Numerical methods for stochastic Allen-Cahn equation and stochastic subdiffusion and superdiffusion

Thesis submitted in accordance with the requirements of the University of Chester for the degree of Doctor in Philosophy

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April 2023

Introduction

In this Thesis, we consider the numerical solution of stochastic partial differential equations with particular interest on the $\varepsilon-$dependent Allen-Cahn equation, and the stochastic time fractional partial differential equations in both subdiffusion and superdiffusion cases.

In Chapter one, we present some of the basic concepts of weak (variational) formulations, and the finite elements approximation of weak solutions. We discuss in the context of ordinary differential equations the two-point boundary value problem, and introduce the definitions of strong and weak solutions. We then present how existence and uniqueness of a weak solution is established by the application of Lax-Milgram Lemma. We also present the finite elements formulation to spaces of finite dimensions. Moreover, by using a Green's function, we provide the analytical solution of the elliptic problem. We then focus on partial differential equations and consider the finite element method for the numerical approximation of parabolic equations. We show existence, uniqueness and stability of a solution. We discuss the semidiscrete Galerkin method and the fully discrete scheme.

In Chapter two, a discontinuous Galerkin method is presented. The purpose of this method is to construct approximate solutions as piece-wise polynomial functions in the time variable which are discontinuous at the nodal points of the time partition. We consider existence and uniqueness of solutions. When the method is applied for the (linear) parabolic equation, we present the stability of the scheme and some error estimates. Moreover, we discuss the method when applied for the (nonlinear) Allen-Cahn equation which stands as a model for the phase separation of a multi-component alloy including order-disordered transitions. We review some of the results of [15] for the $\varepsilon-$dependent stochastic Allen-Cahn equation with mild noise, such as existence and uniqueness of a solution and as well as stability for the nonlinear space-time discontinuous in time Galerkin method introduced there.

Chapter three is devoted to the derivation of *a posteriori* error estimates of the space-time discontinuous Galerkin finite element method for the $\varepsilon-$dependent stochastic Allen-

Cahn equation with mild noise. The noise depends on ε, it is once differentiable in time and smooth in space, rapidly oscillating, and becomes white in time on the sharp interface limit as $\varepsilon \to 0$. The numerical solution is in general discontinuous at the nodal points of the time partition. We describe analytically how the numerical scheme is implemented in the special case of finite element spaces of piece-wise constant or piece-wise linear functions in time. We then use a certain interpolant in order to derive, when general finite element spaces are considered, *a posteriori* error estimates in the space-time H^1 norm and discuss the estimation of the time derivative of the error by the initial data, the noise and the numerical solution which is transferred to the estimator.

In Chapters four and five, we discuss the Galerkin method for approximating the solution of semi-linear stochastic space-time fractional subdiffusion and superdiffusion problems with the Caputo fractional derivative of order $\alpha \in (0,1)$ and $\alpha \in (1,2)$ respectively, driven by fractionally integrated additive noise. After discussing the existence, uniqueness and regularity results, we approximate the noise and obtain regularized stochastic fractional subdiffusion and superdiffusion problems. The regularized problems are then approximated by using the finite element method in spatial direction. The mean squared errors proven are based on the sharp estimates of the various Mittag-Leffler functions involved in the integrals. Numerical experiments are conducted to show that the numerical results are consistent with the theoretical findings.

Chapter six includes some conclusive remarks, while an Appendix with the numerical codes used for the simulations presented in this Thesis is provided in the end of the Thesis.

Declaration

No part of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other institution of learning. However some parts of the materials contained herein have been published previously.

Publications

- Antonopoulou, D.C., **Egwu, B.A.**, & Yan, Y. (2023). *A posteriori* analysis of space-time discontinuous Galerkin methods for the $\varepsilon-$dependent Allen-Cahn equation with mild noise. IMA J. Numer. Anal. **00**, 1-41.

  Chapter 3 of this thesis is derived from the content presented in this paper

- Kang, W., **Egwu, B.A.**, Yan, Y., & Pani, K.A. (2022). Galerkin finite element approximation of a stochastic semilinear fractional subdiffusion with fractionally integrated additive noise, IMA J. Numer. Anal. **42**, 2301-2335.

  Chapter 4 of this thesis is derived from the content presented in this paper.

- **Egwu, B.A.**, & Yan, Y. (2023). Galerkin finite element approximation of a stochastic semilinear fractional superdiffusion with fractionally integrated additive noise. Foundations **3**, 290-322.

  Chapter 5 of this thesis is derived from the content presented in this paper.

- Yan, Y., **Egwu, B.A.**, Liang, Z., & Yan, Y. (2021). Error estimates of a continuous Galerkin time stepping method for subdiffusion problem, Journal of Scientific Computing **88**(3) 68.

Conference Attended

- LMS Lecture Series May 2020, Brunel University London (Online).

Seminar

- Department of Physical, Mathematical and Engineering, University of Chester weekly Seminar, 2022.

- One World Stochastic Numeric and Inverse Problem Seminar (Online) April, 2020-present, organized by the University of Edinburgh, Scotland.

Presentation

- Title: Galerkin finite element approximation of a stochastic semilinear fractional superdiffusion with fractionally integrated additive noise, 24th May, 2023 at the mathematics seminar, University of Chester, UK.
- Title: Numerical methods for stochastic superdiffusion, 23rd June, 2023 at the University of Chester Postgraduate Research Symposium, 2023. I won the overall 'best presentation' of the symposium with certificate of awards.

Acknowledgement

I am eternally grateful to God for His all-round help these past years. I am forever indebted to my supervisors Dr. Yubin Yan and Dr. D.C. Antonopoulou for their relentless help and patience. No words can describe how I feel. You are both rare gems. Just to let you know that your reassurances and encouragements are the ladder I have used to be where I am now in this journey. To my siblings; Catherine, Joseph, Ene and Enenche you are the best. Thank you for your support. Catherine is my co-labourer in prayer God bless you. All my friends who have stood by me I appreciate you and thank you. I want to express my gratitude to the Head of Physical, Mathematical and Engineering Department and all the staff for your support. Thank you University of Chester I am happy to be part of this family.

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Chapter 1

Basic definitions and some fundamental results

In this chapter, we shall introduce some basic definitions in functional analysis and discuss the analytic solutions of the two-point boundary value problem and consider the finite element approximation of the parabolic problem. We also introduce some basic notations in stochastic processes and fractional calculus which we need in the subsequent chapters.

1.1 Basic definitions

Let us present some basic definitions, see for example in [99].

Definition 1.1.1. A norm $\|\cdot\|$ is a function from a real vector space $X \to \mathbb{R}$ such that

(i) $\|u\| \geq 0$, $\forall u \in X$, $\quad \|u\| = 0$, if and only if $u = 0$.
(ii) $\|\alpha u\| = |\alpha|\|u\|, \forall u \in X$, $\alpha \in \mathbb{R}$.
(iii) $\|u+v\| \leq \|u\| + \|v\|, \forall u, v \in X$. (triangle inequality)

Definition 1.1.2. An inner product on a real vector space X is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ such that

1. $\langle u, u\rangle \geq 0$, $\forall u \in X$, and $\langle u, u\rangle = 0$ if and only if $u = 0$ (positive definite).

2. $\langle u, v\rangle = \langle v, u\rangle$, $\forall\, u, v \in X$. (symmetry)

3. $\langle \lambda u + \mu v, w\rangle = \lambda\langle u, w\rangle + \mu\langle v, w\rangle, \forall\, \lambda, \mu \in \mathbb{R}$ and $\forall\, u, v, w \in X$ (linearity with respect to the first argument).

Definition 1.1.3. A normed vector space $(X, \|\cdot\|)$ is called Banach space if it is complete, i.e., if every Cauchy sequence $\{v_k\}_{k=1}^{\infty} \in X$ converges to some $v \in X$, in the norm $\|\cdot\|$ of the space.

Definition 1.1.4. A Hilbert space is a vector space equipped with an inner product $\langle\cdot,\cdot\rangle$ which is a Banach space with respect to the norm $\|v\| :=\langle v, v\rangle^{\frac{1}{2}}$.

Definition 1.1.5. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_*)$ be two normed spaces, then

(a) $A : X \to Y$ is a linear operator provided that $A(\lambda u + \mu v)=\lambda Au + \mu Av$, $\forall\, u, v \in X$, $\forall \lambda, \mu \in \mathbb{R}$.

(b) A linear operator $A : X \to Y$ is called bounded if and only if there exists $C > 0$ such that $\|Ax\|_* \le C\|x\|, \forall\, x \in X$.

Definition 1.1.6. A bilinear form $a(\cdot,\cdot)$ on a vector space V is a function $a : V \times V \to \mathbb{R}$ which is linear in each argument separately, i.e., $\forall\; u, v, w \in V$ and $\lambda, \mu \in \mathbb{R}$

$$a\big(\lambda u + \mu v, w\big) = \lambda a\big(u, w\big) + \mu a\big(v, w\big),$$
$$a\big(w, \lambda u + \mu v\big) = \lambda a\big(w, u\big) + \mu a\big(w, v\big).$$

The bilinear form $a(\cdot,\cdot)$ is said to be symmetric if $a(w, v) = a(v, w),\ \forall\; v, w \in V$ and positive definite if $a(v, v) > 0, \forall\; 0 \ne v \in V$.

Definition 1.1.7. Let $p \in [1, \infty)$. The $L^p(a, b)$ space is defined as

$$L^p(a, b) = \Big\{v : (a, b) \to \mathbb{R} : \int_a^b |v(x)|^p\, dx < \infty\Big\}.$$

The norm of $L^p(a, b)$ space is defined by

$$\|v\|_{L^p(a,b)} = \Big(\int_a^b |v(x)|^p\, dx\Big)^{\frac{1}{p}}.$$

We also recall some Sobolev spaces:

$$H^1(a, b) := \{v \in L^2(a, b) : \quad v' \in L^2(a, b)\},$$

$$H_0^1(a,b) := \{v \in L^2(a,b) : \ \ v' \in L^2(a,b), \text{ and } v(a) = v(b) = 0\},$$

and for any $\mathbb{N} \ni k \geq 2$

$$H^k(a,b) := \{v \in L^2(a,b) : \ \ v', v'', \cdots, v^{(k)} \in L^2(a,b)\},$$

where $v^{(k)}$ denotes the k-order derivative.

1.2 The two-point boundary value problem

We will discuss some basic results for the two-point boundary value problem from [85, 99, 60, 121]. We shall introduce the definitions of strong and weak solutions of the problem [85, 99]. Existence and uniqueness of weak solutions is proven by using Lax-Milgram Lemma [85, 100]. We also present the finite element formulation of approximate solutions to the problem. Finally, we present the Green's function of the relevant elliptic problem, and its analytic solution, [85].

1.2.1 Weak formulation

We seek a function $u \in H^2(a,b) \cap H_0^1(a,b)$ such that

$$\begin{aligned} &-(p(x)u'(x))' + q(x)u(x) = f(x) \ \ \forall\, x \in (a,b), \\ &u(a) = u(b) = 0, \end{aligned} \tag{1.2.1}$$

for some given functions:

$$f \in L^2(a,b),\ p \in C^1[a,b], \ \ q \in C[a,b],$$

$$p(x) \geq \beta > 0, \ \ q(x) \geq 0,\ \forall\, x \in [a,b],$$

where β is a positive constant. If such a u exists for almost every $x \in (a,b)$, then it is called a strong solution of (1.2.1).

Multiplying (1.2.1) by $v \in H_0^1(a,b)$, integrating in (a,b), applying integration by parts and using the boundary condition, we obtain

$$a(u,v) := (pu', v') + (qu, v) = (f, v),$$

where $(\cdot, \cdot)$ is the $L^2(a,b)$ inner product and

$$a(w,v) := (pw', v') + (qw, v).$$

We then get the variational formulation of the problem whose solution $u \in H_0^1(a,b)$ is called a weak solution.

Remark 1. Obviously a strong solution of (1.2.1) is a weak solution. Moreover, if a weak solution of (1.2.1) is in $H^2(a,b) \cap H_0^1(a,b)$, then it satisfies (1.2.1).

By C in this thesis we denote a positive constant independent of the functions and parameters concerned, but not necessarily the same at different occurrences.

Definition 1.2.1. Let H be a real Hilbert space. A bilinear form $a : H \times H \to \mathbb{R}$ is called coercive if there exists a constant $C > 0$ such that $a(v,v) \geq C\|v\|_H^2$, $\forall\, v \in H$.

We shall present Lax-Milgram Lemma below which is a useful tool on establishing weak solutions for initial and boundary value problems.

Lemma 1.2.1. [85] (Lax-Migram Lemma) Let H be a real Hilbert space and let L be a bounded linear functional on H. Let $a : H \times H \to \mathbb{R}$ be a bilinear form that is bounded and coercive, then there exists a unique $u \in H$ such that $a(u,v) = L(v)$, $\forall v \in H$.

We denote for simplicity $\|\cdot\| := \|\cdot\|_{L^2(a,b)}$. Let us consider the two-point boundary value problem (1.2.1), with the bilinear form a and the linear operator L as defined previously, and set $H := H_0^1(a,b)$, equipped with the $H^1(a,b)$ norm given by

$$\|v\|_{H^1(a,b)} := (\|v\|^2 + \|v'\|^2)^{1/2}.$$

Let $v \in H$. By using that $v(a) = 0$, and the Cauchy-Schwarz inequality, we have that for all $x \in [a,b]$

$$|v(x)|^2 = \Big| \int_a^x v'(y)d(y) \Big|^2 \leq \int_a^x (v'(y))^2 d(y) \int_a^x 1^2 d(y) \leq c\|v'\|^2,$$

for a constant $c > 0$. Therefore, integrating in (a,b), we obtain that $\|v\| \leq c\|v'\|$, for all $v \in H$.

Using the above, we have for any $v \in H$

$$a(v,v) = (pv', v') + (qv, v) \geq \beta\|v'\|^2 \geq C\|v\|_{H^1(a,b)}^2, \tag{1.2.2}$$

since $p(x) \geq \beta > 0$ and $q(x) \geq 0$ for all $x \in [a, b]$. The inequality (1.2.2) expresses that the bilinear form $a(\cdot, \cdot)$ is coercive in $H_0^1(a, b)$.

Further, for any $u, v \in H$, we have, since p, q are continuous in $[a, b]$, that

$$|a(u, v)| = |(pu', v') + (qu, v)| \leq \max_{[a,b]} p(x)\|u'\|\|v'\| + \max_{[a,b]} q(x)\|u\|\|v\| \leq c\|u\|_{H^1(a,b)}\|v\|_{H^1(a,b)},$$

which yields the boundedness of the bilinear form with respect to the norm in H.

Moreover, for any $v \in H$, it holds that

$$|L(v)| = |(f, v)| \leq \|f\| \cdot \|v\| \leq c\|v\| \leq c\|v\|_{H^1(a,b)},$$

since $f \in L^2(a, b)$. This yields that the linear functional L is bounded in H.

Therefore, by Lax-Milgram Lemma, (1.2.1) has a unique weak solution.

Remark 2. The same Lemma can be applied to a finite dimensional Hilbert space $H := V_h$, where the test function and the solution will belong to V_h that could be defined as a Finite Element Space by keeping the same definition for the bilinear form a and the linear functional L. In such a case the weak formulation and the finite basis of H, (i.e., the specific selection of H), induces the so-called finite elements formulation. The unique solution of the scheme stands as an approximate solution of the continuous problem.

In particular, the finite element formulation takes the form: find $u_h \in V_h \subset H_0^1(a, b)$ where V_h such that

$$a(u_h, v_h) := (pu_h', v_h') + (qu_h, v_h) = (f, v_h), \quad \forall\, v_h \in V_h.$$

It is sufficient for the above to be valid for all the elements of the basis which is the simplest choice for the test functions v_h.

1.2.2 Analytic solutions

Denote

$$Au := -(pu')' + qu,$$

then Problem (1.2.1) can be written as the following operator form:

$$Au = f.$$

We shall derive a representation of a solution in terms of a so called Green's function $G(x, y)$.

Let V_0, V_1 be two solutions of the homogeneous equation i.e.

$$AV_0 = AV_1 = 0 \text{ in } (a, b),$$

with

$$V_0(a) = 1, \quad V_0(b) = 0,$$

and

$$V_1(a) = 0, \quad V_1(b) = 1.$$

The next theorem holds.

Theorem 3. *[85] The solution of* (1.2.1) *is given by*

$$u(x) = \int_a^b G(x, y) f(y)\, dy,$$

where

$$G(x, y) := \begin{cases} \frac{1}{k} V_0(x) V_1(y), & for \ a \le y \le x \le b \\ \frac{1}{k} V_1(x) V_0(y), & for \ a \le x \le y \le b \end{cases}$$

for some fixed constant $k := p(x)\big(V_0(x)V_1'(x) - V_0'(x)V_1(x)\big)$.

Proof. Let $x, \ y \in (a, b)$, we first observe that

$$G(a, y) = \frac{1}{k} V_1(a) V_0(y) = 0 = G(b, y) = \frac{1}{k} V_0(b) V_1(y),$$

which implies that

$$u(a) = u(b) = 0.$$

Since $(pV_j')' = qV_j$, $j = 1, 2$, we have

$$k' = V_0(pV_1')' - V_1(pV_0')' = V_0 q V_1 - V_1 q V_0 = 0,$$

and so k is a well defined constant. (Further one may show $k > 0$, see [85, Page 19])

Let $x \in (a, b)$. We write

$$\begin{aligned} u(x) &= \int_a^x G(x, y) f(y)\, dy + \int_x^b G(x, y) f(y) dy \\ &= \frac{1}{k} V_0(x) \int_a^x V_1(y) f(y)\, dy + \frac{1}{k} V_1(x) \int_x^b V_0(y) f(y) dy. \end{aligned}$$

Hence by differentiating, we get

$$\begin{aligned} u'(x) =& \frac{1}{k}\left(V_0'(x)\int_a^x V_1(y)f(y)\,dy + V_0(x)V_1(x)f(x)\right) \\ &+ \frac{1}{k}\left(V_1'(x)\int_x^b V_0(y)f(y)\,dy - V_0(x)V_1(x)f(x)\right) \\ =& \frac{1}{k}\left(V_0'(x)\int_a^x V_1(y)f(y)\,dy + V_1'(x)\int_x^b V_0(y)f(y)\,dy\right). \end{aligned}$$

Multiplying by $-p(x)$ and differentiating, we obtain

$$\begin{aligned} -\big(p(x)u'(x)\big)' =& -\frac{1}{k}\left(p(x)V_0'(x)\right)'\int_a^x V_1(y)f(y)\,dy - \frac{1}{k}p(x)V_0'(x)V_1(x)f(x) \\ &- \frac{1}{k}(p(x)V_1'(x))'\int_x^b V_0(y)f(y)\,dy + \frac{1}{k}p(x)V_1'(x)V_0(x)f(x). \end{aligned}$$

Using once again that $\big(pV_j'\big)' = qV_j$, we arrive at

$$\begin{aligned} -\big(p(x)u'(x)\big)' &= -\frac{1}{k}\left(q(x)V_0(x)\int_a^x V_1(y)f(y)\,dy + q(x)V_1(x)\int_x^b V_0(y)f(y)\,dy\right) + f(x) \\ &= -q(x)\int_a^b G(x,y)f(y)\,dy + f(x) \\ &= -q(x)u(x) + f(x). \end{aligned}$$

This completes the proof of the theorem.

□

1.3 The parabolic equation

In this section, we discuss the finite element method for solving the parabolic equation, [85, 100]. Firstly, we will show the existence and uniqueness, stability and energy estimates of solution. Secondly, we shall discuss the semidiscrete Galerkin finite element method for its numerical solution. Furthermore, we will consider time discretisation schemes.

1.3.1 The general framework

Let H be a Hilbert space. A sequence of vectors $\{e_1, e_2, e_3, ...\} \in H$ is called orthonormal if and only if

$$\langle e_i, e_j\rangle = \begin{cases} 1, \ i = j, \\ 0, \ i \neq j, \ \forall\, i,j \ \in \ \mathbb{N} = \{1,2,3,...\}. \end{cases}$$

An orthonormal basis for H is an orthonormal sequence such that each $v \in H$ admits a unique representation as a convergent series $v = \sum_{j=1}^{\infty} c_j e_j$ with $c_j \in \mathbb{R}$.

The eigenfunctions of certain differential operators may form orthonormal sequences with respect to the L^2 inner product denoted by $(\cdot, \cdot)$. In particular, we have the following lemma.

Lemma 1.3.1. [85, 121] The negative Laplacian operator $-\Delta$ on a bounded domain $D \subset \mathbb{R}^d, d = 1, 2, 3$ subject to Dirichlet or Neumann boundary conditions admits real eigenvalues $\{\lambda_j\}$, and there exists an orthonormal basis of $L^2(D)$ of corresponding eigenfunctions $e_j \in C^2(\overline{D})$ satisfying the same boundary conditions. Furthermore, $\lambda_j > 0$ in the case of Dirichlet conditions, while $\lambda_j \geq 0$ when Neumann conditions are posed along the boundary.

Proof. We only consider the case with homogeneous Dirichlet boundary condition. Let a sequence $\{e_j\} \in C^2(\overline{D})$ satisfy

$$-\Delta e_j = \lambda_j e_j \text{ in } D,$$

with Dirichlet condition.

(The existence of such e_j can be obtained by solving the corresponding boundary value problem, for example, in one dimensional case, $\lambda_j = j^2\pi^2,\ e_j = \sqrt{2}\sin j\pi x,\ x \in (0, 1)$)

We then have

$$(\Delta e_i, e_j) = (e_i, \Delta e_j).$$

By the eigenvalue property, this reduces to

$$(\lambda_i - \overline{\lambda_j})(e_i, e_j) = 0.$$

When $i = j$, the inner product satisfies $\|e_i\|^2 > 0$, as $e_i \neq 0$ which implies that $\lambda_i = \overline{\lambda_i} \in \mathbb{R}\ \forall\ i$. If $\lambda_i \neq \lambda_j$, then $(e_i, e_j) = 0$, which means that the eigenfunctions corresponding to different eigenvalues are pairwise orthogonal. Dividing the eigenfunctions by their measure the orthonormal basis of eigenfunctions is then derived.

The divergence theorem yields

$$\lambda_i = (-\Delta e_i, e_i) = \int\limits_D |\nabla e_i|^2 dx,$$

implying that $\lambda_i \geq 0$. If $\lambda_i = 0$ then the equation also shows that $\nabla e_i = 0$, and so e_i is constant. In the Dirichlet case the only constant solution is the trivial zero solution $e_i = 0$ which is excluded since an eigenfunction can never be the zero function, and therefore $\lambda_i > 0$ in the Dirichlet case. $\square$

Now we turn to the following parabolic problem (heat equation)

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \text{ in } D \times \mathbb{R}_+, \\ &u(x,t) = 0, \text{ in } \partial D \times \mathbb{R}_+, \\ &u(x,0) = v(x), \text{ in } D, \end{aligned} \tag{1.3.1}$$

where D is a bounded interval in $\mathbb{R}$. Assume that

$$u(x,t) = \sum_{i=1}^{\infty} \widehat{u}_i(t) e_i(x), \tag{1.3.2}$$

where $\widehat{u}_i : \mathbb{R}_+ \to \mathbb{R}$ are coefficients to be determined. This approach is also called the method of separation of variables.

Inserting (1.3.2) into the differential equation (1.3.1) we obtain

$$\sum_{i=1}^{\infty} \big(\widehat{u}_i'(t) + \lambda_i \widehat{u}_i(t)\big) e_i(x) = 0,$$

for $x \in D, \quad t \in \mathbb{R}_+$. Since $e_i(x)$ form an orthonormal basis, we have

$$\widehat{u}_i'(t) + \lambda_i \widehat{u}_i(t) = 0,$$

for $t \in \mathbb{R}_+$, $i = 1, 2, 3, \ldots$, which has the solution

$$\widehat{u}_i(t) = \widehat{u}_i(0) e^{-\lambda_i t} = \widehat{v}_i e^{-\lambda_i t}, \tag{1.3.3}$$

where $u(x,0) = \sum_{i=1}^{\infty} \widehat{u}_i(0) e_i(x) = \sum_{i=1}^{\infty} \widehat{v}_i e_i(x)$. We thus see that, at least formally, the solution of the equation has the form,

$$u(x,t) = \sum_{i=1}^{\infty} \widehat{v}_i e^{-\lambda_i t} e_i(x).$$

Theorem 4. *[85] Let $u(t) \in H_0^1(D)$ satisfy the heat equation* (1.3.1). *Then, there is a constant $C > 0$ such that for any $t > 0$,*

$$\|u(t)\|^2 + \int_0^t |u(s)|_1^2 ds \leqslant \|v\|^2 + C\int_0^t \|f(s)\|^2 ds, \tag{1.3.4}$$

$$|u(t)|_1^2 + \int_0^t \|u_t(s)\|^2 ds \leqslant |v|_1^2 + \int_0^t \|f(s)\|^2 ds, \tag{1.3.5}$$

where for a function $w \in H^1(D)$, $|w|_1 := \|\nabla w\|$ (which is in general a seminorm).

Proof. Let $a(u,v) := (\nabla u, \nabla v)$. The variational formulation of the heat equation is

$$(u_t, \varphi) + a(u,\varphi) = (f,\varphi), \forall \varphi \in H_0^1(D),\ t \in \mathbb{R}_+,$$

where u_t denotes the derivative with respect to the time variable.

We first prove (1.3.4). Let $\varphi = u$, then

$$(u_t, u) + a(u,u) = (f,u), \quad t > 0.$$

But it holds that

$$(u_t, u) = \int_D u_t u dx = \int_D \frac{1}{2}(u^2)_t dx = \frac{1}{2}\frac{d}{dt}\|u\|^2.$$

Note that by Poincaré inequality, i.e.,

$$\|u\| \leq C|u|_1, \forall\ u \in H_0^1(D),$$

we get

$$(f,u) \leq |(f,u)| \leqslant \|f\|\|u\| \leqslant C\|f\||u|_1 \leqslant \frac{1}{2}|u|_1^2 + \frac{1}{2}C^2\|f\|^2,$$

while

$$a(u,u) = \int_D \nabla u \nabla u dx = \int_D |\nabla u|^2 dx = |u|_1^2,$$

so we have, with some suitable constant $C > 0$,

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + |u|_1^2 \leqslant \frac{1}{2}|u|_1^2 + \frac{1}{2}C\|f\|^2. \tag{1.3.6}$$

Multiplying (1.3.6) by 2, we get

$$\frac{d}{dt}\|u\|^2 + 2|u|_1^2 \leqslant |u|_1^2 + C\|f\|^2,$$

which implies that

$$\frac{d}{dt}\|u\|^2 + |u|_1^2 \leqslant C\|f\|^2.$$

Integrating over $(0,t)$ we have

$$\int_0^t \frac{d}{dt}\|u(s)\|^2 ds + \int_0^t |u(s)|_1^2 ds \leqslant C\int_0^t \|f\|^2 ds,$$

which yields

$$\|u(t)\|^2 + \int_0^t |u(s)|_1^2 ds \leqslant \|v\|^2 + C\int_0^t \|f\|^2 ds.$$

Thus we proved (1.3.4).

Next we show (1.3.5). We choose

$$\varphi = u_t, \quad (u_t,\varphi) + a(u,\varphi) = (f,\varphi), \quad \varphi \in H_0^1, \quad t \in \mathbb{R}_+,$$

and get

$$\|u_t\|^2 + \frac{1}{2}\frac{d}{dt}|u|_1^2 \le \frac{1}{2}\|f\|^2 + \frac{1}{2}\|u_t\|^2. \tag{1.3.7}$$

Multiplying (1.3.7) by 2, we have

$$2\|u_t\|^2 + \frac{d}{dt}|u|_1^2 \le \|f\|^2 + \|u_t\|^2,$$

which implies

$$\frac{d}{dt}|u|_1^2 + \|u_t\|^2 \le \|f\|^2. \tag{1.3.8}$$

Integrating (1.3.8) over $(0,t)$, we obtain

$$\int_0^t \frac{d}{dt}|u(s)|_1^2 ds + \int_0^t \|u_t\|^2 ds \le \int_0^t \|f\|^2 ds,$$

and so

$$|u(t)|_1^2 + \int_0^t \|u_t\|^2 ds \le |v|_1^2 + \int_0^t \|f\|^2 ds.$$

These estimates complete the proof of (1.3.5). $\square$

1.3.2 The semidiscrete Galerkin Finite Element Method

Let $D \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary ∂D, and consider the initial boundary value problem

$$
\begin{aligned}
&u_t - \Delta u = f, \text{ in } D \times \mathbb{R}_+, \\
&u = 0, \quad \text{on } \partial D \times \mathbb{R}_+, \\
&u(0) = v, \quad \text{in } D,
\end{aligned}
\tag{1.3.9}
$$

where $u_t = \frac{\partial u}{\partial t}$ is the partial derivative with respect to the time, and $\Delta = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$ is the Laplacian operator.

Let $\mathcal{T}_h$ denote the triangle partitions of the domain $D \in \mathbb{R}^2$, where h denotes the maximum diameter of the triangles. We shall approximate the solution $u(x,t)$ by means of a function $u_h(x,t)$ which, for each fixed t, is a piecewise linear function of x over a triangulation $\mathcal{T}_h$ of D, thus depending on a finite number of parameters.

Let $S_h \subset H_0^1(D)$ denote the linear finite element space which consists of all the piecewise continuous linear functions defined on $\mathcal{T}_h$.

The weak form of (1.3.9) is to find $u \in H_0^1(D)$ such that, with $a(v,w) = (\nabla v, \nabla w),\ \forall v, w \in H_0^1(D)$,

$$
(u_t, \varphi) + a(u, \varphi) = (f, \varphi), \quad \forall\, \varphi \in H_0^1(D),\ t > 0.
$$

The semidiscrete problem is to find $u_h \in S_h$ such that

$$
\begin{aligned}
(u_{h,t}, \chi) + a(u_h, \chi) = (f, \chi), &\quad \forall\, \chi \in S_h,\ t > 0, \\
u_h(0) = v_h, &
\end{aligned}
$$

where $v_h \in S_h$ is some approximation of v. Since we have discretised only the space variable, this is referred to as a spatially semidiscrete problem.

Let $\{\varphi_j\}_{j=1}^{N_h}$ be the finite element basis functions and let

$$
u_h(x,t) = \sum_{j=1}^{N_h} \alpha_j(t) \varphi_j(x),
$$

we then have

$$\sum_{j=1}^{N_h} \alpha_j'(t)(\varphi_j, \varphi_k) + \sum_{j=1}^{N_h} \alpha_j(t) a(\varphi_j, \varphi_k) = (f(t), \varphi_k),\ k = 1, 2, ..., N_h,$$

$$\alpha_k(0) = (u(0), \varphi_k).$$

In matrix notation, this may be expressed as

$$B\alpha'(t) + A\alpha(t) = b(t), for\ t > 0,$$

where $B = (b_{kj})$ is the mass matrix with $b_{kj} = (\varphi_j, \varphi_k),\ j, k = 1, 2, \cdots, N_h$, and $A = (a_{kj})$ is the stiffness matrix with $a_{kj} = a(\varphi_j, \varphi_k)$, and $b = (b_k)$ is the vector with $b_k = (f, \varphi_k)$.

We recall that the stiffness matrix A is symmetric positive definite and this holds also for the mass matrix B since $(\cdot, \cdot)$ is an inner product and φ_j for $j = 1, 2, \cdots, N_h$ belong to the finite element basis. In particular, B is invertible and therefore

$$B\alpha'(t) + A\alpha(t) = b(t),\ t > 0,$$

may be written as the linear ordinary differential system

$$\alpha'(t) + B^{-1}A\alpha(t) = B^{-1}b(t), \quad t > 0, \quad \alpha(0) \text{ is given},$$

which has a unique solution.

Since $u_h(t) \in S_h$, we may choose $\chi = u_h(t)$ in the semidiscrete problem, to obtain

$$(u_{h,t}, u_h) + a(u_h, u_h) = (f, u_h), \quad t > 0.$$

Note that the first term equals $\frac{1}{2}\frac{d}{dt}\|u_h\|^2$ and the second term is non-negative, so we get

$$\|u_h\|\frac{d}{dt}\|u_h\| = \frac{1}{2}\frac{d}{dt}\|u_h\|^2 \leq |(f, u_h)| \leq \|f\|\|u_h\|,$$

which implies that

$$\frac{d}{dt}\|u_h\| \leqslant \|f\|.$$

Integrating over $(0, t)$, we arrive at

$$\|u_h(t)\| \leqslant \|v_h\| + \int_0^t \|f\| ds.$$

To write

$$(u_{h,t}, \chi) + a(u_h, \chi) = (f, \chi), \quad \forall\, \chi \in S_h, \quad t > 0,$$

$$u_h(0) = v_h, \quad v_h \in S_h,$$

into an operator form, we shall introduce the discrete Laplacian

$$\Delta_h : S_h \to S_h,$$

defined by

$$(-\Delta_h \varphi, \chi) = a(\varphi, \chi), \quad \forall\, \varphi, \chi \in S_h.$$

Assume that

$$\Delta_h \varphi = \sum_{j=1}^{N_h} d_j \varphi_j,$$

we then have

$$\sum_{j=1}^{N_h} d_j(\varphi_j, \varphi_k) = -a(\varphi, \varphi_k), \quad k = 1, ..., N_h.$$

Since the matrix of this system is the positive definite mass matrix encountered above, the operator Δ_h is easily seen to be self adjoint and $-\Delta_h$ is positive definite in S_h with respect to the L^2 inner product, and well defined.

Let P_h denote the L^2 projection onto S_h satisfying

$$(f - P_h f, \chi) = 0, \quad \forall\, \chi \in S_h,$$

then we have

$$(u_{h,t} - \Delta_h u_h - P_h f, \chi) = 0, \quad \forall\, \chi \in S_h.$$

Note that the first factor is in S_h, so that χ may be chosen equal to it. It follows that

$$u_{h,t} - \Delta_h u_h = P_h f, \text{ for } t > 0, \text{ with } u_h(0) = v_h.$$

1.3.3 Error estimates of the spatially semi-discrete scheme

In this section, we consider the error estimates of the semidiscrete problem.

Theorem 5. *[85] Let u_h be the solution of*

$$(u_{h,t}, \chi) + a(u_h, \chi) = (f, \chi), \ \forall\, \chi \in S_h, \ t > 0,$$

$$u_h(0) = v_h,$$

where $v_h \in S_h$ *is some approximation of* v*. Let* u *be the solution of*

$$\begin{aligned} u_t - \Delta u &= f, \text{ in } D \times \mathbb{R}_+, \\ u &= 0, \quad \text{on } \partial D \times \mathbb{R}_+, \\ u(.,0) &= v, \quad \text{in } D. \end{aligned}$$

Then it holds that

$$\|u_h(t) - u(t)\| \leq \|v_h - v\| + Ch^2\left(\|v\|_2 + \int_0^t \|u_t\|_2 ds\right), \quad t \geq 0.$$

Here, we require as usual, that the solution of the continuous problem has the regularity assumed by the presence of the norms on the right of the previous error inequality.

Let $v \in H^2(D) \cap H_0^1(D)$. Let $v_h = I_h v$, where $I_h : C[a,b] \to S_h$ is the interpolation operator. Then there exists a constant $C > 0$ such that [85]

$$\|v_h - v\| \leq Ch^2\|v\|_2,$$

where $\|\cdot\|_2$ denotes the norm in $H^2(D)$, that is, $\|v\|_2 := \|v\|_{H^2(D)}$.

The same holds true if $v_h = P_h v$, where P_h is the orthogonal projection of $L^2(D)$ onto S_h. Note that this choice is the best approximation of v *in* S_h with respect to the L^2 -norm and so

$$\|P_h v - v\| \leq \|I_h v - v\| \leq Ch^2\|v\|_2, \forall\, v \in H^2(D) \cap H_0^1(D).$$

Another choice of v_h is $v_h = R_h v$, where R_h is the elliptic (or Ritz) projection onto S_h defined by

$$a(R_h v - v, \chi) = 0, \quad \forall\, \chi \in S_h, v \in H_0^1(D).$$

Thus, $R_h v$ is the finite element approximation of the solution of the elliptic problem whose exact solution is v.

We finally recall the error estimates

$$\|R_h v - v\| + h|R_h v - v|_1 \leq ch^s\|v\|_s, \quad \text{for } s = 1, 2. \tag{1.3.10}$$

Proof of Theorem 5. We write

$$u_h - u = (u_h - R_h u) + (R_h u - u) = \theta + \rho.$$

The second term is easily bounded using

$$\|R_h v - v\| + h|R_h v - v|_1 \le ch^2\|v\|_2,$$

so we have

$$\|\rho(t)\| \le ch^2\|u(t)\|_2 \le ch^2\Big(\|v\|_2 + \int_0^t \|u_t\| ds\Big).$$

Observe that the operator R_h commutes with time differentiation i.e

$$R_h u_t = (R_h u)_t.$$

Also

$$a(R_h v, \chi) = a(v, \chi),\ \forall \chi \in S_h.$$

In order to bound θ, we note that, $\forall \chi \in S_h$,

$$\begin{aligned}(\theta_t, \chi) + a(\theta, \chi) &= (u_{h,t}, \chi) + a(u_h, \chi) - (R_h u_t, \chi) - a(R_h u, \chi) \\ &= (f, \chi) - (R_h u_t, \chi) - a(u, \chi) = (u_t - R_h u_t, \chi) = -(\rho_t, \chi).\end{aligned} \tag{1.3.11}$$

Applying the stability estimate, we obtain

$$\|\theta(t)\| \le \|\theta(0)\| + \int_0^t \|\rho_t\| ds,$$

where

$$\|\theta(0)\| = \|v_h - R_h v\| \le \|v_h - v\| + \|R_h v - v\| \le \|v_h - v\| + ch^2\|v\|_2.$$

These estimates together with the estimate $\|\rho_t\| \le ch^2\|u_t\|_2$ complete the proof of Theorem 5. □

Theorem 6. *[85] Under the assumptions of Theorem 5, and for u sufficiently smooth, we have for $t \ge 0$*

$$|u_h(t) - u(t)|_1 \le |v_h - v|_1 + ch\Big(\|v\|_2 + \|u(t)\|_2 + \big(\int_0^t \|u_t\|_2^2 ds\big)^{\frac{1}{2}}\Big).$$

Proof. As before, we write the error in the form

$$u_h - u = (u_h - R_h u) + (R_h u - u) = \theta + \rho.$$

Recall that $\|R_h v - v\| + h|R_h v - v|_1 \leq ch^s \|v\|_s, \quad for\ s = 1, 2$. Thus we have

$$|\rho(t)|_1 = |R_h u(t) - u(t)|_1 \leq ch\|u(t)\|_2.$$

In order to estimate $\nabla\theta$ we set $\chi = \theta_t$, in

$$(\theta_t, \chi) + a(\theta, \chi) = -(\rho_t, \chi), \quad \forall\ \chi \in S_h.$$

and obtain

$$\|\theta_t\|^2 + \frac{1}{2}\frac{d}{dt}|\theta|_1^2 = -(\rho_t, \theta_t) \leq \frac{1}{2}(\|\rho_t\|^2 + \|\theta_t\|^2).$$

So, we get

$$\frac{d}{dt}|\theta|_1^2 \leq \|\rho_t\|^2,$$

and by integration, noting $\theta(0) = u_h(0) - R_h v = v_h - R_h v$,

$$|\theta(t)|_1^2 \leq |\theta(0)|_1^2 + \int_0^t \|\rho_t\|^2 ds \leq (|v_h - v|_1 + |R_h v - v|_1)^2 + \int_0^t \|\rho_t\|^2 ds.$$

Hence, since $a^2 + b^2 \leq (|a| + |b|)^2$ and in view of

$$\|R_h v - v\| + h|R_h v - v|_1 \leq ch^s \|v\|_s, \quad s = 1, 2,$$

and the estimate of $\|\rho_t\|$, we conclude that,

$$|\theta(t)|_1 \leq |v_h - v|_1 + ch\big(\|v\|_2 + (\int_0^t \|u_t\|_2^2 ds)^{\frac{1}{2}}\big).$$

The above estimates complete the proof of Theorem 6. □

1.3.4 Time discretization

In this section, we consider time discretization schemes. We shall turn our attention to some simple schemes for discretization with respect to the time variable.

Let S_h be the space of piece-wise linear finite element functions as before. We begin with the backward Euler-Galerkin method. Let k be the time step and $u^n \in S_h$ the approximation of $u(t)$ at $t = t_n = nk$. This method is defined by replacing the time derivative in

$$(u_{h,t}, \chi) + a(u_h, \chi) = (f, \chi), \quad \forall\ \chi \in S_h\ t > 0,$$
$$u_h(0) = v_h,$$

by a backward difference quotient given as

$$\bar{\partial} u^n = \frac{u^n - u^{n-1}}{k},$$

and obtain

$$\begin{aligned} (\bar{\partial} u^n, \chi) + a(u^n, \chi) = (f(t_n), \chi), \quad \forall\, \chi \in S_h,\ n \geq 1, \\ u^0 = v_h. \end{aligned} \tag{1.3.12}$$

Given u^{n-1} this defines u^n implicitly from the discrete elliptic problem

$$(u^n, \chi) + ka(u^n, \chi) = (u^{n-1} + kf(t_n), \chi), \quad \forall \chi \in S_h.$$

Expressing u^n in terms of the basis $\{\varphi_j\}_{j=1}^{N_h}$ as

$$u^n(x) = \sum_{j=1}^{N_h} \alpha_j^n \varphi_j(x),$$

we may write this equation in matrix notation as

$$B\alpha^n + kA\alpha^n = B\alpha^{n-1} + kb^n, \text{ for } n \geq 1,$$

where b^n denotes the vector with components $(f(t_n), \varphi_j), j = 1, 2, \ldots, N$, and α^n is the vector with components α_j^n defined by

$$\alpha^n = (B + kA)^{-1} B\alpha^{n-1} + k(B + kA)^{-1} b^n, \text{ for } n \geq 1, \text{ with } \alpha^0, \text{ is given.}$$

Here the existence of $(B + kA)^{-1}$ follows from the positivity of the matrices A and B [85].

We begin our analysis of the backward Euler method by showing that it is unconditionally stable, i.e., that it is stable independently of the relation between h and k.

Choosing $\chi = u^n$ in (1.3.12), we have, since $a(u^n, u^n) \geq 0$ that

$$(\bar{\partial} u^n, u^n) \leq \|f^n\| \|u^n\|, \text{ where } f^n = f(t_n),$$

or

$$\|u^n\|^2 - (u^{n-1}, u^n) \leq k\|f^n\| \|u^n\|.$$

Since

$$(u^{n-1}, u^n) \leq \|u^{n-1}\| \|u^n\|,$$

this shows that

$$\|u^n\| \le \|u^{n-1}\| + k\|f^n\|, \text{ for } n \ \ge 1,$$

and hence

$$\|u^n\| \le \|u^0\| + k\sum_{j=1}^{n} \|f^j\|. \qquad \text{(stability estimate)}$$

Theorem 7. *[85] Let u^n and u be the solutions of*

$$(\overline{\partial} u^n, \chi) + a(u^n, \chi) = (f(t_n), \chi), \quad \forall\ \chi \in S_h,\ n \ge 1$$
$$u^0 = v_h.$$

and

$$u_t - \Delta u = f, \quad \text{in } D \times \mathbb{R}_+,$$
$$u = 0, \quad \text{on } \partial D \times \mathbb{R}_+,$$
$$u(0) = v, \quad \text{in } D,$$

respectively, where v_h is chosen so that

$$\|v_h - v\| \le ch^2\|v\|_2.$$

Then for any $n = 1, 2, \cdots$, it holds that

$$\|u^n - u(t_n)\| \le ch^2\big(\|v\|_2 + \int_0^t \|u_t\|_2 ds\big) + ck\int_0^t \|u_{tt}\|ds.$$

Proof. We write

$$u^n - u(t_n) = (u^n - R_h u(t_n)) + (R_h u(t_n) - u(t_n)) = \theta^n + \rho^n.$$

As before, by the estimates for the Ritz projection in (1.3.10) we get

$$\|\rho^n\| \le Ch^2\|u(t_n)\|_2 \le Ch^2\big(\|v\|_2 + \int_0^t \|u_t\|_2 ds\big).$$

For θ^n, we have,

$$(\overline{\partial}\theta^n, \chi) + a(\theta^n, \chi) = -(w^n, \chi),$$

where

$$w^n = R_h\overline{\partial_t} u(t_n) - u_t(t_n) = (R_h - I)\overline{\partial} u(t_n) + (\overline{\partial} u(t_n) - u_t(t_n)) = w_1^n + w_2^n.$$

By the estimate of Riesz projection, we get

$$\|\theta^0\| = \|v_h - R_h v\| \leq \|v_h - v\| + \|v - R_h v\| \leq ch^2\|v\|_2.$$

Note now that

$$w_1^j = (R_h - I)k^{-1}\int_{t_{j-1}}^{t_j} u_t ds = k^{-1}\int_{t_{j-1}}^{t_j} (R_h - I)u_t ds,$$

which implies that

$$k\sum_{j=1}^{n} \|w_1^j\| \leq \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} ch^2\|u_t\|_2 ds = ch^2\int_0^{t_n} \|u_t\|_2 ds.$$

Further, by Taylor's formula, it follows that

$$w_2^j = k^{-1}(u(t_j) - u(t_{j-1})) - u(t_j) = -k^{-1}\int_{t_{j-1}}^{t_j} (s - t_{j-1})u_{tt}(s)ds,$$

and so

$$k\sum_{j=1}^{n} \|w_2^j\| \leq \|\int_{t_{j-1}}^{t_j} (s - t_{j-1})u_{tt}(s)ds\| \leq k\int_0^{t_n} \|u_{tt}\|ds.$$

These estimates complete the proof. □

Replacing the backward difference quotient with respect to time in (1.3.12), by the forward difference quotient

$$\partial u^n = \frac{(u^{n+1} - u^n)}{k},$$

we arrive at the forward Euler-Galerkin method which, in matrix form, may be expressed as

$$B\alpha^{n+1} = (B - kA)\alpha^n + kb^n, \text{for } n \geq 0.$$

Using the discrete Laplacian defined by

$$(-\Delta_h\varphi, \chi) = a(\varphi, \chi), \forall\varphi, \chi \in S_h,$$

the forward Euler method may also be given as

$$u^{n+1} = (I + k\Delta_h)u^n + kP_h f(t_n), \text{for } n \geq 0, \text{with } u^0 = v_h.$$

This method is not unconditionally stable unlike the backward Euler method. Note that because of the non-symmetric choice of the discretization in time, the method is only first order accurate in time.

We therefore now turn to the Crank-Nicolson Galerkin method, in which the semidiscrete equation is discretised in a symmetric manner around the point

$$t_{n-\frac{1}{2}} := \Big(n - \frac{1}{2}\Big)k,$$

which yields a method with second order of accuracy in time. More precisely, we define $u^n \in S_h$ recursively for $n \geq 1$, by

$$(\bar{\partial} u^n, \chi) + a(\frac{1}{2}(u^n + u^{n-1}), \chi) = (f(t_{n-\frac{1}{2}}), \chi), \quad \forall\, \chi \in S_h, \tag{1.3.13}$$
$$u^0 = v_h.$$

In matrix notation, it takes the form

$$B\alpha^n + \frac{1}{2}kA\alpha^n = B\alpha^{n-1} - \frac{1}{2}kA\alpha^{n-1} + kb^{n-\frac{1}{2}}, \text{for } n \geq 1,$$

with α^0 given, which yields

$$\alpha^n = (B + \frac{1}{2}kA)^{-1}(B - \frac{1}{2}kA)\alpha^{n-1} + k(B + \frac{1}{2}kA)^{-1}b^{n-\frac{1}{2}}, n \geq 1.$$

This method is also unconditionally stable which may be shown by choosing

$$\chi = u^n + u^{n-1}$$

in (1.3.13)

Note that

$$k(\bar{\partial} u^n, u^n + u^{n+1}) = \|u^n\|^2 - \|u^{n-1}\|^2 = (\|u^n\| - \|u^{n-1}\|)(\|u^n\| + \|u^{n-1}\|).$$

The positivity of $a(u^n, u^n)$ yields

$$\|u^n\| \leq \|u^{n-1}\| + k\|f^{n-\frac{1}{2}}\|, \text{where} \quad f^{n-\frac{1}{2}} = f(t_{n-\frac{1}{2}}),$$

and after summation,

$$\|u^n\| \leq \|v_h\| + k\sum_{j=1}^{n} \|f^{j-\frac{1}{2}}\|.$$

This shows the stability of the Crank-Nicolson scheme.

1.4 Basic definitions from stochastic processes

Definition 1.4.1. A set Ω containing all the simple events of a random experiment is called sample space.

Definition 1.4.2. Let $\Omega \neq \emptyset$. A collection $\mathcal{F}$ of subsets of Ω satisfying

1. the empty set $\emptyset$ belongs to $\mathcal{F}$,
2. if $A \in \mathcal{F}$ then its complement $A^c \in \mathcal{F}$,
3. if $A_1, A_2, \cdots \in \mathcal{F}$ then $A_1 \cup A_2 \cdots \in \mathcal{F}$,

is called σ-algebra on Ω.

Definition 1.4.3. A function P defined on a σ-algebra $\mathcal{F}$ satisfying

1. $P : \mathcal{F} \to [0, 1]$,
2. for any collection $A_1, A_2, \cdots$ of pairwise disjoint sets in $\mathcal{F}$ it holds that

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots ,$$

is called probability measure.

Definition 1.4.4. Let $\Omega \neq \emptyset$ be a sample space, $\mathcal{F}$ a σ-algebra on Ω, and $P : \mathcal{F} \to [0, 1]$ a probability measure defined on $\mathcal{F}$. The triple $(\Omega, \mathcal{F}, P)$ is called probability space.

Definition 1.4.5. A collection of random variables with index set $\mathcal{T}$ is called stochastic process and is denoted by

$$\{\xi(t) : \ t \in \mathcal{T}\} \ \text{ or } \ \{\xi_t : \ t \in \mathcal{T}\}.$$

Definition 1.4.6. Let $\{\xi(t) : \ t \in \mathcal{T}\}$ be a stochastic process taking values in a set H. For any fixed $w \in \Omega$, a path or sample path of ξ denotes the function $\xi(t) \equiv \xi(t; w)$, $t \in \mathcal{T}$, [31], [99].

Remark 8. When the process ξ takes real values, and $\mathcal{T} \subseteq \mathbb{R}$, then the path can be viewed as the graph of ξ as a function of t.

Definition 1.4.7. A stochastic process $\{\xi(t):\ t \in \mathbb{R}\}$ is called stationary if the mean value of $\xi(t)$ is independent of t, while for any $s, t \in \mathbb{R}$ the covariance of $\xi(t)$, $\xi(s)$ depends only on $t - s$, [99].

Definition 1.4.8. Let $\{\xi(t):\ t \in \mathcal{T}\}$ be an H-valued stochastic process, where H is a linear space, and let $t, s \in \mathcal{T}$. The new random variable $\xi(t) - \xi(s)$ is called increment of the process. Usually we consider $t \geq s$ when $\mathcal{T} \subseteq \mathbb{R}$.

Definition 1.4.9. We say that a stochastic process $\{\xi(t):\ t \in \mathcal{T}\}$ with values in a linear space H has stationary increments if for any $s, t \in \mathcal{T}$ the probability distribution function of the increment $\xi(t+h) - \xi(s+h)$ is independent of h for all h such that $s+h, t+h \in \mathcal{T}$, [31]. Here, we need to consider index sets $\mathcal{T}$ where some addition is defined, for example subsets of a linear space.

Definition 1.4.10. The smallest σ-algebra containing all the intervals of $\mathbb{R}$ (of finite or infinite length), is called a Borel σ-algebra on $\mathbb{R}$ and is denoted by $\mathcal{B}(\mathbb{R})$, while any set in $\mathcal{B}(\mathbb{R})$ is called a Borel set of $\mathbb{R}$. Note that an analogous definition can be given on $\mathbb{R}^n$, where we define $\mathcal{B}(\mathbb{R}^n)$ as the smallest σ-algebra containing all the Cartesian products in $\mathbb{R}^n$ of all intervals of $\mathbb{R}$. [99]

Definition 1.4.11. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A family $\{\mathcal{F}_t:\ t \geq 0\}$ of sub σ-algebras $\mathcal{F}_t$ of $\mathcal{F}$ is called filtration of $\mathcal{F}$, where a σ-algebra $\mathcal{F}_t$ on Ω satisfying $\mathcal{F}_t \subseteq \mathcal{F}$ is called sub σ-algebra of $\mathcal{F}$, while a family is defined as increasing when it satisfies for any $s < t$ that $\mathcal{F}_s \subseteq \mathcal{F}_t$. The quadruple $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is called filtered probability space, [99].

Definition 1.4.12. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space, a stochastic process $\{\xi(t):\ t \in [0, T]\}$ is called $\mathcal{F}_t$-adapted if for any $t \in [0, T]$ the random variable $\xi(t)$ is $\mathcal{F}_t$-measurable, [99].

Remark 9. If the process is real valued, then $\xi : [0, T] \to \mathbb{R}$ and the $\mathcal{F}_t$-measurability of $\xi(t)$ of the above definition is considered in the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, i.e., for any Borel set B of $\mathbb{R}$ in $\mathcal{B}(\mathbb{R})$ it holds that $\xi^{-1}(B) \in \mathcal{F}_t$. In case of an $\mathcal{F}_t$-adapted stochastic process ξ, it follows that the inverse image through ξ for example of the Borel set $B := (a, b)$ given by $\xi^{-1}(B) := \{w \in \Omega :\ a < \xi(s; w) < b\}$ that belongs to $\mathcal{F}_s$ it belongs also to $\mathcal{F}_t$ for any $t > s$ as well, and is thus $\mathcal{F}_t$-measurable also.

Definition 1.4.13. Let $(\Omega, \mathcal{F}, P)$ be the probability space on which the process $\{\xi(z) : z \in \mathbb{Z}\}$ is realized and define for $-\infty \leq m \leq l \leq \infty$ the sub σ-algebra $\mathcal{F}_{m,l} := \sigma\{\xi(z) : m \leq z \leq l\}$ generated by $\xi(z)$ for any $z \in \mathbb{Z}$ with $m \leq z \leq l$. If

$$\lim_{l\to\infty}\Big(\sup_{m\geq-\infty}\ \sup_{A\in\mathcal{F}_{-\infty,m},\ B\in\mathcal{F}_{m+l,\infty}}|P(A\cap B)-P(A)P(B)|\Big)=0,$$

the process ξ is called strongly mixing, see in [29], or in [28] for the case of index set $\mathcal{T} := \mathbb{N}$.

Remark 10. The previous definition can be extended for stochastic processes with index set $\mathcal{T} := \mathbb{R}$ or $\mathbb{R}^+$.
Let $(\Omega, \mathcal{F}, P)$ be the probability space on which the process $\{\xi(t) : \ t \geq 0\}$ is realized and define for $0 \leq s \leq t \leq \infty$ the sub σ-algebra $\mathcal{F}_{s,t} := \sigma\{\xi(r) : \ s \leq r \leq t\}$ generated by $\xi(r)$ for any $r \in [s, t]$. If $\lim_{t\to\infty}\Big(\sup_{s\geq 0}\ \sup_{A\in\mathcal{F}_{0,s},\ B\in\mathcal{F}_{s+t,\infty}}|P(A\cap B)-P(A)P(B)|\Big)=0$, the process ξ is called strongly mixing. Such a strongly mixing process appears in [57] in the definition of a mild noise. We will return in next chapter for the detailed definition of all the additional properties of this process and of the mild noise of [57].

Brownian motion has been introduced to describe the random movement of a particle in the water in the absence of friction. Its mathematical definition as a Wiener stochastic process is given in the sequel.

Definition 1.4.14. [31] A Wiener process (Brownian Motion) is defined as a real stochastic process

$$\{W(t) : \ t \in \mathcal{T} := [0, \infty)\},$$

such that

1. $W(0) = 0$ almost surely (a.s.),
2. the sample paths $t \to W(t; w)$ are almost surely (a.s.) continuous,
3. for any finite sequence of times $0 < t_1 < t_2 < \cdots < t_n$ and any Borel sets of $\mathbb{R}$ $A_1, A_2, \cdots, A_n$, it holds that

$$P(\{W(t_1) \in A_1 \text{ and } \ W(t_2) \in A_2 \ \ \cdots \ \ \text{and } \ W(t_n) \in A_n\}) :=$$
$$:= \int_{A_1}\int_{A_2}\cdots\int_{A_n} p(t_1, 0, x_1)p(t_2 - t_1, x_1, x_2)\cdots p(t_n - t_{n-1}, x_{n-1}, x_n)dx_n dx_{n-1}\cdots dx_1,$$

for p defined for any $x, y \in \mathbb{R}$ and any $t > 0$ by

$$p(t,x,y) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}},$$

the so-called transition density, and $P : \mathcal{B}(\mathbb{R}) \to [0,1]$ a probability measure defined on the Borel σ-algebra on $\mathbb{R}$.

Remark 11. By the above definition it follows that the transition density p of the Brownian Motion satisfies $p(t,x,y) = f(x-y)$ for f the density function of the Normal distribution $N(0,t)$ of mean value 0 and variance t. Also, the density function of $W(t)$ is given by $f(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-0)^2}{2t}}$, [31], which coincides with the density function uniquely defining the Normal distribution $N(0,t)$, and so $W(t)$ follows $N(0,t)$ and has thus variance t and mean 0.
Moreover, for any $t, s \geq 0$ the increment $W(t) - W(s)$ follows the Normal distribution $N(0, |t-s|)$, while for any $0 \leq s \leq r \leq t$ the increments $W(t) - W(r)$, $W(r) - W(s)$ have zero covariance and being normally distributed they are stochastically independent, [31].

In stochastic equations, the stochastic quantities appearing in their differential equations statement may be smooth, or non smooth and noisy. These are stochastic processes, or, in the noise case, may be defined through formal differentiation of a stochastic process corresponding after integration of the equation to a well defined stochastic integral.
Let us give some examples of such processes.
As it is known, [31], the Brownian Motion $W(t)$ is almost surely (a.s.) nowhere differentiable, in fact the values of its rate of change between s, t approach $\pm\infty$ as $|t-s| \to 0$. However, a basic choice for a noise term is $\dot{W}(t)$ denoting the formal derivative of W. A main idea implemented in the numerical approximation of this noise term is the use of the rate of change

$$\dot{W}(t) \simeq \frac{W(t^n) - W(t^{n-1})}{t^n - t^{n-1}},$$

for $t \in (t^{n-1}, t^n)$ where $0 = t^0 < t^1 < \cdots < t^{n-1} < t^n < \cdots < t^N = T$ is a partition of $[0,T]$. The increment property yields

$$\dot{W}(t) \simeq \frac{W(t^n - t^{n-1})}{t^n - t^{n-1}} \equiv \frac{W(1)}{\sqrt{t^n - t^{n-1}}} = h^{-1/2} W(1),$$

where $h := t^n - t^{n-1}$, and $W(1) \sim N(0,1)$. Computational realizations of pseudo-random numbers from the Standard Normal distribution $N(0,1)$ are in frequent use.

A classic definition of an infinite dimensional noise, cf. [45], is given by the formal differentiation of a Fourier Brownian series. Let γ_i be the eigenvalues of a positive definite symmetric operator Q taking values on a Hilbert space $H := L^2(D)$, and a complete orthonormal basis of H of eigenfunctions $\{e_i\}_{i=1}^{\infty}$. A Q-Wiener process $W_Q : D \times \mathbb{R}^+ \to H$ is defined by

$$W_Q(x,t) := \sum_{i=1}^{\infty} \gamma_i^{1/2} e_i(x)\beta_i(t),$$

for $\beta_i(t)$ stochastically independent Brownian Motions. An infinite dimensional noise is given by

$$\dot{W}_Q(x,t) = \sum_{i=1}^{\infty} \gamma_i^{1/2} e_i(x)\dot{\beta}_i(t),$$

for $\cdot$ denoting the formal differentiation in t. In practice such a noise is numerically approximated by cutting off first the series to the first N modes, for some N, and then proceed to some approximation of the N modes involving the Brownian Motions formal derivatives.

The concepts of mild noise and of mild noise approximation of a rough noise will be discussed in more detail in a later chapter.

1.5 Basic definitions from fractional calculus

In this section we will introduce some of the fundamental definitions of fractional derivatives and integrals, such as Riemann-Liouville integral, Riemann-Liouville fractional derivatives, Caputo derivative, etc. We will also discuss some theorems and facts related to fractional calculus that we will apply in our research.

1.5.1 Riemann-Liouville (R-L) fractional integral

Let $n \in \mathbb{Z}^+$. The operator J_a^n defined on $L^1(a,b)$ by

$$J_a^n f(t) := \frac{1}{\Gamma(n)} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau, \tag{1.5.1}$$

for $a \le t \le b$, is called the Riemann-Liouville fractional integral operator of order n.

For $n = 0$ we set $J_a^0 := I$, the identity operator and in this case the operator is quite convenient for further manipulations. Moreover, for $n \ge 1$ it is obvious that the integral

$J_a^n f(t)$ exists for every $t \in [a,b]$ because the integrand is the product of an integrable function f and the continuous function $(t-\cdot)^{n-1}$,[50]. We may extend (1.5.1) for any $n \in \mathbb{R}^+$, one of the most important property of Riemman-Liouville integral is as follows

Theorem 12. *[50] Let $\alpha, \beta \geq 0$ and $f \in L^1(a,b)$. Then*

$$J_a^\alpha J_a^\beta f = J_a^{\alpha+\beta} f \tag{1.5.2}$$

holds almost everywhere on $[a,b]$. If additionally $f \in C[a,b]$ or $\alpha+\beta \geq 1$, then the identity holds everywhere on $[a,b]$.

1.5.2 Riemann-Liouville fractional derivative

Suppose $p > 0$ we define the following Riemann-Liouville fractional derivative as [50]

$${}_0^R D_t^p f(t) = D^n [{}_0^R D_t^{p-n} f(t)] = D^n \frac{1}{\Gamma(n-p)} \int_0^t (t-\tau)^{n-p-1} f(\tau) d\tau, \quad p > 0 \tag{1.5.3}$$

where $D^n = \frac{d^n}{dt^n}$ and $n-1 < p < n$. Recall that $D^n = \frac{d^n}{dt^n}$ is the derivative part while $[{}_0^R D_t^{p-n} f(t)] = J_0^{n-p} f(t)$ is Riemann-Liouville integral part.

Example 13. *Suppose $f(t) = t^2$, find the value of ${}_0^R D_t^{\frac{1}{2}} f(t)$?*

Solution: *Here $p = \frac{1}{2}$ and lies on the interval $0 < p < 1$ such that $n = 1$. Using (1.5.3) gives*

$${}_0^R D_t^{\frac{1}{2}} f(t) = D^1 [{}_0^R D_t^{-\frac{1}{2}} f(t)] = \frac{d}{dt} \left[\frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-\tau)^{-\frac{1}{2}} \tau^2 d\tau \right] \tag{1.5.4}$$

1.5.3 Caputo fractional derivative

Suppose $n-1 < p < n$ and $p > 0$ we define the following Caputo's fractional derivative as [50]

$${}_0^C D_t^p f(t) = {}_0^C D_t^{p-n} [D^n f(t)] = \frac{1}{\Gamma(n-p)} \int_0^t (t-\tau)^{n-p-1} \left[D^n f(\tau) \right] d\tau, \tag{1.5.5}$$

Example 14. *Suppose $f(t) = t^2$, find the value of ${}_0^C D_t^{\frac{1}{2}} f(t)$?*

Solution: *Here $p = \frac{1}{2}$ and lies on the interval $0 < p < 1$ such that $n = 1$. Using (1.5.5) gives*

$$ {}_0^C D_t^{\frac{1}{2}} f(t) = {}_0^C D_t^{\frac{1}{2}-1}[D^1 f(t)] = \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-\tau)^{-\frac{1}{2}} \left[\frac{d}{d\tau} f(\tau)\right] d\tau \tag{1.5.6}$$

Remark 15. Suppose $p > 0$ and $n - 1 < p < n$, then the relation between Riemman-Liouville and Caputo fractional derivative can be expressed by the theorem [50] below.

Theorem 16. *Let $p > 0$ and $n - 1 < p < n$, we have,*

$$ {}_0^R D_t^p f(t) = {}_0^C D_t^p f(t) + \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(-p+k+1)} t^{k-p} \tag{1.5.7}$$

Proof. We only consider the case for $n = 1$ and $0 < p < 1$

$$\begin{aligned}
{}_0^R D_t^p f(t) &= D^1[{}_0^R D_t^{p-1} f(t)] = \frac{d}{dt}\left[\frac{1}{\Gamma(1-p)} \int_0^t (t-\tau)^{-p} f(\tau) d\tau\right] \\
&= \frac{d}{dt}\left(\frac{1}{\Gamma(1-p)} \left[-f(\tau)\frac{(t-\tau)^{-p+1}}{-p+1}\right]_{\tau=0}^{\tau=t} + \int_0^t \frac{(t-\tau)^{-p+1}}{-p+1} f'(\tau) d\tau\right) \\
&= \frac{d}{dt}\left(\frac{1}{\Gamma(1-p)} \left[f(0)\frac{t^{1-p}}{1-p} + \int_0^t \frac{(t-\tau)^{1-p}}{1-p} f'(\tau) d\tau\right]\right) \\
&= \frac{1}{\Gamma(1-p)} f(0) t^{-p} + \frac{d}{dt}\frac{1}{\Gamma(1-p)} \int_0^t \frac{(t-\tau)^{1-p}}{1-p} f'(\tau) d\tau \\
&= \frac{1}{\Gamma(1-p)} f(0) t^{-p} + \frac{1}{\Gamma(1-p)} \int_0^t \left[\frac{\partial}{\partial t}\left(\frac{(t-\tau)^{1-p}}{1-p} f'(\tau)\right)\right] d\tau \\
&= \frac{1}{\Gamma(1-p)} f(0) t^{-p} + \frac{1}{\Gamma(1-p)} \left[\int_0^t (t-\tau)^{-p} f'(\tau) d\tau\right] \\
&= {}_0^C D_t^p f(t) + \frac{1}{\Gamma(1-p)} f(0) t^{-p}
\end{aligned}$$

Similarly, we can prove the case for $n - 1 < p < n$, $n > 1$, i.e,

$$ {}_0^R D_t^p f(t) = {}_0^C D_t^p f(t) + \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(-p+k+1)} t^{k-p} \tag{1.5.8}$$

□

Chapter 2

The Discontinuous in time Galerkin method

The Discontinuous Galerkin method was first introduced in [113, 88]. Jamet, [68], analyzed a discontinuous in the time variable Galerkin method for parabolic equations when posed in a variable domain. Later, Delfour et al. in 1981, [48], used the method for the discretization of ordinary differential equations. Such schemes construct approximate solutions as piecewise polynomial functions of degree at most $n-1$ [6, 26, 27, 42, 5, 67, 41, 110]. For the combination of the method with refinement and adaptivity, see [26], [27]. In [20], the scheme of [68] was applied on the linear Schrödinger equation of under-water acoustics on naval environments of variable topography. Considering Discontinuous Galerkin methods on nonlinear problems, like compressible flows, and compressible Navier–Stokes equations, see in [122, 123], or in [15] for the stochastic Allen-Cahn equation. We also refer to the results of [119, 120] for equations from linear elasticity and Navier-Stokes equations, and in [103] for scalar hyperbolic conservation laws. See also in [19] the *a posteriori* error analysis of the scheme of [68].

In this chapter, we present the scheme of [68] for parabolic equations, the main arguments for the existence and uniqueness of numerical solution, stability, and some error estimates. In [15], the space-time discontinuous in time Galerkin method was introduced for the $\varepsilon-$dependent stochastic Allen-Cahn equation with mild noise. We will discuss briefly the nonlinear scheme and some of the results proven there such as existence, uniqueness, and the abstract error of the numerical solution. Such discontinuous in time schemes are fully discrete, and involve adaptive high order finite element methods with space-time variational formulation and partition. So any Runge-Kutta method or finite difference approximation in time is avoided. Improved temporal accuracy can be achieved by elevating the order of the piecewise polynomial approximation over time. Within each time subinterval, it is also possible to employ high-order piecewise polynomials across multiple spatial variables to approximate the solution, resulting in enhanced spatial convergence. Also, the initial condition of the continuous problem coincides with the initial condition of the discrete scheme and therefore we do not need to approximate the initial value.

2.1 Approximation of the parabolic equation

Let D be a bounded domain in $\mathbb{R}^d$, and consider the parabolic equation,

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u = f, \qquad x \in D, \quad 0 \leq t \leq T, \\
u = 0, \quad x \in \partial D, \\
u(0) = u_0, \quad x \in D,
\end{aligned}
\tag{2.1.1}
$$

for $f \in L^2(D \times (0,T))$, [68]. The term f is inserted so that the analysis will cover the more general non-homogeneous problem which, among other, is useful for testing the efficiency of the numerical scheme for exact solutions leaving a residual f to the linear homogeneous problem. Let $\tau \leq t$, we denote by $(\cdot,\cdot)_{(\tau,t)\times D}$ the L^2 inner product on $(\tau,t)\times D$ and by $\|\cdot\|_{(\tau,t)\times D}$ the induced L^2 norm. In the same sense $(\cdot,\cdot)_D$ will denote the L^2 inner product on D and $\|\cdot\|_D$ the induced L^2 norm.

We also consider a partition in time $0 = t_0 < t_1 < \cdots < t_N = T$.

The variational form of (2.1.1) is given as follows: we seek $u : (0,T)\times D \to \mathbb{R}$ in H^1 where

$$H^1 := H^1((0,T)\times D) = \{u \in L^2((0,T)\times D) : u_t,\ \nabla u \in L^2((0,T)\times D)\},$$

such that for all $n = 0, 1, 2, ..., N-1$,

$$\Big(\frac{\partial u}{\partial t}, \varphi\Big)_{(t_n,t_{n+1})\times D} + \big(\nabla u, \nabla \varphi\big)_{(t_n,t_{n+1})\times D} = (f,\varphi)_{(t_n,t_{n+1})\times D}, \quad \forall\, \varphi \in H^1. \tag{2.1.2}$$

Integrating the first term of the variational form by parts with respect to time t, we have

$$
\begin{aligned}
(\frac{\partial u}{\partial t},\varphi)_{(t_n,t_{n+1})\times D} &= \int\limits_D \int\limits_{t_n}^{t_{n+1}} \frac{\partial u}{\partial t}\varphi dtdx = \int\limits_D \Big(\varphi u|_{t_n}^{t_{n+1}} - \int\limits_{t_n}^{t_{n+1}} \frac{\partial \varphi}{\partial t} udt\Big)dx \\
&= \int\limits_D u(t_{n+1})\varphi(t_{n+1})dx - \int\limits_D u(t_n)\varphi(t_n)dx - \int\limits_D \int\limits_{t_n}^{t_{n+1}} u\frac{\partial \varphi}{\partial t} dtdx \\
&= \big(u(t_{n+1}),\varphi(t_{n+1})\big)_D - \big(u(t_n),\varphi(t_n)\big)_D - \big(u,\frac{\partial \varphi}{\partial t}\big)_{(t_n,t_{n+1})\times D}.
\end{aligned}
$$

Substituting $\big(\frac{\partial u}{\partial t},\varphi\big)_{(t_n,t_{n+1})\times D}$ into (2.1.2), we have,

$$
\begin{aligned}
-\big(u,\frac{\partial \varphi}{\partial t}\big)_{(t_n,t_{n+1})\times D} &+ (\nabla u,\nabla\varphi)_{(t_n,t_{n+1})\times D} + (u(t_{n+1}),\varphi(t_{n+1}))_D \\
&- (u(t_n),\varphi(t_n))_D = (f,\varphi)_{(t_n,t_{n+1})\times D},\ \forall\ \varphi\in H^1.
\end{aligned}
\tag{2.1.3}
$$

We define now the bilinear form

$$
\begin{aligned}
B_n(u,\varphi) = &- \big(u,\frac{\partial \varphi}{\partial t}\big)_{(t_n,t_{n+1})\times D} + (\nabla u,\nabla\varphi)_{(t_n,t_{n+1})\times D} \\
&+ (u(t_{n+1}),\varphi(t_{n+1}))_D - (u(t_n),\varphi(t_n+0))_D,
\end{aligned}
$$

where $\varphi(t_n+0) := \lim_{\epsilon\to 0^+}\varphi(t_n+\epsilon)$. Thus, in the variational formulation of (2.1.1) we seek $u\in H^1$ with $u(0)=u_0$, such that for all $n=0,1,2,...,N-1$,

$$
B_n(u,\varphi) = (f,\varphi)_{(t_n,t_{n+1})\times D},\ \forall\ \varphi\in H^1. \tag{2.1.4}
$$

2.1.1 The Discontinuous Galerkin Method

The finite element space V_h will consist of continuous in space functions $v_h(t,x)$ defined on $[0,T]\times D$ piece-wisely at each sub-interval of the time partition (for example as polynomials) that may be discontinuous in time at the nodal points t_n.

Let $v_h^{n+0} := \lim_{\epsilon\to 0^+} v_h(t_n+\epsilon,x)$, $v_h^{n-0} = \lim_{\epsilon\to 0^+} v_h(t_n-\epsilon,x)$. We assume that $v_h(t,x)$ is left continuous on each t_n, $n=1,2,\cdots,N$ i.e., $v_h(t_n,x)=v_h^{n-0}$, $n=1,2,\cdots,N$.

The discontinuous Galerkin method: we seek $u_h\in V_h$, with $u_h(0)=u_0$, such that

$$
B_n(u_h,\chi) = (f,\chi)_{(t_n,t_{n+1})\times D}, \qquad \forall\ \chi\in V_h, \tag{2.1.5}
$$

where

$$
\begin{aligned}
B_n(u_h,\chi) = &- (u_h,\frac{\partial \chi}{\partial t})_{(t_n,t_{n+1})\times D} + (\nabla u_h,\nabla\chi)_{(t_n,t_{n+1})\times D} \\
&+ (u_h^{n+1},\chi^{n+1})_D - (u_h^{n},\chi^{n+0})_D.
\end{aligned}
$$

Let $\chi = u_h$. We have

$$\begin{aligned}B_n(u_h,u_h) =& -(u_h,\frac{\partial u_h}{\partial t})_{(t_n,t_{n+1})\times D} + (\nabla u_h,\nabla u_h)_{(t_n,t_{n+1})\times D}\\ &+ (u_h^{n+1},u_h^{n+1})_D - (u_h^n,u_h^{n+0})_D.\end{aligned}$$

Note that

$$\begin{aligned}\big(u_h,\frac{\partial u_h}{\partial t}\big)_{(t_n,t_{n+1})\times D} &= \int_D \big(u_h(t_{n+1})u_h(t_{n+1}) - u_h^{n+0}u_h^{n+0}\big)dx - \int_D\int_{t_n}^{t_{n+1}} u_h\frac{\partial u_h}{\partial t}dtdx\\ &= \big(u_h(t_{n+1}),u_h(t_{n+1})\big)_D - \big(u_h^{n+0},u_h^{n+0}\big)_D - (u_h,\frac{\partial u_h}{\partial t})_{(t_n,t_{n+1})\times D}\\ &= \|u_h^{n+1}\|_D^2 - \|u_h^{n+0}\|_D^2 - (u_h,\frac{\partial u_h}{\partial t})_{(t_n,t_{n+1})\times D},\end{aligned}$$

which implies

$$2(u_h,\frac{\partial u_h}{\partial t})_{(t_n,t_{n+1})\times D} = \|u_h^{n+1}\|_D^2 - \|u_h^{n+0}\|_D^2.$$

Thus, we arrive at

$$(u_h,\frac{\partial u_h}{\partial t})_{(t_n,t_{n+1})\times D} = \frac{1}{2}\|u_h^{n+1}\|_D^2 - \frac{1}{2}\|u_h^{n+0}\|_D^2.$$

Substituting this into the bilinear form to obtain:

$$B_n(u_h,u_h) = -\frac{1}{2}\big(\|u_h^{n+1}\|_D^2 - \|u_h^{n+0}\|_D^2\big) + (\nabla u_h,\nabla u_h)_{(t_n,t_{n+1})\times D} + \|u_h^{n+1}\|_D^2 - (u_h^n,u_h^{n+0})_D.$$

An application of

$$(u_h^n,u_h^{n+0})_D = \frac{1}{2}\|u_h^n\|_D^2 + \frac{1}{2}\|u_h^{n+0}\|_D^2 - \frac{1}{2}\|u_h^{n+0} - u_h^n\|_D^2$$

yields

$$\begin{aligned}B_n(u_h,u_h) =& -\frac{1}{2}\big[\|u_h^{n+1}\|_D^2 - \|u_h^{n+0}\|_D^2\big] + (\bigtriangledown u_h,\bigtriangledown u_h)_{(t_n,t_{n+1})\times D}\\ &+ \|u_h^{n+1}\|_D^2 - \frac{1}{2}\|u_h^n\|_D^2 - \frac{1}{2}\|u_h^{n+0}\|_D^2 + \frac{1}{2}\|u_h^{n+0} - u_h^n\|_D^2\\ =&(\bigtriangledown u_h,\bigtriangledown u_h)_{(t_n,t_{n+1})\times D} + \frac{1}{2}\|u_h^{n+1}\|_D^2 - \frac{1}{2}\|u_h^n\|_D^2 + \frac{1}{2}\|u_h^{n+0} - u_h^n\|_D^2.\end{aligned} \tag{2.1.6}$$

2.1.2 Stability

Theorem 17. *[68] Let u_h be the solution of* (2.1.1). *Then there exists a constant $C > 0$ such that for all $n = 1, 2, \cdots, N$*

$$\|\nabla u_h\|_{(0,t_n)\times D}^2 + \frac{1}{2}\|u_h^n\|_D^2 \le C\big(\|u^0\|_D^2 + \|f\|_{(0,t_n)\times D}^2\big).$$

Proof. Note that for $n \le N-1$

$$B_n(u_h,\chi) = (f,\chi)_{(t_n,t_{n+1})\times D}, \quad \forall\, \chi \in V_h. \tag{2.1.7}$$

Letting $\chi = u_h$, we get

$$(\nabla u_h, \nabla u_h)_{(t_n,t_{n+1})\times D} + \frac{1}{2}\|u_h^{n+1}\|^2 - \frac{1}{2}\|u_h^n\|^2 + \frac{1}{2}\|u_h^{n+0} - u_h^n\|^2 = (f,u_h)_{(t_n,t_{n+1})\times D}.$$

Summing from $n=0$ to $N-1$ we get

$$\begin{aligned}\sum_{n=0}^{N-1}(\nabla u_h, \nabla u_h)_{(t_n,t_{n+1})\times D} &+ \frac{1}{2}\sum_{n=0}^{N-1}\left(\|u_h^{n+1}\|^2 - \|u_h^n\|^2\right)\\ &+ \frac{1}{2}\sum_{n=0}^{N-1}\|u_h^{n+0} - u_h^n\|^2 = \sum_{n=0}^{N-1}(f,u_h)_{(t_n,t_{n+1})\times D}\end{aligned} \tag{2.1.8}$$

Thus we obtain

$$\|\nabla u_h\|^2_{(0,t_N)\times D} + \frac{1}{2}\|u_h^N\|^2 \le \frac{1}{2}\|u_h^0\|^2 + (f,u_h)_{(0,t_N)\times D}.$$

By Cauchy-Schwarz inequality, we have, with some suitable chosen small $\varepsilon > 0$,

$$\begin{aligned}(f,u_h)_{(0,t_N)\times D} &\le \|f\|_{(0,t_N)\times D}\|u_h\|_{(0,t_N)\times D} \le C\|f\|_{(0,t_N)\times D}\|\nabla u_h\|_{(0,t_N)\times D}\\ &\le C_\varepsilon\|f\|^2_{(0,t_N)\times D} + \varepsilon\|\nabla u_h\|^2_{(0,t_N)\times D},\end{aligned}$$

where we used the Poincaré inequality, i.e., $\|u_h\|_D \le C\|\nabla u_h\|_D$. Thus, it follows that

$$\|\nabla u_h\|^2_{(0,t_N)\times D} + \frac{1}{2}\|u_h^N\|^2 \le \frac{1}{2}\|u_h^0\|^2 + C_\varepsilon\|f\|^2_{(0,t_N)\times D} + \varepsilon\|\nabla u_h\|^2_{(0,t_N)\times D}.$$

The application of a kick-back argument (that is, moving the term $\varepsilon\|\nabla u_h\|^2_{(0,t_N)\times D}$ in the right side to the left side to get the estimates of $\|\nabla u_h\|^2_{(0,t_N)\times D}$) yields, e.g., with $\varepsilon = 1/2$,

$$\|\nabla u_h\|^2_{(0,t_N)\times D} + \frac{1}{2}\|u_h^N\|^2 \le C\left(\|u_h^0\|^2 + \|f\|^2_{(0,t_N)\times D}\right),$$

and the proof of theorem is then complete. □

2.1.3 Error estimate

Theorem 18. *[68] Let u be the solution of* (2.1.1)*, and u_h the solution of* (2.1.7)*. It holds that*

$$\begin{aligned}&\|\nabla(u-u_h)\|_{(0,t_N)\times D}+\frac{1}{\sqrt{2}}\|u^N-u_h^N\|_D\\ \leq & C\Big(\sum_{n=0}^{N-1}\Big\|\frac{\partial}{\partial t}(u-\chi)\Big\|^2_{(t_n,t_{n+1})\times D}\Big)^{\frac{1}{2}}+C\|\nabla(u-\chi)\|_{(0,t_N)\times D}\\ &+C\max_{1\leq n\leq N}\|u^n-\chi\|_D+2\sum_{n=1}^{N-1}\|\chi^{n+0}-\chi^n\|_D,\quad \forall\,\chi\in V_h.\end{aligned}$$

Remark 19. Theorem 18 establishes error bounds for $\|\nabla(u-u_h)\|_{(0,t_N)\times D}$ and $\|u^N - u_h^N\|_D$. These error bounds depend on the choice of $\chi \in V_h$, where χ represents an arbitrary function. To derive a priori error estimates for $\|\nabla(u-u_h)\|_{(0,t_N)\times D}$ and $\|u^N-u_h^N\|_D$, one can opt for a specific $\chi \in V_h$. As an illustrative example, selecting χ as the interpolation function of u is a viable choice.

To prove Theorem 18, we need the following
lemma, see [68, Lemma 4.1].

Lemma 2.1.1. Let a^n and b^n, $1 \leq n \leq N$, be two sequences of nonnegative real numbers which satisfy

$$(a^n)^2+(b^n)^2\leq \alpha a^n+\beta b^n+\sum_{\kappa=1}^{n-1}\gamma^\kappa b^\kappa, \tag{2.1.9}$$

where, α, β and γ^κ for $1 \leq \kappa \leq N-1$ are nonnegative real numbers. Then

$$a^n+b^n\leq\sqrt{2}\big(\alpha+\beta\sum_{\kappa=1}^{n-1}\gamma^\kappa\big),$$

[68].

Proof. Let $c^n = \big((a^n)^2+(b^n)^2\big)^{\frac{1}{2}}$. Then (2.1.9) yields

$$(c^n)^2\leq(\alpha+\beta)c^n+\sum_{\kappa=1}^{n-1}\gamma^\kappa c^\kappa,\quad \text{for } 1\leq n\leq N.$$

Let $d^n, 1 \leq n \leq N$, satisfy

$$(d^n)^2\leq(\alpha+\beta)d^n+\sum_{\kappa=1}^{n-1}\gamma^\kappa d^\kappa,\quad for\ 1\leq n\leq N. \tag{2.1.10}$$

We have $c^1 \leq d^1 = \alpha + \beta$ and $d^n \geq \alpha + \beta$ for all n. By mathematical induction we prove that $c^n \leq d^n$ for all n: assume $c^\kappa \leq d^\kappa$ for $\kappa = 1, 2, \cdots, n-1$; then $g(c^n) \leq g(d^n)$ with $g(y) = y^2 - (\alpha+\beta)y$; since $g(y)$ is increasing for $y > \alpha + \beta$, we deduce $c^n \leq d^n$. On the other hand, we have $g(d^n) \leq g(d^{n+1})$ for all n, therefore $d^n \leq d^{n+1}$ for all n and (2.1.10) yields after replacing d^κ *by* d^n,

$$d^n \leq \alpha + \beta + \sum_{\kappa=1}^{n-1} \gamma^\kappa. \tag{2.1.11}$$

Finally, we have $a^n + b^n \leq \sqrt{2}c^n \leq \sqrt{2}d^n$, which completes the proof of the lemma. □

We now turn to the proof of Theorem 18.

Proof of Theorem 18. Note that $u \in H^1$, and satisfies

$$B_n(u, \varphi) = (f, \varphi)_{(t_n, t_{n+1})\times D}, \quad \forall\, \varphi \in H^1, \tag{2.1.12}$$

while $u_h \in V_h$, and satisfies

$$B_n(u_h, \chi) = (f, \chi)_{(t_n, t_{n+1})\times D}, \quad \forall\, \chi \in V_h, \tag{2.1.13}$$

where

$$\begin{aligned} B_n(u, \varphi) = &- (u, \frac{\partial \varphi}{\partial t})_{(t_n, t_{n+1})\times D} \\ &+ (\nabla u, \nabla \varphi)_{(t_n, t_{n+1})\times D} + (u^{n+1}, \varphi^{n+1})_D - (u^n, \varphi^{n+0})_D, \quad \forall\, \varphi \in H^1, \end{aligned}$$

and

$$\begin{aligned} B_n(u_h, \chi) = &- (u_h, \frac{\partial \chi}{\partial t})_{(t_n, t_{n+1})\times D} \\ &+ (\nabla u_h, \nabla \chi)_{(t_n, t_{n+1})\times D} + (u_h^{n+1}, \chi^{n+1})_D - (u_h^n, \chi^{n+0})_D, \quad \forall\, \chi \in V_h. \end{aligned}$$

Subtracting (2.1.13) from (2.1.12), we arrive at

$$B_n(u - u_h, \chi) = 0, \quad \forall\, \chi \in V_h,$$

which is called the orthogonality of the discontinuous Galerkin Method. We then have

$$B_n(u - u_h, u - u_h) = B_n(u - u_h, u - \chi + \chi - u_h) = B_n(u - u_h, u - \chi) + B_n(u - u_h, \chi - u_h).$$

By orthogonality, it holds that

$$B_n(u - u_h, \chi - u_h) = 0, \forall\, \chi \in V_h, u_h \in V_h.$$

By the definition of B_n, we have,

$$\begin{aligned} B_n(u-u_h, u-\chi) =& - \big(u-u_h, \frac{\partial}{\partial t}(u-\chi)\big)_{(t_n,t_{n+1})\times D} \\ &+ \big(\nabla(u-u_h), \nabla(u-\chi)\big)_{(t_n,t_{n+1})\times D} \\ &+ \big((u-u_h)^{n+1}, (u-\chi)^{n+1}\big)_D - \big((u-u_h)^n, (u-\chi)^{n+0}\big)_D, \end{aligned}$$

and, noting that

$$(v, v_t)_{(t_n,t_{n+1}\times D)} = \frac{1}{2}\big(\|v^{n+1}\|^2 - \|v^{n+0}\|^2\big),$$

we get

$$\begin{aligned} B_n(u-u_h, u-u_h) =& - \big(u-u_h, \frac{\partial}{\partial t}(u-u_h)\big)_{(t_n,t_{n+1})\times D} \\ &+ \big(\nabla(u-u_h), \nabla(u-u_h))_{(t_n,t_{n+1})\times D} \\ &+ \big((u-u_h)^{n+1}, (u-u_h)^{n+1}\big)_D - \big((u-u_h)^n, (u-u_h)^{n+0}\big)_D \\ =& \|\nabla(u-u_n)\|^2_{(t_n,t_{n+1})\times D} + \frac{1}{2}\|(u-u_h)^{n+1}\|^2_D - \frac{1}{2}\|(u-u_h)^n\|^2_D \\ &+ \frac{1}{2}\|(u-u_h)^{n+0} - (u-u_h)^n\|^2_D. \end{aligned}$$

Summing from $n=0$ to $N-1$ we get, using the kick-back inequality,

$$\begin{aligned} &\|\nabla(u-u_h)\|^2_{(0,t_N)\times D} + \frac{1}{2}\|u^N - u_h^N\|^2_D - \frac{1}{2}\|u^0 - u_h^0\|^2_D + \frac{1}{2}\sum_{n=0}^{N-1}\|u_h^{n+0} - u_h^n\|^2_D \\ &= \sum_{n=0}^{N-1}(u-u_h, \frac{\partial}{\partial t}(u-\chi))_{(t_n,t_{n+1})\times D} + \sum_{n=0}^{N-1}(\nabla(u-u_h), \nabla(u-\chi))_{(t_n,t_{n+1})\times D} \\ &+ (u^N - u_h^N, u^N - \chi^N)_D - (u^0 - u_h^0, u^0 - \chi^0)_D + \sum_{n=0}^{N-1}(u^n - u_h^n, \chi^{n+0} - \chi^n)_D =: \mathcal{A} \end{aligned}$$

and, using the inequality $\sum_{n=0}^{N-1} a_n b_n \leq (\sum_{n=0}^{N-1} a_n^2)^{\frac{1}{2}} (\sum_{n=0}^{N-1} b_n^2)^{\frac{1}{2}}$,

$$\begin{aligned} \mathcal{A} \leq& \Big(\sum_{n=0}^{N-1}\|u-u_h\|^2_{(t_n,t_{n+1})\times D}\Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{N-1}\|\frac{\partial}{\partial t}(u-\chi)\|^2_{(t_n,t_{n+1})\times D}\Big)^{\frac{1}{2}} \\ &+ (\nabla(u-u_h), \nabla(u-\chi))_{(0,t_n)\times D} + (u^N - u_h^N, u^n - \chi^N)_D - 0 \\ &+ \sum_{n=0}^{N-1}(u^n - u_h^n, \chi^{n+0} - \chi^n)_D, \end{aligned}$$

where in the last inequality we used that $u_h^0 = u^0$ in (2.1.5). An application of Cauchy-Schwarz inequality yields

$$\mathcal{A} \le \|u - u_h\|_{(0,t_N)\times D}\Big(\sum_{n=0}^{N-1} \|\frac{\partial}{\partial t}(u-\chi)\|^2_{(t_n,t_{n+1})\times D}\Big)^{\frac{1}{2}}$$

$$+ \|\nabla(u-u_h)\|_{(0,t_N)\times D}\|\nabla(u-\chi)\|_{(0,t_N)\times D}$$

$$+ \|u^N - u_h^N\|_D \times \|u^N - \chi^N\|_D + \sum_{n=0}^{N-1} \|u^n - u_h^n\|_D\|\chi^{n+0} - \chi^n\|_D.$$

We now define

$$a^N = \|\nabla(u-u_h)\|_{(0,t_N)\times D},$$

$$b^N = \frac{1}{\sqrt{2}}\|u^N - u_h^N\|_D,$$

$$\alpha^N = \Big(\sum_{n=0}^{N-1} \|\frac{\partial}{\partial t}(u-\chi)\|^2_{(t_n,t_{n+1})\times D}\Big)^{\frac{1}{2}} + \|\nabla(u-\chi)\|_{(0,t_N)\times D},$$

$$\beta^N = \sqrt{2}\max_{1\le n\le N}\|u^N - \chi^N\|_D,$$

$$\gamma^N = \sqrt{2}\|\chi^{N+0} - \chi^N\|_D,$$

and obtain

$$(a^N)^2 + (b^N)^2 \le \alpha^N a^N + \beta^N b^N + \sum_{n=0}^{N-1}\gamma^n b^n.$$

An application of Lemma 4.1 in [68] yields

$$a^N + b^N \le \sqrt{2}\Big(a^N + \beta^N + \sum_{n=0}^{N-1}\gamma^n\Big).$$

This completes the rest of the proof. □

2.2 Approximation of the stochastic Allen-Cahn

The Allen-Cahn equation is a nonlinear reaction-diffusion equation that models the phase separation of multi-component mixtures. This equation finds applications in diverse fields such as materials science and, more recently, mathematical biology. It governs the temporal evolution of scalar state variables, such as the concentration of one of the phases, as documented in various references [10, 7, 9, 41, 59, 64, 110, 113, 24, 137, 25, 30, 8]. The introduction of noise into the system can be attributed to sources like thermal fluctuations, the system's free energy, or impurities within the mixture. Meanwhile, the nonlinearity in the equation is characterized by the derivative of a double equal-well potential. In this section, we delve into some of the findings presented in [15], focusing on the numerical approximation of the stochastic ε-Allen-Cahn equation. This equation exhibits mild noise and involves nonlinearity in both the problem formulation and the discontinuous time-stepping scheme.

2.2.1 The Allen-Cahn equation

The Allen-Cahn equation falls within the category of Ginzburg-Landau equations, which are employed to describe phase transition phenomena in materials science. It captures the dynamic evolution of phase concentrations within a binary alloy undergoing phase separation. Imagine a two-phase mixture confined within a vessel D which, for instance, could be a substance transitioning from a melted state to two well-separated phases due to forced homogenization. As the alloy departs from equilibrium, phase evolution commences. Rapidly, transitional layers form around the phases, thinning over time. The parameter $\varepsilon > 0$ characterizes the width of these layers. As ε diminishes, signifying increasingly thinner layers, the evolution decelerates. It's at this juncture that the ε-dependent Allen-Cahn equation takes effect.

The "sharp interface limit" emerges as $\varepsilon \to 0$, marking the point where the layers reduce to infinitely sharp interfaces with zero width, and the solution adopts a two-valued step function. Here, our focus shifts to the shape and behavior of these interfaces over extended periods, delineating the regions where concentration takes on distinct values. In the context of the Allen-Cahn equation, the sharp interface limit problem pertains to the evolution of the sharp interfaces' velocity, influenced by their mean curvature.

The phases of the phase separation process can be summarized as follows:

Homogenization: During this phase, the concentration remains constant. Spinodal Decomposition: As $\varepsilon > 0$, intricate snake-like patterns emerge. Coarsening: Sharp interfaces evolve, and as ε approaches zero, this phase is reached. Equilibrium: The system ultimately settles into an equilibrium state. The physical system is often open and may incorporate additional factors such as thermal fluctuations, external fields, mass supply, or impurities within the alloy. These factors are typically modeled in the equation through deterministic forcing or additive/multiplicative noise components, as described in [18, 64].

The Allen-Cahn equation, a third-order nonlinear equation, differs from the Cahn-Hilliard equation, which is fourth-order, in its mass-preserving behavior. However, this mass conservation property can be achieved by considering a modified version of the Allen-Cahn equation that incorporates an average integral term. In scenarios involving binary alloys where both phases exhibit a proclivity to separate, the nonlinearity $-f$ in the Allen-Cahn equation is defined as the derivative of a double equal-well potential F. A common choice for this potential is:

$$-f = F'(u), \quad F(u) := \frac{1}{4}(1 - w^2)^2.$$

The physical scale of this problem typically involves $\varepsilon \ll 1$. When we delve into numerical methods, it's imperative to incorporate this parameter into the formulation of the continuous problem and, consequently, into the numerical scheme. Ensuring a rigorous error analysis that takes into account the influence of ε on the numerical error is essential. This allows us to steer clear of various schemes that are prone to significant rounding machine errors when $\varepsilon \ll 1$ is a significant factor.

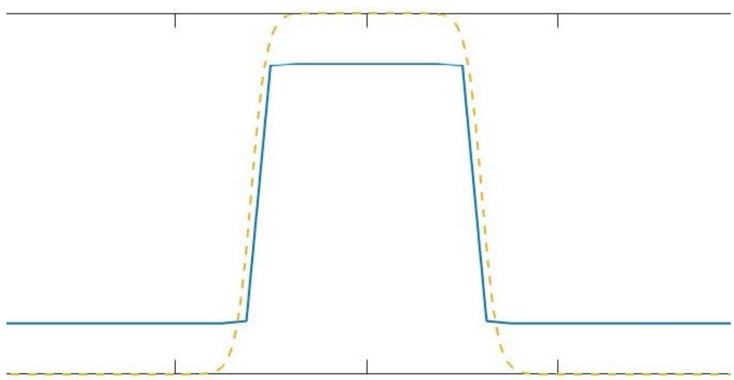

Figure 2.2.1: The solution of the one-dimensional Allen-Cahn equation with two transitional layers. Then dash line denotes the two-layered initial condition.

Remark 20. We stress that the discontinuous in time Galerkin method has been successfully applied for the numerical approximation of the linear Heat and linear Schrödinger equations [68, 20, 19], and more recently for the stochastic Allen-Cahn equation [15]. It admits a high order of accuracy and it is adaptive, and continuous in space. The solution of the ε-Allen-Cahn equation is continuous in space, however on the sharp interface limit as $\varepsilon \to 0$ the solution is discontinuous and ε enables the description of very steep layers just before discontinuity in space occurs (near the sharp-interface limit). The discontinuity of the scheme in time permits adaptivity in case of new layers generation or annihilation.

2.2.2 The problem

Let D be a bounded domain in $\mathbb{R}^d$ and consider the following ε-dependent stochastic Allen-Cahn equation with additive noise and a Neumann boundary condition [15]

$$\begin{aligned}\frac{\partial w}{\partial t}(t,x) - \Delta w = \frac{f(w)}{\varepsilon^2} + \frac{\dot{W}(t,x;\varepsilon)}{\varepsilon}, \quad t \in (0,T], \ \ x \in D, \qquad (2.2.1)\\ \frac{\partial w}{\partial n} = 0, \ x \in \partial D, \ t \in (0,T],\\ w(x,0) = w_0(x), \ x \in D,\end{aligned}$$

where $f(w) = w - w^3$. The ε-dependent noise $\dot{W}(t,x;\varepsilon)$ is mild, being smooth in space and in time, but rapidly oscillating for $\varepsilon \ll 1$ and tending to a white noise in time at the sharp interface when $\varepsilon \to 0^+$. The small parameter $\varepsilon > 0$ gives the order of the width of the transitional layers in D.

We summarize the properties of the smooth in space mild noise $\dot{W}^\varepsilon(x,t) = \dot{W}(x,t;\varepsilon)$ from [57, 87, 129, 15]. Let $0 < \gamma < \frac{1}{3}$, and define

$$\dot{W}^\varepsilon(x,t) := \varepsilon^{-\gamma}\xi(x, \varepsilon^{-2\gamma}t), \quad x \in D, \quad t > 0,$$

where $\xi(x,t)$ denotes a stationary and strongly mixing stochastic process in t on a probability space $(\Omega, \mathcal{F}, P)$, defined according to the Definitions 1.4.7, 1.4.13, and this of Remark 10, [57, 87, 15]. We also assume that there exists a deterministic constant M independent of ε such that $|\xi| \leq M$, $|\dot{\xi}| \leq M$ uniformly for any $x \in D$, and any $t \in [0,T]$, almost surely (a.s.), where $\dot{\xi} := \frac{d\xi}{dt}$, while $E[\xi] = 0$, [57, 87, 15]. Observe that $|\dot{W}^\varepsilon| \leq c\varepsilon^{-\gamma} \leq c\varepsilon^{-\frac{1}{3}}$, uniformly for any $x \in D$, and any $t \in [0,T]$ almost surely (a.s.), and that the noise $\dot{W}^\varepsilon(x,t)$ is at least one-time differentiable in time a.s. Considering the smoothness of the noise in space which can be as high as we wish, we shall assume that the noise is in $C^\infty(\overline{D})$.

When the noise depends only on t, an alternative definition for $\dot{W}^\varepsilon$, see in [129], is given by the differentiation of an approximated Brownian motion W^ε in time. In detail, let $\rho : \mathbb{R} \to \mathbb{R}^+$, be compactly supported and symmetric around zero, satisfying $\int_{-\infty}^{\infty} \rho(x)dx = 1$, and vanishing outside $[-1, 1]$. Let $W(t) \sim N(0, t)$ for any $t \geq 0$ be a Brownian motion, and consider a stochastically independent to $W(t)$ Brownian motion $\widetilde{W}(t)$ defined for any $t \geq 0$. The domain of definition of W is extended to the negative axes by setting $W(t) := \widetilde{W}(-t)$ for any $t < 0$, and so $(W(t), \mathbb{R})$ is a Gaussian process, [129]. Then $W^\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ is defined as the convolution

$$W^\varepsilon(t) := \int_{-\infty}^{\infty} \rho^\varepsilon(t-s)W(s)ds,$$

for $\rho^\varepsilon := \varepsilon^{-\gamma}\rho$, and $0 < \gamma < \frac{2}{3}$. Then W^ε approximates the Brownian motion $W(t)$ for any $t \geq 0$, [129]. The time derivative $\dot{W}^\varepsilon$ admits higher regularity than the minimum regularity assumed in [57].

Let $w = e^{b(\varepsilon)t}u$, for an arbitrarily large $b(\varepsilon)$ chosen to satisfy $(b(\varepsilon) - \varepsilon^{-2} - c_0) > 0$ and c_0 is some constant obtained by the application of Young's inequality on the integration of the noise with respect to time. Then, it follows that $\frac{\partial w}{\partial t} = b(\varepsilon)e^{b(\varepsilon)t}u + e^{b(\varepsilon)t}u_t$, and the equation (2.2.1) reduces to

$$b(\varepsilon)e^{b(\varepsilon)t}u + e^{b(\varepsilon)t}u_t = \Delta u e^{b(\varepsilon)t} + \frac{e^{b(\varepsilon)t}u - (e^{b(\varepsilon)t}u)^3}{\varepsilon^2} + \frac{\dot{W}(x,t;\varepsilon)}{\varepsilon}.$$

Thus, the problem is transformed to

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) - \Delta u &= -b(\varepsilon)u + \frac{g(u,\varepsilon;t)}{\varepsilon^2} + \frac{m(\varepsilon,t)\dot{W}(t,x;\varepsilon)}{\varepsilon}, \quad x \in D, \\ \frac{\partial u}{\partial n} &= 0, \quad x \in \partial D, \\ u(x,0) &= u_0(x), \quad x \in D, \end{aligned} \tag{2.2.2}$$

where

$$g(u,\varepsilon;t) := u - e^{2b(\varepsilon)t}u^3, \quad m(\varepsilon,t) := e^{-b(\varepsilon)t}.$$

2.2.3 The Discontinuous Galerkin scheme

Let $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ be a partition of $[0,T]$, and let, as in the previous sections, V_h consist of continuous in space functions $v_h(t,x)$ defined on $[0,T] \times D$ piecewisely at each sub-interval of the time partition that may be discontinuous in time at t_n,

$n \geq 1$. The discontinuous Galerkin method for (2.2.2): we seek $u_h \in V_h$, with $u_h(0) = u_0$, such that for $0 \leq n \leq N-1$

$$B_n(u_h, \chi) = (\tilde{f}, \chi)_{(t_n, t_n+1)\times D}, \ \forall \, \chi \in V_h, \tag{2.2.3}$$

where

$$B_n(u_h, \chi) = -\Big(u_h, \frac{\partial \chi}{\partial t}\Big)_{(t_n, t_{n+1})\times D} + (\nabla u_h, \nabla \chi)_{(t_n, t_{n+1})\times D} + b(\varepsilon)(u_h, \chi)_{(t_n, t_{n+1})\times D}$$
$$- \varepsilon^{-2}(u_h, \chi)_{(t_n, t_{n+1})\times D} + \varepsilon^{-2}(e^{2b(\varepsilon)t}u_h^3, \chi)_{(t_n, t_{n+1})\times D} + (u_h^{n+1}, \chi^{n+1}) - (u_h^n, \chi^{n+0}),$$

and $\tilde{f} := \varepsilon^{-1} m(\varepsilon, t)\dot{W}$.

We note that the scheme is nonlinear, cf. the term u_h^3 at the right hand-side.

2.2.4 Existence-Uniqueness

Lemma 2.2.1. Let $u_h \in V_h$ be the solution of (2.2.2), it holds that

for any $c_0 > 0$ as small as we wish

$$(b(\varepsilon) - \varepsilon^{-2} - c_0)\|u_h\|^2_{(0,t_n)\times D} + \|\nabla u_h\|^2_{(0,t_n)\times D} + \varepsilon^{-2}\sum_{i=0}^{n-1}(e^{2b(\varepsilon)t}u_h^3, u_h)_{(t_i, t_{i+1})\times D} + \frac{1}{2}\|u_h^n\|^2_D$$
$$\leq \frac{1}{2}\|u_0\|^2_D + \frac{\varepsilon^{-2}}{4c_0}\|\dot{W}\|^2_{(0,t_n)\times D} \quad n = 1, 2, 3, \cdots, N.$$

Proof. Step 1: By (2.2.3), we have,

$$B_n(u_h, \chi) = (\tilde{f}, \chi)_{(t_n, t_{n+1})\times D}, \quad \forall \, \chi \in V_h.$$

Set $\chi = u_h$, and obtain

$$B_n(u_h, u_h) = -\,(u_h, \frac{\partial u_h}{\partial t})_{(t_n, t_{n+1})\times D} + (\nabla u_h, \nabla u_h)_{(t_n, t_{n+1})\times D} + b(\varepsilon)(u_h, u_h)_{(t_n, t_{n+1})\times D}$$
$$- \varepsilon^{-2}(u_h, u_h)_{(t_n, t_n+1)\times D} + \varepsilon^{-2}(e^{2b(\varepsilon)t}u_h^3, u_h)_{(t_n, t_{n+1})\times D}$$
$$+ (u_h^{n+1}, u_h^{n+1})_D - (u_h^n, u_h^{n+0})_D.$$

Integrating by parts the first term, and using that

$$(u_h^n, u_h^{n+0}) = \frac{1}{2}\|u_h^n\|^2_D + \frac{1}{2}\|u_h^{n+0}\|^2_D + \frac{1}{2}\|u_h^{n+0} - u_h^0\|^2_D,$$

yields

$$
\begin{aligned}
B_n(u_h,u_h) =& -\frac{1}{2}\big[\|u_h^{n+1}\|_D^2 - \|u_h^{n+0}\|_D^2\big] + (\nabla u_h, \nabla u_h)_{(t_n,t_{n+1})\times D} \\
&+ b(\varepsilon)(u_h,u_h)_{(t_n,t_{n+1})\times D} - \varepsilon^{-2}(u_h,u_h)_{(t_n,t_{n+1})\times D} + \varepsilon^{-2}(e^{2b(\varepsilon)t}u_h^3,u_h)_{(t_n,t_{n+1})\times D} \\
&+ (u_h^{n+1},u_h^{n+1})_D - (u_h^n,u_h^{n+0})_D \\
=& -\frac{1}{2}[\|u_h^{n+1}\|_D^2 - \|u_h^{n+0}\|_D^2] + (\nabla u_h,\nabla u_h)_{(t_n,t_{n+1})\times D} + b(\varepsilon)(u_h,u_h)_{(t_n,t_{n+1})\times D} \\
&- \varepsilon^{-2}(u_h,u_h)_{(t_n,t_{n+1})\times D} + \varepsilon^{-2}(e^{2b(\varepsilon)t}u_h^3,u_h)_{(t_n,t_{n+1})\times D} + \|u_h^{n+1}\|_D^2 \\
&- \frac{1}{2}\|u_h^n\|_D^2 - \frac{1}{2}\|u_h^{n+0}\|_D^2 + \frac{1}{2}\|u_h^{n+0} - u_h^0\|_D^2,
\end{aligned}
$$

and finally

$$
\begin{aligned}
B_n(u_h,u_h) =& \|\nabla u_h\|_{(t_n,t_{n+1})\times D}^2 + b(\varepsilon)\|u_h\|_{(t_n,t_{n+1})\times D}^2 - \varepsilon^{-2}\|u_h\|_{(t_n,t_{n+1})\times D}^2 \\
&+ \varepsilon^{-2}(e^{2b(\varepsilon)t}u_h^3,u_h)_{(t_n,t_{n+1})\times D} \quad + \frac{1}{2}\|u_h^{n+1}\|_D^2 - \frac{1}{2}\|u_h^n\|_D^2 + \frac{1}{2}\|u_h^{n+0} - u_h^n\|_D^2,
\end{aligned}
$$

where

$$
B_n(u_h,u_h) = (\tilde{f},u_h)_{(t_n,t_{n+1})\times D} = (\varepsilon^{-1}m(\varepsilon,t)\dot{W},u_h)_{(t_n,t_{n+1})\times D}.
$$

Step 2: Summing from $n=0$ to $N-1$, we get,

$$
\begin{aligned}
&(b(\varepsilon)-\varepsilon^{-2})\int_0^{t_N}\|u_h\|_D^2dt + \int_0^{t_N}\|\nabla u_h\|_D^2dt + \varepsilon^{-2}\sum_{n=0}^{N-1}(e^{2b(\varepsilon)t}u_h^3,u_h)_{(t_n,t_{n+1})\times D} \\
&+ \frac{1}{2}\|u_h^N\|_D^2 \le \frac{1}{2}\|u^0\|_D^2 + \varepsilon^{-1}\int_0^{t_N}\|\dot{W}\|_D\|u_h\|_D dt.
\end{aligned}
$$

By using that for any $c_0>0$ it holds that for arbitrary $a,b\in\mathbb{R}$, $ab = 2(\sqrt{c_0}a)(\frac{1}{2\sqrt{c_0}}b) \le c_0a^2 + \frac{1}{4c_0}b^2$, which is applied on the term $\|u_h\|_D\varepsilon^{-1}\|\dot{W}\|_D$ of the above, for $a := \|u_h\|_D$, $b := \varepsilon^{-1}\|\dot{W}\|_D$, we obtain

$$
\begin{aligned}
&(b(\varepsilon)-\varepsilon^{-2}-c_0)\int_0^{t_N}\|u_h\|_D^2dt + \int_0^{t_N}\|\nabla u_h\|_D^2dt + \varepsilon^{-2}\sum_{n=0}^{N-1}(e^{2b(\varepsilon)t}u_h^3,u_h)_{(t_n,t_{n+1})\times D} \\
&+ \frac{1}{2}\|u_h^N\|_D^2 \le \frac{1}{2}\|u^0\|_D^2 + \frac{\varepsilon^{-2}}{4c_0}\int_0^{t_N}\|\dot{W}\|_D^2.
\end{aligned}
$$

This completes the proof of Lemma 2.2.1. □

Remark 21. When linear numerical schemes are considered, obviously uniqueness establishes existence as well. In contrast, this is not the case when the existence of the nonlinear scheme is analyzed. [85]

In order to ease notation, let us denote the $L^2(D)$ inner product by $(\cdot,\cdot)$ coinciding to $(\cdot,\cdot)_D$ of the notation used so far for the discontinuous Galerkinn (DG) schemes. We also denote the $L^2(D)$-norm by $\|\cdot\|$ coinciding to $\|\cdot\|_D$, used in the previous sections.

Theorem 22. *There exists a unique solution of the discontinuous Galerkin method* (2.2.3)*.*

Proof. Step 1: Uniqueness.
Let D and z be solutions of (2.2.3), then for any $0 \le n \le N-1$ and for any $\chi \in V_h$ it holds that

$$\begin{aligned}
&-\int_{t_n}^{t_{n+1}} (D,\chi')dt + \int_{t_n}^{t_{n+1}} (\nabla D,\nabla\chi)dt + (D^{n+1},\chi^{n+1}) - (D^n,\chi^{n+0}) \\
&+ b(\varepsilon)\int_{t_n}^{t_{n+1}} (D,\chi)dt - \varepsilon^{-2}\int_{t_n}^{t_{n+1}} (D,\chi)dt + \varepsilon^{-2}\int_{t_n}^{t_{n+1}} (e^{2b(\varepsilon)t}D^3,\chi)dt = \int_{t_n}^{t_{n+1}} (\tilde{f},\chi)dt,
\end{aligned} \tag{2.2.4}$$

and

$$\begin{aligned}
&-\int_{t_n}^{t_{n+1}} (z,\chi')dt + \int_{t_n}^{t_{n+1}} (\nabla z,\nabla\chi)dt + (z^{n+1},\chi^{n+1}) - (z^n,\chi^{n+0}) \\
&+ b(\varepsilon)\int_{t_n}^{t_{n+1}} (z,\chi)dt - \varepsilon^{-2}\int_{t_n}^{t_{n+1}} (z,\chi)dt + \varepsilon^{-2}\int_{t_n}^{t_{n+1}} (e^{2b(\varepsilon)t}z^3,\chi)dt = \int_{t_n}^{t_{n+1}} (\tilde{f},\chi)dt.
\end{aligned} \tag{2.2.5}$$

Subtracting equation (2.2.5) from (2.2.4), it follows

$$\begin{aligned}
&(b(\varepsilon)-\varepsilon^{-2})\int_{t_n}^{t_{n+1}} (D-z,\chi)dt + \int_{t_n}^{t_{n+1}} (\nabla(D-z),\nabla\chi)dt + \varepsilon^{-2}\int_{t_n}^{t_{n+1}} (e^{2b(\varepsilon)t}(D^3-z^3),\chi)dt \\
&+ (D^{n+1}-z^{n+1},\chi^{n+1}) - (D^n-z^n,\chi^{n+0}) - \int_{t_n}^{t_{n+1}} (D-z,\chi')dt = 0.
\end{aligned}$$

Setting $\chi = D - z$, we obtain

$$(b(\varepsilon)-\varepsilon^{-2})\int_{t_n}^{t_{n+1}}\|D-z\|^2dt+\int_{t_n}^{t_{n+1}}\|\nabla(D-z)\|^2dt+\varepsilon^{-2}\int_{t_n}^{t_{n+1}}(e^{2b(\varepsilon)t}(D^3-z^3),(D-z))dt$$

$$+(D^{n+1}-z^{n+1},D^{n+1}-z^{n+1})-(D^n-z^n,D^n-z^{n+0})-\int_{t_n}^{t_{n+1}}(D-z,(D-z)')dt=0.$$

Thus we get,

$$(b(\varepsilon)-\varepsilon^{-2})\int_{t_n}^{t_{n+1}}\|D-z\|^2dt+\int_{t_n}^{t_{n+1}}\|\nabla(D-z)\|^2dt+\varepsilon^{-2}\int_{t_n}^{t_{n+1}}(e^{2b(\varepsilon)t}(D^3-z^3),(D-z))dt$$

$$+\|D^{n+1}-z^{n+1}\|^2-\frac{1}{2}\|D^n-z^n\|^2+\frac{1}{2}\|(D^{n+0}-z^{n+0})-(D^n-z^n)\|^2=0.$$

Summation for all n yields

$$(b(\varepsilon)-\varepsilon^{-2})\int_0^{t_N}\|D-z\|^2dt+\int_0^{t_N}\|\nabla(D-z)\|^2dt$$

$$+\varepsilon^{-2}\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}(e^{2b(\varepsilon)t}(D^3-z^3),(D-z))dt$$

$$+\frac{1}{2}\|D^N-z^N\|^2-\frac{1}{2}\|D^0-z^0\|^2+\frac{1}{2}\sum_{n=0}^{N-1}\|(D^{0+}-z^{0+})-(D^0-z^0)\|^2=0. \quad (2.2.6)$$

Since $b(\varepsilon)-\varepsilon^{-2}>0$, the difference between the two solutions equals to zero, which implies uniqueness of solution.

Step 2: Existence

In [15], an operator $\phi : V_h \to V_h$ is defined, which satisfies for all $0 \le i \le N-1$ and all $\chi \in V_h$

$$\int_{t_i}^{t_{i+1}}(\phi(\chi),\chi)=\frac{1}{2}\|\chi^{i+0}-\chi^i\|^2+(b(\varepsilon)-\varepsilon^{-2})\int_{t_i}^{t_{i+1}}\|\chi\|^2dt$$

$$+\int_{t_i}^{t_{i+1}}\|\nabla\chi\|^2dt+\varepsilon^{-2}\int_{t_i}^{t_{i+1}}(e^{2b(\varepsilon)t}(\chi)^3,\chi)dt$$

$$-\varepsilon^{-1}\int_{t_i}^{t_{i+1}}(e^{2b(\varepsilon)t}\dot{W},\chi)dt+\frac{1}{2}\|\chi^{i+1}\|^2-\frac{1}{2}\|\chi^i\|.$$

By summation for all $i \le n-1$, it follows that for all $n \le N$

$$\int_0^{t_n}(\phi(\chi),\chi)dt\ge(b(\varepsilon)-\varepsilon^{-2}-c_0)\int_0^{t_n}\|\chi\|^2dt-\frac{1}{2}\|u_0\|^2-\frac{\varepsilon^{-2}}{4c_0}\int_0^{t_n}\|\dot{W}\|^2dt.$$

The right hand-side of the above inequality is strictly positive for these $\chi \in V_h$ with properly defined values in the $L^2((0,t^n) \times D)$-norm depending only on n, the initial data and the mild noise. Brouwer's fixed point Theorem is then applied for the operator ϕ that establishes existence of numerical solution; see in [15] the detailed proof of existence. □

2.2.5 Error estimates

We present briefly the error estimate holding true when the finite element space V_h is generic, [15].

Theorem 23. *Let u and u_h be the solutions of* (2.2.2) *and* (2.2.3) *respectively, then for any $\chi \in V_h$, we have*

$$(b(\varepsilon) - \varepsilon^{-2} - c_0) \int_0^{t_N} \|u - u_h\|^2 dt + \frac{1}{2} \int_0^{t_N} \|\nabla(u - u_h)\|^2 dt$$

$$+ \frac{1}{2} \sum_{n=0}^{N-1} \|u_h^{n+0} - u_h^n\|^2 + \frac{1}{8} \max_{1 \le n \le N} \|u(t_n) - u_h^n\|^2$$

$$\le (-b(\varepsilon) + \varepsilon^{-2})^2 \int_0^{t_N} \|u - \chi\|^2 dt + (\sum_{n=0}^{N-1} \|\chi^{n+0} - \chi^n\|^2)^2$$

$$+ \max_{1 \le n \le N} \|u(t_n) - \chi^n\|^2 + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \|\partial_t(u - \chi)\|^2 dt + \int_0^{t_N} \|\nabla(u - \chi)\|^2 dt$$

$$+ \varepsilon^{-2-\gamma} \max_{0 \le n \le T} (e^{2b(\varepsilon)t}) \|u - \chi\|^2_{L^4((0,t_N) \times D)}.$$

Proof. Step 1: Error equation.

Let us denote the error as $e := u - u_h$, and its time derivative $\partial_t e$ by e'.

We have

$$- \int_{t_n}^{t_{n+1}} (e, e')dt + \int_{t_n}^{t_{n+1}} (\nabla e, \nabla e)dt + (-b(\varepsilon) + \varepsilon^{-2}) \int_{t_n}^{t_{n+1}} (e, e)dt$$

$$+ (e^{n+1}, e^{n+1}) - (e^n, e^{n+0}) + B^n = - \int_{t_n}^{t_{n+1}} (e, (u - u_h)')dt + \int_{t_n}^{t_{n+1}} (\nabla e, \nabla(u - u_h))dt$$

$$- (-b(\varepsilon) + \varepsilon^{-2}) \int_{t_n}^{t_{n+1}} (e, (u - u_h))dt + (e^{n+1}, u^{n+1} - u_h^{n+1}) - (e^n, u^n - u_h^{n+0}),$$

where

$$B^n = \varepsilon^{-2} \int_{t_n}^{t_{n+1}} (e^{2b(\varepsilon)t}((u)^3 - (u_h)^3), e)dt - \varepsilon^{-2} \int_{t_n}^{t_{n+1}} (e^{2b(\varepsilon)t}((u)^3 - (u_h)^3), (u - u_h))dt. \tag{2.2.7}$$

In fact, u satisfies

$$- \int_{t_n}^{t_{n+1}} (u, \partial_t \chi)dt + \int_{t_n}^{t_{n+1}} (\nabla u, \nabla \chi)dt - (-b(\varepsilon) + \varepsilon^{-2}) \int_{t_n}^{t_{n+1}} (u, \chi)dt$$

$$+ \varepsilon^{-2} \int_{t_n}^{t_{n+1}} (e^{2b(\varepsilon)t} u^3, \chi)dt + (u^{n+1}, \chi^{n+1}) - (u^n, \chi^{n+0}) = \varepsilon^{-1} \int_{t_n}^{t_{n+1}} (m\dot{W}, \chi)dt,$$

while the same equation holds for u_h in place of u, i.e.,

$$- \int_{t_n}^{t_{n+1}} (u_h, \partial_t \chi)dt + \int_{t_n}^{t_{n+1}} (\nabla u_h, \nabla \chi)dt - (-b(\varepsilon) + \varepsilon^{-2}) \int_{t_n}^{t_{n+1}} (u_h, \chi)dt$$

$$+ \varepsilon^{-2} \int_{t_n}^{t_{n+1}} (e^{2b(\varepsilon)t} (u_h)^3, \chi)dt + (u_h^{n+1}, \chi^{n+1}) - (u_h^n, \chi^{n+0}) = \varepsilon^{-1} \int_{t_n}^{t_{n+1}} (m\dot{W}, \chi)dt.$$

Thus we obtain by subtraction

$$0 = - \int_{t_n}^{t_{n+1}} (e, \partial_t \chi)dt + \int_{t_n}^{t_{n+1}} (\nabla e, \nabla \chi)dt - (-b(\varepsilon) + \varepsilon^{-2}) \int_{t_n}^{t_{n+1}} (e, \chi)dt$$

$$+ \varepsilon^{-2} \int_{t_n}^{t_{n+1}} (e^{2b(\varepsilon)t}((u)^3 - (u_h)^3), \chi)dt + (e^{n+1}, \chi^{n+1}) - (e^n, \chi^{n+0}).$$

Note that $\chi = e + (\chi - u) + u_h$, we arrive at

$$0 = - \int_{t_n}^{t_{n+1}} (e, \partial_t e)dt + \int_{t_n}^{t_{n+1}} (\nabla e, \nabla e)dt - (-b(\varepsilon) + \varepsilon^{-2}) \int_{t_n}^{t_{n+1}} (e, e)dt$$

$$+ \varepsilon^{-2} \int_{t_n}^{t_{n+1}} (e^{2b(\varepsilon)t}((u)^3 - (u_h)^3), e)dt + (e^{n+1}, e^{n+1}) - (e^n, e_+^n)$$

$$+ \int_{t_n}^{t_{n+1}} (e, \partial_t (u - \chi))dt - \int_{t_n}^{t_{n+1}} (\nabla e, \nabla (u - \chi))dt + (-b(\varepsilon) + \varepsilon^{-2}) \int_{t_n}^{t_{n+1}} (e, u - \chi)dt$$

$$- \varepsilon^{-2} \int_{t_n}^{t_{n+1}} (e^{2b(\varepsilon)t}((u)^3 - (u_h)^3), u - \chi)dt - (e^{n+1}, u^{n+1} - \chi^{n+1}) + (e^n, u^n - \chi_+^n) + 0.$$

Step 2: Notice that $\int\limits_{t_n}^{t_{n+1}} (e, e')dt = \frac{1}{2}[(e^{n+1}, e^{n+1}) + (e^{n+0}, e^{n+0})]$. Let us denote for simplicity the time derivative $\partial_t(u - \chi)$ by $(u - \chi)'$. We then obtain

$$\begin{aligned}
&(b(\varepsilon) - \varepsilon^{-2}) \int\limits_{t_n}^{t_{n+1}} \|e\|^2 dt + \int\limits_{t_n}^{t_{n+1}} \|\nabla e\|^2 dt \\
&+ \frac{1}{2}\|e^{n+1}\|^2 - \frac{1}{2}\|e^n\|^2 + \frac{1}{2}\|u_h^{n+0} - u_h^n\|^2 + B^n \\
&= - \int\limits_{t_n}^{t_{n+1}} (e, (u - \chi)')dt + \int\limits_{t_n}^{t_{n+1}} (\nabla e, \nabla(u - \chi))dt \\
&+ (b(\varepsilon) - \varepsilon^{-2}) \int\limits_{t_n}^{t_{n+1}} (e, u - \chi)dt + (e^{n+1}, u^{n+1} - \chi^{n+1}) \\
&- (e^n, u^n - \chi^n) - (e^n, u^n - \chi^n) + (e^n, u^n - \chi^n),
\end{aligned} \tag{2.2.8}$$

where

$$B^n = \varepsilon^{-2} \int\limits_{t_n}^{t_{n+1}} (e^{2b(\varepsilon)t}((u)^3 - (u_h)^3), e)dt - \varepsilon^{-2} \int\limits_{t_n}^{t_{n+1}} (e^{2b(\varepsilon)t}((u)^3 - (u_h)^3), u - \chi)dt.$$

Summing for $n = 0, \cdots, N - 1$, we get

$$\begin{aligned}
&(b(\varepsilon) - \varepsilon^{-2}) \int\limits_{0}^{t_N} \|e\|^2 dt + \int\limits_{0}^{t_N} \|\nabla e\|^2 dt + \frac{1}{2}\|e^N\|^2 + \frac{1}{2}\sum_{n=0}^{N-1} \|u_h^{n+0} - u_h^n\|^2 \\
&+ \sum_{n=0}^{N-1} B^n = - \sum_{n=0}^{N-1} \int\limits_{t_n}^{t_{n+1}} (e, (u - \chi)')dt + \sum_{n=0}^{N-1} \int\limits_{t_n}^{t_{n+1}} (\nabla e, \nabla(u - \chi))dt \\
&+ (b(\varepsilon) - \varepsilon^{-2}) \sum_{n=0}^{N-1} \int\limits_{t_n}^{t_{n+1}} (e, (u - \chi))dt + (e^N, u^N - \chi^N) - \sum_{n=0}^{N-1} (e^n, \chi^n - \chi^{n+0}) \\
&\leq \frac{1}{4}\|e^N\|^2 + \|u^N - \chi^N\|^2.
\end{aligned}$$

Application of a kick-back argument yields the result. □

In [15], the previous theorem is used in order to derive the error estimate of the scheme by properly constructing V_h and selecting χ. In particular, an optimal error is proven when $d = 2$, and V_h is specified as a tensorial finite element space of piece-wise polynomial functions in space and time of separated variables.

Chapter 3

A posteriori analysis of space-time discontinuous Galerkin methods for the ε-dependent stochastic Allen-Cahn equation with mild noise

In this chapter, we develop an *a posteriori* error analysis for the space-time, discontinuous in time, Galerkin scheme proposed in [15] for the ε-dependent stochastic Allen-Cahn equation with mild noise [21, 75, 76, 88, 101, 110, 113, 124, 137, 130, 84, 36, 95, 98].

3.1 The problem

We consider the transformed problem (2.2.2), presented in the previous chapter, i.e.,

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) - \Delta u &= -b(\varepsilon)u + \frac{g(u,\varepsilon;t)}{\varepsilon^2} + \frac{m(\varepsilon,t)\dot{W}(t,x;\varepsilon)}{\varepsilon}, \quad x \in D, \\ \frac{\partial u}{\partial n} &= 0, \quad x \in \partial D, \\ u(x,0) &= u_0(x), \quad x \in D, \end{aligned}$$

where recall that $g(u,\varepsilon;t) := u - e^{2b(\varepsilon)t}u^3, \quad m(\varepsilon,t) := e^{-b(\varepsilon)t}$.

The exponential transformation there was chosen so that

$$\inf_{\varepsilon\in(0,1)} (b(\varepsilon) - \varepsilon^{-2}) \geq \hat{c}_0 > 0, \tag{3.1.1}$$

for some fixed positive constant $\hat{c}_0$.

Let us recall the smooth in space and in time mild noise $\dot{W}$ properties stated in the previous Section 2.2.2; for a more detailed presentation we refer to [57, 87, 129, 15].

Let $0 < \gamma < \frac{1}{3}$, then

$$\dot{W} := \dot{W}(t, x; \varepsilon) := \varepsilon^{-\gamma} \xi(\varepsilon^{-2\gamma} t, x),$$

where $\xi(t, x)$ is a stationary and strongly mixing stochastic process in t on a probability space $(\Omega, \mathcal{F}, P)$, satisfying

$$|\xi(t, x)| \leq M, \qquad |\frac{\partial \xi(t, x)}{\partial t}| \leq M, \qquad \mathbf{E}(\xi(t, x)) = 0,$$

for a constant M independent from the realization, and ε.

The noise $\xi(t, x)$ is smooth in space, while it is also sufficiently smooth in time, i.e.,

$$\frac{\partial \xi(t, x)}{\partial t} \in L^2((0, T) \times D),$$

which implies that $\xi(\cdot, x) \in H^1(0, T)$ for any fixed $x \in D$. Thus, $\xi(\cdot, x)$ is almost surely (a.s.) continuous with respect to $t \in [0, T]$. The smoothness of the noise, for a sufficiently smooth initial condition, yields that the solution u is almost surely (a.s.) continuous in time, which is essential for the application of the numerical scheme that we will consider [119, 63]. The smoothness of u in space is induced from the smoothness of the noise in space, which can be as high as we wish. For an alternative definition of $\dot{W}$ see for example in [129]. This is given by the formal differentiation of an approximated Brownian motion in time. Nevertheless, the solution w of the initial problem even if smooth in space or continuous in time a.s., has bounds in various Sobolev norms of negative polynomial order in ε, and as $\varepsilon \to 0^+$, it converges to the irregular step function ± 1; see for the case of time-dependent noise in [57, 129] when the problem is posed in the unbounded domain $\mathbb{R}^d$, and in [87] for mild noise depending on x as well.

Remark 24. The mild noise definitions from [57, 87, 129], obviously exclude the Gaussian noise in t as they require a minimum regularity in t (one-time differentiability in t) which is a property not holding true for Gaussians, as they are a.s. nowhere differentiable. Considering the definition of [57, 87], such processes exist.

The alternative mild noise definition given in [129] is not comparable with the definition of [57], and it is not a special case of [57]; we refer to Proposition 1.2. in [129] for its properties, which include that it is a stationary centred Gaussian process. This mild noise admits a C^∞ regularity in t can be given by a specific example, for a compactly supported ρ in its convolution definition as follows.

Let the compactly supported bump function $r : \mathbb{R} \to \mathbb{R}^+$, with

$$r(x) := \begin{cases} e^{-\frac{1}{1-x^2}} & x \in (-1,1) \\ 0 & \text{otherwise.} \end{cases}$$

We define

$$\rho(x) := \begin{cases} e^{-\frac{1}{1-x^2}} \Big[\int_{-1}^{1} e^{-\frac{1}{1-x^2}} dx \Big]^{-1} & x \in (-1,1), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\rho : \mathbb{R} \to \mathbb{R}^+$ is compactly supported and symmetric around zero, it satisfies $\int_{-\infty}^{\infty} \rho(x)dx = 1$, while it vanishes outside $[-1,1]$. We define for $0 < \gamma < \frac{2}{3}$ as needed in [129]

$$\rho^\varepsilon(x) := \varepsilon^{-\gamma}\rho\Big(\frac{x}{\varepsilon^\gamma}\Big) = \begin{cases} \varepsilon^{-\gamma} e^{-\frac{1}{1-\varepsilon^{-2\gamma}x^2}} \Big[\int_{-1}^{1} e^{-\frac{1}{1-x^2}} dx \Big]^{-1} & x\varepsilon^{-\gamma} \in (-1,1), \\ 0 & \text{otherwise.} \end{cases}$$

For any $t \geq 0$ we consider $W(t), \widetilde{W}(t) \sim N(0,t)$ two stochastically independent Brownian Motions and set $W(t) := \widetilde{W}(-t)$ for any $t < 0$, and so $(W(t), \mathbb{R})$ is a Gaussian process. We then define the convolution

$$W^\varepsilon(t) = \int_{-\infty}^{\infty} \rho^\varepsilon(t-s)W(s)ds,$$

and the noise by

$$\dot{W}^\varepsilon(t) := \int_{-\infty}^{\infty} \partial_t(\rho^\varepsilon(t-s))W(s)ds.$$

The above integral can be numerically approximated for all $t \in [0,T]$ by using for example the composite trapezoidal rule.

3.2 Discontinuous Galerkin method

Let us give more details for the discontinuous in time Galerkin scheme which was presented briefly at the previous chapter. As mentioned there, apart from the fact that this scheme by its construction is of high order of accuracy, it is also continuous in space and ε-dependent as the solution of the ε-Allen-Cahn equation is. On the sharp interface limit the solution becomes discontinuous in space admitting very steep layers when close to the sharp interface limit. This scheme permits the approximation of solution up to just before discontinuity in space occurs. By the other hand, the discontinuity in time makes the scheme adaptive in t so that new layers generation or annihilation can be captured by the numerical solution.

Let $T > 0$, we define $S_T := (0,T) \times D$, and consider $0 = t_0 < t_1 < \cdots < t_N = T$, a partition of $[0,T]$. Let $G^n := (t_n, t_{n+1}) \times D$, $\widetilde{G}^n := (t_n, t_{n+1}] \times \overline{D}$, while for $0 \leq \tau_0 < \tau_1 \leq T$ let $G(\tau_0, \tau_1) := (\tau_0, \tau_1) \times D$. For each $0 \leq n \leq N-1$, let $\{V_h^n\}$ be a family of finite dimensional subspaces of $H^1(G^n)$, parameterized by some $0 < h \leq 1$. V_h will denote the space of all functions u_h defined on $\overline{S}_T$ such that their restriction to each $\widetilde{G}^n$ coincides with the restriction to $\widetilde{G}^n$ of a function $v_h \in V_h^n$; note that the functions of V_h are in general discontinuous at the interior nodes t^n, $n = 1, \cdots, N-1$. We also define $v_h^n := v_h(\cdot, t^n)$ for any $0 \leq n \leq N$, and $v_h^{n+0} := \lim_{\alpha \to 0^+} v_h(\cdot, t^n + \alpha)$ for any $0 \leq n \leq N-1$, and observe that obviously $v_h^n = \lim_{\alpha \to 0^+} v_h(\cdot, t^n - \alpha)$ for any $1 \leq n \leq N$, see [15].

Let $w_h \in V_h$. For any fixed x, $w_h(\cdot, x)$ is a piece-wise polynomial function defined on $[0,T]$. On each $(t_n, t_{n+1}]$, $n = 0, 1, 2, \cdots, N-1$, $w_h(\cdot, x)$ is a polynomial, and $w_h(t_{n+1}, x) = w_h(t_{n+1} - 0, x)$, $n = 0, 1, 2, \cdots, N-1$, while $w_h(t_0, x)$ for $t_0 = 0$ is the starting value. In general $w_h(t, x)$ is not continuous at $t = t_n$, $n = 1, 2, \cdots, N-1$.

We recall the discontinuous Galerkin method (2.2.3) written after replacing the integral norms there: we seek $u_h \in V_h$ with $u_h^0 = u_0$, such that

$$
\begin{aligned}
&- \int_{t_n}^{t_{n+1}} \int_D u_h \partial_t v_h dx dt + \int_{t_n}^{t_{n+1}} \int_D \nabla u_h \nabla v_h dx dt + b(\varepsilon) \int_{t_n}^{t_{n+1}} \int_D u_h v_h dx dt \\
&- \varepsilon^{-2} \int_{t_n}^{t_{n+1}} \int_D u_h v_h dx dt + \varepsilon^{-2} \int_{t_n}^{t_{n+1}} \int_D e^{2b(\varepsilon)t} (u_h)^3 v_h dx dt \\
&+ \int_D u_h^{n+1} v_h^{n+1} dx - \int_D u_h^n v_h^{n+0} dx = \varepsilon^{-1} \int_{t_n}^{t_{n+1}} \int_D (m(\varepsilon, t) \dot{W}) v_h dx dt, \quad \forall\, v_h \in V_h^n,
\end{aligned}
$$

for $n = 0, 1, 2, \cdots, N-1$.

We define now V_h^n as the space of all functions $v_h : (t_n, t_{n+1}] \times \bar{D} \to \mathbb{R}$, such that for any fixed $t \in (t_n, t_{n+1}]$, $v_h(t, \cdot)$ is a piece-wise polynomial function defined on $\bar{D}$, and it is continuous with respect to x. Moreover, for any fixed $x \in \bar{D}$, $v_h(\cdot, x)$ is a polynomial function defined on $(t_n, t_{n+1}]$.

Let us consider the algorithms when V_h is a space of piece-wise constant or piece-wise linear functions with respect to time.

When V_h is a space of piece-wise constant functions with respect to time, we may use the following steps to construct the approximate solutions.

Step 1: Use the initial value $u_h(0) = u_0(x)$.

Step 2: Find $u_h : (t_0, t_1] \times \bar{D} \to \mathbb{R}$. Since u_h is a piece-wise constant with respect to t, we assume that $u_h(t, x) = U^1 \in S_h$, where S_h is the space of piece-wise linear continuous functions with respect to x and independent of t. In other words, $u_h(t, x)$ is independent of t and takes values in S_h.

Now, the discontinuous Galerkin method is written as: find $u_h(t, x) = U^1 \in S_h$ for $t \in (t_0, t_1]$ with $u_h(0, x) = u_0(x)$, such that

$$
\begin{aligned}
& -\int_{t_0}^{t_1}\int_D U^1 \partial_t \chi dx dt - \int_{t_0}^{t_1}\int_D \nabla U^1 \nabla \chi dx dt + (b(\varepsilon) - \varepsilon^{-2}) \int_{t_0}^{t_1}\int_D U^1 \chi dx dt \\
& + \varepsilon^{-2} \int_{t_0}^{t_1}\int_D e^{2b(\varepsilon)t} (U^1)^3 \chi dx dt + \int_D U^1 \chi dx - \int_D u_0(x) \chi dx \\
& = \varepsilon^{-1} \int_{t_0}^{t_1}\int_D (m(\varepsilon, t)\dot{W}) \chi dx dt, \quad \forall\, \chi \in V_h^0,\ n = 0, 1, 2, \cdots, N-1.
\end{aligned} \tag{3.2.1}
$$

Here, V_h^0 is the space of all functions $v_h : (t_0, t_1] \times \bar{D} \to \mathbb{R}$, such that for any fixed $t \in (t_0, t_1]$, $v_h(t, \cdot)$ is a piece-wise linear function defined on $\bar{D}$ and it is continuous with respect to x. Moreover, for any fixed $x \in \bar{D}$, $v_h(\cdot, x)$ is a constant function defined on $(t_0, t_1]$.

Note that, $v_h \in V_h^0$ is a constant function with respect to $t \in (t_0, t_1]$, and takes the values in S_h. Thus, we see that $V_h^0 = S_h$. Here, $\chi \in V_h^0$ is independent of t.

It is important to note that in (3.2.1), $\partial_t \chi$ denotes the time derivative of χ, therefore, $\partial_t \chi = 0$ since χ is independent of t. Thus, the discontinuous Galerkin method takes the form: find $u_h(t,\chi) = U^1 \in S_h$, $t \in (t_0, t_1]$ such that

$$
\begin{aligned}
&-k\int_D \nabla U^1 \nabla \chi dx + (b(\varepsilon)-\varepsilon^{-2})k\int_D U^1\chi dx + \varepsilon^{-2}\int_{t_0}^{t_1}\int_D e^{2b(\varepsilon)t}(U^1)^3\chi dxdt \\
&+\int_D U^1\chi dx - \int_D u_0\chi dx = \varepsilon^{-1}\int_{t_0}^{t_1}\int_D (m(\varepsilon,t)\dot{W})\chi dxdt, \quad \forall\, \chi \in S_h,
\end{aligned}
\tag{3.2.2}
$$

where k denotes time step size. We can solve this equation above by using the standard finite element method.

Step 3: Use the same way as in **Step 2** to find $U^2, U^2, \cdots, U^N$.

We next consider the algorithm when V_h is a space of piece-wise linear functions with respect to time.

Let V_h be the space of functions $w_h : [0,T]\times\bar{D} \to \mathbb{R}$ such that for any fixed $t \in [0,T]$, $w_h(t,\cdot)$ is a piece-wise linear continuous function defined on $\bar{D}$. Moreover, for any fixed $x \in \bar{D}$, $w_h(\cdot,x)$ is a piece-wise linear function defined on $[0,T]$ and it is left continuous on $(t_n, t_{n+1}]$, for $n = 1,2,\cdots,N-1$. The discontinuous Galerkin method takes the form: find $u_h \in V_h$ with $u_h(0) = u_0$, such that

$$
\begin{aligned}
&-\int_{t_n}^{t_{n+1}}\int_D u_h(\partial_t v_h)dxdt + \int_{t_n}^{t_{n+1}}\int_D (\nabla u_h)(\nabla v_h)dxdt + (b(\varepsilon)-\varepsilon^{-2})\int_{t_n}^{t_{n+1}}\int_D u_h v_h dxdt \\
&+\varepsilon^{-2}\int_{t_n}^{t_{n+1}}\int_D e^{2b(\varepsilon)t}(u_h)^3 v_h dxdt + \int_D u_h^{n+1}v_h^{n+1}dx - \int_D u_h^n v_h^{n+0}dx \\
&=\varepsilon^{-1}\int_{t_n}^{t_{n+1}}\int_D (m(\varepsilon,t)\dot{W})v_h dxdt, \quad \forall\, v_h \in V_h^n,\ n = 0,1,2,\cdots,N-1.
\end{aligned}
$$

Here, V_h^n is the space of functions $v_h : [t_n, t_{n+1}]\times\bar{D} \to \mathbb{R}$, such that for any fixed $t \in [t_n, t_{n+1}]$, $v_h(t,\cdot)$ is a piece-wise linear function defined on $\bar{D}$ and it is continuous with respect to x. Moreover, for any fixed $x \in \bar{D}$, $v_h(\cdot,x)$ is a linear function defined on $[t_n, t_{n+1}]$.

The algorithm is as follows.

Step 1: Use initial value $u_h(0) := u_0(x)$.

Step 2: Find the approximate solution $u_h : (t_0, t_1]\times\bar{D} \to \mathbb{R}$. Since u_h is a piece-wise linear function with respect to t on $[0,T]$, we may assume that $u_h(t,x)$ has the following form

$$u_h(t,x) = U_0^1 + U_1^1\big(\frac{t-t_0}{k}\big), \quad U_0^1, U_1^1 \in S_h,\ t \in (t_0, t_1].$$

Similarly, we have

$$u_h(t_1,x) = U_0^1 + U_1^1\big(\frac{t_1-t_0}{k}\big) = U_0^1 + U_1^1.$$

Let us see now how to specify U_0^1, U_1^1. The discontinuous Galerkin method takes the form: find $u_h(t,x) = U_0^1 + U_1^1\big(\frac{t_1-t_0}{k}\big)$, $t \in (t_0, t_1]$, such that

$$\begin{aligned}
&-\int_{t_0}^{t_1}\int_D u_h(\partial_t v_h)dxdt + \int_{t_0}^{t_1}\int_D (\nabla u_h)(\nabla v_h)dxdt + (b(\varepsilon) - \varepsilon^{-2})\int_{t_0}^{t_1}\int_D u_h v_h dxdt \\
&+ \varepsilon^{-2}\int_{t_0}^{t_1}\int_D e^{2b(\varepsilon)t}(u_h)^3 v_h dxdt + \int_D u_h^1 v_h^1 dx - \int_D u_h^0 v_h^{n+0} dx \\
&= \varepsilon^{-1}\int_{t_0}^{t_1}\int_D (m(\varepsilon,t)\dot{W})v_h dxdt, \quad \forall\, v_h \in V_h^0.
\end{aligned} \tag{3.2.3}$$

Here, V_h^0 is the space of functions $v_h : [t_0, t_1] \times \bar{D} \to \mathbb{R}$, such that for any fixed $t \in [t_0, t_1]$, $v_h(t, \cdot)$ is a piece-wise linear function defined on $\bar{D}$ and it is continuous with respect to x. Moreover, for any fixed $x \in \bar{D}$, $v_h(\cdot, x)$ is a linear function defined on $[t_0, t_1]$.

Since v_h is a linear function with respect to t, we may choose two different test functions $v_h = \chi \in S_h$ and $v_h = (t - t_0)\eta$, $\eta \in S_h$, where S_h is the finite element space of piece-wise linear continuous functions with respect to the space variable.

Choosing the test function $v_h = \chi \in S_h$, we get

$$\begin{aligned}
&-\int_{t_0}^{t_1}\int_D (U_0^1 + U_1^1(\frac{t_1-t_0}{k}))\partial_t\chi dxdt - \int_{t_0}^{t_1}\int_D \nabla\Big(U_0^1 + U_1^1(\frac{t_1-t_0}{k})\Big)\nabla\chi dxdt \\
&+ (b(\varepsilon) - \varepsilon^{-2})\int_{t_0}^{t_1}\int_D (U_0^1 + U_1^1(\frac{t_1-t_0}{k}))\chi dxdt + \varepsilon^{-2}\int_{t_0}^{t_1}\int_D e^{2b(\varepsilon)t}(U_0^1 + U_1^1(\frac{t_1-t_0}{k}))^3\chi dxdt \\
&= \varepsilon^{-1}\int_{t_0}^{t_1}\int_D (m(\varepsilon,t)\dot{W})\chi dxdt, \quad \forall\, \chi \in U_h^0.
\end{aligned}$$

Choosing the test function $v_h = (t - t_0)\eta$, $\eta \in S_h$, we get

$$
\begin{aligned}
&- \int_{t_0}^{t_1} \int_D \Big(U_0^1 + U_1^1 \big(\frac{t_1 - t_0}{k}\big) \Big) \big(\partial_t (t - t_0)\eta \big) dxdt \\
&- \int_{t_0}^{t_1} \int_D \nabla \Big(U_0^1 + U_1^1 \big(\frac{t_1 - t_0}{k}\big) \Big) \big(\nabla (t - t_0)\eta \big) dxdt \\
&+ (b(\varepsilon) - \varepsilon^{-2}) \int_{t_0}^{t_1} \int_D \Big(U_0^1 + U_1^1 \big(\frac{t_1 - t_0}{k}\big) \Big) (t - t_0)\eta dxdt \\
&+ \varepsilon^{-2} \int_{t_0}^{t_1} \int_D e^{2b(\varepsilon)t} \Big(U_0^1 + U_1^1 \big(\frac{t_1 - t_0}{k}\big) \Big)^3 (t - t_0)\eta dxdt \\
&= \varepsilon^{-1} \int_{t_0}^{t_1} \int_D (m(\varepsilon, t)\dot{W}) \big((t - t_0)\eta \big) dxdt, \quad \forall\, \eta \in S_h.
\end{aligned}
$$

Note that $\partial_t \chi = 0$ for all $\chi \in S_h$, and $\partial_t((t - t_0)\eta) = \eta$ for all $\eta \in S_h$.

We then get the following two equations for U_0^1 and U_1^1

$$
\begin{aligned}
&- \int_{t_0}^{t_1} \int_D \nabla \Big(U_0^1 + U_1^1 \big(\frac{t - t_0}{k}\big) \Big) \nabla \chi dxdt - (b(\varepsilon) - \varepsilon^{-2}) \int_{t_0}^{t_1} \int_D \Big(U_0^1 + U_1^1 \big(\frac{t_1 - t_0}{k}\big) \Big) \chi dxdt \\
&+ \varepsilon^{-2} \int_{t_0}^{t_1} \int_D e^{2b(\varepsilon)t} \Big(U_0^1 + U_1^1 \big(\frac{t_1 - t_0}{k}\big) \Big)^3 \chi dxdt \\
&= \varepsilon^{-1} \int_{t_0}^{t_1} \int_D (m(\varepsilon, t)\dot{W}) \chi dxdt, \quad \forall\, \chi \in S_h,
\end{aligned}
\tag{3.2.4}
$$

and

$$
\begin{aligned}
&- \int_{t_0}^{t_1} \int_D \Big(U_0^1 + U_1^1 (\frac{t_1 - t_0}{k}) \Big) \eta dx dt - \int_{t_0}^{t_1} \int_D \nabla \Big(U_0^1 + U_1^1 (\frac{t_1 - t_0}{k}) \Big) \big(\nabla (t - t_0) \eta \big) dx dt \\
&+ (b(\varepsilon) - \varepsilon^{-2}) \int_{t_0}^{t_1} \int_D \Big(U_0^1 + U_1^1 (\frac{t_1 - t_0}{k}) \Big) (t - t_0) \eta dx dt \\
&+ \varepsilon^{-2} \int_{t_0}^{t_1} \int_D e^{2b(\varepsilon)t} \Big(U_0^1 + U_1^1 (\frac{t_1 - t_0}{k}) \Big)^3 (t - t_0) \eta dx dt \\
&= \varepsilon^{-1} \int_{t_0}^{t_1} \int_D (m(\varepsilon, t) \dot{W}) \big(t - t_0) \eta \big) dx dt, \quad \forall \, \eta \in S_h.
\end{aligned}
\tag{3.2.5}
$$

From (3.2.4) and (3.2.5), we obtain $U_0^1 \in S_h$ and $U_1^1 \in S_h$.

Then we get

$$
u_h(x, t_1) = U_0^1 + U_1^1 (\frac{t_1 - t_0}{k}) = U_0^1 + U_1^1.
$$

After we obtain $u_h^1 := u_h(x, t_1)$, we may go to the next step.

Step 3: Use the same way as in **Step 2** and find the approximate solutions $u_h : (t_n, t_{n+1}] \times \bar{D} \to \mathbb{R}$, $n = 1, 2, \ldots N - 1$.

3.3 A posteriori Error estimates

In this section, completing the error analysis in the *a posteriori* sense, we derive the *a posteriori* error estimation of the scheme, where the error will be bounded above by using the discrete solution, the initial data and the mild noise.

In order to facilitate notation, we write the continuous problem as

$$
\begin{aligned}
&u_t - \Delta u + b(\varepsilon) u - \varepsilon^{-2} (u - e^{2b(\varepsilon)t} u^3) = g, \quad x \in D, \quad t > 0, \\
&u(0, x) = u_0(x), \\
&\frac{\partial u(0, x)}{\partial n} = 0, \quad x \in \partial D,
\end{aligned}
\tag{3.3.1}
$$

where $\frac{\partial}{\partial n}$ denotes the normal derivative on ∂D and g is some function depending on t and x, and assume that $u_0 \in H^3(D)$. We also return to the notation $(\cdot, \cdot)$ for the $L^2(D)$ inner product.

The variational formulation is written as: find for all $n = 0, \cdots, N-1$, $u \in H^1((t_n, t_{n+1}) \times D)$ such that

$$a(u, v) = \int_{t_n}^{t_{n+1}} (g, v)dt, \quad \forall\, v \in H^1((t_n, t_{n+1}) \times D), \tag{3.3.2}$$

where the nonlinear form is defined by

$$\begin{aligned} a(u, v) = & -\int_{t_n}^{t_{n+1}} (u, v_t)dt + \int_{t_n}^{t_{n+1}} (\nabla u, \nabla v)dt + \int_{t_n}^{t_{n+1}} ((b(\varepsilon) - \varepsilon^{-2})u, v)dt \\ & + \int_{t_n}^{t_{n+1}} (\varepsilon^{-2} e^{2b(\varepsilon)t} u^3, v)dt + (u^{n+1}, v^{n+1}) - (u^n, v^{n+0}). \end{aligned} \tag{3.3.3}$$

Note that a is linear at the second argument.

The Discontinuous Galerkin finite element method is written as: find $u_h \in V_h^n$ such that

$$a(u_h, v_h) = \int_{t_n}^{t_{n+1}} (g, v_h)dt, \quad \forall\, v_h \in V_h^n, \;\; n = 0, \cdots, N-1. \tag{3.3.4}$$

Let the error be defined by $e = u - u_h$.

We have the following lemma.

Lemma 3.3.1. Let $e = u - u_h$ where u *and* u_h are the solutions of (3.3.2) and (3.3.4) respectively. Then it holds that

$$\begin{aligned} & \int_{t_n}^{t_{n+1}} (\nabla e, \nabla e)dt + \int_{t_n}^{t_{n+1}} \Big((b(\varepsilon) - \varepsilon^{-2})e, e \Big) dt + \varepsilon^{-2} \int_{t_n}^{t_{n+1}} \Big(e^{2b(\varepsilon)t}(u^3 - (u_h)^3), e \Big) dt \\ & + \Big((e^{n+1}, e^{n+1}) - (e^n, e^{n+0}) \Big) - \frac{1}{2} \Big((e^{n+1}, e^{n+1}) - (e^{n+0}, e^{n+0}) \Big) \\ & = \int_{t_n}^{t_{n+1}} (g, e - v_h)dt - \int_{t_n}^{t_{n+1}} \Big(\nabla u_h, \nabla(e - v_h) \Big) dt + \int_{t_n}^{t_{n+1}} (-b(\varepsilon) + \varepsilon^{-2})(u_h, e - v_h)dt \\ & - \varepsilon^{-2} \int_{t_n}^{t_{n+1}} e^{2b(\varepsilon)t}(u_h^3, e - v_h)dt - (u_h^{n+1}, (e - v_h)^{n+1}) + (u_h^n, (e - v_h)^{n+0}) + \int_{t_n}^{t_{n+1}} (u_h, \partial_t(e - v_h))dt. \end{aligned} \tag{3.3.5}$$

Proof. We observe that using the Neumann boundary condition for u the solution of (3.3.1), it follows that

$$
\begin{aligned}
&-\int_{t_n}^{t_{n+1}} (u, v_{ht})dt + \int_{t_n}^{t_{n+1}} (\nabla u, \nabla v_h)dt + \int_{t_n}^{t_{n+1}} ((b(\varepsilon) - \varepsilon^{-2})u, v_h)dt \\
&+ \int_{t_n}^{t_{n+1}} (\varepsilon^{-2} e^{2b(\varepsilon)t} u^3, v_h)dt + (u^{n+1}, v_h^{n+1}) - (u^n, v_h^{n+0}) = \int_{t_n}^{t_{n+1}} (g, v_h),
\end{aligned}
\tag{3.3.6}
$$

i.e.,

$$a(u, v_h) = \int_{t_n}^{t_{n+1}} (g, v_h).$$

But we have by (3.3.4) and replacing a

$$
\begin{aligned}
&-\int_{t_n}^{t_{n+1}} (u_h, v_{ht})dt + \int_{t_n}^{t_{n+1}} (\nabla u_h, \nabla v_h)dt + \int_{t_n}^{t_{n+1}} ((b(\varepsilon) - \varepsilon^{-2})u_h, v_h)dt \\
&+ \int_{t_n}^{t_{n+1}} (\varepsilon^{-2} e^{2b(\varepsilon)t} (u_h)^3, v_h)dt + (u_h^{n+1}, v_h^{n+1}) - (u_h^n, v_h^{n+0}) = a(u_h, v_h) = \int_{t_n}^{t_{n+1}} (g, v_h)dt.
\end{aligned}
\tag{3.3.7}
$$

Substracting (3.3.6), (3.3.7), we obtain

$$
\begin{aligned}
&-\int_{t_n}^{t_{n+1}} (u - u_h, v_{ht})dt + \int_{t_n}^{t_{n+1}} (\nabla(u - u_h), \nabla v_h)dt + \int_{t_n}^{t_{n+1}} ((b(\varepsilon) - \varepsilon^{-2})(u - u_h), v_h)dt \\
&+ \int_{t_n}^{t_{n+1}} (\varepsilon^{-2} e^{2b(\varepsilon)t} (u^3 - (u_h)^3), v_h)dt + (u^{n+1} - u_h^{n+1}, v_h^{n+1}) - (u^n - u_h^n, v_h^{n+0}) \\
&= \int_{t_n}^{t_{n+1}} (g, v_h) - \int_{t_n}^{t_{n+1}} (g, v_h) = 0.
\end{aligned}
\tag{3.3.8}
$$

We have,

$$
\begin{aligned}
&-\int_{t_n}^{t_{n+1}}(e,e_t)dt+\int_{t_n}^{t_{n+1}}(\nabla e,\nabla e)dt+\int_{t_n}^{t_{n+1}}\Big((b(\varepsilon)-\varepsilon^{-2})e,e\Big)dt\\
&+\int_{t_0}^{t_1}\Big(\varepsilon^{-2}e^{2b(\varepsilon)t}(u^3-(u_h)^3),e\Big)dt+(e^{n+1},e^{n+1})-(e^n,e^{n+0})\\
&=-\int_{t_n}^{t_{n+1}}(e,(e-v_h)_t)dt+\int_{t_n}^{t_{n+1}}(\nabla e,\nabla(e-v_h))dt+\int_{t_n}^{t_{n+1}}\Big((b(\varepsilon)-\varepsilon^{-2})e,e-v_h\Big)dt\\
&+\int_{t_0}^{t_1}\Big(\varepsilon^{-2}e^{2b(\varepsilon)t}(u^3-(u_h)^3),e-v_h\Big)dt+(e^{n+1},(e-v_h)^{n+1})-(e^n,(e-v_h)^{n+0})\\
&-\int_{t_n}^{t_{n+1}}(e,v_{ht})dt+\int_{t_n}^{t_{n+1}}(\nabla e,\nabla v_h)dt+\int_{t_n}^{t_{n+1}}\Big((b(\varepsilon)-\varepsilon^{-2})e,v_h\Big)dt\\
&+\int_{t_0}^{t_1}\Big(\varepsilon^{-2}e^{2b(\varepsilon)t}(u^3-(u_h)^3),v_h\Big)dt+(e^{n+1},v_h^{n+1})-(e^n,v_h^{n+0})\\
=&-\int_{t_n}^{t_{n+1}}(e,(e-v_h)_t)dt+\int_{t_n}^{t_{n+1}}(\nabla e,\nabla(e-v_h))dt+\int_{t_n}^{t_{n+1}}\Big((b(\varepsilon)-\varepsilon^{-2})e,e-v_h\Big)dt\\
&+\int_{t_0}^{t_1}\Big(\varepsilon^{-2}e^{2b(\varepsilon)t}(u^3-(u_h)^3),e-v_h\Big)dt+(e^{n+1},(e-v_h)^{n+1})-(e^n,(e-v_h)^{n+0})+0,
\end{aligned}
$$

as $a(u,v_h)=\int_{t_n}^{t_{n+1}}(g,v_h)dt=a(u_h,v_h)$ (in detail the last zero term appears due to (3.3.8)).

So, we get

$$
\begin{aligned}
& -\int_{t_n}^{t_{n+1}} (e, e_t)dt + \int_{t_n}^{t_{n+1}} (\nabla e, \nabla e)dt + \int_{t_n}^{t_{n+1}} \Big((b(\varepsilon) - \varepsilon^{-2})e, e\Big)dt \\
& + \int_{t_0}^{t_1} \Big(\varepsilon^{-2} e^{2b(\varepsilon)t}(u^3 - (u_h)^3), e\Big)dt + (e^{n+1}, e^{n+1}) - (e^n, e^{n+0}) \\
= & -\int_{t_n}^{t_{n+1}} (e, (e - v_h)_t)dt + \int_{t_n}^{t_{n+1}} (\nabla e, \nabla(e - v_h))dt + \int_{t_n}^{t_{n+1}} \Big((b(\varepsilon) - \varepsilon^{-2})e, e - v_h\Big)dt \\
& + \int_{t_0}^{t_1} \Big(\varepsilon^{-2} e^{2b(\varepsilon)t}(u^3 - (u_h)^3), e - v_h\Big)dt + (e^{n+1}, (e - v_h)^{n+1}) - (e^n, (e - v_h)^{n+0}) \\
= & -\int_{t_n}^{t_{n+1}} (u, (e - v_h)_t)dt + \int_{t_n}^{t_{n+1}} (\nabla u, \nabla(e - v_h))dt + \int_{t_n}^{t_{n+1}} \Big((b(\varepsilon) - \varepsilon^{-2})u, e - v_h\Big)dt \\
& + \int_{t_0}^{t_1} \Big(\varepsilon^{-2} e^{2b(\varepsilon)t}u^3, e - v_h\Big)dt + (u^{n+1}, (e - v_h)^{n+1}) - (u^n, (e - v_h)^{n+0}) \\
& + \int_{t_n}^{t_{n+1}} (u_h, (e - v_h)_t)dt - \int_{t_n}^{t_{n+1}} (\nabla u_h, \nabla(e - v_h))dt - \int_{t_n}^{t_{n+1}} \Big((b(\varepsilon) - \varepsilon^{-2})u_h, e - v_h\Big)dt \\
& - \int_{t_0}^{t_1} \Big(\varepsilon^{-2} e^{2b(\varepsilon)t}(u_h)^3, e - v_h\Big)dt - (u_h^{n+1}, (e - v_h)^{n+1}) + (u_h^n, (e - v_h)^{n+0}) \\
= & \int_{t_n}^{t_{n+1}} (g, e - v_h)dt \\
& + \int_{t_n}^{t_{n+1}} (u_h, (e - v_h)_t)dt - \int_{t_n}^{t_{n+1}} (\nabla u_h, \nabla(e - v_h))dt - \int_{t_n}^{t_{n+1}} \Big((b(\varepsilon) - \varepsilon^{-2})u_h, e - v_h\Big)dt \\
& - \int_{t_0}^{t_1} \Big(\varepsilon^{-2} e^{2b(\varepsilon)t}(u_h)^3, e - v_h\Big)dt - (u_h^{n+1}, (e - v_h)^{n+1}) + (u_h^n, (e - v_h)^{n+0}).
\end{aligned}
$$

This gives

$$
\begin{aligned}
&- \int_{t_n}^{t_{n+1}} (e, e_t)dt + \int_{t_n}^{t_{n+1}} (\nabla e, \nabla e)dt + \int_{t_n}^{t_{n+1}} \Big((b(\varepsilon) - \varepsilon^{-2})e, e \Big) dt \\
&+ \int_{t_0}^{t_1} \Big(\varepsilon^{-2} e^{2b(\varepsilon)t} (u^3 - (u_h)^3), e \Big) dt + (e^{n+1}, e^{n+1}) - (e^n, e^{n+0}) \\
= &\int_{t_n}^{t_{n+1}} (g, e - v_h)dt \\
&+ \int_{t_n}^{t_{n+1}} (u_h, (e - v_h)_t)dt - \int_{t_n}^{t_{n+1}} (\nabla u_h, \nabla (e - v_h))dt - \int_{t_n}^{t_{n+1}} \Big((b(\varepsilon) - \varepsilon^{-2}) u_h, e - v_h \Big) dt \\
&- \int_{t_0}^{t_1} \Big(\varepsilon^{-2} e^{2b(\varepsilon)t} (u_h)^3, e - v_h \Big) dt - (u_h^{n+1}, (e - v_h)^{n+1}) + (u_h^n, (e - v_h)^{n+0}).
\end{aligned}
$$

Replacing

$$
\int_{t_n}^{t_{n+1}} (e, e_t)dt = \frac{1}{2} \int_{t_n}^{t_{n+1}} \frac{d}{dt}(e, e)dt = \frac{1}{2} \Big[(e^{n+1}, e^{n+1}) - (e^{n+0}, e^{n+0}) \Big].
$$

we obtain the result. □

Lemma 3.3.2. Let u and u_h be the solutions of (3.3.2) and (3.3.4), respectively.

Let $e = u - u_h$. It holds that

$$
\begin{aligned}
&(b(\varepsilon) - \varepsilon^{-2}) \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (e, e) dt + \int_0^{t_n} (\nabla e, \nabla e) dt \\
&+ \varepsilon^{-2} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \Big(e^{2b(\varepsilon)t} (u^3 - (u_h)^3), e \Big) dt + \frac{1}{2} \|e^n\|^2 + \frac{1}{2} \sum_{i=0}^{n-1} \|u_h^{i+0} - u_h^i\|^2 \\
= &\int_0^{t_n} (g, e - v_h) dt + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (u_h, \partial_t (e - v_h)) dt \\
&- \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (\nabla u_h, \nabla (e - v_h)) dt - (b(\varepsilon) - \varepsilon^{-2}) \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (u_h, e - v_h) dt \\
&- \varepsilon^{-2} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \Big(e^{2b(\varepsilon)t} (u_h)^3, e - v_h \Big) dt + (u^0, (e - v_h)^0) - (u_h^n, (e - v_h)^n) \\
&- \sum_{i=0}^{n-1} (u_h^i, (e - v_h)^i - (e - v_h)^{i+0}).
\end{aligned}
\tag{3.3.9}
$$

Proof. Summation of (3.3.5) implies that

$$
\begin{aligned}
&\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (\nabla e, \nabla e) dt + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (b(\varepsilon) - \varepsilon^{-2})(e, e) dt \\
&+ \varepsilon^{-2} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \Big(e^{2b(\varepsilon)t} (u^3 - (u_h)^3), e \Big) dt \\
&+ \sum_{i=0}^{n-1} \big\{ (e^{i+1}, e^{i+1}) - (e^i, e^{i+0}) - \frac{1}{2} [(e^{i+1}, e^{i+1}) - (e^{i+0}, e^{i+0})] \big\}. \\
= &\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (g, e - v_h) dt + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (u_h, \partial_t (e - v_h)) dt - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (\nabla u_h, \nabla (e - v_h)) dt \\
&+ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (-b(\varepsilon) + \varepsilon^{-2})(u_h, e - v_h) dt - \varepsilon^{-2} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \Big(e^{2b(\varepsilon)t} (u_h^3), e - v_h \Big) dt \\
&+ \sum_{i=0}^{n-1} \big\{ - (u_h^{i+1}, (e - v_h)^{i+1}) + (u_h^i, (e - v_h)^{i+0} \big\}.
\end{aligned}
$$

Note that

$$\sum_{i=0}^{n-1}\{(e^{i+1},e^{i+1})-(e^i,e^{i+0})-\frac{1}{2}[(e^{i+1},e^{i+1})-(e^i,e^{i+0})]\}$$

$$=\sum_{i=0}^{n-1}\big[\frac{1}{2}(e^{i+1},e^{i+1})-(e^i,e^{i+0})+\frac{1}{2}(e^{i+0},e^{i+0})\big]$$

$$=\sum_{i=0}^{n-1}\Big\{\big[\frac{1}{2}(e^{i+1},e^{i+1})-\frac{1}{2}(e^i,e^{i+0})\big]-\big[\frac{1}{2}(e^i,e^{i+0})-\frac{1}{2}(e^{i+0},e^{i+0})\big]\Big\}$$

$$=\frac{1}{2}\Big[(e^n,e^n)-(e^{n-1},e^{n-1+0})-(e^{n-1},e^{n-1+0})+(e^{n-1+0},e^{n-1+0})$$

$$+(e^{n-1},e^{n-1})-(e^{n-2},e^{n-2+0})-(e^{n-2},e^{n-2+0})+(e^{n-2+0},e^{n-2+0})+\cdots+(e^2,e^2)$$

$$-(e^1,e^{1+0})+(e^1,e^{1+0})+(e^{1+0},e^{1+0})$$

$$+(e^1,e^1)-(e^0,e^{0+0})-(e^0,e^{0+0})+(e^{0+0},e^{0+0})\Big]$$

$$=\frac{1}{2}(e^n,e^n)+\frac{1}{2}\|e^{n-1}-e^{n-1+0}\|^2+...+\frac{1}{2}\|e^1-e^{1+0}\|^2+\|e^{0+}\|^2$$

$$=\frac{1}{2}\|e^n\|^2+\frac{1}{2}\sum_{i=0}^{n-1}\|u_h^{i+1}-u_h^i\|^2,$$

where $e^{0+}=u^{0+}-u_h^{0+}=u^0-u_h^{0+}=u_h^0-u_h^{0+}$.

Note also that, for $\eta:=e-v_h$,

$$\sum_{i=0}^{n-1}\{-(u_h^{i+1},\eta^{i+1})+(u_h^i,\eta^{i+0})\}=-(u_h^n,\eta^n)+(u_h^{n-1},\eta^{n-1+0})-(u_h^{n-1},\eta^{n-1})$$

$$+(u_h^{n-2},\eta^{n-2+0})+\cdots-(u_h^1,\eta^1)+(u_h^0,\eta^{0+0})$$

$$=(u_h^0,\eta^{0+})-(u_h^n,\eta^n)-\sum_{i=0}^{n-1}(u_h^i,\eta^i-\eta^{i+0})$$

$$=(u^0,\eta^{0+})-(u_h^n,\eta^n)-\sum_{i=0}^{n-1}(u_h^i,\eta^i-\eta^{i+0})+(u_h^0,\eta^0-\eta^{0+})$$

$$=(u^0,\eta^0)-(u_h^n,\eta^n)-\sum_{i=0}^{n-1}(u_h^i,\eta^i-\eta^{i+0}).$$

Together these estimates complete the proof of the lemma. □

Let us now derive a useful identity given by

$$\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(u_h,\partial_t\eta)dt = -\sum_{i=0}^{n-1}(\partial_t u_h,\eta)dt + (u_h^n,\eta^n) - (u_h^0,\eta^0)$$
$$+\sum_{i=0}^{n-1}(u_h^i,\eta^i-\eta^{i+0}) + \sum_{i=0}^{n-1}(u_h^i-u_h^{i+0},\eta^{i+0}). \tag{3.3.10}$$

We observe that $\eta = e - v_h$ for arbitrary $v_h \in V_h^n$, satisfies

$$\sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}(u_h,\partial_t\eta)dt = -\sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}(\partial_t u_h,\eta)dt + \sum_{i=1}^{n-1}\big[(u_h^{i+1},\eta^{i+1}) - (u_h^{i+0},\eta^{i+0})\big].$$

Further, we have,

$$\sum_{i=1}^{n-1}\big[(u_h^{i+1}-\eta^{i+1}) - (u_h^{i+0},\eta^{i+0})\big] = (u_h^n,\eta^n) - (u_h^{n-1+0},\eta^{n-1+0})$$
$$= (u_h^n,\eta^n) - (u_h^{n-1+0},\eta^{n-1+0}) + (u_h^{n-1},\eta^{n-1}) - (u_h^{n-2+0},\eta^{n-2+0})$$
$$+\cdots+(u_h^1,\eta^1) - (u_h^{0+0},\eta^{0+0})$$
$$= (u_h^n,\eta^n) + [(u_h^{n-1},\eta^{n-1}) - (u_h^n,\eta^n) - (u_h^{n-1+0},\eta^{n-1+0})] + \cdots$$
$$+(u_h^1,\eta^1) - (u_h^{1+0},\eta^{1+0}) - (u_h^{0+},\eta^{0+})$$
$$= \mathcal{C},$$

where

$$\begin{aligned}
\mathcal{C} =&(u_h^n,\eta^n) + (u_h^{n-1},\eta^{n-1}-\eta^{n-1+0}) + (u_h^{n-1}-u_h^{n-1+0},\eta^{n-1+0})\\
&+\cdots+(u_h^1,\eta^1-\eta^{1+0}) + (u_h^1-u_h^{1+0},\eta^{1+0}) - (u_h^{0+},\eta^{0+})\\
=&(u_h^n,\eta^n) + (u_h^{n+1},\eta^{n-1}-\eta^{n-1+0}) + (u_h^{n-1}-u_h^{n-1+0},\eta^{n-1+0})\\
&+\cdots+(u_h^1,\eta^1-\eta^{1+0}) + (u_h^1-u_h^{1+0},\eta^{1+0}) + (u_h^{0+},\eta^0-\eta^{0+})\\
&+(u_h^0-u_h^{0+},\eta^{0+}) - (u_h^0,\eta^0-\eta^{0+}) - (u_h^0-u_h^{0+},\eta^{0+}) - (u_h^{0+},\eta^{0+})\\
&= (u_h^n,\eta^n) - (u_h^0,\eta^0) + \sum_{i=1}^{n-1}(u_h^i-\eta^i-\eta^{i+0}) + \sum_{i=1}^{n-1}(u_h^i-u^{i+1},\eta^{i+0}).
\end{aligned}$$

Thus, we get (3.3.10).

Let $\mathcal{T}_h^n$ be a partition of $\overline{G^n}$ and

$$V_h^n = \{z_h \in H^1(G^n):\ z_h|_K \in P_{\rho-1}(K), \forall K \in \mathcal{T}_h^n\}, \tag{3.3.11}$$

where $P_{\rho-1}$ is the space of polynomials of total degree at most $\rho - 1 \geq 1$ in the time and space variables. We let h_n denote the maximum element diameter in the partition $\mathcal{T}_h^n$ and define $h := \max_n h_n$. Further, if ℓ is an interior edge of $\mathcal{T}_h^n$ we let

$$[\nabla u_h \cdot \mathbf{n}]_\ell := \nabla u_h \cdot \mathbf{n}|_{\ell^+} - \nabla u_h \cdot \mathbf{n}|_{\ell^-}$$

denote the jump of $\nabla u_h \cdot \mathbf{n}$ across the edge ℓ, where $\mathbf{n}$ is the normal direction. We also denote by E_K^n, $E_{K\text{in}}^n$, and $E_{K\text{b}}^n$, the set of all edges, the set of interior edges, and the set of boundary edges of an element K of the partition $\mathcal{T}_h^n$ respectively.
It holds that

$$\begin{aligned}\sum_{i=0}^{n-1}((\nabla u_h, \nabla\eta))_{G^i} = &-\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}((\Delta u_h, \eta))_K \\ &+\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\sum_{\ell\in E_{K\text{in}}^i}\int_\ell \eta[\nabla u_h\cdot\mathbf{n}]_\ell ds \\ &+\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\sum_{\ell\in E_{K\text{b}}^i}\int_\ell \eta\nabla u_h\cdot\mathbf{n} ds,\end{aligned} \tag{3.3.12}$$

where we used the fact that η is continuous in space variables. Also note that v_h and thus η are not vanishing at the boundary, this gives the trace term

$$\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\sum_{\ell\in E_{K\text{b}}^i}\int_\ell \eta\nabla u_h\cdot\mathbf{n} ds$$

in the above equation. This term would not exist in case of a Dirichlet initial and boundary value problem, see for example in [19]; here, we treat the stochastic Allen-Cahn which satisfies Neumann boundary conditions.
Let $\rho \geq 2$, we define as l_n the minimum diameter of elements in the partition $\mathcal{T}_h^n$ and $l := \min_n l_n$. Let $\dim(D) = m$, then $\dim \overline{G^n} = m+1$, the interior elements of the partition are $(m+1)$-simplices, and the boundary elements may posses a possibly curved boundary. We shall assume that partition is regular, i.e., there exists a $c_0 > 0$ independent of n such that $h \leq c_0 l$. We select in the definition of η, v_h as follows: For $n \geq 1$, let $v_h|_{G^n}$ restricted in every element K of $\mathcal{T}_h^n$ be the Clément's interpolant $\pi_h^n e$ [41] of the error e in $P_{\rho-1}(K)$. We recall that there exists a constant $C > 0$, depending only on c_0, such that,

$$\begin{aligned}&\|e - \pi_h^n e\|_{L^2(K)} \leq Ch\|e\|_{H^1(\Delta K)} \leq Ch(\|e\|_{L^2(\Delta K)} + \|\nabla e\|_{L^2(\Delta K)} + \|e_t\|_{L^2(\Delta K)}), \\ &\|e - \pi_h^n e\|_{L^2(\ell)} \leq Ch^{1/2}\|e\|_{H^1(\Delta K)} \leq Ch^{1/2}(\|e\|_{L^2(\Delta K)} + \|\nabla e\|_{L^2(\Delta K)} + \|e_t\|_{L^2(\Delta K)}).\end{aligned}$$

(3.3.13)

In the above, ℓ is an edge of K, and ΔK denotes the set of elements having an edge or vertex common with K. We note that the L^2, H^1 norms, are space-time norms as K and ℓ are space-time elements. So for $\nabla = \nabla_x$, we then have

$$\|e\|_{H^1(\Delta K)} = \Big(\|e\|^2_{L^2(\Delta K)} + \|\nabla e\|^2_{L^2(\Delta K)} + \|e_t\|^2_{L^2(\Delta K)}\Big)^{1/2}.$$

We present now the following theorem.

Theorem 3.3.3. Let u and u_h be the solutions of (3.3.2) and (3.3.4), respectively, $e = u - u_h$, and h small enough. Then for any given p, α, with $p \geq 2$, $\alpha \geq 0$, there exists positive constant $C > 0$ independent of ε and h, such that for any $1 \leq n \leq N$,

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\|\nabla e\|^2_{L^2(K)} + \sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\|e\|^2_{L^2(K)} + \varepsilon^{-2}\sum_{i=0}^{n-1}((\exp(2b(\varepsilon)t)(u^3-(u_h)^3),e))_{G^i}\\
&\quad+\|e^n\|^2_{L^2(D)} + \sum_{i=0}^{n-1}\|u_h^{i+0}-u_h^i\|^2_{L^2(D)}\Big]\\
&\leq \mathbb{E}\Big[Ch^{\frac{2p-2}{p}}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\|\varepsilon^{-1}e^{-b(\varepsilon)t}\xi_t^\varepsilon - \partial_t u_h + \Delta u_h - (b(\varepsilon)-\varepsilon^{-2})u_h\|_{L^p(K)}\\
&\quad+Ch^{(2-\alpha)p-2}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\|\varepsilon^{-1}e^{-b(\varepsilon)t}\xi_t^\varepsilon - \partial_t u_h + \Delta u_h - (b(\varepsilon)-\varepsilon^{-2})u_h\|^p_{L^p(K)}\\
&\quad+Ch^{\frac{p-1}{p}}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\sum_{\ell\in E^i_{K\mathrm{in}}}\Big(\int_\ell|[\nabla u_h\cdot\mathbf{n}]_\ell|^p ds\Big)^{1/p}\\
&\quad+Ch^{(1-\alpha)p-1}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\sum_{\ell\in E^i_{K\mathrm{in}}}\int_\ell|[\nabla u_h\cdot\mathbf{n}]_\ell|^p ds\\
&\quad+Ch^{\frac{p-1}{p}}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\sum_{\ell\in E^i_{K\mathrm{b}}}\Big(\int_\ell|\nabla u_h\cdot\mathbf{n}|^p ds\Big)^{1/p}\\
&\quad+Ch^{(1-\alpha)p-1}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\sum_{\ell\in E^i_{K\mathrm{b}}}\int_\ell|\nabla u_h\cdot\mathbf{n}|^p ds\\
&\quad+[Ch+Ch^{\frac{\alpha p}{p-1}}]\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\|\partial_t e\|^2_{L^2(K)} + C\Big|\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}((\varepsilon^{-2}\exp(2b(\varepsilon)t)(u_h)^3,\eta))_K\Big|\Big],
\end{aligned}$$

for η given through the Clément's interpolant. Here, $((\cdot,\cdot))_{G^i}$, $((\cdot,\cdot))_K$ denote the $L^2(G^i)$ and $L^2(K)$ inner products respectively, while note that $\varepsilon^{-2}\sum_{i=0}^{n-1}((e^{2b(\varepsilon)t}(u^3-(u_h)^3),e))_{G^i} \geq 0$.

Proof. Let a function $v : A \to \mathbb{R}$, for A an arbitrary connected set with Lipschitz boundary, and consider any $p \geq 2$. We have

$$
\begin{aligned}
\Big|\int_A v\eta ds\Big| &\leq \Big(\int_A |v|^p ds\Big)^{\frac{1}{p}}\Big(\int_A |\eta|^{\frac{p}{p-1}} ds\Big)^{\frac{p-1}{p}} \\
&\leq C\|v\|_{L^p(A)}\mathrm{Vol(A)}^{\frac{p-1}{p}} + C\|v\|_{L^p(A)}\|\eta\|_{L^2(A)}^{\frac{2p-2}{p}},
\end{aligned}
\tag{3.3.14}
$$

where we use Hölder's inequality and, since $p \geq 2$ which yields that $\frac{p}{p-1} \leq 2$, Young's inequality, i.e., $ab \leq \frac{a^{p_1}}{p_1} + \frac{b^{q_1}}{q_1}$ for any $a, b > 0$ and $p_1, q_1 > 1$ with $\frac{1}{p_1} + \frac{1}{q_1} = 1$.
Note that if A is $m+1$-dimensional with $m \geq 1$ with diameter of order h, then $\mathrm{Vol(A)} \leq Ch^2$ for all $m \geq 1$, recalling that the volume of a ball in $\mathbb{R}^{m+1}$ with radius $h < 1$ is equal to $Ch^{m+1} \leq Ch^2$. We obtain, by the above for $A := K$ in (3.3.14) and bounding η by using the error e

$$
\begin{aligned}
|((v,\eta))_K| =& \Big|\int_K v\eta ds\Big| \leq C\|v\|_{L^p(K)}\mathrm{Vol(K)}^{\frac{p-1}{p}} + C\|v\|_{L^p(K)}\|\eta\|_{L^2(K)}^{\frac{2p-2}{p}} \\
\leq& Ch^{\frac{2p-2}{p}}\|v\|_{L^p(K)} + C\|v\|_{L^p(K)}[Ch(\|e\|_{L^2(\Delta K)} \\
&+ \|\nabla e\|_{L^2(\Delta K)} + \|e_t\|_{L^2(\Delta K)})]^{\frac{2p-2}{p}} \\
\leq& Ch^{\frac{2p-2}{p}}\|v\|_{L^p(K)} + Ch^{\frac{2p-2}{p}}\|v\|_{L^p(K)}\|e\|_{L^2(\Delta K)}^{\frac{2p-2}{p}} \\
&+ Ch^{\frac{2p-2}{p}}\|v\|_{L^p(K)}\|\nabla e\|_{L^2(\Delta K)}^{\frac{2p-2}{p}} + Ch^{\frac{2p-2}{p}}\|v\|_{L^p(K)}\|e_t\|_{L^2(\Delta K)}^{\frac{2p-2}{p}} \\
\leq& Ch^{\frac{2p-2}{p}}\|v\|_{L^p(K)} + Ch^{2p-2}\|v\|_{L^p(K)}^p + C_0\|e\|_{L^2(\Delta K)}^2 \\
&+ Ch^{2p-2}\|v\|_{L^p(K)}^p + C_0\|\nabla e\|_{L^2(\Delta K)}^2 \\
&+ Ch^{(2-\alpha)p-2}\|v\|_{L^p(K)}^p + \hat{C}_0 h^{\frac{2\alpha p}{2p-2}}\|e_t\|_{L^2(\Delta K)}^2 \\
\leq& Ch^{\frac{2p-2}{p}}\|v\|_{L^p(K)} + Ch^{(2-\alpha)p-2}\|v\|_{L^p(K)}^p + C_0\|e\|_{L^2(\Delta K)}^2 \\
&+ C_0\|\nabla e\|_{L^2(\Delta K)}^2 + \hat{C}_0 h^{\frac{\alpha p}{p-1}}\|e_t\|_{L^2(\Delta K)}^2,
\end{aligned}
\tag{3.3.15}
$$

for $C_0 > 0$ as small enough we wish, and $\hat{C}_0 > 0$, and for any $\alpha \geq 0$. Here, we used Young's inequality since $\frac{2p-2}{p} < 2$, with $\frac{2p-2}{2p} + \frac{1}{p} = 1$, for the bounds

$$
Ch^{\frac{2p-2}{p}}\|v\|_{L^p(K)}\|e\|_{L^2(\Delta K)}^{\frac{2p-2}{p}} \leq Ch^{2p-2}\|v\|_{L^p(K)}^p + C_0\|e\|_{L^2(\Delta K)}^2,
$$

$$
Ch^{\frac{2p-2}{p}}\|v\|_{L^p(K)}\|\nabla e\|_{L^2(\Delta K)}^{\frac{2p-2}{p}} \leq Ch^{2p-2}\|v\|_{L^p(K)}^p + C_0\|\nabla e\|_{L^2(\Delta K)}^2
$$

and

$$
Ch^{\frac{2p-2}{p}}\|v\|_{L^p(K)}\|e_t\|_{L^2(\Delta K)}^{\frac{2p-2}{p}} \leq Ch^{(2-\alpha)p-2}\|v\|_{L^p(K)}^p + \hat{C}_0 h^{\frac{2\alpha p}{2p-2}}\|e_t\|_{L^2(\Delta K)}^2,
$$

for any $\alpha \geq 0$.

If the space-time partition element K is $m+1$-dimensional with $m \geq 1$ then since ℓ is an edge of K, it follows that ℓ is m-dimensional, and so $\mathrm{Vol}(\ell) \leq Ch$ for all $m \geq 1$. The same calculation as before, setting now $A := \ell$ in (3.3.14), yields

$$\begin{aligned}\Big|\int_\ell v\eta ds\Big| \leq & Ch^{\frac{p-1}{p}}\|v\|_{L^p(\ell)} + Ch^{\frac{p-1}{p}}\|v\|_{L^p(\ell)}\|e\|_{L^2(\Delta K)}^{\frac{2p-2}{p}} \\ & + Ch^{\frac{p-1}{p}}\|v\|_{L^p(\ell)}\|\nabla e\|_{L^2(\Delta K)}^{\frac{2p-2}{p}} + Ch^{\frac{p-1}{p}}\|v\|_{L^p(\ell)}\|e_t\|_{L^2(\Delta K)}^{\frac{2p-2}{p}} \\ \leq & Ch^{\frac{p-1}{p}}\|v\|_{L^p(\ell)} + Ch^{(1-\alpha)p-1}\|v\|_{L^p(\ell)}^p + C_0\|e\|_{L^2(\Delta K)}^2 \\ & + C_0\|\nabla e\|_{L^2(\Delta K)}^2 + \hat{C}_0 h^{\frac{\alpha p}{p-1}}\|e_t\|_{L^2(\Delta K)}^2.\end{aligned} \tag{3.3.16}$$

Since any element of $\mathcal{T}_h^i$ has a bounded number of edges, independent of i, this yields for arbitrary z the equivalence condition

$$C_1\|z\|_{L^2(G^i)} \leq \sum_{K\in\mathcal{T}_h^i}\|z\|_{L^2(\Delta K)} \leq C_2\|z\|_{L^2(G^i)}. \tag{3.3.17}$$

Additionally, see in [19]

$$\begin{aligned}\|\eta^{i+0}\|_{L^2(D)}^2 \leq & Ch\sum_{K\in\mathcal{T}_h^i}\|e\|_{H^1(\Delta K)}^2 \leq Ch\sum_{K\in\mathcal{T}_h^i}\|e\|_{H^1(K)}^2 \\ & = Ch\sum_{K\in\mathcal{T}_h^i}[\|e\|_{L^2(K)}^2 + \|\nabla e\|_{L^2(K)}^2 + \|e_t\|_{L^2(K)}^2].\end{aligned} \tag{3.3.18}$$

So, by the above, we have

$$\begin{aligned}(u_h^i - u_h^{i+0}, \eta^{i+0})_D \leq & \|u_h^i - u_h^{i+0}\|_{L^2(D)}\|\eta^{i+0}\|_{L^2(D)} \\ \leq & C_0\|u_h^i - u_h^{i+0}\|_{L^2(D)}^2 + C\|\eta^{i+0}\|_{L^2(D)}^2 \\ \leq & C_0\|u_h^i - u_h^{i+0}\|_{L^2(D)}^2 + Ch\sum_{K\in\mathcal{T}_h^i}\|e\|_{L^2(K)}^2 \\ & + Ch\sum_{K\in\mathcal{T}_h^i}\|\nabla e\|_{L^2(K)}^2 + Ch\sum_{K\in\mathcal{T}_h^i}\|e_t\|_{L^2(K)}^2,\end{aligned} \tag{3.3.19}$$

for $C_0 > 0$ as small as we wish.

Using (3.3.15), (3.3.16) in summation and then the equivalence condition (3.3.17) on the

$\|\cdot\|_{L^2(\Delta K)}$ terms, and (3.3.19), we derive

$$
\begin{aligned}
&(b(\varepsilon)-\varepsilon^{-2})\sum_{i=0}^{n-1}\|e\|^2_{L^2(G^i)} + \|\nabla e\|^2_{L^2(G(0,t^n))} + \varepsilon^{-2}\sum_{i=0}^{n-1}((\exp(2b(\varepsilon)t)(u^3-(u_h)^3),e))_{G^i} \\
&\quad + \frac{1}{2}\|e^n\|^2_{L^2(D)} + \frac{1}{2}\sum_{i=0}^{n-1}\|u_h^{i+0}-u_h^i\|^2_{L^2(D)} \\
&\quad \le \mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3+\mathcal{A}_4+\mathcal{R}_0+\mathcal{R}_1+\mathcal{R}_2,
\end{aligned}
$$

where

$$
\mathcal{A}_1 := Ch^{\frac{2p-2}{p}}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\|\varepsilon^{-1}e^{-b(\varepsilon)t}\xi_t^\varepsilon - \partial_t u_h + \Delta u_h - (b(\varepsilon)-\varepsilon^{-2})u_h\|_{L^p(K)},
$$

$$
\mathcal{A}_2 := Ch^{(2-\alpha)p-2}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\|\varepsilon^{-1}e^{-b(\varepsilon)t}\xi_t^\varepsilon - \partial_t u_h + \Delta u_h - (b(\varepsilon)-\varepsilon^{-2})u_h\|^p_{L^p(K)},
$$

$$
\begin{aligned}
\mathcal{A}_3 :&= Ch^{\frac{p-1}{p}}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\sum_{\ell\in E^i_{K\mathrm{in}}}\Big(\int_\ell |[\nabla u_h\cdot\mathbf{n}]_\ell|^p ds\Big)^{1/p} \\
&\quad + Ch^{(1-\alpha)p-1}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\sum_{\ell\in E^i_{K\mathrm{in}}}\int_\ell |[\nabla u_h\cdot\mathbf{n}]_\ell|^p ds,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_4 :&= Ch^{\frac{p-1}{p}}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\sum_{\ell\in E^i_{K\mathrm{b}}}\Big(\int_\ell |\nabla u_h\cdot\mathbf{n}|^p ds\Big)^{1/p} \\
&\quad + Ch^{(1-\alpha)p-1}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\sum_{\ell\in E^i_{K\mathrm{b}}}\int_\ell |\nabla u_h\cdot\mathbf{n}|^p ds,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_0 :&= CC_0\sum_{i=0}^{n-1}\|u_h^i - u_h^{i+0}\|^2_{L^2(D)} + [Ch + CC_0]\sum_{i=0}^{n-1}\|e\|^2_{L^2(G^i)} \\
&\quad + [Ch + CC_0]\|\nabla e\|^2_{L^2(G(0,t^n))},
\end{aligned}
$$

and

$$
\mathcal{R}_1 := C\Big|\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}((\varepsilon^{-2}e^{2b(\varepsilon)t}(u_h)^3,\eta))_K\Big|,
$$

$$
\mathcal{R}_2 := [Ch + Ch^{\frac{\alpha p}{p-1}}]\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\|\partial_t e\|^2_{L^2(K)}.
$$

Since C_0 is as small we wish, and for h small enough, then all the terms involving $\|u_h^i - u_h^{i+0}\|^2_{L^2(D)}$, $\|e^n\|^2_{L^2(D)}$, $\|e\|_{L^2(G_i)}$ and $\|\nabla e\|_{L^2(G(0,t^n))}$ will be hidden at the left, and the result follows after taking expectation at both sides. □

Remark 25. In view of right hand-side of the estimate of the previous theorem, we can bound first η, by using e, ∇e, e_t. Then, we can bound e_t by using e, the initial data, the mild noise, and arrive at the final *a posteriori* estimate. In particular,

$$\|e_t\|_{L^2(K)} = \|u_t - \partial_t u_h\|_{L^2(K)} \leq \|u_t\|_{L^2(K)} + \|\partial_t u_h\|_{L^2(K)},$$

where the term $\|u_t\|_{L^2(K)}$ is bounded explicitly by a function $\mathcal{F} = \mathcal{F}(u_0, \mathrm{Vol}(D), T, \xi_t^\varepsilon, b(\varepsilon), \varepsilon)$, while $\|\partial_t u_h\|_{L^2(K)}$ is transferred to the estimator.

Using the previous theorem, and the upper bound of $\|\eta\|_{L^2(K)}$ in terms of e, ∇e, e_t we derive the next estimate.

Theorem 3.3.4. Let u and u_h be the solutions of (3.3.2) and (3.3.4), respectively, $e = u - u_h$, and h small enough. Then for any given p, α, with $p \geq 2$, $\alpha \geq 0$, there exists positive constant $c > 0$ independent of ε and h, such that

$$\begin{aligned}\mathbb{E}\Big[\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\|\nabla e\|^2_{L^2(K)} + \sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\|e\|^2_{L^2(K)} + \varepsilon^{-2}\sum_{i=0}^{n-1}((\exp(2b(\varepsilon)t)(u^3 - (u_h)^3), e))_{G^i} \\ + \|e^n\|^2_{L^2(D)} + \sum_{i=0}^{n-1}\|u_h^{i+0} - u_h^i\|^2_{L^2(D)}\Big] \\ \leq \mathbb{E}[\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{A}_5],\end{aligned}$$

for

$$\mathcal{A}_1 := ch^{\frac{2p-2}{p}}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\|\varepsilon^{-1}m(\varepsilon,t)\xi_t^\varepsilon - \partial_t u_h + \Delta u_h - (b(\varepsilon)-\varepsilon^{-2})u_h - \varepsilon^{-2}\exp(2b(\varepsilon)t)(u_h)^3\|_{L^p(K)},$$

$$\mathcal{A}_2 := ch^{(2-\alpha)p-2}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\|\varepsilon^{-1}m(\varepsilon,t)\xi_t^\varepsilon - \partial_t u_h + \Delta u_h - (b(\varepsilon)-\varepsilon^{-2})u_h - \varepsilon^{-2}\exp(2b(\varepsilon)t)(u_h)^3\|^p_{L^p(K)},$$

$$\mathcal{A}_3 := ch^{\frac{p-1}{p}}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\sum_{\ell\in E^i_{K\mathrm{in}}}\Big(\int_\ell |[\nabla u_h\cdot\mathbf{n}]_\ell|^p ds\Big)^{1/p} + ch^{(1-\alpha)p-1}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\sum_{\ell\in E^i_{K\mathrm{in}}}\int_\ell |[\nabla u_h\cdot\mathbf{n}]_\ell|^p ds,$$

$$\mathcal{A}_4 := ch^{\frac{p-1}{p}}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\sum_{\ell\in E^i_{K\mathrm{b}}}\Big(\int_\ell |\nabla u_h\cdot\mathbf{n}|^p ds\Big)^{1/p} + ch^{(1-\alpha)p-1}\sum_{i=0}^{n-1}\sum_{K\in\mathcal{T}_h^i}\sum_{\ell\in E^i_{K\mathrm{b}}}\int_\ell |\nabla u_h\cdot\mathbf{n}|^p ds,$$

and

$$\mathcal{A}_5 := [ch + ch^{\frac{\alpha p}{p-1}}] \sum_{i=0}^{n-1} \sum_{K \in \mathcal{T}_h^i} \|\partial_t e\|_{L^2(K)}^2.$$

Remark 26. Considering the previous theorem, the terms $\mathcal{A}_i$ for $i = 1, \cdots, 4$ involve the approximate solution u_h and normally depend on h, and consist thus parts of the *a posteriori* estimator, that as it is expected depends on u_h. The term $\mathcal{A}_5$ involves $\|\partial_t e\|_{L^2(K)}^2$, for which as mentioned, it holds that $\|\partial_t e\|_{L^2(K)}^2 = \|u_t - \partial_t u_h\|_{L^2(K)} \leq \|u_t\|_{L^2(K)} + \|\partial_t u_h\|_{L^2(K)}$, which obviously depends on u_h. Moreover, cf. in [17], the u_t term there is bounded with bounds depending only on the initial data, the volume of the space domain, $b(\varepsilon)$, the noise, T, and ε which all consist defined from the start parameters of the continuous problem and do not depend on h or u.

Various interesting numerical schemes have been applied so far for the stochastic Allen-Cahn equation with noise rougher than the mild noise analyzed in this Thesis, see in [30, 54, 101, 95, 127]. In [44] strong and weak error estimates have been established in space for finite element approximation on stochastic equations with one-sided Lipschitz coefficients with additive noise, including as a special case the stochastic Allen-Cahn equation. We refer also to [96] for an optimal strong error analysis when the noise is multiplicative. The possible advantages of our scheme are briefly summarized as follows: It is adaptive in time and of high order of accuracy, it avoids any RK method or finite differences approximation in time; in case of tensor finite elements, higher accuracy can be reached by just elevating the order of the piece-wise polynomial approximations in time. The initial condition of the continuous problem is used as the initial condition of the discrete scheme without any approximation. The presence of ε permits the numerical approximation near the sharp interface limit, while the time adaptivity can capture in the numerical solution new layers generation and annihilation which occur in the physical problem.

Chapter 4

Galerkin finite element approximation of a stochastic semilinear space-time fractional subdiffusion with fractionally integrated noise

4.1 Introduction

This chapter discusses the Galerkin finite element method applied to approximate the solution of a semilinear stochastic space-time fractional subdiffusion problem with the Caputo fractional derivative of order $\alpha \in (0, 1)$ driven by fractionally integrated additive noise, [135, 138, 116, 115, 114, 56, 55, 52, 49, 72, 117]. After discussing the existence, uniqueness and regularity of the solution, we approximate the noise with a piecewise constant function in time in order to obtain a regularized stochastic fractional subdiffusion problem. The regularized problem is then approximated by using the finite element method in spatial direction. The mean squared errors are proved based on the sharp estimates of the various Mittag-Leffler functions involved in the integrals.

4.2 Physical model defined in whole $\mathbb{R}^d, d = 1, 2, 3$

Consider the transport of particles in medium with memory (e.g, heat conduct) and let $u(t,x), e(t,x), \vec{F}(t,x)$ denote the temperature of materials, internal energy and heat flow (heat flux) respectively. Then we have, [40],

$$\frac{\partial e(t,x)}{\partial t} = -div(\vec{F}),$$
$$e(t,x) = \beta u(t,x),$$
$$\vec{F}(t,x) = -\lambda \nabla u(t,x),$$

where $\beta,\ \lambda > 0$ are some positive constants. The temperature of the materials then satisfies the classical heat equation

$$\beta \frac{\partial u}{\partial t} = \lambda \Delta u.$$

In the medium with memory, the internal energy satisfies

$$e(t,x) = \bar{\beta} u(t,x) + \int_0^t n(t-s) u(s,x)\, ds,$$

here $\bar{\beta} \geq 0$, n denotes the kernel function, with $\gamma_1 \in (0,1)$,

$$n(t) = \frac{1}{\Gamma(1-\gamma_1)} t^{-\gamma_1}.$$

The convolution means the internal energy $e(t,x)$ depends on the temperature $u(s,x)$ of the materials for the past time $0 < s < t$.

In the real problem, the internal energy $e(t,x)$ depends on the temperature in past time randomly. We introduce the noise

$$e(t,x) = \bar{\beta} u(t,x) + \int_0^t n(t-s) u(s,x)\, ds + \int_0^t l(t-s) h(s, u(s,x))\, dW(s), \qquad (4.2.1)$$

where W denotes the random effect of the heat source, l denotes a kernel function.

Choose, with $\gamma_2 \in (0,1)$,

$$l(t) = \frac{t^{1-\gamma_2}}{\Gamma(2-\gamma_2)}.$$

Assume that $\bar{\beta} = 0$ and taking the derivative of (4.2.1), we get

$$\lambda \Delta u = -\, div \vec{F} = \frac{\partial e(t,x)}{\partial t} = \frac{1}{\Gamma(1-\gamma_1)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\gamma_1} u(s,x)\, ds$$
$$+ \frac{1}{\Gamma(2-\gamma_2)} \frac{\partial}{\partial t} \int_0^t (t-s)^{1-\gamma_2} h(s, u(s,x))\, dW(s).$$

Thus we have

$$ {}_0^C D_t^{\gamma_1} u(t,x) = \lambda \Delta u(t,x) - {}_0^R D^{\gamma_2} \int_0^t h(s,u(s,x))\, dW(s). $$

That is, with $\gamma_1, \gamma_2 \in (0,1)$,

$$ {}_0^C D_t^{\gamma_1} u(t,x) = \lambda \Delta u(t,x) - {}_0^R D_t^{\gamma_2 - 1} h(s,u(s,x))\, \dot{W}(t,x), $$

which is the fractional model we are interested in. Here $\dot{W}(t,x)$ denotes the noise and ${}_0^C D_t^{\gamma_1} w$ and ${}_0^R D_t^{\gamma_2-1} w$ denote the Caputo fractional derivative and Riemann-Liouville fractional integral, respectively.

4.3 Physical model defined on a bounded domain $D \subset \mathbb{R}^d$, $d = 1, 2, 3$

In this chapter, we shall consider the finite element approximation of the following stochastic semilinear space-time fractional subdiffusion problem driven by fractionally integrated additive noise [40, 13, 94, 38] with $0 < \alpha < 1$, $\frac{1}{2} < \beta \leqslant 1$, $0 \leqslant \gamma \leqslant 1$,

$$ \begin{aligned} & {}_0^C D_t^{\alpha} u(t,x) + (-\Delta)^{\beta} u(t,x) = f(t, u(t,x)) + {}_0^R D_t^{-\gamma} \dot{W}(t,x),\ 0 < t < T,\ x \in D, \\ & u(t,x) = 0,\ 0 < t < T,\ x \in \partial D, \qquad (4.3.1) \\ & u(0,x) = u_0(x),\ x \in D. \end{aligned} $$

Here ${}_0^C D_t^{\alpha} w$ and ${}_0^R D_t^{-\gamma} w$ denote the Caputo fractional derivative and Riemann-Liouville fractional integral, respectively. We assume that the noise takes the following form

$$ \dot{W}(t,x) = \sum_{j=1}^{\infty} \sigma_j(t) e_j(x) \dot{\beta}_j(t), \qquad (4.3.2) $$

where $\sigma_j(t), j = 1, 2, \dots$ decay rapidly with respect to j. For example, if $\sigma_j(t) = \gamma_j^{1/2}$ and $\mathrm{Tr}(Q) = \sum_{j=1}^{\infty} \gamma_j < \infty$, then $W(t)$ is called a trace class noise. If $\sigma_j(t) = 1$, then $W(t)$ is called a white noise. Here, $e_j(x), j = 1, 2, \dots$ are eigen functions of the elliptic operator $A = -\Delta$, $D(A) = H_0^1(D) \cap H^2(D)$, and $\beta_j(t),\ j = 1, 2, \dots,$ denote the Brownian motions.

Limited literature addresses the numerical approximation of the stochastic subdiffusion problem (4.3.1). Cao et al. [34] explored numerical methods for a stochastic semilinear time fractional partial differential equation driven by fractional Brownian motion. Jin et al. [73] developed and analyzed a numerical approach for the linear variant of (4.3.1), specifically when $f = 0$ and $\beta = 1$, with noise of the form ${}_0^R D_t^{-\gamma} \dot{W}(t), \gamma \in [0,1]$, where $W(t)$ denotes a Wiener process with covariance operator Q. This method employed linear finite element approximation in space and the classical Grünwald-Letnikov method in time, along with L^2-projection for handling the noise, see also Zou [139]. Wu et al. [131] tackled the time discretization of a stochastic linear subdiffusion problem, presenting error estimates for their proposed scheme. Li et al. [89] explored the finite element Galerkin approximation for a stochastic space-time fractional linear wave equation in one space dimension, driven by additive space-time noise, within the parameter range $\alpha \in (1,2)$, $\gamma = 1$, and $f = 0$. More recently, Li et al. [90] analyzed a numerical method for the stochastic semilinear space fractional superdiffusion problem driven by fractional noise with $\alpha \in (1,2)$.

This study centers on the finite element approximation of (4.3.1) with a time fractional derivative order $\alpha \in (0,1)$. Unlike the case of stochastic superdiffusion with $\alpha \in (1,2)$ investigated by Li et al. (2017), (2019), the stochastic subdiffusion problem with $\alpha \in (0,1)$ poses greater challenges due to the singularity of the solution near $t = 0$.

To the best of our knowledge, the work in this chapter represents the first attempt at devising numerical techniques for approximating the solution of a stochastic semilinear space-time fractional subdiffusion problem with $\alpha \in (0,1)$ driven by fractionally integrated additive noise. The exact solution is explicitly expressed using Mittag-Leffler functions, and the existence of a unique solution is established via the Banach contraction mapping theorem. Following a method similar to recent articles by Li et al. (2017), (2019) for $\alpha \in (1,2)$, we approximate the noise using piecewise constant functions in time to regularize the problem. Subsequently, we apply a finite element method to discretize the spatial direction of the regularized problem, deriving mean squared error estimates. The final error estimate encompasses contributions from errors due to regularization and the finite element Galerkin approximation of the regularized problem.

Let $H = L^2(D)$ with norm $\| \cdot \|$ and the inner product $(\cdot, \cdot)$. Let $H_0^1 = \{v \in H^1 : v =$

0 on $\partial D\}$. Let $A = -\Delta : D(A) = H^2(D) \cap H_0^1(D) \to H$ be a closed linear self-adjoint positive definite operator with compact inverse and assume that $(\lambda_k, e_k), k = 1, 2, 3, \cdots$ is a sequence of the eigenpairs of $-\Delta$. The sequence $\{e_k\}_{k=1}^{\infty}$ forms an orthonormal basis of H.

Set $\dot{H}^s(D)$ or simply $\dot{H}^s$ for any $s \in \mathbb{R}$, as a Hilbert space induced by the norm

$$|\psi|_s := \big(\sum_{k=1}^{\infty} \lambda_k^s (\psi, e_k)^2\big)^{\frac{1}{2}}. \tag{4.3.3}$$

For $s = 0$ we denote $\dot{H}^0$ by H.

Let $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\big)$ be a complete filtered probability space with $\mathcal{F}_0$ containing all P-null sets of $\mathcal{F}$ and let $L^2(\Omega; \dot{H}^s)$ be a separable Hilbert space of all strongly measurable square-integrable random variables ϕ with values in $\dot{H}^s$ such that $\|\phi\|_{L^2(\Omega;\dot{H}^s)} := (\mathbf{E}|\phi|_s^2)^{\frac{1}{2}} < \infty$, where $\mathbf{E}$ denotes the expectation.

Lemma 4.3.1. [99, Theorem 10.16] (Itô isometry property) Let $\{\psi(s) : s \in [0, T]\}$ be a real-valued predictable process such that $\int_0^T \mathbf{E}|\psi(s)|^2 ds < \infty$. Let $B(t)$ denote a real-valued standard Brownian motion. Then, the following isometry equality holds for $t \in (0, T]$,

$$\mathbf{E}\Big|\int_0^t \psi(s) dB(s)\Big|^2 = \int_0^t \mathbf{E}|\psi(s)|^2 ds. \tag{4.3.4}$$

Mittag-Leffler function plays a very important role in the error estimate of our problem. Now let us introduce the Mittag-Leffler function.

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \ z \in \mathbb{C}, \alpha > 0.$$

A two parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ \alpha > 0, \ \beta \in \mathbb{R}, z \in \mathbb{C}. \tag{4.3.5}$$

Lemma 4.3.2. (Mittag-Leffler function property) [111]
Let $0 < \bar{\alpha} < 1$ and $\bar{\beta} \in \mathbb{R}$. Let $E_{\bar{\alpha},\bar{\beta}}$ be defined by (4.3.5). Suppose that μ is an arbitrary real number such that $\frac{\pi\bar{\alpha}}{2} < \mu < \min(\pi, \pi\bar{\alpha})$. Then there exists a constant $C = C(\bar{\alpha}, \bar{\beta}, \mu) > 0$ such that

$$|E_{\bar{\alpha},\bar{\beta}}(z)| \leq \frac{C}{1 + |z|}, \ \mu \leq |\arg(z)| \leq \pi. \tag{4.3.6}$$

In particular,

$$|E_{\bar{\alpha},\bar{\beta}}(z)| \le C,\ \mu \le |\arg(z)| \le \pi. \tag{4.3.7}$$

Moreover, for $\lambda > 0,\ \bar{\alpha} > 0,\ \bar{\beta} > 0,\ \bar{\gamma} > 0, \bar{\gamma} \ne 1$,

$$\frac{d}{dt}\Big(t^{\bar{\gamma}-1}E_{\bar{\alpha},\bar{\gamma}}(-\lambda^{\bar{\beta}}t^{\bar{\alpha}})\Big) = t^{\bar{\gamma}-2}E_{\bar{\alpha},\bar{\gamma}-1}(-\lambda^{\bar{\beta}}t^{\bar{\alpha}}),\ t > 0. \tag{4.3.8}$$

To show (4.3.8) we note that $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}$, thus we have the following $\frac{d}{dt}\big(t^{\bar{\gamma}-1}\sum_{k=0}^{\infty}\frac{(-\lambda^{\bar{\beta}}t^{\bar{\alpha}})^k}{\Gamma(\bar{\alpha}k+\bar{\gamma})}\big) = t^{\bar{\gamma}-2}\sum_{k=0}^{\infty}\frac{(-\lambda^{\bar{\beta}}t^{\bar{\alpha}})^k}{\Gamma(\alpha k+\bar{\gamma}-1)}$. For $k = 0, 1, 2, 3, \cdots$ we have the following expansion

$$\begin{aligned}
&\frac{d}{dt}\left(t^{\bar{\gamma}-1}\left[\frac{(-\lambda^{\bar{\beta}}t^{\bar{\alpha}})^0}{\Gamma(\bar{\gamma})} + \frac{(-\lambda^{\bar{\beta}}t^{\bar{\alpha}})}{\Gamma(\bar{\alpha}+\bar{\gamma})} + \frac{(-\lambda^{\bar{\beta}}t^{\bar{\alpha}})^2}{\Gamma(2\bar{\alpha}+\bar{\gamma})} + \frac{(-\lambda^{\bar{\beta}}t^{\bar{\alpha}})^3}{\Gamma(3\bar{\alpha}+\bar{\gamma})} + \cdots\right]\right) \\
&= t^{\bar{\gamma}-2}\left[\frac{(-\lambda^{\bar{\beta}}t^{\bar{\alpha}})^0}{\Gamma(\bar{\gamma}-1)} + \frac{(-\lambda^{\bar{\beta}}t^{\bar{\alpha}})}{\Gamma(\bar{\alpha}+\bar{\gamma}-1)} + \frac{(-\lambda^{\bar{\beta}}t^{\bar{\alpha}})^2}{\Gamma(2\bar{\alpha}+\bar{\gamma}-1)} + \frac{(-\lambda^{\bar{\beta}}t^{\bar{\alpha}})^3}{\Gamma(3\bar{\alpha}+\bar{\gamma}-1)} + \cdots\right],
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{dt}\left(t^{\bar{\gamma}-1}\left[\frac{1}{\Gamma(\bar{\gamma})} - \frac{\lambda^{\bar{\beta}}t^{\bar{\alpha}}}{\Gamma(\bar{\alpha}+\bar{\gamma})} + \frac{\lambda^{2\bar{\beta}}t^{2\bar{\alpha}}}{\Gamma(2\bar{\alpha}+\bar{\gamma})} - \frac{\lambda^{3\bar{\beta}}t^{3\bar{\alpha}}}{\Gamma(3\bar{\alpha}+\bar{\gamma})} + \cdots\right]\right) \\
&= t^{\bar{\gamma}-2}\left[\frac{1}{\Gamma(\bar{\gamma}-1)} - \frac{\lambda^{\bar{\beta}}t^{\bar{\alpha}}}{\Gamma(\bar{\alpha}+\bar{\gamma}-1)} + \frac{\lambda^{2\bar{\beta}}t^{2\bar{\alpha}}}{\Gamma(2\bar{\alpha}+\bar{\gamma}-1)} - \frac{\lambda^{3\bar{\beta}}t^{3\bar{\alpha}}}{\Gamma(3\bar{\alpha}+\bar{\gamma}-1)} + \cdots\right].
\end{aligned}$$

Removing the brackets on the right hand side yields

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{t^{\bar{\gamma}-1}}{\Gamma(\bar{\gamma})} - \frac{\lambda^{\bar{\beta}}t^{\bar{\alpha}+\bar{\gamma}-1}}{\Gamma(\bar{\alpha}+\bar{\gamma})} + \frac{\lambda^{2\bar{\beta}}t^{2\bar{\alpha}+\bar{\gamma}-1}}{\Gamma(2\bar{\alpha}+\bar{\gamma})} - \frac{\lambda^{3\bar{\beta}}t^{3\bar{\alpha}+\bar{\gamma}-1}}{\Gamma(3\bar{\alpha}+\bar{\gamma})} + \cdots\right) \\
&= \frac{t^{\bar{\gamma}-2}}{\Gamma(\bar{\gamma}-1)} - \frac{\lambda^{\bar{\beta}}t^{\bar{\alpha}+\bar{\gamma}-2}}{\Gamma(\bar{\alpha}+\bar{\gamma}-1)} + \frac{\lambda^{2\bar{\beta}}t^{2\bar{\alpha}+\bar{\gamma}-2}}{\Gamma(2\bar{\alpha}+\bar{\gamma}-1)} - \frac{\lambda^{3\bar{\beta}}t^{3\bar{\alpha}+\bar{\gamma}-2}}{\Gamma(3\bar{\alpha}+\bar{\gamma}-1)} + \cdots.
\end{aligned}$$

Differentiating the left hand side with respect to t, we arrive at

$$\begin{aligned}
&(\bar{\gamma}-1)\frac{t^{\bar{\gamma}-2}}{\Gamma(\bar{\gamma})} - (\bar{\alpha}+\bar{\gamma}-1)\frac{\lambda^{\bar{\beta}}t^{\bar{\alpha}+\bar{\gamma}-2}}{\Gamma(\bar{\alpha}+\bar{\gamma})} + (2\bar{\alpha}+\bar{\gamma}-1)\frac{\lambda^{2\bar{\beta}}t^{2\bar{\alpha}+\bar{\gamma}-2}}{\Gamma(2\bar{\alpha}+\bar{\gamma})} \\
&- (3\bar{\alpha}+\bar{\gamma}-1)\frac{\lambda^{3\bar{\beta}}t^{3\bar{\alpha}+\bar{\gamma}-2}}{\Gamma(3\bar{\alpha}+\bar{\gamma})} + \cdots = \frac{t^{\bar{\gamma}-2}}{\Gamma(\bar{\gamma}-1)} - \frac{\lambda^{\bar{\beta}}t^{\bar{\alpha}+\bar{\gamma}-2}}{\Gamma(\bar{\alpha}+\bar{\gamma}-1)} + \frac{\lambda^{2\bar{\beta}}t^{2\bar{\alpha}+\bar{\gamma}-2}}{\Gamma(2\bar{\alpha}+\bar{\gamma}-1)} \\
&- \frac{\lambda^{3\bar{\beta}}t^{3\bar{\alpha}+\bar{\gamma}-2}}{\Gamma(3\bar{\alpha}+\bar{\gamma}-1)} + \cdots
\end{aligned}$$

Note that $\Gamma(\gamma) = (\gamma-1)\Gamma(\gamma-1)$. Therefore we show (4.3.8).

Assumption 4.3.1. [125] There is a positive constant C such that the nonlinear function $f : \mathbb{R}^+ \times H \to H$ satisfies

$$\|f(t_1, u_1) - f(t_2, u_2)\| \leq C(|t_1 - t_2| + \|u_1 - u_2\|),$$

and

$$\|f(t, u)\| \leq C(1 + \|u\|).$$

For example, the following functions f all satisfy Assumption 4.3.1: $f(t, u) = u,\ f(t, u) = \frac{Cu}{1+u^2},\ C > 0$ and $f(t, u) = \sin(u)$.

Assumption 4.3.2. [51] The sequence $\sigma_k(t)$ with its derivative is uniformly bounded by $|\sigma_k(t)| \leq \mu_k,\ |\sigma_k'(t)| \leq \gamma_k,\quad \forall\ t \in [0, T],$ where $\sum_{k=1}^{\infty} \mu_k$ and $\sum_{k=1}^{\infty} \gamma_k$ are convergent.

Assumption 4.3.3. (Regularity of the noise) Let $\frac{1}{2} < \alpha < 1,\ \frac{1}{2} < \beta \leq 1,\ 0 \leq \gamma \leq 1$. We assume, with $0 \leq r \leq \kappa$,

$$\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^{r-\kappa} < \infty,$$

where κ is defined by

$$\kappa = \begin{cases} 2\beta,\ 2\gamma > 1, \\ (2 - \frac{1-2\gamma}{\alpha})\beta - \epsilon,\ 2\gamma \leq 1, \end{cases},$$

and $\lambda_k, k = 1, 2, 3, \cdots$ are eigenvalues of the operator $A = -\Delta$ with $D(A) = H_0^1(D) \cap H^2(D)$.

Definition 4.3.1. (Lemma 2.4 in [89] or (4.1) in [69])

An adapted process $u(t)_{t \geq 0}$ is called a mild solution to (4.3.1) if it satisfies the following integral equation

$$u(t) = \mathbb{E}_{\alpha,\beta}(t) u_0(x) + \int_0^t \bar{\mathbb{E}}_{\alpha,\beta}(t-s) f(s, u(s)) ds + \int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s) dW(s), \quad (4.3.9)$$

where $dW(s)$ denotes

$$dW(s) = \sum_{k=1}^{\infty} \sigma_k(s) e_k d\beta_k(s), \quad (4.3.10)$$

and

$$\mathbb{E}_{\alpha,\beta}(t)u_0(x) := \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k^{\beta}t^{\alpha})(u_0(x),e_k)e_k,$$

$$\bar{\mathbb{E}}_{\alpha,\beta}(t)u_0(x) := t^{\alpha-1}\sum_{k=1}^{\infty} E_{\alpha,\alpha}(-\lambda_k^{\beta}t^{\alpha})(u_0(x),e_k)e_k,$$

$$\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t)u_0(x) := t^{\alpha+\gamma-1}\sum_{k=1}^{\infty} E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}t^{\alpha})(u_0(x),e_k)e_k.$$

The operators $\mathbb{E}_{\alpha,\beta}(t)$, $\bar{\mathbb{E}}_{\alpha,\beta}(t)$ and $\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t)$ have smoothing properties. The solution operator $\mathbb{E}_{\alpha,\beta}(t)$ satisfies the following smoothing properties for $t>0$, see e.g. Lemma 4.1 in [69] or Lemma 2.5 in [89],

$$|\mathbb{E}_{\alpha,\beta}(t)u_0|_p \le Ct^{-\alpha\frac{p-q}{2\beta}}|u_0|_q,\ 0\le p-q\le 2\beta,\ p>q. \tag{4.3.11}$$

Next we consider the smoothing properties of the solution operator $\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t)$.

Lemma 4.3.3. Let $0<\alpha<1,\ \frac{1}{2}<\beta\le 1,\ 0\le\gamma\le 1$. For any $t>0$ and $0\le p-q\le 2\beta$, there holds

$$|\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t)u_0|_p \le Ct^{-1+(\alpha+\gamma)-\alpha\frac{p-q}{2\beta}}|u_0|_q. \tag{4.3.12}$$

Proof. From the definition of $\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t)$, it follows that

$$\begin{aligned}
|\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t)u_0|_p^2 &= \sum_{k=1}^{\infty}\lambda_k^p|t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}t^{\alpha})|^2|(u_0,e_k)|^2\\
&\le Ct^{2(-1+(\alpha+\gamma)-\alpha\frac{p-q}{2\beta})}\sum_{k=1}^{\infty}\frac{(\lambda_k^{\beta}t^{\alpha})^{\frac{p-q}{\beta}}}{(1+\lambda_k^{\beta}t^{\alpha})^2}\lambda_k^q|(u_0,e_k)|^2\\
&\le Ct^{2(-1+(\alpha+\gamma)-\alpha\frac{p-q}{2\beta})}|u_0|_q^2.
\end{aligned}$$

Note that, by the boundedness of the Mittage-Lefler function: $\sup_k \frac{(\lambda_k^{\beta}t^{\alpha})^{\frac{p-q}{\beta}}}{(1+\lambda_k^{\beta}t^{\alpha})^2}\le C$ for $0\le p-q\le 2\beta$. This completes the rest of the proof. □

When $\gamma=0$, we obtain the estimate of $\bar{\mathbb{E}}_{\alpha,\beta}(t)$ as

$$|\bar{\mathbb{E}}_{\alpha,\beta}(t)u_0|_p \le Ct^{-1+\alpha-\frac{\alpha(p-q)}{2\beta}}|u_0|_q. \tag{4.3.13}$$

Theorem 27. ***(Existence and Uniqueness Theorem)***
Let $\frac{1}{2}<\alpha<1,\ \frac{1}{2}<\beta\le 1$ and $0\le\gamma\le 1$. Assume that the Assumptions 4.3.1, 4.3.2, 4.3.3 hold. Let $v\in L^2(\Omega;H)$. Then, there exists a unique mild solution $u\in C([0,T];L^2(\Omega;H))$ given by (4.3.9).

Proof. Set $C([0,T];L^2(\Omega;H))_\lambda,\ \lambda>0$ as the set of functions in $C([0,T];L^2(\Omega;H))$ with the following weighted norm $\|\phi\|_\lambda^2 := \sup_{t\in[0,T]}\mathbf{E}(\|e^{-\lambda t}\phi(t)\|^2),\ \forall\,\phi\in C([0,T];L^2(\Omega;H))$. Note that for any fixed $\lambda>0$ this norm is equivalent to the standard norm on $C([0,T];L^2(\Omega;H))$. We now define a nonlinear map $\mathcal{T}: C([0,T];L^2(\Omega;H))_\lambda \to C([0,T];L^2(\Omega;H))_\lambda$ by

$$\mathcal{T}u(t) = \mathbb{E}_{\alpha,\beta}(t)u_0 + \int_0^t \bar{\mathbb{E}}_{\alpha,\beta}(t-s)f(s,u(s))ds + \int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s). \qquad (4.3.14)$$

In order to apply the Banach fixed point theorem, it is sufficient to show that for an appropriately chosen $\lambda>0$, $\mathcal{T}$ is a contraction.

We first show that $\mathcal{T}u \in C([0,T];L^2(\Omega;H))$ for any $u\in C([0,T];L^2(\Omega;H))$. By Cauchy-Schwarz inequality we arrive with $u\in C([0,T];L^2(\Omega;H))$ at

$$\begin{aligned}\mathbf{E}\|\mathcal{T}u(t)\|^2 &\le 3\mathbf{E}\|\mathbb{E}_{\alpha,\beta}(t)u_0\|^2 + 3\mathbf{E}\|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta}(t-s)f(s,u(s))ds\|^2\\ &\quad + 3\mathbf{E}\|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s)\|^2\\ &\le 3\mathbf{E}\|\mathbb{E}_{\alpha,\beta}(t)u_0\|^2 + 3t\int_0^t \mathbf{E}\|\bar{\mathbb{E}}_{\alpha,\beta}(t-s)f(s,u(s))\|^2 ds\\ &\quad + 3\mathbf{E}\|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s)\|^2.\end{aligned}$$

Using the smoothing property (4.3.11) for $\bar{\mathbb{E}}_{\alpha,\beta}(t)$ and Assumption 4.3.1, it follows that

$$\begin{aligned}\mathbf{E}\|\mathcal{T}u(t)\|^2 &\le C\mathbf{E}\|u_0\|^2 + Ct\int_0^t (t-s)^{2(-1+\alpha)}(1+\mathbf{E}\|u(s)\|^2)ds\\ &\quad + C\mathbf{E}\|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s)\|^2. \qquad (4.3.15)\end{aligned}$$

For the stochastic integral, by Isometry property Lemma 4.3.1, Assumptions 4.3.2 and 4.3.3 and the smoothing property (4.3.12), there holds with $0\le r\le\kappa$,

$$\begin{aligned}&\mathbf{E}\|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s)\|^2 = \mathbf{E}\|\int_0^t A^{\frac{\kappa-r}{2}}\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)\sum_{k=1}^\infty \sigma_k(s)A^{\frac{r-\kappa}{2}}e_k d\beta_k(s)\|^2\\ &= \sum_{k=1}^\infty\int_0^t \|A^{\frac{\kappa-r}{2}}\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)\sigma_k(s)A^{\frac{r-\kappa}{2}}e_k\|^2 ds \le C\Big(\int_0^t \|A^{\frac{\kappa-r}{2}}\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(s)\|^2 ds\Big)\Big(\sum_{k=1}^\infty \mu_k^2\lambda_k^{r-\kappa}\Big)\\ &\le C\Big(\int_0^t \|A^{\frac{\kappa-r}{2}}\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(s)\|^2 ds\Big)\Big(\sum_{k=1}^\infty \mu_k^2\lambda_k^{r-\kappa}\Big) \le C\Big(\sum_{k=1}^\infty \mu_k^2\lambda_k^{r-\kappa}\Big)\int_0^t (s^{\alpha+\gamma-1-\frac{\kappa-r}{2\beta}\alpha})^2 ds\\ &\le C\Big(\sum_{k=1}^\infty \mu_k^2\lambda_k^{r-\kappa}\Big)\int_0^t s^{\alpha(2-\frac{\kappa-r}{\beta})+2\gamma-2}ds \le C\Big(\sum_{k=1}^\infty \mu_k^2\lambda_k^{r-\kappa}\Big) < \infty.\end{aligned}$$

Note that $u_0 \in L^2(\Omega; H)$ and $u \in C([0,T]; L^2(\Omega; H))$, we then obtain $\sup_{t\in[0,T]} \mathbf{E}\|\mathcal{T}u\|^2 < \infty$. That is

$$\mathbf{E}\|\mathcal{T}u(t)\|^2 \le C\mathbf{E}\|u_0(x)\|^2 + Ct\int_0^t (t-s)^{2(-1+\alpha)}\Big(1+\mathbf{E}\|u(s)\|^2\Big)ds + C\Big(\sum_{k=1}^{\infty}\mu_k^2\lambda_k^{r-\kappa}\Big), \tag{4.3.16}$$

which implies that $\mathcal{T}u \in C([0,T]; L^2(\Omega; H))$. Next we consider the contraction property of the mapping $\mathcal{T}$. For any given functions u_1 and u_2 in $C([0,T]; L^2(\Omega; H))_\lambda$ it follows that

$$\begin{aligned}
&\mathbf{E}\|e^{-\lambda t}(\mathcal{T}u_1(t) - \mathcal{T}u_2(t))\|^2 \\
&= \mathbf{E}\|e^{-\lambda t}\int_0^t \bar{\mathbb{E}}_{\alpha,\beta}(t-s)\Big(f(s,u_1(s)) - f(s,u_2(s))\Big)ds\|^2 \\
&\le t\mathbf{E}\Big(\int_0^t e^{-2\lambda(t-s)}\|\bar{\mathbb{E}}_{\alpha,\beta}(t-s)e^{-\lambda s}\Big(f(s,u_1(s)) - f(s,u_2(s))\Big)\|^2 ds\Big) \\
&\le Ct\int_0^t (t-s)^{2(-1+\alpha)}e^{-2\lambda(t-s)}\mathbf{E}\|e^{-\lambda s}\Big(f(s,u_1(s)) - f(s,u_2(s))\Big)\|^2 ds.
\end{aligned}$$

A use of $s = ty$ with Lipschitz condition for the nonlinear term from Assumption 4.3.1 and $\alpha > \frac{1}{2}$ yields

$$\begin{aligned}
&\mathbf{E}\|e^{-\lambda t}(\mathcal{T}u_1(t) - \mathcal{T}u_2(t))\|^2 \le C\lambda^{-2\alpha}\int_0^1 (1-y)^{2(-1+\alpha)}(\lambda t)^{2\alpha}e^{-2\lambda t(1-y)}dy\|u_1 - u_2\|_\lambda^2 \\
&\le C \sup_{\lambda>0, t\in[0,T], y\in[0,1]} \Big((\lambda t(1-y))^{\alpha-\frac{1}{2}}e^{-2\lambda t(1-y)}\Big)\Big[(\frac{t}{\lambda})^{\alpha-\frac{1}{2}}t\Big]\int_0^1 (1-y)^{\alpha-\frac{3}{2}}dy\|u_1 - u_2\|_\lambda^2 \\
&\le CT \sup_{\lambda>0, t\in[0,T], y\in[0,1]} \Big((\lambda t(1-y))\Big)^{\alpha-\frac{1}{2}}e^{-2\lambda t(1-y)}\Big(\frac{T}{\lambda}\Big)^{\alpha-\frac{1}{2}}\int_0^1 (1-y)^{\alpha-\frac{3}{2}}dy\|u_1 - u_2\|_\lambda^2 \\
&\le C(T)\Big(\frac{T}{\lambda}\Big)^{\alpha-\frac{1}{2}}\|u_1 - u_2\|_\lambda^2.
\end{aligned}$$

Taking maximum over $t \in [0,T]$ and choose $\lambda > 0$ appropriately so that $C(T)\Big(\frac{T}{\lambda}\Big)^{\alpha-\frac{1}{2}} = \delta$ with $\delta \in (0,1)$, that is $\|\mathcal{T}u_1 - \mathcal{T}u_2\|_\lambda \le \delta\|u_1 - u_2\|_\lambda$.

The proof of this Theorem 27 is now complete. □

In Theorem 27, we require that $\frac{1}{2} < \alpha < 1$. This condition can be relaxed to $0 < \alpha < 1$ in the following theorem.

Theorem 28. *Let $0 < \alpha < 1$, $\frac{1}{2} < \beta \le 1$, $\alpha + \gamma > \frac{1}{2}$ and $0 \le \gamma \le 1$. Assume that the Assumptions 4.3.1, 4.3.2, 4.3.3 hold. Let $v \in L^2(\Omega; H)$. Then, there exists a unique mild solution $u \in C([0,T]; L^2(\Omega; H))$ given by (4.3.9).*

Proof. The proof is based on the Banach fixed point theorem.

Step 1: Introduce the following space, with $\lambda > 0$ to be determined later,

$$C([0,T], L^2(\Omega; H))_\lambda := \{u \in C([0,T], L^2(\Omega; H)),\ \|u\|_\lambda^2 := \sup_{t\in[0,T]} \mathbf{E}\|e^{-\lambda t}u(t)\|^2 < \infty\}.$$

For any $\lambda > 0$, the following two norms are equivalent, that is, $\|u\|_\lambda \approx \|u\|_0$.

Step 2: Define a nonlinear map $\mathcal{T}$:

$$\mathcal{T}u(t) = \mathbb{E}_{\alpha,\beta}(t-s)u_0 + \int_0^t \bar{\mathbb{E}}_{\alpha,\beta}(t-s)f(u(s))\,ds + \int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)\,dW(s).$$

Step 3: Show that

$$\mathcal{T}u \in C([0,T], L^2(\Omega; H)), \quad \forall\, u \in C([0,T], L^2(\Omega; H)),$$

which follows from Cauchy-Schwarz inequality, linear growth, Isometry property.

Step 4: Contraction property. We consider two cases.

Case 1. $\alpha \in (1/2, 1)$. In this case, we have

$$\begin{aligned}
&\mathbf{E}\Big\|e^{-\lambda t}(\mathcal{T}u_1 - \mathcal{T}u_2)(t)\Big\|^2\\
&= \mathbf{E}\Big\|e^{-\lambda t}\int_0^t \bar{\mathbb{E}}_{\alpha,\beta}(t-s)\big(f(u_1(s)) - f(u_2(s))\big)\,ds\Big\|^2\\
&= \mathbf{E}\Big\|\int_0^t 1\cdot\Big[e^{-\lambda(t-s)}\bar{\mathbb{E}}_{\alpha,\beta}(t-s)\Big]\Big[e^{-\lambda s}\big(f(u_1(s)) - f(u_2(s))\big)\Big]\,ds\Big\|^2\\
&\le Ct\mathbf{E}\int_0^t \Big\|\Big[e^{-\lambda(t-s)}\bar{\mathbb{E}}_{\alpha,\beta}(t-s)\Big]\Big[e^{-\lambda s}\big(f(u_1(s)) - f(u_2(s))\big)\Big]\Big\|^2\,ds\\
&\le Ct\Big[\int_0^t \|e^{-\lambda(t-s)}\bar{\mathbb{E}}_{\alpha,\beta}(t-s)\|^2\,ds\Big]\cdot\|u_1 - u_2\|_\lambda^2.
\end{aligned}$$

Note that

$$\begin{aligned}
Ct\int_0^t \|e^{-\lambda(t-s)}\bar{\mathbb{E}}_{\alpha,\beta}(t-s)\|^2\,ds &\le Ct\int_0^t e^{-2\lambda(t-s)}(t-s)^{2(\alpha-1)}\,ds\\
&= Ct\int_0^t e^{-2\lambda\tau}\tau^{2(\alpha-1)}\,d\tau = Ct\Big[\int_0^t e^{-2x}x^{2\alpha-2}\,dx\Big]\lambda^{1-2\alpha}\\
&\le Ct\Big[\int_0^\infty e^{-2x}x^{2\alpha-2}\,dx\Big]\lambda^{1-2\alpha} \le C(T)\lambda^{1-2\alpha}.
\end{aligned}$$

Choose sufficiently large λ, we get

$$\mathbf{E}\Big\|e^{-\lambda t}(\mathcal{T}u_1 - \mathcal{T}u_2)(t)\Big\|^2 \le C(T)\lambda^{1-2\alpha}\|u_1 - u_2\|_\lambda^2 \le \delta\|u_1 - u_2\|_\lambda^2, \text{ for some } 0 < \delta < 1.$$

Hence $\|\mathcal{T}(u_1) - \mathcal{T}(u_2)\|_\lambda \le \delta \|u_1 - u_2\|_\lambda$.

Case 2. $\alpha \in (0, 1/2),\ \alpha + \gamma > 1/2$. In this case, we have

$$\begin{aligned}
&\mathbf{E}\Big\|e^{-\lambda t}(\mathcal{T}u_1 - \mathcal{T}u_2)(t)\Big\|^2 \\
&= \mathbf{E}\Big\|e^{-\lambda t}\int_0^t \bar{\mathbb{E}}_{\alpha,\beta}(t-s)\big(f(u_1(s)) - f(u_2(s))\big)\,ds\Big\|^2 \\
&= \mathbf{E}\Big\|\int_0^t \Big[e^{-\lambda(t-s)}\bar{\mathbb{E}}_{\alpha,\beta}(t-s)\Big]\Big[e^{-\lambda s}\big(f(u_1(s)) - f(u_2(s))\big)\Big]\,ds\Big\|^2 \\
&= \mathbf{E}\Big(\int_0^t \|e^{-\lambda(t-s)}\bar{\mathbb{E}}_{\alpha,\beta}(t-s)\|\|e^{-\lambda s}\big(f(u_1(s)) - f(u_2(s))\big)\|\,ds\Big)^2.
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\mathbf{E}\Big\|e^{-\lambda t}(\mathcal{T}u_1 - \mathcal{T}u_2)(t)\Big\|^2 \\
&= \mathbf{E}\Big(\int_0^t \|e^{-\lambda(t-s)}\bar{\mathbb{E}}_{\alpha,\beta}(t-s)\|\|e^{-\lambda s}\big(f(u_1(s)) - f(u_2(s))\big)\|\,ds\Big)^2 \\
&= \mathbf{E}\Big(\int_0^t \big\|e^{-\lambda(t-s)}\bar{\mathbb{E}}_{\alpha,\beta}(t-s)\big\|^{1/2} \\
&\quad \cdot \big\|e^{-\lambda(t-s)}\bar{\mathbb{E}}_{\alpha,\beta}(t-s)\big\|^{1/2}\|e^{-\lambda s}\big(f(u_1(s)) - f(u_2(s))\big)\|\,ds\Big)^2 \\
&= \Big[\int_0^t e^{-\lambda(t-s)}\|\bar{\mathbb{E}}_{\alpha,\beta}(t-s)\|\,ds\Big] \\
&\cdot \mathbf{E}\Big[\int_0^t \big\|e^{-\lambda(t-s)}\bar{\mathbb{E}}_{\alpha,\beta}(t-s)\big\|\big\|e^{-\lambda s}\big(f(u_1(s)) - f(u_2(s))\big)\big\|^2\,ds\Big],
\end{aligned}$$

which implies that

$$\begin{aligned}
&\mathbf{E}\Big\|e^{-\lambda t}(\mathcal{T}u_1 - \mathcal{T}u_2)(t)\Big\|^2 \\
&= \Big[\int_0^t e^{-\lambda(t-s)}\|\bar{\mathbb{E}}_{\alpha,\beta}(t-s)\|\,ds\Big] \\
&\cdot \mathbf{E}\Big[\int_0^t \big\|e^{-\lambda(t-s)}\bar{\mathbb{E}}_{\alpha,\beta}(t-s)\big\|\big\|e^{-\lambda s}\big(f(u_1(s)) - f(u_2(s))\big)\big\|^2\,ds\Big] \\
&\le C\Big[\int_0^t e^{-\lambda(t-s)}(t-s)^{\alpha-1}\,ds\Big]^2\|u_1 - u_2\|_\lambda^2 \le C\lambda^{-2\alpha}\|u_1 - u_2\|_\lambda^2.
\end{aligned}$$

With $\alpha \in (0, 1/2)$, choose sufficiently large λ, we get

$$\mathbf{E}\Big\|e^{-\lambda t}(\mathcal{T}u_1 - \mathcal{T}u_2)(t)\Big\|^2 \le \lambda^{-2\alpha}\|u_1 - u_2\|_\lambda^2 \le \delta\|u_1 - u_2\|_\lambda^2, \text{ for some } 0 < \delta < 1.$$

Hence $\|\mathcal{T}(u_1) - \mathcal{T}(u_2)\|_\lambda \le \delta\|u_1 - u_2\|_\lambda$. The proof of Theorem 28 is complete. □

Theorem 29. *(Regularity) Let* $\frac{1}{2} < \alpha < 1$, $\frac{1}{2} < \beta \leq 1$, $0 \leq \gamma \leq 1$. *Assume that Assumptions 4.3.1, 4.3.2 and 4.3.3 hold. Let* $u_0 \in L^2(\Omega; \dot{H}^q)$ *with* $q \in [0, 2\beta]$. *Then, the following regularity result holds for the solution* u *of* (4.3.9) *with* $r \in [0, \kappa]$ *and* $0 \leq q \leq r \leq 2\beta$,

$$\mathbf{E}|u(t)|_r^2 \leq Ct^{-\frac{(r-q)\alpha}{\beta}}\mathbf{E}|u_0|_q^2 + C\mathbf{E}(\sup_{s\in[0,T]} \|u(s)\|^2).$$

Proof. From the definition of the mild solution (4.3.9) and $t \in (0, T]$ with $0 \leq q \leq r \leq 2\beta$ it follows that, with $r \in [0, \kappa]$,

$$\begin{aligned}\mathbf{E}|u(t)|_r^2 \leq & 3\Big(\mathbf{E}|\mathbb{E}_{\alpha,\beta}(t)u_0|_r^2 + \mathbf{E}|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta}(t-s)f(s,u(s))ds|_r^2 \\ & + \mathbf{E}|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s)|_r^2\Big) \leq 3(I_1 + I_2 + I_3).\end{aligned}$$

For I_1 a use of (4.3.11) with $p = r$ and $0 \leq q \leq r \leq 2\beta$ yields

$$\mathbf{E}|\mathbb{E}_{\alpha,\beta}(t)u_0|_r^2 \leq Ct^{-\alpha(\frac{r-q}{\beta})}\mathbf{E}|u_0|_q^2.$$

For I_2 we arrive from (4.3.3) and Assumption 4.3.1 that

$$\begin{aligned}I_2 = \mathbf{E}|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta}(t-s)f(s,u(s))ds|_r^2 &\leq \mathbf{E}\Big(\int_0^t |\bar{\mathbb{E}}_{\alpha,\beta}(t-s)f(s,u(s))ds|_r\Big)^2 \\ \leq C\mathbf{E}\Big(\int_0^t (t-s)^{\alpha-1+\frac{(r-0)\alpha}{\beta}}\|f(s,u(s))\|ds\Big)^2 \\ \leq C\Big(\int_0^t (t-s)^{\alpha-1+\frac{(r-0)\alpha}{\beta}}ds\Big)^2\mathbf{E}\Big[\sup_{s\in[0,T]}\|f(s,u(s))\|\Big]^2 \leq C\mathbf{E}\Big(\sup_{s\in[0,T]}\|u(s)\|^2\Big).\end{aligned} \tag{4.3.17}$$

For I_3, by Isometry property Lemma 4.3.1, Assumptions 4.3.2 and 4.3.3 and the smoothing property of the operator in Lemma 4.3.3, we have with, $0 \leq r \leq \kappa$,

$$
\begin{aligned}
I_3 &= \mathbf{E}\| \int_0^t A^{\frac{r}{2}} \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s) dW(s)\|^2 \\
&= \mathbf{E}\| \int_0^t A^{\frac{\kappa}{2}} \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s) \sum_{k=1}^{\infty} \sigma_k(s) A^{\frac{r-\kappa}{2}} e_k d\beta_k(s)\|^2 \\
&= \sum_{k=1}^{\infty} \int_0^t \|A^{\frac{\kappa}{2}} \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)\sigma_k(s) A^{\frac{r-\kappa}{2}} e_k\|^2 ds \\
&\leq C\Big(\int_0^t \|A^{\frac{\kappa}{2}} \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(s)\|^2 ds\Big)\Big(\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^{r-\kappa}\Big) \\
&\leq C\Big(\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^{r-\kappa}\Big) \int_0^t \Big(s^{\alpha+\gamma-1-\frac{(\kappa-0)\alpha}{2\beta}}\Big)^2 ds \\
&\leq C\Big(\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^{r-\kappa}\Big) \int_0^t s^{\alpha(2-\frac{\kappa}{\beta})+2\gamma-2} ds \leq C\Big(\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^{r-\kappa}\Big) < \infty.
\end{aligned} \tag{4.3.18}
$$

Therefore we have

$$
\mathbf{E}|u(t)|_r^2 = Ct^{-\alpha(\frac{r-q}{\beta})} \mathbf{E}|u_0|_q^2 + C\mathbf{E}\Big(\sup_{s\in[0,T]} \|u(s)\|^2\Big) + C\Big(\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^{r-\kappa}\Big)
$$

This completes the proof of Theorem 29. □

Assumption 4.3.4. There is a positive constant C such that the nonlinear function $f : \mathbb{R}^+ \times H \to H$ satisfies, with $u_1, u_2 \in \dot{H}^q$, $0 \leq q \leq 2\beta$ and $\frac{1}{2} < \beta \leq 1$,

$$
\|(-\Delta)^{\frac{q}{2}} \Big(f(t_1, u_1) - f(t_2, u_2)\Big)\| \leq L\Big(|t_1 - t_2| + \|(-\Delta)^{\frac{q}{2}} (u_1 - u_2)\|\Big),
$$

and

$$
\|(-\Delta)^{\frac{q}{2}} f(t, u)\| \leq C\Big(1 + \|(-\Delta)^{\frac{q}{2}} u\|\Big).
$$

Theorem 30. *Let* $\frac{1}{2} < \alpha < 1$, $\frac{1}{2} < \beta \leq 1$, $\frac{1}{2} \leq \gamma \leq 1$. *Assume that Assumptions 4.3.2, 4.3.3 and 4.3.4 hold. Let* $u_0 \in L^2(\Omega; \dot{H}^{2\beta})$. *Then there exists a unique mild solution* $u \in C([0, T]; L^2(\Omega; \dot{H}^{2\beta}))$ *given by* (4.3.9).

Proof. We proceed in a similar fashion as in the proof of Theorem 27 and only indicate the changes in the proof. Set $C([0, T]; L^2(\Omega; \dot{H}^{2\beta}))_\lambda$, $\lambda > 0$ as the set of functions in $C([0, T]; L^2(\Omega; \dot{H}^{2\beta}))$ with the following weighted norm

$$
\|\phi\|_{\lambda,\beta}^2 := \sup_{t\in[0,T]} \mathbf{E}\big(|e^{-\lambda t} \phi(t)|_{2\beta}^2\big), \ \forall \ \phi \in C([0, T]; L^2(\Omega; \dot{H}^{2\beta})). \tag{4.3.19}
$$

For the proof, it is now enough to show that the map $\mathcal{T} : C([0,T]; L^2(\Omega; \dot{H}^{2\beta}))_\lambda \to C([0,T]; L^2(\Omega; \dot{H}^{2\beta}))_\lambda$ is a contraction. We first show that $\mathcal{T}u \in C([0,T]; L^2(\Omega; \dot{H}^{2\beta}))$ for any $u \in C([0,T]; L^2(\Omega; \dot{H}^{2\beta}))$. By (4.3.12) and the Cauchy-Schwarz inequality we obtain with $u \in C([0,T]; L^2(\Omega; \dot{H}^{2\beta}))$,

$$\begin{aligned}
\mathbf{E}|\mathcal{T}u(t)|_{2\beta}^2 &\leq 3\mathbf{E}|\mathbb{E}_{\alpha,\beta}(t)u_0|_{2\beta}^2 + 3\mathbf{E}|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta}(t-s)f(s,u(s))ds|_{2\beta}^2 \\
&\quad + 3\mathbf{E}|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s)|_{2\beta}^2 \\
&\leq 3\mathbf{E}|\mathbb{E}_{\alpha,\beta}(t)u_0|_{2\beta}^2 + 3t\int_0^t \mathbf{E}|\bar{\mathbb{E}}_{\alpha,\beta}(t-s)f(s,u(s))|_{2\beta}^2 ds \\
&\quad + 3\mathbf{E}|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s)|_{2\beta}^2. \qquad (4.3.20)
\end{aligned}$$

By (4.3.11) and (4.3.12) with $p = q$, and using Assumption 4.3.1, it follows that

$$\begin{aligned}
\mathbf{E}|\mathcal{T}u(t)|_{2\beta}^2 &\leq C\mathbf{E}|u_0|_{2\beta}^2 + Ct\int_0^t (t-s)^{2(-1+\alpha)}(1 + \mathbf{E}|u(s)|_{2\beta}^2)ds \\
&\quad + C\mathbf{E}|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s)|_{2\beta}^2. \qquad (4.3.21)
\end{aligned}$$

For the stochastic integral in (4.3.21) a use of the Isometry property, Assumptions 4.3.3 and 4.3.4, the smoothing property (4.3.12), yields, with, $0 \leq r \leq \kappa$,

$$\begin{aligned}
&\mathbf{E}|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s)|_{2\beta}^2 \\
&= \mathbf{E}\|\int_0^t A^{\frac{\kappa-r+2\beta}{2}}\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)\sum_{k=1}^\infty \sigma_k(s)A^{\frac{r-\kappa}{2}}e_k d\beta_k(s)\|^2 \\
&= \sum_{k=1}^\infty \int_0^t \|A^{\frac{\kappa-r+2\beta}{2}}\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)\sigma_k(s)A^{\frac{r-\kappa}{2}}e_k\|^2 ds \\
&\leq C\Big(\int_0^t \|A^{\frac{\kappa-r+2\beta}{2}}\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(s)\|^2 ds\Big)\Big(\sum_{k=1}^\infty \mu_k^2\lambda_k^{r-\kappa}\Big). \qquad (4.3.22)
\end{aligned}$$

To make the integral $\int_0^t \|A^{\frac{\kappa-r+2\beta}{2}}\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(s)\|^2 ds < \infty$, we have to choose $r = 2\beta$, which implies that $\kappa = r = \beta$ since $0 \leq r \leq \kappa$. Hence, we need to restrict $\gamma > \frac{1}{2}$ in order to get $\kappa = 2\beta$ by Assumption 4.3.3. With such choices of κ and r and by noting that $\frac{1}{2} < \gamma \leq 1$, we arrive at

$$\mathbf{E}|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s)|_{2\beta}^2 \leq C\Big(\int_0^t \|A^{\frac{\kappa}{2}}\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(s)\|^2 ds\Big)\Big(\sum_{k=1}^{\infty}\mu_k^2\lambda_k^{r-\kappa}\Big)$$
$$\leq C\Big(\sum_{k=1}^{\infty}\mu_k^2\lambda_k^{r-\kappa}\Big)\int_0^t \Big(s^{\alpha+\gamma-1-\frac{(\kappa-0)\alpha}{2\beta}}\Big)^2 ds$$
$$\leq C\Big(\sum_{k=1}^{\infty}\mu_k^2\Big)\int_0^t s^{2\gamma-2}ds \leq C\Big(\sum_{k=1}^{\infty}\mu_k^2\Big) < \infty. \quad (4.3.23)$$

Note that $u_0 \in L^2(\Omega;\dot{H}^{2\beta})$ and $u \in C([0,T];L^2(\Omega;\dot{H}^{2\beta}))$, we then obtain $\sup_{t\in[0,T]}\mathbf{E}|\mathcal{T}u|_{2\beta}^2 < \infty$, which implies that $\mathcal{T}u \in C([0,T];L^2(\Omega;\dot{H}^{2\beta}))$.

We next consider the contraction property of the mapping $\mathcal{T}$. For any given functions u_1 and u_2 in $C([0,T];L^2(\Omega;\dot{H}^{2\beta}))_\lambda$, it follows from (4.3.12) and the estimate in (4.3.13) with $p = 2\beta$ that

$$\mathbf{E}|e^{-\lambda t}(\mathcal{T}u_1(t)-\mathcal{T}u_2(t))|_{2\beta}^2 = \mathbf{E}|e^{-\lambda t}\int_0^t \bar{\mathbb{E}}_{\alpha,\beta}(t-s)(f(s,u_1(s))-f(s,u_2(s)))ds|_{2\beta}^2$$
$$\leq \mathbf{E}\Big(\int_0^t e^{-\lambda(t-s)}|\bar{\mathbb{E}}_{\alpha,\beta}(t-s)e^{-\lambda s}(f(s,u_1(s))-f(s,u_2(s)))|_{2\beta}ds\Big)^2$$
$$\leq C\mathbf{E}\Big(\int_0^t (t-s)^{\frac{\alpha q}{2\beta}-1}e^{-\lambda(t-s)}|e^{-\lambda s}(f(s,u_1(s))-f(s,u_2(s)))|_{2\beta}ds\Big)^2$$
$$\leq C\mathbf{E}\Big(\int_0^t (t-s)^{\frac{\alpha q}{2\beta}-1}e^{-\lambda(t-s)}|e^{-\lambda s}(u_1(s)-u_2(s))|_{2\beta}ds\Big)^2$$
$$\leq C\mathbf{E}\Big(\int_0^t 1\cdot\Big[(t-s)^{\frac{\alpha q}{2\beta}-1}e^{-\lambda(t-s)}\Big]\Big[|e^{-\lambda s}\big(u_1(s)-u_2(s)|_{2\beta}\big)\Big]ds\Big)^2$$
$$\leq Ct\int_0^t (t-s)^{2(\frac{\alpha q}{2\beta}-1)}e^{-2\lambda(t-s)}ds \sup_{s\in[0,T]}\mathbf{E}|e^{-\lambda s}\big(u_1(s)-u_2(s)\big)|_{2\beta}^2$$
$$\leq Ct\int_0^t \tau^{\frac{\alpha q}{\beta}-2}e^{-2\lambda\tau}d\tau \sup_{s\in[0,T]}\mathbf{E}|e^{-\lambda s}\big(u_1(s)-u_2(s)\big)|_{2\beta}^2$$
$$\leq Ct\int_0^t \big(\frac{x}{\lambda}\big)^{\frac{\alpha q}{\beta}-2}e^{-2x}dx\lambda^{-1}\big[\sup_{s\in[0,T]}\mathbf{E}|u_1(s)-u_2(s)|_{2\beta}^2\big]$$
$$\leq Ct\Big[\int_0^t x^{\frac{\alpha q}{\beta}-2}e^{-2x}dx\Big]\lambda^{1-\frac{\alpha q}{\beta}}\big[\sup_{s\in[0,T]}\mathbf{E}|u_1(s)-u_2(s)|_{2\beta}^2\big]$$
$$\leq C(T)\lambda^{1-\frac{\alpha q}{\beta}}\sup_{s\in[0,T]}\mathbf{E}|u_1(s)-u_2(s)|_{2\beta}^2. \quad (4.3.24)$$

By the contraction property of the Banach fixed point theorem, the fact that $\frac{\alpha q}{\beta}-2>1$ and following the same argument as in the proof of Theorem 27 and by (4.3.19) the rest of the proof follows and this concludes the proof the theorem. □

4.4 Approximation of the fractionally integrated noise

Let $0 = t_1 < t_2 < t_3 < \cdots < t_N < t_{N+1} = T$ be a partition of $[0, T]$ and $\triangle t = \frac{T}{N}$ the time step size. We will approximate $\frac{d\beta_k(s)}{ds}$ by using Euler method

$$\frac{d\beta_k(s)}{ds} \simeq \frac{\beta_k(t_{i+1}) - \beta_k(t_1)}{\triangle t} =: \partial\beta_k^i \text{ on } [t_i, t_{i+1}],\ i = 1, 2, \cdots, N.$$

Here, $\beta_k(t_{i+1}) - \beta_k(t_i) = \sqrt{\triangle t} \cdot \mathcal{N}(0, 1)$, where $\mathcal{N}(0, 1)$ denotes the normally distributed random variable.

Let $\sigma_k^n(s)$ be some approximation of $\sigma_k(s)$. In order to obtain an approximation of

$$\dot{W}(t, x) = \sum_{k=1}^{\infty} \sigma_k(t)\dot{\beta}_k(t)e_k(x),$$

we replace it by

$$\dot{W}_n(t, x) = \sum_{k=1}^{\infty} \sigma_k^n(t)e_k(x)\big(\sum_{k=1}^{N}(\partial\beta_k^i)\chi_i(t)\big),$$

where $\chi_i(t)$ is the characteristic function on the i^{th} time interval $[t_i, t_{i+1}], i = 1, 2, 3, \cdots, N$ and σ_k^n is some approximation of $\sigma_k(t)$ which will be specified below. Then, we derive the following regularized stochastic space time fractional subdiffusion problem: Find u_n such that

$$\begin{aligned} &{}_0^C D_t^\alpha u_n(t, x) + (-\Delta)^\beta u_n(t, x) = f(t, u_n(t, x)) + {}_0^R D_t^{-\gamma} \dot{W}_n(t, x),\ (t, x) \in (0, T] \times D, \\ &u_n(t, x) = 0,\ (t, x) \in (0, T] \times \partial D,\ 0 < t < T, \\ &u_n(0, x) = u_0(x),\ x \in D. \end{aligned} \tag{4.4.1}$$

As it is in the continuous case, that is (4.3.9), the solution of (4.4.1) takes the following form

$$u_n(t) = \mathbb{E}_{\alpha,\beta}(t)u_0 + \int_0^t \bar{\mathbb{E}}_{\alpha,\beta}(t-s)f(s, u_n(s))ds + \int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW_n(s), \tag{4.4.2}$$

where $dW_n(s)$ denotes, with $\chi(s)$ the characteristics function defined on $[t_i, t_{i+1}], i = 1, 2, 3, \cdots, N$,

$$dW_n(s) = \sum_{k=1}^{\infty} \sigma_k^n(s)e_k\big(\sum_{i=1}^{N}(\partial\beta_k^i)\chi_i(s)\big)ds. \tag{4.4.3}$$

Assumption 4.4.1. Assume that the coefficients $\sigma_k^n(t)$ are constructed in such a way that

$$|\sigma_k(t) - \sigma_k^n(t)| \le \eta_k^n, \ |\sigma_k^n(t)| \le \mu_k^n, \ |(\sigma_k^n)'(t)| \le \gamma_k^n, \ \forall\, t \in [0,T].$$

We also need the following assumption for the regularity of the regularized noise $\dot{W}_n(s)$.

Assumption 4.4.2. Let $\frac{1}{2} < \alpha < 1, \ \frac{1}{2} < \beta \le 1, \ 0 \le \gamma \le 1$. There holds for $0 \le r \le \kappa$, $\sum_{k=1}^{\infty} (\mu_k^n)^2 \lambda_k^{r-\kappa} < \infty$, where κ is defined by

$$\kappa = \begin{cases} 2\beta, \ 2\gamma > 1, \\ (2 - \frac{1-2\gamma}{\alpha})\beta - \epsilon, \ 2\gamma \le 1, \end{cases}$$

and $\lambda_k, k = 1, 2, 3, \cdots$, are the eigenvalues of the operator $A = -\Delta$ with $D(A) = H_0^1(D) \cap H^2(D)$.

Theorem 31. *(Existence and Uniqueness) Let $\frac{1}{2} < \alpha < 1, \frac{1}{2} < \beta \le 1, 0 \le \gamma \le 1$. Assume that Assumptions 4.3.1, 4.3.2, 4.4.1, and 4.4.2 hold. Let $u_0 \in L^2(\Omega; H)$. Then, there exists a unique mild solution $u_n \in C([0,T]; L^2(\Omega; H))$ given by* (4.4.2) *to the problem* (4.4.1).

Proof. Since the proof is similar to that of Theorem 27 we omit the proof here. □

Theorem 32. *(Regularity) Let $\frac{1}{2} < \alpha < 1, \frac{1}{2} < \beta \le 1, 0 \le \gamma \le 1$. Assume that Assumptions 4.3.1, 4.3.2, 4.4.1 and 4.4.2 hold. Let $u_0 \in L^2(\Omega; \dot{H}^q)$ with $q \in [0, 2\beta]$. Then, the following regularity result for the solution u_n of the equation* (4.4.2) *holds with $r \in [0, \kappa]$ and $0 \le q \le r \le 2\beta$,*

$$\mathbf{E}|u_n(t)|_r^2 \le Ct^{-\frac{r-q}{\beta}\alpha} \mathbf{E}|u_0|_q^2 + C\mathbf{E}\Big(\sup_{s\in[0,T]} \|u_n(s)\|^2 \Big).$$

Proof. From the definition of the mild solution (4.4.2) and for $t \in (0,T]$ with $0 \le q \le r \le 2\beta$, it follows with $r \in [0, \kappa]$ that

$$\begin{aligned} \mathbf{E}|u_n(t)|_r^2 \le 3\Big(& \mathbf{E}|\mathbb{E}_{\alpha,\beta}(t)u_0|_r^2 + \mathbf{E}\Big|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta}(t-s) f(s, u_n(s)) ds\Big|_r^2 \\ & + \mathbf{E}\Big|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s) dW_n(s)\Big|_r^2 \Big) \\ = {} & 3(I_1 + I_2 + I_3). \end{aligned}$$

The terms I_1 and I_2 can be estimated as in the proof of Theorem 29.

It remains to estimate I_3. From the definition of $\dot{H}^r-$norm it follows that

$$\begin{aligned} I_3 &= \mathbf{E}|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW_n(s)|_r^2 \\ &= \mathbf{E}|\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\sum_{k=1}^{\infty}(t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha) \\ &\Big(\sum_{m=1}^{\infty}\sigma_m^n(s)\frac{\beta_m(t_{\ell+1})-\beta_m(t_\ell)}{\triangle t}e_m,e_k\Big)e_k ds|_r^2 \\ &= C\sum_{k=1}^{\infty}\lambda_k^r\Big\{\int_{t_\ell}^{t_{\ell+1}}\frac{1}{\triangle t}\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}(t-\bar{s})^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-\bar{s})^\alpha)\sigma_k^n(\bar{s})d\bar{s}d\beta_k(s)\Big\}^2 \\ &= C\sum_{k=1}^{\infty}\lambda_k^r\sum_{\ell=1}^{N}\frac{1}{(\triangle t)^2}\int_{t_\ell}^{t_{\ell+1}}\Big\{\int_{t_\ell}^{t_{\ell+1}}(t-\bar{s})^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-\bar{s})^\alpha)\sigma_k^n(\bar{s})d\bar{s}\beta_k(s)\Big\}^2 ds \\ &= C\sum_{k=1}^{\infty}\lambda_k^r\sum_{\ell=1}^{N}\frac{1}{\triangle t}\Big\{\int_{t_\ell}^{t_{\ell+1}}(t-\bar{s})^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-\bar{s})^\alpha)\sigma_k^n(\bar{s})d\bar{s}\Big\}^2 ds. \end{aligned}$$

A use of the Cauchy-Schwarz inequality shows

$$\begin{aligned} I_3 &\leq C\Big(\sum_{k=1}^{\infty}\lambda_k^r(\mu_k^n)^2\Big)\Big[\int_0^t(t-s)^{2(\alpha+\gamma-1)}E_{\alpha,\alpha+\gamma}^2(-\lambda_k^\beta(t-s)^\alpha)ds\Big] \\ &= C\Big(\sum_{k=1}^{\infty}\lambda_k^{r-\kappa}(\mu_k^n)^2\Big)\Big[\int_0^t\lambda_k^\kappa(t-s)^{2(\alpha+\gamma-1)}E_{\alpha,\alpha+\gamma}^2(-\lambda_k^\beta(t-s)^\alpha)ds\Big] \\ &= C\Big(\sum_{k=1}^{\infty}\lambda_k^{r-\kappa}(\mu_k^n)^2\Big)\int_0^t\|A^{\frac{\kappa}{2}}\bar{\mathbf{E}}_{\alpha,\beta,\gamma}(t-s)\|^2 \\ &= C\Big(\sum_{k=1}^{\infty}\lambda_k^{r-\kappa}(\mu_k^n)^2\Big)\int_0^t(s^{\alpha+\gamma-1-\alpha(\frac{\kappa-0}{2\beta})})^2 ds \leq C\Big(\sum_{k=1}^{\infty}\lambda_k^{r-\kappa}(\mu_k^n)^2\Big) < \infty. \end{aligned} \tag{4.4.4}$$

Together with the above estimates we complete the rest of the proof. □

Theorem 33. *Let $\frac{1}{2} < \alpha < 1, \frac{1}{2} < \beta \leq 1, 0 \leq \gamma \leq 1$. Assume that Assumptions 4.3.1, 4.3.2, 4.4.1, and 4.4.2 hold. Let u and u_n be solutions of* (4.3.1) *and* (4.4.1) *respectively. Then we have for any $\epsilon > 0$,*

1. *for $\frac{1}{2} < \alpha+\gamma < 1$,*

$$\begin{aligned} \mathbf{E}\|u(t)-u_n(t)\|^2 \leq Ct^{2(\alpha+\gamma)-1}\sum_{k=1}^{\infty}(\eta_k^n)^2 + Ct^{2(\alpha+\gamma)-1}(\triangle t)^2\sum_{k=1}^{\infty}(\gamma_k^n)^2 \\ + Ct^{2\epsilon}(\triangle t)^{2(\alpha+\gamma)-1-2\epsilon}\sum_{k=1}^{\infty}(\mu_k^n)^2, \end{aligned} \tag{4.4.5}$$

2. for $1 \leq \alpha+\gamma < \frac{3}{2}$,

$$\mathbf{E}\|u(t)-u_n(t)\|^2 \leq C\sum_{k=1}^{\infty}\lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\eta_k^n)^2 + C(\triangle t)^2\sum_{k=1}^{\infty}\lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\gamma_k^n)^2$$
$$+ Ct^{2\epsilon}(\triangle t)^{2(\alpha+\gamma)-1-2\epsilon}\sum_{k=1}^{\infty}(\mu_k^n)^2, \qquad (4.4.6)$$

3. for $\frac{3}{2} \leq \alpha+\gamma < 2$,

$$\mathbf{E}\|u(t)-u_n(t)\|^2 \leq C\sum_{k=1}^{\infty}\lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\eta_k^n)^2 + C(\triangle t)^2\sum_{k=1}^{\infty}\lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\gamma_k^n)^2$$
$$+ Ct^{2\epsilon}(\triangle t)^2\sum_{k=1}^{\infty}(\mu_k^n)^2. \qquad (4.4.7)$$

Proof. Subtracting (4.4.2) from (4.3.9) we obtain

$$\begin{aligned} u(t)-u_n(t) &= \int_0^t \bar{\mathbf{E}}_{\alpha,\beta}(t-s)\Big(f(s,u(s)) - f(s,u_n(s))\Big)ds \\ &\quad + \int_0^t \bar{\mathbf{E}}_{\alpha,\beta,\gamma}(t-s)\Big(dW(s) - dW_n(s)\Big) \qquad (4.4.8) \\ &= G_1 + G_2. \end{aligned}$$

By definition of dW and dW_n given by (4.3.10) and (4.4.3) respectively, we now rewrite G_2 as

$$\begin{aligned} G_2 &= \int_0^t \sum_{k=1}^{\infty}(t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-s)^{\alpha})\Big(\sum_{k=1}^{\infty}(\sigma_m(s)-\sigma_m^n(s))(e_m,e_k)d\beta_m(s)\Big)e_k \\ &\quad + \Big\{\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}(t-s)^{\alpha+\gamma-1}\sum_{k=1}^{\infty}E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-s)^{\alpha})\Big(\sum_{m=1}^{\infty}\sigma_m^n(s)(e_m,e_k)d\beta_m(s)\Big)e_k \\ &\quad - \sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}(t-s)^{\alpha+\gamma-1}\sum_{k=1}^{\infty}E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-s)^{\alpha})\Big(\sum_{m=1}^{\infty}\sigma_m^n(s)(e_m,e_k)(\partial\beta_m^{\ell})ds\Big)e_k\Big\} \\ &= G_{21} + G_{22}. \qquad (4.4.9) \end{aligned}$$

We first estimate $\mathbf{E}\|G_1\|$. From the form of G_1 and using (4.3.13) with $p = q = 0$, and the Assumption 4.3.2, we arrive with $\frac{1}{2} < \alpha < 1$ at,

$$\begin{aligned}
\mathbf{E}\|G_1\|^2 &= \mathbf{E}\Big(\int_0^t (t-s)^{\alpha-1}\|f(s,u(s))-f(s,u_n(s))\|ds\Big)^2\\
&\le C\mathbf{E}\big(\int_0^t (t-s)^{\alpha-1}\|u(s)-u_n(s)\|ds\big)^2\\
&\le C\int_0^t (t-s)^{\alpha-\frac{3}{2}}ds\mathbf{E}\int_0^t (t-s)^{\alpha-\frac{1}{2}}\|u(s)-u_n(s)\|^2ds\\
&\le Ct^{2\alpha-1}\int_0^t \mathbf{E}\|u(s)-u_n(s)\|^2ds. \qquad (4.4.10)
\end{aligned}$$

For the estimate of $\mathbf{E}\|G_{21}\|^2$, using the Itõ Isometry property and the Assumption 4.3.3, we obtain

$$\begin{aligned}
\mathbf{E}\|G_{21}\|^2 &= \mathbf{E}\|\int_0^t \sum_{k=1}^{\infty}(t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-s)^{\alpha})\\
&\quad\Big(\sum_{m=1}^{\infty}(\sigma_m(s)-\sigma_m^n(s))(e_m,e_k)d\beta_m(s)\Big)e_k\|^2\\
&= \int_0^t \sum_{k=1}^{\infty}(t-s)^{2(\alpha+\gamma-1)}|E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-s)^{\alpha})|^2(\sigma_k(s)-\sigma_k^n(s))^2ds\\
&\le \sum_{k=1}^{\infty}(\eta_k^n)^2\int_0^t (t-s)^{2(\alpha+\gamma-1)}|E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-s)^{\alpha})|^2ds. \qquad (4.4.11)
\end{aligned}$$

Note that, for $\frac{1}{2}<\alpha+\gamma<1$, a use of the boundedness property of Mittag-Lefler function (4.3.7) yields,

$$\int_0^t (t-s)^{2(\alpha+\gamma-1)}|E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-s)^{\alpha})|^2ds \le C\int_0^t (t-s)^{2(\alpha+\gamma-1)}ds = Ct^{2(\alpha+\gamma)-1}. \qquad (4.4.12)$$

For $1\le\alpha+\gamma<2$, by using the asymptotic property of Mittag-Lefler function (4.3.6), we have,

$$\begin{aligned}
&\int_0^t (t-s)^{2(\alpha+\gamma-1)}|E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-s)^{\alpha})|^2ds\\
&\le \int_0^t \Big|\frac{(t-s)^{\alpha+\gamma-1}}{1+\lambda_k^{\beta}(t-s)^{\alpha}}\Big|^2ds = \int_0^t \Big|\frac{(\lambda_k^{\beta}(t-s)^{\alpha})^{\frac{\alpha+\gamma-1}{\alpha}}\lambda_k^{-\frac{\beta(\alpha+\gamma-1)}{\alpha}}}{1+\lambda_k^{\beta}(t-s)^{\alpha}}\Big|^2ds\\
&= \lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}\int_0^t \Big|\frac{(\lambda_k^{\beta}(t-s)^{\alpha})^{\frac{\alpha+\gamma-1}{\alpha}}}{1+\lambda_k^{\beta}(t-s)^{\alpha}}\Big|^2ds \le C\lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}. \qquad (4.4.13)
\end{aligned}$$

Thus, we now arrive at,

$$\mathbf{E}\|G_{21}\|^2 \leq \begin{cases} Ct^{2(\alpha+\gamma-1)}\sum_{k=1}^{\infty}(\eta_k^n)^2, \text{ for } \frac{1}{2}<\alpha+\gamma<1, \\ C\sum_{k=1}^{\infty}\lambda^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\eta_k^n)^2, \text{ for } 1\leq\alpha+\gamma<2. \end{cases} \tag{4.4.14}$$

We now estimate G_{22}. We first denote $\frac{\beta_m(t_{\ell+1})-\beta_m(t_\ell)}{\triangle t}$ by $\frac{1}{\triangle t}\int_{t_\ell}^{t_{\ell+1}} d\beta_m(s)$ and replace the variable s with $\bar{s}$ in the second term in G_{22}. Using the orthogonality property of $e_k, k = 1,2,3,\cdots$ we obtain

$$\begin{aligned}
\mathbf{E}\|G_{22}\|^2 &= \mathbf{E}\|\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}(t-s)^{\alpha+\gamma-1}\sum_{k=1}^{\infty}E_{\alpha,\alpha+\gamma}\big(-\lambda_k^\beta(t-s)^\alpha\big) \\
&\left(\sum_{m=1}^{\infty}\sigma_k^n(s)(e_m,e_k)d\beta_m(s)\right)e_k \\
&-\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}(t-\bar{s})^{\alpha+\gamma-1}\sum_{k=1}^{\infty}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-\bar{s})^\alpha) \\
&\left(\sum_{m=1}^{\infty}\sigma_m^n(\bar{s})\frac{1}{\triangle t}\int_{t_\ell}^{t_{\ell+1}}(e_m,e_k)d\beta_m(s)\right)e_k ds\|^2 \\
&= \mathbf{E}\|\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}(t-s)^{\alpha+\gamma-1}\sum_{k=1}^{\infty}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha)\sigma_k^n(s)e_k d\beta_k(s) \\
&-\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\frac{1}{\triangle t}\int_{t_\ell}^{t_{\ell+1}}(t-\bar{s})^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-\bar{s})^\alpha)\sigma_k^n(\bar{s})e_k d\bar{s}d\beta_k(s)\|^2 \\
&= \mathbf{E}\sum_{k=1}^{\infty}|\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}(t-s)^{\alpha+\gamma-1}\sum_{k=1}^{\infty}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha)\sigma_k^n(s)d\beta_k(s) \\
&-\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\frac{1}{\triangle t}\int_{t_\ell}^{t_{\ell+1}}(t-\bar{s})^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-\bar{s})^\alpha)\sigma_k^n(\bar{s})d\bar{s}d\beta_k(s)|^2 \\
&= \mathbf{E}\sum_{k=1}^{\infty}|\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\left[\frac{1}{\triangle}\int_{t_\ell}^{t_{\ell+1}}(t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha)\sigma_k^n(s)d\bar{s}\right]d\beta_k(s) \\
&-\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\left[\frac{1}{\triangle t}\int_{t_\ell}^{t_{\ell+1}}(t-\bar{s})^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-\bar{s})^\alpha)\sigma_k^n(\bar{s})d\bar{s}\right]d\beta_k(s)|^2.
\end{aligned}$$

Thus, a use of the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\mathbf{E}\|G_{22}\|^2 = \sum_{k=1}^{\infty}\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}} \frac{1}{(\triangle t)^2}\bigg(\int_{t_\ell}^{t_{\ell+1}} (t-s)^{\alpha+\gamma-1} \\
E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha)\big(\sigma_k^n(s)-\sigma_k^n(\bar{s})\big)d\bar{s} \\
+\int_{t_\ell}^{t_{\ell+1}}\Big[(t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha) \\
-(t-\bar{s})^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-\bar{s})^\alpha)\Big]\sigma_k^n(\bar{s})d\bar{s}\bigg)^2 ds \\
\leq 2\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\frac{1}{\triangle t}\sum_{k=1}^{\infty}\int_{t_\ell}^{t_{\ell+1}}(t-s)^{2(\alpha+\gamma-1)}|E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha)|^2 \\
|\sigma_k^n(s)-\sigma_k^n(\bar{s})|^2 d\bar{s}ds \\
+2\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\frac{1}{\triangle t}\sum_{k=1}^{\infty}\int_{t_\ell}^{t_{\ell+1}}\bigg((t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha)- \\
(t-\bar{s})^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-\bar{s})^\alpha)\bigg)^2|\sigma_k^n(\bar{s})|^2 d\bar{s}ds \\
=2I_1+2I_2.
\end{aligned}$$

For I_1, using the mean value theorem and the Assumption 4.3.3, we arrive with ℓ lying between s and $\bar{s}$ at

$$\begin{aligned}
I_1 &\leq (\triangle t)^2\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\sum_{k=1}^{\infty}(t-s)^{2(\alpha+\gamma-1)}|E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha)|^2(\gamma_k^n)^2 ds \\
&= (\triangle t)^2\sum_{k=1}^{\infty}(\gamma_k^n)^2\int_0^t (t-s)^{2(\alpha+\gamma-1)}|E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^2)|^2 ds.
\end{aligned}$$

Now following the same estimates as in (4.4.14), we find that

$$I_1 \leq \begin{cases} Ct^{2(\alpha+\gamma)-1}(\triangle t)^2\sum_{k=1}^{\infty}(\gamma_k^n)^2, & \text{for } \frac{1}{2}<\alpha+\gamma<1, \\ C(\triangle t)^2\sum_{k=1}^{\infty}\lambda^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\gamma_k^n)^2, & \text{for } 1\leq\alpha+\gamma<2. \end{cases}$$

For I_2, we note by lemma 4.3.2 that

$$\begin{aligned}
&(t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha)-(t-\bar{s})^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-\bar{s})^\alpha) \\
&=\int_{\bar{s}}^{s}\frac{d}{d\tau}\big[(t-\tau)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-\tau)^\alpha)\big]d\tau \\
&=\int_{\bar{s}}^{s}-(t-\tau)^{\alpha+\gamma-2}E_{\alpha,\alpha+\gamma-1}(-\lambda_k^\beta(t-\tau)^\alpha)d\tau \\
&\leq C\Big|\int_{\bar{s}}^{s}(t-\tau)^{\alpha+\gamma-2}d\tau\Big| \qquad (4.4.15)
\end{aligned}$$

and hence,

$$I_2 \le C \sum_{\ell=1}^{N} \int_{t_\ell}^{t_{\ell+1}} \frac{1}{\triangle t} \sum_{k=1}^{\infty} \mu_k^2 \int_{t_\ell}^{t_{\ell+1}} \Big(\int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-2} d\tau \Big)^2 d\bar{s} ds.$$

Now we estimate $\int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-2} d\tau$ for the different α and γ. We shall show that with $0 < \epsilon < \frac{1}{2}$,

$$|\int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-2} d\tau| \le \begin{cases} C(t-\max(s-\bar{s}))^{\frac{-1}{2}+\epsilon} (\triangle t)^{\alpha+\gamma-\frac{1}{2}-\epsilon}, \ \frac{1}{2} < \alpha+\gamma < \frac{3}{2} \\ C(t-\max(s-\bar{s}))^{\alpha+\gamma-2} \triangle t, \ \frac{3}{2} \le \alpha+\gamma < 2. \end{cases}$$

Case 1. We first consider the case $\frac{1}{2} < \alpha+\gamma < \frac{3}{2}$. If $\bar{s} < s$, then with $0 < \epsilon < \frac{1}{2}$, it follows that,

$$\begin{aligned} |\int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-2} d\tau| &= \int_{\bar{s}}^{s} (t-\tau)^{-\frac{1}{2}+\epsilon} (t-\tau)^{\alpha+\gamma-\frac{3}{2}-\epsilon} d\tau \\ &\le (t-s)^{-\frac{1}{2}+\epsilon} \int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-\frac{3}{2}-\epsilon} d\tau \\ &= -(t-s)^{-\frac{1}{2}+\epsilon} \frac{1}{\alpha+\gamma-\frac{1}{2}-\epsilon} (t-\tau)^{\alpha+\gamma-\frac{1}{2}-\epsilon} |_{\tau=\bar{s}}^{\tau=s}. \end{aligned}$$

Since $a^\theta - b^\theta \le (a-b)^\theta$, for $a > b > 0$ and $0 < \theta < 1$, then for $\frac{1}{2} < \alpha+\gamma < \frac{3}{2}$

$$-(t-\tau)^{\alpha+\gamma-\frac{1}{2}-\epsilon} |_{\tau=\bar{s}}^{\tau=s} \le (s-\bar{s})^{\alpha+\gamma-\frac{1}{2}-\epsilon} \le (\triangle t)^{\alpha+\gamma-\frac{1}{2}-\epsilon},$$

and this implies that,

$$|\int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-2} d\tau| \le C(t-s)^{-\frac{1}{2}+\epsilon} (\triangle t)^{\alpha+\gamma-\frac{1}{2}-\epsilon},$$

similarly, we may show that for $s < \bar{s}$ with $0 < \epsilon < \frac{1}{2}$,

$$|\int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-2} d\tau| \le C(t-\bar{s})^{-\frac{1}{2}+\epsilon} (\triangle t)^{\alpha+\gamma-\frac{1}{2}-\epsilon}.$$

Therefore, we arrive for $\frac{1}{2} < \alpha+\gamma < \frac{3}{2}$ at

$$|\int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-2} d\tau| \le C(t-\max(s-\bar{s}))^{-\frac{1}{2}+\epsilon} (\triangle t)^{\alpha+\gamma-\frac{1}{2}-\epsilon}.$$

Case 2. We next consider the case $\frac{3}{2} \le \alpha+\gamma < 2$. If $\bar{s} < s$ then we obtain,

$$|\int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-2} d\tau| \le (t-s)^{\alpha+\gamma-2} (s-\bar{s}) \le (t-s)^{\alpha+\gamma-2} \triangle t.$$

Therefore, we arrive for $\frac{3}{2} \le \alpha+\gamma < 2$ at

$$|\int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-2} d| \le C(t-\max(s-\bar{s}))^{\alpha+\gamma-2} \triangle t.$$

Thus, we derive the following estimate for I_2, for $\frac{1}{2} < \alpha+\gamma < \frac{3}{2}$,

$$I_2 \le C\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\frac{1}{\triangle t}\sum_{k=1}^{\infty}(\mu_k^n)^2\int_{t_\ell}^{t_{\ell+1}}(t-\max(s-\bar{s}))^{-1+2\epsilon}(\triangle t)^{2(\alpha+\gamma)-1-2\epsilon}d\bar{s}ds$$
$$\le C(\triangle t)^{2(\alpha+\gamma)-1-2\epsilon}\sum_{k=1}^{\infty}(\mu_k^n)^2\int_0^t(t-s)^{-1+2\epsilon}ds \le Ct^{2\epsilon}(\triangle t)^{2(\alpha+\gamma)-1-2\epsilon}\sum_{k=1}^{\infty}(\mu_k^n)^2.$$

For $\frac{3}{2} \le \alpha+\gamma < 2$,

$$I_2 \le C\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\frac{1}{\triangle t}\sum_{k=1}^{\infty}(\mu_k^n)^2\int_{t_\ell}^{t_{\ell+1}}(t-\max(s-\bar{s}))^{2(\alpha+\gamma)-4}(\triangle t)^2 d\bar{s}ds$$
$$\le C(\triangle t)^2\int_0^t(t-s)^{2(\alpha+\gamma)-4}ds\sum_{k=1}^{\infty}(\mu_k^n)^2 \le Ct^{2(\alpha+\gamma)-3}(\triangle t)^2\sum_{k=1}^{\infty}(\mu_k^n)^2.$$

Together with the estimates we obtain the following results:

1. For $\frac{1}{2} < \alpha+\gamma < 1$, there holds for $t > 0$,

$$\mathbf{E}\|u(t)-u_n(t)\|^2 \le Ct^{2(\alpha+\gamma)-1}\sum_{k=1}^{\infty}(\eta_k^n)^2 + Ct^{2(\alpha+\gamma)-1}(\triangle t)^2\sum_{k=1}^{\infty}(\gamma_k^n)^2$$
$$+ Ct^{2\epsilon}(\triangle t)^{2(\alpha+\gamma)-1-2\epsilon}\sum_{k=1}^{\infty}(\mu_k^n)^2 + Ct^{2\alpha-1}\int_0^t\mathbf{E}\|u(s)-u_n(s)\|^2ds.$$

2. For $1 \le \alpha+\gamma < \frac{3}{2}$, it follows that for $t > 0$,

$$\mathbf{E}\|u(t)-u_n(t)\|^2 \le C\sum_{k=1}^{\infty}\lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\eta_k^n)^2 + C(\triangle t)^2\sum_{k=1}^{\infty}\lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\gamma_k^n)^2$$
$$+ Ct^{2\epsilon}(\triangle t)^{2(\alpha+\gamma)-1-2\epsilon}\sum_{k=1}^{\infty}(\mu_k^n)^2 + Ct^{2\alpha-1}\int_0^t\mathbf{E}\|u(s)-u_n(s)\|^2ds.$$

3. For $\frac{3}{2} \le \alpha+\gamma < 2$, we arrive for $t > 0$ at

$$\mathbf{E}\|u(t)-u_n(t)\|^2 \le C\sum_{k=1}^{\infty}\lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\eta_k^n)^2 + C(\triangle t)^2\sum_{k=1}^{\infty}\lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\gamma_k^n)^2$$
$$+ Ct^{2(\alpha+\gamma)-3}(\triangle t)^2\sum_{k=1}^{\infty}(\mu_k^n)^2 + Ct^{2\alpha-1}\int_0^t\mathbf{E}\|u(s)-u_n(s)\|^2ds.$$

An application of the Gronwall's lemma completes the rest of the proof of theorem 33.

□

4.5 Finite element approximation

Let $\mathcal{T}_h$ be a shape regular and quasi-uniform triangulation of the domain D with spatial discretization parameter $h = \max_{K\in\mathcal{T}_h} h_K$, where h_K denotes the diameter of K. Let $V_h \subset \dot{H}^\beta, \frac{1}{2} < \beta \leq 1$ be the piecewise linear finite element space associated with the triangulation $\mathcal{T}_h$, that is

$$V_h := \{v_h \in \dot{H}^\beta(D) : v_h|_K \in P_1(K),\ \forall K \in \mathcal{T}_h\},$$

where $P_1(K)$ is the space of linear polynomials defined on K. On the space V_h we define the following L_2 projection P_h, the fractional Ritz projection R_h and the fractional discrete Laplacian $(-\Delta_h)^\beta$ respectively.

Definition 4.5.1. [121] The L_2 projection $P_h : L_2(D) \to V_h$ is defined by

$$(P_h v, \chi) = (v, \chi),\ \forall\ \chi \in V_h.$$

Definition 4.5.2. (Fractional Ritz projection) [121], [2] Let $\frac{1}{2} < \beta \leq 1$. The fractional Ritz projection $R_h : \dot{H}^\beta \to V_h$ is defined by, with $v \in \dot{H}^\beta$,

$$\Big((-\Delta)^{\frac{\beta}{2}} R_h v, (-\Delta)^{\frac{\beta}{2}}\chi\Big) = \Big((-\Delta)^{\frac{\beta}{2}} v, (-\Delta)^{\frac{\beta}{2}}\chi\Big),\ \forall\ \chi \in V_h.$$

Below, we discuss the approximation properties of P_h and R_h.

Lemma 4.5.1. ([121], [2]) The operators P_h and R_h satisfy

$$\|P_h v - v\| + h^\beta \|(-\Delta)^{\frac{\beta}{2}}(P_h v - v)\| \leq C h^r |v|_r,\ \forall\ v \in \dot{H}^r,\ r \in [\beta, 2\beta],$$

and

$$\|R_h v - v\| + h^\beta \|(-\Delta)^{\frac{\beta}{2}}(R_h v - v)\| \leq C h^r |v|_r,\ \forall\ v \in \dot{H}^r,\ r \in [\beta, 2\beta].$$

Let $-\Delta_h : V_h \to V_h$ be the discrete Laplacian defined by see [1]

$$\Big((-\Delta_h)\psi, \chi\Big) = \Big(\nabla\psi, \nabla\chi\Big), \forall\ \chi \in V_h.$$

Further, let $(\lambda_k^h, e_k^h)_{k=1}^{N_h}$ be the eigenpairs of the discrete Laplacian i.e.,

$$\begin{cases} \Big(-\Delta_h\Big) e_k^h(x) = \lambda_k^h e_k^h(x),\ x \in D, \\ e_k^h(x) = 0,\ x \in \partial D, \end{cases}$$

such that $(e_k^h)_{k=1}^{N_h}$ forms an orthonormal basis of $V_h \subset H$, i.e.,

$$(e_k^h, e_\ell^h)_{L^2(D)} = \begin{cases} 1, \ k = \ell \\ 0, \ k \neq \ell. \end{cases} \tag{4.5.1}$$

Definition 4.5.3. (Fractional discrete Laplacian) Let $\frac{1}{2} < \beta \leq 1$. The fractional discrete Laplacian $(-\Delta_h)^\beta : V_h \to V_h$ is defined by, with $\psi \in V_h$

$$((-\Delta_h)^\beta \psi, \chi) = ((-\Delta)^{\frac{\beta}{2}} \psi, (-\Delta)^{\frac{\beta}{2}} \chi), \ \forall \ \chi \in V_h.$$

Definition 4.5.4. (Discrete norm) For $\chi \in V_h$ we may define the discrete norm by

$$|\chi|_{p,h}^2 = \sum_{k=1}^{N_h} (\lambda_k^h)^p (\chi, e_k^h)^2, \ p \in \mathbb{R},$$

where N_h is the dimension of the finite element space V_h.

The semidiscrete finite element method approximation of the equation (4.4.1) is to seek $u_n^h(t, \cdot) \in V_h$ for $t \in [0, T]$ such that

$$\begin{cases} {}_0^C D_t^\alpha u_n^h(t) + \big(- \Delta_h \big)^\beta u_n^h(t) = P_h f\big(t, u_n^h(t)\big) + P_h\big(D_t^{-\gamma} dW_n(t)\big), \\ u_n^h(0) = v^h, \end{cases} \tag{4.5.2}$$

where $v^h = P_h v$ is chosen as L^2 projection of the initial function v into V_h.
As in the continuous case the solution of (4.5.2) takes the form

$$u_n^h(t) = \mathbb{E}_{\alpha,\beta}^h(t) P_h v + \int_0^t \bar{\mathbb{E}}_{\alpha,\beta}^h(t-s) P_h f(s, u_n^h(s)) ds + \int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}^h(t-s) P_h dW_n(s), \tag{4.5.3}$$

where for each $t \in [0, T]$, the operators $\mathbb{E}_{\alpha,\beta}^h(t)$, $\bar{\mathbb{E}}_{\alpha,\beta}^h(t)$ and $\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t)$ are defined from $V_h \to V_h$ by

$$\mathbb{E}_{\alpha,\beta}^h(t) v^h := \sum_{k=1}^m E_{\alpha,1}\big((-\lambda_k^h)^\beta t^\alpha\big)(v^h, e_k^h) e_k^h,$$

$$\bar{\mathbb{E}}_{\alpha,\beta}^h(t) v^h := \sum_{k=1}^m t^{\alpha-1} E_{\alpha,\alpha}\big((-\lambda_k^h)^\beta t^\alpha\big)(v^h, e_k^h) e_k^h,$$

$$\bar{\mathbb{E}}_{\alpha,\beta,\gamma}^h(t) v^h := \sum_{k=1}^m t^{\alpha+\gamma-1} E_{\alpha,\alpha+\gamma}\big((-\lambda_k^h)^\beta t^\alpha\big)(v^h, e_k^h) e_k^h.$$

For the discrete analogue of (4.3.11), the following Lemma shows the smoothing property of the discrete solution operator $\bar{\mathbb{E}}_{\alpha,\beta,\gamma}^h$.

Lemma 4.5.2. For any $t > 0$ and $0 \le p - q \le 2\beta$ there holds for $v^h \in V_h$

$$|\bar{\mathbb{E}}^h_{\alpha,\beta,\gamma}(t)v^h|_{p,h} \le Ct^{-1+(\alpha+\gamma)-\alpha\frac{(p-q)}{2\beta}}|v^h|_{q,h}.$$

Similar conclusion can be drawn for $\bar{\mathbb{E}}^h_{\alpha,\beta}$, that is for $\bar{\mathbb{E}}^h_{\alpha,\beta,\gamma}$ with $\gamma = 0$.

Lemma 4.5.3. (Inverse estimate in V_h For any $\ell > s$, there exists a constant C independent of h such that

$$|v^h|_{\ell,h} \le Ch^{s-\ell}|v^h|_{s,h}, \ \forall \ v^h \in V_h.$$

4.6 Error estimates

Write $u - u^h_n := (u - u_n) + (u_n - u^h_n)$. Since estimate $\mathbf{E}\|u(t) - u_n(t)\|^2$ is known from Theorem 30. It remains to show the estimate $\mathbf{E}\|u_n(t) - u^h_n(t)\|^2$.

Theorem 34. *Let* $\frac{1}{2} < \alpha < 1$, $\frac{1}{2} < \beta \le 1$, $0 \le \gamma \le 1$. *Assume that Assumptions 4.3.1, 4.3.2, 4.3.3 4.4.1, and 4.4.2 hold. Let* u_n *and* u^h_n *be the solutions of* (4.4.1) *and* (4.5.2), *respectively. Let* $v \in L^2(\Omega; \dot{H}^\beta)$ *with* $0 \le q \le 2\beta$. *Then, there exists a positive constant* C *such that, for any* $\epsilon > 0$, *with* $r \in [0, \kappa]$ *and* $0 \le \max(q, \beta) \le r \le 2\beta$,

$$\begin{aligned}&\mathbf{E}\|u_n(t) - u^h_n(t)\|^2 + h^{2\beta}\mathbf{E}\|(-\Delta)^{\frac{\beta}{2}}\big(u_n(t) - u^h_n\big)\|^2 \\ &\le Ch^{-2\epsilon+2r}\left[\mathbf{E}|v|^2_q + \mathbf{E}\Big(\sup_{s\in[0,T]}\|u_n(s)\|^2\Big)\right] + Ch^{2r}t^{-\alpha\frac{r-q}{\beta}}\mathbf{E}|v|^2_q.\end{aligned} \tag{4.6.1}$$

Proof. Introducing $\tilde{u}^h_n(t) \in V_h$ as a solution of an intermediate discrete system,

$$\begin{cases} {}^C_0D^\alpha_t\tilde{u}^h_n(t) + (-\Delta_h)^\beta\tilde{u}^h_n(t) = P_hf(t, \tilde{u}_n(t)) + P_h(D^{-\gamma}_t dW_n(t)), \\ u^h_n(0) = P_hv. \end{cases} \tag{4.6.2}$$

We split the error $u^h_n - u_n(t) := (u^h_n(t) - \tilde{u}^h_n(t)) + (\tilde{u}^h_n(t) - u_n(t)) = \zeta(t) + \eta(t)$. Again, using P_hu_n we split $\eta(t)$ as

$$\eta(t) := (\tilde{u}^h_n(t) - P_hu_n) + (P_hu_n - u_n) =: \theta + \rho.$$

From Lemma 4.5.1 it follows that with $r \in [\beta, 2\beta]$,

$$\mathbf{E}\|\rho(t)\|^2 + h^{2\beta}\mathbf{E}\|(-\Delta)^{\frac{\beta}{2}}\rho(t)\|^2 \le Ch^{2r}\mathbf{E}|u_n(t)|^2_r,$$

which implies that, by Theorem 32,

$$
\begin{aligned}
&\mathbf{E}\|\rho(t)\|^2 + h^{2\beta}\mathbf{E}\|(-\Delta)^{\frac{\beta}{2}}\rho(t)\|^2 \\
&\le Ch^{2r}\Big(Ct^{-\alpha\frac{r-q}{\beta}}\mathbf{E}|v|_q^2 + C\mathbf{E}\Big[\sup_{s\in[0,T]}\|f(s,u_n(s))\|\Big]^2 + C\sum_{m=1}^{\infty}\mu_m^2\lambda_m^{r-\kappa}\Big) \\
&\le Ch^{2r}\Big(Ct^{-\alpha\frac{r-q}{\beta}}\mathbf{E}|v|_q^2 + C\mathbf{E}\Big(\sup_{s\in[0,T]}\|u_n(s)\|^2\Big) + C\sum_{m=1}^{\infty}\mu_m^2\lambda_m^{r-\kappa}\Big).
\end{aligned}
\tag{4.6.3}
$$

We now estimate θ. Note that θ satisfies the following equation,

$$
\begin{cases}
{}_0^C D_t^{\alpha}\theta(t) + (-\Delta_h)^{\beta}\theta(t) = (-\Delta_h)^{\beta}(R_h u_n - P_h u_n), \\
\theta(0) = 0,
\end{cases}
$$

and hence the representation of solution θ is written as

$$
\theta(t) = \int_0^t \bar{\mathbb{E}}_{\alpha,\beta}^h(t-s)(-\Delta_h)^{\beta}\Big(R_n u_n(s) - P_h u_n(s)\Big)ds. \tag{4.6.4}
$$

In fact,

$$
\begin{aligned}
&{}_0^C D_t^{\alpha}\theta + (-\Delta_h)^{\beta}\theta \\
&= \big({}_0^C D_t^{\alpha}\bar{u}_n^h - {}_0^C D_t^{\alpha}P_h u_n\big) + (-\Delta_h)^{\beta}(\bar{u}_h - P_h u_n) \\
&= (-\Delta_h)^{\beta}\bar{u}_h - (-\Delta_h)^{\beta}P_h u_n \\
&= {}_0^C D_t^{\alpha}\bar{u}_n^h - {}_0^C D_t^{\alpha}P_h u_n + (-\Delta_h)^{\beta}\bar{u}_n - {}_0^C D_t^{C}P_h u_n - (-\Delta_h)^{\beta}P_h u_n \\
&= {}_0^C D_t^{\alpha}\bar{u}_n^h - {}_0^C D_t^{\alpha}P_h u_n - (-\Delta_h)^{\beta}{}_0^C D_t^{\alpha}P_h u_n \\
&= (P_h f - {}_0^C D_t^{\alpha}P_h u_n) - (-\Delta_h)^{\beta}P_h u_n = P_h(-\Delta)u_n - (-\Delta_h)P_h u_n \\
&= (-\Delta_h)^{\beta}R_h u - (-\Delta_h)P_h u_n = (-\Delta_h)^{\beta}(R_h u_n - P_h u_n).
\end{aligned}
\tag{4.6.5}
$$

Choose $p = 0$ and $p = \beta$ separately, from Lemma 4.5.2 with $\gamma = 0$ and Lemma 4.5.3, it follows for $q = \epsilon - 2\beta + p$ for $0 < \epsilon < 2\beta$ that

$$
\begin{aligned}
\mathbf{E}|\theta(t)|_{p,h}^2 &\le \epsilon\Big(\int_0^t |\bar{\mathbb{E}}_{\alpha,\beta}^h(t-s)(-\Delta_h)^{\beta}(R_h u_n(s) - P_h u_n(s))|_{p,h}ds\Big)^2 \\
&\le C\mathbf{E}\Big(\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1}|(-\Delta_h)^{\beta}(R_h u_n - P_h u_n)(s)|_{\epsilon-2\beta+p,h}ds\Big)^2 \\
&\le C\mathbf{E}\Big(\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1}|(R_h u_n - P_h u_n)(s)|_{\epsilon+p,h}ds\Big)^2 \\
&\le Ch^{2r-2p-2\epsilon}\big(\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1}ds\big)\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1}\mathbf{E}|u_n(s)|_r^2 ds \\
&\le Ch^{2r-2p-2\epsilon}t^{\frac{\alpha\epsilon}{2\beta}}\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1}\mathbf{E}|u_n(s)|_r^2 ds.
\end{aligned}
\tag{4.6.6}
$$

Using Lemma 4.5.1 we have with $p = 0$ and β, and $0 \leq p \leq r \leq 2\beta$,

$$\begin{aligned}
\mathbf{E}|\theta(t)|_{p,h}^2 &\leq Ch^{2r-2p-2\epsilon}\mathbf{E}\Big(\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1}|u_n(s)|_r ds\Big)^2 \\
&\leq Ch^{2r-2p-2\epsilon}\mathbf{E}\Big(\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1} ds\Big)\Big(\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1}|u_n(s)|_r^2 ds\Big) \\
&\leq Ch^{2r-2p-2\epsilon}\Big(\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1} ds\Big)\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1}|u_n(s)|_r^2 ds \\
&\leq Ch^{2r-2p-2\epsilon}t^{\frac{\alpha\epsilon}{2\beta}}\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1} ds\mathbf{E}|u_n(s)|_r^2 ds. \qquad (4.6.7)
\end{aligned}$$

Now, an application of regularity Theorem 32 shows

$$\begin{aligned}
\mathbf{E}|\theta(t)|_{p,h}^2 \leq Ch^{2r-2p-2\varepsilon} &\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1}\Bigg[s^{-\alpha\frac{r-q}{\beta}}\|v\|_{L^2(\Omega;\dot{H}^q)}^2 \\
&+ \mathbf{E}\Big(\sup_{s\in[0,T]}\|f(s,u_n(s))\|\Big)^2\Bigg] ds \\
&\leq Ch^{2r-2p-2\varepsilon}\Bigg[\mathbf{E}|v|_q^2 + \mathbf{E}\big(\sup_{s\in[0,T]}\|u_n(s)\|\big)^2\Bigg], \qquad (4.6.8)
\end{aligned}$$

where we use the fact $\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1}s^{-\alpha\frac{r-q}{\beta}}ds < \infty$ and also by the assumptions that $0 < \frac{\alpha\epsilon}{2\beta} < 1$ and $0 \leq \alpha\frac{r-q}{\beta} < 1$. We now combine the estimates (4.6.3), (4.6.7) and (4.6.8) to arrive at an estimate for η as, with $p = 0$ and β, and $0 \leq p \leq r \leq 2\beta$,

$$\mathbf{E}|\eta(t)|_{p,h}^2 \leq Ch^{2r-2p-2\epsilon}\Bigg[\|v\|_{L^2(\Omega;\dot{H}^q)} + \mathbf{E}\big(\sup_{s\in[0,T]}\|u_n(s)\|^2\big)\Bigg] + Ch^{2r-2p}t^{-\alpha\frac{r-q}{\beta}}\|v\|_{L^2(\Omega;\dot{H}^q)}. \qquad (4.6.9)$$

Now, it remains to estimate ζ. Note that $\zeta(t) \in V_h$ satisfies,

$${}_0^C D_t^\alpha \zeta(t) + (-\Delta_h)^\beta \zeta(t) = P_h\big(f(u_n^h) - f(u_n)\big), \qquad (4.6.10)$$

and therefore, we now write $\zeta(t)$ in the integral form as

$$\zeta(t) = \int_0^t \bar{\mathbb{E}}_{\alpha,\beta}(t-s)P_h\big(f(u_n^h(s)) - f(u_n(s))\big)ds. \qquad (4.6.11)$$

Again, choose $p = 0, \beta$. From Lemma 4.5.2 with $\gamma = 0$ and Lemma 4.5.3, it follows that for $q = p$ and for $\frac{1}{2} < \alpha < 1$,

$$\begin{aligned}\mathbf{E}|\zeta|_{p,h}^2 &\le \mathbf{E}\Big(\int_0^t |\bar{\mathbb{E}}_{\alpha,\beta}(t-s)P_h(f(u_n^h(s)) - f(u_n(s))|_{p,h} ds\Big)^2\\ &\le \mathbf{E}\Big(\int_0^t (t-s)^{\alpha-1}|P_h(f(u_n^h(s)) - f(u_n(s))|_{p,h} ds\Big)^2\\ &\le C\mathbf{E}\Big(\int_0^t (t-s)^{\alpha-1}|u_n(s) - u_n^h(s)|_p ds\Big)^2\\ &\le C(\int_0^t (t-s)^{\alpha-1} ds)\Big(\int_0^t (t-s)^{\alpha-1}\mathbf{E}|u_n(s) - u_n^h(s)|_p^2 ds\Big)\\ &\le Ct^\alpha \int_0^t (t-s)^{\alpha-1}\mathbf{E}|u_n(s) - u_n^h(s)|_p^2 ds. \end{aligned} \tag{4.6.12}$$

Combining (4.6.9) and (4.6.12) it follows for $p = 0$ and β and $0 \le p \le r \le 2\beta$ that

$$\begin{aligned}\mathbf{E}|u_n(t) - u_n^h(t)|_p^2 \le Ch^{-2\epsilon+2(r-p)}&\Big[\mathbf{E}|v|_q^2 + \mathbf{E}\big(\sup_{s\in[0,T]} \|u_n(s)\|^2\big)\Big]\\ &+ Ch^{2(r-p)}t^{-\alpha\frac{r-q}{\beta}}|v|_q^2 + C\int_0^t (t-s)^{\alpha-1}\mathbf{E}|u_n(s) - u_n^h(s)|_p^2 ds. \end{aligned} \tag{4.6.13}$$

An application of Gronwall's Lemma completes the rest of the proof. □

Now we state our main theorem in this chapter.

Theorem 35. *Let $\frac{1}{2} < \alpha < 1$, $\frac{1}{2} < \beta \le 1$, $0 \le \gamma \le 1$. Assume that Assumptions 4.3.1, 4.3.2, 4.3.3 4.4.1, and 4.4.2 hold. Let u_n and u_n^h be the solutions of (4.4.1) and (4.5.2) respectively. Let $v \in L^2(\Omega; \dot{H}^q)$ with $0 \le q \le 2\beta$. Then, there exists a positive constant C such that, for any $\epsilon > 0$ and $0 \le \max(q, \beta) \le r \le 2\beta$, the following hold.*

1. *For $\frac{1}{2} < \alpha + \gamma < 1$,*

$$\begin{aligned}\mathbf{E}\|u(t) - u_n^h(t)\|^2 &\le Ct^{2(\alpha+\gamma)-1}\sum_{k=1}^{\infty}(\eta_k^n)^2 + Ct^{2(\alpha+\gamma)-1}(\triangle t)^2\sum_{k=1}^{\infty}(\gamma_k^n)^2\\ &+ Ct^{2\epsilon}(\triangle t)^{2(\alpha+\gamma)-1-2\epsilon}\sum_{k=1}^{\infty}(\mu_k^n)^2 + Ch^{-2\epsilon+2r}\Big[\mathbf{E}|v|_q^2 + \mathbf{E}\big(\sup_{s\in[0,T]}\|u_n(s)\|^2\big)\Big]\\ &+ Ch^{2r}t^{-\alpha\frac{r-q}{\beta}}\mathbf{E}|v|_q^2. \end{aligned} \tag{4.6.14}$$

2. For $1 \leq \alpha + \gamma \leq \frac{3}{2}$,

$$\begin{aligned}\mathbf{E}\|u(t) - u_n^h(t)\|^2 \leq & C\sum_{k=1}^{\infty}\lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\eta_k^n)^2 + C(\Delta t)^2\sum_{k=1}^{\infty}\lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\gamma_k^n)^2 \\ & + Ct^{2\epsilon}(\triangle t)^{2(\alpha+\gamma)-1-2\epsilon}\sum_{k=1}^{\infty}(\mu_k^n)^2 + Ch^{-2\epsilon+2r}\left[\mathbf{E}|v|_q^2 + \mathbf{E}\big(\sup_{s\in[0,T]}\|u_n(s)\|^2\big)\right] \\ & + Ch^{2r}t^{-\alpha\frac{r-q}{\beta}}\mathbf{E}|v|_q^2. \end{aligned} \tag{4.6.15}$$

3. For $\frac{3}{2} \leq \alpha + \gamma < 2$,

$$\begin{aligned}\mathbf{E}\|u(t) - u_n^h(t)\|^2 \leq & C\sum_{k=1}^{\infty}\lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\eta_k^n)^2 + C(\triangle t)^2\sum_{k=1}^{\infty}\lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\gamma_k^n)^2 \\ & + Ct^{2(\alpha+\gamma)-3}(\triangle t)^2\sum_{k=1}^{\infty}(\mu_k^n)^2 + Ch^{-2\epsilon+2r}\left[\mathbf{E}|v|_q^2 + \mathbf{E}\big(\sup_{s\in[0,T]}\|u_n(s)\|^2\big)\right] \\ & + Ch^{2r}t^{-\alpha\frac{r-q}{\beta}}\mathbf{E}|v|_q^2. \end{aligned}$$

Remark 36. In particular, when the noise is the trace class noise i.e.,

$$\dot{W}(t,x) = \sum_{k=1}^{\infty}\gamma_k^{\frac{1}{2}}\dot{\beta}_k(t)e_k(x),\ Tr(Q) = \sum_{k=1}^{\infty}\gamma_k < \infty,$$

we obtain, with $\epsilon > 0$,

$$\mathbf{E}\|u(t) - u_n^h(t)\|^2 = \mathcal{O}(h^{4-\epsilon} + (\Delta t)^{1-\epsilon}),$$

which is consistent with the results obtained in [133] for stochastic heat equation.

Remark 37. The primary importance of Theorem 35 lies in achieving the upper bounds on the error in both spatial and temporal domains for the finite element approximation of the regularized stochastic time-space fractional subdiffusion equation. This approximation involves discretizing the temporal noise using piecewise constant functions. The outcomes provide precise insights into the interplay between the convergence rates in time and the specific parameters $\alpha \in (0,1)$ and $\gamma \in (0,1)$.

4.7 Numerical simulations

In this section, we shall consider the L1 scheme [71, 117, 72, 82] for solving the following stochastic time fractional PDEs: with $\alpha \in (0,1)$,

$$ {}_0^C D_t^{\alpha} u(t,x) - \Delta u(t,x) = f(t,x) + g(t,x), \quad 0 \le t \le T, \ 0 < x < 1, \tag{4.7.1} $$

$$ u(0,x) = u_0(x), \tag{4.7.2} $$

$$ u(t,0) = u(t,1) = 0, \tag{4.7.3} $$

where $\Delta = \frac{\partial^2}{\partial x^2}$ denotes the Laplacian and ${}_0^C D_t^{\alpha} u$ denotes the Caputo fractional derivative. Here $f(t,x), u_0(x)$ are given data. Here, with $\gamma \in [0,1]$,

$$ g(t,x) := {}_0^R D_t^{-\gamma} \frac{dW(t,x)}{dt} = {}_0^R D_t^{-\gamma} \sum_{m=1}^{\infty} \gamma_m^{1/2} e_m(x) \frac{d\beta_m^H(t)}{dt}, \tag{4.7.4} $$

where $\beta_m^H(t)$, $m = 1, 2, \cdots$ are the fractional Brownian motions with Hurst number $H \in [1/2, 1]$. In particular, when $H = 1/2$, $\beta_m^H(t)$, $m = 1, 2, \cdots$ are reduced to the standard Brownian motions. Here $e_m(x) = \sqrt{2} \sin m\pi x$ denote the eigenfunctions of the operator $A = -\frac{\partial^2}{\partial x^2}$ with $D(A) = H_0^1(0,1) \cap H^2(0,1)$. Further $\gamma_m, m = 1, 2, \cdots$ are the eigenvalues of the covariance operator Q of the stochastic process $W(t)$, that is

$$ Qe_m = \gamma_m e_m. $$

We shall consider two cases in our numerical simulations.

Case 1: the white noise case, e.g., $\gamma_m = m^{-\beta}$ with $\beta = 0$ which implies that

$$ \mathrm{tr}(Q) = \sum_{m=1}^{\infty} \gamma_m = \sum_{m=1}^{\infty} m^{-\beta} = \sum_{m=1}^{\infty} 1 = \infty. $$

Case 2: The trace class case, e.g., $\gamma_m = m^{-\beta}$ with $\beta > 1$, which implies that

$$ \mathrm{tr}(Q) = \sum_{m=1}^{\infty} \gamma_m = \sum_{m=1}^{\infty} m^{-\beta} < \infty. $$

The numerical methods for solving stochastic time fractional partial differential equations are similar to the numerical methods for solving deterministic time fractional partial differential equations. The only difference is that we have the extra term g in stochastic case and we need to consider how to approximate g (please refer to the numerical methods for solving time fractional partial differential equations). Let $v = u - u_0$. Then (4.7.1)-(4.7.3) can be written as the following

$$ {}_0^C D_t^\alpha v(t,x) - \Delta v(t,x) = \Delta u_0(x) + f(t,x) + g(t,x), \quad 0 < x < 1, \tag{4.7.5} $$

$$ v(0,x) = 0, \tag{4.7.6} $$

$$ v(t,0) = v(t,1) = 0. \tag{4.7.7} $$

Since the initial value $v(0,x) = 0$ in (4.7.5)-(4.7.7), it is easier to consider the numerical analysis for the time discretization scheme of (4.7.5)-(4.7.7). From now on, we shall consider the fully discrete schemes for solving (4.7.5)-(4.7.7). Let $A = -\frac{\partial}{\partial x^2}$ with $D(A) = H_0^1(0,1) \cap H^2(0,1)$. Then (4.7.5)-(4.7.7) can be written as the following abstract form

$$ {}_0^C D_t^\alpha v + Av = -Au_0 + f(t) + g(t), \quad v(0) = 0, \tag{4.7.8} $$

Let $0 < t_0 < t_1 < \cdots < t_N = T$ be a partition of the time interval $[0,T]$ and τ the time step size. Let $0 = x_0 < x_1 < \cdots < x_M = 1$ be a partition of the space interval $[0,1]$ and h the space step size. Let $S_h \subset H_0^1(0,1)$ be the piecewise linear finite element space defined by $S_h = \{\chi \in C[0,1] : \chi$ is the piecewise linear function defined on $[0,1]$ and $\chi(0) = \chi(1) = 0\}$.

The finite element method of (4.7.5)-(4.7.7) is to find $v_h(t) \in S_h$ such that, with $\chi \in S_h$,

$$ \big({}_0^C D_t^\alpha v_h(t), \chi\big) + (\nabla v_h(t), \nabla \chi) = -(\nabla P_h u_0, \nabla \chi) + (f(t), \chi) + (g(t), \chi), \tag{4.7.9} $$

$$ v_h(0) = 0, \tag{4.7.10} $$

where $P_h : H \to S_h$ denotes the L_2 projection operator defined by

$$ (P_h v, \chi) = (v, \chi), \quad \forall\, \chi \in S_h. $$

Let $V^n \approx v_h(t_n), n = 0, 1, \dots, N$ be the approximation of $v_h(t_n)$. The L1 scheme is to find $V^n \in S_h$, with $n = 1, 2, \dots, N$, such that, with $V^0 = 0$, [71]

$$ \Big(\tau^{-\alpha} \sum_{j=1}^{n} w_{n-j} V^j, \chi\Big) + (\nabla V^n, \nabla \chi) = -(\nabla u_0, \nabla \chi) + (f(t_n), \chi) + (g(t_n), \chi), \tag{4.7.11} $$

where the weights are defined in [132], [97], [71]. Let $A_h : S_h \to S_h$ be the discrete analogue of the operator A defined by

$$ (A_h \psi, \chi) = (\nabla \psi, \nabla \chi), \quad \forall\, \chi \in S_h. \tag{4.7.12} $$

Then (4.7.11) can be written as the following abstract form

$$\tau^{-\alpha}\sum_{j=1}^{n} w_{n-j}V^j + A_h V^n = -A_h u_0 + f(t_n) + g(t_n), \quad V^0 = 0. \tag{4.7.13}$$

Let $\varphi_1(x), \varphi_2(x), \cdots, \varphi_{M-1}(x)$ be the linear finite element basis functions defined by, with $j = 1, 2, \cdots, M-1$,

$$\varphi_j(x) = \begin{cases} \frac{x-x_{j-1}}{x_j - x_{j-1}}, & x_{j-1} < x < x_j, \\ \frac{x-x_{j+1}}{x_j - x_{j+1}}, & x_j < x < x_{j+1}, \\ 0, & \text{otherwise.} \end{cases}$$

To find the solution $V^n \in S_h,\ n = 0, 1, \cdots, N$, we assume that

$$V^n = \sum_{m=1}^{M-1} \alpha_m^n \varphi_m,$$

for some coefficients α_k^n, $k = 1, 2, \cdots, M-1$. Choose $\chi = \varphi_l$, $l = 1, 2, \cdots, M-1$ in (4.7.11), we have with $n = 1, 2, \cdots, N$,

$$\begin{aligned} &\tau^{-\alpha}\sum_{j=1}^{n} w_{n-j}\Big[\sum_{m=1}^{M-1}\big(\varphi_m, \varphi_l\big)\alpha_m^j\Big] + \sum_{m=1}^{M-1}\big(\nabla\varphi_m, \nabla\varphi_l\big)\alpha_m^n \\ &= -\sum_{m=1}^{M-1}\big(\nabla\varphi_m, \nabla\varphi_l\big)\alpha_m^0 + \big(f(t_n), \varphi_l\big) + \big(g(t_n), \varphi_l\big), \end{aligned} \tag{4.7.14}$$

To get $\alpha_m^n, n = 1, 2, \cdots, N$ from (4.7.14), we also need the inital α_m^0 which can be obtained by

$$u_0 \approx P_h u_0 = \sum_{m=1}^{M-1} \alpha_m^0 \varphi_m.$$

To solve (4.7.14) by MATLAB, we need to write (4.7.14) into the matrix form which we shall do now.

Denote

$$\alpha^n = \begin{pmatrix} \alpha_1^n \\ \alpha_2^n \\ \vdots \\ \alpha_{M-1}^n \end{pmatrix}_{(M-1)\times 1}, \quad \mathbf{f}^n = \begin{pmatrix} \big(f(t_n), \varphi_1\big) \\ \big(f(t_n), \varphi_2\big) \\ \vdots \\ \big(f(t_n), \varphi_{M-1}\big) \end{pmatrix}_{(M-1)\times 1},$$

and

$$\mathbf{g}^n = \begin{pmatrix} \big(g(t_n), \varphi_1\big) \\ \big(g(t_n), \varphi_2\big) \\ \vdots \\ \big(g(t_n), \varphi_{M-1}\big) \end{pmatrix}_{(M-1)\times 1},$$

After some simple calculations, we may get the following mass and stiffness metrics

$$\mathbf{M} = \Big((\varphi_m, \varphi_l)\Big)_{m,l=1}^{M-1} = h \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & & 0 \\ \frac{1}{6} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{1}{6} \\ 0 & & \frac{1}{6} & \frac{2}{3} \end{pmatrix}_{(M-1)\times(M-1)},$$

and

$$\mathbf{S} = \Big((\nabla\varphi_m, \nabla\varphi_l)\Big)_{m,l=1}^{M-1} = \frac{1}{h} \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}_{(M-1)\times(M-1)},$$

respectively. Then (4.7.14) can be written as the following matrix form, $n = 1, 2, \cdots, N$,

$$\tau^{-\alpha} \sum_{j=1}^{n} w_{n-j} \mathbf{M} \alpha^j + \mathbf{S}\alpha^n = -\mathbf{S}\alpha^0 + \mathbf{f}^n + \mathbf{g}^n, \quad \alpha^0 \text{ given}, \tag{4.7.15}$$

Denote $\mathbf{A}_h = \mathbf{M}^{-1}\mathbf{S}$. Then (4.7.15) can be written as, with $n = 1, 2, \cdots, N$,

$$\tau^{-\alpha} \sum_{j=1}^{n} w_{n-j} \alpha^j + \mathbf{A}_h \alpha^n = -\mathbf{A}_h \alpha^0 + \mathbf{M}^{-1}\mathbf{f}^n + \mathbf{M}^{-1}\mathbf{g}^n, \quad \alpha^0 \text{ given}, \tag{4.7.16}$$

which is the matrix approximation form of (4.7.13). Hence $\alpha^n, n = 1, 2, \cdots, N$ can be calculated by the following formula

$$\alpha^n = (w_0 + \tau^\alpha \mathbf{A}_h)^{-1} \Big(-\tau^\alpha \mathbf{A}_h \alpha^0 + \tau^\alpha \mathbf{M}^{-1}\mathbf{f}^n + \tau^\alpha \mathbf{M}^{-1}\mathbf{g}^n - \sum_{j=1}^{n-1} w_{n-j} \alpha^{n-j} \Big), \quad \alpha^0 \text{ given}. \tag{4.7.17}$$

We now consider how to calculate $\mathbf{f}^n$. The lth term $\big(f(t_n), \varphi_l\big)$ in $\mathbf{f}^n$ can be approximated by using the midpoint quadrature formula

$$\begin{aligned}\big(f(t_n), \varphi_l\big) &= \int_0^1 f(t_n)\varphi_l \, dx = \int_{x_{l-1}}^{x_l} f(t_n)\varphi_l \, dx + \int_{x_l}^{x_{l+1}} f(t_n)\varphi_l \, dx \\ &\approx f(t_n, \frac{x_{l-1}+x_l}{2})\varphi_l(\frac{x_{l-1}+x_l}{2})h + f(t_n, \frac{x_l+x_{l+1}}{2})\varphi_l(\frac{x_l+x_{l+1}}{2})h \\ &= \frac{h}{2}\Big(f(t_n, \frac{x_{l-1}+x_l}{2}) + f(t_n, \frac{x_l+x_{l+1}}{2})\Big).\end{aligned}$$

In MATLAB, we use the following code to calculate $\mathbf{f}^n$ with some given $f(t,x)$.

```
% find (f, phi)
    function y=f_phi(x,n,tau,alpha)
        % case 1: f(t, x) = x^2 (1-x)^2 exp(t)-(2-12 x+12 x^2) exp(t)
        tn=n*tau;
        h=x(2)-x(1);
        x0=[0;x(1:end-1)]; x1=x; x2=[x(2:end); x(end)+h];
        x=(x0+x1)/2;
        y1=(x.^2).*((1-x).^2)*exp(tn)-(2-12*x+12*(x.^2))*exp(tn);
        x=(x1+x2)/2;
        y2=(x.^2).*((1-x).^2)*exp(tn)-(2-12*x+12*(x.^2))*exp(tn);
        y=h/2*(y1+y2);


       %case 2: f(t, x)=0
        y=zeros(size(x));    %f=0
```

Remark 38. One may modify the MATLAB function

```
 f_{phi}(x,n,tau,alpha)
```

to consider other f such as $f(u) = u^3 - u$.

We next consider how to calculate $\mathbf{g}^n$ which is more complicated than $\mathbf{f}^n$. Approximating the Riemann-Liouville fractional integral by the Lubich first order convolution quadrature formula and truncating the noise term to $M-1$ terms, we obtain the lth element of $\mathbf{g}^n$ by, with $l = 1, 2, \cdots, M-1$,

$$\mathbf{g}^n(l) = \big(g(t_n), \varphi_l\big) = {}_0^R D_t^{-\gamma} \sum_{m=1}^{\infty} \gamma_m^{1/2} e_m(x) \frac{d\beta_m^H(t)}{dt}$$
$$\approx \tau^{\gamma} \sum_{j=1}^{n} w_{n-j}^{(-\gamma)} \Big[\sum_{m=1}^{M-1} \gamma_m^{1/2} (e_m, \varphi_l) \frac{\beta_m^H(t_j) - \beta_m^H(t_{j-1})}{\tau} \Big], \qquad (4.7.18)$$

where $w_j^{(-\gamma)}, j = 0, 1, 2, \cdots, n$ are generated by the Lubich first order method, with $\gamma \in [0, 1]$,

$$(1-\zeta)^{-\gamma} = \sum_{j=0}^{\infty} w_j^{(-\gamma)} \zeta^j.$$

To solve (4.7.18), we first need to generate $M-1$ Brownian motions $\beta_m^H(t), m = 1, 2, \cdots, M-1$ which can be done by using MathWorks MATLAB function **fbm1d.m**.

Let **Nref** $= 2^7$ and $T = 1$ and let **dtref** $= T/Nref$ denote the reference time step size. Let $0 = t_0 < t_1 < \cdots < t_{Nref} = T$ be the time partition of $[0, T]$. We generate the fractional Brownian motions $\beta_m^H(t_0), \beta_m^H(t_1), \cdots \beta_m^H(t_{Nref})$, $m = 1, 2, \cdots, M-1$ with the Hurst number $H \in [1/2, 1]$ by using the following code:

```
% Fractional Brownian paths  with Hurst number 1/2 \leq H \leq 1
     W=[];
     for j=1:M-1
         [Wj,t]=fbm1d(H,Nref,T);
         W=[W Wj];
     end
     W(1,:)=zeros(1, M-1);
```

Remark 39. When $H = 1/2$, $\mathbf{fbm1d}(\mathbf{H}, \mathbf{Nref}, \mathbf{T})$ generates the standard Brownian motions. The standard Brownian motions can also be generated by the following code

```
% Standard Brownian paths
    dW=sqrt(dtref)*randn(Nref,M-1);
    W=cumsum(dW,1);
    W=[zeros(1, M-1); W];
```

Since we do not know the exact solution of the system, we shall use the reference time step size $dtref$ and the space step size $h = 2^{-7}$ to calculate the reference solution **vref**. The spacial discretization is based on the linear finite element method.

We then choose $kappa = 2^5, 2^4, 2^3, 2^2$ and consider the different time step size $\tau = dtref * kappa$ to obtain the approximate solutions V^n at $t_n = n\tau$.

Let us discuss how to calculate the lth element of $\mathbf{g}^n$ in MATLAB. Denote

$$\mathbf{w}_\gamma = [w_0^{(-\gamma)},\, w_1^{(-\gamma)},\, \dots,\, w_{n-1}^{(-\gamma)}]_{1\times(M-1)},$$

and

$$\mathbf{dWdt} = \begin{pmatrix} \sum_{m=1}^{M-1} \gamma_m^{1/2}(e_m, e_l)\frac{\beta_m(t_n)-\beta_m(t_{n-1})}{\tau} \\ \sum_{m=1}^{M-1} \gamma_m^{1/2}(e_m, e_l)\frac{\beta_m(t_{n-1})-\beta_m(t_{n-2})}{\tau} \\ \vdots \\ \sum_{m=1}^{M-1} \gamma_m^{1/2}(e_m, e_l)\frac{\beta_m(t_1)-\beta_m(t_0)}{\tau} \end{pmatrix}_{(M-1)\times 1}.$$

The lth element of the vector $\mathbf{g}^n$ satisfies

$$\mathbf{g}^n(l) = \mathbf{w}_\gamma * \mathbf{dWdt}, \quad l = 1, 2, \cdots, M-1.$$

Based on this idea, we use the following MATLAB function

```
g_{phi}(x,n,tau,ga,kappa,W)
```

to calculate $\mathbf{g}^n(l)$ in our numerical simulations.

```
% find (g, phi)
   function y=g_phi(x,n,tau,ga,kappa,W)
       y=[];
       M=length(x)+1;
       %Find w_ga=[w_{0}^{-ga} w_{1}^{-ga}  w_{n-1}^{-ga}]
       w_ga=[];
       for nn=0:n-1
           w_ga=[w_ga w_gru(nn,-ga)];
       end
```

```
        for k=1:M-1
            A=dWdt_k(x,n,tau,kappa,W,k);
            y1=tau^(ga)*w_ga*A;
            y=[y;y1];
        end


    % Find dWdt_k
 function y= dWdt_k(x,n,tau,kappa,W,k)
        y=zeros(n,1);
        M=length(x)+1;
        for m=1:M-1
            beta=2; % white noise beta=0, trace class beta=2
            ga_m=m^(-beta);
            k1=n:-1:1;  %tn=n*tau=(n*kappa)*dtref
            dW_k1=W(k1*kappa+1,m)-W((k1-1)*kappa+1,m); %dW_k is a vector
            h=x(2)-x(1);
            x1=((k-1)*h+k*h)/2; x2= (k*h+(k+1)*h)/2;
            e_phi=h/2*(sqrt(2)*sin(pi*m*x1)+sqrt(2)*sin(pi*m*x2));
            y=y+ga_m^(1/2)*e_phi*(dW_k1/tau);
        end
```

Finally we shall consider how to calculate the L_2 projection $P_h u_0$ of u_0. Assume that

$$P_h u_0 = \sum_{m=1}^{M-1} \alpha_m^0 \varphi_m.$$

By the definition of P_h, we obtain

$$\sum_{m=1}^{M-1} \alpha_m^0 (\varphi_m, \varphi_l) = (u_0, \varphi_l).$$

Hence α^0 can be calculated by

$$\alpha^0 = \mathbf{M}^{-1}\mathbf{u}^0, \tag{4.7.19}$$

where

$$\mathbf{u}^0 = \begin{pmatrix} (u_0, \varphi_1) \\ (u_0, \varphi_2) \\ \vdots \\ (u_0, \varphi_{M-1}) \end{pmatrix}_{(M-1)\times 1}.$$

Remark 40. When we use (4.7.19) to calculate α^0, we have to calculate $\mathbf{M}^{-1}$ which will produce some computational errors. In our numerical examples, we shall simply choose $\alpha^0(l) = u_0(x_l)$, $l = 1, 2, \cdots, M-1$ (instead of (4.7.19)) which also give the required accuracy for our numerical simulations.

Example 41. *Consider the following stochastic time fractional PDE, with $\alpha \in (0,1)$,*

$$ {}_0^C D_t^\alpha u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x) + g(t,x), \quad 0 \le t \le T, \ 0 < x < 1, \tag{4.7.20}$$

$$u(0,x) = u_0(x), \tag{4.7.21}$$

$$u(t,0) = u(t,1) = 0, \tag{4.7.22}$$

where $f(t,x) = x^2(1-x)^2 e^t - (2 - 12x + 12x^2)e^t$ and the initial value $u_0(x) = x^2(1-x)^2$ and $g(t,x)$ is defined by (4.7.4).

Let $v(t,x) = u(t,x) - u_0(x)$ and transform the system (4.7.20)-(4.7.22) *of u into the system of v. We shall consider the approximation of v at $T = 1$. We choose the space step size $h = 2^{-6}$ and the time step size $dtref = 2^{-7}$ to get the reference solution vref. To observe the time convergence orders, we consider the different time step sizes $\tau = kappa * dtref$ with $kappa = [2^5, 2^4, 2^3, 2^2]$ to obtain the approximate solution V. We choose $M1 = 50$ simulations to calculate the following L2 error at $T = 1$ with the different time step sizes*

$$\|vref - V\|_{L^2(\Omega;H)} = \sqrt{\mathbb{E}\|vref - V\|_H^2}.$$

By Theorem 35, the convergence order should be

$$\|vref - V\|_{L^2(\Omega;H)} = O(\tau^{\min\{1, \alpha+\gamma-1/2\}}). \tag{4.7.23}$$

In Table 4.7.1, we consider the trace class noise, that is $\gamma_m = m^{-2}, m = 1, 2, \dots$ and we observe that the experimentally determined time convergence orders are slightly

α	γ	$\tau = 1/4$	$\tau = 1/8$	1/16	1/32	order
0.5	0.0	7.4317e-01	6.7559e-01	6.4005e-01	5.1914e-01	
			0. 3021	0.1375	0.078	0.1725 (0.00)
0.5	0.4	4.4679e-02	2.8694e-02	2.3152e-02	1.5683e-02	
			0.5619	0.6389	0.3096	0.5035 (0.40)
0.5	0.6	9.0785e-03	4.359e-03	2.282e-03	1.5110e-03	
			0.8524	1.0582	1.6763	0.8623 (0.60)
0.9	0.0	6.5226e-02	3.4896e-02	2.3907e-02	1.4621e-02	
			0.7093	0.9024	0. 5457	0.7191 (0.40)
0.9	0.6	5.4063e-03	2.7815e-03	1.9341e-03	1.1227e-03	
			0.7847	0.9588	0.5241	0.7559 (1.00)
0.9	0.8	4.0498e-03	1.8511e-03	1.1434e-03	6.2200e-04	
			0.8783	1.1294	0.6951	0.9010 (1.00)

Table 4.7.1: Time convergence orders in Example 41 at $T = 1$ with trace class noise $\gamma_m = m^{-2}, m = 1, 2, \dots$

better than the theoretical convergence orders. The numbers in the brackets denote the theoretical convergence orders.

In Figure 4.7.1, we plot the experimentally determined orders of convergence with $\gamma = 0.6$ and $\alpha = 0.5$ in Table 4.7.1. The expected convergence order is $O(\tau^{\min\{1,\alpha+\gamma-1/2\}}) = O(\tau)$. We indeed observe this in the figure where the reference line is for the order $O(\tau)$.

In Figure 4.7.2, we plot one approximate solution with $\alpha = 0.9$ and $\gamma = 0$ for all $x \in (0, 1)$ and $t \in (0, 1)$ in Example 41. In Figure 4.7.3, we plot one approximate solution with $\alpha = 0.9$ and $\gamma = 0$ at time $T = 1$ in Example 41.

In Figure 4.7.4, we plot one approximate solution with $\alpha = 0.9$ and $\gamma = 0.9$ for all $x \in (0, 1)$ and $t \in (0, 1)$ in Example 41. In Figure 4.7.5, we plot one approximate solution with $\alpha = 0.9$ and $\gamma = 0.9$ at time $T = 1$ in Example 41.

We observe that the solution with $\alpha = 0.9, \gamma = 0.9$ is much smoother than the solution with $\alpha = 0.9, \gamma = 0$ as we expected.

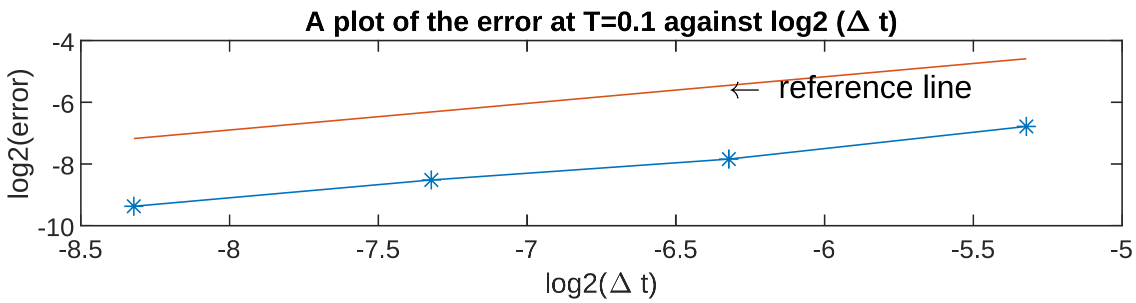

Figure 4.7.1: The experimentally determined orders of convergence with $\gamma = 0.6$ and $\alpha = 0.5$ in Table 4.7.1

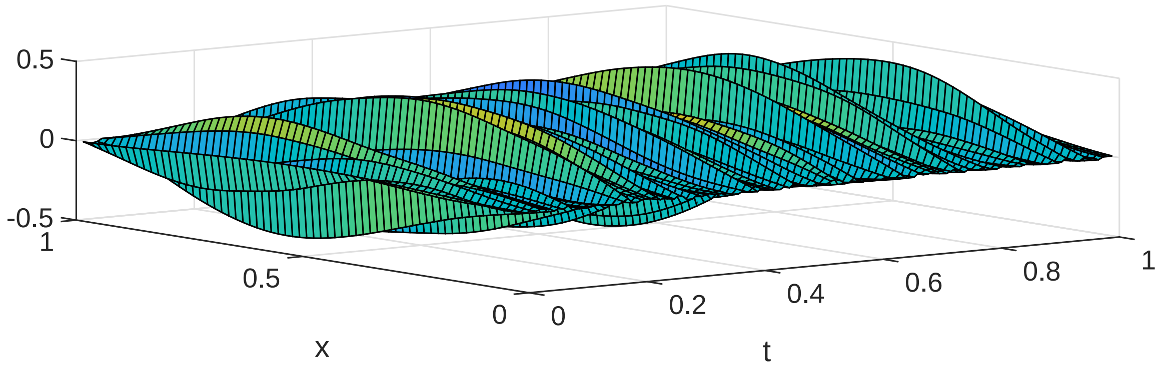

Figure 4.7.2: Approximate realisation of the solution with $\alpha = 0.9$ and $\gamma = 0$ for $x \in (0, 1)$ and $t \in (0, 1)$ in Example 41

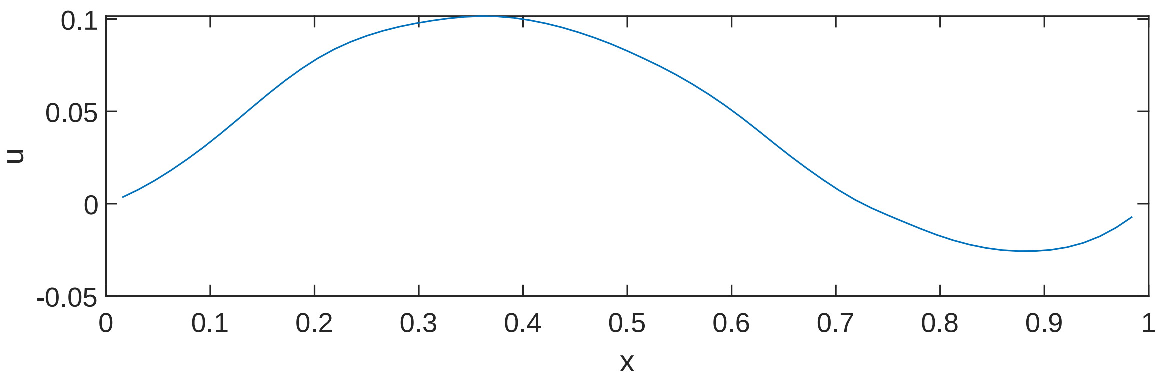

Figure 4.7.3: Approximate realisation of the solution at time $T = 1$ with $\alpha = 0.9$ and $\gamma = 0$ in Example 41

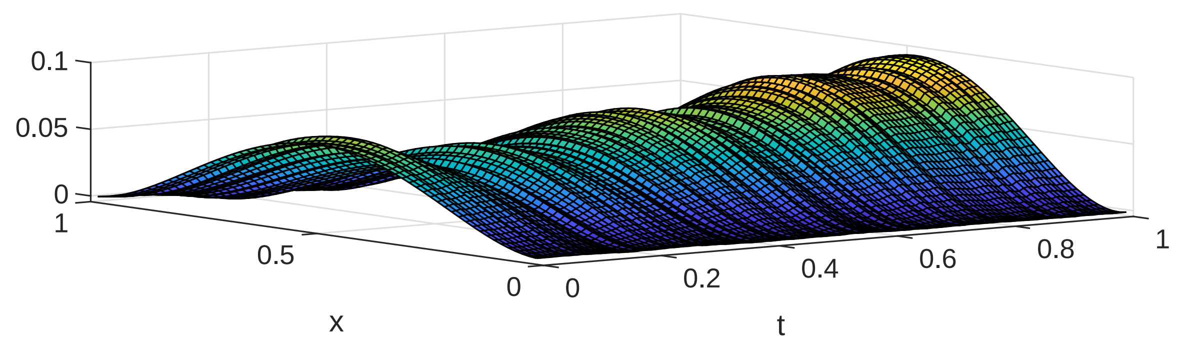

Figure 4.7.4: Approximate realisation of the solution with $\alpha = 0.9$ and $\gamma = 0.9$ for $x \in (0, 1)$ and $t \in (0, 1)$ in Example 41

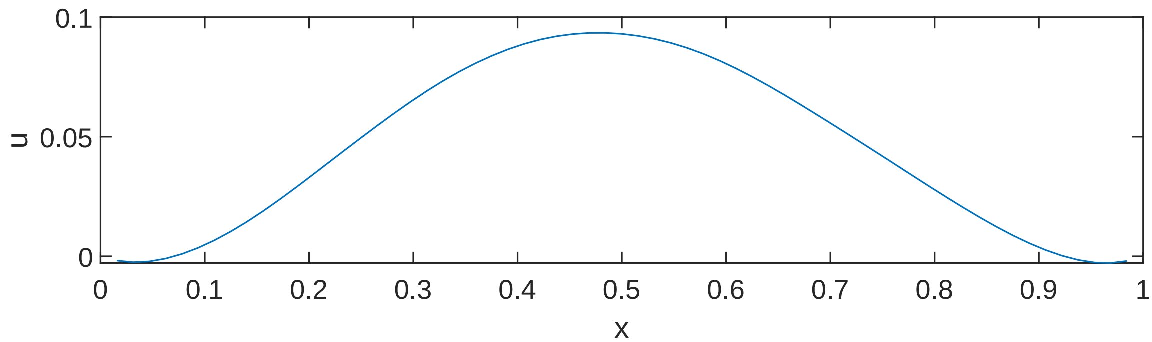

Figure 4.7.5: Approximate realisation of the solution at time $T = 1$ with $\alpha = 0.9$ and $\gamma = 0$ in Example 41

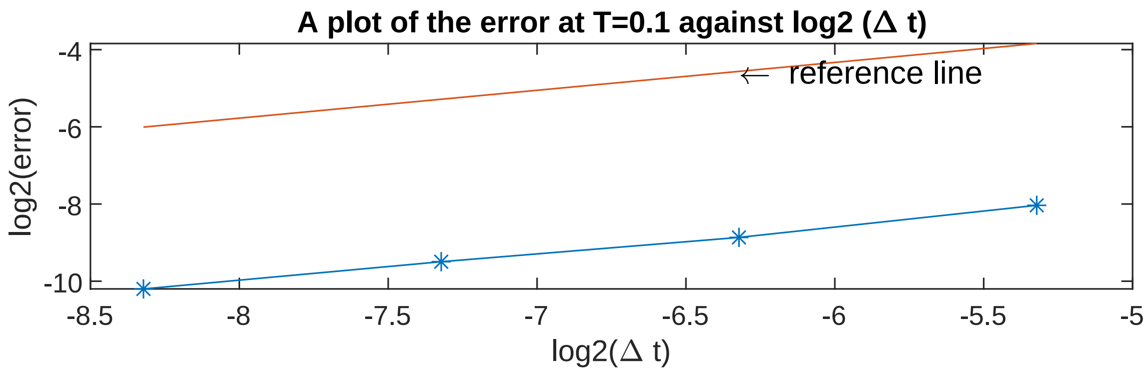

Figure 4.7.6: The experimentally determined orders of convergence with $\gamma = 0.8$ and $\alpha = 0.5$ in Table 4.7.2

Example 42. *Consider the following stochastic time fractional PDE, with $\alpha \in (0,1)$,*

$$ {}_0^C D_t^{\alpha} u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = f(u(t,x)) + g(t,x), \quad 0 \leq t \leq T, \ 0 < x < 1, \tag{4.7.24} $$

$$ u(0,x) = u_0(x), \tag{4.7.25} $$

$$ u(t,0) = u(t,1) = 0, \tag{4.7.26} $$

where $f(u) = \sin(u)$ and the initial values $u_0(x) = x^2(1-x)^2$ and $g(t,x)$ is defined by (4.7.4).

We use the same notations as in Example 41. In Table 4.7.1, we consider the trace class noise, that is $\gamma_m = m^{-2}, m = 1, 2, \dots$ and we observe that the experimentally determined time convergence orders are consistent with our theoretical convergence orders. The numbers in the brackets denote the theoretical convergence orders.

In Table 4.7.2, we consider the trace class noise, that is $\gamma_m = m^{-2}, m = 1, 2, \dots$ and we observe that the experimentally determined time convergence orders are consistent with our theoretical convergence orders. The numbers in the brackets denote the theoretical convergence orders.

In Figure 4.7.6, we plot the experimentally determined orders of convergence with $\gamma = 0.8$ and $\alpha = 0.5$ in Table 4.7.2.

The expected convergence order is $O(\tau^{\min\{1,\alpha+\gamma-1/2\}}) = O(\tau)$. We indeed observe this in the figure where the reference line is for the order $O(\tau)$.

α	γ	$\tau = 1/4$	$\tau = 1/8$	1/16	1/32	order
0.5	0.0	7.4680e-01	6.7394e-3	6.3525e-01	5.1577e-01	
			0.3006	0.1481	0.0853	0.1780 (0.00)
0.5	0.4	4.8982e-02	3.0706e-02	2.3346e-02	1.547e-02	
			0.5934	0.6737	0.3954	0.5542 (0.40)
0.5	0.6	1.1733e-02	6.1260e-03	3.5130e-03	1.8018e-03	
			0.9632	0.9376	0.8023	0.9010 (0.60)
0.5	0.8	3.8149e-03	2.1440e-03	1.3858e-03	8.5028e-04	
			0.7047	0.9023	0.6296	0.7219 (0.80)
0.9	0.0	6.7694e-02	3.6219e-02	2.4353e-02	1.4678e-02	
			0.7305	0.9023	0.5726	0.7351 (0.40)
0.9	0.4	9.2713e-03	5.3332e-03	3.4346e-03	2.0406e-03	
			0.7511	0.7978	0.6349	0.7279 (0.80)
0.9	0.6	9.8655e-03	5.0390e-03	2.7109e-03	1.4385e-03	
			0.9142	0.9693	0.8944	0.9260 (1.00)
0.9	0.8	9.8791e-03	4.9514e-03	2.3681e-03	1.1356e-03	
			1.0602	0.9965	1.0641	1.0403 (1.00)

Table 4.7.2: Time convergence orders in Example 42 at $T = 1$ with trace class noise $\gamma_m = m^{-2}, m = 1, 2, \ldots$

Example 43. *Consider the following stochastic time fractional PDE, with* $\alpha \in (0, 1)$,

$$ {}_0^C D_t^{\alpha} u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = f(u(t,x)) + g(t,x), \quad 0 \le t \le T,\ 0 < x < 1, \tag{4.7.27} $$

$$ u(0,x) = u_0(x), \tag{4.7.28} $$

$$ u(t,0) = u(t,1) = 0, \tag{4.7.29} $$

where $f(u) = -u^3 + u$ *and the initial values* $u_0(x) = x^2(1-x)^2$ *and* $g(t,x)$ *is defined by* (4.7.4).

We use the same notations as in Example 41. In Table 4.7.1, we consider the trace class noise, that is $\gamma_m = m^{-2}, m = 1, 2, \ldots$ *and we observe that the experimentally* *determined time convergence orders are consistent with our theoretical convergence orders. The numbers in the brackets denote the theoretical convergence orders.*

In Table 4.7.3, we consider the white noise, that is $\gamma_m = 1, m = 1, 2, \ldots$ *and we observe that the experimentally determined time convergence orders are slightly less than the orders in the trace class noise case as we expected.*

In Figure 4.7.7, we plot the experimentally determined orders of convergence with $\gamma = 0.6$ *and* $\alpha = 0.9$ *in Table 4.7.3. The expected convergence order is* $O(\tau^{\min\{1,\alpha+\gamma-1/2\}}) = O(\tau)$*. We indeed observe this in the figure where the reference line is for the order* $O(\tau)$*.*

α	γ	$\tau = 1/4$	$\tau = 1/8$	1/16	1/32	order
0.5	0.0	7.1957e-01	6.7388e-01	6.4560e-01	5.3525e-01	
			0.2704	0.0946	0.0619	0.1423 (0.00)
0.5	0.4	4.8614e-02	3.0505e-02	2.3321e-02	1.5482e-02	
			0.5911	0.6723	0.3874	0.5503 (0.40)
0.5	0.6	1.1684e-02	6.0865e-03	3.4920e-03	1.7909e-03	
			0.9634	0.9409	0.8016	0.9019 (0.60)
0.5	0.8	3.7847e-03	2.1331e-03	1.3813e-03	8.4861e-04	
			0.7029	0.8273	0.6269	0.7190 (0.80)
0.9	0.0	6.7242e-02	3.5923e-02	1.4633e-02	1.4633e-02	
			0.7311	0.9045	0.5646	0.7334 (0.40)
0.9	0.4	9.2195e-03	5.3122e-03	3.4275e-03	2.0384e-4	
			0.7498	0.7954	0.6321	0.7258 (0.80)
0.9	0.6	9.8077e-03	5.0123e-03	2.7012e-03	1.4346e-03	
			0.9129	0.9684	0.8919	0.9244 (1.00)
0.9	0.8	9.8145e-03	4.9210e-03	2.3553e-03	1.1303e-03	
			1.0591	0.9960	1.0630	1.0394 (1.00)

Table 4.7.3: Time convergence orders in Example 42 at $T = 1$ with trace class noise $\gamma_m = m^{-2}, m = 1, 2, \ldots$

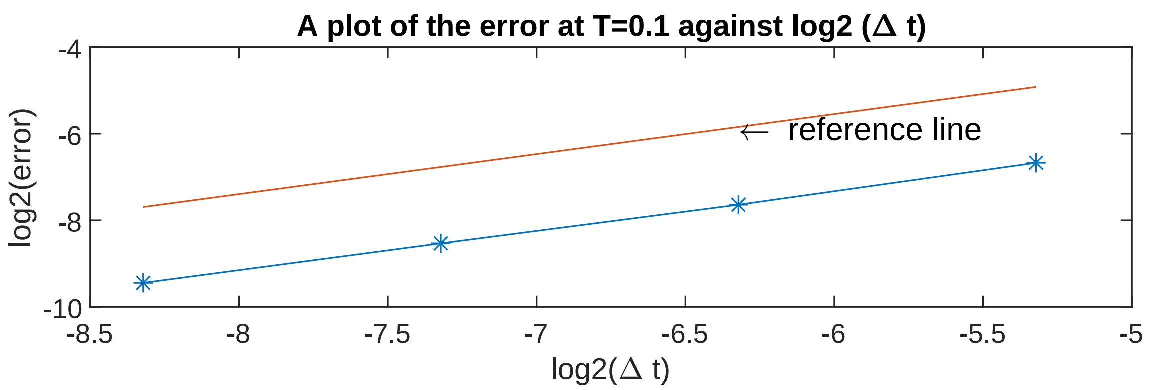

Figure 4.7.7: The experimentally determined orders of convergence with $\gamma = 0.6$ and $\alpha = 0.9$ in Table 4.7.3

Chapter 5

Galerkin finite element approximation of a stochastic semilinear fractional superdiffusion with fractionally integrated additive noise

5.1 Introduction

This chapter discusses the Galerkin finite element method applied to approximate the solution of a semilinear stochastic space and time fractional superdiffusion problem with the Caputo fractional derivative of order $\alpha \in (1, 2)$ driven by fractionally integrated additive noise [107, 47, 46, 43, 94, 104, 62, 102, 106, 78]. After discussing the existence, uniqueness and regularity results, we approximate the noise with a piecewise constant function in time in order to obtain a regularized stochastic fractional superdiffusion problem. The regularized problem is then approximated by using the finite element method in spatial direction. The mean squared errors are proved based on the sharp estimates of the various Mittag-Leffler functions involved in the integrals. Numerical experiments are conducted to show that the numerical results are consistent with the theoretical findings.

Model problem

Consider the following stochastic semilinear superdiffusion problem driven by fractionally integrated additive noise with, $1 < \alpha < 2$, $\frac{1}{2} < \beta \leq 1$, $0 \leq \gamma \leq 1$, see [40, 13, 94, 38, 108, 109],

$$
\begin{aligned}
{}_0^C D_t^{\alpha} u(t,x) + (-\Delta)^{\beta} u(t,x) &= f(t,u(t,x)) + {}_0^R D_t^{-\gamma} \dot{W}(t,x), \quad 0 < t < T, \quad x \in D,\\
u(t,x) &= 0, \quad 0 < t < T, \quad x \in \partial D,\\
u(0,x) &= v_1(x), \quad x \in D,\\
\frac{\partial u(0,x)}{\partial t} &= v_2(x), \quad x \in D,
\end{aligned} \tag{5.1.1}
$$

where D is a bounded domain in $\mathbb{R}^d$, $d = 1,2,3$ with smooth boundary ∂D and ${}_0^C D_t^{\alpha} u(t)$ and ${}_0^R D_t^{-\gamma} u(t)$ represent the Caputo fractional derivative of order $\alpha \in (1,2)$ and the Riemann-Liouville fractional integral of order $\gamma \in [0,1]$ respectively. In addition, $(-\Delta)^{\beta}$ is the fractional Laplacian and $\dot{W}(t,x)$ denotes the space-time noise defined on a complete filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\big)$. The initial values v_1 and v_2 and the nonlinear function (source term) f are given functions in their respective domain of definitions.

The non-stochastic case of our model problem (5.1.1) known as superdiffusion equation has been well-studied by several researchers because of its numerous applications in engineering, physics and biology. The noise term $\dot{W}(t,x)$ in (5.1.1) describes random effects on the movement of particles in a medium with memory or particles subject to sticking and trapping [39]. The fractionally integrated noise ${}_0^R D_t^{-\gamma} \dot{W}(t,x)$ is a typical example of the case where the internal energy depends as well on the past random effects. For the physical system the different stochastic perturbations are basically from many natural sources which sometimes cannot be ignored and hence we need to put those into the corresponding deterministic model and consequently we obtain stochastic partial differential equations. The following researchers among others have recently studied stochastic partial differential equations theoretically [37, 65, 79, 61] and numerically [112, 54, 80, 83, 133, 66, 81]. The stochastic subdiffusion with $0 < \alpha < 1$ has also been very actively investigated, see [12, 38, 39, 40]. [12] discussed sufficient conditions for a Gaussian solution (in the mean-squared sense) and derived temporal, spatial and spatio-temporal Hölder continuity of the solution. [38] analyzed moments Hölder continuity and intermittency of the solution of one-dimensional nonlinear stochastic subdiffusion problem.

It is not possible to find the analytic solution of the space-time fractional equation

(5.1.1). Therefore one needs to introduce and analyze some efficient numerical methods for solving (5.1.1). Li et al. [89] considered the Galerkin finite element method of (5.1.1) for the linear case with the additive Gaussian noise, that is, $f = 0$ and $\gamma = 0$ and obtain the error estimates. In [91], the authors studied the Galerkin finite element method for approximating the semilinear stochastic time-tempered fractional wave equations with multiplicative Gaussian noise and additive fractional Gaussian noise, but they only established error estimates for $\alpha \in (\frac{3}{2}, 2)$.

In this chapter, our focus lies on the application of the Galerkin finite element method to solve (5.1.1). Firstly, we establish the existence of a unique solution for (5.1.1) using the Banach fixed point theorem. Additionally, we analyze the spatial and temporal regularities of the solution. To approximate the noise, we employ a piecewise constant function in time, resulting in a stochastic regularized equation. This equation is then tackled using the Galerkin finite element method. We provide corresponding error estimates, utilizing the various smoothing properties exhibited by the Mittag-Leffler functions. We extend the error estimates in [91] from the linear case of (5.1.1) with Gaussian additive noise to the semilinear case with the more general integrated additive noise. We also extend the error estimates of [91] for the stochastic semilinear time fractional wave equation from $\alpha \in (\frac{3}{2}, 2)$ to $\alpha \in (1, 2)$.

To establish our error estimates, we employ a similar argument as developed in our recent work [91], which focused on approximating the stochastic semilinear subdiffusion equation with $\alpha \in (0, 1)$. We demonstrate that the solution's spatial and temporal regularities for (5.1.1) with $\alpha \in (1, 2)$ surpass those with $\alpha \in (0, 1)$. Moreover, we observe that the convergence orders of the Galerkin finite element method for (5.1.1) with $\alpha \in (1, 2)$ are higher than those with $\alpha \in (0, 1)$, as expected.

5.2 Notation and preliminaries

This section deals with some notations and preliminary results to be used in our subsequent sections. Let $\dot{H}^s(D)$ or simply $\dot{H}^s$ be the standard Sobolev Hilbert space of index $s \in \mathbb{R}^+$ with usual norm and inner product. Also, let $H = L^2(D)$(Lesbegue measurable function or square integrable function) with norm $|\cdot|$ and the inner product $(\cdot, \cdot)$ and let

$H_0^1 = \{v \in H^1 : v = 0 \; on \; \partial D\}$. Note that $A = -\Delta$ with domain $D(A) = H^2(D) \cap H_0^1(D)$ is a closed linear self-adjoint positive definite operator with compact inverse and has the eigenpairs $(\lambda_k, e_k), \quad k = 1, 2, 3, \cdots$ subject to the homogeneous Dirichlet boundary conditions. Further we assume that $(\lambda_k, e_k), k = 1, 2, 3, \cdots$, is a sequence of eigenpairs of $A : D(A) \subset H \to H$.

Set $\dot{H}^s(D)$ or simply $\dot{H}^s$ for any $s \in \mathbb{R}$ as a Hilbert space induced by the norm

$$|v|_s^2 := \sum_{k=1}^{\infty} \lambda_k^s (v, e_k)^2.$$

For $s = 0$, we denote $\dot{H}^0$ by H. For any function $\psi \in \dot{H}^{2\beta}$, $\frac{1}{2} < \beta \leq 1$, define $(-\Delta)^\beta \psi := \sum_{k=1}^{\infty} \lambda_k^\beta (\psi, e_k) e_k$. Let $L^2(\Omega; \dot{H}^s)$, $s \in \mathbb{R}$ be a separable Hilbert space of all measurable square-integrable random variables ϕ with values in $\dot{H}^s$ such that $\|\phi\|_{L^2(\Omega;\dot{H}^s)} := (\mathbf{E}|\phi|_s^2)^{\frac{1}{2}} < \infty$, where $\mathbf{E}$ denotes the expectation.

We define the space-time noise $\dot{W}(t, x)$ by, see [51] and [89],

$$\dot{W}(t, x) = \sum_{k=1}^{\infty} \sigma_k(t) \dot{\beta}_k(t) e_k(x), \tag{5.2.1}$$

where $\sigma_k(\cdot), k = 1, 2, 3, \cdots$, are some real-valued continuous function rapidly decaying with respect to k so that the series converges. Here, the sequence $\{\beta_k\}_{k=1}^{\infty}$ is mutually independent and identically distributed one-dimensional standard Brownian motions and the white noise $\dot{\beta}_k(t) = \frac{d\beta_k(t)}{dt}, k = 1, 2, 3, \cdots$, is the formal derivative of the Brownian motion $\beta_k(t)$.

5.3 Existence, uniqueness and regularity results

This section focuses on the existence, uniqueness and regularity results of the mild solution of the stochastic semilinear space-time fractional superdiffusion model (5.1.1).

Assumption 5.3.1. [125] There is a positive constant C such that the non linear function $f : \mathbb{R}^+ \times H \to H$ satisfies

$$\|f(t_1, u_1) - f(t_2, u_2)\| \leq C\big(|t_1 - t_2| + \|u_1 - u_2\|\big), \tag{5.3.1}$$

and

$$\|f(t,u)\| \le C(1+\|u\|). \tag{5.3.2}$$

Assumption 5.3.2. [74, 89] The sequence $\sigma_k(t)$ with its derivative is uniformly bounded by μ_k and γ_k respectively, i.e.,

$$|\sigma_k(t)| \le \mu_k, \tag{5.3.3}$$

$$|\sigma_k'(t)| \le \gamma_k,\ \forall\ t \in [0,T], \tag{5.3.4}$$

where the series $\sum_{k=1}^{\infty}\mu_k$ and $\sum_{k=1}^{\infty}\gamma_k$ are convergent.

Assumption 5.3.3. [89] Let $1<\alpha<2,\ \frac{1}{2}<\beta\le 1,\ 0\le\gamma\le 1$. It holds, with $0\le r\le\kappa$,

$$\sum_{k=1}^{\infty}\mu_k^2\lambda_k^{r-\kappa}<\infty,$$

where

$$\kappa=\begin{cases}2\beta, & \gamma>\frac{1}{2},\\ (2-\frac{1-2\gamma}{\alpha})\beta-\epsilon, & \gamma\le\frac{1}{2}.\end{cases},$$

and $\lambda_k, k=1,2,\cdots$ are the eigenvalues of the Laplacian $A=-\Delta$, with $D(A)=H_0^1(D)\cap H^2(D)$.

Lemma 5.3.1. [89, Lemma 2.4] An adapted process $\{u(t)\}_{t\ge 0}$ is called a mild solution to (5.1.1) if it satisfies the following integral equation with $1<\alpha<2,\ \frac{1}{2}<\beta\le 1,\ 0\le\gamma\le 1$,

$$u(t,x)=\mathbb{E}_{\alpha,\beta}(t)v_1+\tilde{\mathbb{E}}_{\alpha,\beta}(t)v_2+\int_0^t\bar{\mathbb{E}}_{\alpha,\beta}(t-s)f(s,u(s))ds+\int_0^t\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s), \tag{5.3.5}$$

where $dW(s)$ denotes

$$dW(s)=\sum_{k=1}^{\infty}\sigma_k(s)e_k d\beta_k(s),$$

and

$$\mathbb{E}_{\alpha,\beta}(t)v_1 := \sum_{k=1}^{\infty}E_{\alpha,1}(-\lambda_k^{\beta}t^{\alpha})(v_1,e_k)e_k,$$

$$\tilde{\mathbb{E}}_{\alpha,\beta}(t)v_2 := \sum_{k=1}^{\infty}tE_{\alpha,2}(-\lambda_k^{\beta}t^{\alpha})(v_2,e_k)e_k,$$

$$\bar{\mathbb{E}}_{\alpha,\beta}(t)v := \sum_{k=1}^{\infty}t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k^{\beta}t^{\alpha})(v,e_k)e_k,$$

$$\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t)v := \sum_{k=1}^{\infty}t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}t^{\alpha})(v,e_k)e_k.$$

Lemma 5.3.2. [89, Lemma 2.5] The solution $u(t)$ of the homogeneous problem of (5.1.1) satisfies, for $t > 0$

$$|u(t)|_p \leq \begin{cases} Ct^{-\frac{\alpha(p-q)}{2\beta}}|v_1|_q + Ct^{1-\frac{\alpha(p-r)}{2\beta}}|v_2|_r, \ 0 \leq q, r \leq p \leq 2\beta, \\ Ct^{-\alpha}|v_1|_q + Ct^{1-\alpha}|v_2|_r, \ q, r > p, \end{cases} \tag{5.3.6}$$

and it also implies that

$$|\partial_t^\alpha u(t)|_p \leq Ct^{-\alpha-\alpha\frac{p-q}{2\beta}}|v_1|_q + Ct^{-\alpha+1-\alpha\frac{p-r}{2\beta}}|v_2|_r. \tag{5.3.7}$$

Proof. By the boundedness property of the Mittag-Leffler function, we get that

$$\begin{aligned} |\mathbb{E}_{\alpha,\beta}(t)v_1|_p^2 &= \sum_{k=1}^{\infty} \lambda_k^p |E_{\alpha,1}(-\lambda_k^\beta t^\alpha)|^2 (v_1, e_k)^2 \\ &\leq t^{\frac{\alpha(q-p)}{\beta}} \sum_{k=1}^{\infty} \frac{C(\lambda_k^\beta t^\alpha)^{\frac{p-q}{\beta}}}{(1+\lambda_k^\beta t^\alpha)^2} \lambda_k^q (v_1, e_k)^2 \leq Ct^{\frac{\alpha(q-p)}{\beta}} |v_1|_q^2, \end{aligned} \tag{5.3.8}$$

where we have used $\frac{(\lambda_k^\beta t^\alpha)^{\frac{p-q}{\beta}}}{(1+\lambda_k^\beta t^\alpha)^2} \leq C$ for $0 \leq \frac{p-q}{\beta} \leq 2$.
Also

$$|\tilde{\mathbb{E}}_{\alpha,\beta}(t)v_2|_p^2 \leq Ct^{2-\frac{\alpha(p-r)}{\beta}}|v_2|_r^2, \text{ for } 0 \leq \frac{p-r}{\beta} \leq 2, \ p > r. \tag{5.3.9}$$

Note that $q > p$, we obtain from Lemma 2.1 and 2.4 in [89] that

$$|\mathbb{E}_{\alpha,\beta}(t)v_1|_p^2 \leq \sum_{k=1}^{\infty} \frac{C}{\lambda_k^{q-p}(1+\lambda_k^\beta t^\alpha)^2} \lambda_k^q (v_1, e_k)^2 \leq Ct^{-2\alpha}|v_1|_q^2, \tag{5.3.10}$$

and in a similar way

$$|\tilde{\mathbb{E}}_{\alpha,\beta}(t)v_2|_p^2 \leq Ct^{2-2\alpha}|v_2|_r^2, \ r > p. \tag{5.3.11}$$

Thus, (5.3.7) follows immediately by the triangle inequality. On the other hand, it follows that

$$\begin{aligned} |\partial_t^\alpha \mathbb{E}_{\alpha,\beta}(t)v_1|_p^2 &= \sum_{\ell=1}^{\infty} \lambda_\ell^{p+2\beta} \big(\mathbb{E}_{\alpha,\beta}(t)v_1, e_\ell\big)^2 = \sum_{\ell=1}^{\infty} \lambda_\ell^{p+2\beta} |E_{\alpha,1}(-\lambda_\ell^\beta t^\alpha)|^2 (v_1, e_\ell)^2 \\ &\leq t^{-\alpha(2+\frac{p-q}{\beta})} \sum_{\ell=1}^{\infty} \frac{C(\lambda_\ell^\beta t^\alpha)^{\frac{2\beta+p-q}{\beta}}}{(1+\lambda_\ell^\beta t^\alpha)} \lambda_\ell^q (v_1, e_\ell)^2 \leq Ct^{-\alpha(2+\frac{p-q}{\beta})}|v_1|_q^2. \end{aligned} \tag{5.3.12}$$

A similar estimate for $|\partial_t^\alpha \tilde{\mathbb{E}}_{\alpha,\beta}(t)v_2|_p^2$ holds and this completes the proof. □

Lemma 5.3.3. [74] Let $1 < \alpha < 2$, $\frac{1}{2} < \beta \leq 1$, $0 \leq \gamma \leq 1$. For any $t > 0$ and $0 \leq p - q \leq 2\beta$, there holds,

$$|\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t)v|_p \leq Ct^{-1+(\alpha+\gamma)-\alpha\frac{(p-q)}{2\beta}}|v|_q. \tag{5.3.13}$$

Proof. By definition,

$$\begin{aligned}
|\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t)v|_p^2 &= \sum_{k=1}^{\infty} \lambda_k^p |t^{\alpha+\gamma-1}\mathbb{E}_{\alpha,\alpha+\gamma}(-\lambda_k^\beta t^\alpha)|^2 |(v,e_k)|^2 \\
&\leq Ct^{(-1+(\alpha+\gamma)-\frac{\alpha}{2\beta}(p-q))} \sum_{k=1}^{\infty} \frac{(\lambda_k^\beta t^\alpha)^{\frac{p-q}{\beta}}}{(1+\lambda_k^\beta t^\alpha)^2} \lambda_k^q |(v,e_k)|^2 |v|_q^2. \\
&\leq Ct^{2(-1+(\alpha+\gamma)-\frac{\alpha}{2\beta}(p-q))}|v|_q^2,
\end{aligned} \tag{5.3.14}$$

which completes the proof. □

To establish the proof of the existence and uniqueness of the mild solution of (5.1.1), we shall apply the Banach contraction mapping theorem.

Theorem 44. ***(Existence and uniqueness theorem)*** *Let $1 < \alpha < 2$, $\frac{1}{2} < \beta \leq 1$ and $0 \leq \gamma \leq 1$. Let Assumptions 5.3.1 and 5.3.2, 5.3.3 hold. Let $v_1, v_2 \in L^2(\Omega; H)$. Then, there exists a unique mild solution $u \in C\big([0,T]; L^2(D;H)\big)$ given by (5.3.5) to the problem (5.1.1) for all $t \in [0,T]$.*

Proof. The proof of this theorem is similar as the proof of Theorem 27 and one only need to replace $\alpha \in (0,1)$ by $\alpha \in (1,2)$. We omit the proof here.

□

Theorem 45. *(Regularity) Let $1 < \alpha < 2$, $\frac{1}{2} < \beta \leq 1$, $0 \leq \gamma \leq 1$. Assume that Assumptions 5.3.1-5.3.3 hold. Let $v_1 \in L^2(\Omega; \dot{H}^q)$, $v_2 \in L^2(\Omega; \dot{H}^p)$ with $p, q \in [0, 2\beta]$. Then, the following regularity results hold for the solution u of (5.3.5) with $r \in [0,\kappa]$ and $0 \leq p \leq r \leq 2\beta$, $0 \leq q \leq r \leq 2\beta$,*

$$\mathbf{E}|u(t)|_r^2 \leq Ct^{\alpha\frac{(q-r)}{\beta}}|v_1|_q^2 + Ct^{2-\alpha\frac{(r-p)}{\beta}}|v_2|_p^2 + C\mathbf{E}\Big(\sup_{s\in[0,T]} \|u(s)\|^2\Big) + C\Big(\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^{r-\kappa}\Big). \tag{5.3.15}$$

Proof. The proof of this theorem is similar as the proof of Theorem 29 and one only need to replace $\alpha \in (0,1)$ by $\alpha \in (1,2)$. We omit the proof here.

□

Assumption 5.3.4. There is a positive constant C such that the nonlinear function $f:\mathbb{R}\times H\to H$ satisfies, with $u_1,u_2\in \dot{H}^q$ with $0\le q\le 2\beta$ and $\frac{1}{2}<\beta\le 1$.

$$\|(-\Delta)^{\frac{q}{2}}(f(t_1,u_1)-f(t_2,u_2))\|\le L\big(|t_1-t_2|+\|(-\Delta)^{\frac{q}{2}}(u_1-u_2)\|\big) \tag{5.3.16}$$

and

$$\|(-\Delta)^{\frac{q}{2}}f(t,u)\|\le C\big(1+\|(-\Delta)^{\frac{q}{2}}u\|\big). \tag{5.3.17}$$

Theorem 46. *Let* $1<\alpha<2$, $\frac{1}{2}<\beta\le 1$, $\frac{1}{2}<\gamma\le 1$. *Assume that Assumptions 5.3.2-5.3.4 hold. Let* $v_1,\ v_2\in L^2(\Omega;\dot{H}^{2\beta})$. *Then, there exists a unique mild solution* $u\in C\big([0,T];L^2(\Omega;\dot{H}^{2\beta})\big)$ *given by* (5.3.5) *to the model problem for all* $t\in[0,T]$.

Proof. Set $C\big([0,T];L^2(\Omega;\dot{H}^{2\beta})\big)_\lambda$, $t>0$ as the set of functions in $C\big([0,T];L^2(\Omega;\dot{H}^{2\beta})\big)$ with the following weighted norm

$$\|\phi\|^2_{\lambda,\beta}:=\sup_{t\in[0,T]}\mathbf{E}\bigg(|e^{-\lambda t}\phi(t)|^2_{2\beta}\bigg),\forall\ \phi\in C\bigg([0,T];L^2(\Omega;\dot{H}^{2\beta})\bigg). \tag{5.3.18}$$

For the proof, it is now enough to show that the map $\mathcal{T}:C\big([0,T];L^2(\Omega;\dot{H}^{2\beta})\big)_\lambda\to C\big([0,T];L^2(\Omega;\dot{H}^{2\beta})\big)_\lambda$ is a contraction. We first show that $\mathcal{T}u\in C\big([0,T];L^2(\Omega;\dot{H}^{2\beta})\big)$ for any $u\in C\big([0,T];L^2(\Omega;\dot{H}^{2\beta})\big)$. By Cauchy-Schwarz inequality, we obtain with $u\in C\big([0,T];L^2(\Omega;\dot{H}^{2\beta})\big)$,

$$\begin{aligned}\mathbf{E}|\mathcal{T}u(t)|^2_{2\beta}&\le 4\mathbf{E}|\mathbb{E}_{\alpha,\beta}(t)v_1|^2_{2\beta}+4\mathbf{E}|\tilde{\mathbb{E}}_{\alpha,\beta}(t)v_2|^2_{2\beta}+4\mathbf{E}|\int_0^t\bar{\mathbb{E}}_{\alpha,\beta}(t-s)f(s,u(s))ds|^2_{2\beta}\\&\quad+4\mathbf{E}|\int_0^t\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s)|^2_{2\beta}\\&\le 4\mathbf{E}|\mathbb{E}_{\alpha,\beta}(t)v_1|^2_{2\beta}+4\mathbf{E}|\tilde{\mathbb{E}}_{\alpha,\beta}(t)v_2|^2_{2\beta}+4t\int_0^t\mathbf{E}|\bar{\mathbb{E}}_{\alpha,\beta}(t-s)f(s,u(s))|^2_{2\beta}ds\\&\quad+4\mathbf{E}|\int_0^t\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s)|^2_{2\beta}.\end{aligned} \tag{5.3.19}$$

By the smoothing properties of $\mathbb{E}_{\alpha,\beta}$ and $\tilde{\mathbb{E}}_{\alpha,\beta}$ with $p=q$, and using the Assumption 5.3.1, it follows that

$$\begin{aligned}\mathbf{E}|\mathcal{T}u(t)|^2_{2\beta}&\le C\mathbf{E}|v_1|^2_{2\beta}+C\mathbf{E}|v_2|^2_{2\beta}+Ct\int_0^t(t-s)^{2(-1+\alpha)}\big(1+\mathbf{E}|u(s)|^2_{2\beta}\big)ds\\&\quad+4\mathbf{E}|\int_0^t\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s)|^2_{2\beta}.\end{aligned} \tag{5.3.20}$$

For the integral $\mathbf{E}|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s)|_{2\beta}^2$, a use of the isometry property and Assumption 5.3.3 and 5.3.4 with the smoothing property of the operator $\bar{\mathbb{E}}_{\alpha,\beta,\gamma}$, for $A=-\Delta$, $D(A)=H_0^1(D)\cap H^2(D)$ and $0\le r\le\kappa$, yields

$$\begin{aligned}\mathbf{E}|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s)|_{2\beta}^2 &= \mathbf{E}\|\int_0^t A^{\frac{\kappa-r+2\beta}{2}}\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)\sum_{k=1}^{\infty}\sigma_k(s)A^{\frac{r-\kappa}{2}}e_k d\beta_k(s)\|^2\\ &= \sum_{k=1}^{\infty}\int_0^t \|A^{\frac{\kappa-r+2\beta}{2}}\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)\sigma_k(s)A^{\frac{r-\kappa}{2}}e_k\|^2 ds\\ &\le C\Big(\int_0^t \|A^{\frac{\kappa-r+2\beta}{2}}\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(s)\|^2 ds\Big)\Big(\sum_{k=1}^{\infty}\mu_k^2\lambda_k^{r-\kappa}\Big).\end{aligned}$$

To resolve the integral $\int_0^t \|A^{\frac{\kappa-r+2\beta}{2}}\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(s)\|^2 ds<\infty$, it is enough to choose $r=2\beta$, which means that $k=r=2\beta$ since $0\le r\le\kappa$. Hence, we need to restrict $2\gamma>1$ in order to get $\kappa=2\beta$ by Assumption 5.3.3. With such choices of κ and r and by noting that $\frac{1}{2}<\gamma\le 1$, we arrive at

$$\begin{aligned}\mathbf{E}|\int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)dW(s)|_{2\beta}^2 &\le C\big(\int_0^t \|A^{\frac{\kappa}{2}}\bar{\mathbb{E}}_{\alpha,\beta,\gamma}(s)\|^2 ds\big)\big(\sum_{k=1}^{\infty}\mu_k^2\lambda_k^{r-\kappa}\big)\\ &\le C\Big(\sum_{k=1}^{\infty}\mu_k^2\lambda_k^{r-\kappa}\Big)\int_0^t\Big(s^{\alpha+\gamma-1-\alpha\frac{(\kappa-0)}{2\beta}}\Big)^2 ds\\ &= C\Big(\sum_{k=1}^{\infty}\mu_k^2\Big)\int_0^t s^{2\gamma-2}ds\le C\Big(\sum_{k=1}^{\infty}\mu_k^2\Big)<\infty. \quad (5.3.21)\end{aligned}$$

We note that $v_1,\ v_2\in L^2(\Omega;\dot{H}^{2\beta})$ and $u\in C\big([0,T];L^2(\Omega;\dot{H}^{2\beta})\big)$, we obtain $\sup_{t\in[0,T]}\mathbf{E}|\mathcal{T}u|_{2\beta}^2<\infty$, which implies that $\mathcal{T}u\in C\big([0,T];L^2(\Omega;\dot{H}^{2\beta})\big)$.

Next, we look at the contraction property of the mapping $\mathcal{T}$. For any given two functions u_1 and u_2 in $C\big([0,T];L^2(\Omega;\dot H^{2\beta})\big)_\lambda$, it follows that

$$\mathbf{E}|e^{-\lambda t}(\mathcal{T}u_1(t)-\mathcal{T}u_2(t))|^2_{2\beta}=\mathbf{E}|e^{-\lambda t}\int_0^t\bar{\mathbb{E}}_{\alpha,\beta}(t-s)(f(s,u_1(s))-f(s,u_2(s)))ds|^2_{2\beta}$$
$$\le \mathbf{E}\Big(\int_0^t e^{-\lambda(t-s)}|\bar{\mathbb{E}}_{\alpha,\beta}(t-s)e^{-\lambda s}(f(s,u_1(s))-f(s,u_2(s)))|_{2\beta}ds\Big)^2$$
$$\le C\mathbf{E}\Big(\int_0^t (t-s)^{\frac{\alpha q}{2\beta}-1}e^{-\lambda(t-s)}|e^{-\lambda s}(f(s,u_1(s))-f(s,u_2(s)))|_{2\beta}ds\Big)^2$$
$$\le C\mathbf{E}\Big(\int_0^t (t-s)^{\frac{\alpha q}{2\beta}-1}e^{-\lambda(t-s)}|e^{-\lambda s}(u_1(s)-u_2(s))|_{2\beta}ds\Big)^2$$
$$\le C\mathbf{E}\Big(\int_0^t 1\cdot\Big[(t-s)^{\frac{\alpha q}{2\beta}-1}e^{-\lambda(t-s)}\Big]\Big[|e^{-\lambda s}\big(u_1(s)-u_2(s)|_{2\beta}\big)\Big]ds\Big)^2$$
$$\le Ct\int_0^t (t-s)^{2(\frac{\alpha q}{2\beta}-1)}e^{-2\lambda(t-s)}ds\sup_{s\in[0,T]}\mathbf{E}|e^{-\lambda s}\big(u_1(s)-u_2(s)\big)|^2_{2\beta}$$
$$\le Ct\int_0^t \tau^{\frac{\alpha q}{\beta}-2}e^{-2\lambda\tau}d\tau\sup_{s\in[0,T]}\mathbf{E}|e^{-\lambda s}\big(u_1(s)-u_2(s)\big)|^2_{2\beta}$$
$$\le Ct\int_0^t \tau^{2(\frac{\alpha 2\beta}{2\beta}-1)}e^{-2\lambda\tau}d\tau\sup_{s\in[0,T]}\mathbf{E}|e^{-\lambda s}\big(u_1(s)-u_2(s)\big)|^2_{2\beta}$$
$$\le Ct\int_0^t \big(\frac{x}{\lambda}\big)^{2\alpha-2}e^{-2x}dx\lambda^{-1}\big[\sup_{s\in[0,T]}\mathbf{E}|u_1(s)-u_2(s)|^2_{2\beta}\big]$$
$$\le Ct\Big[\int_0^t x^{2\alpha-2}e^{-2x}dx\Big]\lambda^{1-2\alpha}\big[\sup_{s\in[0,T]}\mathbf{E}|u_1(s)-u_2(s)|^2_{2\beta}\big]$$
$$\le C(T)\lambda^{1-2\alpha}\sup_{s\in[0,T]}\mathbf{E}|u_1(s)-u_2(s)|^2_{2\beta}. \tag{5.3.22}$$

Based on the same argument of the existence and uniqueness theorem proof, the rest of the proof follows and this concludes the proof. □

5.4 Approximation of fractionally integrated noise

Let $0=t_1<t_2<\cdots<t_N<t_{N+1}=T$ be the discretization of $[0,T]$ and $\triangle t=\frac{T}{N}$ be the time step size. The noise $\frac{d\beta_k(s)}{ds}$ can be approximated by using Euler method,

$$\frac{d\beta_k(s)}{ds}\approx\frac{\beta_k^{i+1}-\beta_k^i}{\triangle t}:=\partial\beta_k^i,$$

with $\beta_k^i=\beta_k(t_i),\ \ i=1,2,\cdots,N$, where $\beta_k(t_{i+1})-\beta_k(t_i)=\sqrt{\triangle t}\cdot\mathcal{N}(0,1)$ and $\mathcal{N}(0,1)$ is the normally distributed random variable with mean 0 and variance 1. Assume that $\sigma_k^n(s)$ is some approximation of $\sigma_k(s)$. To be able to obtain an approximation of

$$\dot W(t,x)=\sum_{k=1}^\infty\sigma_k(t)\dot\beta_k(t)e_k(x),\ \text{in } [t_i,t_{i+1}],\quad i=1,2,\cdots,N,$$

in (5.1.1), we replace it with

$$\dot{W}_n(t,x) = \sum_{k=1}^{\infty} \sigma_k^n(t) e_k(x) \Big(\sum_{k=1}^{N} (\partial \beta_k^i) \chi_i(t) \Big).$$

Here, $\chi_i(t)$ is the characteristic function for the i^{th} time step length $[t_i, t_{t_{i+1}}],\ i = 1, 2, \cdots, N$ and σ_k^n is some approximations of σ_k. The following is the regularised stochastic space-time fractional superdiffusion problem. Let u_n be an approximation of u defined by

$$\begin{aligned}
&{}_0^C D_t^{\alpha} u_n(t,x) + (-\Delta)^{\beta} u_n(t,x) = f(t, u_n(t,x)) + {}_0^R D_t^{-\gamma} \dot{W}_n(t,x),\ (t,x) \in (0,T] \times D, \\
&u_n(t,x) = 0,\ 0 < t < T,\ x \in \partial D, \qquad (5.4.1)\\
&u_n(0,x) = v_1(x), \quad \frac{\partial u_n(0,x)}{\partial t} = v_2(x).
\end{aligned}$$

The solution of (5.4.1) takes the following form:

$$u_n(t) = \mathbb{E}_{\alpha,\beta}(t) v_1 + \tilde{\mathbb{E}}_{\alpha,\beta}(t) v_2 + \int_0^t \bar{\mathbb{E}}_{\alpha,\beta}(t-s) f(s, u_n(s)) ds + \int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s) dW_n(s). \quad (5.4.2)$$

Here $dW_n(s) = \sum_{k=1}^{\infty} \sigma_k^n(s) e_k \big(\sum_{i=1}^{N} (\partial \beta_k^i) \chi_i(s) \big) ds$ where $\chi_i(s)$ is the characteristic function defined on $[t_i, t_{i+1}],\ i = 1, 2, \cdots, N$.

Assumption 5.4.1. [51] Suppose that the coefficients $\sigma_k^n(t)$ are generated in such a way that,

$$\begin{aligned}
&|\sigma_k(t) - \sigma_k^n(t)| \le \eta_k^n, \\
&|\sigma_k^n(t)| \le \mu_k^n, \\
&|(\sigma_k^n)'(t)| \le \gamma_k^n,\ \forall\ t \in [0,T].
\end{aligned}$$

To regularize the noise $\frac{dW_n(s)}{ds}$, we need the following regularity assumption.

Assumption 5.4.2. Let $1 < \alpha < 2,\ \frac{1}{2} < \beta \le 1,\ 0 \le \gamma \le 1$. It holds, with $0 \le r \le \kappa$,

$$\sum_{k=1}^{\infty} (\mu_k^n)^2 \lambda_k^{r-\kappa} < \infty,$$

where κ is defined by

$$\kappa = \begin{cases} 2\beta,\ \gamma > \frac{1}{2} \\ (2 - \frac{1-2\gamma}{\alpha})\beta - \epsilon,\ \gamma \le \frac{1}{2}, \end{cases}$$

and $\lambda_k, k = 1, 2, \cdots$, are the eigenvalues of the Laplacian $-\Delta$ with $D(-\Delta) = H_0^1(D) \cap H^2(D)$.

Theorem 47. *(Existence and Uniqueness) Let* $1 < \alpha < 2,\ \frac{1}{2} < \beta \leq 1,\ 0 \leq \gamma \leq 1$. *Suppose that Assumptions 5.3.1-5.3.4, 5.4.1-5.4.2 hold. And let* $v_1, v_2 \in L^2(\Omega; H)$. *There exists a unique mild solution* $u_n \in C\big([0,T]; L^2(\Omega; H)\big)$ *given by* (5.4.2) *to the problem* (5.4.1)*, for all* $t \in [0,T]$.

Proof. The proof of this theorem is similar as the proof of Theorem 27, we omit the proof here.

□

Theorem 48. *(Regularity) Let* $1 < \alpha < 2,\ \frac{1}{2} < \beta \leq 1,\ 0 \leq \gamma \leq 1$. *Suppose that Assumptions 5.3.1-5.3.4, 5.4.1-5.4.2 hold. Let* $v_1 \in L^2(\Omega; \dot{H}^q)$ *with* $q \in [0, 2\beta]$ *and* $v_2 \in L^2(\Omega; \dot{H}^p)$ *with* $p \in [0, 2\beta]$. *Then the following regularity result for the solution* u_n *of the equation* (5.4.2) *holds with* $r \in [0, \kappa]$ *and* $0 \leq q, p \leq r \leq 2\beta$*,*

$$\mathbf{E}|u_n(t)|_r^2 \leq C t^{\alpha \frac{(q-r)}{\beta}} \mathbf{E}|v_1|_q^2 + C t^{2-\alpha\frac{r-p}{\beta}} |v_2|_p^2 + C\mathbf{E}\big(\sup_{s\in[0,T]} \|u_n(s)\|^2\big). \tag{5.4.3}$$

Proof. The proof of this theorem is similar as the proof of Theorem 29, we omit the proof here.

□

5.5 Error estimates

We now give the error estimates between u and u_n, where u and u_n are the solutions of the equations (5.1.1) and (5.4.1) respectively.

Theorem 49. *Let* $1 < \alpha < 2,\ \frac{1}{2} < \beta \leq 1,\ 0 \leq \gamma \leq 1$. *Suppose that Assumptions 5.3.1-5.3.4, 5.4.1-5.4.2 hold. Let* u *and* u_n *be the solutions of the equations* (5.1.1) *and* (5.4.1) *respectively. Then we have for any given* $\epsilon > 0$,

1. *for* $\alpha + \gamma \leq \frac{3}{2}$,

$$\mathbf{E}\|u(t) - u_n(t)\|^2 \leq C \sum_{k=1}^{\infty} \lambda^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}} (\eta_k^n)^2 + C(\triangle t)^2 \sum_{k=1}^{\infty} \lambda^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}} (\gamma_k^n)^2$$
$$+ C t^{2\epsilon} (\triangle t)^{2(\alpha+\gamma)-1-2\epsilon} \sum_{k=1}^{\infty} (\mu_k^n)^2 ds, \tag{5.5.1}$$

2. for $\frac{3}{2} \le \alpha+\gamma < 3$,

$$\mathbf{E}\|u(t)-u_n(t)\|^2 \le C\sum_{k=1}^{\infty}\lambda^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\eta_k^n)^2 + C(\triangle t)^2\sum_{k=1}^{\infty}\lambda^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\gamma_k^n)^2 + Ct^{2(\alpha+\gamma)-3}(\triangle t)^2\sum_{k=1}^{\infty}(\mu_k^n)^2. \tag{5.5.2}$$

Proof. Subtracting (5.4.2) from (5.3.5) we obtain

$$\begin{aligned} u(t)-u_n(t) &= \int_0^t \bar{\mathbb{E}}_{\alpha,\beta}(t-s)\big(f(s,u(s))-f(s,u_n(s))\big)ds \\ &+ \int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)\big(dW(s)-dW_n(s)\big) \\ &= G_1+G_2, \end{aligned} \tag{5.5.3}$$

where

$$G_1 = \int_0^t \bar{\mathbb{E}}_{\alpha,\beta}(t-s)\Big(f(s,u(s))-f(s,u_n(s))\Big)ds,$$

and

$$G_2 = \int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}(t-s)\Big(dW(s)-dW_n(s)\Big).$$

By the definitions of dW and dW_n, we now rewrite G_2 as

$$\begin{aligned} G_2 &= \int_0^t \sum_{k=1}^{\infty}(t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}\big(-\lambda_k^\beta(t-s)^\alpha\big)\Big(\sum_{m=1}^{\infty}(\sigma_m(s)-\sigma_m^n(s))(e_m,e_k)d\beta_m(s)\Big)e_k ds \\ &+\Big\{\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}(t-s)^{\alpha+\gamma-1}\sum_{k=1}^{\infty}E_{\alpha,\alpha+\gamma}\big(-\lambda_k^\beta(t-s)^\alpha\big)\Big(\sum_{m=1}^{\infty}\sigma_m^n(s)(e_m,e_k)d\beta_m(s)\Big)e_k \\ &-\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}(t-s)^{\alpha+\gamma-1}\sum_{k=1}^{\infty}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha)\Big(\sum_{m=1}^{\infty}\sigma_m^n(s)(e_m,e_k)(\partial\beta_m^i)ds\Big)e_k\Big\} \\ &= G_{21}+G_{22}, \end{aligned}$$

where

$$G_{21} = \int_0^t \sum_{k=1}^{\infty}(t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}\big(-\lambda_k^\beta(t-s)^\alpha\big)\Big(\sum_{m=1}^{\infty}(\sigma_m(s)-\sigma_m^n(s))(e_m,e_k)d\beta_m(s)\Big)ds,$$

and

$$\begin{aligned} G_{22} &= \Big\{\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}(t-s)^{\alpha+\gamma-1}\sum_{k=1}^{\infty}E_{\alpha,\alpha+\gamma}\big(-\lambda_k^\beta(t-s)^\alpha\big)\Big(\sum_{m=1}^{\infty}\sigma_m^n(s)(e_m,e_k)d\beta_m(s)\Big)e_k \\ &-\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}(t-s)^{\alpha+\gamma-1}\sum_{k=1}^{\infty}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha)\Big(\sum_{m=1}^{\infty}\sigma_m^n(s)(e_m,e_k)(\partial\beta_m^i)ds\Big)e_k\Big\}. \end{aligned}$$

We first estimate $\mathbf{E}\|G_1\|^2$. From the form of G_1, using the smoothing property of the operator $\bar{\mathbb{E}}_{\alpha,\beta}(t-s)$ and Assumption 5.3.1, we arrive with $1<\alpha<2$ at

$$\begin{aligned}
\mathbf{E}\|G_1\|^2 &= \mathbf{E}\Big(\int_0^t (t-s)^{\alpha-1}\|f(s,u(s))-f(s,u_n(s))\|ds\Big)^2\\
&\le \mathbf{E}\Big(\int_0^t (t-s)^{\alpha-1}\|u(s)-u_n(s)\|ds\Big)^2\\
&\le C\int_0^t (t-s)^{\alpha-\frac{3}{2}}ds\mathbf{E}\int_0^t (t-s)^{\alpha-\frac{1}{2}}\|u(s)-u_n(s)\|^2ds\\
&\le Ct^{2\alpha-1}\int_0^t \mathbf{E}\|u(s)-u_n(s)\|^2ds. \qquad (5.5.4)
\end{aligned}$$

For the estimate of $\mathbf{E}\|G_{21}\|^2$, using the Ito isometry property and the Assumption 5.4.2, we obtain

$$\begin{aligned}
\mathbf{E}\|G_{21}\|^2 &= \mathbf{E}\|\int_0^t \sum_{k=1}^{\infty}(t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}\big(-\lambda_k^{\beta}(t-s)^{\alpha}\big)\\
&\Big(\sum_{m=1}^{\infty}(\sigma_m(s)-\sigma_m^n(s))(e_m,e_k)d\beta_m(s)\Big)ds\|^2\\
&= \int_0^t \sum_{k=1}^{\infty}(t-s)^{2(\alpha+\gamma-1)}|E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-s)^{\alpha})|^2\big(\sigma_k(s)-\sigma_k^n(s)\big)^2ds\\
&\le \sum_{k=1}^{\infty}(\eta_k^n)^2\int_0^t (t-s)^{2(\alpha+\gamma-1)}|E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-s)^{\alpha})|^2ds.
\end{aligned}$$

Note that, for $\alpha+\gamma<\frac{3}{2}$, a use of the boundedness property of Mittag-Lefler function yields

$$\int_0^t (t-s)^{2(\alpha+\gamma-1)}|E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-s)^{\alpha})|^2ds \le C\int_0^t (t-s)^{2(\alpha+\gamma-1)}ds = Ct^{2(\alpha+\gamma)-1}. \qquad (5.5.5)$$

Also, for $1<\alpha+\gamma<3$, by using the asymptotic property of Mittag-Lefler function, we have

$$\begin{aligned}
&\int_0^t (t-s)^{2(\alpha+\gamma-1)}|E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-s)^{\alpha})|^2ds\\
&\le \int_0^t \Big|\frac{(t-s)^{\alpha+\gamma-1}}{1+\lambda_k^{\beta}(t-s)^{\alpha}}\Big|^2ds = \int_0^t \Big|\frac{(\lambda_k^{\beta}(t-s)^{\alpha})^{\frac{\alpha+\gamma-1}{\alpha}}}{1+\lambda_k^{\beta}(t-s)^{\alpha}}\lambda_k^{-\frac{\beta(\alpha+\gamma-1)}{\alpha}}\Big|^2ds\\
&= \lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}\int_0^t \Big|\frac{(\lambda_k^{\beta}(t-s)^{\alpha})^{\frac{\alpha+\gamma-1}{\alpha}}}{1+\lambda_k^{\beta}(t-s)^{\alpha}}\Big|^2ds \le C\lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}. \qquad (5.5.6)
\end{aligned}$$

Thus, we now arrive at

$$\mathbf{E}\|G_{21}\|^2 \leq C\sum_{k=1}^{\infty}\lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\eta_k^n)^2, \text{ for } 1\leq\alpha+\gamma<3. \tag{5.5.7}$$

We now estimate G_{22}. We first denote $\frac{\beta_m(t_{\ell+1})-\beta_m(t_\ell)}{\triangle t}$ by $\frac{1}{\triangle t}\int_{t_\ell}^{t_{\ell+1}} d\beta_m(s)$ and replace the variable s and $\bar{s}$ in the second term of G_{22}. Using the orthogonality property of $e_k,\ k=1,2,\cdots$, we obtain

$$\mathbf{E}\|G_{22}\|^2 = \mathbf{E}\|\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}(t-s)^{\alpha+\gamma-1}\sum_{k=1}^{\infty}E_{\alpha,\alpha+\gamma}$$
$$\big(-\lambda_k^{\beta}(t-s)^{\alpha}\big)\Big(\sum_{m=1}^{\infty}\sigma_m^n(s)(e_m,e_k)d\beta_m(s)\Big)e_k$$
$$-\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}(t-\bar{s})^{\alpha+\gamma-1}\sum_{k=1}^{\infty}E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-\bar{s})^{\alpha})$$
$$\Big(\sum_{m=1}^{\infty}\sigma_m^n(\bar{s})\frac{1}{\triangle t}\int_{t_\ell}^{t_{\ell+1}}(e_m,e_k)d\beta_m(s)\Big)e_k d\bar{s}\|^2$$
$$= \mathbf{E}\|\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}(t-s)^{\alpha+\gamma-1}\sum_{k=1}^{\infty}E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-s)^{\alpha})\sigma_k^n(s)e_k d\beta_k(s)$$
$$-\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\frac{1}{\triangle t}\int_{t_\ell}^{t_{\ell+1}}(t-\bar{s})^{\alpha+\gamma-1}\sum_{k=1}^{\infty}E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-\bar{s})^{\alpha})\sigma_k^n(\bar{s})e_k d\bar{s}d\beta_k(s)\|^2$$
$$= \mathbf{E}\sum_{k=1}^{\infty}\Big|\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}(t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-s)^{\alpha})\sigma_k^n(s)d\beta_k(s)$$
$$-\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\frac{1}{\triangle t}\int_{t_\ell}^{t_{\ell+1}}(t-\bar{s})^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-\bar{s})^{\alpha})\sigma_k^n(\bar{s})d\bar{s}d\beta_k(s)\Big|^2$$
$$= \mathbf{E}\sum_{k=1}^{\infty}\Big|\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\Big[\frac{1}{\triangle t}\int_{t_\ell}^{t_{\ell+1}}(t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-s)^{\alpha})\sigma_k^n(s)d\bar{s}\Big]d\beta_k(s)$$
$$-\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\Big[\frac{1}{\triangle t}\int_{t_\ell}^{t_{\ell+1}}(t-\bar{s})^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^{\beta}(t-\bar{s})^{\alpha})\sigma_k^n(\bar{s})d\bar{s}\Big]d\beta_k(s)\Big|^2.$$

Thus, a use of the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\mathbf{E}\|G_{22}\|^2 &= \sum_{k=1}^{\infty}\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\frac{1}{(\triangle t)^2}\bigg(\int_{t_\ell}^{t_{\ell+1}}(t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha)\\
&\quad (\sigma_k^n(s)-\sigma_k^n(\bar{s}))d\bar{s}+\int_{t_\ell}^{t_{\ell+1}}\Big((t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha)\\
&\quad -(t-\bar{s})^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-\bar{s})^\alpha)\Big)\sigma_k^n(\bar{s})d\bar{s}\bigg)^2 ds\\
&\le 2\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\frac{1}{\triangle t}\sum_{k=1}^{\infty}\int_{t_\ell}^{t_{\ell+1}}(t-s)^{2(\alpha+\gamma-1)}\big|E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha)\big|^2\\
&\quad \Big|\sigma_k^n(s)-\sigma_k^n(\bar{s})\Big|^2 d\bar{s}ds\\
&\quad +2\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\frac{1}{\triangle t}\int_{t_\ell}^{t_{\ell+1}}\bigg((t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha)\\
&\quad -(t-\bar{s})^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-\bar{s})^\alpha)\bigg)^2|\sigma_k^n(\bar{s})|^2 d\bar{s}ds\\
&= 2I_1+2I_2.
\end{aligned}$$

For I_1, using the mean value theorem and the Assumption 5.4.1 we arrive at

$$\begin{aligned}
I_1 &\le (\triangle t)^2\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\sum_{k=1}^{\infty}(t-s)^{2(\alpha+\gamma-1)}\Big|E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha)\Big|^2(\gamma_k^n)^2ds\\
&= (\triangle t)^2\sum_{k=1}^{\infty}(\gamma_k^n)^2\int_0^t(t-s)^{2(\alpha+\gamma-1)}\big|E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha)\big|^2ds.
\end{aligned}$$

Now, following the same estimates as in (5.5.7)

$$I_1 \le C(\triangle t)^2\sum_{k=1}^{\infty}\lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\gamma_k^n)^2,\ \text{for } 1<\alpha+\gamma<3. \tag{5.5.8}$$

For I_2, we note by the Mittage-Leffler function property [111] that

$$\begin{aligned}
&(t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-s)^\alpha)-(t-\bar{s})^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-\bar{s})^\alpha)\\
&=\int_{\bar{s}}^{s}\frac{d}{d\tau}\Big[(t-\tau)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-\lambda_k^\beta(t-\tau)^\alpha)\Big]d\tau\\
&=\int_{\bar{s}}^{s}(t-\tau)^{\alpha+\gamma-2}E_{\alpha,\alpha+\gamma-1}(-\lambda_k^\beta(t-\tau)^\alpha)d\tau\\
&\le C|\int_{\bar{s}}^{s}(t-\tau)^{\alpha+\gamma-2}d\tau|,
\end{aligned}$$

hence,

$$I_2 \le C \sum_{\ell=1}^{N} \int_{t_\ell}^{t_{\ell+1}} \frac{1}{\triangle t} \sum_{k=1}^{\infty} \mu_k^2 \int_{t_\ell}^{t_{\ell+1}} \Big(\int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-2} d\tau \Big)^2 d\bar{s} ds. \tag{5.5.9}$$

Now we estimate $\int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-2} d\tau$ for the different α and γ. We shall show that, with $0 < \epsilon < \frac{1}{2}$,

$$\Big| \int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-2} d\tau \Big| \le \begin{cases} C(t-\max(s,\bar{s}))^{-\frac{1}{2}+\epsilon} (\triangle t)^{\alpha+\gamma-\frac{1}{2}-\epsilon}, \ \alpha+\gamma < \frac{3}{2}, \\ \\ C(t-\max(s,\bar{s}))^{\alpha+\gamma-2} \triangle t, \ \frac{3}{2} \le \alpha+\gamma < 3. \end{cases} \tag{5.5.10}$$

Case 1. We now consider the case $\alpha+\gamma < \frac{3}{2}$. If $\bar{s} < s$, then with $0 < \epsilon < \frac{1}{2}$, it implies that

$$\begin{aligned} \Big| \int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-2} d\tau \Big| &= \int_{\bar{s}}^{s} (t-\tau)^{-\frac{1}{2}+\epsilon} (t-\tau)^{\alpha+\gamma-\frac{3}{2}-\epsilon} d\tau \\ &\le (t-s)^{-\frac{1}{2}+\epsilon} \int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-\frac{3}{2}-\epsilon} d\tau \\ &= -(t-s)^{-\frac{1}{2}+\epsilon} \frac{1}{\alpha+\gamma-\frac{1}{2}-\epsilon} (t-\tau)^{\alpha+\gamma-\frac{1}{2}-\epsilon} \Big|_{\tau=\bar{s}}^{\tau=s}. \end{aligned}$$

Since $a^\theta - b^\theta \le (a-b)^\theta$, for $a > b > 0$ and $0 < \theta < 1$, then for $\alpha+\gamma < \frac{3}{2}$,

$$-(t-\tau)^{\alpha+\gamma-\frac{1}{2}-\epsilon} \Big|_{\tau=\bar{s}}^{\tau=s} \le (s-\bar{s})^{\alpha+\gamma-\frac{1}{2}-\epsilon} \le (\triangle t)^{\alpha+\gamma-\frac{1}{2}-\epsilon},$$

and this implies that

$$\Big| \int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-2} d\tau \Big| \le C(t-s)^{-\frac{1}{2}+\epsilon} (\triangle t)^{\alpha+\gamma-\frac{1}{2}-\epsilon}. \tag{5.5.11}$$

Similarly, we may show that for $s < \bar{s}$, with $0 < \epsilon < \frac{1}{2}$,

$$\Big| \int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-2} d\tau \Big| \le C(t-\bar{s})^{-\frac{1}{2}+\epsilon} (\triangle t)^{\alpha+\gamma-\frac{1}{2}-\epsilon}. \tag{5.5.12}$$

Therefore, for $\alpha+\gamma < \frac{3}{2}$ obtain,

$$\Big| \int_{\bar{s}}^{s} (t-\tau)^{\alpha+\gamma-2} d\tau \Big| \le C(t-\max(s,\bar{s}))^{-\frac{1}{2}+\epsilon} (\triangle t)^{\alpha+\gamma-\frac{1}{2}-\epsilon}. \tag{5.5.13}$$

Case 2. Next, consider the case $\frac{3}{2} \le \alpha+\gamma < 3$. If $\bar{s} < s$ then we get,

$$|\int_{\bar{s}}^{s}(t-\tau)^{\alpha+\gamma-2}d\tau| \leq C(t-\bar{s})^{\alpha+\gamma-2}(s-\bar{s}) \leq (t-s)^{\alpha+\gamma-2}\triangle t. \tag{5.5.14}$$

Similarly, for $s < \bar{s}$, it follows that

$$|\int_{\bar{s}}^{s}(t-\tau)^{\alpha+\gamma-2}d\tau| \leq C(t-\bar{s})^{\alpha+\gamma-2}(\triangle t) \leq C(t-\bar{s})^{\alpha+\gamma-2}\triangle t. \tag{5.5.15}$$

Therefore, for $\frac{3}{2} \leq \alpha+\gamma < 3$ we get,

$$|\int_{\bar{s}}^{s}(t-\tau)^{\alpha+\gamma-2}d\tau| \leq C(t-\max(s,\bar{s}))^{\alpha+\gamma-2}\triangle t. \tag{5.5.16}$$

Note that

$$|\int_{\bar{s}}^{s}(t-\tau)^{\alpha+\gamma-2}d\tau| \leq C(t-\max(s,\bar{s}))^{\alpha+\gamma-2}\triangle t, \text{ for } \alpha+\gamma < 2,$$

and

$$|\int_{\bar{s}}^{s}(t-\tau)^{\alpha+\gamma-2}d\tau| \leq C(t-\min(s,\bar{s}))^{\alpha+\gamma-2}\triangle t, \text{ for } \alpha+\gamma > 2.$$

Thus, we derive the following estimate for I_2. For $\alpha+\gamma < \frac{3}{2}$,

$$\begin{aligned}
I_2 &\leq C\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\frac{1}{\triangle t}\sum_{k=1}^{\infty}(\mu_k^n)^2\int_{t_\ell}^{t_{\ell+1}}(t-\max(s,\bar{s}))^{-1+2\epsilon}(\triangle t)^{2(\alpha+\gamma)-1-2\epsilon}d\bar{s}ds \\
&\leq C(\triangle t)^{2(\alpha+\gamma)-1-2\epsilon}\sum_{k=1}^{\infty}(\mu_k^n)^2\int_0^t(t-s)^{-1+2\epsilon}ds \\
&\leq Ct^{2\epsilon}(\triangle t)^{2(\alpha+\gamma)-1-2\epsilon}\sum_{k=1}^{\infty}(\mu_k^n)^2.
\end{aligned}$$

For $\frac{3}{2} \leq \alpha+\gamma < 3$,

$$\begin{aligned}
I_2 &\leq C\sum_{\ell=1}^{N}\int_{t_\ell}^{t_{\ell+1}}\frac{1}{\triangle t}\sum_{k=1}^{\infty}(\mu_k^n)^2\int_{t_\ell}^{t_{\ell+1}}(t-\max(s,\bar{s}))^{2(\alpha+\gamma)-4}(\triangle t)^2 d\bar{s}ds \\
&\leq C(\triangle t)^2\int_0^t(t-s)^{2(\alpha+\gamma)-4}ds\sum_{k=1}^{\infty}(\mu_k^n)^2 \leq Ct^{2(\alpha+\gamma)-3}(\triangle t)^2\sum_{k=1}^{\infty}(\mu_k^n)^2.
\end{aligned}$$

Together with the estimates we obtain the following results.

1. For $\alpha + \gamma < \frac{3}{2}$, it follows that for $t > 0$,

$$\mathbf{E}\|u(t) - u_n(t)\|^2 \leq C \sum_{k=1}^{\infty} \lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}} (\eta_k^n)^2 + C(\triangle t)^2 \sum_{k=1}^{\infty} \lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{k}} (\gamma_k^n)^2$$
$$+ Ct^{2\epsilon}(\triangle t)^{2(\alpha+\gamma)-1-2\epsilon} \sum_{k=1}^{\infty} (\mu_k^n)^2. \tag{5.5.17}$$

2. For $\frac{3}{2} \leq \alpha + \gamma < 3$,

$$\mathbf{E}\|u(t) - u_n(t)\|^2 \leq C \sum_{k=1}^{\infty} \lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}} (\eta_k^n)^2 + C(\triangle t)^2 \sum_{k=1}^{\infty} \lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{k}} (\gamma_k^n)^2$$
$$+ Ct^{2(\alpha+\gamma)-3}(\triangle t)^2 \sum_{k=1}^{\infty} (\mu_k^n)^2. \tag{5.5.18}$$

An application of the Gronwall's Lemma completes the rest of the proof. □

5.6 Finite element approximation and error analysis

Let D be the spatial domain and let $\mathcal{T}_h$ be a shape regular and quasi-uniform triangulation of the domain D with spatial discretization parameter $h = \max_{K \in \mathcal{T}_h} h_K$, where h_K is the diameter of K. Let $V_h \subset \dot{H}^\beta$, $\frac{1}{2} < \beta \leq 1$ be the piecewise linear finite element space with respect to the triangulation $\mathcal{T}_h$, that is

$$V_h := \{v_h \in \dot{H}^\beta(D) : v_h|_K \in P_1(K),\ \forall\ K \in \mathcal{T}_h\}. \tag{5.6.1}$$

Recall that $\Delta_h : V_h \to V_h$ is the discrete Laplacian operator defined by $\big((-\Delta_h)\psi, \chi\big) = \big(\nabla\psi, \nabla\chi\big),\ \forall\ \chi \in V_h$.

The semi-discrete finite element method approximation of the equation (5.4.1) is to seek $u_n^h(t) \in V_h$, for $t \in [0, T]$ such that

$$\begin{aligned}
&{}_0^C D_t^\alpha u_n^h(t) + (-\Delta_h)^\beta u_n^h(t) = P_h f(t, u_n^h(t)) + P_h(D_t^{-\gamma} dW_n(t)),\ t \in (0, T),\\
&u_n^h(0) = v_1^h,\\
&\partial_t u_n^h(0) = v_2^h,
\end{aligned} \tag{5.6.2}$$

where $v_1^h = P_h v_1,\ v_2^h = P_h v_2$ are chosen as L^2 projection of the initial functions $v_1^h,\ v_2^h \in V_h$.

As it is in the continuous case, the solution of (5.6.2) takes the form

$$u_n^h(t) = \mathbb{E}_{\alpha,\beta}^h(t)P_hv_1+\tilde{\mathbb{E}}_{\alpha,\beta}^h(t)P_hv_2+\int_0^t \bar{\mathbb{E}}_{\alpha,\beta}^h(t-s)P_hf(s,u_n^h(s))ds+\int_0^t \bar{\mathbb{E}}_{\alpha,\beta,\gamma}^h(t-s)P_hdW_n(s), \tag{5.6.3}$$

where for each $t \in [0,T]$, the operators $\mathbb{E}_{\alpha,\beta}^h(t)$, $\tilde{\mathbb{E}}_{\alpha,\beta}^h(t)$ and $\bar{\mathbb{E}}_{\alpha,\beta,\gamma}^h(t)$ are defined from $V_h \to V_h$ by

$$\mathbb{E}_{\alpha,\beta}^h(t)v_h = \sum_{k=1}^m E_{\alpha,1}\big((-\lambda_k^h)^\beta t^\alpha\big)\big(v_h, e_k^h\big)e_k^h,$$
$$\tilde{\mathbb{E}}_{\alpha,\beta}^h(t)v_h = \sum_{k=1}^m tE_{\alpha,2}\big((-\lambda_k^h)^\beta t^\alpha\big)\big(v_h, e_k^h\big)e_k^h,$$
$$\bar{\mathbb{E}}_{\alpha,\beta,\gamma}^h(t)v_h = \sum_{k=1}^m t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}\big((-\lambda_k^h)^\beta t^\alpha\big)\big(v_h, e_k^h\big)e_k^h.$$

Lemma 5.6.1. For any $t > 0$ and $0 < r, q < p \le 2\beta$, there hold for $v_h \in V_h$,

$$|\mathbb{E}_{\alpha,\beta}^h(t)v_h|_{p,h} \le Ct^{\alpha\frac{(q-p)}{2\beta}}|v_h|_q,\ 0 \le q \le p \le 2\beta,$$
$$|\tilde{\mathbb{E}}_{\alpha,\beta}^h(t)v_h|_{p,h} \le Ct^{1-\alpha\frac{(p-r)}{2\beta}}|v_h|_r,\ 0 \le r \le p \le 2\beta,$$
$$|\bar{\mathbb{E}}_{\alpha,\beta,\gamma}^h(t)v_h|_{p,h} \le Ct^{-1+(\alpha+\gamma)-\alpha\frac{(p-q)}{2\beta}}|v_h|_q,\ 0 \le q \le p \le 2\beta,$$
$$|\bar{\mathbb{E}}_{\alpha,\beta}^h(t)v_h|_{p,h} \le Ct^{-1+\alpha-\alpha\frac{(p-q)}{2\beta}}|v_h|_q,\ 0 \le q \le p \le 2\beta.$$

Lemma 5.6.2. [89] (Inverse Estimate in V_h) For any $\ell > s$, there exists a constant C independent of h such that

$$|v_h|_{\ell,h} \le Ch^{s-\ell}|v_h|_{s,h},\ \forall\ v_h \in V_h.$$

We now consider the error estimate. Let $u - u_n^h := (u - u_n) + (u_n - u_n^h)$ since the estimate $\mathbf{E}\|u(t) - u_n(t)\|^2$ is known. We will show the estimate $\mathbf{E}\|u_n(t) - u_n^h(t)\|^2$.

Theorem 50. *Let $1 < \alpha < 2$, $\frac{1}{2} < \beta \le 1$, $0 \le \gamma \le 1$. Suppose that Assumptions 5.3.1-5.3.4, 5.4.1 hold. Let u_n and u_n^h be the solutions of* (5.4.1) *and* (5.6.2) *respectively. Let $v_1, v_2 \in L^2(\Omega; \dot{H}^q)$ with $0 \le q \le 2\beta$. Then, there exists a positive constant C such that for any $\epsilon > 0$, with $r \in [0, \kappa]$ and $0 \le \max(q, \beta) \le r \le 2\beta$*

$$\begin{aligned}
&\mathbf{E}\|u_n(t) - u_n^h(t)\|^2 + h^{2\beta}\mathbf{E}\|(-\triangle)^{\frac{\beta}{2}}(u_n(t) - u_n^h(t))\|^2 \\
&\le Ch^{-2\epsilon+2r}\Big[\mathbf{E}|v_1|_q^2 + \mathbf{E}|v_2|_q^2 + \mathbf{E}\big(\sup_{s\in[0,T]}\|u_n(s)\|^2\big)\Big] \\
&\quad + Ch^{2r}t^{-\alpha\frac{(r-q)}{\beta}}\mathbf{E}|v_1|_q^2 + Ch^{2r}t^{-2+(\alpha+\gamma)-\alpha\frac{(r-q)}{\beta}}\mathbf{E}|v_2|_q^2.
\end{aligned} \tag{5.6.4}$$

Proof. Introducing $\tilde{u}_n^h(t) \in V_h$ as a solution of an intermediate discrete system

$$\begin{aligned}
&{}_0^C D_t^{\alpha} \tilde{u}_n^h(t) + (-\Delta_h)^{\beta} \tilde{u}_n^h(t) = P_h f(t, \tilde{u}_n(t)) + P_h(D_t^{-\gamma} dW_n(t)), \ t \in (0, T], \\
&\tilde{u}_n^h(0) = P_h v_1, \\
&\partial_t \tilde{u}_n^h(0) = P_h v_2.
\end{aligned} \tag{5.6.5}$$

We split the error $u_n^h(t) - u_n(t) := (u_n^h(t) - \tilde{u}_n^h(t)) + (\tilde{u}_n^h(t) - u_n(t)) := \zeta(t) + \eta(t)$ Again using $P_h u_n$ we split $\eta(t)$,

$$\eta(t) := (\tilde{u}_n^h - P_h u_n) + (P_h u_n - u_n) := \theta + \rho. \tag{5.6.6}$$

From Lemma **??** it follows that, with $r \in [\beta, 2\beta]$,

$$\mathbf{E}\|\rho(t)\|^2 + h^{2\beta}\mathbf{E}\|(-\Delta)^{\frac{\beta}{2}}\rho(t)\|^2 \le Ch^{2r}\mathbf{E}|u_n(t)|_r^2, \tag{5.6.7}$$

which means that

$$\begin{aligned}
&\mathbf{E}\|\rho(t)\|^2 + h^{2\beta}\mathbf{E}\|(-\Delta)^{\frac{\beta}{2}}\rho(t)\|^2 \\
&\le Ch^{2r}\Bigg(Ct^{-\alpha\frac{(r-q)}{\beta}}\mathbf{E}|v_1|_q^2 + Ct^{2-\alpha\frac{(r-q)}{\beta}}\mathbf{E}|v_2|_q^2 \\
&\quad + C\mathbf{E}\big[\sup_{s\in[0,T]} \|f(s, u_n(s))\|\big]^2 + C\sum_{m=1}^{\infty} \mu_m^2 \lambda_k^{r-\kappa}\Bigg) \\
&\le Ch^{2r}\Bigg(Ct^{-\alpha\frac{(r-q)}{\beta}}\mathbf{E}|v_1|_q^2 + Ct^{2-\alpha\frac{(r-q)}{\beta}}\mathbf{E}|v_2|_q^2 \\
&\quad + C\mathbf{E}\big(\sup_{s\in[0,T]} \|u_n(s)\|^2\big) + C\sum_{m=1}^{\infty} \mu_m^2 \lambda_k^{r-\kappa}\Bigg).
\end{aligned} \tag{5.6.8}$$

To estimate θ, note that θ satisfies the following equation

$$\begin{aligned}
&{}_0^C D_t^{\alpha} \theta(t) + (-\Delta_h)^{\beta}\theta(t) = (-\Delta_h)^{\beta}(R_h u_n - P_h u_n), \\
&\theta(0) = 0,
\end{aligned} \tag{5.6.9}$$

and hence, the representation of solution θ is written as

$$\theta(t) = \int_0^t \bar{\mathbb{E}}_{\alpha,\beta}^h(t-s)(-\Delta_h)^{\beta}(R_h u_n(s) - P_h u_n(s))ds. \tag{5.6.10}$$

Choose $p = 0$ and $p = \beta$ separately, from Lemma **??** with $\gamma = 0$ and Lemma 5.6.1, it follows that for $q=\epsilon-2\beta+p$ and $0<\epsilon<2\beta$ that

$$\begin{aligned}
\mathbf{E}|\theta(t)|_{p,h}^2 &\le \mathbf{E}\Big(\int_0^t |\bar{\mathbb{E}}_{\alpha,\beta}^h(t-s)(-\Delta_h)^\beta(R_hu_n(s)-P_hu_n(s))|_{p,h}ds\Big)^2\\
&\le C\mathbf{E}\Big(\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1}|(-\Delta_h)^\beta(R_hu_n-P_hu_n)(s)|_{\epsilon-2\beta+p,h}\Big)^2\\
&\le C\mathbf{E}\Big(\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1}|(R_hu_n-P_hu_n)(s)|_{\epsilon+p,h}ds\Big)^2\\
&\le Ch^{2r-2p-2\epsilon}\Big(\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1}ds\Big)\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1}\mathbf{E}|u_n(s)|_r^2ds\\
&\le Ch^{2r-2p-2\epsilon}t^{\frac{\alpha\epsilon}{2\beta}}\mathbf{E}|u_n(s)|_r^2ds. \qquad (5.6.11)
\end{aligned}$$

Now an application of regularity shows

$$\begin{aligned}
\mathbf{E}|\theta(t)|_{p,h}^2 &\le Ch^{2r-2p-2\epsilon}\int_0^t (t-s)^{\frac{\alpha\epsilon}{2\beta}-1}\Big[s^{-\alpha\frac{(r-q)}{\beta}}\|v_1\|^2_{L^2(\Omega;\dot{H}^q)}+s^{2-\alpha\frac{(r-q)}{\beta}}\|v_2\|^2_{L^2(\Omega;\dot{H}^q)}\\
&+\mathbf{E}\big(\sup_{s\in[0,T]}\|f(s,u_n(s))\|^2\big)\Big]ds\\
&\le Ch^{2r-2p-2\epsilon}\Big[\mathbf{E}|v_1|_q^2+\mathbf{E}|v_2|_q^2+\mathbf{E}\big(\sup_{s\in[0,T]}\|u_n(s)\|^2\big)\Big], \qquad (5.6.12)
\end{aligned}$$

where we used the fact that $\int_0^t(t-s)^{\frac{\alpha\epsilon}{2\beta}-1}s^{-\alpha\frac{(r-q)}{\beta}}ds<\infty$, $\int_0^t(t-s)^{\frac{\alpha\epsilon}{2\beta}-1}s^{2-\alpha\frac{(r-q)}{\beta}}ds<\infty$ since $0<\frac{\alpha\epsilon}{2\beta}<2$ and $0\le\alpha(\frac{r-q}{\beta})<2$.
We now combine (5.6.8), (5.6.11) and (5.6.12) to arrive at an estimate for η as, with $p=0$ and β, $0\le p\le r\le 2\beta$,

$$\begin{aligned}
\mathbf{E}|\eta(t)|_{p,h}^2 \le Ch^{2r-2p-2\epsilon}&\Big[\|v_1\|^2_{L^2(\Omega;\dot{H}^q)}+\|v_2\|^2_{L^2(\Omega;\dot{H}^q)}+\mathbf{E}\big(\sup_{s\in[0,T]}\|u_n(s)\|^2\big)\Big]\\
&Ch^{2r-2p}\big(t^{-\alpha\frac{(r-q)}{\beta}}\|v_1\|^2_{L^2(\Omega;\dot{H}^q)}+t^{2-\alpha\frac{(r-q)}{\beta}}\|v_2\|^2_{L^2(\Omega;\dot{H}^q)}\big). \qquad (5.6.13)
\end{aligned}$$

Now to estimate ζ, note that $\zeta(t)\in V_h$ satisfies

$$ {}_0^CD_t^\alpha\zeta(t)+(-\Delta_h)^\beta\zeta(t)=P_h(f(u_n^h)-f(u_n)), \qquad (5.6.14)$$

and therefore we now write $\zeta(t)$ in the integral form as

$$\zeta(t)=\int_0^t\bar{\mathbb{E}}_{\alpha,\beta}(t-s)P_h(f(u_n^h(s))-f(u_n(s)))ds. \qquad (5.6.15)$$

Again, choose $p=0,\beta$. From Lemma **??** with $\gamma=0$ and Lemma 5.6.1, it follows for $q=p$

and for $1 < \alpha < 2$, that

$$\begin{aligned}
\mathbf{E}|\zeta|_{p,h}^2 &\leq \mathbf{E}\Big(\int_0^t |\bar{\mathbb{E}}_{\alpha,\beta}(t-s)P_h\big(f(u_n^h(s)) - f(u_n(s))\big)|_{p,h} ds\Big)^2 \\
&\leq \mathbf{E}\Big(\int_0^t (t-s)^{\alpha-1}|P_h(f(u_n^h(s)) - f(u_n(s)))|_{p,h} ds\Big)^2 \\
&\leq C\mathbf{E}\Big(\int_0^t (t-s)^{\alpha-1}|u_n(s) - u_n^h(s)|_p ds\Big)^2 \\
&\leq C\Big(\int_0^t (t-s)^{\alpha-1} ds \big(\int_0^t (t-s)^{\alpha-1}\mathbf{E}|u_n(s) - u_n^h(s)|_p^2 ds\big)\Big) \\
&\leq Ct^\alpha \int_0^t (t-s)^{\alpha-1}\mathbf{E}|u_n(s) - u_n^h(s)|_p^2 ds. \qquad (5.6.16)
\end{aligned}$$

Combining (5.6.13) and (5.6.16) it follows for $p = 0$ and β, and $0 \leq p \leq r \leq 2\beta$ that

$$\begin{aligned}
\mathbf{E}|u_n(t) - u_n^h(t)|_p^2 \leq\ & Ch^{-2\epsilon+2(r-p)}\Big[\mathbf{E}|v_1|_q^2 + \mathbf{E}|v_2|_q^2 + \mathbf{E}\big(\sup_{s\in[0,T]}\|u_n(s)\|^2\big)\Big] \\
& + Ch^{2(r-p)}\Big(t^{-\alpha\frac{(r-q)}{\beta}}|v_1|_q^2 + t^{2-\alpha\frac{(r-q)}{\beta}}|v_2|_q^2\Big) \\
& + C\int_0^t (t-s)^{\alpha-1}\mathbf{E}|u_n(s) - u_n^h(s)|_p^2 ds. \qquad (5.6.17)
\end{aligned}$$

An application of the Gronwall's Lemma completes the rest of the proof. □

Theorem 51. *Let $1 < \alpha < 2$, $\frac{1}{2} < \beta \leq 1$, $0 \leq \gamma \leq 1$. Assume that Assumptions 5.3.1, 5.3.2, 5.3.3, 5.3.4 and 5.4.1 hold. Let u and u_n^h be the solutions of* (5.1.1) *and* (5.6.2) *respectively. Let $v_1,\ v_2 \in L^2(\Omega; \dot{H}^q)$ with $0 \leq q \leq 2\beta$. Then, there exists a positive constant C such that, for any $\epsilon > 0$ with $r \in [0, \kappa]$ and $0 \leq \max(q, \beta) \leq r \leq 2\beta$,*

1. *for $\alpha + \gamma < \frac{3}{2}$,*

$$\begin{aligned}
\mathbf{E}\|u(t) - u_n^h(t)\|^2 \leq\ & C\sum_{k=1}^\infty \lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\eta_k^n)^2 + C(\triangle t)^2\sum_{k=1}^\infty \lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}}(\gamma_k^n)^2 \\
& + Ct^{2\epsilon}(\triangle t)^{2(\alpha+\gamma)-1-2\epsilon}\sum_{k=1}^\infty(\mu_k^n)^2 + Ch^{-2\epsilon+2r}\big[\mathbf{E}|v_1|_q^2 + \mathbf{E}|v_2|_q^2 \\
& + \mathbf{E}\big(\sup_{s\in[0,T]}\|u_n(s)\|^2\big)\big] + Ch^{2r}t^{-\alpha\frac{(r-q)}{\beta}}\mathbf{E}|v_1|_q^2 + Ch^{2r}t^{2-\alpha\frac{(r-q)}{\beta}}\mathbf{E}|v_2|_q^2, \qquad (5.6.18)
\end{aligned}$$

2. for $\frac{3}{2} \leq \alpha + \gamma < 3$

$$\mathbf{E}\|u(t) - u_n^h(t)\|^2 \leq C \sum_{k=1}^{\infty} \lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}} (\eta_k^n)^2 + C(\triangle t)^2 \sum_{k=1}^{\infty} \lambda_k^{-\frac{2\beta(\alpha+\gamma-1)}{\alpha}} (\gamma_k^n)^2$$
$$+ Ct^{2(\alpha+\gamma)-3}(\triangle t)^2 \sum_{k=1}^{\infty} (\mu_k^n)^2 + Ch^{-2\epsilon+2r}\big[\mathbf{E}|v_1|_q^2 + \mathbf{E}|v_2|_q^2$$
$$+ \mathbf{E}\big(\sup_{s\in[0,T]} \|u_n(s)\|^2\big)\big] + Ch^{2r}t^{-\alpha\frac{(r-q)}{\beta}}\mathbf{E}|v_1|_q^2 + Ch^{2r}t^{2-\alpha\frac{(r-q)}{\beta}}\mathbf{E}|v_2|_q^2. \qquad (5.6.19)$$

Remark In particular, when the noise is the trace class noise i.e.,

$$\frac{\partial^2 W(t,x)}{\partial t \partial x} = \sum_{k=1}^{\infty} \gamma_k^{\frac{1}{2}} \dot{B}_k(t) e_k(x),$$
$$Tr(Q) = \sum_{k=1}^{\infty} \gamma_k < \infty.$$

Under the Assumption 5.4.1, we have $\eta_k^n = 0$, $\gamma_k^n = 0$, $\sum_{k=1}^{\infty} (\mu_k^n)^2 = \sum_{k=1}^{\infty} \gamma_k < \infty$, where $\alpha \to 1$, $\beta = 1$, $\gamma = 0$, we obtain with $\epsilon > 0$,

$$\mathbf{E}\|u(t) - u_n^t(t)\|^2 = \mathcal{O}(h^{4-\epsilon} + (\Delta t)^{1-\epsilon}),$$

which are constant with the results obtained in [133] for the stochastic heat equation.

5.7 Numerical simulations

In this section, we shall consider numerical simulations for the following stochastic semi-linear fractional wave equation, with $\alpha \in (1, 2)$,

$${}_0^C D_t^{\alpha} u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = f(u(t,x)) + g(t,x), \quad 0 \leq t \leq T, \ 0 < x < 1, \qquad (5.7.1)$$
$$u(0,x) = u_0(x), \quad \frac{\partial u(0,x)}{\partial t} = u_1(x), \quad 0 < x < 1, \qquad (5.7.2)$$
$$u(t,0) = u(t,1) = 0, \quad 0 < x < 1, \qquad (5.7.3)$$

where $f(r)$, $r \in \mathbb{R}, u_0(x)$ and $u_1(x)$ are given smooth functions. Here, with $\gamma \in [0, 1]$,

$$g(t,x) := {}_0^R D_t^{-\gamma} \frac{\partial^2 W(t,x)}{\partial t \partial x} = {}_0^R D_t^{-\gamma} \sum_{m=1}^{\infty} \gamma_m^{1/2} e_m(x) \frac{d\beta_m(t)}{dt}, \qquad (5.7.4)$$

where $\beta_m(t)$, $m = 1, 2, \ldots$ are the Brownian motions. Here $e_m(x) = \sqrt{2} \sin m\pi x$ denote the eigenfunctions of the operator $A = -\frac{\partial^2}{\partial x^2}$ with $D(A) = H_0^1(0,1) \cap H^2(0,1)$. Further

$\gamma_m, m = 1, 2, \ldots$ are the eigenvalues of the covariance operator Q of the stochastic process $W(t)$, that is

$$Qe_m = \gamma_m e_m.$$

We shall consider two cases in our numerical simulations.

Case 1: the white noise case, e.g., $\gamma_m = m^{-\beta}$ with $\beta = 0$ which implies that

$$\mathrm{tr}(Q) = \sum_{m=1}^{\infty} \gamma_m = \sum_{m=1}^{\infty} m^{-\beta} = \sum_{m=1}^{\infty} 1 = \infty.$$

Case 2: The trace class case, e.g., $\gamma_m = m^{-\beta}$ with $\beta > 1$, which implies that

$$\mathrm{tr}(Q) = \sum_{m=1}^{\infty} \gamma_m = \sum_{m=1}^{\infty} m^{-\beta} < \infty.$$

The numerical methods for solving stochastic time fractional partial differential equations are similar to the numerical methods for solving deterministic time fractional partial differential equations. The only difference is that we have the extra term g in stochastic case and we need to consider how to approximate g.

Let $v(t,x) = u(t,x) - u_0(x) - tu_1(x)$. Then (5.7.1)-(5.7.3) can be written as the following

$$ {}_0^C D_t^{\alpha} v(t,x) - \Delta v(t,x) = \Delta u_0(x) + t\Delta u_0(x) + f(u(t,x)) + g(t,x), \quad 0 \leq t \leq T,\ 0 < x < 1, \tag{5.7.5}$$

$$v(0,x) = 0, \quad \frac{\partial v(0,x)}{\partial t} = 0, \tag{5.7.6}$$

$$v(t,0) = v(t,1) = 0. \tag{5.7.7}$$

Since the initial values $v(0,x) = 0$, $\frac{\partial v(0,x)}{\partial t} = 0$ in (5.7.5)-(5.7.7), it is easier to consider the numerical analysis for the time discretization scheme of (5.7.5)-(5.7.7). For simplicity, we assume that the initial values $u_0(x), u_1(x)$ are sufficiently smooth, then we may write (5.7.5)-(5.7.7) into the following abstract form: with $v'(t) = \frac{dv(t)}{dt}$, $F(t) = f(u(t))$,

$${}_0^C D_t^{\alpha} v(t) + Av(t) = -Au_0 - tAu_1 + F(t) + g(t), \quad v(0) = 0,\ v'(0) = 0. \tag{5.7.8}$$

Let $0 < t_0 < t_1 < \cdots < t_N = T$ be a partition of the time interval $[0, T]$ and τ the time step size. Let $0 = x_0 < x_1 < \cdots < x_M = 1$ be a partition of the space interval $[0, 1]$ and h the space step size.

Let $S_h \subset H_0^1(0, 1)$ be the piecewise linear finite element space defined by

$$S_h = \{\chi \in C[0, 1] : \ \chi \text{ is a piecewise linear function defined on } [0, 1] \text{ and } \chi(0) = \chi(1) = 0\}.$$

The finite element method of (5.7.5)-(5.7.7) is to find $v_h(t) \in S_h$ such that, $\forall\, \chi \in S_h$,

$$\begin{aligned} &\big({}_0^C D_t^{\alpha} v_h(t), \chi\big) + (\nabla v_h(t), \nabla \chi) \\ &\qquad = -(\nabla P_h u_0, \nabla \chi) - t(\nabla P_h u_1, \nabla \chi) + (F(t), \chi) + (g(t), \chi), \qquad (5.7.9) \\ &v_h(0) = v_h'(0) = 0, \qquad (5.7.10) \end{aligned}$$

where $P_h : H \to S_h$ denotes the L_2 projection operator.

Let $A_h : S_h \to S_h$ be the discrete analogue of the operator A defined by

$$(A_h \psi, \chi) = (\nabla \psi, \nabla \chi), \quad \forall\, \chi \in S_h. \qquad (5.7.11)$$

Then we may write (5.7.9)-(5.7.10) into the following abstract form:

$$\begin{aligned} &{}_0^C D_t^{\alpha} v_h(t) + A_h v_h(t) = -A_h P_h u_0 - t A_h P_h u_1 + P_h F(t) + P_h g(t), \\ &v_h(0) = 0, \ v_h'(0) = 0. \qquad (5.7.12) \end{aligned}$$

Remark 52. When we consider the abstract form of the finite element approximation of (5.7.8), we may choose $u_{0h} = P_h u_0$ and $u_{1h} = P_h u_1$ as the initial approximations of $u_0, u_1 \in H$ and replace the elliptic operator A in (5.7.8) by the discrete analogue $A_h : S_h \to S_h$. In other words, for any initial values $u_0, u_1 \in H$, the abstract form (5.7.12) is well defined.

Let $V^n \approx v_h(t_n), n = 0, 1, \ldots, N$ be the approximation of $v_h(t_n)$. We define the following time discretization scheme: find $V^n \in S_h$, with $n = 1, 2, \ldots, N$, such that, $\forall\, \chi \in S_h$,

$$\begin{aligned} &\Big(\tau^{-\alpha} \sum_{j=1}^{n} w_{n-j} V^j, \chi\Big) + (\nabla V^n, \nabla \chi) \\ &= -(\nabla P_h u_0, \nabla \chi) - (\nabla P_h u_1, \nabla \chi) + (F(t_n), \chi) + (g(t_n), \chi), \qquad (5.7.13) \\ &V^0 = 0, \qquad (5.7.14) \end{aligned}$$

where the weights are generated by the Lubich's convolution quadrature formula, with $\alpha \in (1, 2)$,

$$(1-z)^{\alpha} = \sum_{j=0}^{\infty} w_j z^j.$$

Hence (5.7.13)-(5.7.14) can be written as the following abstract form

$$\tau^{-\alpha} \sum_{j=1}^{n} w_{n-j} V^j + A_h V^n = -A_h P_h u_0 - t_n A_h P_h u_1 + F(t_n) + g(t_n), \quad V^0 = 0. \tag{5.7.15}$$

Let $\varphi_1(x), \varphi_2(x), \ldots, \varphi_{M-1}(x)$ be the linear finite element basis functions defined by, with $j = 1, 2, \ldots, M-1$,

$$\varphi_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}}, & x_{j-1} < x < x_j \\ \frac{x - x_{j+1}}{x_j - x_{j+1}}, & x_j < x < x_{j+1}, \\ 0, & \text{otherwise.} \end{cases}$$

To find the solution $V^n \in S_h,\ n = 0, 1, \ldots, N$, we assume that

$$V^n = \sum_{m=1}^{M-1} \alpha_m^n \varphi_m,$$

for some coefficients α_k^n, $k = 1, 2, \ldots, M-1$. Choose $\chi = \varphi_l$, $l = 1, 2, \ldots, M-1$ in (5.7.13), we have, with $n = 1, 2, \ldots, N$,

$$\begin{aligned} &\tau^{-\alpha} \sum_{j=1}^{n} w_{n-j} \Big[\sum_{m=1}^{M-1} (\varphi_m, \varphi_l) \alpha_m^j \Big] + \sum_{m=1}^{M-1} (\nabla \varphi_m, \nabla \varphi_l) u_m^n \\ &= - \sum_{m=1}^{M-1} (\nabla \varphi_m, \nabla \varphi_l) u_m^0 - t_n \sum_{m=1}^{M-1} (\nabla \varphi_m, \nabla \varphi_l) u_m^1 + (F(t_n), \varphi_l) + (g(t_n), \varphi_l), \end{aligned} \tag{5.7.16}$$

where we assume the initial values $P_h u_0$ and $P_h u_1$ have the following expressions:

$$P_h u_0 = \sum_{m=1}^{M-1} u_m^0 \varphi_m, \qquad P_h u_1 = \sum_{m=1}^{M-1} u_m^1 \varphi_m.$$

To solve (5.7.16) by MATLAB, we need to write (5.7.16) into the matrix form which we shall do now.

Denote

$$\alpha^n = \begin{pmatrix} \alpha_1^n \\ \alpha_2^n \\ \vdots \\ \alpha_{M-1}^n \end{pmatrix}_{(M-1)\times 1}, \quad \mathbf{F}^n = \begin{pmatrix} \big(F(t_n), \varphi_1\big) \\ \big(F(t_n), \varphi_2\big) \\ \vdots \\ \big(F(t_n), \varphi_{M-1}\big) \end{pmatrix}_{(M-1)\times 1},$$

and

$$\mathbf{g}^n = \begin{pmatrix} \big(g(t_n), \varphi_1\big) \\ \big(g(t_n), \varphi_2\big) \\ \vdots \\ \big(g(t_n), \varphi_{M-1}\big) \end{pmatrix}_{(M-1)\times 1}, \quad \mathbf{u}^0 = \begin{pmatrix} u_1^0 \\ u_2^0 \\ \vdots \\ u_{M-1}^0 \end{pmatrix}_{(M-1)\times 1},$$

and

$$\mathbf{u}^1 = \begin{pmatrix} u_1^1 \\ u_2^1 \\ \vdots \\ u_{M-1}^1 \end{pmatrix}_{(M-1)\times 1},$$

After some simple calculations, we may get the following mass and stiffness metrics

$$\mathbf{M} = \Big((\varphi_m, \varphi_l)\Big)_{m,l=1}^{M-1} = h \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & & 0 \\ \frac{1}{6} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{1}{6} \\ 0 & & \frac{1}{6} & \frac{2}{3} \end{pmatrix}_{(M-1)\times(M-1)},$$

and

$$\mathbf{S} = \Big((\nabla\varphi_m, \nabla\varphi_l)\Big)_{m,l=1}^{M-1} = \frac{1}{h} \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}_{(M-1)\times(M-1)},$$

respectively. Then (5.7.16) can be written as the following matrix form, $n = 1, 2, \ldots, N$,

$$\tau^{-\alpha} \sum_{j=1}^{n} w_{n-j} \mathbf{M} \alpha^j + \mathbf{S}\alpha^n = -\mathbf{S}\mathbf{u}^0 - t_n \mathbf{S}\mathbf{u}^1 + \mathbf{F}^n + \mathbf{g}^n, \quad \alpha^0 \text{ given,} \tag{5.7.17}$$

Denote $\mathbf{A}_h = \mathbf{M}^{-1}\mathbf{S}$. Then (5.7.17) can be written as, with $n = 1, 2, \dots, N$,

$$\tau^{-\alpha} \sum_{j=1}^{n} w_{n-j} \alpha^j + \mathbf{A}_h \alpha^n = -\mathbf{A}_h \mathbf{u}^0 - t_n \mathbf{A}_h \mathbf{u}^1 + \mathbf{M}^{-1}\mathbf{F}^n + \mathbf{M}^{-1}\mathbf{g}^n, \quad \alpha^0 \text{ given}, \tag{5.7.18}$$

which is the matrix approximation form of (5.7.15). Hence $\alpha^n, n = 1, 2, \dots, N$ can be calculated by the following formula

$$\alpha^n = (w_0 + \tau^\alpha \mathbf{A}_h)^{-1}\Big(-\tau^\alpha \mathbf{A}_h \mathbf{u}^0 - \tau^\alpha t_n \mathbf{A}_h \mathbf{u}^1 + \tau^\alpha \mathbf{M}^{-1}\mathbf{f}^n + \tau^\alpha \mathbf{M}^{-1}\mathbf{g}^n - \sum_{j=1}^{n-1} w_{n-j} \alpha^{n-j}\Big). \tag{5.7.19}$$

We now consider how to calculate $\mathbf{F}^n$.

Case 1. Assume that $F(t)$ is independent of u, that is, $F(t) = f(t)$. Then the kth components $\big(F(t_n), \varphi_k\big), k = 1, 2, \dots, M-1$ in $\mathbf{F}^n$ can be approximated by using the midpoint quadrature formula

$$\begin{aligned}
\big(F(t_n), \varphi_k\big) &= \int_0^1 f(t_n)\varphi_k \, dx = \int_{x_{k-1}}^{x_k} f(t_n)\varphi_k \, dx + \int_{x_k}^{x_{k+1}} f(t_n)\varphi_k \, dx \\
&\approx f(t_n, \frac{x_{k-1}+x_k}{2})\varphi_k(\frac{x_{k-1}+x_k}{2})h + f(t_n, \frac{x_k+x_{k+1}}{2})\varphi_k(\frac{x_k+x_{k+1}}{2})h \\
&= \frac{h}{2}\Big(f(t_n, \frac{x_{k-1}+x_k}{2}) + f(t_n, \frac{x_k+x_{k+1}}{2})\Big).
\end{aligned}$$

In MATLAB, we use the following code to calculate $\mathbf{f}^n$ with some given $f(t, x)$.

```
% find (f, phi)
  function y=f_phi(x,n,tau,alpha)
    % case 1: f(t, x) = x^2 (1-x)^2 exp(t)-(2-12 x+12 x^2) exp(t)
    tn=n*tau;
    h=x(2)-x(1);
    x0=[0;x(1:end-1)]; x1=x; x2=[x(2:end); x(end)+h];
    x=(x0+x1)/2;
    y1=(x.^2).*((1-x).^2)*exp(tn)-(2-12*x+12*(x.^2))*exp(tn);
    x=(x1+x2)/2;
    y2=(x.^2).*((1-x).^2)*exp(tn)-(2-12*x+12*(x.^2))*exp(tn);
    y=h/2*(y1+y2);
```

```
%case 2: f(t, x)=0
 y=zeros(size(x));   %f=0
```

Case 2. Assume that $F(t)$ depends on $u(t)$, that is, $F(t) = f(u(t))$. Then kth element $\big(F(t_n), \varphi_k\big), k = 1, 2, \ldots, M-1$ in $\mathbf{F}^n$ can be approximated by using the following formula:

$$
\begin{aligned}
&\big(F(t_n), \varphi_k\big) = \int_0^1 f(u(t_n))\varphi_k \, dx \approx \int_0^1 f(u(t_{n-1}))\varphi_k \, dx \\
&= \int_{x_{k-1}}^{x_k} f(u(t_{n-1}))\varphi_k \, dx + \int_{x_k}^{x_{k+1}} f(u(t_{n-1}))\varphi_k \, dx \\
&\approx \frac{h}{2}\Big[f(u(t_{n-1}, \frac{x_{k-1}+x_k}{2})) + f(u(t_{n-1}, \frac{x_k + x_{k+1}}{2}))\Big] \\
&\approx \frac{h}{2}\Big[\frac{F(u(t_{n-1}, x_{k-1})) + F(u(t_{n-1}, x_k))}{2} + \frac{F(u(t_{n-1}, x_k)) + F(u(t_{n-1}, x_{k+1}))}{2}\Big] \\
&= \frac{h}{4}\Big[F(u(t_{n-1}, x_{k-1})) + 2F(u(t_{n-1}, x_k)) + F(u(t_{n-1}, x_{k+1}))\Big] \\
&= \frac{h}{4}\Big[F_{-1} + F_0 + F_1\Big],
\end{aligned}
$$

where, with $k = 1, 2, \ldots, M-1$,

$$
\begin{aligned}
F_{-1} &= F(u(t_{n-1}, x_{k-1})) = F(v(t_{n-1}, x_{k-1}) + u_0(x_{k-1}) + t_{n-1}u_1(x_{k-1})), \\
F_0 &= F(u(t_{n-1}, x_k)) = F(v(t_{n-1}, x_k) + u_0(x_k) + t_{n-1}u_1(x_k)), \\
F_1 &= F(u(t_{n-1}, x_{k+1})) = F(v(t_{n-1}, x_{k+1}) + u_0(x_{k+1}) + t_{n-1}u_1(x_{k+1})).
\end{aligned}
$$

In MATLAB, we use the following code to calculate the kth element of $(f(u(t_n)), \varphi_k)$ in $\mathbf{F}^n$.

```
% find (fu, phi)
  function y=fu_phi(x,n,tau,alpha,v,Ph_u0,Ph_u1)
       tn=n*tau;
       h=x(2)-x(1);
       U0=v+Ph_u0+tn*Ph_u1;
       U_1=[0;U0(1:end-1)];
       U1=[U0(2:end);0];
       % f(u)= sin(u)
```

```
F0=sin(U0); F_1=sin(U_1); F1=sin(U1);
y=h/4*(F_1+2*F0+F1);
```

We next consider how to calculate $\mathbf{g}^n$ which is more complicated than $\mathbf{F}^n$. Approximating the Riemann-Liouville fractional integral by the Lubich first order convolution quadrature formula and truncating the noise term to $M-1$ terms, we obtain the lth element of $\mathbf{g}^n$ by, with $l = 1, 2, \dots, M-1$,

$$\mathbf{g}^n(l) = \big(g(t_n), \varphi_l\big) = {}_0^R D_t^{-\gamma} \sum_{m=1}^{\infty} \gamma_m^{1/2} (e_m(x), \varphi_l) \frac{d\beta_m^H(t)}{dt}$$
$$\approx \tau^\gamma \sum_{j=1}^{n} w_{n-j}^{(-\gamma)} \Big[\sum_{m=1}^{M-1} \gamma_m^{1/2} (e_m, \varphi_l) \frac{\beta_m^H(t_j) - \beta_m^H(t_{j-1})}{\tau} \Big], \qquad (5.7.20)$$

where $w_j^{(-\gamma)}, j = 0, 1, 2, \dots, n$ are generated by the Lubich first order method, with $\gamma \in [0, 1]$,

$$(1-\zeta)^{-\gamma} = \sum_{j=0}^{\infty} w_j^{(-\gamma)} \zeta^j.$$

To solve (5.7.20), we first need to generate $M-1$ Brownian motions $\beta_m^H(t), m = 1, 2, \dots, M-1$ which can be done by using MathWorks MATLAB function **fbm1d.m**.

Let $Nref = 2^7$ and $T = 1$ and let $dtref = T/Nref$ denote the reference time step size. Let $0 = t_0 < t_1 < \dots < t_{Nref} = T$ be the time partition of $[0, T]$. We generate the fractional Brownian motions $\beta_m^H(t_0), \beta_m^H(t_1), \dots \beta_m^H(t_{Nref})$, $m = 1, 2, \dots, M-1$ with the Hurst number $H \in [1/2, 1]$ by using the following code:

```
% Fractional Brownian paths  with Hurst number 1/2 \leq H \leq 1
    W=[];
    for j=1:M-1
        [Wj,t]=fbm1d(H,Nref,T);
        W=[W Wj];
    end
    W(1,:)=zeros(1, M-1);
```

Remark 53. When $H = 1/2$, $\mathbf{fbm1d(H, Nref, T)}$ generates the standard Brownian motions. The standard Brownian motions can also be generated by the following code

```
% Standard Brownian paths
    dW=sqrt(dtref)*randn(Nref,M-1);
    W=cumsum(dW,1);
    W=[zeros(1, M-1); W];
```

Since we do not know the exact solution of the system, we shall use the reference time step size $dtref$ and the space step size $h = 2^{-7}$ to calculate the reference solution **vref**. The spacial discretization is based on the linear finite element method.

We then choose $kappa = 2^5, 2^4, 2^3, 2^2$ and consider the different time step size $\tau = dtref * kappa$ to obtain the approximate solutions V^n at $t_n = n\tau$.

Let us discuss how to calculate the lth element of $\mathbf{g}^n$ in MATLAB. Denote

$$\mathbf{w}_\gamma = [w_0^{(-\gamma)},\, w_1^{(-\gamma)},\, \ldots,\, w_{n-1}^{(-\gamma)}]_{1\times(M-1)},$$

and

$$\mathbf{dWdt} = \begin{pmatrix} \sum_{m=1}^{M-1} \gamma_m^{1/2}(e_m, e_l)\frac{\beta_m(t_n)-\beta_m(t_{n-1})}{\tau} \\ \sum_{m=1}^{M-1} \gamma_m^{1/2}(e_m, e_l)\frac{\beta_m(t_{n-1})-\beta_m(t_{n-2})}{\tau} \\ \vdots \\ \sum_{m=1}^{M-1} \gamma_m^{1/2}(e_m, e_l)\frac{\beta_m(t_1)-\beta_m(t_0)}{\tau} \end{pmatrix}_{(M-1)\times 1}.$$

The lth element of the vector $\mathbf{g}^n$ satisfies

$$\mathbf{g}^n(l) = \mathbf{w}_\gamma * \mathbf{dWdt}, \quad l = 1, 2, \ldots, M-1.$$

Based on this idea, we use the following MATLAB function

```
g_{phi}(x,n,tau,ga,kappa,W)
```

to calculate $\mathbf{g}^n(l)$ in our numerical simulations.

```
 % find (g, phi)
    function y=g_phi(x,n,tau,ga,kappa,W)
        y=[];
        M=length(x)+1;
        %Find w_ga=[w_{0}^{-ga} w_{1}^{-ga}  w_{n-1}^{-ga}]
        w_ga=[];
```

```
        for nn=0:n-1
            w_ga=[w_ga w_gru(nn,-ga)];
        end


        for k=1:M-1
            A=dWdt_k(x,n,tau,kappa,W,k);
            y1=tau^(ga)*w_ga*A;
            y=[y;y1];
        end


    % Find dWdt_k
 function y= dWdt_k(x,n,tau,kappa,W,k)
        y=zeros(n,1);
        M=length(x)+1;
        for m=1:M-1
            beta=2; % white noise beta=0, trace class beta=2
            ga_m=m^(-beta);
            k1=n:-1:1;  %tn=n*tau=(n*kappa)*dtref
            dW_k1=W(k1*kappa+1,m)-W((k1-1)*kappa+1,m); %dW_k is a vector
            h=x(2)-x(1);
            x1=((k-1)*h+k*h)/2; x2= (k*h+(k+1)*h)/2;
            e_phi=h/2*(sqrt(2)*sin(pi*m*x1)+sqrt(2)*sin(pi*m*x2));
            y=y+ga_m^(1/2)*e_phi*(dW_k1/tau);
        end
```

Finally we shall consider how to calculate the L_2 projections $P_h u_0$ and $P_h u_1$ of u_0 and u_1, respectively. Here we only consider the case $P_h u_0$. The calculation of $P_h u_1$ is similar. Assume that

$$P_h u_0 = \sum_{m=1}^{M-1} u_m^0 \varphi_m.$$

By the definition of P_h, we obtain

$$\sum_{m=1}^{M-1} u_m^0 (\varphi_m, \varphi_l) = (u_0, \varphi_l).$$

Hence $\mathbf{u}^0$ can be calculated by

$$\mathbf{u}^0 = \mathbf{M}^{-1}\mathbf{v}^0, \tag{5.7.21}$$

where

$$\mathbf{v}^0 = \begin{pmatrix} (u_0, \varphi_1) \\ (u_0, \varphi_2) \\ \vdots \\ (u_0, \varphi_{M-1}) \end{pmatrix}_{(M-1)\times 1}.$$

Remark 54. When we use (5.7.21) to calculate $\mathbf{u}^0$, we have to calculate $\mathbf{M}^{-1}$ which will produce some computational errors. In our numerical examples, we shall simply choose $\mathbf{u}^0(l) = u_0(x_l)$, $l = 1, 2, \ldots, M-1$ (instead of (5.7.21)) which also give the required accuracy in our numerical simulations.

Example 55. *Consider the following stochastic time fractional PDE, with $\alpha \in (1,2)$,*

$$ {}_0^C D_t^\alpha u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x) + g(t,x), \quad 0 \le t \le T,\ 0 < x < 1, \tag{5.7.22}$$

$$u(0,x) = u_0(x),\ \frac{\partial u(0,x)}{\partial t} = u_1(x), \tag{5.7.23}$$

$$u(t,0) = u(t,1) = 0, \tag{5.7.24}$$

where $f(t,x) = x^2(1-x)^2 e^t - (2 - 12x + 12x^2)e^t$ and the initial value $u_0(x) = x^2(1-x)^2, u_1(x) = x$ and $g(t,x)$ is defined by (5.7.4).

Let $v(t,x) = u(t,x) - u_0(x) - tu_1(x)$ and transform the system (5.7.22)-(5.7.24) of u into the system of v. We shall consider the approximation of v at $T = 1$. We choose the space step size $h = 2^{-6}$ and the time step size $dtref = 2^{-7}$ to get the reference solution vref. To observe the time convergence orders, we consider the different time step sizes $\tau = kappa * dtref$ with $kappa = [2^5, 2^4, 2^3, 2^2]$ to obtain the approximate solution V. We choose $M1 = 20$ simulations to calculate the following L2 error at $T = 1$ with the different time step sizes

$$\|vref - V\|_{L^2(\Omega;H)} = \sqrt{\mathbb{E}\|vref - V\|_H^2}.$$

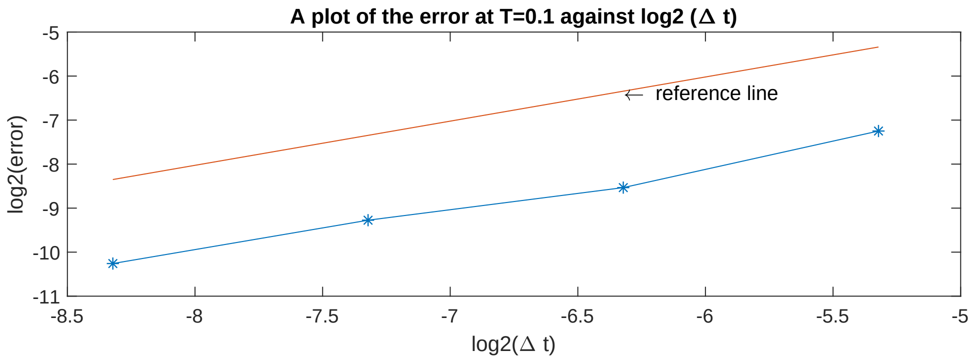

Figure 5.7.1: The experimentally determined orders of convergence with $\gamma = 0.6$ and $\alpha = 1.1$ in Table 5.7.1

By Theorem 51, the convergence order should be

$$\|vref - V\|_{L^2(\Omega;H)} = O(\tau^{\min\{1,\alpha+\gamma-1/2\}}). \tag{5.7.25}$$

In Table 5.7.1, we consider the trace class noise, that is $\gamma_m = m^{-2}, m = 1, 2, \ldots$ *and we observe that the experimentally determined time convergence orders are consistent with our theoretical convergence orders. The numbers in the brackets denote the theoretical convergence orders.*

In Table 5.7.2, we consider the white noise, that is $\gamma_m = 1, m = 1, 2, \ldots$ *and we observe that the experimentally determined time convergence orders are slightly less than the orders in the trace class noise case as we expected.*

In Figure 5.7.1, we plot the experimentally determined orders of convergence with $\gamma = 0.6$ *and* $\alpha = 1.1$ *in Table 5.7.1. The expected convergence order is* $O(\tau^{\min\{1,\alpha+\gamma-1/2\}}) = O(\tau)$*. We indeed observe this in the figure where the reference line is for the order* $O(\tau)$*.*

In Figure 5.7.2, we plot the experimentally determined orders of convergence with $\gamma = 0.6$ *and* $\alpha = 1.1$ *in Table 5.7.2. We observe that the convergence order is almost* $O(\tau)$ *in the figure where the reference line is for the order* $O(\tau)$*.*

In Figure 5.7.3, we plot one approximate solution with $\alpha = 1.5$ *and* $\gamma = 0$ *for all* $x \in (0, 1)$ *and* $t \in (0, 1)$ *in Example 55. In Figure 5.7.4, we plot one approximate solution with* $\alpha = 1.5$ *and* $\gamma = 0$ *at time* $T = 1$ *in Example 55.*

In Figure 5.7.5, we plot one approximate solution with $\alpha = 1.5$ *and* $\gamma = 0.9$ *for all* $x \in (0, 1)$ *and* $t \in (0, 1)$ *in Example 55. In Figure 5.7.6, we plot one approximate solution with* $\alpha = 1.5$ *and* $\gamma = 0.9$ *at time* $T = 1$ *in Example 55.*

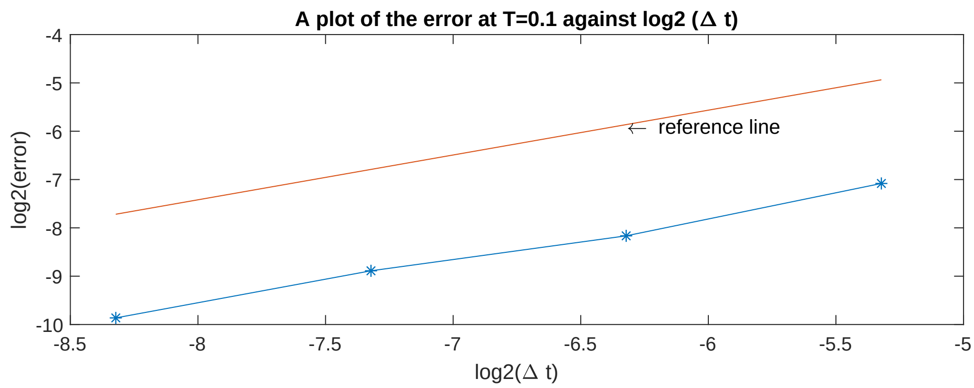

Figure 5.7.2: The experimentally determined orders of convergence with $\gamma = 0.6$ and $\alpha = 1.1$ in Table 5.7.2

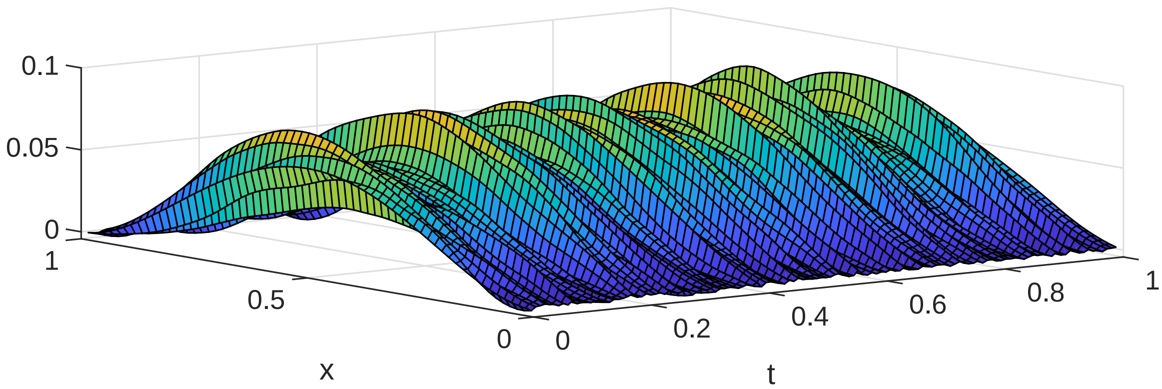

Figure 5.7.3: Approximate realisation of the solution with $\alpha = 1.5$ and $\gamma = 0$ for $x \in (0, 1)$ and $t \in (0, 1)$ in Example 55

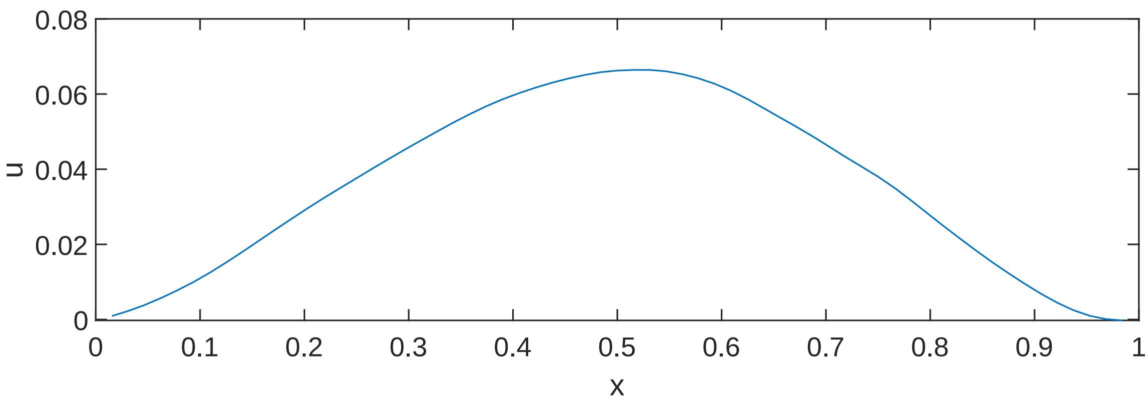

Figure 5.7.4: Approximate realisation of the solution at time $T = 1$ with $\alpha = 1.5$ and $\gamma = 0$ in Example 55

α	γ	$\tau = 1/4$	$\tau = 1/8$	1/16	1/32	order
1.1	0.0	1.91e-2	1.07e-3	6.76e-3	3.75e-3	
			0.82	0.67	0.85	0.78 (0.60)
1.1	0.4	1.15e-2	5.25e-3	3.33e-3	1.63e-3	
			1.13	0.565	1.03	0.94 (0.80)
1.1	0.6	1.01e-2	4.54e-3	2.43e-3	1.16e-3	
			1.15	0.90	1.05	1.03 (1.00)
1.1	0.8	8.54e-3	3.96e-3	1.93e-3	9.07e-4	
			1.10	1.03	1.09	1.07 (1.00)
1.6	0.0	1.38e-2	6.34e-3	3.50e-3	1.68e-3	
			1.12	0.85	1.05	1.01 (1.00)
1.6	0.4	7.76e-3	3.70e-3	1.82e-3	8.07e-4	
			1.06	1.02	1.17	1.08 (1.00)
1.6	0.6	6.73e-3	3.33e-3	1.61e-3	6.96e-4	
			1.01	1.04	1.21	1.09 (1.00)
1.6	0.8	6.33e-3	3.19e-3	1.54e-3	6.61e-4	
			0.98	1.05	1.22	1.08 (1.00)

Table 5.7.1: Time convergence orders in Example 55 at $T = 1$ with trace class noise $\gamma_m = m^{-2}, m = 1, 2, \ldots$

We observe that the solution with $\alpha = 1.5, \gamma = 0.9$ is much smoother than the solution with $\alpha = 1.5, \gamma = 0$ as we expected.

Example 56. *Consider the following stochastic time fractional PDE, with $\alpha \in (1, 2)$,*

$$ {}_0^C D_t^{\alpha} u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = f(u(t,x)) + g(t,x), \quad 0 \le t \le T, \ 0 < x < 1, \tag{5.7.26}$$

$$ u(0,x) = u_0(x), \ \frac{\partial u(0,x)}{\partial t} = u_1(x), \tag{5.7.27}$$

$$ u(t,0) = u(t,1) = 0, \tag{5.7.28}$$

where $f(u) = \sin(u)$ and the initial values $u_0(x) = x^2(1-x)^2, u_1(x) = 2x(1-x)(1-2x)$ and $g(t,x)$ is defined by (5.7.4).

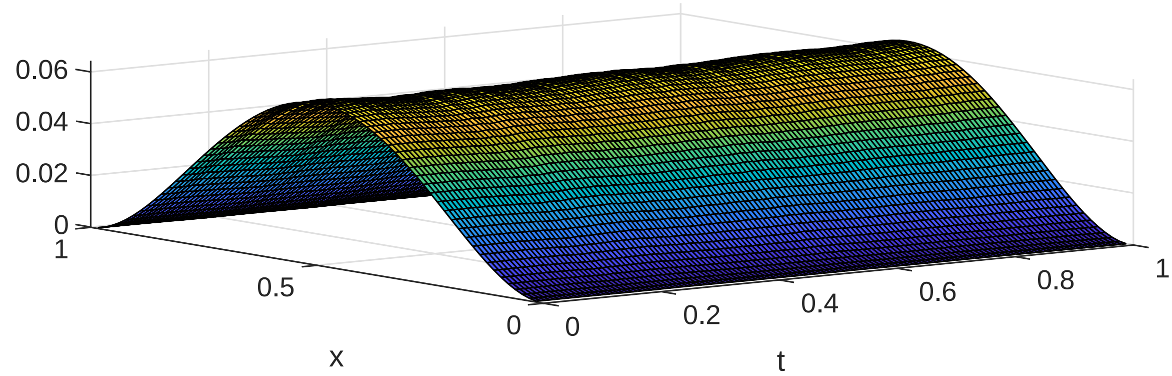

Figure 5.7.5: Approximate realisation of the solution with $\alpha = 1.5$ and $\gamma = 0.9$ for $x \in (0, 1)$ and $t \in (0, 1)$ in Example 55

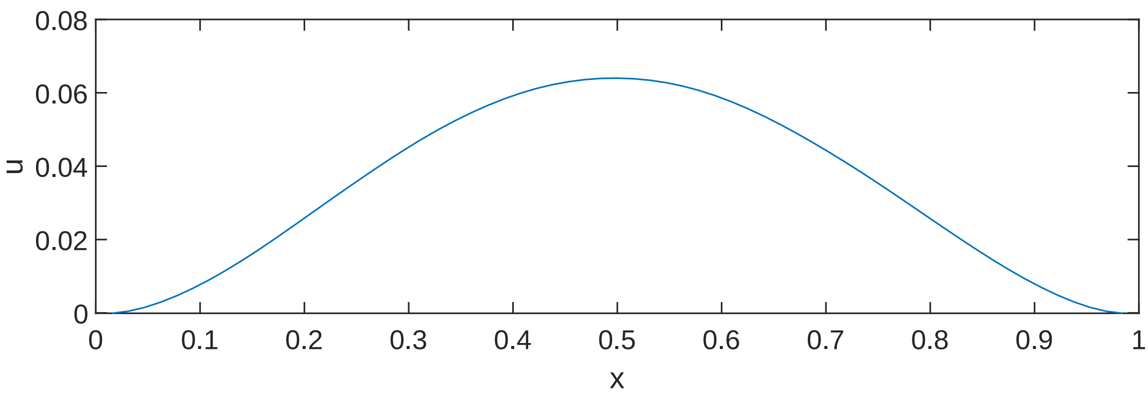

Figure 5.7.6: Approximate realisation of the solution at time $T = 1$ with $\alpha = 1.5$ and $\gamma = 0.9$ in Example 55

α	γ	$\tau = 1/4$	$\tau = 1/8$	1/16	1/32	order
1.1	0.0	6.58e-2	4.86e-2	3.61e-2	2.37e-2	
			0.43	0.43	0.60	0.49 (0.60)
1.1	0.4	1.32e-2	7.00e-3	4.45e-3	2.29e-3	
			0.92	0.65	0.95	0.84 (1.00)
1.1	0.6	1.06e-2	5.01e-3	2.75e-3	1.34e-3	
			1.08	0.86	1.03	0.99 (1.00)
1.1	0.8	8.75e-3	4.10e-3	2.03e-3	9.59e-4	
			1.09	1.01	1.08	1.06 (1.00)
1.6	0.0	2.69e-2	1.64e-2	1.02e-2	5.58e-3	
			0.70	0.68	0.87	0.75 (1.00)
1.6	0.4	9.71e-3	5.07e-3	2.68e-3	1.26e-3	
			0.93	0.91	1.08	0.98 (1.00)
1.6	0.6	7.40e-3	3.75e-3	1.87e-3	8.24e-4	
			0.97	1.00	1.18	1.05 (1.00)
1.6	0.8	6.54e-3	3.30e-3	1.60e-3	6.88e-4	
			0.98	1.04	1.22	1.08 (1.00)

Table 5.7.2: Time convergence orders in Example 55 at $T = 0.1$ with white noise $\gamma_m = 1, m = 1, 2, \ldots$

We use the same notations as in Example 55. In Table 5.7.3, we consider the trace class noise, that is $\gamma_m = m^{-2}, m = 1, 2, \ldots$ *and we observe that the experimentally determined time convergence orders are consistent with our theoretical convergence orders. The numbers in the brackets denote the theoretical convergence orders.*

In Table 5.7.4, we consider the white noise, that is $\gamma_m = 1, m = 1, 2, \ldots$ *and we observe that the experimentally determined time convergence orders are slightly less than the orders in the trace class noise case as we expected.*

In Figure 5.7.7, we plot the experimentally determined orders of convergence with $\gamma = 0.6$ *and* $\alpha = 1.6$ *in Table 5.7.3. The expected convergence order is* $O(\tau^{\min\{1, \alpha+\gamma-1/2\}}) =$

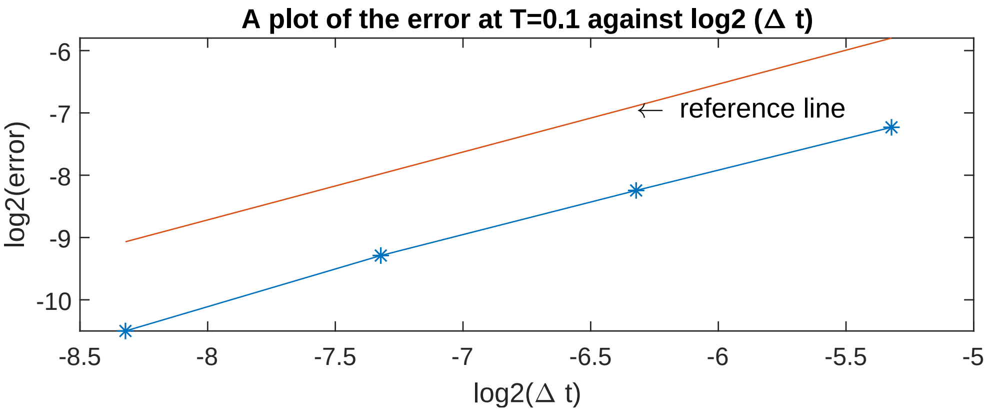

Figure 5.7.7: The experimentally determined orders of convergence with $\gamma = 0.6$ and $\alpha = 1.6$ in Table 5.7.3

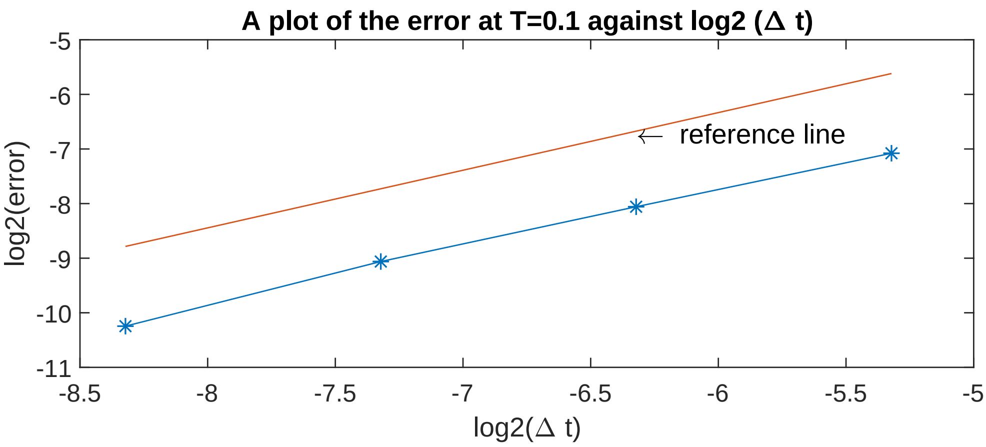

Figure 5.7.8: The experimentally determined orders of convergence with $\gamma = 0.6$ and $\alpha = 1.6$ in Table 5.7.4

$O(\tau)$. *We indeed observe this in the figure where the reference line is for the order* $O(\tau)$.

In Figure 5.7.8, we plot the experimentally determined orders of convergence with $\gamma = 0.6$ *and* $\alpha = 1.6$ *in Table 5.7.4. We observe that the converegnce order is almost* $O(\tau)$ *in the figure where the reference line is for the order* $O(\tau)$.

Example 57. *Consider the following stochastic time fractional PDE, with* $\alpha \in (1, 2)$,

$$ {}_0^C D_t^{\alpha} u(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} = f(u(t,x)) + g(t,x), \quad 0 \le t \le T, \ 0 < x < 1, \tag{5.7.29} $$

$$ u(0,x) = u_0(x), \ \frac{\partial u(0,x)}{\partial t} = u_1(x), \tag{5.7.30} $$

$$ u(t,0) = u(t,1) = 0, \tag{5.7.31} $$

α	γ	$\tau = 1/4$	$\tau = 1/8$	1/16	1/32	order
1.1	0.0	1.48e-2	8.28e-3	4.96e-3	2.77e-3	
			0.84	0.73	0.84	0.80 (0.60)
1.1	0.4	8.46e-3	3.86e-3	2.68e-3	1.34e-3	
			1.13	0.52	0.99	0.88 (1.00)
1.1	0.6	6.57e-3	2.69e-3	1.61e-3	8.16e-4	
			1.28	0.74	0.98	1.00 (1.00)
1.1	0.8	4.57e-3	1.85e-3	9.91e-3	5.00e-3	
			1.30	0.90	0.98	1.06 (1.00)
1.6	0.0	1.32e-2	5.77e-3	3.18e-3	1.54e-3	
			1.19	0.85	1.04	1.03 (1.00)
1.6	0.4	7.73e-3	3.62e-3	1.75e-3	7.90e-4	
			1.09	1.04	1.15	1.09 (1.00)
1.6	0.6	6.64e-3	3.24e-3	1.55e-3	6.77e-4	
			1.03	1.06	1.19	1.09 (1.00)
1.6	0.8	6.17e-3	3.07e-3	1.46e-3	6.34e-4	
			1.00	1.06	1.20	1.09 (1.00)

Table 5.7.3: Time convergence orders in Example 56 at $T = 1$ with trace class noise $\gamma_m = m^{-2}, m = 1, 2, \ldots$

where $f(u) = -u^3 + u$ *and the initial values* $u_0(x) = x^2(1-x)^2, u_1(x) = 2x(1-x)(1-2x)$ *and* $g(t, x)$ *is defined by* (5.7.4).

We use the same notations as in Example 55. In Table 5.7.5, we consider the trace class noise, that is $\gamma_m = m^{-2}, m = 1, 2, \ldots$ *and we observe that the experimentally determined time convergence orders are consistent with our theoretical convergence orders. The numbers in the brackets denote the theoretical convergence orders.*

In Table 5.7.6, we consider the white noise, that is $\gamma_m = 1, m = 1, 2, \ldots$ *and we observe that the experimentally determined time convergence orders are slightly less than the orders in the trace class noise case as we expected.*

α	γ	$\tau = 1/4$	$\tau = 1/8$	1/16	1/32	order
1.1	0.0	6.57e-2	4.86e-2	3.60e-2	2.36e-2	
			0.43	0.43	0.60	0.49 (0.60)
1.1	0.4	1.08e-2	6.14e-3	4.06e-3	2.14e-3	
			0.81	0.59	0.92	0.77 (1.00)
1.1	0.6	7.38e-3	3.48e-3	2.11e-3	1.07e-3	
			1.08	0.72	0.97	0.92 (1.00)
1.1	0.8	4.97e-3	2.16e-3	1.18e-3	5.95e-4	
			1.19	0.87	0.99	1.02 (1.00)
1.6	0.0	2.70e-2	1.64e-2	1.02e-2	5.58e-3	
			0.71	0.68	0.87	0.75 (1.00)
1.6	0.4	9.84e-3	5.10e-3	2.68e-3	1.27e-3	
			0.94	0.92	1.07	0.98 (1.00)
1.6	0.6	7.40e-3	3.72e-3	1.83e-3	8.17e-4	
			0.99	1.02	1.16	1.05 (1.00)
1.6	0.8	6.42e-3	3.21e-3	1.54e-3	6.66e-4	
			0.99	1.05	1.20	1.08 (1.00)

Table 5.7.4: Time convergence orders in Example 56 at $T = 0.1$ with white noise $\gamma_m = 1, m = 1, 2, \dots$

In Figure 5.7.9, we plot the experimentally determined orders of convergence with $\gamma = 0.6$ *and* $\alpha = 1.6$ *in Table 5.7.5. The expected convergence order is* $O(\tau^{\min\{1,\alpha+\gamma-1/2\}}) = O(\tau)$*. We indeed observe this in the figure where the reference line is for the order* $O(\tau)$*.*

In Figure 5.7.10, we plot the experimentally determined orders of convergence with $\gamma = 0.6$ *and* $\alpha = 1.6$ *in Table 5.7.6. We observe that the convergence order is almost* $O(\tau)$ *in the figure where the reference line is for the order* $O(\tau)$*.*

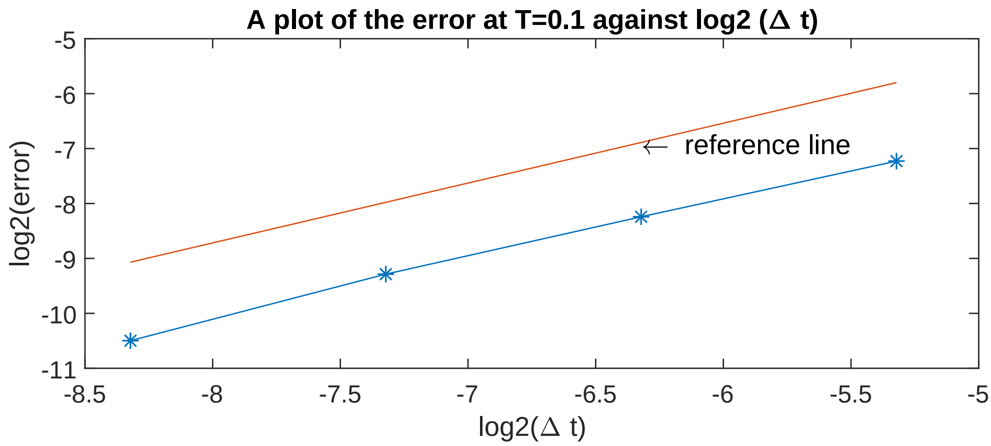

Figure 5.7.9: The experimentally determined orders of convergence with $\gamma = 0.6$ and $\alpha = 1.6$ in Table 5.7.5

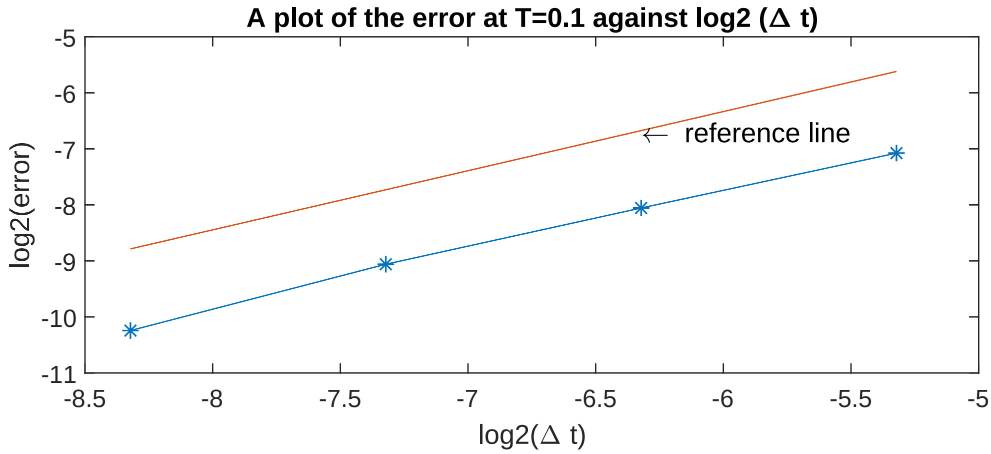

Figure 5.7.10: The experimentally determined orders of convergence with $\gamma = 0.6$ and $\alpha = 1.6$ in Table 5.7.6

α	γ	$\tau = 1/4$	$\tau = 1/8$	1/16	1/32	order
1.1	0.0	1.91e-2	1.07e-2	6.75e-3	3.74e-3	
			0.82	0.67	0.84	0.78 (0.60)
1.1	0.4	1.15e-2	5.24e-3	3.32e-3	1.62e-3	
			1.13	0.65	1.03	0.94 (0.80)
1.1	0.6	1.01e-2	4.53e-3	2.42e-3	1.16e-3	
			1.15	0.90	1.05	1.03 (1.00)
1.1	0.8	8.50e-3	3.94e-3	1.93e-3	9.04e-4	
			1.10	1.03	1.09	1.07 (1.00)
1.6	0.0	1.38e-2	6.34e-3	3.50e-3	1.68e-3	
			1.12	0.85	1.05	1.01 (1.00)
1.6	0.4	7.77e-3	3.70e-3	1.82e-3	8.08e-4	
			1.06	1.02	1.17	1.08 (1.00)
1.6	0.6	6.74e-3	3.33e-3	1.61e-3	6.98e-4	
			1.01	1.04	1.21	1.09 (1.00)
1.6	0.8	6.35e-3	3.20e-3	1.54e-3	6.62e-4	
			0.98	1.05	1.22	1.08 (1.00)

Table 5.7.5: Time convergence orders in Example 57 at $T = 1$ with trace class noise $\gamma_m = m^{-2}, m = 1, 2, \ldots$

α	γ	$\tau = 1/4$	$\tau = 1/8$	$1/16$	$1/32$	order
1.1	0.0	6.58e-2	4.86e-2	3.61e-2	2.37e-2	
			0.43	0.43	0.60	0.49 (0.60)
1.1	0.4	1.32e-2	6.99e-3	4.45e-3	2.29e-3	
			0.92	0.65	0.95	0.84 (1.00)
1.1	0.6	1.06e-2	5.00e-3	2.75e-3	1.34e-3	
			1.08	0.86	1.03	0.99 (1.00)
1.1	0.8	8.71e-3	4.08e-3	2.02e-3	5.56e-4	
			1.09	1.01	1.08	1.06 (1.00)
1.6	0.0	2.69e-2	1.64e-2	1.02e-2	5.57e-3	
			0.70	0.68	0.87	0.75 (1.00)
1.6	0.4	9.71e-3	5.07e-3	2.68e-3	1.26e-3	
			0.93	0.91	1.08	0.98 (1.00)
1.6	0.6	7.41e-3	3.75e-3	1.87e-3	8.25e-4	
			0.97	1.00	1.18	1.05 (1.00)
1.6	0.8	6.55e-3	3.31e-3	1.60e-3	6.90e-4	
			0.98	1.04	1.22	1.08 (1.00)

Table 5.7.6: Time convergence orders in Example 57 at $T = 0.1$ with white noise $\gamma_m = 1, m = 1, 2, \ldots$

Chapter 6

Conclusion

This thesis considers the numerical methods for approximating the ε-dependent stochastic Allen-Cahn equation, along with stochastic semilinear space-time fractional subdiffusion and superdiffusion problems.

For the ε-dependent stochastic Allen-Cahn equation, the noise exhibits smoothness both in time and space, characterized as mild noise. To tackle this, we propose the space-time Galerkin method (discontinuous in time and continuous in space) to effectively approximate the equation. By using finite element approximation, we derive a posteriori error estimates in the H^1 norm.

In the context of the stochastic semilinear time-space fractional subdiffusion problem, the noise maintains smoothness in space but acquires nonsmoothness in time. Its representation involves eigenfunctions and Brownian motions, allowing us to express it as a series. Through an innovative approach that involves approximating the time-dependent noise using piecewise constant functions, we regularize the problem. Employing the finite element method, we proceed to approximate the regularized version, ultimately establishing optimal convergence orders with respect to time and space. These convergence rates are contingent upon $\alpha \in (0, 1)$ and $\gamma \in [0, 1]$, where α signifies the fractional derivative order and γ represents the fractional integral order.

In the context of the stochastic semilinear time-space fractional superdiffusion problem, we encounter a similar scenario where the noise maintains spatial smoothness but temporal nonsmoothness. Again utilizing the representation based on eigenfunctions and Brownian motions, we facilitate regularization by approximating the time-dependent noise using piecewise constant functions. Applying the finite element method, we approximate the regularized problem and establish optimal convergence orders concerning time and space. Remarkably, these convergence orders surpass those obtained in existing literature under similar assumptions.

This thesis opens avenues for further exploration:

1. Investigate adaptive methods for solving the ϵ-dependent stochastic Allen-Cahn equation, capitalizing on the a posteriori error estimates furnished in this study.

2. Undertake the fully discretization of the stochastic semilinear time-space fractional subdiffusion problem addressed herein.

3. Undertake the fully discretization of the stochastic semilinear time-space fractional superdiffusion problem examined in this work.

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Appendix

1. Matlab codes in Section 4.7

In this section, we include the MATLAB code for solving stochastic space-time fractional subdiffusion in Section 4.7. One may copy the codes and run them to get the numerical solutions in Section 4.7.

```
function  Y = sfpde( )
% Solves
% D_t^{al} u(x, t)-u''(x,t)= g(x, t)+ D^{-ga} dW/dt,   0<t<1, 0<x < 1
%          u(0)= x^2(1-x)^2
% g(x, t) = x^2 (1-x)^2 e^t -(2-12x+12x^2) e^t
% Consider the time  convergence order

% Time discretization: Grunwall-Letnikov method space discretization: FEM,
% E-M uses 5 different timesteps:   kappa*dt
% where        kappa=[2^2, 2^3, 2^4, 2^5, 2^6];
% Examine strong convergence at T=1:  Xerr_appr= E | Xn - X(T) |.
%Check the convergence order
% Xerr_appr = (E \|Xn -X(T)\|^2)^{1/2}  \leq C tau^{1/2},

%finite element   method
% Let 0= t_0 < t_1 < \dots < t_N=T  be the time partition of [0,T] and tau
% the time step size.
% Let 0< x_0< x_1< \dots x_M =1 be the space partition of [0, 1] and h the
% space step size
% Let u-u0=v
% v satisfies the following abstract form
%  D^{al} v(t)  + A v  = -A u0 + f(t) +g(t), t>0
```

```
%  v(0) = 0,

% The variational form is to find v such that
% (D^{al} v(t), \phi) + (v', \phi')
% =- (u0', \phi')+  (f, \phi) +(g, \phi) ,
%  \forall \phi \in H_{0}^{1}(0,1)
% The finite element method is to find vh  such that
% (D^{alpha} vh(t), \chi) + (vh', \chi')
% = - (u0', \phi') + (f, \chi)+ (g, \chi) , \forall \chi \in S_h
% The L1 scheme is to find V^n such that
% ( \tau^{-\alpha} \sum_{j=1}^{n} w_{n-j} V^{j},  \chi)
% + ( U^{n}', \chi') = - (u0', \phi') +(f^{n}, \chi)
%    +(g^{n}, \chi), \forall \chi \in S_h
% Let
% V^{n} = \sum_{m=1}^{M-1} \alpha_{m}^{n}  \phi_m
% we get
%  w_{0} \sum_{m=1}^{M-1} \alpha_{m}^{n} ( \phi_m, \phi_l)
% + \tau^{\alpha} \sum_{m=1}^{M-1} \alpha_{m}^{n} ( \phi_m', \phi_l')
%= - \tau^{\alpha} \sum_{m=1}^{M-1} \alpha_{m}^{0} ( \phi_m', \phi_l')
% + \tau^{\alpha} (f^n}, \phi_l) + \tau^{\alpha} (g^n}, \phi_l)
% + \sum_{j=1}^{n-1} w_{n-j} \sum_{m=1}^{M-1}
%  \alpha_{m}^{j} ( \phi_m, \phi_l), l=1, 2, \dots, M-1
%
% Matrix form
% (w_{0}I+ tau* Ah) alpha^n =
% - tau^{alpha}*Ah alpha^0
% + Mass^{-1} \tau^{alpha} * f^n + Mass^{-1} \tau^{alpha} * g^n
% - \sum_{j=1}^{n-1} w_{n-j} alpha^{j}
% where
% Ah= Mass^{-1}*Stiffness,
% the integral (f_{n}, phi_{l}) is calculated by the midpoint rule
```

```
% (f^{n})(l)  = (f_{n}, phi_{l})
% = h/2*[ f_{n}((x_{l-1}+x_{l})/2)+ f_{n}((x_{l}+x_{l+1})/2)]
% Here Mass = h * [2/3   1/6   0   0 ...     0
%                  1/6   2/3  1/6  0 ...     0
%                  .....
%                  0       0   ...    2/3    1/6
%                  0       0 ...      1/6    2/3 ]
% Stiffness = 1/h*     [2   -1  0  0 ...  0
%                      -1   2  -1 0 ...  0
%                      .....
%                      0   0 ...    2   -1
%                      0   0 ...    -1   2]

% Remark 1: The algorithm of fpde is similar as the algorithm for pde,
% please see the MATLAB code for parabolic pdes
% Remark 2: Initial value u0 is better than Ph(u0) in experiements,We shall
% choose alpha^0(k) = u0 (x_{k})  instead of alpha^0(k) = Ph(u0) (x_{k})

% correction algorithm for recovering the optimal convergence orders
% Matrix form for the correction algorithm, n=1, 2, 3, ..., N,
% (w_{0}I+ tau* Ah) alpha^n
% =
% - tau^{alpha}*Ah alpha^0 *(1+c0)
% + Mass^{-1} \tau^{alpha} * f^n  + Mass^{-1} \tau^{alpha} * g^n
% - \sum_{j=1}^{n-1} w_{n-j} alpha^{j}
% where
% c0 = 1/2, n=1
%    = 0,   n=2, 3, \dots, N

%Algorithm;
%Step 1, Given initial value U^0 ( U0 =u0, not Ph (u0))
```

```
%Step 2, Find U^1
%Step 3: Find U^2

clear
alpha   =default(' 0 <alpha <=1  (default is alpha=0.3)', 0.3);
ga      =default('ga \in [0, 1],  (default is alpha=0)', 0.0);
c0      =default('c0 = 0 or 0.5 corrections (default is c0=0.0 )',0.0);
M1      =default('M1=No of the simulations (default is M1=20)', 20);
H       =default('H= Hurst number  (default is  H =0.5)', 0.5);

randn('state',100)
%space discretization
x0=0; x1=1; M=2^6;
h=(x1-x0)/M; x=linspace(x0,x1,M+1); x=x(2:end-1); x=x';

%time discretization
t0=0; T=1;
Nref = 2^7;

error_M1=[];
 for s = 1:M1    % M1: the number of the simulations
    s
dtref=T/Nref;
 % Brownian paths W=[B_1(t), B_2(t),...B_{M-1}(t)
 % dW=sqrt(dtref)*randn(Nref,M-1); W=cumsum(dW,1); W=[zeros(1, M-1); W];
% Fractional Brownian paths 1/2 \leq H \leq 1
     W=[];
     for j=1:M-1
         [Wj,t]=fbm1d(H,Nref,T);
         W=[W Wj];
     end
```

```
     W(1,:)=zeros(1, M-1);

% exact solution
kappa =1;
[vref]=sfpde(alpha,Nref,M,c0,ga,T,kappa,W);

%approximate solutions
kappa =[2^5, 2^4, 2^3, 2^2];    % N=Nref/kappa, tau=T/N
error=[];
for i=1:length(kappa)
   [V]=sfpde(alpha,Nref,M,c0,ga,T,kappa(i),W);
    err=vref-V; err_L2norm=sqrt((err'*err)*h);
    error = [error err_L2norm];
end
error_M1=[error_M1; error];  %error_M1 is a matrix
end
error_M1_L2= sqrt(sum(error_M1.*error_M1,1)/M1); % \|e \|_{L^2(D)}



%Find the convergene orders
ratio=[];
for j=1:length(error_M1_L2)-1
    ratio=[ratio error_M1_L2(j)/error_M1_L2(j+1)];
end
format short
ratio= log2(ratio);

ratio
mean(ratio)   % Show the ratios and mean of the rations

format shortE
```

```
error_M1_L2                % Show the errors

%main program
function [vref]=sfpde(alpha,Nref,M,c0,ga,T,kappa,W);
x0=0; x1=1; h=(x1-x0)/M;
x=linspace(x0, x1, M+1); x=x(2:end-1); x=x';

N = Nref/kappa;
tau=T/N;

% construct matrix A for finite element method
Mass=2/3*diag(ones(1,M-1))+1/6*diag(ones(1,M-2),1)+1/6*diag(ones(1,M-2),-1);
Mass=h*Mass;
Stiffness=2*diag(ones(1,M-1))+(-1)*diag(ones(1,M-2),1)+(-1)*diag(ones(1,M-2),-1);
Stiffness=(1/h)*Stiffness;
Ah=inv(Mass)*Stiffness;

% v0
v0=zeros(M-1,1);



%Find L2 projection of u0.
%We may use Ph_u0=Ph1(x,u0,Mass) to calculate Ph_u0, but the results
%are not good.We simply use u0 instead of Ph (u0)
% to avoid calculate Mass^{-1}.

% Case 1: u0 = x^2(1-x)^2;
 Ph_u0=(x.^2).*((1-x).^2);

vN=[v0];
% First time level, n=1
```

```
% Find v1= the value at t1, (correction c0=0.5, No correction c0=0)
   initial =-(1+c0)*tau^alpha*Ah*Ph_u0;
   f1=inv(Mass)*f_phi(x,1,tau,alpha);
   g1=inv(Mass)*g_phi(x,1,tau,ga,kappa,W);
   v=(w(0,1,alpha)*eye(M-1)+tau^alpha*Ah)\(tau^(alpha)*f1 ...
       +tau^(alpha)*g1 + initial);
    vN=[v vN];
% find v2, v3, .... vN
for n=2:N
    sum1=0;
    for j=1:n
        sum1=sum1+w(j,n,alpha)*vN(:,j);  %summation
    end
    initial =-tau^alpha*Ah*Ph_u0;       %u0
    fn=inv(Mass)*f_phi(x,n,tau,alpha);  %f
    gn=inv(Mass)*g_phi(x,n,tau,ga,kappa,W); %g
  v=(w(0,n,alpha)*eye(M-1)+tau^alpha*Ah)\(tau^(alpha)*fn ...
      +tau^(alpha)*gn - sum1 + initial);
    vN=[v vN];
end
vref=v;

%weights L1 scheme
    function y=w(k,j,alpha)
        if k==0
            y=1/gamma(2-alpha);
        elseif j==1 && k==j
            y=-alpha/gamma(2-alpha);
        elseif  k==1 && j>=2
            y=(2^(1-alpha)-2)/gamma(2-alpha);
        elseif k>=2 && k<=j-1
```

```
            y=((k-1)^(1-alpha)+(k+1)^(1-alpha)-2*k^(1-alpha))/gamma(2-alpha);
        else  k==j && j>=2;
            y=((k-1)^(1-alpha)-(alpha-1)*k^(-alpha)-k^(1-alpha))/gamma(2-alpha);
        end

% find the L2 projection of u0
function y=Ph1(x,u0,Mass)
        h=x(1)-x(2);
        x0=[0;x(1:end-1)]; x1=x; x2=[x(2:end); x(end)+h];
        x=(x0+x1)/2;
        y1=u0;
        x=(x1+x2)/2;
        y2=u0;
        y=h/2*(y1+y2);
        y=inv(Mass)*y;

% nonsmooth initial data u0
 function y=Ph2(x,Mass)
     h=x(1)-x(2);
     M=length(x)+1;
     y= zeros(M-1,1);
     for j=1:length(M-1)
         if j <= ceil(M/2)
             y(j)=0;
         else
             y(j)=h;
         end
     end
     y=inv(Mass)*y;

% find (f, phi)
```

```
function y=f_phi(x,n,tau,alpha)
    % case 1: f(t, x) = x^2 (1-x)^2 exp(t)-(2-12 x+12 x^2) exp(t)
    tn=n*tau;
    h=x(2)-x(1);
    x0=[0;x(1:end-1)]; x1=x; x2=[x(2:end); x(end)+h];
    x=(x0+x1)/2;
    y1=(x.^2).*((1-x).^2)*exp(tn)-(2-12*x+12*(x.^2))*exp(tn);
    x=(x1+x2)/2;
    y2=(x.^2).*((1-x).^2)*exp(tn)-(2-12*x+12*(x.^2))*exp(tn);
    y=h/2*(y1+y2);

   %case 2: f(t, x)=0
    y=zeros(size(x));   %f=0

% find (g, phi)
  function y=g_phi(x,n,tau,ga,kappa,W)
    y=[];
    M=length(x)+1;
    %Find w_ga=[w_{0}^{-ga} w_{1}^{-ga}  w_{n-1}^{-ga}]
    w_ga=[];
    for nn=0:n-1
        w_ga=[w_ga w_gru(nn,-ga)];
    end

    for k=1:M-1
        A=dWdt_k(x,n,tau,kappa,W,k);
        y1=tau^(ga)*w_ga*A;
        y=[y;y1];
    end

  % Find dWdt_k
```

```
 function y= dWdt_k(x,n,tau,kappa,W,k)
        y=zeros(n,1);
        M=length(x)+1;
        for m=1:M-1
            beta=2; % white noise beta=0, trace class beta=2
            ga_m=m^(-beta);
            k1=n:-1:1;  %tn=n*tau=(n*kappa)*dtref
            dW_k1=W(k1*kappa+1,m)-W((k1-1)*kappa+1,m); %dW_k is a vector
            h=x(2)-x(1);
            x1=((k-1)*h+k*h)/2; x2= (k*h+(k+1)*h)/2;
            e_phi=h/2*(sqrt(2)*sin(pi*m*x1)+sqrt(2)*sin(pi*m*x2));
            y=y+ga_m^(1/2)*e_phi*(dW_k1/tau);
        end

 function [ y ] =  w_gru(k,al)
% Grunwald-Letnikov weights


if k==0;
    y=1;
elseif k==1;
    y=-al;
else
y=al;
for l=1:k-1
y=y*(al-l);
end
y=(-1)^k/factorial(k)*y;
end

function [W,t]=fbm1d(H,n,T)
%
```

```
%https://de.mathworks.com/matlabcentral/fileexchange
%/38935-fractional-brownian-motion-generator

%fast one dimensional fractional Brownian motion (FBM) generator
% output is 'W_t' with t in [0,T] using 'n' equally spaced grid points;
% code uses Fast Fourier Transform (FFT) for speed.
% INPUT:
%           - Hurst parameter 'H' in [0,1]
%           - number of grid points 'n', where 'n' is a power of 2;
%             if the 'n' supplied is not a power of two,
%             then we set n=2^ceil(log2(n)); default is n=2^12;
%           - final time 'T'; default value is T=1;
% OUTPUT:
%           - Fractional Brownian motion 'W_t' for 't';
%           - time 't' at which FBM is computed;
%             If no output it invoked, then function plots the FBM.
% Example: plot FBM with hurst parameter 0.95 on the interval [0,10]
% [W,t]=fbm1d(0.95,2^12,10); plot(t,W)
% Reference:
% Kroese, D. P., & Botev, Z. I. (2015). Spatial Process Simulation.
% In Stochastic Geometry, Spatial Statistics and Random Fields(pp. 369-404)
% Springer International Publishing, DOI: 10.1007/978-3-319-10064-7_12
if (H>1)|(H<0) % Hurst parameter error check
    error('Hurst parameter must be between 0 and 1')
end
if nargin<2
    n=2^12;  % grid points
else
    n=2^ceil(log2(n));
end
if nargin<3
```

```
    T=1;
end
r=nan(n+1,1);r(1) = 1;idx=1:n;
r(idx+1) = 0.5*((idx+1).^(2*H) - 2*idx.^(2*H) + (idx-1).^(2*H));
r=[r; r(end-1:-1:2)]; % first rwo of circulant matrix
lambda=real(fft(r))/(2*n); % eigenvalues
W=fft(sqrt(lambda).*complex(randn(2*n,1),randn(2*n,1)));
W = n^(-H)*cumsum(real(W(1:n+1))); % rescale
W=T^H*W; t=(0:n)/n; t=t*T; % scale for final time T
if nargout==0
    plot(t,W); title('Fractional Brownian motion');
    xlabel('time $t$','interpreter','latex')
    ylabel('$W_t$','interpreter','latex')
end
```

```
function reply = default(query,value)
%default   gets response to IFISS prompt
%   reply = default(query,value);
%   input
%          query   character string: asks a question
%          value   integer: the default response
%
%   IFISS function: AR; 31 August 2005.
% Copyright (c) 2005 D.J. Silvester, H.C. Elman, A. Ramage (see readme.m)
global BATCH FID
if exist('BATCH') & BATCH==1,
   replycell=textscan(FID,'%f%*[^\n]',1);
   reply=deal(replycell{:});
   disp(query)
   disp(reply)
```

```
else
   reply=input([query,' : ']);
   if isempty(reply), reply=value; end
end
return
```

2. Matlab codes in Section 5.6

In this section, we include the MATLAB code for solving stochastic space-time fractional superdiffusion in Section 5.6. One may copy the codes and run them to get the numerical solutions in Section 5.6.

```
function  Y = sfpde( )
% Solve
% D_t^{al} u(x,t)-u''(x,t)= f(x, t)+ D^{-ga} dW/dt,   0<t<1, 0<x < 1
%          u(0,x)=u0(x), u'(0,x)=u1(x)
%          u(t,0)= u(t,1)=0,
% f(x, t) = x^2 (1-x)^2 e^t -(2-12x+12x^2) e^t
% 1 < al < 2
% Consider the time  convergence order

% Time discretization: Grunwall-Letnikov method
% space discretization: FEM
%
% E-M uses 5 different timesteps:   kappa*dt
% See lord's book
% where        kappa=[2^2, 2^3, 2^4, 2^5, 2^6];
%
% Examine strong convergence at T=1:  Xerr_appr= E | Xn - X(T) |.
% Check the convergence order
% Xerr_appr = (E \|Xn -X(T)\|^2)^{1/2}  \leq C tau^{1/2},
```

```
%finite element   method
% Let 0= t_0 < t_1 < \dots < t_N=T  be the time partition of [0,T] and tau
% the time step size.
% Let 0< x_0< x_1< \dots x_M =1 be the space partition of [0, 1] and h the
% space step size
% Let u-u0- t*u1=v
% v satisfies the following abstract form
%  D^{al} v(t)  + A v  = -A u0 - t A u1 + f(t) +g(t), t>0
%  v(0) = 0,

% The variational form is to find v such that
% (D^{al} v(t), \phi) + (v', \phi')
% =- (u0', \phi') - t*(u1', \phi') +  (f, \phi) +(g, \phi) ,
%  \forall \phi \in H_{0}^{1}(0,1)
% The finite element method is to find vh  such that
% (D^{alpha} vh(t), \chi) + (vh', \chi')
% = - (u0', \phi')-(u1',\phi')+(f, \chi)+ (g, \chi) , \forall \chi \in S_h
% The numerical scheme is to find V^n such that
% ( \tau^{-\alpha} \sum_{j=1}^{n} w_{n-j} V^{j},  \chi)
% + ( U^{n}', \chi') = - (u0', \phi')- t_n * (u1', \phi') +(f^{n}, \chi)
%    +(g^{n}, \chi), \forall \chi \in S_h
% Let
% V^{n} = \sum_{m=1}^{M-1} \alpha_{m}^{n}  \phi_m
% we get
%  w_{0} \sum_{m=1}^{M-1} \alpha_{m}^{n} ( \phi_m, \phi_l)
% + \tau^{\alpha} \sum_{m=1}^{M-1} \alpha_{m}^{n} ( \phi_m', \phi_l')
%= - \tau^{\alpha} \sum_{m=1}^{M-1} \alpha_{m}^{0} ( \phi_m', \phi_l')
% + \tau^{\alpha} (f^n}, \phi_l) + \tau^{\alpha} (g^n}, \phi_l)
% + \sum_{j=1}^{n-1} w_{n-j} \sum_{m=1}^{M-1}
```

```
%  \alpha_{m}^{j} ( \phi_m, \phi_l), l=1, 2, \dots, M-1
%
% Matrix form
% (w_{0}I+ tau* Ah) alpha^n =
% - tau^{alpha}*Ah alpha^0
% + Mass^{-1} \tau^{alpha} * f^n + Mass^{-1} \tau^{alpha} * g^n
% - \sum_{j=1}^{n-1} w_{n-j} alpha^{j}
% where
% Ah= Mass^{-1}*Stiffness,
% the integral (f_{n}, phi_{l}) is calculated by the midpoint rule
% (f^{n})(l)  = (f_{n}, phi_{l})
% = h/2*[ f_{n}((x_{l-1}+x_{l})/2)+ f_{n}((x_{l}+x_{l+1})/2)]
% Here Mass = h * [2/3   1/6   0   0 ...     0
%                  1/6   2/3  1/6  0 ...     0
%                  .....
%                  0       0   ...    2/3    1/6
%                  0       0 ...      1/6    2/3 ]
% Stiffness = 1/h*     [2   -1  0  0 ...  0
%                      -1   2  -1 0 ...  0
%                      .....
%                      0   0 ...    2   -1
%                      0   0 ...    -1   2]

% Remark 1: The algorithm of fpde is similar as the algorithm for pde,
% please see the MATLAB code for parabolic pdes
% Remark 2: Initial value u0 is better than Ph(u0) in experiements,We shall
% choose alpha^0(k) = u0 (x_{k})  instead of alpha^0(k) = Ph(u0) (x_{k})

% correction algorithm for recovering the optimal convergence orders
% Matrix form for the correction algorithm, n=1, 2, 3, ..., N,
% (w_{0}I+ tau* Ah) alpha^n
```

```
% =
% - tau^{alpha}*Ah alpha^0 *(1+c0)
% + Mass^{-1} \tau^{alpha} * f^n  + Mass^{-1} \tau^{alpha} * g^n
% - \sum_{j=1}^{n-1} w_{n-j} alpha^{j}
% where
% c0 = 1/2, n=1
%    = 0,   n=2, 3, \dots, N




%Algorithm;
%Step 1, Given initial value U^0 ( U0 =u0, not Ph (u0))
%Step 2, Find U^1
%Step 3: Find U^2

clear
alpha   =default(' 1 <alpha <=2  (default is alpha=1.6)', 1.6);
ga      =default('ga \in [0, 1],  (default is alpha=0.6)', 0.6);
M1      =default('M1=No of the simulations (default is M1=20)', 20);
beta =default('beta=3(trace class)or 0(white noise)(default is beta=3)',3);

randn('state',100)

%some parameters
c0=0;   %correction parameter
H=0.5;   %Standard Brownian motion

%space discretization
x0=0; x1=1; M=2^6;
h=(x1-x0)/M; x=linspace(x0,x1,M+1); x=x(2:end-1); x=x';
```

```
%time discretization
t0=0; T=0.1;
Nref = 2^7;

error_M1=[];
 for s = 1:M1    % M1: the number of the simulations
    s
    dtref=T/Nref;

   % Brownian paths W=[B_1(t), B_2(t),...B_{M-1}(t)
   % dW=sqrt(dtref)*randn(Nref,M-1); W=cumsum(dW,1); W=[zeros(1, M-1); W];

     % Fractional Brownian paths 1/2 \leq H \leq 1
     W=[];
     for j=1:M-1
         [Wj,t]=fbm1d(H,Nref,T);
         W=[W Wj];
     end
     W(1,:)=zeros(1, M-1);

% exact solution
kappa =1;
[vref]=sfpde(alpha,Nref,M,c0,ga,T,kappa,W,beta);

%approximate solutions
kappa =[2^5, 2^4, 2^3, 2^2];    % N=Nref/kappa, tau=T/N
error=[];
for i=1:length(kappa)
   [V]=sfpde(alpha,Nref,M,c0,ga,T,kappa(i),W,beta);
    err=vref-V; err_L2norm=sqrt((err'*err)*h);
    error = [error err_L2norm];
```

```
end
error_M1=[error_M1; error];  %error_M1 is a matrix
end
error_M1_L2= sqrt(sum(error_M1.*error_M1,1)/M1); % \|e \|_{L^2(D)}



%Find the convergene orders
ratio=[];
for j=1:length(error_M1_L2)-1
    ratio=[ratio error_M1_L2(j)/error_M1_L2(j+1)];
end
format short
ratio= log2(ratio);

ratio
mean(ratio)   % Show the ratios and mean of the rations

format shortE
error_M1_L2               % Show the errors



%plot
figure(1)

Dt=dtref*kappa; y=error_M1_L2;
plot(log2(Dt), log2(y),'*-')
hold on
r=mean(ratio);
plot(log2(Dt),  r*log2(Dt))    %order is 1
xlabel('log2(\Delta t)')
ylabel('log2(error)')
```

```
title('A plot of the error at T=0.1 against log2 (\Delta t)')
x=log2(Dt); x1=x(2); y1= r*x1;
% text(x1,y1,'\leftarrow  reference line with slope 1',...
%     'HorizontalAlignment','left')
text(x1,y1,'\leftarrow  reference line',...
     'HorizontalAlignment','left')
```

```
%main program
function [vref]=sfpde(alpha,Nref,M,c0,ga,T,kappa,W,beta);
x0=0; x1=1; h=(x1-x0)/M;
x=linspace(x0, x1, M+1); x=x(2:end-1); x=x';

N = Nref/kappa;
tau=T/N;

% construct matrix A for finite element method
Mass=2/3*diag(ones(1,M-1))+1/6*diag(ones(1,M-2),1)+1/6*diag(ones(1,M-2),-1);
Mass=h*Mass;
Stiffness=2*diag(ones(1,M-1))+(-1)*diag(ones(1,M-2),1)+(-1)*diag(ones(1,M-2),-1);
Stiffness=(1/h)*Stiffness;
Ah=inv(Mass)*Stiffness;


% v0
v0=zeros(M-1,1);
```

```
%Find L2 projection of u0.
%We may use Ph_u0=Ph1(x,u0,Mass) to calculate Ph_u0, but the results
%are not good.We simply use u0 instead of Ph (u0)
% to avoid calculate Mass^{-1}.

% Case 1: u0 = x^2(1-x)^2;
 Ph_u0=(x.^2).*((1-x).^2);
 %Ph_u1=(x.^2).*((1-x).^2);
 Ph_u1=sin(2*pi*x);

 % Case 2: u0 = x(1-x);
 %Ph_u0=(x.^1).*((1-x).^1);
 %Ph_u1=sin(2*pi*x);


%case 3: u0 = ones(M-1, 1);
% Ph_u0=ones(M-1,1);

%case 4: u0 = sin(pi x);
%Ph_u0 =sin(pi*x);

%case 4: u0 = 0;
% Ph_u0 =zeros(M-1,1);


vN=[v0];
% First time level, n=1
% Find v1= the value at t1, (correction c0=0.5, No correction c0=0)
   t1= 1*tau;
```

```
   initial =-(1+c0)*tau^alpha*Ah*Ph_u0 -tau^alpha*t1*Ah*Ph_u1;
   f1=inv(Mass)*f_phi(x,1,tau,alpha);
   g1=inv(Mass)*g_phi(x,1,tau,ga,kappa,W,beta);
   v=(w_gru(0,alpha)*eye(M-1)+tau^alpha*Ah)\(tau^(alpha)*f1 ...
       +tau^(alpha)*g1 + initial);
    vN=[v vN];
% find v2, v3, .... vN
for n=2:N
    sum1=0;
    for j=1:n
        sum1=sum1+w_gru(j,alpha)*vN(:,j);  %summation
    end
    tn=n*tau;
    initial =-tau^alpha*Ah*Ph_u0-tau^alpha*tn*Ah*Ph_u1;       %u0
    fn=inv(Mass)*f_phi(x,n,tau,alpha);  %f
    gn=inv(Mass)*g_phi(x,n,tau,ga,kappa,W,beta); %g
  v=(w_gru(0,alpha)*eye(M-1)+tau^alpha*Ah)\(tau^(alpha)*fn ...
      +tau^(alpha)*gn - sum1 + initial);
    vN=[v vN];
end
vref=v;



%weights L1 scheme
    function y=w(k,alpha)
        if k==0
            y=1/gamma(2-alpha);
        elseif j==1 && k==j
            y=-alpha/gamma(2-alpha);
        elseif  k==1 && j>=2
```

```
            y=(2^(1-alpha)-2)/gamma(2-alpha);
        elseif k>=2 && k<=j-1
            y=((k-1)^(1-alpha)+(k+1)^(1-alpha)-2*k^(1-alpha))/gamma(2-alpha);
        else  k==j && j>=2;
            y=((k-1)^(1-alpha)-(alpha-1)*k^(-alpha)-k^(1-alpha))/gamma(2-alpha);
        end
```

```
% find the L2 projection of u0
function y=Ph1(x,u0,Mass)
        h=x(1)-x(2);
        x0=[0;x(1:end-1)]; x1=x; x2=[x(2:end); x(end)+h];
        x=(x0+x1)/2;
        y1=u0;
        x=(x1+x2)/2;
        y2=u0;
        y=h/2*(y1+y2);
        y=inv(Mass)*y;

% nonsmooth initial data u0
 function y=Ph2(x,Mass)
     h=x(1)-x(2);
     M=length(x)+1;
     y= zeros(M-1,1);
     for j=1:length(M-1)
         if j <= ceil(M/2)
             y(j)=0;
         else
             y(j)=h;
         end
```

```
  end
  y=inv(Mass)*y;



% find (f, phi)
  function y=f_phi(x,n,tau,alpha)
      % case 1: f(t, x) = x^2 (1-x)^2 exp(t)-(2-12 x+12 x^2) exp(t)
      tn=n*tau;
      h=x(2)-x(1);
      x0=[0;x(1:end-1)]; x1=x; x2=[x(2:end); x(end)+h];
      x=(x0+x1)/2;
      y1=(x.^2).*((1-x).^2)*exp(tn)-(2-12*x+12*(x.^2))*exp(tn);
      x=(x1+x2)/2;
      y2=(x.^2).*((1-x).^2)*exp(tn)-(2-12*x+12*(x.^2))*exp(tn);
      y=h/2*(y1+y2);

     %case 2: f(t, x)=0
      %y=zeros(size(x));   %f=0


 % find (g, phi)
  function y=g_phi(x,n,tau,ga,kappa,W,beta)
      y=[];
      M=length(x)+1;
      %Find w_ga=[w_{0}^{-ga} w_{1}^{-ga}  w_{n-1}^{-ga}]
      w_ga=[];
      for nn=0:n-1
          w_ga=[w_ga w_gru(nn,-ga)];
      end
```

```
        for k=1:M-1
            A=dWdt_k(x,n,tau,kappa,W,k,beta);
            y1=tau^(ga)*w_ga*A;
            y=[y;y1];
        end

    % Find dWdt_k
 function y= dWdt_k(x,n,tau,kappa,W,k,beta)
        y=zeros(n,1);
        M=length(x)+1;
        for m=1:M-1
           % beta=0; white noise beta=0, trace class beta=3
            ga_m=m^(-beta);
            k1=n:-1:1;  %tn=n*tau=(n*kappa)*dtref
            dW_k1=W(k1*kappa+1,m)-W((k1-1)*kappa+1,m); %dW_k is a vector
            h=x(2)-x(1);
            x1=((k-1)*h+k*h)/2; x2= (k*h+(k+1)*h)/2;
            e_phi=h/2*(sqrt(2)*sin(pi*m*x1)+sqrt(2)*sin(pi*m*x2));
            y=y+ga_m^(1/2)*e_phi*(dW_k1/tau);
        end

 function [ y ] =  w_gru(k,al)
% Grunwald-Letnikov weights


if k==0;
    y=1;
elseif k==1;
    y=-al;
else
y=al;
for l=1:k-1
```

```
y=y*(al-1);
end
y=(-1)^k/factorial(k)*y;
end
```

```
 function [W,t]=fbm1d(H,n,T)
%
%https://de.mathworks.com/matlabcentral/fileexchange
%/38935-fractional-brownian-motion-generator

%fast one dimensional fractional Brownian motion (FBM) generator
% output is 'W_t' with t in [0,T] using 'n' equally spaced grid points;
% code uses Fast Fourier Transform (FFT) for speed.
% INPUT:
%          - Hurst parameter 'H' in [0,1]
%          - number of grid points 'n', where 'n' is a power of 2;
%            if the 'n' supplied is not a power of two,
%            then we set n=2^ceil(log2(n)); default is n=2^12;
%          - final time 'T'; default value is T=1;
% OUTPUT:
%          - Fractional Brownian motion 'W_t' for 't';
%          - time 't' at which FBM is computed;
%            If no output it invoked, then function plots the FBM.
% Example: plot FBM with hurst parameter 0.95 on the interval [0,10]
% [W,t]=fbm1d(0.95,2^12,10); plot(t,W)
% Reference:
% Kroese, D. P., & Botev, Z. I. (2015). Spatial Process Simulation.
% In Stochastic Geometry, Spatial Statistics and Random Fields(pp. 369-404)
% Springer International Publishing, DOI: 10.1007/978-3-319-10064-7_12
```

```
if (H>1)|(H<0) % Hurst parameter error check
    error('Hurst parameter must be between 0 and 1')
end
if nargin<2
    n=2^12;  % grid points
else
    n=2^ceil(log2(n));
end
if nargin<3
    T=1;
end
r=nan(n+1,1);r(1) = 1;idx=1:n;
r(idx+1) = 0.5*((idx+1).^(2*H) - 2*idx.^(2*H) + (idx-1).^(2*H));
r=[r; r(end-1:-1:2)]; % first rwo of circulant matrix
lambda=real(fft(r))/(2*n); % eigenvalues
W=fft(sqrt(lambda).*complex(randn(2*n,1),randn(2*n,1)));
W = n^(-H)*cumsum(real(W(1:n+1))); % rescale
W=T^H*W; t=(0:n)/n; t=t*T; % scale for final time T
if nargout==0
    plot(t,W); title('Fractional Brownian motion');
    xlabel('time $t$','interpreter','latex')
    ylabel('$W_t$','interpreter','latex')
end
```

```
function reply = default(query,value)
%default   gets response to IFISS prompt
%   reply = default(query,value);
%   input
%          query   character string: asks a question
%          value   integer: the default response
```

```
%
%   IFISS function: AR; 31 August 2005.
% Copyright (c) 2005 D.J. Silvester, H.C. Elman, A. Ramage (see readme.m)
global BATCH FID
if exist('BATCH') & BATCH==1,
   replycell=textscan(FID,'%f%*[^\n]',1);
   reply=deal(replycell{:});
   disp(query)
   disp(reply)
else
   reply=input([query,' : ']);
   if isempty(reply), reply=value; end
end
return
```