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On zeros of an entire function coinciding with exponential typequasi-polynomials, associated with a regular third-order differential operator on an interval

In this paper, we consider the question on study of zeros of an entire function of one class, which coincides with quasi-polynomials of exponential type. Eigenvalue problems for some classes of differential operators on a segment are reduced to a similar problem. In particular, the studied problem is led by the eigenvalue problem for a linear differential equation of the third order with regular boundary value conditions in the space $W_2^3(0,1)$. The studied entire function is adequately characteristic determinant of the spectral problem for a third-order linear differential operator with periodic boundary value conditions. An algorithm to construct a conjugate indicator diagram of an entire function of one class is indicated, which coincides with exponential type quasi-polynomials with comparable exponents according to the monograph by A.F. Leontyev. Existence of a countable number of zeros of the studied entire function in each series is proved, which are simultaneously eigenvalues of the above-mentioned third-order differential operator with regular boundary value conditions. We determine distance between adjacent zeros of each series, which lies on the rays perpendicular to sides of the conjugate indicator diagram, that is a regular hexagon on the complex plane. In this case, zero is not an eigenvalue of the considered operator, that is, zero is a regular point of the operator. Fundamental difference of this work is finding the corresponding eigenfunctions of the operator. System of eigenfunctions of the operator corresponding in each series is found. Adjoint operator is constructed.

Keywords: entire function, zeros, quasi-polynomials, indicator diagram, series, operator, regular periodic boundary value conditions, eigenvalues, system of eigenfunctions.

Introduction and Formulation of the problem

We consider the question on distribution of zeros of an entire function of the following form:

$$\Delta(\lambda) = \sqrt[3]{\lambda}((k_2-k_3)e^{k_1\sqrt[3]{\lambda}} + (k_1-k_2)e^{(k_2+k_1)\sqrt[3]{\lambda}} +$$

$$+(k_3-k_1)e^{k_2\sqrt[3]{\lambda}} + (k_3-k_1)e^{(k_3+k_1)\sqrt[3]{\lambda}} + (k_1-k_2)e^{k_3\sqrt[3]{\lambda}} + (k_2-k_3)e^{(k_2+k_3)\sqrt[3]{\lambda}}),$$

where $k_1 = 1$, $k_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $k_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

Eigenvalue problems for some classes of differential operators on a segment are reduced to a similar problem. In particular, the following problem on eigenvalues in the space $W_2^3(0,1)$ leads to the studied question:

$$L_0 u \equiv l(u) = u'''(x) = -\lambda u(x),\ 0 < x < 1, \tag{1}$$

$$U_1(u) = u(0) = 0,\ U_2(u) = u(1) = 0,\ U_3(u) = u'(0) = u'(1), \tag{2}$$

where $U_1(u)$, $U_2(u)$, $U_3(u)$ are linear forms, which are regular, according to J.D. Birkhoff [1, 2]. An important result established by Birkhoff was to estimate resolvent of a regular differential operator and to establish asymptotics of the spectrum. In the monograph by M.A. Naimark [3; 67], a subclass of regular boundary conditions, so-called strongly regular boundary conditions, was singled out, where it was noted that for an odd order of the equation all regular conditions are strongly regular.

Connection between zeros of quasi-polynomials and spectral problems was reflected in [3–15]. Zeros of entire functions having an integral representation were studied in [16–23].

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Main Results

We consider the question on distribution of zeros of the entire function $\Delta_1(\lambda) = \frac{\Delta(\lambda)}{\sqrt[3]{\lambda}}$ on the complex plane λ.

$$\Delta_1(\lambda) = (k_2 - k_3)e^{k_1\sqrt[3]{\lambda}} + (k_1 - k_2)e^{(k_2+k_1)\sqrt[3]{\lambda}} +$$
$$+(k_3 - k_1)e^{k_2\sqrt[3]{\lambda}} + (k_3 - k_1)e^{(k_3+k_1)\sqrt[3]{\lambda}} + (k_1 - k_2)e^{k_3\sqrt[3]{\lambda}} + (k_2 - k_3)e^{(k_2+k_3)\sqrt[3]{\lambda}} = 0. \quad (3)$$

In [11, 14] the following was proved:

Proposition 1.

1. There are infinitely many zeros of an entire function $\Delta_1(\lambda)$;

2. Distance between two adjacent zeros of the same series ($j - const$) is exactly $\frac{2\pi}{|d|}$;

3. Zeros of each series lie on the rays perpendicular to the segment, that is, perpendicular to sides of the hexagon containing

$$(\overline{k_1}, \overline{(k_3 + k_1)}); (\overline{k_3}, \overline{k_3 + k_1}); (\overline{k_2 + k_3}; \overline{k_3}); (\overline{k_2}, \overline{k_2 + k_3}); (\overline{k_2}, \overline{k_2 + k_1}); (\overline{k_2 + k_1}, \overline{k_1}).$$

The rays which are perpendicular to the indicator diagram are called critical. According to the result of the monograph [6], there are exactly six critical rays on the plane λ, that is $arg\sqrt[3]{\lambda} = \frac{\pi}{6} + \frac{\pi n}{3}$, $n = 0, 1, 2, 4, 5$;

In [11, 14] the zeros of the entire function $\Delta(\lambda)$:

$$\lambda_{jk} = \frac{(ln|z_j| + i(Arg(z_j) + 2\pi k))^3}{d^3}, \quad k = 0, \pm1, \pm2, ...; \; j = \overline{1, m} \quad (4)$$

were found, and conjugate indicator diagram-hexagon was constructed on the complex plane λ.

Taking $k_1 = 1, k_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, k_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, into account, due to the formula (4) and Proposition 1, we have that along the ray perpendicular to the segment passing through the points $\overline{2}; \overline{1 - i\sqrt{3}}$ there are zeros of the quasi-polynomial $(\sqrt{3} + 3i) \cdot e^{(1+i\sqrt{3})\lambda} - 2\sqrt{3} \cdot e^{2\lambda}$, they are majorizing exponents. In this case, other exponents from (3) do not contribute along this ray. Let's find zeros of the quasi-polynomial:

$$(\sqrt{3} + 3i) \cdot e^{(1+i\sqrt{3})\lambda} - 2\sqrt{3} \cdot e^{2\lambda} = 0$$

$$(\sqrt{3} + 3i) \cdot e^{(1+i\sqrt{3})\lambda} = 2\sqrt{3} \cdot e^{2\lambda}$$

$$\lambda_{k1} = \frac{2ik\pi}{-1 + i\sqrt{3}} + \frac{\ln|\frac{2\sqrt{3}}{\sqrt{3}+3i}| + iarg(\frac{2\sqrt{3}}{\sqrt{3}+3i})}{-3 + i\sqrt{3}}, \quad k = 1, 2, 3, ...$$

Which are zeroes of the first series, where $\ln|\frac{2\sqrt{3}}{\sqrt{3}+3i}| + iArg(\frac{2\sqrt{3}}{\sqrt{3}+3i}) = const$. Similar procedure is performed on the other sides of the hexagon, and along other perpendicular rays we have the corresponding series of zeros of the quasi-polynomials from (3):

- segment $[\overline{-1 - i\sqrt{3}}; \overline{1 - i\sqrt{3}}]$, 2-nd series of zeroes $\lambda_{k2} = \frac{ik\pi}{1+i\sqrt{3}} + \frac{const}{2(1+i\sqrt{3})}, \quad k = 1, 2, ..., (1 + i\sqrt{3})$
- segment $[\overline{-1 + i\sqrt{3}}; \overline{1 + i\sqrt{3}}]$, 3-rd series of zeroes $\lambda_{k3} = ik\pi + const, \quad k = 1, 2, ...,$
- segment $[\overline{-2}; \overline{-1 - i\sqrt{3}}]$, 4-th series of zeroes $\lambda_{k4} = \frac{2ik\pi}{1+i\sqrt{3}} + \frac{const}{1+i\sqrt{3}}, \quad k = 1, 2, ...,$
- segment $[\overline{2}; \overline{1 + i\sqrt{3}}]$, 5-th series of zeroes $\lambda_{k5} = -\frac{2ik\pi}{1+i\sqrt{3}} - \frac{const}{1+i\sqrt{3}}, \quad k = 1, 2, ...,$
- segment $[\overline{-1 + i\sqrt{3}}; \overline{-2}]$, 6-th series of zeroes $\lambda_{k6} = \frac{2ik\pi}{1+i\sqrt{3}} + \frac{const}{2(1+i\sqrt{3})}, \quad k = 1, 2, ...,$

The zeros that were found are adequately eigenvalues of the operator L_0 [11].

Fundamental difference of this section from [11, 14, 22, 23] is the determination of eigenfunctions of the operator L_0. The following theorem takes place.

Theorem. Let the entire function $\Delta_1(\lambda) = \frac{\Delta(\lambda)}{\sqrt[3]{\lambda}}$ in (3), according to [11, 14], be a characteristic polynomial of the spectral problem (1), (2) and all points of Proposition 1. be satisfied, as well as zeros of the characteristic polynomial (4) be the corresponding eigenvalues of the operator L_0. Then the system of eigenfunctions of the operator L_0 of each series:

$$u_{k1}(x) = C_1 e^{-2\frac{k\sqrt{3}}{2}x} \cdot e^{2\frac{\pi}{4\sqrt{3}}x} \left[\cos 2\left(\frac{k\pi}{2} - \frac{\pi}{6}\right)x - i\sin 2\left(\frac{k\pi}{2} - \frac{\pi}{6}\right)x\right] + \left\{C_2\left[\cos\left(\frac{k\pi}{2} - \right.\right.\right.$$

$$\left. - \frac{\pi}{6}\right)x \cdot \cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2} + \frac{\pi}{4\sqrt{3}}\right)x \cdot \mathrm{ch}\,\sqrt{3}\left(\frac{k\pi}{2} - \frac{\pi}{6}\right)x - \sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2} + \frac{\pi}{4\sqrt{3}}\right)x\cdot$$

$$\cdot\,\mathrm{sh}\,\sqrt{3}\left(\frac{k\pi}{2} - \frac{\pi}{6}\right)x \cdot \sin\left(\frac{k\pi}{2} - \frac{\pi}{6}\right)x - i\left(\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2} + \frac{\pi}{4\sqrt{3}}\right)x \cdot \mathrm{ch}\,\sqrt{3}\left(\frac{k\pi}{2} - \right.\right.$$

$$\left. - \frac{\pi}{6}\right)x \sin\left(\frac{k\pi}{2} - \frac{\pi}{6}\right)x + \cos\left(\frac{k\pi}{2} - \frac{\pi}{6}\right)x \cdot \sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2} + \frac{\pi}{4\sqrt{3}}\right)x \cdot \mathrm{sh}\,\sqrt{3}\left(\frac{k\pi}{2} - \right.$$

$$\left.\left.\left. - \frac{\pi}{6}\right)x\right)\right] + C_3\left[\cos\left(\frac{k\pi}{2} - \frac{\pi}{6}\right)x \cdot \sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2} + \frac{\pi}{4\sqrt{3}}\right)x \cdot \mathrm{ch}\,\sqrt{3}\left(\frac{k\pi}{2} - \frac{\pi}{6}\right)x - \right.$$

$$- \sin\left(\frac{k\pi}{2} - \frac{\pi}{6}\right)x \cdot \cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2} + \frac{\pi}{4\sqrt{3}}\right)x \cdot \mathrm{sh}\,\sqrt{3}\left(\frac{k\pi}{2} - \frac{\pi}{6}\right)x - i\left(\sin\left(\frac{k\pi}{2} - \frac{\pi}{6}\right)x\cdot\right.$$

$$\cdot\,\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2} + \frac{\pi}{4\sqrt{3}}\right)x \cdot \mathrm{ch}\,\sqrt{3}\left(\frac{k\pi}{2} - \frac{\pi}{6}\right)x + \cos\left(\frac{k\pi}{2} - \frac{\pi}{6}\right)x \cdot \cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2} + \right.$$

$$\left.\left.\left.\left. + \frac{\pi}{4\sqrt{3}}\right)x \cdot \mathrm{sh}\,\sqrt{3}\left(\frac{k\pi}{2} - \frac{\pi}{6}\right)x\right)\right]\right\} e^{-\frac{k\sqrt{3}}{2}x} \cdot e^{\frac{\pi}{4\sqrt{3}}x};$$

$$u_{k2}(x) = C_1 e^{k\sqrt{3}x} \cdot e^{-\frac{\pi}{4}x}\left[\cos\left(\frac{k\pi}{2} - \frac{\pi}{12}\right)x + i\sin\left(\frac{k\pi}{2} - \frac{\pi}{12}\right)x\right] + \left\{C_2\left[\cos\left(\frac{k\pi}{4} - \right.\right.\right.$$

$$\left. - \frac{\pi}{24}\right)x \cdot \cos\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2} - \frac{\pi}{8}x\right) \cdot \mathrm{ch}\,\sqrt{3}\left(\frac{k\pi}{4} - \frac{\pi}{24}\right)x + \sin\left(\frac{k\pi}{4} - \frac{\pi}{24}\right)x \cdot \sin\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2} - \right.$$

$$\left. - \frac{\pi}{8}\right)x \cdot \mathrm{sh}\,\sqrt{3}\left(\frac{k\pi}{4} - \frac{\pi}{24}\right)x - i\left(\cos\left(\frac{k\pi}{4} - \frac{\pi}{24}\right)x \cdot \sin\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2} - \frac{\pi}{8}\right)x \cdot \mathrm{sh}\,\sqrt{3}\left(\frac{k\pi}{4} - \right.\right.$$

$$\left.\left.\left. - \frac{\pi}{24}\right)x - \sin\left(\frac{k\pi}{4} - \frac{\pi}{24}\right)x \cdot \cos\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2} - \frac{\pi}{8}\right)x \cdot \mathrm{ch}\,\sqrt{3}\left(\frac{k\pi}{4} - \frac{\pi}{24}\right)x\right)\right] +$$

$$+C_3\left[\cos\left(\frac{k\pi}{4} - \frac{\pi}{24}\right)x \cdot \sin\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2} - \frac{\pi}{8}\right) \cdot \mathrm{ch}\,\sqrt{3}\left(\frac{k\pi}{4} - \frac{\pi}{24}\right)x + \sin\left(\frac{k\pi}{4} - \right.\right.$$

$$\left. - \frac{\pi}{24}\right)x\cos\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2} - \frac{\pi}{8}\right)x \cdot \mathrm{sh}\,\sqrt{3}\left(\frac{k\pi}{4} - \frac{\pi}{24}\right)x - i\left(\cos\left(\frac{k\pi}{4} - \frac{\pi}{24}\right)x\cos\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2} - \right.\right.$$

$$\left. - \frac{\pi}{8}\right)x \cdot \mathrm{sh}\,\sqrt{3}\left(\frac{k\pi}{4} - \frac{\pi}{24}\right)x + \sin\left(\frac{k\pi}{4} - \frac{\pi}{24}\right)x \cdot \sin\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2} - \frac{\pi}{8}\right) \cdot \mathrm{ch}\,\sqrt{3}\left(\frac{k\pi}{4} - \right.$$

$$\left.\left.\left.\left. - \frac{\pi}{24}\right)x\right)\right]\right\} e^{\frac{k\pi\sqrt{3}}{2}x} \cdot e^{-\frac{\pi}{8}x};$$

$$u_{k3}(x) = C_1\left(\cos 2\left(k\pi - \frac{\pi}{3}\right)x + i\sin 2\left(k\pi - \frac{\pi}{3}\right)x\right) + \left[C_2\,\mathrm{ch}\,\sqrt{3}\left(k\pi - \frac{\pi}{3}\right)x\cdot\right.$$

$$\cdot\cos\left(k\pi - \frac{\pi}{3}\right)x + C_3\,\mathrm{sh}\,\sqrt{3}\left(k\pi - \frac{\pi}{3}\right)x \cdot \sin\left(k\pi - \frac{\pi}{3}\right)x + i\left(C_3\,\mathrm{sh}\,\sqrt{3}\left(k\pi - \frac{\pi}{3}\right)x\cdot\right.$$

$$\cdot\ \cos\left(k\pi-\frac{\pi}{3}\right)x-C_2\,\mathrm{ch}\,\sqrt{3}\left(k\pi-\frac{\pi}{3}\right)x\cdot\sin\left(k\pi-\frac{\pi}{3}\right)x\Big)\Big];$$

$$u_{k4}\left(x\right)=C_1e^{2\frac{k\pi\sqrt{3}}{2}x}\cdot e^{-2\frac{\pi}{12}x}\left[\cos 2\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x+i\sin 2\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\right]+\Bigg\{C_2\Bigg[\cos\left(\frac{k\pi}{2}+\right.$$

$$\left.+\frac{\pi\sqrt{3}}{12}\right)x\cdot\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{ch}\,\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x-\sin\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\cdot$$

$$\cdot\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{sh}\,\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x-i\left(\sin\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\cdot\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\right.\right.$$

$$\left.+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{ch}\,\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x+\cos\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\cdot\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot$$

$$\cdot\ \mathrm{sh}\,\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\Big)\Big]+C_3\Bigg[\cos\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\cdot\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{ch}\,\sqrt{3}\left(\frac{k\pi}{2}-\right.$$

$$\left.-\frac{\pi}{69}\right)x-\sin\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\cdot\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{sh}\,\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x-i\left(\sin\left(\frac{k\pi}{2}-\right.\right.$$

$$\left.-\frac{\pi}{6}\right)x\cdot\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{ch}\,\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x+\cos\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\cdot\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\right.$$

$$\left.+\ \frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{sh}\,\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{6}\right)x\Big)\Big]\Big\}e^{-\frac{k\pi\sqrt{3}}{2}x}\cdot e^{\frac{\pi}{12}x};$$

$$u_{k5}\left(x\right)=C_1e^{-2\frac{k\pi\sqrt{3}}{2}x}\cdot e^{2\frac{\pi}{12}x}\left[\cos 2\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x+i\sin 2\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\right]+$$

$$+\Bigg\{C_2\Bigg[\cos\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\cdot\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{ch}\,\sqrt{3}\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x+$$

$$+\sin\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{sh}\,\sqrt{3}\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x+i\left(\sin\left(\frac{k\pi}{2}+\right.\right.$$

$$\left.+\frac{\pi\sqrt{3}}{12}\right)x\cdot\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{ch}\,\sqrt{3}\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x-$$

$$-\ \cos\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\Big)\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{sh}\,\sqrt{3}\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\Big]-C_3\Bigg[\cos\left(\frac{k\pi}{2}+\right.$$

$$\left.+\frac{\pi\sqrt{3}}{12}\right)x\cdot\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{ch}\,\sqrt{3}\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x+\sin\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\cdot$$

$$\cdot\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{sh}\,\sqrt{3}\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x+i\left(\sin\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\sin\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\right.\right.$$

$$\left.+\frac{\pi}{4\sqrt{3}}\right)x\cdot\mathrm{ch}\,\sqrt{3}\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x-\cos\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\cdot\cos\sqrt{3}\left(-\frac{k\sqrt{3}}{2}+\frac{\pi}{4\sqrt{3}}\right)x\cdot$$

$$\cdot \ \mathrm{sh}\sqrt{3}\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\Bigg)\Bigg]\Bigg]\Bigg\} e^{-\frac{k\pi\sqrt{3}}{2}x}\cdot e^{-\frac{\pi}{12}x};$$

$$u_{k6}\left(x\right)=C_1 e^{-2\frac{k\pi\sqrt{3}}{2}x}\cdot e^{2\frac{\pi}{12}x}\left[\cos 2\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x+i\sin 2\left(\frac{k\pi}{2}+\frac{\pi\sqrt{3}}{12}\right)x\right]+$$

$$+\Bigg\{C_2\Bigg[\cos\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x\cdot\cos\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{12}\right)x\cdot\mathrm{ch}\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x+$$

$$+\sin\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x\sin\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{12}\right)x\cdot\mathrm{sh}\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x+i\Bigg(\sin\Bigg(\frac{k\pi\sqrt{3}}{2}-$$

$$-\frac{\pi\sqrt{3}}{12}\Bigg)x\cdot\cos\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{12}\right)x\cdot\mathrm{ch}\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x-$$

$$-\cos\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x\sin\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{12}\right)x\cdot\mathrm{sh}\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x\Bigg)\Bigg]+C_3\Bigg[\cos\Bigg(\frac{k\pi\sqrt{3}}{2}-$$

$$-\frac{\pi\sqrt{3}}{12}\Bigg)x\cdot\sin\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{12}\right)x\cdot\mathrm{ch}\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x+\sin\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x\cdot$$

$$\cdot\cos\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{12}\right)x\cdot\mathrm{sh}\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x+i\Bigg(\sin\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x\sin\sqrt{3}\Bigg(\frac{k\pi}{2}-$$

$$-\frac{\pi}{12}\Bigg)x\cdot\mathrm{ch}\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x-\cos\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x\cdot\cos\sqrt{3}\left(\frac{k\pi}{2}-\frac{\pi}{12}\right)x\cdot$$

$$\cdot\ \mathrm{sh}\sqrt{3}\left(\frac{k\pi\sqrt{3}}{2}-\frac{\pi\sqrt{3}}{12}\right)x\Bigg)\Bigg]\Bigg\} e^{-\frac{k\pi\sqrt{3}}{2}x},$$

for each series, where $k = 1, 2, 3, \ldots$.

Here is the scheme of the proof:

General solution of the equation (1) has the form:

$$u(x) = C_1 e^{2\lambda x} + (C_2 cos\sqrt{3}\lambda x + C_3 sin\sqrt{3}\lambda x)e^{-\lambda x}. \quad (5)$$

Substituting the zeros of each series into (5) in order and satisfying the equation (1), as well as the boundary value conditions (2), we obtain the corresponding eigenfunctions of the operator L_0.

Remark. Questions of completeness, uniform minimality, and basis property of systems of eigenfunctions of the operator L_0 remain open. Note that questions of the basis property of systems of root vectors of the multiple differentiation operator with regular, but not strongly regular boundary value conditions were studied in [24–30].

Conjugate problem

By using integration by parts, we obtain the Lagrange formula: $\int\limits_0^1 l(u)\overline{v(x)}dx + \int\limits_0^1 u(x)\overline{l^*(v)}dx = u''(1)\overline{v(1)} - -u''(0)\overline{v(0)} - [\overline{v'(0)} - \overline{v'(1)}]\cdot u'(0) + u(1)\overline{v''(1)} - u(0)\overline{v''(0)}$.

Here $l^*(v)$ is an adjoint differential expression:

$$l^*(v) = -v'''(x), \quad 0 < x < 1. \quad (6)$$

Consequently, an operator L_0^*, adjoint to the operator L_0 is given by the differential expression (6) and boundary value conditions:

$$V_1(v) \equiv v(1) = 0, V_2(v) \equiv v(0) = 0, V_3(v) = v'(0) - v'(1) = 0.$$

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Кесіндідегі үшінші ретті регулярлы дифференциалдық оператормен байланысқан, экспоненциалды типтегі квазикөпмүшеліктермен сәйкес келетін бүтін функцияның нөлдері жайлы

Мақалада көрсеткіштері өлшемді экспоненциалды типтегі квазикөпмүшеліктермен сәйкес келетін бір кластағы бүтін функциялардың нөлдерін зерттеу мәселесі қарастырылды. Мұндағы қарастырылатын мәселе, көп жағдайларда, кейбір кластардағы кесіндідегі дифференциалдық операторлардың меншікті мәндерін зерттеуге берілген есептерден туындайды. Дәлірек айтқанда, қарастырылатын мәселеге $W_2^3(0,1)$ кеңістігіндегі регулярлы шеттік шарттармен берілген үшінші ретті сызықтық дифференциалдық теңдеудің меншікті мәндерін зерттеуге арналған есепке алып келеді. Зерттелетін бүтін функция, тікелей периодтық шеттік шарттармен берілген сызықтық дифференциалдық үшінші ретті оператор үшін аталған спетралдық есептің характеристикалық анықтауышы болып табылады. А.Ф. Леонтьевтің монографиясындағы нәтижесінің негізінде, қарастырылып отырған бір кластағы өлшемді көрсеткіштері бар экспоненциалды типтегі квазикөпмүшеліктермен сәйкес келетін бүтін функцияның түйіндес индикаторлық диаграммасын құрудың алгоритмі көрсетілген. Бүтін функцияның әрбір сериядағы саналымды нөлдерінің бар болуы дәлелденген және олардың кесіндідегі регулярлы периодтық шеттік шарттармен берілген сызықтық үшінші ретті дифференциалдық оператордың меншікті мәндері екендігі сипатталған. Бүтін функцияның әр сериядағы көршілес жатқан нөлдерінің арақашықтығы анықталған және әр серия комплексті жазықтықтағы түйіндес индикаторлық диаграмманың, яғни дұрыс алтыбұрыштың қабырғаларына перпендикуляр, координаталар бас нүктесінен шығатын сәулелер болатындығы көрсетілген. Алайда, нөл нүктесі жоғарыда айтылған қарастырылатын оператордың меншікті мәні болмайтындығы, яғни нөл оператордың регулярлы нүктесі екендігі сипатталған. Бұл жұмыстағы алынған нәтиженің ерекшелігі, оператордың әр сериядағы меншікті мәндеріне сәйкес меншікті функциялар жүйесінің табылуында. Сондай-ақ, осы жұмыстың зерттеу нысанына айналып отырған оператордың түйіндес операторы құрылған.

Кілт сөздер: бүтін функцияның нөлдері, квазикөпмүшеліктер, индикаторлық диаграмма, серия, оператор, регулярлы периодтық шеттік шарттар, меншікті мәндер, меншікті функциялардың жүйесі.

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О нулях целой функции, совпадающей с квазиполиномами экспоненциального типа, связанной с регулярным дифференциальным оператором третьего порядка на отрезке

В статье рассмотрен вопрос распределения нулей целой функции одного класса, которые являются квазиполиномами экспоненциального типа. К подобной проблеме редуцированы задачи на собственные значения для некоторых классов дифференциальных операторов на отрезке. В частности, к изучаемому вопросу приводит задача на собственные значения линейного дифференциального уравнения третьего порядка с регулярными краевыми условиями в пространстве $W_2^3(0,1)$. Исследуемая целая функция адекватно является характеристическим определителем спектральной задачи для линейного дифференциального оператора третьего порядка с периодическими краевыми условиями. Построена сопряженная индикаторная диаграмма целой функции экспоненциального типа соизмеримыми показателями. Доказано существование счетного числа нулей исследуемой целой функции в каждой серии, которые являются одновременно собственными значениями рассматриваемого дифференциального оператора третьего порядка с периодическими краевыми условиями. Определено расстояние между соседними нулями каждой серии, лежащее на лучах, перпендикулярных сторонам сопряженной индикаторной диаграммы, то есть правильного шестиугольника на комплексной плоскости. При этом нуль не является собственным значением рассматриваемого оператора. Принципиальным отличием настоящей работы является нахождение соответствующих собственных функций рассматриваемого оператора. Построен сопряженный оператор.

Ключевые слова: целая функция, нули, квазиполиномы, индикаторная диаграмма, серия, оператор, регулярные периодические краевые условия, собственные значения, система собственных функций.

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