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Ranks, Spectra and Their Dynamics for Families of Constant Expansions of Theories

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Abstract. Constant or nonessential extensions of elementary theories provide a productive tool for the study and structural description of models of these theories, which is widely used in Model Theory and its applications, both for various stable and ordered theories, countable and uncountable theories, algebraic, geometric and relational structures and theories. Families of constants are used in Henkin's classical construction of model building for consistent families of formulas, for the classification of uncountable and countable models of complete theories, and for some dynamic possibilities of countable spectra of ordered Ehrenfeucht theories.

The paper describes the possibilities of ranks and degrees for families of constant extensions of theories. Rank links are established for families of theories with Cantor-Bendixson ranks for given theories. It is shown that the e-minimality of a family of constant expansions of the theory is equivalent to the existence and uniqueness of a nonprincipal type with a given number of variables. In particular, for strongly minimal theories this means that the non-principal 1-type is unique over an appropriate tuple. Relations between e-spectra of families of constant expansions of theories and ranks and degrees are established. A model-theoretic characterization of the existence of the least generating set is obtained.

It is also proved that any inessential finite expansion of an o-minimal Ehrenfeucht theory preserves the Ehrenfeucht property, and this is true for constant expansions of dense spherically ordered theories. For the expansions under consideration, the dynamics of the values of countable spectra is described.

Keywords: family of theories, rank, degree, constant expansion, Ehrenfeucht theory, ordered theory, spherical theory

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Научная статья

Ранги, спектры и их динамика для семейств константных обогащений теорий

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Аннотация. Описаны возможности рангов и степеней для семейств константных расширений теорий. Установлена связь рангов для семейств теорий с рангами Кантора – Бендиксона для данных теорий. Показано, что e-минимальность семейства константных обогащений теории равносильна существованию и единственности неглавного типа с данным числом переменных. В частности, для сильно минимальных теорий это означает единственность неглавного 1-типа над подходящим кортежем. Установлена связь e-спектров семейств константных обогащений теорий с рангами и степенями. Получена теоретико-модельная характеризация существования наименьшего порождающего множества.
Также доказано, что любое несущественное конечное обогащение о-минимальной эренфойхтовой теории сохраняет эренфойхтовость, и это верно для константных обогащений плотных сферически упорядоченных теорий. Для рассматриваемых обогащений описана динамика значений счетных спектров.

Ключевые слова: семейство теорий, ранг, степень, константное расширение, эренфойхтова теория, упорядоченная теория, сферическая теория

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1. Introduction

Constant, or nonessential expansions of elementary theories give a productive tool for the study and structural descriptions of models of given theories. It is broadly used in Model Theory and its applications, both for various stable [5; 21; 22] and ordered [36] theories, countable and uncountable theories [2; 3; 8; 12; 35], algebraic, geometric [20] and relational [4] structures and theories. Families of constants are used in the classical Henkin construction of models for consistent families of formulae [7], for the classification of uncountable [6; 22] and countable models of complete theories [29], and for some possibilities of dynamics for countable spectra of Ehrenfeucht ordered theories [17].

Recall that for a theory T and a cardinality λ the value $I(T, \lambda)$ of *spectrum function* is the number of pairwise non-isomorphic models of T in the cardinality λ. A theory T is *Ehrenfeucht* if T has finitely many but more than one countable models, i.e., $1 < I(T, \omega) < \omega$.

In the present paper, we continue to study families of theories [24–28; 30; 31] and their rank [10; 13–15; 18; 19; 32; 33] describing possibilities of ranks and degrees for families of finite constant expansions of theories, their links with Cantor-Bendixson rank and e-spectra (Section 1). We apply a general approach of constant expansions for ordered Ehrenfeucht theories showing that the Ehrenfeuchtness and o-minimality are preserved under finite constant expansions (Corollary 3.6), and describe the dynamics of countable spectra $I(T, \omega)$ for these theories T (Theorem 3.5). These results are modified for finite constant expansions of Ehrenfeucht spherically ordered theories showing the possibilities for dynamics of countable spectra (Theorem 3.9) and the preservation of Ehrenfeuchtness under these expansions (Corollary 3.10).

Throughout the paper we consider complete first-order theories T in predicate languages $\Sigma(T)$ and use both the terminology on combinations of theories, families of theories, and their ranks in [24–28; 30; 31; 33], and on ordered theories in [1; 9; 11; 16].

2. Expansions theories by tuples of constants and their families

Let T be a theory in a language L, $\overline{a}$ be a tuple of new constant symbols, of length $l(\overline{a}) = n$. We denote by $\mathcal{T}_{T,\overline{a}}$ the set of all expansions $T(\overline{a})$ of T by constants in $\overline{a}$.

Clearly, there is a one-to-one correspondence between $\mathcal{T}_{T,\overline{a}}$ and $S^n(T)$ preserving the topological space.

Notice that $\mathcal{T}_{T,\overline{a}}$ is E-closed since theories in $\mathcal{T}_{T,\overline{a}}$ preserve the theory T and have realizations of all types in $S^{l(\overline{a})}(T)$ so that accumulation points for realizations of types in $S^{l(\overline{a})}(T)$ are again realizations of types in $S^{l(\overline{a})}(T)$.

Clearly, if $T = \mathrm{Th}(\mathcal{M})$ for a finite model $\mathcal{M}$ then $\mathcal{T}_{T,\overline{a}}$ is finite. Moreover, $\mathcal{T}_{T,\overline{a}}$ is finite if T has finitely many $l(\overline{a})$-types, in particular, if T is ω-categorical. Conversely, if $\mathcal{T}_{T,\overline{a}}$ is finite then there are finitely many possibilities to substitute $\overline{a}$ as a realization of a type $p(\overline{x}) \in S^{l(\overline{a})}(T)$. Therefore, $S^{l(\overline{a})}(T)$ is finite.

Since $|\mathcal{T}_{T,\overline{a}}| \geq \omega$ means that $\mathcal{T}_{T,\overline{a}}$ contains an approximating subfamily by [31, Theorem 6.1], we obtain the following:

Proposition 2.1. *For any theory T and a tuple $\overline{a}$ the following conditions are equivalent:*

(1) *$\mathcal{T}_{T,\overline{a}}$ contains an approximating subfamily;*

(2) *$S^{l(\overline{a})}(T)$ is infinite.*

Ryll-Nardzewski Theorem and Proposition 2.1 immediately imply:

Corollary 2.2. *For any theory T the following conditions are equivalent:*

(1) *for some tuple $\overline{a}$, $\mathcal{T}_{T,\overline{a}}$ contains an approximating subfamily;*

(2) *$S(T)$ contains nonisolated types, i.e., T is not ω-categorical and does not have finite models.*

Since $|\mathcal{T}_{T,\overline{a}}| = |S^{l(\overline{a})}(T)|$ and $\mathrm{RS}(\mathcal{T}) = 0$ if and only if $\mathcal{T}$ is finite with $\mathrm{ds}(\mathcal{T}) = |\mathcal{T}|$, we have the following:

Corollary 2.3. *For any theory T and a tuple $\overline{a}$ with finite $S^{l(\overline{a})}(T)$, $\mathrm{RS}(\mathcal{T}_{T,\overline{a}}) = 0$ and $\mathrm{ds}(\mathcal{T}_{T,\overline{a}}) = |S^{l(\overline{a})}(T)|$.*

Following [23], for a type $p \in S^n(T)$, we denote by $\mathrm{CB}_n(p)$ the Cantor–Bendixson rank for the type p in the compact topological space on $S^n(T)$, $\mathrm{CB}_n(T) = \sup\{\mathrm{CB}_n(p) \mid p \in S^n(T)\}$.

Theorem 2.4. (1) *For any type $p \in S^n(T)$ and a tuple $\overline{a}$ with $l(\overline{a}) = n$, $\mathrm{CB}_n(p) = \mathrm{RS}_{\mathcal{T}_{T,\overline{a}}}(T \cup p(\overline{a}))$.*

(2) *For any tuple $\overline{a}$, $\mathrm{CB}_{l(\overline{a})}(T) = \mathrm{RS}(\mathcal{T}_{T,\overline{a}})$.*

(3) *For any tuple $\overline{a}$, if $\mathrm{CB}_{l(\overline{a})}(T) = \mathrm{RS}(\mathcal{T}_{T,\overline{a}})$ is an ordinal then $\mathrm{ds}(\mathcal{T}_{T,\overline{a}})$ equals CB-degree of $S^{l(\overline{a})}(T)$.*

Proof. (1) We show $\mathrm{CB}_n(p) = \mathrm{RS}_{\mathcal{T}_{T,\overline{a}}}(T \cup p(\overline{a}))$ by induction on ordinals. If p is isolated then both $\mathrm{CB}_n(p) = 0$ and $\mathrm{RS}_{\mathcal{T}_{T,\overline{a}}}(T \cup p(\overline{a})) = 0$ witnessed by the principal formula $\varphi(\overline{x}) \in p$: p is an isolated point in $S^n(T)$ by $\varphi(\overline{x})$ and $T \cup p(\overline{a})$ is an isolated point in $\mathcal{T}_{T,\overline{a}}$ by $\varphi(\overline{a})$. For limit ordinals α the equalities $\mathrm{CB}_n(p) = \alpha = \mathrm{RS}_{\mathcal{T}_{T,\overline{a}}}(T \cup p(\overline{a}))$ are followed immediately by induction. Now if we assume that $\mathrm{CB}_n(p) \geq \alpha+1$ then p is an accumulation point for some types $q \in Q$ with $\mathrm{CB}_n(q) \geq \alpha$. Having $\mathrm{RS}_{\mathcal{T}_{T,\overline{a}}}(T \cup q(\overline{a})) \geq \alpha$ by induction hypothesis and since $T \cup p(\overline{a})$ is an accumulation point for $\{T \cup q(\overline{a}) \mid q \in Q\}$, we obtain $\mathrm{RS}_{\mathcal{T}_{T,\overline{a}}}(T \cup p(\overline{a})) \geq \alpha + 1$. And vice versa, if $\mathrm{RS}_{\mathcal{T}_{T,\overline{a}}}(T \cup p(\overline{a})) \geq \alpha + 1$ then $T \cup p(\overline{a})$ is an accumulation point for some $T \cup q(\overline{a})$, $q \in Q$, with $\mathrm{RS}_{\mathcal{T}_{T,\overline{a}}}(T \cup q(\overline{a})) \geq \alpha$. Thus, p is an accumulation point Q, and having $\mathrm{CB}_n(q) \geq \alpha$ by induction hypothesis, we obtain $\mathrm{CB}_n(p) \geq \alpha + 1$.

(2) is immediately implied by (1) since both $\mathrm{CB}_{l(\overline{a})}(T) = \sup\{\mathrm{CB}_{l(\overline{a})}(p) \mid p \in S^{l(\overline{a})}(T)\}$ and $\mathrm{RS}(\mathcal{T}_{T,\overline{a}}) = \sup\{\mathrm{RS}_{\mathcal{T}_{T,\overline{a}}}(T \cup p(\overline{a})) \mid p \in S^{l(\overline{a})}(T)\}$.

(3) is again implied by (1) since it confirms that there are equally many limit points in $S^{l(\overline{a})}(T)$ and in $\mathcal{T}_{T,\overline{a}}$ of rank $\alpha = \mathrm{CB}_{l(\overline{a})}(T) = \mathrm{RS}(\mathcal{T}_{T,\overline{a}})$. □

Corollary 2.5. *For any theory T and a tuple $\overline{a}$ the following conditions are equivalent:*

(1) *$\mathcal{T}_{T,\overline{a}}$ is e-minimal;*

(2) *T has unique nonprincipal $l(\overline{a})$-type.*

Proof. If $\mathcal{T}_{T,\overline{a}}$ is e-minimal then it contains a unique accumulation point by [31, Theorem 7.3], with $\mathrm{CB}_{l(\overline{a})}(T) = \mathrm{RS}(\mathcal{T}_{T,\overline{a}}) = 1$ and $\mathrm{ds}(\mathcal{T}_{T,\overline{a}}) = 1$ in view of Theorem 2.4. Since $\mathrm{CB}_{l(\overline{a})}(T) = 1$ with CB-degree 1, there is a unique nonprincipal type in $S^{l(\overline{a})}(T)$. Conversely, having a unique accumulation point in $S^{l(\overline{a})}(T)$ we again apply Theorem 2.4 obtaining $\mathrm{RS}(\mathcal{T}_{T,\overline{a}}) = 1$ and $\mathrm{ds}(\mathcal{T}_{T,\overline{a}}) = 1$, i.e., $\mathcal{T}_{T,\overline{a}}$ is e-minimal. □

Since strongly minimal theories have at most one non-principal 1-type over finite sets, Corollary 2.5 immediately implies the following criterion for an expansion $T(\overline{a})$ of a strongly minimal theory T by a tuple $\overline{a}$:

Corollary 2.6. *For any strongly minimal theory T, a tuple $\overline{a}$ and an element b the following conditions are equivalent:*

(1) *$\mathcal{T}_{T(\overline{a}),b}$ is e-minimal;*

(2) *T has a nonprincipal 1-type over $\overline{a}$.*

Remark 2.7. For any strongly minimal theory T, a tuple $\overline{a}$ and an element b, either $\mathcal{T}_{T(\overline{a}),b}$ is finite or e-minimal. At the same time, clearly, these conditions do not characterize the strong minimality, since they do not guarantee that the sets of solutions for formulas $\varphi(x, \overline{a})$ are finite or cofinite. For that characterization it suffices to use the requirement that for any formula $\varphi(x, \overline{a})$ and sets A there are finitely many possibilities, with respect to $n \in \omega$, for $T \cup \{\varphi(b, \overline{a}) \mid b \in A\} \cup \{\text{“}|A| = n\text{”}\}$ or $T \cup \{\neg\varphi(b, \overline{a}) \mid b \in A\} \cup \{\text{“}|A| = n\text{”}\}$.

Theorem 2.8. (1) *If $\mathcal{T}_{T,\overline{a}}$ is finite then e-$\mathrm{Sp}(\mathcal{T}_{T,\overline{a}}) = 0$.*

(2) *If* $\mathcal{T}_{T,\overline{a}}$ *is infinite and has finitely many accumulation points then* $e\text{-Sp}(\mathcal{T}_{T,\overline{a}}) = \mathrm{ds}(\mathcal{T}_{T,\overline{a}})$.

(3) *If* $\mathcal{T}_{T,\overline{a}}$ *is infinite and has infinitely many accumulation points then* $e\text{-Sp}(\mathcal{T}_{T,\overline{a}}) \leq \min\{2^{|T|}, \mathrm{RS}(\mathcal{T}_{T,\overline{a}})\}$, *and* $e\text{-Sp}(\mathcal{T}_{T,\overline{a}}) = |\mathrm{RS}(\mathcal{T}_{T,\overline{a}})|$ *if* $\mathrm{RS}(\mathcal{T}_{T,\overline{a}})$ *is an ordinal. Moreover, if* T *is countable, then* $e\text{-Sp}(\mathcal{T}_{T,\overline{a}})=\min\{2^\omega,\mathrm{RS}(\mathcal{T}_{T,\overline{a}})\}$.

Proof. (1) If $\mathcal{T}_{T,\overline{a}}$ is finite then $\mathcal{T}_{T,\overline{a}}$ consists of finitely many isolated points which do not produce new theories. Hence $e\text{-Sp}(\mathcal{T}_{T,\overline{a}}) = 0$.

(2) Assuming that $\mathcal{T}_{T,\overline{a}}$ is infinite and has finitely many accumulation points, $\mathcal{T}_{T,\overline{a}}$ contains the least generating set $\mathcal{T}_0$ consisting of isolating points. Since by Theorem 2.4, $\mathrm{CB}_{l(\overline{a})}(T) = \mathrm{RS}(\mathcal{T}_{T,\overline{a}}) = 1$ and $\mathcal{T}_0$ produces $\mathrm{ds}(\mathcal{T}_{T,\overline{a}})$ new points, we have $e\text{-Sp}(\mathcal{T}_{T,\overline{a}}) = \mathrm{ds}(\mathcal{T}_{T,\overline{a}})$.

(3) Let $\mathcal{T}_{T,\overline{a}}$ be infinite and have infinitely many accumulation points. It implies that $\mathrm{CB}_{l(\overline{a})}(T) \geq 2$. If $\mathrm{CB}_{l(\overline{a})}(T)$ is an ordinal then by Theorem 2.4, $\mathcal{T}_{T,\overline{a}}$ contains the least generating set $\mathcal{T}_0$ consisting of isolating points. The set $\mathcal{T}_0$ generates $|\mathrm{CB}_{l(\overline{a})}(T)| < 2^{|T|}$ accumulation points. Since $\mathrm{CB}_{l(\overline{a})}(T) = \mathrm{RS}(\mathcal{T}_{T,\overline{a}})$, we obtain $e\text{-Sp}(\mathcal{T}_{T,\overline{a}}) = |\mathrm{RS}(\mathcal{T}_{T,\overline{a}})|$. If $\mathrm{CB}_{l(\overline{a})}(T) = \infty$ then the inequality $e\text{-Sp}(\mathcal{T}_{T,\overline{a}}) \leq \min\{2^{|T|}, \mathrm{RS}(\mathcal{T}_{T,\overline{a}})\}$ is obvious, since $\mathcal{T}_{T,\overline{a}}$ can not have more than $2^{|T|}$ accumulation points.

If T is countable then $\mathrm{CB}_{l(\overline{a})}(T) = \mathrm{RS}(\mathcal{T}_{T,\overline{a}})$ is either a countable ordinal, with countably many $l(\overline{a})$-types, of equal infinity, with continuum many $l(\overline{a})$-types. In the latter case, by [33, Theorem 4.5], $e\text{-Sp}(\mathcal{T}_{T,\overline{a}}) = 2^\omega$. □

The following example shows that the inequality in Theorem 2.8, (3) can be strict.

Example 2.9. Let λ be an infinite cardinality and T_λ be a theory of independent unary predicates P_i, $i \in \omega$, expanded by λ empty additional predicates. For any tuple $\overline{a}$ we have $\mathrm{CB}_{l(\overline{a})}(T_\lambda) = \mathrm{RS}(\mathcal{T}_{T_\lambda,\overline{a}}) = \infty$, $e\text{-Sp}(\mathcal{T}_{T_\lambda,\overline{a}}) = 2^\omega$, whereas $\min\{2^{|T|}, \mathrm{RS}(\mathcal{T}_{T,\overline{a}})\} = 2^\lambda$ can be unboundedly large.

At the same time, taking a theory T in a language with λ independent unary predicates we obtain, for a tuple $\overline{a}$, $\mathrm{CB}_{l(\overline{a})}(T) = \mathrm{RS}(\mathcal{T}_{T_\lambda,\overline{a}}) = \infty$ and $e\text{-Sp}(\mathcal{T}_{T_\lambda,\overline{a}}) = 2^{|T|}$.

Now we consider a countable theory T. Since for any $n \in \omega$ the Stone space $S^n(T)$ is either at most countable, with countable CB-rank of each element, or $|S^n(T)| = 2^\omega$ with CB-rank ∞, we deduce the following theorem.

Theorem 2.10. *For any countable theory* T *and a tuple* $\overline{a}$ *the following conditions are equivalent:*

(1) $|S^{l(\overline{a})}(T)| \leq \omega$;

(2) $|\mathcal{T}_{T,\overline{a}}| \leq \omega$;

(3) $\mathrm{RS}(\mathcal{T}_{T,\overline{a}})$ *is a* (*countable*) *ordinal;*

(4) $e\text{-Sp}(\mathcal{T}_{T,\overline{a}}) \leq \omega$.

Proof. (1) $\Leftrightarrow$ (2) follows in view of $|S^{l(\overline{a})}(T)| = |\mathcal{T}_{T,\overline{a}}|$. (2) $\Rightarrow$ (3) holds since if $|\mathrm{RS}(\mathcal{T}_{T,\overline{a}})| > \omega$ then there is a 2-tree of sentences producing $|\mathcal{T}_{T,\overline{a}}| = 2^\omega$. Assuming $|\mathrm{RS}(\mathcal{T}_{T,\overline{a}})| \leq \omega$ we obtain at most countably many accumulation points implying $|\mathcal{T}_{T,\overline{a}}| \leq \omega$ and thus we have (3) $\Rightarrow$ (2). Finally, (2) $\Rightarrow$ (4) is obvious, and (4) $\Rightarrow$ (3) is satisfied in view of [33, Theorem 4.5]. □

Theorem 2.10 immediately implies the following:

Corollary 2.11. *For any countable theory T the following conditions are equivalent:*

(1) *T is small, i.e., $|S(T)| = \omega$;*

(2) *for any tuple $\overline{a}$, $|\mathcal{T}_{T,\overline{a}}| \leq \omega$;*

(3) *for any tuple $\overline{a}$, $\mathrm{RS}(\mathcal{T}_{T,\overline{a}})$ is a (countable) ordinal;*

(4) *for any tuple $\overline{a}$, $e\text{-}\mathrm{Sp}(\mathcal{T}_{T,\overline{a}}) \leq \omega$.*

Recall the following theorem.

Theorem 2.12. [24]. *If $\mathcal{T}_0'$ is a generating set for an E-closed set $\mathcal{T}_0$ then the following conditions are equivalent:*

(1) *$\mathcal{T}_0'$ is the least generating set for $\mathcal{T}_0$;*

(2) *$\mathcal{T}_0'$ is a minimal generating set for $\mathcal{T}_0$;*

(3) *any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0')_\varphi$, i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0')_\varphi = \{T\}$;*

(4) *any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0)_\varphi$, i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0)_\varphi = \{T\}$.*

Applying Theorem 2.12 for the families $\mathcal{T}_{T,\overline{a}}$, which are always E-closed, we obtain the following:

Theorem 2.13. *For any countable theory T the following conditions are equivalent:*

(1) *for any tuple $\overline{a}$, $\mathcal{T}_{T,\overline{a}}$ has the least generating set;*

(2) *T has a prime model.*

Proof. If the families $\mathcal{T}_{T,\overline{a}}$ have the least generating sets then, by Theorem 2.12, each consistent formula $\varphi(\overline{x})$ is implied by a principal one. Therefore, T has a prime model. Conversely, having a prime model we obtain, for each $\overline{a}$, a generating subset $\mathcal{T} \subseteq \mathcal{T}_{T,\overline{a}}$ consisting of isolated points correspondent to principal formulas. Thus, by Theorem 2.12, $\mathcal{T}_{T,\overline{a}}$ has the least generating set. □

Remark 2.14. Since there are many countable theories with prime models and continuum many types (see, for instance [29, Chapter 7]), the items in Theorem 2.13 are not equivalent to the items in Corollary 2.11.

At the same time, arguments of Theorem 2.13 for a fixed $\overline{a}$ imply that $\mathcal{T}_{T,\overline{a}}$ has the least generating set if and only if each consistent T-formula $\varphi(\overline{x})$ with $l(\overline{x}) = l(\overline{a})$ is forced by some principal one.

3. Constant expansions of ordered theories and dynamics of their countable spectra

Definition. Let M is a weakly o-minimal structure, $A \subseteq M$, M is $|A|^+$-saturated, and $p, q \in S_1(A)$ are non-algebraic types. We say that p is not *weakly orthogonal* to q ($p \not\perp^w q$) if there are an L_A-formula $H(x, y)$, $\alpha \in p(M)$, and $\beta_1, \beta_2 \in q(M)$ such that $\beta_1 \in H(M, \alpha)$ and $\beta_2 \notin H(M, \alpha)$.

In other words, p is *weakly orthogonal* to q ($p \perp^w q$) if $p(x) \cup q(y)$ has a unique extension to a complete 2-type over A.

Observe that in the o-minimal case $p \not\perp^w q$ iff there exists an A-definable strictly monotonic bijection $f : p(M) \to q(M)$.

Lemma 3.1. [1] *Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$. Then the non-weak orthogonality relation $\not\perp^w$ is an equivalence relation on $S_1(A)$.*

Let A be an arbitrary subset of a linearly ordered structure M. We denote by A^+ (respectively, A^-) the set of elements $b \in M$ with $A < b$ ($b < A$).

Definition [1]. Let M be a weakly o-minimal structure, $A \subseteq M$, $p \in S_1(A)$ be non-algebraic. We say p is *quasirational-to-right (left)* if there is a convex L_A-formula $U_p(x) \in p$ such that for any sufficiently saturated model $N \succ M$, $U_p(N)^+ = p(N)^+$ ($U_p(N)^- = p(N)^-$). A non-isolated 1-type is called *quasirational* if it either quasirational-to-right or quasirational-to-left. A non-quasirational non-isolated 1-type is called *irrational*.

Obviously, an 1-type being simultaneously quasirational-to-right and quasirational-to-left is *isolated*. We say a quasirational-to-right (left) type p is said to be *rational-to-right (left)* if $U_p(x) = x < b$ ($U_p(x) = x > b$) for some $b \in \mathrm{dcl}(A) \cup \{\infty, -\infty\}$. Observe that in an o-minimal structure any quasirational 1-type is rational.

Proposition 3.2 [1]. *Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$, $p, q \in S_1(A)$ be non-algebraic, $p \not\perp^w q$. Then:*

(1) *p is irrational $\Leftrightarrow$ q is irrational;*

(2) *p is quasirational $\Leftrightarrow$ q is quasirational.*

Example 3.3. We consider the known Ehrenfeucht's example: $M = \langle \mathbb{Q}, <, c_k \rangle_{k \in \omega}$, where $c_k < c_{k+1}$ for each $k \in \omega$ and $\lim_{k \to \infty} c_k = \infty$. Let $p(x)$ be a type closed under deducibility and isolated by the set $\{c_k < x \mid k \in \omega\}$ of formulas. Clearly, $p \in S_1(\emptyset)$ and p is non-isolated. It is known that $\mathrm{Th}(M)$ has exactly three countable models: in the first case the type p is omitted; in the second case there is a countable model $M_1 \succ M$ such that $p(M_1)$ has the order type $[0, 1) \cap \mathbb{Q}$; in the third case there is a countable model $M_2 \succ M$ such that $p(M_2)$ has the order type $(0, 1) \cap \mathbb{Q}$. Clearly, M is an o-minimal structure, and p is rational-to-right.

Now we expand the given theory by a constant $a \in M_1$ distinguishing the least element of $p(M_1)$. Then $T_1 = Th(\langle M_1, a\rangle) \supset T = Th(M)$. Obviously, T_1 is also o-minimal and $I(T_1, \omega) = 3$, i.e., T_1 is also Ehrenfeucht. Here $p_1(x) := p(x) \cup \{x < a\} \in S_1(T_1)$ is also rational-to-right.

Example 3.4. Consider Example 1.1.4.2 of [29]: let

$$M = \langle \mathbb{Q}, <, c_k, c'_k\rangle_{k\in\omega},$$

where $<$ is an ordinary relation of strict order on the set of rational numbers $\mathbb{Q}$, the constants c_k form a strictly increasing sequence and the constants c'_k form a strictly decreasing sequence, $c_k < c'_l$ for all $k, l < \omega$, and $\lim_{k\to\infty} c_k = \sqrt{2} = \lim_{l\to\infty} c'_l$. The theory has 6 pairwise non-isomorphic countable models, and M is also o-minimal.

Clearly, $p(x) := \{c_k < x < c'_l \mid k, l \in \omega\} \in S_1(\emptyset)$, and p is irrational.

Now we expand the given theory by a constant $a \in M_1$, where M_1 is a model of $T = Th(M)$ with $p(M_1) = \{a\}$. Let $T_1 = Th(\langle M_1, a\rangle)$. Obviously, $T \subset T_1$ and T_1 is o-minimal. Consider the following sets of formulas:

$$p_1(x) := p(x) \cup \{x < a\}, \quad p_2(x) := p(x) \cup \{x > a\}.$$

Clearly, $p_1, p_2 \in S_1(T_1)$ and they are rational 1-types. It can easily check that $I(T_1, \omega) = 9$, i.e. T_1 is also Ehrenfeucht.

We say $\Gamma \subseteq S_1(\emptyset)$ is *independent* if for any set Γ' consisting of exactly one realization of each type in Γ for every $c' \in \Gamma'$, $c' \notin \mathrm{dcl}(\Gamma' \setminus \{c'\})$. We say $p \in S_1(\emptyset)$ *depends on* Γ (or p and Γ *are dependent*) if $\Gamma \cup \{p\}$ is not independent. *The dimension* of Γ (denoted by $\dim(\Gamma)$) equals the cardinality of a maximal independent subset of Γ. Obviously, if $\Gamma = \{p_1, p_2, \ldots, p_s\}$ be a set of pairwise weakly orthogonal 1-types over $\emptyset$ then in the o-minimal case $\dim(\Gamma) = s$.

For an arbitrary o-minimal theory T introduce the following notations:

$$m_T = \dim\{p \in S_1(\emptyset) \mid p \text{ is irrational}\}, k_T = \dim\{p \in S_1(\emptyset) \mid p \text{ is rational}\}.$$

Theorem 3.5. *Let T be an o-minimal theory. If T is Ehrenfeucht then for any $\mathcal{M} \models T$, for any $n < \omega$ and for any $\bar{a} = \langle a_1, \ldots, a_n\rangle \in M$ the theory $T_1 = Th(\langle \mathcal{M}, \bar{a}\rangle)$ is also Ehrenfeucht.*

Moreover,

(1) *if each a_i is a realization of an isolated or a rational 1-type over $\emptyset$ then $I(T_1, \omega) = I(T, \omega)$;*

(2) *if there exist $1 \leq s \leq n$ and $1 \leq i_1 < i_2 < \ldots < i_s \leq n$ such that a_{i_t} is a realization of an irrational type p_{i_t} over $\emptyset$ for every $1 \leq t \leq s$, where $l = \dim\{p_{i_1}, p_{i_2}, \ldots, p_{i_s}\}$, then $I(T_1, \omega) = 6^{m_T - l} 3^{k_T + 2l}$.*

Proof of Theorem 3.5. Firstly, $\langle \mathcal{M}, \bar{a}\rangle$ is also o-minimal.

Suppose that T is Ehrenfeucht. Then both m_T and k_T are finite. Laura Mayer proved in [16] that $I(T,\omega) = 6^{m_T}3^{k_T}$.

Case $n = 1$. Obviously, $a_1 \in p(M)$ for some $p \in S_1(\emptyset)$.

Firstly, suppose that p is isolated. By Proposition 3.2, $p \perp^w q$ for any non-isolated $q \in S_1(\emptyset)$, i.e. $\mathrm{dcl}(a_1) \cap q(M) = \emptyset$. Therefore, each non-isolated 1-type q is uniquely extended to an 1-type q' over $\{a_1\}$ so that $q(M') = q'(M')$ for any $M' \succeq M$. Thus, $m_{T_1} = m_T$ and $k_{T_1} = k_T$.

Suppose now that p is non-isolated. Then p is either rational or irrational. If p is rational then p is either rational-to-right or rational-to-left. Without loss of generality, suppose that p is rational-to-right, i.e. there exists $b \in M \cup \{\infty\}$ such that for any $M' \succeq M$ with $p(M') \neq \emptyset$ we have $\sup p(M') = b$. Then we have that the following set of formulas

$$p'(x) := p(x) \cup \{x < a_1\}$$

also determines a rational-to-right type over $\{a_1\}$. And the formula $a < x < b$ determines an isolated 1-type over $\{a_1\}$. By Proposition 3.2, $p \perp^w q$ for any irrational $q \in S_1(\emptyset)$, and whence we have $m_{T_1} = m_T$. If $p \not\perp^w r$ for some rational $r \in S_1(\emptyset)$ then there exists an $\emptyset$-definable strictly monotonic bijection $f : p(M') \to r(M')$ for any $M' \succeq M$. If f is strictly increasing then $r'(x) := r(x) \cup \{x < f(a_1)\}$ is rational-to-right. If f is strictly decreasing then $r''(x) := r(x) \cup \{x > f(a_1)\}$ is rational-to-left. If $p \perp^w r$ for some rational $r \in S_1(\emptyset)$ then r is uniquely extended to an 1-type r' over $\{a_1\}$ so that $r(M') = r'(M')$ for any $M' \succeq M$. Thus, $k_{T_1} = k_T$.

Suppose now that p is irrational. Then the following sets of formulas

$$p'(x) := p(x) \cup \{x < a_1\}, \quad p''(x) := p(x) \cup \{x > a_1\}$$

determine rational 1-types over $\{a_1\}$. By Lemma 2.17 [9] in the o-minimal case $p(M')$ is indiscernible over $\emptyset$ for any $M' \succeq M$, and therefore $p' \perp^w p''$. By Proposition 3.2, $p \perp^w q$ for any rational $q \in S_1(\emptyset)$. Thus, we obtain that $m_{T_1} = m_T - 1$ and $k_{T_1} = k_T + 2$, i.e. $I(T_1,\omega) = 6^{m_T-1}3^{k_T+2}$.

Case $n > 1$. If for any $1 \leq i \leq n$ a_i is a realization of an isolated or rational 1-type over $\emptyset$ then $I(T_1,\omega) = I(T,\omega)$.

Suppose now that there exist $1 \leq s \leq n$ and $1 \leq i_1 < i_2 < \ldots < i_s \leq n$ such that a_{i_t} is a realization of an irrational type p_{i_t} over $\emptyset$ for every $1 \leq t \leq s$. Let $l = \dim\{p_{i_1}, p_{i_2}, \ldots, p_{i_s}\}$. Then we assert that $I(T_1,\omega) = 6^{m_T-l}3^{k_T+2l}$.

Corollary 3.6. *Let $\mathcal{T}$ be the family of all o-minimal Ehrenfeucht theories, $\mathcal{T}_{\bar{a}}$ be the family of all expansions $T(\bar{a})$ of T by constants in $\bar{a}$ for each $T \in \mathcal{T}$, where $\bar{a}$ is a tuple of new constant symbols. Then $\mathcal{T}_{\bar{a}}$ preserves o-minimality and Ehrenfeuchtness.*

Now we consider possibilities for expansions of dense spherical orders with countably many constants [11], generalizing possibilities for linear and circular orders.

Definition [11;34]. An n-ary relation $K_n \subseteq A^n$ is called a *n-ball*, or *n-spherical*, or *n-circular* order relation, for $n \geq 3$, if it satisfies the following conditions:

(nso1) $\forall x_1, \ldots, x_n (K_n(x_1, x_2, \ldots, x_n) \to K_n(x_2, \ldots, x_n, x_1))$;

(nso2) $\forall x_1, \ldots, x_n \Bigg((K_n(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \wedge$

$$\wedge K_n(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)) \leftrightarrow \bigvee_{1 \leq k < l \leq n} x_k \approx x_l \Bigg)$$

for any $1 \leq i < j \leq n$;

(nso3) $\forall x_1, \ldots, x_n \Bigg(K_n(x_1, \ldots, x_n) \to$

$$\to \forall t \left(\bigvee_{i=1}^{n} K_n(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n) \right) \Bigg);$$

(nso4) $\forall x_1, \ldots, x_n (K_n(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \vee$

$$\vee K_n(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)),\ 1 \leq i < j \leq n.$$

The structure $\langle A, K_n \rangle$ is called the *n-spherically ordered set*, or the *n-spherical order*, too.

An n-spherically ordered set $\langle A, K_n \rangle$, where $n \geq 3$, is called *dense* if it contains at least two elements and for each $(a_1, a_2, a_3, \ldots, a_n) \in K_n$ with $a_1 \neq a_2$ there is $b \in A \setminus \{a_1, a_2, \ldots, a_n\}$ such that

$$\models K_n(a_1, b, a_3, \ldots, a_n) \wedge K_n(b, a_2, a_3, \ldots, a_n).$$

For a (dense) n-spherically ordered set $\langle A, K_n \rangle$ its elementary theory and any expansion are called (dense) *n-spherical*.

The following theorem describes possibilities for countable spectra of constant expansions of dense n-spherical theories T_n.

Theorem 3.7. [11]. *Let T be a countable constant expansion of the dense n-spherical theory T_n, $n \geq 3$. Then either T has 2^ω countable models or T has exactly $\prod\limits_{k \in n \setminus \{1\}} (2^k + 2)^{r_k}$ countable models, where r_k are natural numbers. Moreover, for any $r_0, \ldots, r_{n-1} \in \omega$ there is an aforesaid theory T with exactly $\prod\limits_{k \in n \setminus \{1\}} (2^k + 2)^{r_k}$ countable models.*

Remark 3.8. In view of Theorem 3.7, taking a dense Ehrenfeucht 3-spherical theory T with countably many constants we have $I(T_3, \omega) = 3^{r_1} 6^{r_2}$ for some $r_1, r_2 \in \omega$, $r_1 + r_2 > 0$, and obtain the same dynamics, as in Theorem 3.5, for the values $I(T,\omega)$ under expansions by finitely many new constants.

For a dense Ehrenfeucht 4-spherical theory T with countably many constants we have $I(T,\omega)=3^{r_1}6^{r_2}10^{r_3}$ for some $r_1,r_2,r_3\in\omega$, $r_1+r_2+r_3>0$, [11]. Adding a constant which realizes an isolated 1-type or a nonisolated *rational* 1-type, i.e., a type responsible for 3 countable models we preserve 3 countable models as in Theorem 3.5, (1).

If new constant realizes a nonisolated "irrational" 1-type, which is responsible for 6 countable models, we obtain two rational 1-types each of which is responsible for 3 countable models as in Theorem 3.5, (2), producing $3^2=9$ countable models. If new constant realizes a nonisolated "irrational" 1-type responsible for 10 countable models we obtain three 1-types each of which is responsible for 6 countable models, i.e., there are $6^3=108$ countable models instead of 10 ones.

Continuing the process for dense n-spherical theories, $n\geq 5$, and taking, for instance, an "irrational" 1-type $p(x)$ responsible for 18 countable models we can add a realization of $p(x)$ which produces $10^4=10000$ countable models instead of 18.

In general case, if a 1-type $q(x)$ is responsible for 2^k+2 countable models, $k\in\omega\setminus\{0,1\}$, then new constant realizing $q(x)$ produces $(2^{k-1}+2)^n$ countable models.

Having several new constants realizing same 1-type we obtain a chain of replacements of some $2^{k_i}+2$ countable models by $(2^{k_i-1}+2)^n$ countable models, where i are the indexes for these replacements.

In view of Remark 3.8 we have the following modification of Theorem 3.5.

Theorem 3.9. *Let T be an Ehrenfeucht constant expansion of a dense n-spherical theory. Then for any $\mathcal{M}\models T$, for any $m<\omega$ and for any $\bar{a}=\langle a_1,\ldots,a_m\rangle\in M$ the theory $T_1=Th(\langle\mathcal{M},\bar{a}\rangle)$ is also Ehrenfeucht.*

Moreover,

(1) *if each a_i is a realization of an isolated 1-type or a rational 1-type over $\emptyset$, i.e., a 1-type responsible for 3 countable models, then $I(T_1,\omega)=I(T,\omega)$;*

(2) *if there exist $1\leq s\leq m$ and $1\leq i_1<i_2<\ldots<i_s\leq m$ such that a_{i_t} is a realization of an irrational type p_{i_t} over $\emptyset$, i.e., a 1-type, responsible for 2^k+2 countable models, with $k\geq 2$ for every $1\leq t\leq s$, then each addition of the constant a_{i_t} replaces its multiplier 2^k+2 in $I(T,\omega)$ by $(2^{k-1}+2)^n$ for $I(T_1,\omega)$.*

Corollary 3.10. *Let $\mathcal{T}$ be the family of all Ehrenfeucht constant expansions of dense n-spherical theories, $\mathcal{T}_{\bar{a}}$ be the family of all expansions $T(\bar{a})$ of T by constants in $\bar{a}$ for each $T\in\mathcal{T}$, where $\bar{a}$ is a tuple of new constant symbols. Then $\mathcal{T}_{\bar{a}}$ preserves Ehrenfeuchtness.*

4. Conclusion

We studied possibilities of rank and e-spectrum values for families of constant expansions of theories as well as their links with Cantor-Bendixson rank and degree. Criteria for smallness of a theory and of the existence of a prime model are obtained. A general approach of constant expansions is applied for ordered Ehrenfeucht theories. It is shown that Ehrenfeuchtness and o-minimality are preserved under finite constant expansions. The dynamics of countable spectra $I(T,\omega)$ for Ehrenfeucht o-minimal theories T is described. These results are modified for finite constant expansions of Ehrenfeucht spherically ordered theories showing the possibilities for dynamics of countable spectra and the preservation of Ehrenfeuchtness under these expansions. It would be interesting to apply the general approach studying constant expansions and their characteristics for other natural classes of theories.

References

1. Baizhanov B.S. Expansion of a model of a weakly o-minimal theory by a family of unary predicates. *The Journal of Symbolic Logic*, 2001, vol. 66, no. 3, pp. 1382–1414. https://doi.org/10.2307/2695114
2. Chang C.C., Keisler H.J. *Model Theory.* Mineola, New York, Dover Publications, Inc., 2012. 650 p.
3. Ershov Yu.L., Palyutin E.A. *Mathematical Logic.* Moscow, Fizmatlit Publ., 2011. 356 p. (in Russian)
4. Fraïssé R. *Theory of relations.* Amsterdam, North-Holland, 1986, 451 p.
5. *Handbook of Mathematical Logic*, ed. J. Barwise. Moscow, Nauka Publ., 1982, vol. 1: Model Theory, 392 p. (in Russian)
6. Hart B., Hrushovski E.,Laskowski M.S. The uncountable spectra of countable theories. *Annals of Mathematics*, 2000, vol. 152, no. 1, pp. 207–257. https://doi.org/10.2307/2661382
7. Henkin L. The completeness of the first-order functional calculus. *The Journal of Symbolic Logic*, 1949, vol. 14, no. 3, pp. 159–166. https://doi.org/10.2307/2267044
8. Hodges W. *Model Theory.* Cambridge, Cambridge University Press, 1993, 772 p.
9. Kulpeshov B.Sh., Sudoplatov S.V. Vaught's conjecture for quite o-minimal theories. *Annals of Pure and Applied Logic*, 2017, vol. 168, iss. 1, pp. 129–149. https://doi.org/10.1016/j.apal.2016.09.002
10. Kulpeshov B.Sh., Sudoplatov S.V. Properties of ranks for families of strongly minimal theories. *Siberian Electronic Mathematical Reports*, 2022, vol. 19, no. 1, pp. 120–124. https://doi.org/10.33048/semi.2022.19.011
11. Kulpeshov B.Sh., Sudoplatov S.V. Spherical orders, properties and countable spectra of their theories. *Siberian Electronic Mathematical Reports*, 2023, vol. 20, no. 2, pp. 588–599. https://doi.org/10.33048/semi.2023.20.034
12. Marker D. *Model Theory: An Introduction*, New York, Springer-Verlag, 2002. — Graduate texts in Mathematics. Vol. 217. 342 p.
13. Markhabatov N.D., Sudoplatov S.V. Ranks for families of all theories of given languages. *Eurasian Mathematical Journal*, 2021, vol. 12, no. 2, pp. 52–58. https://doi.org/10.32523/2077-9879-2021-12-2-52-58

14. Markhabatov N.D., Sudoplatov S.V. Definable subfamilies of theories, related calculi and ranks. *Siberian Electronic Mathematical Reports*, 2020, vol. 17, pp. 700–714. https://doi.org/10.33048/semi.2020.17.048
15. Markhabatov N.D. Ranks for families of permutation theories. *The Bulletin of Irkutsk State University. Series Mathematics*, 2019, vol. 28, pp. 86–95. https://doi.org/10.26516/1997-7670.2019.28.85
16. Mayer L.L. Vaught's conjecture for o-minimal theories. *The Journal of Symbolic Logic*, 1988, vol. 53, no. 1, pp. 146–159. https://doi.org/10.2307/2274434
17. Omarov B. Nonessential extensions of complete theories. *Algebra and Logic*, 1983, vol. 22, no. 5, pp. 390–397. https://doi.org/10.1007/BF01982116
18. Pavlyuk In.I., Sudoplatov S.V. Ranks for families of theories of abelian groups. *The Bulletin of Irkutsk State University. Series Mathematics*, 2019, vol. 28, pp. 96–113. https://doi.org/10.26516/1997-7670.2019.28.95
19. Pavlyuk In.I., Sudoplatov S.V. Formulas and properties for families of theories of abelian groups. *The Bulletin of Irkutsk State University. Series Mathematics*, 2021, vol. 36, pp. 95–109. https://doi.org/10.26516/1997-7670.2021.36.95
20. Pillay A. *Geometric Stability Theory.* Oxford, Clarendon Press, 1996, 361 p.
21. Poizat B.P. *Cours de théorie des modèles.* Villeurbane, Nur Al-Mantiq Wal-Ma'rifah, 1985, 444 p.
22. Shelah S. *Classification theory and the number of non-isomorphic models.* Amsterdam, North-Holland, 1990, 705 p.
23. Sudoplatov S.V., Tanović P. Semi-isolation and the strict order property. *Notre Dame Journal of Formal Logic*, 2015, vol. 56, no. 4, pp. 555–572. https://doi.org/10.1215/00294527-3153579
24. Sudoplatov S.V. Closures and generating sets related to combinations of structures. *The Bulletin of Irkutsk State University. Series Mathematics*, 2016, vol. 16, pp. 131–144.
25. Sudoplatov S.V. Families of language uniform theories and their generating sets. *The Bulletin of Irkutsk State University. Series Mathematics*, 2016, vol. 17, pp. 62–76.
26. Sudoplatov S.V. Combinations related to classes of finite and countably categorical structures and their theories. *Siberian Electronic Mathematical Reports*, 2017, vol. 14, pp. 135–150. https://doi.org/10.17377/semi.2017.14.014
27. Sudoplatov S.V. Relative e-spectra and relative closures for families of theories. *Siberian Electronic Mathematical Reports*, 2017, vol. 14, pp. 296–307. https://doi.org/10.17377/semi.2017.14.027
28. Sudoplatov S.V. On semilattices and lattices for families of theories. *Siberian Electronic Mathematical Reports*, 2017, vol. 14, pp. 980–985. https://doi.org/10.17377/semi.2017.14.082
29. Sudoplatov S.V. *Classification of Countable Models of Complete Theories.* Novosibirsk, NSTU Publ., 2018.
30. Sudoplatov S.V. Combinations of structures. *The Bulletin of Irkutsk State University. Series Mathematics*, 2018, vol. 24, pp. 82–101. https://doi.org/10.26516/1997-7670.2018.24.82
31. Sudoplatov S.V. Approximations of theories. *Siberian Electronic Mathematical Reports*, 2020, vol. 17, pp. 715–725. https://doi.org/10.33048/semi.2020.17.049
32. Sudoplatov S.V. Hierarchy of families of theories and their rank characteristics. *The Bulletin of Irkutsk State University. Series Mathematics*, 2020, vol. 33, pp. 80–95. https://doi.org/10.26516/1997-7670.2020.33.80
33. Sudoplatov S.V. Ranks for families of theories and their spectra. *Lobachevskii Journal of Mathematics*, 2021, vol. 42, no. 12, pp. 2959–2968. https://doi.org/10.1134/S1995080221120313

34. Sudoplatov S.V. Arities and aritizabilities of first-order theories. *Siberian Electronic Mathematical Reports*, 2022, vol. 19, no. 2, pp. 889–901. https://doi.org/10.33048/semi.2022.19.075
35. Tent K., Ziegler M. *A Course in Model Theory*. Cambridge, Cambridge University Press, 2012, 248 p.
36. van den Dries L.P.D. *Tame Topology and O-minimal Structures*. Cambridge, Cambridge University Press, 1998, 182 p. https://doi.org/10.1017/CBO9780511525919

Список источников

1. Baizhanov B.S. Expansion of a model of a weakly o-minimal theory by a family of unary predicates // The Journal of Symbolic Logic. 2001. Vol. 66, No. 3. P. 1382–1414. https://doi.org/10.2307/2695114
2. Chang C.C., Keisler H.J. Model Theory, Mineola, New York : Dover Publications, Inc., 2012, 650 p.
3. Ершов Ю.Л., Палютин Е.А. Математическая логика, М. : Физматлит, 2011, 356 с.
4. Fraïssé R. Theory of relations, Amsterdam : North-Holland, 1986, 451 p.
5. Справочная книга по математической логике / Под ред. Дж. Барвайса. М. : Наука, 1982. — Ч. 1. Теория моделей. — 392 с.
6. Hart B., Hrushovski E.,Laskowski M.S. The uncountable spectra of countable theories // Annals of Mathematics. 2000. Vol. 152, No. 1. P. 207–257. https://doi.org/10.2307/2661382
7. Henkin L. The completeness of the first-order functional calculus // The Journal of Symbolic Logic. 1949. Vol. 14, No. 3. P. 159–166. https://doi.org/10.2307/2267044
8. Hodges W. Model Theory, Cambridge : Cambridge University Press, 1993, 772 p.
9. Kulpeshov B.Sh., Sudoplatov S.V. Vaught's conjecture for quite o-minimal theories // Annals of Pure and Applied Logic. 2017. Vol. 168, Iss. 1. P. 129–149. https://doi.org/10.1016/j.apal.2016.09.002
10. Kulpeshov B.Sh., Sudoplatov S.V. Properties of ranks for families of strongly minimal theories // Siberian Electronic Mathematical Reports. 2022. Vol. 19, No. 1. P. 120–124. https://doi.org/10.33048/semi.2022.19.011
11. Kulpeshov B.Sh., Sudoplatov S.V. Spherical orders, properties and countable spectra of their theories // Siberian Electronic Mathematical Reports. 2023. Vol. 20, No. 2. P. 588–599. https://doi.org/10.33048/semi.2023.20.034
12. Marker D. Model Theory: An Introduction, New York : Springer-Verlag, 2002. — Graduate texts in Mathematics. Vol. 217. 342 p.
13. Markhabatov N.D., Sudoplatov S.V. Ranks for families of all theories of given languages // Eurasian Mathematical Journal. 2021. Vol. 12, No. 2. P. 52–58. https://doi.org/10.32523/2077-9879-2021-12-2-52-58
14. Markhabatov N.D., Sudoplatov S.V. Definable subfamilies of theories, related calculi and ranks // Siberian Electronic Mathematical Reports. 2020. Vol. 17. P. 700–714. https://doi.org/10.33048/semi.2020.17.048
15. Markhabatov N.D. Ranks for families of permutation theories // The Bulletin of Irkutsk State University. Series Mathematics. 2019. Vol. 28. P. 86–95. https://doi.org/10.26516/1997-7670.2019.28.85
16. Mayer L.L. Vaught's conjecture for o-minimal theories // The Journal of Symbolic Logic. 1988. Vol. 53, No. 1. P. 146–159. https://doi.org/10.2307/2274434
17. Omarov B. Nonessential extensions of complete theories // Algebra and Logic. 1983. Vol. 22, No. 5. P. 390–397. https://doi.org/10.1007/BF01982116

18. Pavlyuk In.I., Sudoplatov S.V. Ranks for families of theories of abelian groups // The Bulletin of Irkutsk State University. Series Mathematics. 2019. Vol. 28. P. 96–113. https://doi.org/10.26516/1997-7670.2019.28.95
19. Pavlyuk In.I., Sudoplatov S.V. Formulas and properties for families of theories of abelian groups // The Bulletin of Irkutsk State University. Series Mathematics. 2021. Vol. 36. P. 95–109. https://doi.org/10.26516/1997-7670.2021.36.95
20. Pillay A. Geometric Stability Theory, Oxford : Clarendon Press, 1996, 361 p.
21. Poizat B.P. Cours de théorie des modèles, Villeurbane : Nur Al-Mantiq Wal-Ma'rifah, 1985, 444 p.
22. Shelah S. Classification theory and the number of non-isomorphic models, Amsterdam : North-Holland, 1990, 705 p.
23. Sudoplatov S.V., Tanović P. Semi-isolation and the strict order property // Notre Dame Journal of Formal Logic. 2015. Vol. 56, No. 4. P. 555–572. https://doi.org/10.1215/00294527-3153579
24. Sudoplatov S.V. Closures and generating sets related to combinations of structures // The Bulletin of Irkutsk State University. Series Mathematics. 2016. Vol. 16. P. 131–144.
25. Sudoplatov S.V. Families of language uniform theories and their generating sets // The Bulletin of Irkutsk State University. Series Mathematics. 2016. Vol. 17. P. 62–76.
26. Sudoplatov S.V. Combinations related to classes of finite and countably categorical structures and their theories // Siberian Electronic Mathematical Reports. 2017. Vol. 14. P. 135–150. https://doi.org/10.17377/semi.2017.14.014
27. Sudoplatov S.V. Relative e-spectra and relative closures for families of theories // Siberian Electronic Mathematical Reports. 2017. Vol. 14. P. 296–307. https://doi.org/10.17377/semi.2017.14.027
28. Sudoplatov S.V. On semilattices and lattices for families of theories // Siberian Electronic Mathematical Reports. 2017. Vol. 14. P. 980–985. https://doi.org/10.17377/semi.2017.14.082
29. Sudoplatov S.V. Classification of Countable Models of Complete Theories, Novosibirsk : NSTU, 2018.
30. Sudoplatov S.V. Combinations of structures // The Bulletin of Irkutsk State University. Series Mathematics. 2018. Vol. 24. P. 65–84. https://doi.org/10.26516/1997-7670.2018.24.82
31. Sudoplatov S.V. Approximations of theories // Siberian Electronic Mathematical Reports. 2020. Vol. 17. P. 715–725. https://doi.org/10.33048/semi.2020.17.049
32. Sudoplatov S.V. Hierarchy of families of theories and their rank characteristics // The Bulletin of Irkutsk State University. Series Mathematics. 2020. Vol. 33. P. 80–95. https://doi.org/10.26516/1997-7670.2020.33.80
33. Sudoplatov S.V. Ranks for families of theories and their spectra // Lobachevskii Journal of Mathematics. 2021. Vol. 42, No. 12. P. 2959–2968. https://doi.org/10.1134/S1995080221120313
34. Sudoplatov S.V. Arities and aritizabilities of first-order theories // Siberian Electronic Mathematical Reports. 2022. Vol. 19, No. 2. P. 889–901. https://doi.org/10.33048/semi.2022.19.075
35. Tent K., Ziegler M. A Course in Model Theory, Cambridge : Cambridge University Press, 2012, 248 p.
36. van den Dries L.P.D. Tame Topology and O-minimal Structures, Cambridge : Cambridge University Press, 1998, 182 p. https://doi.org/10.1017/CBO9780511525919

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