ implies that $\Theta'_j \cap S = \Lambda'_j \cap S = \varnothing$: since $\mathsf{EH}_S(\Gamma \dashv G)$ is non-contradictory it is sufficient to guarantee that either $\Phi_i \neq \varnothing$ or $\Psi_i \neq \varnothing$, then we have that either $(\Theta'_j \cap \Lambda_i) \subseteq \Psi_i$ or $\left(\Theta_i \cap \Lambda'_j\right) \subseteq \Phi_i$. This implies that either $\Psi_i \cap \Theta'_j \neq \varnothing$ or $\Phi_i \cap \Lambda'_j \neq \varnothing$ – which is enough to conclude that $\overline{\overline{\mathsf{G4}}}$ proves $\Phi_i, \Theta'_j \vdash \Lambda'_j, \Psi_i$.

If $\overline{\overline{\mathsf{G4}}}$ proves $\Phi_i, \Theta'_j \vdash \Lambda'_j, \Psi_i$, with $1 \leq i \leq m$ and $1 \leq j \leq n$, then it also proves each sequent $\Phi_i, \bigwedge \Theta'_j \vdash \bigvee \Lambda'_j, \Psi_i$, and, by m applications of $\rightarrow_{\mathcal{R}}$ and $m-1$ applications of $\wedge_{\mathcal{R}}$, each sequent $\Theta'_j \vdash \Lambda'_j, \mathsf{EH}_S(\Gamma \dashv G)$. As an immediate consequence, we have that $\overline{\overline{\mathsf{G4}}}$ proves $A \vdash \mathsf{EH}_S(\Gamma \dashv G)$. □

The following example illustrates that even when we restrict ourselves to abductive problems where the LCH-hypothesis serves as an *explanans*, there is no guarantee that the EH_S-hypothesis is also an *explanans*, for some set of atoms $S \neq \varnothing$.

Example 4.3 *Consider the problem* $(p \wedge \neg q) \rightarrow r, q \rightarrow \neg r, ⓘ \vdash r$*: if* $S = \{q, r\}$*, Procedure 4.1 yields* $\top \rightarrow \bot$ *as an optimized version of* $\mathsf{EH}_S((p \wedge \neg q) \rightarrow r, q \rightarrow$

$\neg r$, ⓘ $\vdash r$) *- and the sequents* $\top \to \bot \vdash r$ *and* $(p \wedge \neg q) \to r, q \to \neg r, (\top \to \bot) \vdash$ *are clearly provable in* $\overline{\overline{\mathsf{G4}}}$.

Once more, closer examination shows that the EH_S-hypothesis satisfies conditions A2 and A3 in a restricted number of cases, which is characterized by the following result:

Theorem 4.2 *For any problem* Γ, ⓘ $\vdash G$ *and any* $S \subseteq \mathsf{ID}(\Gamma \dashv G)$, $\mathsf{EH}_S(\Gamma \dashv G)$ *is an* explanans *just in case*

- (i) *there is at least one* $\Theta \dashv \Lambda \in \mathsf{top}_{\dashv}(\mathsf{EH}_S(\Gamma \dashv G) \dashv)$ *such that, for any* $\Theta' \subseteq \Theta$, $\Lambda' \subseteq \Lambda$, $\Theta' \dashv \Lambda' \notin \mathsf{top}_{\dashv}(G \dashv)$*;*
- (ii) *there is at least one* $\Theta \dashv \Lambda \in \mathsf{top}_{\dashv}(\mathsf{EH}_S(\Gamma \dashv G) \dashv)$ *such that, for any* $\Theta' \subseteq \Theta$, $\Lambda' \subseteq \Lambda$, $\Theta' \dashv \Lambda' \notin \mathsf{top}_{\dashv}(\dashv \bigwedge \Gamma)$.

Proof Note that if $\mathsf{EH}_S(\Gamma \dashv G)$ serves as an *explanans*, it cannot be contradictory. According to Facts 2.1 and 2.3, this means that $\mathsf{top}_{\dashv}(\mathsf{EH}_S(\Gamma \dashv G) \dashv) \neq \varnothing$. Furthermore, observe that if $\overline{\overline{\mathsf{G4}}}$ proves either $G \vdash$ or $\vdash \bigwedge \Gamma$, then by Corollary 4.2 and closure of G4 under Cut, it also proves either $\Gamma, \mathsf{EH}_S(\Gamma \dashv G)$ or $\mathsf{EH}_S(\Gamma \dashv G) \vdash G$. Thus, if $\mathsf{EH}_S(\Gamma \dashv G)$ is an explanans, then $\mathsf{top}_{\dashv}(G \dashv)$ and $\mathsf{top}_{\dashv}(\dashv \bigwedge \Gamma)$ are both non-empty.
We can focus on case (i), since case (ii) is analogous.

(i) Let us assume by contradiction that $\mathsf{EH}_S(\Gamma \dashv G)$ is an *explanans* and, for any $\Theta \dashv \Lambda \in \mathsf{top}_{\dashv}(\mathsf{EH}_S(\Gamma \dashv G) \dashv)$, there exist $\Theta' \subseteq \Theta$ and $\Lambda' \subseteq \Lambda$ such that $\Theta' \dashv \Lambda' \in \mathsf{top}_{\dashv}(G \dashv)$. Since $\overline{\overline{\mathsf{G4}}}$ refutes $\Gamma \vdash G$ and thus $\mathsf{top}_{\dashv}(G \dashv) \neq\ \dashv$, we have $(\Theta' \cup \Lambda') \neq \varnothing$. It is easy to show that for any $\Theta' \dashv \Lambda' \in \mathsf{top}_{\dashv}(G \dashv)$ and any $\Theta'' \dashv \Lambda'' \in \mathsf{top}_{\dashv}(\dashv G)$, either $\Theta' \cap \Lambda'' \neq \varnothing$ or $\Lambda' \cap \Theta'' \neq \varnothing$ (cf. the proof of Proposition 4.1). As a result, $\overline{\overline{\mathsf{G4}}}$ proves $\mathsf{EH}_S(\Gamma \dashv G) \vdash G$, and thus $\mathsf{EH}_S(\Gamma \dashv G)$ does not satisfy condition A2 – a contradiction.
Now, let us assume by contradiction that there exists at least one $\Theta \dashv \Lambda \in \mathsf{top}_{\dashv}(\mathsf{EH}_S(\Gamma \dashv G) \dashv)$ such that, for any $\Theta' \subseteq \Theta$, $\Lambda' \subseteq \Lambda$, $\Theta' \dashv \Lambda' \notin \mathsf{top}_{\dashv}(G \dashv)$, and $\mathsf{EH}_S(\Gamma \dashv G)$ does not satisfy condition A2. Since $\mathsf{top}_{\dashv}(\mathsf{EH}_S(\Gamma \dashv G) \dashv) \neq \varnothing$, we must conclude that $\overline{\overline{\mathsf{G4}}}$ proves $\vdash G$ – another contradiction.

□

Corollary 4.3 *For any problem* Γ, ⓘ $\vdash G$ *and any non-empty* $S \subseteq \mathsf{ID}(\Gamma \dashv G)$*, if* $\mathsf{LCH}(\Gamma \dashv G)$ *is an* explanans*, then* $\mathsf{EH}_S(\Gamma \dashv G)$ *is an* explanans *just in case*

- (i) *there is at least one* $\Theta \dashv \Lambda \in \mathsf{top}_{\dashv}(\mathsf{LCH}(\Gamma \dashv G) \dashv)$ *such that, for any* $\Theta' \subseteq \Theta$, $\Lambda' \subseteq \Lambda$, $\Theta' \dashv \Lambda' \notin \mathsf{top}_{\dashv}(G \dashv)$ *and* $(\Theta \cup \Lambda) \not\subseteq S$*;*
- (ii) *there is least one* $\Theta \dashv \Lambda \in \mathsf{top}_{\dashv}(\mathsf{LCH}(\Gamma \dashv G) \dashv)$ *such that, for any* $\Theta' \subseteq \Theta$, $\Lambda' \subseteq \Lambda$, $\Theta' \dashv \Lambda' \notin \mathsf{top}_{\dashv}(\dashv \bigwedge \Gamma)$ *and* $(\Theta \cup \Lambda) \not\subseteq S$.

Theorem 4.2 and Corollary 4.3 provide some important insights into the nature of $\mathsf{EH}_S(\Gamma \dashv G)$ as an *explanans*. Specifically, they state that $\mathsf{EH}_S(\Gamma \dashv G)$ is an *explanans* if the $\mathsf{LCH}(\Gamma \dashv G)$-hypothesis respects conditions A2 and A3, regardless of whether or not there are abductively redundant atoms present. This is important because it shows that the number of (contingent) novelties against the (contingent) theoretical background can be reduced without necessarily depending on abductively redundant atoms. Furthermore, any EH_S-*explanans* can be used to reduce the number of novelties in a way that approximates the abductively optimal one. Additionally, we can establish that $\mathsf{EH}_S(\Gamma \dashv G)$ and $\bigwedge \Gamma$ are deductively independent of each other, and G is deductively independent of $\mathsf{EH}_S(\Gamma \dashv G)$ (cf. Proposition 3.1).

The cases where an EH_S-hypothesis fails to be maximally supported by evidence can be characterized as follows:

Proposition 4.4 *For any problem* $\Gamma, ⓘ \vdash G$ *and any* $S \subseteq \mathsf{ID}(\mathsf{top}(\Gamma \dashv G))$, *if* $\mathsf{LCH}(\Gamma \dashv G)$ *is an* explanans *then* $\overline{\overline{\mathsf{G4}}}$ *refutes* $G \vdash \mathsf{EH}_S(\Gamma \dashv G)$ *if and only if* $\mathsf{AT}(\Theta \dashv \Lambda) \cap S \neq \varnothing$ *for some* $\Theta \dashv \Lambda \in \mathsf{top}_{\dashv}(\dashv G)$.

Proof Analogous to the proof of Proposition 4.2. □

We are now ready to give a formal rendition of the intuitive notion of 'expected hypothesis' we started this section with:

Procedure 4.2 *(Expected hypothesis). For any problem* $\Gamma, ⓘ \vdash G$ *such that* $\mathsf{LCH}(\Gamma \dashv G)$ *is an* explanans, *and for any subset* S *of* $\mathsf{ID}(\Gamma \dashv G)$, *the set of* expected hypotheses $\mathsf{EH}\big(\Gamma, ⓘ \vdash G\big)$ *is obtained according to the following steps:*

(1) Decompose the antisequent $\Gamma \dashv G$ *till (a reduct under Weakening and Cut of) the set of clauses* $\mathsf{top}_{\dashv}(\Gamma \dashv G)$ *is fully accomplished.*
(2) For each S *apply steps (2) – (4) of Procedure 4.1 so as to get the set* $\mathcal{E}$ *of all (optimized)* EH_S*-hypotheses.*
(3) Take the greatest $\mathcal{E}' \subseteq \mathcal{E}$ *such that* $\mathcal{E}'$ *does not include any formula* A *for which each clause of* $\mathsf{top}_{\dashv}(A \dashv)$ *is both a weakened version of a clause in* $\mathsf{top}_{\dashv}(G \dashv)$ *and a weakened version of a clause in* $\mathsf{top}_{\dashv}(\dashv \bigwedge \Gamma)$ *(cf. Theorem 4.2).*
(4) Finally, take the least $\mathcal{E}'' \subseteq \mathcal{E}'$ *which contains the maximal elements of* $\mathcal{E}'$ *w.r.t.* $\leq$.

We give some examples of how Procedure 4.2 works.

Example 4.4 *For any problem in Fig. 2 it is easy to verify that the set of* EH*-hypotheses produced according to Procedure 4.2 contains only the hypothesis reported in the rightmost column. Take e.g. the abductive problem* $p \to q, r \to s, ⓘ \vdash q \vee s$:

(1) the only reduct under Weakening and Cut of $\mathsf{top}_{\dashv}(\Gamma \dashv G)$ *is* $\{\dashv q, s, r, p\}$, *and thus* $\mathsf{LCH}(\Gamma \dashv G) = q \vee s \vee r \vee p$;
(2) $\mathsf{ID}(\Gamma \dashv G) = \{q, s\}$, *and thus the greatest* EH_S*-hypothesis w.r.t* $\leq$ *is* $r \vee p$;
(3) since the greatest EH_S*-hypothesis w.r.t* $\leq$ *is an* explanans, *we have that the only* EH*-hypothesis is* $r \vee p$.

Example 4.5 *Take the abductive problem* $(p \wedge \neg q) \to r, q \to \neg r, ⓘ \vdash r$ *Example 4.3:*

(1) the only reduct under Weakening and Cut of $\mathsf{top}_{\dashv}(\Gamma \dashv G)$ *is* $\{\dashv r, p \;\; ; \;\; q \dashv r\}$, *and thus* $\mathsf{LCH}(\Gamma \dashv G) = (r \vee p) \wedge (r \vee \neg q)$;

(2) $\mathsf{ID}(\Gamma \dashv G) = \{r, q\}$*, and thus the greatest* $\mathsf{EH}_S(\Gamma \dashv G)$ *w.r.t* $\leq$ *is* $\top \to \bot$*, which is not an* explanans*;*

(3) the greatest EH_S*-hypothesis w.r.t.* $\leq$ *which is an* explanans *is* $p \wedge \neg q$*: the only* EH*-hypothesis is* $p \wedge \neg q$*, as expected.*

Example 4.6 *Take the abductive problem* $(p \wedge \neg q) \to r, q \to \neg r, ⓘ \vdash \neg r$*:*

(1) the only reduct under Weakening and Cut of $\mathsf{top}_\dashv(\Gamma \dashv G)$ *is* $\{r \dashv q\}$*, and thus* $\mathsf{LCH}(\Gamma \dashv G) = r \to q$*;*

(2) $\mathsf{ID}(\Gamma \dashv G) = \{r, q\}$*, and thus the greatest* EH_S*-hypothesis w.r.t.* $\leq$ *is* $\top \to \bot$*, which is not an* explanans*;*

(3) the greatest EH_S*-hypothesis w.r.t.* $\leq$ *which is an* explanans *is* q*: the only* EH*-hypothesis is* q*, as expected.*

Procedure 4.2 is an effective tool for tracking intuitively expected hypotheses in familiar examples of abductive problem: we propose to take it as a normative standard for the rational agent – even in cases where we lack equally strong intuitions.

5 Beyond Analyticity

As we have seen, given a problem $\Gamma, ⓘ \vdash G$, analytic decomposition can be used as a tool for generating formulas, possibly stronger than $\mathsf{LCH}(\Gamma \dashv G)$, which satisfy conditions A1 and, possibly, A2 – A3: since any A among these formulas is such that $\mathsf{AT}(A) \subseteq \mathsf{AT}(\Gamma \dashv G)$ we say that they are *analytic abductive hypotheses* (possibly, *analytic explanantes*). In order to track formulas obtained through decomposition in full generality, we modify Procedure 3.1 as follows:

Procedure 5.1 (Strengthened Least Compromising Hypothesis). *For any problem* $\Gamma, ⓘ \vdash G$*, the* $\vec{S}$-strengthened least compromising hypothesis $\mathsf{SLCH}_{\vec{S}}\big(\Gamma, ⓘ \vdash G\big)$ *is the formula obtained as follows:*

(1) Decompose the antisequent $\Gamma \dashv G$ *till the non-empty set of complementary clauses* $\mathsf{top}_\dashv(\Gamma \dashv G) = \big\{\Theta_1 \dashv \Lambda_1, \ldots, \Theta_n \dashv \Lambda_n\big\}$ *is fully accomplished.*

(2) Define a sequence $\vec{S} = \langle S_1, \ldots, S_n\rangle$ *of subsets of* $\mathsf{AT}(\Gamma \dashv G)$*, and, for each clause* $\Theta_i \dashv \Lambda_i \in \mathsf{top}_\dashv(\Gamma \dashv G)$*, take the largest clause* $\Theta_i' \dashv \Lambda_i'$ *such that* $\Theta_i' \subseteq \Lambda_i$*,* $\Theta_i' \subseteq \Lambda_i$ *and* $\Theta_i' \cap S_i = \Lambda_i' \cap S_i = \varnothing$*.*

(3) For each clause $\Theta_i' \dashv \Lambda_i'$ *thus obtained consider the formula* $C_i \equiv \bigwedge \Theta_i' \to \bigvee \Lambda_i'$*.*

(4) Finally set $\mathsf{SLCH}_S\big(\Gamma, ⓘ \vdash G\big) = C_1 \wedge \cdots \wedge C_n$ *(avoiding repetition of conjuncts).*

Notice that, if $\vec{S} = \langle S_1, \ldots, S_n\rangle$, $S_i = S$ for any $1 \leq i \leq n$ and $S \subseteq \mathsf{ID}(\Gamma \dashv G)$, then $\mathsf{SLCH}_{\vec{S}}(\Gamma \dashv G) = \mathsf{EH}_S(\Gamma \dashv G)$; if it is the case that $S = \varnothing$, then $\mathsf{SLCH}_{\vec{S}}(\Gamma \dashv G) = \mathsf{LCH}(\Gamma \dashv G)$.

Example 5.1 *We apply Procedure 5.1 to compute the formula* $\mathsf{SLCH}_{\vec{S}}\big(p \to q, ⓘ \vdash p \to r\big)$ *for any* $\vec{S}$*:*

(1) By performing decomposition we get

$$\mathsf{top}_{\dashv}(p \to q, \textcircled{?} \vdash p \to r) = \{q, p \dashv r\}$$

and

$$\mathsf{AT}(p \to q, \textcircled{?} \vdash p \to r) = \{p, q, r\}$$

(2) It is trivial to set an enumeration of the elements of $\mathsf{top}_{\dashv}(p \to q, \textcircled{?} \vdash p \to r)$.

(3) For any $\vec{S}$ *we obtain a single clause, which we turn into its corresponding formula to obtain the corresponding* $\mathsf{SLCH}_{\vec{S}}$*-hypothesis:*

$$
\begin{aligned}
q, p \dashv r \quad &\Rightarrow \quad (q \wedge p) \to r \\
p \dashv r \quad &\Rightarrow \quad p \to r \\
q \dashv r \quad &\Rightarrow \quad q \to r \\
q, p \dashv \quad &\Rightarrow \quad \neg q \vee \neg p \\
q \dashv \quad &\Rightarrow \quad \neg q \\
p \dashv \quad &\Rightarrow \quad \neg p \\
\dashv r \quad &\Rightarrow \quad r \\
\dashv \quad &\Rightarrow \quad \top \to \bot
\end{aligned}
$$

Remark that, for any problem $\Gamma, \textcircled{?} \vdash G$*, if* $\mathsf{top}_{\dashv}(\Gamma \dashv G) = \{\Theta_1 \dashv \Lambda_1, \ldots, \Theta_n \dashv \Lambda_n\}$ *and* $\sum_{i=1}^{n} |\mathsf{AT}(\Theta_i \dashv \Lambda_i)| = k$*, then there are (at most)* 2^k $\mathsf{SLCH}_{\vec{S}}$*-hypotheses.*

We can optimize Procedure 5.1, similarly to how we did for Procedure 4.1. In particular, if the set of clauses $\mathcal{S}$ generated by step (2) in Procedure 5.1 includes the empty antisequent, then the only reduct of $\mathcal{S}$ under Weakening and Cut is the singleton of the empty antisequent. As a result, the refined Procedure 5.1 sets an upper bound on the number of all $\mathsf{SLCH}_{\vec{S}}$-hypotheses to $2^k - 2(2^{n-1} - 1)$.

It is immediate to verify that any $\mathsf{SLCH}_{\vec{S}}$-hypothesis satisfies condition A1, as shown in Theorem 4.1 and Corollaries 4.1 and 4.2. Moreover, a $\mathsf{SLCH}_{\vec{S}}$-hypothesis satisfies conditions A2 – A3 if $\mathsf{top}_{\dashv}(\mathsf{SLCH}_{\vec{S}}(\Gamma \dashv G) \dashv)$ contains at least one clause that is not a (possibly) weakened version of a clause in $\mathsf{top}_{\dashv}(G \dashv)$, and at least one clause which is not a (possibly) weakened version of a clause in $\mathsf{top}_{\dashv}(\dashv \bigwedge \Gamma)$, as shown in Theorem 4.2.

Let us introduce a bit more of terminology: for any formula A, if $\mathsf{top}(\models^{*} A) = \{\Theta_1 \models^{*} \Theta_m, \ldots, \Theta_n \models^{*} \Lambda_m\}$ then we use $cnf(A)$ to refer to $\bigwedge_{i=1}^{m} (\bigwedge \Theta_i \to \bigvee \Lambda_i)$. The following result shows that any analytic *explanans* A logically implies some $\mathsf{SLCH}_{\vec{S}}$-*explanans*:

Theorem 5.1 *For any problem* $\Gamma, \textcircled{?} \vdash G$*, if a formula is an analytic* explanans A*, then* A *is logically equivalent to* $B \wedge C$*, where*

(i) $B = \mathsf{SLCH}_{\vec{S}_1}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S}_n}(\Gamma \dashv G)$, *with* $\mathsf{SLCH}_{\vec{S}_i}(\Gamma \dashv G)$ *being an* explanans *for any* $1 \leq i \leq n$*;*

(ii) $\overline{\overline{\mathsf{G4}}}$ *refutes* $\Gamma, C \vdash G$*.*

Proof First, notice that if A is an (analytic) *explanans*, then it is a contingent formula. This is because if A were a tautology, then the refutability of $\Gamma \vdash G$ would imply the refutability of $\Gamma, A \vdash G$ (against condition A1). Similarly, if A were a contradiction, then $\overline{\overline{\mathsf{G4}}}$ would prove both $A \vdash G$ and $\Gamma, A \vdash G$ (against conditions A2 and A3, respectively).
If A is a contingent formula, then by Fact 2.1, $\mathsf{top}_{\dashv}(\dashv A) \neq \varnothing$ and $\mathsf{top}_{\dashv}(A \dashv) \neq \varnothing$. If $\mathsf{top}_{\vdash}(A \dashv) \neq \varnothing$, then we can always consider a formula A' that is logically equivalent to A and such that $\mathsf{top}(A' \dashv) = \mathsf{top}_{\dashv}(A \dashv)$. Let us assume that $\mathsf{top}(A' \dashv) = \{\Theta_1 \dashv \Lambda_1, \ldots, \Theta_m \dashv \Lambda_m\}$ and that $\mathsf{top}_{\dashv}(\Gamma \dashv G) = \{\Phi_1 \dashv \Psi_1, \ldots, \Phi_n \dashv \Psi_n\}$: if $\Gamma, A \vdash G$, and thus $\Gamma, A' \vdash G$, is provable, then for any $\Theta_i \dashv \Lambda_i \in \mathsf{top}(A' \dashv)$ and any $\Phi_j \dashv \Psi_j \in \mathsf{top}_{\dashv}(\Gamma \dashv G)$ there is either one non-empty $\Theta_{i_j} \subseteq \Theta_i$ such that $\Theta_{i_j} = \Theta_i \cap \Psi_j$ or one non-empty $\Lambda_{i_j} \subseteq \Lambda_i$ such that $\Lambda_{i_j} = \Lambda_i \cap \Phi_j$.
Bearing these facts in mind, we can proceed to prove the two statements separately.

(i) Consider any $\Theta' \mathrel{|\!\!\!\overset{*}{-}} \Lambda' \in \mathsf{top}(\dashv A')$ such that, for a given j such that $1 \leq j \leq n$, if $p, q \in \Theta'$, then $p \in \Lambda_{i_j}$ and $q \in \Lambda_{i'_j}$ and, if $r, s \in \Lambda'$, then $r \in \Theta_{i_j}$ and $s \in \Theta_{i'_j}$ – with $1 \leq i \neq i' \leq m$: it is easy to see that there is (at least) one $\vec{S}$ such that $\Theta' \dashv \Lambda' \in \mathsf{top}_{\dashv}(\dashv \mathsf{SLCH}_{\vec{S}}(\Gamma \dashv G))$. Since the set of all $\vec{S}$-strengthenings of a given clause $\Phi_j \dashv \Psi_j \in \mathsf{top}_{\dashv}(\Gamma \dashv G)$ cannot be totally ordered with respect to deductive strength (cf. Theorem 4.1), it may be the case that (a reduct under Weakening and Cut of) the set of the $\vec{S}$-strengthenings of $\Phi_j \dashv \Psi_j \in \mathsf{top}_{\dashv}(\Gamma \dashv G)$ included in $\mathsf{top}(\dashv A')$ does not narrow down to a singleton. This holds for any $1 \leq j \leq n$, and therefore, there exist $\vec{S}_1, \ldots, \vec{S}_N$ such that $\mathsf{top}_{\dashv}\big(\dashv \mathsf{SLCH}_{\vec{S}_1}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S}_N}(\Gamma \dashv G)\big) \subseteq \mathsf{top}(\dashv A')$.
At this point, we must consider two possibilities: either (a) for any $\Theta_i \dashv \Lambda_i \in \mathsf{top}(A' \dashv)$ we have that $\Theta_i = (\Theta_{i_1} \cup \cdots \cup \Theta_{i_n})$ and $\Lambda_i = (\Lambda_{i_1} \cup \cdots \cup \Lambda_{i_n})$, or (b) there is at least one $\Theta_i \dashv \Lambda_i \in \mathsf{top}(A' \dashv)$ such that either $\Theta_i \supset (\Theta_{i_1} \cup \cdots \cup \Theta_{i_n})$ or $\Lambda_i \supset (\Lambda_{i_1} \cup \cdots \cup \Lambda_{i_n})$. In the first case we have $\mathsf{top}_{\dashv}\big(\dashv \mathsf{SLCH}_{\vec{S}_1}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S}_N}(\Gamma \dashv G)\big) = \mathsf{top}(\dashv A')$: since $cnf(A') \equiv A'$ by Fact 2.3, we have that A is logically equivalent to $\mathsf{SLCH}_{\vec{S}_1}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S}_N}(\Gamma \dashv G) \wedge \top$. In the second case, there must exist a formula C such that $\mathsf{top}(\mathrel{|\!\!\!\overset{*}{-}} C) = \mathsf{top}(\dashv A') \setminus \mathsf{top}_{\dashv}\big(\dashv \mathsf{SLCH}_{\vec{S}_1}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S}_N}(\Gamma \dashv G)\big)$. Therefore, A must be logically equivalent to $\mathsf{SLCH}_{\vec{S}_1}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S}_N}(\Gamma \dashv G) \wedge C$. We can reach the conclusion by noticing that $\mathsf{SLCH}_{\vec{S}_1}(\Gamma \dashv G), \ldots, \mathsf{SLCH}_{\vec{S}_N}(\Gamma \dashv G)$ necessarily satisfy conditions A2 – A3.

(ii) Since $\overline{\overline{\mathsf{G4}}}$ refutes $\Gamma, \top \vdash G$, we focus on case (b) and assume by contradiction that $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma, C \vdash G$. The provability of $\Gamma, C \vdash G$ implies that for any $\Pi \mathrel{|\!\!\!\overset{*}{-}} \Sigma \in \mathsf{top}(C \mathrel{|\!\!\!\overset{*}{-}})$ and any $\Phi_j \vdash \Psi_j \in \mathsf{top}_{\dashv}(\Gamma \dashv G)$, either $\overline{\overline{\mathsf{G4}}}$ proves $\Pi \vdash \Sigma$, or there exists a non-empty $\Pi_j \subseteq \Pi$ such that $\Pi_j = \Pi \cap \Psi_j$, or a non-empty $\Sigma_j \subseteq \Sigma$ such that $\Sigma_j = \Sigma \cap \Phi_j$. If $\Pi \vdash \Sigma$ were always provable, then C would be contradictory by Fact 2.1, leading to a contradiction. As a consequence, there must be (at least) one $\Pi \dashv \Sigma \in \mathsf{top}(C \dashv)$ such that for any $1 \leq j \leq n$, there is either one non-empty $\Pi_j \subseteq \Pi$ such that $\Pi_j = \Pi \cap \Psi_j$ or one non-empty $\Sigma_j \subseteq \Sigma$ such that $\Sigma_j = \Sigma \cap \Phi_j$. This means that there is (at least) one $\vec{S}$ such that $\mathsf{top}_{\dashv}(\dashv \mathsf{SLCH}_{\vec{S}}(\Gamma \dashv G)) \subseteq \mathsf{top}(\models^{*} C)$. By construction, for any clause $\Pi' \models^{*} \Sigma' \in \mathsf{top}(\models^{*} C)$, we have that there is (at least) one atom p such that if $p \in \Pi'$, then $p \notin \Phi_j$, and if $p \in \Sigma'$, then $p \notin \Psi_j$, for any $1 \leq j \leq n$. As a result, for any $\vec{S}$, $\mathsf{top}_{\dashv}(\dashv \mathsf{SLCH}_{\vec{S}}(\Gamma \dashv G)) \not\subseteq \mathsf{top}(\models^{*} C)$, which leads to a contradiction.

□

Corollary 5.1 *For any problem* $\Gamma, ? \vdash G$, *if a formula* A *is an analytic* explanans *and* $\mathsf{CUT}(\dashv A) = \varnothing$ *then* $cnf(A) = B \wedge C$, *where*

(*i*) $B = \mathsf{SLCH}_{\vec{S}_1}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S}_n}(\Gamma \dashv G)$, *with* $\mathsf{SLCH}_{\vec{S}_i}(\Gamma \dashv G)$ *being an* explanans *for any* $1 \leq i \leq n$;

(*ii*) $\mathsf{AT}(C) \subseteq \mathsf{AT}(\Gamma \dashv G)$ *and* $\overline{\overline{\mathsf{G4}}}$ *refutes* $\Gamma, C \vdash G$.

Example 5.2 *Take the problem* $p \to q, ? \vdash p \to r$ *of Example 5.1*: $(\neg q \vee \neg p) \vee (r \wedge \neg r)$ *is an analytic* explanans, *and it is logically equivalent to* $((q \wedge p) \to r) \wedge (\neg q \vee \neg p \vee \neg r)$, *with* $(q \wedge p) \to r$ *being an* $\mathsf{SLCH}_{\vec{S}}$-explanans *and the sequent* $p \to q, \neg q \vee \neg p \vee \neg r \vdash p \to r$ *being refutable.*

Theorem 5.1 establishes that, for any problem $\Gamma, ? \vdash G$, each analytic *explanans* A can be decomposed into a conjunction of $\mathsf{SLCH}_{\vec{S}}$-*explanantes* for $\Gamma, ? \vdash G$ and a 'derived' problem $\Gamma, C, ? \vdash G$. The following proposition shows that any $\mathsf{SLCH}_{\vec{S}}$-*explanans* for $\Gamma, ? \vdash G$ is an $\mathsf{SLCH}_{\vec{S}}$-*explanans* for the derived problem $\Gamma, C, ? \vdash G$:

Proposition 5.1 *For any problem* $\Gamma, ? \vdash G$ *and each analytic* explanans A, *if* $A \equiv B \wedge C$, $B = \bigwedge_{i=1}^{n} \mathsf{SLCH}_{\vec{S}_i}(\Gamma \dashv G)$ *and* $\overline{\overline{\mathsf{G4}}}$ *refutes* $\Gamma, C \vdash G$, *then any* $\mathsf{SLCH}_{\vec{S}}$-*abductive hypothesis for* $\Gamma, ? \vdash G$ *is an* $\mathsf{SLCH}_{\vec{S}}$-*abductive hypothesis for* $\Gamma, C, ? \vdash G$.

Proof It is routine to show that $\Theta \dashv \Lambda \in \mathsf{top}_{\dashv}(\Gamma, C \dashv G)$ if and only if there exist $\Theta' \dashv \Lambda' \in \mathsf{top}_{\dashv}(\Gamma \dashv G)$ and $\Theta'' \dashv \Lambda'' \in \mathsf{top}_{\dashv}(C \dashv)$ such that $\Theta' \cup \Theta'' = \Theta$ and $\Lambda' \cup \Lambda'' = \Lambda$. As a result, if there is an atom $p \in (\Theta \cup \Lambda)$ such that $p \in (\Theta' \cup \Lambda') \cap (\Theta'' \cup \Lambda'')$, then $p \in \Theta'$ if and only if $p \in \Theta''$, and $p \in \Lambda'$ if and only if $p \in \Lambda''$: this means that if p is erased from $\Theta' \dashv \Lambda'$ then it is also erased from $\Theta \dashv \Lambda$. By construction, any $\mathsf{SLCH}_{\vec{S}}$-abductive hypothesis is obtained by non-uniformly deleting atoms from clauses in $\mathsf{top}_{\dashv}(\Gamma \dashv G)$: this suffices to get the conclusion. □

For any problem $\Gamma, ? \vdash G$ we say that a set of *explanantes* $A_1, \ldots, A_n$ is a set of *alternative* abductive solutions just if $A_1, \ldots, A_n$ are pairwise mutual exclusive and jointly exhaustive – i.e., such that $\overline{\overline{\mathsf{G4}}}$ proves $A_i, A_j \vdash$ and $\vdash A_1, \ldots, A_n$ respectively, for any $1 \leq i \neq j \leq n$ (cf. [10], pp. 45-46): the following proposition shows that the set of all analytic *explanantes* is not a set of alternative abductive solutions.

Proposition 5.2 *For any problem* $\Gamma, ? \vdash G$, *if* $\mathsf{LCH}(\Gamma \dashv G), A_1, \ldots, A_n$ *are distinct formulas respecting condition* A1 – A3, *then*

(i) $\mathsf{LCH}(\Gamma \dashv G)$ *and* A_i *are not mutually exclusive, for any* $1 \leq i \leq n$*;*
(ii) $\mathsf{LCH}(\Gamma \dashv G), A_1, \ldots, A_n$ *are not jointly exhaustive.*

Proof We treat each case separately.

(i) Assume, by contrast, that $\mathsf{LCH}(\Gamma \dashv G), A_i$ respect conditions A1 – A3, for any $1 \leq i \leq n$, while being mutually exclusive. This means that $\overline{\overline{\mathsf{G4}}}$ proves $\mathsf{LCH}(\Gamma \dashv G), A_i \vdash$: since Theorem 3.2 guarantees that $\overline{\overline{\mathsf{G4}}}$ proves $A_i \vdash \mathsf{LCH}(\Gamma \dashv G)$, we have that $\overline{\overline{\mathsf{G4}}}$ proves $A_i \vdash$ by closure under Cut – i.e. a contradiction.
(ii) Again, assume by contrast that $\mathsf{LCH}(\Gamma \dashv G), A_1, \ldots, A_n$ respect condition A1 and that $\mathsf{LCH}(\Gamma \dashv G), A_1, \ldots, A_n$ are jointly exhaustive. We have that $\overline{\overline{\mathsf{G4}}}$ proves both $\Gamma, \mathsf{LCH}(\Gamma \dashv G) \vdash G$ and $\vdash \mathsf{LCH}(\Gamma \dashv G), A_1, \ldots, A_n$: by closure of G4 under Cut it follows that $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma \vdash G, A_1, \ldots, A_n$. We can iterate n times this reasoning step, by taking at each step exactly one among $A_1, \ldots, A_n$ as cut formula: finally, we reach the conclusion that $\overline{\overline{\mathsf{G4}}}$ proves $\Gamma \vdash G$ – a contradiction. □

At this point it is natural to ask whether our framework can be used to investigate the (infinite) set of *non-analytic abductive hypotheses* (*explanantes*) – i.e., formulas A obeying condition A1 (conditions A1 – A3, respectively) such that $\mathsf{AT}(A) \not\subseteq \mathsf{AT}(\Gamma \dashv G)$.

Example 5.3 *Take the problem* $p \to q, ⓠ \vdash p \to r$ *of Example 5.1:* $(\neg p \wedge s) \vee (\neg q \wedge t)$ *is a non-analytic abductive hypothesis, whereas* $\neg q \wedge (s \vee t)$ *is a non-analytic* explanans.

First, let us notice that non-analytic *explanantes* enjoy the same kind of ‘abductive normal form’ as analytic ones:

Theorem 5.2 *For any problem* $\Gamma, ⓠ \vdash G$*, if a formula is a non-analytic* explanans A*, then* A *is logically equivalent to* $B \wedge C$*, where*

(i) $B = \mathsf{SLCH}_{\vec{S_1}}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S_n}}(\Gamma \dashv G)$*, with* $\mathsf{SLCH}_{\vec{S_i}}(\Gamma \dashv G)$ *being an* explanans *for any* $1 \leq i \leq n$*;*
(ii) $\overline{\overline{\mathsf{G4}}}$ *refutes* $\Gamma, C \vdash G$.

Proof Analogous to the proof of Theorem 5.1. □

Corollary 5.2 *For any problem* $\Gamma, ⓠ \vdash G$*, if a formula* A *is a non-analytic* explanans *and* $\mathsf{CUT}(\dashv A) = \varnothing$ *then* $cnf(A) = B \wedge C$*, where*

(i) $B = \mathsf{SLCH}_{\vec{S_1}}(\Gamma \dashv G) \wedge \cdots \wedge \mathsf{SLCH}_{\vec{S_n}}(\Gamma \dashv G)$*, with* $\mathsf{SLCH}_{\vec{S_i}}(\Gamma \dashv G)$ *being an* explanans *for any* $1 \leq i \leq n$*;*
(ii) $\mathsf{AT}(C) \not\subseteq \mathsf{AT}(\Gamma \dashv G)$ *and* $\overline{\overline{\mathsf{G4}}}$ *refutes* $\Gamma, C \vdash G$.

Example 5.4 *Take the problem* $p \to q, ⓠ \vdash p \to r$ *of Example 5.1:* $((q \wedge p) \vee s) \to (r \wedge t)$ *is a non-analytic* explanans*, and it is logically equivalent to* $((q \wedge p) \to r) \wedge ((q \wedge p) \to t) \wedge (s \to r) \wedge (s \to t)$*, with* $(q \wedge p) \to r$ *being an* $\mathsf{SLCH}_{\vec{S}}$-explanans. *Then, the sequent* $p \to q, ((q \wedge p) \to t) \wedge (s \to r) \wedge (s \to t) \vdash p \to r$ *being refutable and* $\mathsf{AT}\big(((q \wedge p) \to t) \wedge (s \to r) \wedge (s \to t)\big) \not\subseteq \mathsf{AT}\big(p \to q, ⓠ \vdash p \to r\big)$.

Theorem 5.2 states that for any problem of the form $\Gamma, ⓘ \vdash G$ and any non-analytic *explanans* A there exists a "derived" problem of the form $\Gamma, C, ⓘ \vdash G$, which possibly makes all new information in A explicit in the theoretical background. Corollary 5.2 further refines this result for a specific class of non-analytic *explanantes*. It is easy to show that any $\mathsf{SLCH}_{\vec{S}}$-*explanans* for $\Gamma, ⓘ \vdash G$ is also an $\mathsf{SLCH}_{\vec{S}}$-*explanans* for $\Gamma, C, ⓘ \vdash G$ (as per Proposition 5.1). Therefore, we can conclude that the deductive saturation of a problem $\Gamma, ⓘ \vdash G$ through a non-analytic *explanans* A can always be understood as the deductive saturation of a (possibly) distinct problem of the form $\Gamma, C, ⓘ \vdash G$ through an analytic *explanans*. This implies that the set of analytic abductive solutions enjoys a certain "completeness": in the end, deductive saturation can always be performed via analytic *explanantes* including $\mathsf{SLCH}_{\vec{S}}$-*explanantes*.

Let us end this section by proposing the following conjecture: for any problem of the form $\Gamma, ⓘ \vdash G$ and any *explanans* A, A is *candidate for the best explanans* only if A is logically equivalent to $B \wedge C$, where C is such that $\overline{\overline{\mathsf{G4}}}$ refutes $\Gamma, C \vdash G$ and B is a conjunction of EH-hypotheses for the problem $\Gamma, C, ⓘ \vdash G$ (as described in Procedure 4.2).

6 Conclusion

In this work, we presented a proof-theoretic framework to analyze abductive reasoning in classical propositional logic by reading abduction as an *enthymematic* deductive argument in reverse. We assumed the minimal set of logical conditions A1-A3 for abductive explanations, though we acknowledge that the literature suggests additional conditions ([13, 33]) that could be explored in combination with the ones we focussed on in these pages. We also highlighted certain discrepancies between the deductively minimal solution and the expected solution. This led us to design an effective procedure (Procedure 4.2) which recovers what seems to better approximate the reasoner's expectations by pruning the leaves of the deduction-tree from the redundant information.

It should be noticed that, when presented in a standard natural deduction calculus, achieving deductive saturation through an expected hypothesis often requires fewer steps than achieving it through the minimal hypothesis. This suggests that a better understanding of the notion of expected explanation could be gained by aiming for minimality in terms of derivation length. As shown in Fig. 4, consider the abductive problem $p \rightarrow q, ⓘ \vdash p \rightarrow r$. It can be observed that inserting the expected hypothesis $q \rightarrow r$ results in a simpler derivation compared to assuming the deductively minimal formula $(p \wedge q) \rightarrow r$. However, such a characterization is inherently arbitrary because the complexity of a derivation depends on the specific formalism used as a measuring device.

We believe it would be valuable to broaden the application of our proof-theoretic framework to include conservative extensions of classical propositional logic, such as modal logics ([19, 20]), supraclassical logics ([16, 24]), non-monotonic logics ([2, 9, 26, 30]), and a logic for exception and typicality ([26]). Moreover, a proof-theoretic setting that unifies aspects of default reasoning and abductive reasoning could provide

Fig. 4 LCH and EH from the natural deduction point of view

$$\dfrac{\dfrac{\dfrac{[p]^1 \qquad \dfrac{p \to q \qquad [p]^1}{q}\,{\to}\mathcal{E}}{p \wedge q}\,\wedge\mathcal{I} \qquad (p \wedge q) \to r}{r}\,{\to}\mathcal{E}}{p \to r}\,{\to}\mathcal{I}\;(1)$$

$$\dfrac{\dfrac{\dfrac{p \to q \qquad [p]^1}{q}\,{\to}\mathcal{E} \qquad q \to r}{r}\,{\to}\mathcal{E}}{p \to r}\,{\to}\mathcal{I}\;(1)$$

fresh insight into the relationship between the two ([11, 27, 34]). Additionally, it appears that modifications of this framework could work for other non-classical logics. Moreover, the refutation-based approach presented in our work can, in theory, be extended to decidable fragments of predicate logic, with monadic first-order logic presenting an interesting case study, particularly in relation to the traditional topic of *inventio medii* (see e.g. [17]). A broader perspective could involve taming full first-order logic by utilizing an appropriate notion of *approximated* refutation and *approximated* deductive saturation.

Procedure 4.2 provides a proof-theoretic account of the process whereby a rational agent produces an optimal *analytic* solution for a given abductive problem. However, there has been an increasing emphasis among philosophers of science on cases of *creative* abduction, that is situations in which the reasoner formulates abductive hypotheses by incorporating pieces of information not deducible from the original problem [31]. By its very nature, deductive logic cannot anticipate the specific non-analytic information that a rational agent will utilize to solve the abductive problem. Nonetheless, the technical results presented in Section 5 offer a comprehensive approach to effectively distinguish analytic components from non-analytic ones within any non-analytical solution. This methodical treatment of non-analytic solutions seems to suggest that *supraclassical* analytic calculi may offer the appropriate proof-theoretic framework for tackling the challenge of creative abduction (cf. [24]).

Finally, it is widely accepted that the best *explanans* should be chosen based on its higher degree of truthlikeness or verisimilitude ([10], p. 48; [5, 21]). It would be interesting to examine our approach for identifying candidates for the best *explanans* in relation to the definitions of truthlikeness proposed in the literature ([22]), and explore the possibility of using a fractional approach ([25]) to further refine our method.

Legend of the Symbols

$\mathsf{top}(\Gamma \overset{*}{\vDash} \Delta)$ = set of clauses obtained after decomposing $\Gamma \overset{*}{\vDash} \Delta$
$\mathsf{top}_{\vdash}(\Gamma \overset{*}{\vDash} \Delta)$ = set of identity clauses obtained after decomposing $\Gamma \overset{*}{\vDash} \Delta$
$\mathsf{top}_{\dashv}(\Gamma \overset{*}{\vDash} \Delta)$ = set of complementary clauses obtained after decomposing $\Gamma \overset{*}{\vDash} \Delta$
$\mathsf{AT}(\Gamma \overset{*}{\vDash} \Delta)$ = set of all atoms occurring in the clauses of $\mathsf{top}(\Gamma \overset{*}{\vDash} \Delta)$
$\mathsf{ID}(\Gamma \overset{*}{\vDash} \Delta)$ = set of all identity atoms occurring in the clauses of $\mathsf{top}(\overset{*}{\vDash} A)$
$\mathsf{CUT}(\Gamma \overset{*}{\vDash} \Delta)$ = set of all cut atoms occurring in the clauses of $\mathsf{top}(\overset{*}{\vDash} A)$

$\mathsf{AT}(A)$ = set of all atomic subformulas of A
$cnf(A)$ = conjunction of formula translations of clauses in $\mathsf{top}(\models^* A)$
$\Gamma, ⓘ \vdash G$ = abductive problem with G as *explanandum* and Γ as theoretical background
$\mathsf{LCH}(\Gamma, ⓘ \vdash G)$ = the least compromising hypothesis for $\Gamma, ⓘ \vdash G$
$\mathsf{EH}_S(\Gamma, ⓘ \vdash G)$ = the S-approximation to an expected hypothesis for $\Gamma, ⓘ \vdash G$
$\mathsf{EH}(\Gamma, ⓘ \vdash G)$ = an expected hypothesis for $\Gamma, ⓘ \vdash G$
$\mathsf{SLCH}_{\vec{S}}(\Gamma, ⓘ \vdash G)$ = the $\vec{S}$-strengthened least compromising hypothesis for $\Gamma, ⓘ \vdash G$

Author Contributions Gabriele Pulcini conceived of some core ideas. All the three authors developed the framework, discussing the results and contributing to the final manuscript at each step of the writing process.

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Declarations

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