Type-theoretic weak factorization systems

Paige Randall North*

June 4, 2019

Abstract

This article presents three characterizations of the weak factorization systems on finitely complete categories that interpret intensional dependent type theory with Σ -, Π -, and Id-types. The first characterization is that the weak factorization system (L, R) has the properties that L is stable under pullback along R and that all maps to a terminal object are in R. We call such weak factorization systems type-theoretic. The second is that the weak factorization system has an Id-presentation: roughly, it is generated by Id-types in the empty context. The third is that the weak factorization system (L, R) is generated by a Moore relation system, a generalization of the notion of Moore paths.

Contents

1	Inti	roduction	2
2	\mathbf{Pre}	liminaries	3
	2.1	Relations, relational factorizations, and factorizations	4
	2.2	Weak factorization structures and systems	
	2.3	Display map categories	
	2.4	Summary	16
3	Typ tion	pe-theoretic weak factorization systems from Moore rela- ns	16
	3.1	Strict Moore relation systems	17
		3.1.1 Strictly transitive functorial relations	17
		3.1.2 Strictly homotopical functorial relations	20
		3.1.3 Strictly symmetric functorial relations	23
		3.1.4 Summary	25
	3.2	Moore relation systems	26
		3.2.1 Transitive relations	26
		3.2.2 Homotopical relations	28

 $^{^*}$ This material is based upon work supported by the Air Force Office of Scientific Research under award number FA9550-16-1-0212.

	3.2.3 Symmetric relations	
4	Id-presentations from type-theoretic weak factorization systems 4.1 The main tool	34
5	${\bf Moore\ relations\ from\ type-theoretic\ weak\ factorization\ systems}$	41
6	Models of Id-types	45
7	The main result	47
Re	eferences	49

1 Introduction

This paper is the second in a series (based upon the author's thesis [Nor17]) in which we study categorical interpretations of dependent type theory of a certain species: display map categories. It has long been known that categorical interpretations of dependent type theory induce weak factorization systems [GG08], and so our goal is to characterize the weak factorization systems which harbor such interpretations.

In the first paper [Nor19], we considered display map categories modeling Σ and Id-types whose underlying category is Cauchy complete. We showed there
that the induced weak factorization system is *itself* a display map category
modeling Σ - and Id-types. This simplifies our problem: if we want to decide
whether a given weak factorization system is induced by such a display map
category modeling Σ - and Id-types, then we only need to decide whether the
weak factorization system itself is a display map category modeling Σ - and
Id-types.

In the present paper, we turn to this problem: deciding whether a weak factorization system is a display map category modeling Σ - and Id-types. Our main theorem is the following characterization:

Theorem 7.2. Consider a category C with finite limits. The following properties of any weak factorization system (L,R) on C are equivalent:

- 1. it has an Id-presentation;
- 2. it is type-theoretic;
- 3. it is generated by a Moore relation system;
- 4. (C,R) is a display map category modeling Σ and Id-types.

A weak factorization system (L,R) is $\mathit{type\text{-}theoretic}$ when all morphisms to the terminal object are in R (a necessary condition for (C,R) to be a display map

category) and when L is stable under pullback along R (a necessary condition for (C,R) to model Π -types). Thus, the equivalence between (2) and (4) tells us that a category with a weak factorization system is a display map category modeling Σ - and Id-types just when these two conditions hold.

To prove this equivalence, we introduce the notions of Id-presentations and, perhaps more interestingly, of Moore relation systems. Roughly, a weak factorization has an Id-presentation if it is induced by a model of Id-types. A Moore relation system is an explicit algebraic presentation of the weak factorization systems under consideration. These are closely related to the path object categories of [vdBG12] which are used to model identity types. However, a Moore relation system is a weaker notion than that of path object category, and what we lose in strictness, we make up for in the equivalence above.

This paper is organized as follows. Throughout, we fix a finitely complete category C. In Section 2, we introduce the objects of study and the categories that contain them. In particular, we describe the categories Fact(C) of factorizations and Rel(C) of relations, their full subcategories $ttWFS(C) \hookrightarrow Fact(C)$ of type-theoretic weak factorization systems and $IdPres(C) \hookrightarrow Rel(C)$ of Idpresentations, and functors $\mathfrak{F}: Rel(C) \hookrightarrow Fact(C): \mathfrak{R}$. We then show in the following sections that these functors restrict to an equivalence $|\mathfrak{F}|: |IdPres(C)| \simeq$ $|ttWFS(C)|: |\Re|$ where | - | denotes the proset truncation. We use the proset truncation because it makes isomorphism in |ttWFS(C)| the usual notion of sameness between weak factorization systems (that is, having the same left and right classes of morphisms). In Section 3, we describe Moore relation systems and show for that every Moore relation system R, the factorization $\mathfrak{F}(R)$ that it produces is a type-theoretic weak factorization system. In Section 4, we show that \Re restricts to a functor ttWFS(C) \rightarrow IdPres(C) and that \Re restricts to a functor $ttWFS(C) \rightarrow ttWFS(C)$ which is isomorpic to the identity functor under the proset truncation. In Section 5, we show that a relation is a Moore relation system if and only if it is an Id-presentation and that $\Re \mathfrak{F}: IdPres(C) \to IdPres(C)$ is isomorphic to the identity functor under proset truncation. We show in Section 6 that having an Id-presentation is equivalent to modeling Id-types. In Section 7, we put these results together to obtain Theorem 7.1.

2 Preliminaries

This section is devoted to developing the concepts that we will study in the following sections. Our main theorem, 7.1, is a comparison of a certain kind of relation (models of Id-types) and a certain kind of factorization (weak factorization systems). We start with a categorical analysis of such relations and factorizations, and then we define the particular instances in which we are interested.

As mentioned above, we fix a finitely complete category C throughout this paper.

2.1 Relations, relational factorizations, and factorizations

In this subsection, we define the fundamental objects: relations on C, factorizations on C, relational factorizations on C, and functors between them.

Definition 2.1.

a) Let R denote the category generated by the graph

$$\bigcirc \stackrel{\epsilon_0}{\underset{\epsilon_1}{\longleftarrow}} \Psi$$

and the equations $\epsilon_0 \eta = \epsilon_1 \eta = 1_{\text{O}}$. A relation on an object X of a category C is a functor $R : \mathbb{R} \to \mathbb{C}$ such that $R(\mathbb{O}) = X$.

b) Let P denote the category generated by the graph

$$0 \xrightarrow{\kappa} \Phi \xrightarrow{\rho} 0$$

and the equation $\kappa\lambda = 1_{\text{O}}$. A relational factorization of a morphism $f: X \to Y$ in a category C is a functor $P: P \to C$ such that $P(\rho\lambda) = f$.

c) Let ${\sf F}$ denote the category generated by the following graph and no equations.

$$\bigcirc \xrightarrow{\lambda} \Phi \xrightarrow{\rho} \bigcirc$$

A factorization of a morphism $f: X \to Y$ in a category C is a functor $F: \mathsf{F} \to \mathsf{C}$ such that $F(\rho\lambda) = f$.

Remark 2.2. What we have defined above as a *relation* could more descriptively be called an internal reflexive pseudo-relation. However, since all relations will be of this type, we will just call them relations.

Example 2.3. Consider an exponentiable object I and morphisms $0, 1 : * \rightrightarrows I$ of C where * is a terminal object. Let $! : I \to *$ denote the unique morphism to the terminal object.

a) On any object X of $\mathsf{C},$ there is a relation whose image is the following diagram.

$$X \underset{X^1}{\overset{X^0}{\rightleftharpoons}} X^I$$

b) Consider a morphism $f: X \to Y$ in C . Let $X \times_Y Y^I$ denote the pullback

$$\begin{array}{ccc} X \times_Y Y^I \longrightarrow Y^I \\ \downarrow & & \downarrow_{Y^0} \\ X & \xrightarrow{f} & Y \end{array}$$

of $f: X \to Y$ and $Y^0: Y^I \to Y$. Then

$$X \xrightarrow[1_X \times Y^! f]{\pi_X} X \times_Y Y^I \xrightarrow{Y^1 \pi_{Y^I}} Y$$

is the image of a relational factorization of f, which we denote by $P:\mathsf{R}\to\mathsf{C}.$

c) We also obtain a factorization $F:\mathsf{F}\to\mathsf{C}$ of f whose image is depicted below.

$$X \xrightarrow{1_X \times Y^! f} X \times_Y Y^I \xrightarrow{Y^1 \pi_{Y^I}} Y$$

Notation 2.4. Let M denote the category generated by the graph $0 \to 1$. Then C^M is the category of morphisms of C. For a commutative square in C as shown below, let $\langle \alpha, \beta \rangle : f \to g$ denote the morphism that this produces in C^M .

$$X \xrightarrow{\alpha} W$$

$$f \downarrow \qquad \qquad \downarrow g$$

$$Y \xrightarrow{\beta} Z$$

Definition 2.5.

- a) A functorial relation on C is a section of the functor $C^{\circ}: C^{R} \to C$. Let fRel(C) denote the category of such sections.
- b) A functorial relational factorization on C is a section of the functor $C^{\rho\lambda}$: $C^P \to C^M$. Let fRelFact(C) denote the category of such sections.
- c) A functorial factorization on C is a section of the functor $C^{\rho\lambda}: C^F \to C^M$. Let fFact(C) denote the category of such sections.

Example 2.6. Consider Example 2.3. Since the relations, relational factorizations, and functorial factorizations given there are assembled from functors, these generate a functorial relation, a functorial relational factorization, and a functorial factorization on the category C.

Now we take pains to consider variants of these concepts that are not functorial. This is because the Id-types of Martin-Löf type theory are not given functorially, and we aim to model these.

Definition 2.7. Let C and D be categories. An *afunctor* $U: \mathsf{C} \dashrightarrow \mathsf{D}$ consists of an object U(X) of D for every object X of C and a morphism $U(f): U(X) \to U(Y)$ for every morphism $f: X \to Y$ in C.

Let $U, V : \mathsf{C} \dashrightarrow \mathsf{D}$ be afunctors. An (unnatural) transformation $\tau : U \dashrightarrow V$ consists of a morphism $\tau_X : U(X) \to V(X)$ in D for every X in C .

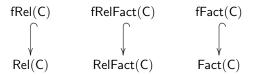
Categories and afunctors comprise a 1-category aCat which contains the 1-category Cat of categories and functors as a wide subcategory. We will use the fact that the unnatural transformations give every hom-set in aCat the structure of a category. Note, however, that this does not make aCat a 2-category.

Definition 2.8.

- a) A relation on C is a section of the functor $C^{O}: C^{R} \to C$ in aCat. Let Rel(C) denote the category of such sections.
- b) A relational factorization on C is a section of the functor $C^{\rho\lambda}: C^P \to C^M$ in aCat. Let RelFact(C) denote the category of such sections.
- c) A functorial factorization on C is a section of the functor $C^{\rho\lambda}: C^F \to C^M$ in aCat. Let Fact(C) denote the category of such sections.

Remark 2.9. We are abusing terminology by speaking of relations, relational factorizations, and factorizations both on an object of a category and on the whole of a category.

There are the following natural inclusions.



We now describe the functors that fit horizontally into this diagram.

There are functors

$$F \xrightarrow{\iota} P \xrightarrow{\sigma} R$$

where ι is the only injection $\mathsf{F} \hookrightarrow \mathsf{P}$ and σ is the surjection sending κ to ϵ_0 , ρ to ϵ_1 , and η to λ .

The functor ι induces a functor $\iota_*: \mathsf{RelFact}(\mathsf{C}) \to \mathsf{Fact}(\mathsf{C})$ given by post-composition with $\mathsf{C}^\iota: \mathsf{C}^\mathsf{P} \to \mathsf{C}^\mathsf{F}$. This restricts to a functor $\iota_*: \mathsf{fRelFact}(\mathsf{C}) \to \mathsf{fFact}(\mathsf{C})$ making the right-hand square in the diagram below (2.10) commute. This functor takes a relational factorization on C to its underlying factorization on C .

Let $!: M \to *$ denote the unique morphism from the category M to the terminal category *. Since the following is a pullback diagram, pulling back along $C^!$ produces a functor $\sigma^* : RelFact(C) \to Rel(C)$.

$$\begin{array}{ccc}
C^{R} & \xrightarrow{C^{\sigma}} & C^{P} \\
C^{\circ} & & \downarrow^{C^{\rho\lambda}} \\
C & \xrightarrow{C^{!}} & C^{M}
\end{array}$$

This restricts to a functor σ^* : fRelFact(C) \rightarrow fRel(C) making the left-hand square in diagram (2.10) commute.

Construction 2.11. Consider an $R \in \text{Rel}(C)$ and an $f: X \to Y$ in C. We construct a relational factorization $\sigma_*(R)(f) \in C^P$ of f. If we denote R(Y) by the following diagram,

$$Y \stackrel{\epsilon_0}{\underset{\epsilon_1}{\longleftarrow}} \Psi Y$$

then we let $\sigma_*(R)(f)$ be the following diagram

$$X \xrightarrow[1_X \times \eta f]{\pi_X} X \times_Y \Psi Y \xrightarrow{\epsilon_1 \pi_{\Psi Y}} Y$$

where $X \times_Y \Psi Y$ is the following pullback.

$$\begin{array}{cccc} X \times_Y \Psi Y \longrightarrow \Psi Y \\ & \downarrow & & \downarrow \epsilon_0 \\ X & \xrightarrow{f} & Y \end{array}$$

Lemma 2.12. The construction above (2.11) assembles into a functor σ_* : $Rel(C) \rightarrow RelFact(C)$ with functions

$$i: \hom_{\mathsf{Rel}(\mathsf{C})}(\sigma^*P, R) \leftrightarrows \hom_{\mathsf{RelFact}(\mathsf{C})}(P, \sigma_*R) : j$$

for all $P \in \mathsf{RelFact}(\mathsf{C})$ and $R \in \mathsf{Rel}(\mathsf{C})$. The functor σ_* restricts to a functor σ_* : $\mathsf{fRel}(\mathsf{C}) \to \mathsf{fRelFact}(\mathsf{C})$, and the functions i, j restrict to a bijection

$$i : \hom_{\mathsf{fRel}(\mathsf{C})}(\sigma^* P, R) \cong \hom_{\mathsf{fRelFact}(\mathsf{C})}(P, \sigma_* R) : j$$

natural in P and R, making $\sigma_*: \mathsf{fRel}(\mathsf{C}) \to \mathsf{fRelFact}(\mathsf{C})$ right adjoint to $\sigma^*: \mathsf{fRelFact}(\mathsf{C}) \to \mathsf{fRel}(\mathsf{C})$.

Remark 2.13. The universal property of σ_* can be interpreted as saying that $\sigma_* R$ is the right Kan extension of $\mathsf{C}^\sigma R : \mathsf{C} \to \mathsf{C}^\mathsf{P}$ along $\mathsf{C}^! : \mathsf{C} \to \mathsf{C}^\mathsf{M}$ in $\mathcal{C}at/\mathsf{C}^\mathsf{M}$ [Nor17, Thm. 3.1.44].

Proof of Lemma 2.12. The functoriality of σ^* is straightforward to check. We construct i and j and check that, when restricted to functorial relations and relational factorizations, they form a natural bijection.

Consider a $P \in \mathsf{RelFact}(\mathsf{C})$ which takes an $f: X \to Y$ in C to the diagram on the left below, and a $R \in \mathsf{Rel}(\mathsf{C})$ which takes $X \in \mathsf{C}$ to the diagram on the right below.

$$X \xrightarrow[\lambda_f]{\kappa_f} \Phi f \xrightarrow[\rho_f]{\rho_f} Y \qquad X \xrightarrow[\epsilon_{1X}]{\epsilon_{0X}} \Psi X$$

First, we construct a function $i: \hom_{\mathsf{Rel}(\mathsf{C})}(\sigma^*P, R) \to \hom_{\mathsf{Rel}\mathsf{Fact}(\mathsf{C})}(P, \sigma_*R)$. An element $\alpha \in \hom_{\mathsf{Rel}(\mathsf{C})}(\sigma^*P, R)$ has at each $X \in \mathsf{C}$, a component of the form shown on the left below.

$$X \xrightarrow{\kappa_{1_{X}}} \Phi 1_{X} \qquad X \xrightarrow{\kappa_{f}} \Phi f \xrightarrow{\rho_{f}} Y$$

$$\parallel \xrightarrow{\epsilon_{0_{X}}} \downarrow a_{X} \qquad \parallel \xrightarrow{\kappa_{f} \times a_{Y} \Phi \langle f, 1_{Y} \rangle} \parallel$$

$$X \xrightarrow{\epsilon_{0_{X}}} \Psi X \qquad X \xrightarrow{\epsilon_{1_{X}}} X \times_{Y} \Psi Y \xrightarrow{\epsilon_{1} \pi_{\Psi Y}} Y$$

Let $i\alpha \in \text{hom}_{\mathsf{RelFact}(\mathsf{C})}(P, \sigma_* R)$ be the transformation with the component at each $f: X \to Y$ in C shown on the right above.

Now, we construct a function $j: \hom_{\mathsf{RelFact}(\mathsf{C})}(P, \sigma_*R) \to \hom_{\mathsf{Rel}(\mathsf{C})}(\sigma^*P, R)$. An element $\beta: \hom_{\mathsf{RelFact}(\mathsf{C})}(P, \sigma_*R)$ has at each $f: X \to Y$ in C , a component of the form shown on the left below.

Let $j\beta \in \text{hom}_{\mathsf{Rel}(\mathsf{C})}(\sigma^*P, R)$ be the transformation with the component at each $X \in \mathsf{C}$ shown on the right above.

Now suppose that P is in $\mathsf{fRelFact}(\mathsf{C})$. To show that $ji\alpha = \alpha$, we show that the only nontrivial component of α_X for each $X \in \mathsf{C}$ is a_X . We calculate:

$$\pi_{\Psi X}(\kappa_{1_X} \times a_X \Phi \langle 1_X, 1_X \rangle) = a_X \Phi \langle 1_X, 1_X \rangle$$
$$= a_X 1_{\Phi 1_X}$$
$$= a_X.$$

Now to show that $ij\beta = \beta$, we calculate

$$\kappa_f \times \pi_{\Psi Y} b_{1_Y} \Phi \langle f, 1_Y \rangle = \kappa_f \times \pi_{\Psi Y} b_f$$
$$= b_f.$$

Thus, $j = i^{-1}$.

Now we show that j is natural in P and R. Consider natural transformations $p: P' \to P$ and $r: R \to R'$. We want to show that the following diagram commutes.

$$\begin{aligned} \hom_{\mathsf{Rel}(\mathsf{C})}(\sigma^*P,R) & \longleftarrow^{j} \hom_{\mathsf{RelFact}(\mathsf{C})}(P,\sigma_*R) \\ \bigvee_{r \circ - \circ \sigma^*p} & \bigvee_{\sigma_*r \circ - \circ p} \\ \hom_{\mathsf{Rel}(\mathsf{C})}(\sigma^*P',R') & \longleftarrow^{j} \hom_{\mathsf{RelFact}(\mathsf{C})}(P',\sigma_*R') \end{aligned}$$

Denote the component of $p: P' \to P$ at a morphism $f: X \to Y$ in C by the diagram below on the left, and denote the component of $r: R \to R'$ at $X \in \mathsf{C}$ by the diagram below on the right.

$$X \xrightarrow{\kappa'_{f}} \Phi' f \xrightarrow{\rho'_{f}} Y$$

$$\parallel \xrightarrow{\kappa_{f}} \psi \qquad \qquad X \xrightarrow{\epsilon_{0X}} \Psi X$$

$$X \xrightarrow{\kappa_{f}} \psi \qquad \qquad \parallel \xrightarrow{\epsilon'_{0X}} \psi \qquad \qquad X \xrightarrow{\epsilon'_{1X}} \psi X$$

$$X \xrightarrow{\kappa'_{f}} \psi \qquad \qquad X \xrightarrow{\epsilon'_{1X}} \psi' X$$

$$X \xrightarrow{\epsilon'_{1Y}} \psi' X$$

For any $\beta \in \text{hom}_{\mathsf{RelFact}(\mathsf{C})}(P, \sigma_* R)$ (with components as depicted in left-hand diagram of (*) above), $r \circ j\beta \circ \sigma^* p$ and $j(\sigma_* r \circ \beta \circ p)$ both have the following component at an object $X \in \mathsf{C}$.

$$X \xrightarrow{\kappa'_{1X}} \Phi' 1_{X}$$

$$\parallel \xrightarrow{\rho'_{1X}} \Phi' 1_{X}$$

$$\downarrow r_{X} b_{1_{X}} p_{1_{x}}$$

$$X \xrightarrow{\epsilon'_{1X}} \Psi' X$$

Thus, j is natural in P and R.

Example 2.14. Consider Example 2.6, and let R denote the relation discussed there. Then σ_*R is the relational factorization of that example.

Construction 2.15. Consider an $F \in \mathsf{Fact}(\mathsf{C})$ and an $f: X \to Y$ in C . We construct a factorization $\iota^*(F)(f) \in \mathsf{C}^\mathsf{F}$ of f. Consider the morphism $1_X \times f: X \to X \times Y$. Denote $F(1_X \times f)$ by the following diagram.

$$X \xrightarrow{\lambda} \Phi(1_X \times f) \xrightarrow{\rho} X \times Y$$

Then let $\iota^*(F)(f)$ be the following diagram.

$$X \xrightarrow[\lambda]{\pi_X \rho} \Phi(1_X \times f) \xrightarrow[\pi_Y \rho]{} Y$$

Lemma 2.16. The construction above (2.15) assembles into a functor ι^* : Fact(C) \rightarrow RelFact(C) which restricts to a functor ι^* : fFact(C) \rightarrow fRelFact(C).

Remark 2.17. Though ι^* is not adjoint to ι_* , $\iota_*\iota^*$ is a comonad on fFact(C) [Nor17, Cor. 3.1.35].

Now we have described functors

We are most interested in the functors between Rel(C) and Fact(C) (and between fRel(C) and fFact(C)). We abbreviate these as follows.

Notation 2.19. Let \mathfrak{F} denote $\iota_*\sigma_*$, and let \mathfrak{R} denote $\sigma^*\iota^*$.

2.2 Weak factorization structures and systems

In this section, we discuss some fundamentals of weak factorization systems and the perspective on them that we take. Consider a factorization F on C. There are two functors $\lambda, \rho: M \to F$ which take the non-identity morphism of M to λ and ρ , respectively. Using these, we obtain afunctors $\lambda_F, \rho_F: C^M \to C^M$ from F. Then λ_F is copointed and ρ_F is pointed in the sense that for every $f: X \to Y$ of C, there are the following morphisms in C^M .

Let F-coalg denote

$$\{f: X \to Y \text{ in } \mathsf{C} \mid \exists s: Y \to \Phi_F(f): sf = \lambda_F(f), \rho_F(f)s = 1_Y\},$$

the class of coalgebras of the copointed endo-afunctor $(\lambda_F, \langle 1, \rho_F \rangle)$. This is the class of morphisms f for which there is a lift in the square shown on the right above. Let F-alg denote

$$\{f: X \to Y \text{ in } \mathsf{C} \mid \exists s: \Phi_F(f) \to X: s\lambda_F(f) = 1_X, fs = \rho_F f\},\$$

the class of algebras of the pointed endo-afunctor $(\rho_F, \langle \lambda_F, 1 \rangle)$. This is, dually, the class of morphisms f for which there is a lift in the square above on the left. We will say that a morphism in F-coalg has an F-coalgebra structure, and that a morphism in F-algebra structure to simplify vocabulary.

Notation 2.20. For any morphisms ℓ, r in C, write $\ell \boxtimes r$ if ℓ has the left lifting property against r. For two collections L, R of morphisms of C, write L \boxtimes R if every morphism of L has the left-lifting property against every morphism of R. Write L $^{\boxtimes}$ (and dually, $^{\boxtimes}$ R) for the class of morphisms with the right lifting property against L (dually, the left lifting property against R).

Proposition 2.21. Consider a factorization F on C. Then F-coalg \square F-alg.

Proof. This appears in Corollary 2.7 of [GT06]. Though they consider only functorial factorizations, their argument works here without modification.

Definition 2.22. A weak factorization structure on C is a factorization F on C such that for every morphism f of C, $\lambda_F(f) \in F$ -coalg and $\rho_F(f) \in F$ -alg. An algebraic weak factorization structure on C is a functorial factorization F on C such that the copointed endofunctor λ_F underlies a comonad on C^M and the pointed endofunctor ρ_F underlies a monad on C^M .

Let $\mathsf{WFS}(\mathsf{C})$ denote the full subcategory of $\mathsf{Fact}(\mathsf{C})$ spanned by those objects which are weak factorization structures.

Notation 2.23. For any category D, let |D| denote the preordered truncation: the preorder (viewed as a category) which has the same objects as D and a morphism $X \to Y$ when there is a morphism $X \to Y$ in D.

For any object X of D, we will let [X] denote the isomorphism class of X in |D|, and we will say that two objects X and Y of D are equivalent if they are isomorphic in |D|.

Proposition 2.24. The isomorphism classes of |WFS(C)| are the weak factorization systems on C.

Proof. We show that there is a function q from the objects of WFS(C) to the weak factorization systems on C which is surjective and whose fibers are the isomorphism classes of |WFS(C)|.

By Theorem 2.4(2) of [RT02], for any $F \in \mathsf{WFS}(\mathsf{C})$, $(F\text{-}\mathsf{coalg}, F\text{-}\mathsf{alg})$ is a weak factorization system on C with factorization given by F. Let q denote the function which maps a weak factorization structure F to the weak factorization system $(F\text{-}\mathsf{coalg}, F\text{-}\mathsf{alg})$.

By Proposition 5.1 of [RT02], for any $F, G \in WFS(C)$, we have that

$$(F$$
-coalg, F -alg $) = (G$ -coalg, G -alg $)$

if and only if there are morphisms $F \subseteq G$. Therefore, the fibers of q are the isomorphism classes of $|\mathsf{WFS}(\mathsf{C})|$.

Consider a weak factorization system (L, R). There exists a factorization of each morphism $f: X \to Y$ in C which we can denote by the diagram on the left below.

$$X \xrightarrow{\lambda(f)} Mf \xrightarrow{\rho(f)} Y$$

$$X \xrightarrow{\lambda(g)\alpha} Mg$$

$$\lambda(f) \downarrow \qquad \qquad \downarrow \rho(g)$$

$$Mf \xrightarrow{\beta\rho(f)} Z$$

For each morphism $g:W\to Z$ and $\langle\alpha,\beta\rangle:f\to g$, we can obtain a morphism $M\langle\alpha,\beta\rangle:Mf\to Mg$ by considering the lifting diagram above on the right. This assembles into a factorization, say F, on C. By Theorem 2.4(1) of [RT02], L=F-coalg and R=F-alg. Thus, q is surjective.

(Again, though only functorial factorizations are considered in [RT02], their proofs of these results work here without modification.) \Box

2.3 Display map categories

Now we define what we consider in this paper to be a categorical interpretation of dependent type theory.

Definition 2.25. A class D of morphisms of C forms a *display map category* (C, D) when the following hold:

- 1. D contains every isomorphism;
- 2. D contains every morphism whose codomain is a terminal object; and
- 3. D is stable under pullback.

We call the elements of D display maps.

The notion of display map category is closely related to others in the literature [Tay99, Shu15, Joy17]. There is a careful comparison of this notion and of the types described below with others in the literature in [Nor19].

Definition 2.26. Let (L, R) be a weak factorization system on C. We say that an object X of C is *fibrant* if every morphism from X to a terminal object is in R.

Example 2.27. Let (L, R) be a weak factorization system on C in which all objects are fibrant. Since right classes of weak factorization systems always contain all isomorphisms and are stable under pullbacks [MP12, Prop. 14.1.8], (C, R) is a display map category.

Definition 2.28. A display map category (C, D) models Σ -types if D is closed under composition. We call a composition gf of display maps a Σ -type and sometimes denote it by $\Sigma_q f$.

Example 2.29. Let (L,R) be a weak factorization system on C in which all objects are fibrant. Since right classes of weak factorization systems are always closed under composition [MP12, Prop. 14.1.8], (C,R) models Σ -types.

Weak factorization systems are not only examples of display map categories but are also induced by ones which model Id-types.

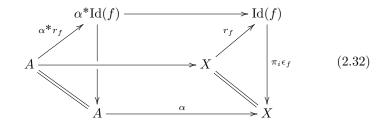
Definition 2.30. Consider a display map category (C, D) which models Σ -types, and consider a display map $f: X \to Y$. An *identity type of* f is a relation on f in the slice C/Y

$$X \xrightarrow{r_f} \operatorname{Id}(f) \xrightarrow{\epsilon_f} X \times_Y X$$

$$\downarrow^{\iota_f} \qquad f \times f \qquad (2.31)$$

such that

- 1. ϵ_f is in D and
- 2. for every morphism $\alpha: A \to X$ in C, the pullback $\alpha^* r_f$, as shown below, is in $\square D$ for i = 0, 1.



We will call the morphism $\iota_f : \mathrm{Id}(f) \to Y$ in Diagram (2.31) the Id -type of f in C/Y .

Given a morphism $\alpha: f \to g$ in C/Y and identity types of f and g, there is a natural transformation between these two relations because $r_f \in {}^{\boxtimes}\mathsf{D}$ and $\epsilon_g \in \mathsf{D}$.

$$f \xrightarrow{r_g \alpha} \iota(g)$$

$$\downarrow^{r_f} \qquad \qquad \downarrow^{\epsilon_g}$$

$$\iota(f) \xrightarrow[(\alpha \times \alpha) \in f]{} g \times g$$

Thus, in what follows, when we assume that every object of C has an Id-type, we will assume that there is a relation on C which specifies these Id-types.

Definition 2.33. Consider a display map category (C, D) which models Σ -types. If there is a relation I on C for which I(X) is an identity type on X for each object X of C, then we say that I is a model of Id-types of objects in (C, D) or just that (C, D) models Id-types of objects. If there is a relation I_Y on each slice C/Y for which $I_Y(f)$ is an identity type of f for each display map $f: X \to Y$ of C, then we say that the collection $\{I_Y\}_{Y \in C}$ is a model of Id-types in (C, D) or just that (C, D) models Id-types. If these relations are functorial, then we call the Id-types functorial.

For any display map category (C, D) which models Σ -types and has a model I of Id-types of objects, the factorization $\mathfrak{F}(I)$ (with \mathfrak{F} as defined in Notation 2.19) is a weak factorization structure, and its underlying weak factorization system $[\mathfrak{F}(I)]$ is $(^{\square}D, (^{\square}D)^{\square})$ [Emm14, Thm. 2.8]. Our goal in this paper is to understand which weak factorization systems arise in this way.

Note that $({}^{\square}D){}^{\square}$ is the retract closure of D and so in particular contains D. Thus, to decide whether a weak factorization system (L,R) on a category C does arise from a model of Σ - and Id-types, it seems that we might have to check whether (C,D) is a display map category modeling Σ - and Id-types for all D whose retract closure is R. However, we showed in [Nor19, Thm. 5.12] that

if (C, D) is a display map category modeling Σ - and functorial Id-types, then $(C, (^{\square}D)^{\square})$ is a display map category modeling Σ - and functorial Id-types (and if (C, D) modeled II-types, then so does $(C, (^{\square}D)^{\square})$). Thus, to decide whether a weak factorization system (L, R) does arise from a model of Σ - and functorial Id-types, we only have to decide if (C, R) is a display map category modeling functorial Σ - and Id-types. In this paper, we show how one can decide such a thing, and we will show that we can also drop the requirement of functoriality on Id-types.

In particular, suppose that we want to decide whether a weak factorization system (L, R) arises as $[\mathfrak{F}(I_*)]$ from a model I of (functorial) Id-types in a display map category (C, D) (where * is a terminal object of C, so that I_* is a model of Id-types of objects). If it does, (L, R) itself has a model J of (functorial) Id-types. Whenever (L, R) itself has a model J of Id-types, we have $[\mathfrak{F}(J_*)] = (^{\square}R, (^{\square}R)^{\square}) = (L, R)$. We will also see in Corollary 6.3 that (L, R) models Id-types if and only if it models Id-types of objects. Thus, a weak factorization system arises from a model of Id-types if and only if it models Id-types of objects; we call such a model an Id-presentation of the weak factorization system.

Definition 2.34. We say that a relation I on C is an Id-presentation of the weak factorization system $[\mathfrak{F}(I)]$ if the factorization $\mathfrak{F}(I)$ is a weak factorization structure and R is a model of Id-types of objects in $(C, \mathfrak{F}(R)$ -alg).

Note that for any relation I which generates a weak factorization structure $\mathfrak{F}(I)$, all objects are fibrant. Consider any object X in C . The solution shown in the following lifting problem is a $\rho_{\mathfrak{F}(I)}$ -algebra structure for $!:X\to *$.

Thus, in the definition (2.34) above, $(\mathsf{C},\mathfrak{F}(R)\text{-alg})$ is in fact a display map category, and so it makes sense to talk of models of Id-types of objects in it. We record this fact here.

Proposition 2.35. Let R be a relation on C which produces a weak factorization structure $\mathfrak{F}(R)$. Then every object is fibrant in $[\mathfrak{F}(R)]$.

If a weak factorization system is going to have an Id-presentation, then all objects must be fibrant in it. The only other condition that we will find we need to place on a weak factorization system to ensure that it does have an Id-presentation is often called the *Frobenius property* [vdBG12] and is closely related to modeling Π -types.

Definition 2.36. A display map category (C, D) models pre- Π -types if for every $g: W \to X$ and $f: X \to Y$ in D, there is a morphism $\Pi_f g$ with codomain Y satisfying the universal property

$$i: \mathsf{C}/X(f^*y,g) \cong \mathsf{C}/Y(y,\Pi_f g)$$

natural in y. Call the map $\Pi_f g$ a pre- Π -type.

A display map category (C, D) models Π -types if it models pre- Π -types and each pre- Π -type is a display map.

Proposition 2.37. Consider a weak factorization system (L, R) on C in which all objects of C are fibrant and which models pre- Π -types.

Then (C, R) models Π -types if and only if L is stable under pullback along R.

This proposition is very similar to the fact that a left adjoint between two categories with weak factorization systems preserves the left maps [MP12, §16.2] if and only if the right adjoint preserves the right maps. Our setting, however, is a bit more convoluted.

Proof of Proposition 2.37. Suppose that (C, R) models Π -types. Let i denote the bijection

$$i: \mathsf{C}/X(f^*y,g) \cong \mathsf{C}/Y(y,\Pi_f g)$$

of Definition 2.36, and consider a morphism ℓ of L and a morphism f of R such that $\text{COD}\ell = \text{COD}f$. To show that $f^*\ell$ is in L, we must show that for any factorization $\lambda(f^*\ell), \rho(f^*\ell)$ of $f^*\ell$ for which $\lambda(f^*\ell) \in \mathsf{L}, \rho(f^*\ell) \in \mathsf{R}$, the lifting problem shown on the left below has a solution.

Consider the lifting problem shown on the right above. It is the transpose of the above lifting problem under i. It has a solution σ since ℓ is in L and $\Pi_f \rho(f^*\ell)$ is in R. Then $i^{-1}(\sigma)$ gives us a solution to our original lifting problem.

Now suppose that L is stable under pullback along R. We need to show that $\Pi_f g$ is in R. The morphism $\Pi_f g$ is in R if and only if for any factorization $\lambda(\Pi_f g), \rho(\Pi_f g)$ of $\Pi_f g$ for which $\lambda(\Pi_f g) \in \mathsf{L}$ and $\rho(\Pi_f g) \in \mathsf{R}$, there is a solution to the lifting problem shown below on the left.

$$\lambda(\Pi_f g) \bigvee_{\rho(\Pi_f g)}^{\overbrace{\sigma}} \prod_{f g} f^* \lambda(\Pi_f g) \bigvee_{f * \rho(\Pi_f g)}^{\overbrace{\sigma} \uparrow f} g$$

Consider the lifting problem on the right above. It is the transpose of the original lifting problem under i^{-1} . Since $f^*\lambda(\Pi_f g)$ is in L and g is in R, there is a solution σ to this lifting problem. Then $i(\sigma)$ is a solution to the original lifting problem.

Definition 2.38. A weak factorization system (L,R) on a finitely complete category C is type-theoretic if it has the following two properties:

- 1. all objects are fibrant, and
- 2. L is stable under pullback along R.

A weak factorization structure F is type-theoretic if [F] is.

2.4 Summary

We have described the following diagram of categories of relations and factorizations.

$$fRel(C) \xrightarrow{\mathfrak{F}} fFact(C)$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$Rel(C) \xrightarrow{\mathfrak{F}} Fact(C)$$

We are interested in the relationship between type-theoretic weak factorization structures and Id-presentations of weak factorization systems. The former are a kind of factorization, so they naturally form a full subcategory of Fact(C). The latter are a kind of relation, so they naturally form a full subcategory of Rel(C).

Definition 2.39. Let ttWFS(C) be the full subcategory of Fact(C) spanned by the type-theoretic weak factorization structures on C, and similarly, let fttWFS(C) be the full subcategory of fFact(C) spanned by the type-theoretic weak factorization structures on C.

Definition 2.40. Let IdPres(C) denote the full subcategory of Rel(C) spanned by those relations which are Id-presentations, and similarly let fIdPres(C) denote the full subcategory of fRel(C) spanned by the Id-presentations.

Then we are interested in what relationship the subcategories $\mathsf{ttWFS}(\mathsf{C})$ and $\mathsf{IdPres}(\mathsf{C})$ have in the following diagram.

In the next sections, we show that $\mathfrak{F},\mathfrak{R}$ restrict to functors between IdPres(C) and ttWFS(C) and constitute an equivalence between them under the proset truncation.

3 Type-theoretic weak factorization systems from Moore relations

In this section, we consider a finitely complete category C and a relation R on C . In the first subsection, we describe structure on R which will make $\mathfrak{F}(R)$

a type-theoretic, algebraic weak factorization structure. We call this a *strict Moore relation structure*. In the second subsection we describe structure on R which will make $\mathfrak{F}(R)$ a type-theoretic weak factorization structure. We call this structure a *Moore relation structure*.

In Section 5, we will show that any relation is an Id-presentation of a weak factorization system if and only if it has a Moore relation structure. Then the full subcategory of Rel(C) spanned by Moore relation systems will coincide with IdPres(C). We originally defined the subcategory IdPres(C) by referencing the functor $\mathfrak{F}: Rel(C) \to Fact(C)$. The description of Moore relation structures which follows describes this subcategory more directly, without making reference to \mathfrak{F} . We will need this direct description to connect the category IdPres(C) with the category IdPres(C), the goal of this chapter.

We are mostly interested in the (non-strict) Moore relation structures since these correspond to Id-presentations. However, first we describe strict Moore relation structures. As mentioned in the introduction, these have already been investigated in [vdBG12]. We mention these first because they have many natural examples, and are thus more readily understandable. By contrast, most examples of non-strict Moore relation structures will come from the equivalence between them and type-theoretic weak factorization systems.

3.1 Strict Moore relation systems

In this subsection, we consider a functorial relation R which preserves pullbacks. For any object X in C, denote the image of RX by

$$X \stackrel{\epsilon_0}{\underset{\epsilon_1}{\longleftarrow}} \Psi X$$
.

Note that the requirement that R preserves pullbacks is equivalent to the requirement that Ψ does.

For any morphism $f: X \to Y$ of C, denote the factorization $\mathfrak{F}(R)f$ by the following diagram.

$$X \xrightarrow{\lambda_f} Mf \xrightarrow{\rho_f} Y$$

Recall that λ is a copointed endofunctor on $\mathsf{C}^\mathsf{M},$ and ρ is a pointed endofunctor on $\mathsf{C}^\mathsf{M}.$

In this section, we discuss the structure on R that will produce a comonad structure on the copointed endofunctor λ and a monad structure on the pointed endofunctor ρ .

3.1.1 Strictly transitive functorial relations

Definition 3.1. Say that a functorial relation R on C is *strictly transitive* if there exists a natural transformation μ with components

$$\mu_X: \Psi X_{\epsilon_1} \times_{\epsilon_0} \Psi X \to \Psi X$$

for each object X in C such that:

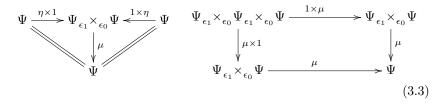
1. $\epsilon_i \mu = \epsilon_i \pi_i$ for i = 0, 1 in the diagram below.

$$\Psi_{\epsilon_{1}} \times_{\epsilon_{0}} \Psi \xrightarrow{\mu} \qquad \Rightarrow \Psi$$

$$\epsilon_{0} \pi_{0} \bigvee_{\downarrow} \epsilon_{1} \pi_{1} \qquad \epsilon_{0} \bigvee_{\downarrow} \epsilon_{1}$$

$$1_{\mathsf{C}} = = 1_{\mathsf{C}}$$
(3.2)

2. $(1_{\mathsf{C}}, \Psi, \epsilon_0, \epsilon_1, \eta, \mu)$ is an internal category in $[\mathsf{C}, \mathsf{C}]$; that is, the following diagrams commute.



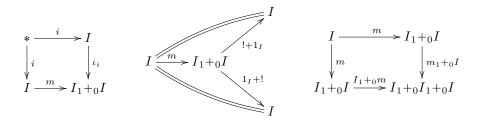
Note that if R is a monic relation, then the existence of μ with the commutativity of the diagram in (3.2) says that the relation R(X) on each object X of C is transitive, and the commutativity of the diagrams in (3.3) is automatic. Thus, the notion of transitivity here is a generalization of the usual one.

Example 3.4. Consider the relation which takes any object X in C to the following diagram

$$X \xrightarrow[X^1]{X^0} X^I$$

as in Example 2.3.

Suppose that there a morphism m making the following diagrams commute.



Then taking $X^m: X^I_{\epsilon_1} \times_{\epsilon_0} X^I \to X^I$ for μ_X makes this relation strictly transitive.

For example, in the category $\mathcal{C}at$, there is such an m when I is M (i.e., the category generated by the graph $0 \to 1$) or the groupoid generated by the graph $0 \to 1$.

Example 3.5. Consider the category \mathcal{T} of topological spaces. Let \mathbb{R}^+ denote the non-negative reals, and let ΓX denote the subspace of $X^{\mathbb{R}^+} \times \mathbb{R}^+$ consisting of pairs (p,r) such that p is constant on $[r,\infty)$. This is called the space of *Moore paths* in X, and it is functorial in X. We think of this as the space of paths in X of finite length.

There is a natural transformation $c: X \to \Gamma X$ which maps $x \in X$ to the constant path of length 0 at x. There are natural transformations $ev_0, ev_\infty : \Gamma X \to X$ which map a pair (p,r) to p(0) and p(r), respectively. These assemble into a functorial relation G on \mathcal{T} .

There is also a natural transformation $\mu_X : \Gamma X_{ev_\infty} \times_{ev_0} \Gamma X \to \Gamma X$ which maps two paths to their concatenation. To be precise, it takes a pair ((p,r),(p',t')) such that p(r) = p'(0) to the pair (q,s) where s = r + r', $q|_{[0,r]} = p|_{[0,r]}$, and $q(x)|_{[r,\infty)} = p'(x-r)$. This makes G a strictly transitive functorial relation.

Proposition 3.6. Let R be a strictly transitive functorial relation on C. Then the functor $\rho: C^M \to C^M$ underlies a monad on C^2 with unit and multiplication components at an object $f: X \to Y$ in C^2 given by the following diagrams

where $1 \times \mu : M\rho_f \to Mf$ is the morphism

$$1_X \times \mu_Y : X_f \times_{\epsilon_0} \Psi Y_{\epsilon_1} \times_{\epsilon_0} \Psi Y \longrightarrow X_f \times_{\epsilon_0} \Psi Y.$$

Proof. We have already seen that the unit square above commutes. The commutativity of the multiplication square above follows from the commutativity of (3.2).

The following diagram displays the unit axioms for the monad.

$$X_f \times_{\epsilon_0} \Psi Y \xrightarrow{1 \times 1 \times \eta} X_f \times_{\epsilon_0} \Psi Y_{\epsilon_1} \times_{\epsilon_0} \Psi Y \xleftarrow{1 \times \eta f \times 1} X_f \times_{\epsilon_0} \Psi Y$$

Its commutativity follows from that of the left-hand diagram in (3.3). This diagram displays the associativity axiom for the monad.

Its commutativity follows from that of the right-hand diagram in (3.3).

3.1.2 Strictly homotopical functorial relations

Definition 3.7. Say that a functorial relation $R: C \to C^R$ is *strictly homotopical* if there exist natural transformations δ and τ with components

$$\delta_X : \Psi X \to \Psi^2 X$$

$$\tau_X : X \times \Psi(*) \to \Psi X$$

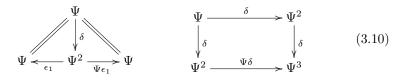
for each object X of C such that:

1. $\eta \eta = \delta \eta$ and $\eta \epsilon_0 = \epsilon_0 \delta$ in the following diagram.

$$\begin{array}{cccc}
\Psi & \xrightarrow{\delta} & \Psi^{2} \\
\epsilon_{0} & & & \downarrow & \uparrow \\
\downarrow & & & & \downarrow & \uparrow \\
1c & \xrightarrow{\eta} & & \Psi
\end{array}$$
(3.8)

2. $\epsilon_i \tau = \pi$ for i = 0, 1 and $\tau(1 \times \eta) = \eta$ in the following diagram.

3. $(\Psi, \epsilon_1, \delta)$ is a comonad on C; that is, the following diagrams commute.



4. τ is a strength for this comonad in the sense that the following diagrams commute.

$$\begin{array}{ccc}
1_{\mathsf{C}} \times \Psi * & & \\
\downarrow^{\tau} & & \\
\Psi & & \Psi^{!} & \\
\end{array}$$

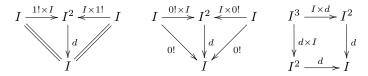
$$\begin{array}{ccc}
\Psi * & & \\
\end{array}$$

$$\begin{array}{cccc}
(3.11)
\end{array}$$

$$\begin{array}{cccc}
\Psi & \xrightarrow{\delta} & \Psi^{2} & & 1_{\mathsf{C}} \times \Psi * \xrightarrow{\tau \times \delta} \Psi(1_{\mathsf{C}} \times \Psi *) \\
\downarrow_{\epsilon_{0} \times \Psi!} & & \downarrow_{\Psi \epsilon_{0}} & & \downarrow_{\tau} & & \downarrow_{\Psi \tau} \\
1_{\mathsf{C}} \times \Psi * \xrightarrow{\tau} & \Psi & & \Psi & \xrightarrow{\delta} & \Psi^{2}
\end{array} (3.12)$$

The word homotopical is used to describe this functorial relation for the following reason. Suppose that we extract from the functorial relation R a notion of homotopy equivalence in the usual way: where two objects X and Y are homotopic if there are morphisms $f: X \to Y$, $g: Y \to X$, $h: X \to \Psi X$, $i: Y \to \Psi Y$ such that $\epsilon_0 h = gf$, $\epsilon_1 h = 1_X$, $\epsilon_0 i = fg$, and $\epsilon_1 i = 1_Y$. Then the data given in the above definition provide a homotopy between every X and ΨX .

Example 3.13. More generally, consider the relation in Example 3.4. Suppose that there is a morphism d making the following diagrams commute.



Then taking $X^d: X^I \to (X^I)^I$ for δ_X and $X^!: X \to X^I$ for τ_X makes this relation strictly homotopical.

For example, in the category Cat, there is such a d when I is M or the groupoid generated by the graph $I: 0 \to 1$. Let the following diagram denote the graph $(I: 0 \to 1)^2$.

$$00 \xrightarrow{0I} 01$$

$$\downarrow_{I0} \qquad \downarrow_{I1}$$

$$10 \xrightarrow{1I} 11$$

Then in either case, d is generated by sending 0I and I0 to the identity morphism on 0, and I1 and II to $I: 0 \to 1$.

Example 3.14. Consider the functorial relation G on topological spaces described in Example 3.5.

There is a natural transformation $\delta_X : \Gamma X \to \Gamma^2 X$ which takes a pair (p, r) to the standard path from c(p(0)) to (p, r). To be precise, it maps (p, r) to (q, r) where $q(t) = (p_t, t) \in \Gamma X$ and $p_t|_{[0,t]} = p|_{[0,t]}$ for each $t \in \mathbb{R}^+$.

There is a natural transformation $\tau_X : X \times \Gamma(*) \to \Gamma X$. The space $\Gamma(*)$ is isomorphic to \mathbb{R}^+ , so it maps a pair $(x,r) \in X \times \mathbb{R}^+$ to the constant path at x of length r.

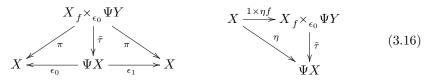
These natural transformations make G into a strictly homotopical functorial relation.

In the following lemma, we record a natural transformation $\tilde{\tau}$ whose existence is equivalent to that of τ , but which will make the proof of the following proposition clearer.

Lemma 3.15. Consider a strictly homotopical functorial relation as above. For any $f: X \to Y$, let $\tilde{\tau}_f: X_f \times_{\epsilon_0} \Psi Y \to \Psi X$ be the composite

$$X \xrightarrow{f} \times_{\epsilon_0} \Psi Y \xrightarrow{1 \times \Psi!} X \times \Psi * \xrightarrow{\tau} \Psi X.$$

It makes the following diagrams commute.



Proof. The commutativity of these diagrams is equivalent to that of the corresponding diagrams in (3.9), (3.11), and (3.12).

Proposition 3.18. Let R be a strictly homotopical functorial relation on C. Then the functor $\lambda: C^{\mathsf{M}} \to C^{\mathsf{M}}$ underlies a comonad on C^{M} where the components of the counit and comultiplication at each object $f: X \to Y$ in C^{M} are given by the following diagrams

$$X = X \qquad X = X$$

$$\downarrow^{\lambda_f} \qquad \downarrow^f \qquad \downarrow^{\lambda_f} \qquad \downarrow^{\lambda_f^2}$$

$$Mf \xrightarrow{-\rho_f} Y \qquad Mf \xrightarrow{1 \times \tilde{\tau} \times \delta} M\lambda_f$$

where the morphism $1 \times \tilde{\tau} \times \delta$ is the composition

$$\boldsymbol{X}_f \times_{\epsilon_0} \Psi \boldsymbol{Y} \xrightarrow{1_{\boldsymbol{X}} \times \tilde{\tau}_f \times \delta_{\boldsymbol{Y}}} \boldsymbol{X}_{\lambda_f} \times_{\epsilon_0} (\Psi \boldsymbol{X}_{\Psi f} \times_{\Psi \epsilon_0} \Psi^2 \boldsymbol{Y}) \cong \boldsymbol{X}_{\lambda_f} \times_{\epsilon_0} \Psi (\boldsymbol{X}_f \times_{\epsilon_0} \Psi \boldsymbol{Y}).$$

Proof. We have already seen that the counit square commutes. To define $1_X \times \tilde{\tau}_f \times \delta_Y$ we make use of the commutativity of (3.8) and the left hand sides of (3.16) and (3.17). The commutativity of the comultiplication square above is given by the commutativity of (3.8) and the right-hand diagram of (3.16).

The following diagrams display the comonad axioms. The commutativity of

$$X_f \times_{\epsilon_0} \Psi Y \xrightarrow{1 \times \Psi(\epsilon_1 \pi_{\Psi Y})} X_{\lambda_f} \times_{\epsilon_0} \Psi(X_f \times_{\epsilon_0} \Psi Y) \xrightarrow{\epsilon_1 \pi_{\Psi(X \times \Psi Y)}} X_f \times_{\epsilon_0} \Psi Y$$

follows from the commutativity of the left-hand diagrams in (3.10) and (3.16), and the commutativity of

$$\begin{split} X_f \times_{\epsilon_0} \Psi Y & \xrightarrow{1 \times \tilde{\tau}_f \times \delta_Y} & X_{\lambda f} \times_{\epsilon_0} \Psi (X_f \times_{\epsilon_0} \Psi Y) \\ & \downarrow^{1 \times \tilde{\tau}_f \times \delta_Y} & \downarrow^{1 \times \tilde{\tau}_\lambda \times \delta_{X \times \Psi Y}} \\ X_{\lambda f} \times_{\epsilon_0} \Psi (X_f \times_{\epsilon_0} \Psi Y) & \xrightarrow{1 \times \Psi (1 \times \tilde{\tau}_f \times \delta_Y)} & X_{\lambda^2 f} \times_{\epsilon_0} \Psi (X_{\lambda f} \times_{\epsilon_0} \Psi (X_f \times_{\epsilon_0} \Psi Y)) \end{split}$$

follows from the right-hand diagrams in (3.10) and (3.17).

3.1.3 Strictly symmetric functorial relations

Definition 3.19. Say that a functorial relation R on C is *strictly symmetric* if there exists a natural isomorphism ι with components

$$\iota_X: \Psi X \to \Psi X$$

for each object X of C such that $\iota \eta = \eta$, $\epsilon_0 \iota = \epsilon_1$, and $\epsilon_1 \iota = \epsilon_0$ in the diagram below.

$$\begin{array}{cccc}
\Psi & \xrightarrow{\iota} & \Psi \\
\epsilon_0 & \uparrow \\
\uparrow & \downarrow \epsilon_1 & & \epsilon_1 \\
1_{\mathsf{C}} & & & \downarrow \uparrow \\
& & \downarrow \epsilon_0
\end{array}$$
(3.20)

If R is a monic relation, then the definition of *strictly symmetric* given here coincides with the usual definition of *symmetric*.

Example 3.21. More generally, consider the relation in Example 3.4.

Suppose that there an isomorphism i making the following diagrams commute.

$$\begin{array}{c}
* \\
 \downarrow n \\
 I \xrightarrow{i} I
\end{array}$$

for $n \in \mathbb{Z}/2$. Then taking $X^i: X^I \to X^I$ for ι_X makes this relation strictly symmetric.

For example, in the category Cat, there is such an i when I is the groupoid generated by the graph $I: 0 \to 1$.

Example 3.22. Consider the functorial relation G on topological spaces described in Example 3.5.

There is a natural transformation $\iota_X: \Gamma X \to \Gamma X$ which takes a pair (p,r) to the pair (q,r) where q(t)=p(r-t) on [0,r].

This makes G into a strictly symmetric functorial relation.

Lemma 3.23. Consider a strictly symmetric, strictly transitive functorial relation R on C. Then for every object X of C, the morphism

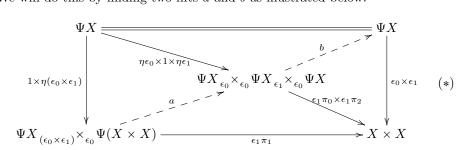
$$\Psi X \xrightarrow{\epsilon_0 \times \epsilon_1} X \times X$$

has a $\mathfrak{F}(R)$ -algebra structure.

Proof. We need to show that there is a solution to the following lifting problem.

$$\begin{array}{c|c} \Psi X = & \Psi X \\ \downarrow & \downarrow \\ \lambda(\epsilon_0 \times \epsilon_1) \\ \Psi X \xrightarrow[(\epsilon_0 \times \epsilon_1)]{} \times_{\epsilon_0} \Psi (X \times X) \xrightarrow{\rho(\epsilon_0 \times \epsilon_1)} X \times X \end{array}$$

We will do this by finding two lifts a and b as illustrated below.



Let $u: \Psi(X \times X) \to \Psi X \times \Psi X$ denote the universal morphism induced by the universal property of $\Psi X \times \Psi X$. It makes the following diagram commute.

$$\begin{array}{c|c} \Psi X \xrightarrow{1 \times \eta \epsilon_0 \times \eta \epsilon_1} & \Psi X_{(\epsilon_0 \times \epsilon_1)} \times_{(\epsilon_0 \times \epsilon_0)} (\Psi X \times \Psi X) \\ \hline \\ 1 \times \eta (\epsilon_0 \times \epsilon_1) & & & & & & \\ \Psi X_{(\epsilon_0 \times \epsilon_1)} \times_{\epsilon_0} \Psi (X \times X) \xrightarrow{\epsilon_1 \pi_{\Psi}(X \times X)} & X \times X \end{array}$$

Note that the outside square of this diagram is isomorphic to the lower-left portion of diagram (*). Therefore, $1 \times u$ is the lift a that we seek.

Now we let $b: \Psi X_{\epsilon_0} \times_{\epsilon_0} \Psi X_{\epsilon_1} \times_{\epsilon_0} \Psi X \to \Psi X$ be the following composite.

$$\Psi X \underset{\epsilon_0}{} \times_{\epsilon_0} \Psi X \underset{\epsilon_1}{} \times_{\epsilon_0} \Psi X \xrightarrow{1 \times \mu} \Psi X \underset{\epsilon_0}{} \times_{\epsilon_0} \Psi X \xrightarrow{\iota \times 1} \Psi X \underset{\epsilon_1}{} \times_{\epsilon_0} \Psi X \xrightarrow{\mu} \Psi X.$$

This b makes the upper right-hand portion of the above diagram commute.

Therefore, we have found a lift in the original diagram, and shown that $\epsilon_0 \times \epsilon_1$ has a $\mathfrak{F}(R)$ -algebra structure.

Theorem 3.24. Consider a strictly symmetric functorial relation $R: C \to C^R$ such that $\mathfrak{F}(R)$ is a weak factorization structure and such that every morphism

$$\Psi X \xrightarrow{\epsilon_0 \times \epsilon_1} X \times X$$

is in $\mathfrak{F}(R)$ -alg. Then the class $\mathfrak{F}(R)$ -coalg is stable under pullback along $\mathfrak{F}(R)$ -alg.

Proof. Consider the following pullback

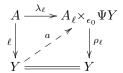
$$\begin{array}{c|c}
A \times_Y X \longrightarrow A \\
 & \downarrow \\
 & \downarrow \\
X \xrightarrow{r} Y
\end{array}$$

where r is in $\mathfrak{F}(R)$ -alg, and ℓ is in $\mathfrak{F}(R)$ -coalg.

The morphism π_X is in $\mathfrak{F}(R)$ -coalg if and only if there is a solution to the following lifting problem.

We will construct such a lift.

Since ℓ is in $\mathfrak{F}(R)$ -coalg, there is a lift a in the following square.



Since r is in $\mathfrak{F}(R)$ -alg, the morphism $r \times 1_X : X \times X \to Y \times X$ is in $\mathfrak{F}(R)$ -alg (as it is a pullback of r), and then the morphism $r\epsilon_0 \times \epsilon_1 : \Psi X \to Y \times X$ is in $\mathfrak{F}(R)$ -alg (as it is the composition of $\epsilon_0 \times \epsilon_1 \in \mathfrak{F}(R)$ -alg and $r \times 1_X \in \mathfrak{F}(R)$ -alg). Thus, there is a lift in the following square.

$$X \xrightarrow{\eta} \Psi X$$

$$\downarrow^{\lambda_r} \downarrow^{b} \downarrow^{r\epsilon_0 \times \epsilon_1}$$

$$X_r \times_{\epsilon_0} \Psi Y \xrightarrow{\epsilon_1 \times 1} Y \times X$$

Now let s be the following composition.

$$X \xrightarrow{ar \times 1_X} A_{\ell} \times_{\epsilon_0} \Psi Y_{\epsilon_1} \times_r X \xrightarrow{\pi_X \times \iota_Y \times \pi_A} X_r \times_{\epsilon_0} \Psi Y_{\epsilon_1} \times_{\ell} A \xrightarrow{\pi_A \times b} A_{\ell} \times_{r\epsilon_0} \Psi X$$

This makes the diagram (*) commute.

Corollary 3.25. Consider a strictly symmetric, strictly transitive relation R on C such that $\mathfrak{F}(R)$ is a weak factorization structure. Then $\mathfrak{F}(R)$ is type-theoretic.

Proof. By the previous two results, we know that $\mathfrak{F}(R)$ -coalg is stable under pullback along $\mathfrak{F}(R)$ -alg. By Proposition 2.35, every object is fibrant. Thus, $\mathfrak{F}(R)$ is type-theoretic.

3.1.4 Summary

We now have the following theorem.

Theorem 3.26. Consider a strictly transitive, strictly homotopical functorial relation R on C. Then the functorial factorization $\mathfrak{F}(R)$ is an algebraic weak factorization structure on C.

Proof. By Proposition 3.18, $\lambda_{\mathfrak{F}(R)}$ underlies a comonad, and by Proposition 3.6, $\rho_{\mathfrak{F}(R)}$ underlies a monad.

Definition 3.27. A strict Moore relation structure on C is a functorial relation R which preserves pullbacks together with the structure described in the definitions of strictly transitive, strictly homotopical, and strictly symmetric. A strict Moore relation system on a category C with finite limits is a functorial relation R which preserves pullbacks and which is strictly transitive, strictly homotopical, and strictly symmetric.

Then we have the following theorem.

Theorem 3.28. Consider a strict Moore relation system R on C. Then the functorial factorization $\mathfrak{F}(R)$ is a type-theoretic, algebraic weak factorization structure on C.

Proof. By the previous theorem, $\mathfrak{F}(R)$ is an algebraic weak factorization structure on C . By Proposition 3.25, it is type-theoretic.

Example 3.29. Consider the relation G on the category \mathcal{T} of topological spaces from Examples 3.5, 3.14, and 3.22. This generates a type-theoretic, algebraic weak factorization structure on C whose factorization of a morphism $f:X\to Y$ is

$$X \xrightarrow{1_X \times cf} X \times_Y \Gamma Y \xrightarrow{\pi_Y} Y$$

whose left class consists of trivial Hurewicz cofibrations, and whose right class consists of Hurewicz fibrations. (This weak factorization system was first described in [Str72] while this particular weak factorization structure was originally described in [May75].)

3.2 Moore relation systems

In this section, we describe the minimal structure that a relation R on \mathbb{C} needs to have so that $\mathfrak{F}(R)$ is a type-theoretic weak factorization structure. The minimality will be justified by Corollary 5.8, and though we do not give any examples in this section, many can be obtained from that corollary.

In what follows, we define what it means for a relation to be *transitive*, *homotopical*, and *symmetric*. Note that while the properties required of a transitive relation can be easily seen to be weaker than the properties required of a *strictly* transitive relation, the definitions of homotopical and symmetric given below differ more significantly from their strict predecessors.

In what follows, we will let λ denote $\lambda_{\mathfrak{F}(R)}$, ρ denote $\rho_{\mathfrak{F}(R)}$, and M denote $COD\lambda = DOM\rho$.

3.2.1 Transitive relations

Definition 3.30. Say that a relation R on C is *transitive* if there exists a morphism

$$\mu_X: \Psi X_{\epsilon_1} \times_{\epsilon_0} \Psi X \to \Psi X$$

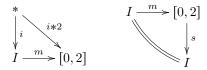
for every object X of C such that the following diagrams commute.

Non-example 3.32. Now we can see why the relation G on the category \mathcal{T} of topological spaces is more useful than the relation H on \mathcal{T} which sends every space X to

$$X \underset{X^1}{\overset{X^0}{\underset{X^1}{\overleftarrow{\times}}}} X^I$$

as in Example 2.6 where I is the usual interval [0, 1].

Suppose that this relation is transitive with a $\mu: X^I_{X^1} \times_{X^0} X^I \to X^I$ of the form $X^m: X^{[0,2]} \to X^{[0,1]}$. Then m would have to make the following diagrams commute for i=0,1



where s is the surjection which maps [0,1] onto [0,1] identically and [1,2] onto the point $\{1\}$. These diagrams say that m(0) = 0, m(1) = 2, and sm = 1. But there is no such continuous function.

Proposition 3.33. Consider a transitive relation R on C as above. Then for every morphism f of C, the morphism ρ_f has a $\mathfrak{F}(R)$ -algebra structure given by

$$M\rho_f \xrightarrow{1 \times \mu} Mf$$

$$\downarrow^{\rho_f^2} \qquad \downarrow^{\rho_f}$$

$$Y = = Y$$

where $1 \times \mu : M\rho_f \to Mf$ is the morphism

$$1_X \times \mu_Y : X_f \times_{\epsilon_0} \Psi Y_{\epsilon_1} \times_{\epsilon_0} \Psi Y \longrightarrow X_f \times_{\epsilon_0} \Psi Y.$$

Proof. The commutativity of the square in the statement follows from the commutativity of the left-hand diagram of (3.31).

It remains to check that the composition of the point with the algebra structure, $(1 \times \mu) \circ \lambda \rho(f)$, is the identity.

$$\begin{array}{c} \boldsymbol{X}_{f} \times_{\epsilon_{0}} \boldsymbol{\Psi} \boldsymbol{Y} \xrightarrow{1 \times 1 \times \eta \epsilon_{1}} \boldsymbol{\rightarrow} \boldsymbol{X}_{f} \times_{\epsilon_{0}} \boldsymbol{\Psi} \boldsymbol{Y}_{\epsilon_{1}} \times_{\epsilon_{0}} \boldsymbol{\Psi} \boldsymbol{Y} \\ & & & & & & & \\ \boldsymbol{X}_{f} \times_{\epsilon_{0}} \boldsymbol{\Psi} \boldsymbol{Y} \end{array}$$

The commutativity of this diagram follows from that of the right-hand diagram in (3.31).

As for the strictly transitive relations of the last section, when a relation R is monic, our definition of transitivity and the usual definition coincide.

3.2.2 Homotopical relations

The definition of *transitive* could immediately be seen to be a weaker version of the definition of strictly transitive. This is not the case for the definition of *homotopical*.

Definition 3.34. Say that a relation R on C is *homotopical* if for each object X of C, there exists an object $\Psi^{\square}X$ of C with morphisms

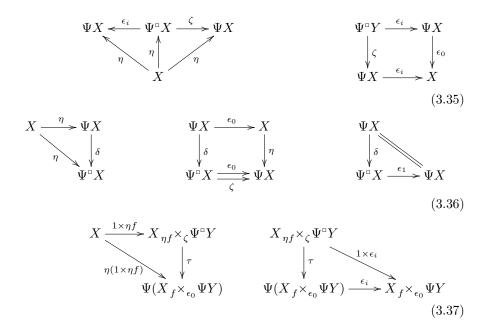
$$X \xrightarrow{\eta} \Psi^{\square} X \xrightarrow{\stackrel{\epsilon_0}{-\epsilon_1}} \Psi X$$

$$\delta_X: \Psi X \to \Psi^{\scriptscriptstyle \square} X$$
,

and for every morphism $f: X \to Y$, a morphism

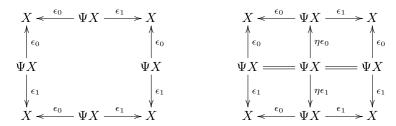
$$\tau_f: X_{\eta f} \times_{\zeta} \Psi^{\square} Y \to \Psi(X_f \times_{\epsilon_0} \Psi Y)$$

which make the following diagrams commute.



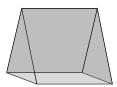
where i ranges over 0, 1.

Example 3.38. The object $\Psi^{\circ}X$ will often (as in Proposition 5.6) be the middle object of the factorization of the morphism $\eta: \Psi X \to \Psi^{\times 4}X$ where $\Psi^{\times 4}X$ is the limit of the diagram below on the left and $\eta: \Psi X \to \Psi^{\times 4}X$ is induced by the cone below on the right



In the category of topological spaces, this might look like the following. We use the relation H here, as described in Non-example 3.32, though we ultimately are interested in the relation G. This is because the description involving H is much easier to write down but still provides intuition to think about G.

Let $\delta(I \times I)$ denote the boundary of the unit square $I \times I$. Let S denote the mapping cylinder of the function $\delta(I \times I) \to I$ which maps (x, y) to x. That is, S is the quotient of $I \times \delta(I \times I)$ obtained by identifying the point (1, x, y) with the point (1, x, y') for any (x, y), (x, y') in $\delta(I \times I)$.



Then let $I^{\circ}X$ denote the space X^{S} of all continuous functions from S into X. The morphism $\eta: X \to I^{\circ}X$ is the precomposition with the map $S \to *$. The projections $\epsilon_{i}, \zeta_{i}: I^{\circ}X \to X^{I}$ are the precompositions of the inclusions of I into each of the bottom edges in the illustration above.

There is a continuous function $S \to I$ which takes the bottom edges associated to ϵ_0 and ζ_0 and the top vertex above their intersection to the point $0 \in I$ and maps the top edge and the edges associated to ϵ_1 and ζ_1 each homeomorphically onto I. Precomposition with this continuous function is the morphism $\delta_X : X^I \to I^{\circ}X$.

There is a homotopy equivalence $h: S \to I^2$ which commutes with the projections to $I^{\times 4}$. Then the composition

$$\boldsymbol{X}_{\eta f} \times_{\zeta_0} I^{\scriptscriptstyle \square} \boldsymbol{Y} \overset{\eta \times h}{\longleftrightarrow} \boldsymbol{X}^I_{f^I} \times_{\epsilon_0^I} \boldsymbol{Y}^{I \times I} \cong (\boldsymbol{X}_f \times_{\epsilon_0} \boldsymbol{Y}^I)^I$$

is the morphism τ_f .

Now we can provide some intuition as to why we have switched from considering $\Psi^2 X$ to $\Psi^{\circ} X$. In a space $\Gamma^2 X$, the lengths of the sides are coupled (e.g., for any $\gamma \in \Gamma^2 X$, $\Gamma \epsilon_0 \gamma$ has the same length as $\Gamma \epsilon_1 \gamma$) but this is not the case for $\Gamma^{\circ} X$. In particular, the middle diagram of 3.36 could not be satisfied

if $\Gamma^{\circ}X = \Gamma^{2}X$. To explain this from a slightly different perspective, when we obtain $\Psi^{\circ}X$ in this way, the morphism $\Psi^{\circ}X \to \Psi^{\times 4}X$ is in the right class of the weak factorization system, giving it better behavior than $\Psi^{2}X \to \Psi^{\times 4}X$.

This intuition will be given mathematical content when we extract this structure from any type-theoretic weak factorization structure in Proposition 5.6.

Proposition 3.39. Let R be a homotopical relation on C. Then for every morphism $f: X \to Y$ in C, the morphism λ_f has a $\mathfrak{F}(R)$ -coalgebra structure given by

$$X = X$$

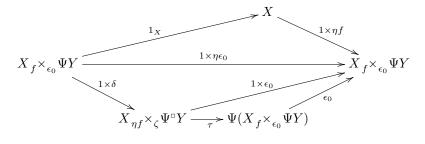
$$\downarrow^{\lambda_f} \qquad \downarrow^{\lambda_f^2}$$

$$Mf \xrightarrow{1 \times \tau \delta} M\lambda_f$$

where $1 \times \tau \delta : Mf \to M\lambda_f$ is

$$1_X \times \tau_f \delta_Y : X_f \times_{\epsilon_0} \Psi Y \to X_{1 \times \eta f} \times_{\epsilon_0} \Psi (X_f \times_{\epsilon_0} \Psi Y).$$

Proof. The morphism $1_X \times \tau_f \delta_Y$ in the statement is induced from the morphisms $\pi_X : X_f \times_{\epsilon_0} \Psi Y \to X$ and $\tau_f (1 \times \delta_Y) : X_f \times_{\epsilon_0} \Psi Y \to \Psi (X_f \times_{\epsilon_0} \Psi Y)$ by the universal property of the pullback $X_{1 \times \eta f} \times_{\epsilon_0} \Psi (X_f \times_{\epsilon_0} \Psi Y)$ because the following diagram commutes.



The upper triangle commutes by the properties of the pullback in its domain. The lower left-hand triangle commutes because of the commutativity of the middle diagram in (3.36). The lower right-hand triangle commutes because of the commutativity of the right-handle diagram in (3.37)

The coalgebra square in the statement can be written more explicitly as

The commutativity of this square follows from the commutativity of the outside of the following diagram by the universal property of the pullback in the lower right-hand corner.

The left-hand square above commutes because the left-hand diagram of (3.36) commutes. The right-hand square commutes because the left-hand diagram of (3.37) commutes.

Now it remains to check that the copoint composed with the coalgebra is the identity.

$$X = X = X$$

$$\downarrow^{\lambda_f} \qquad \downarrow^{\lambda_f^2} \qquad \downarrow^{\lambda_f}$$

$$Mf \xrightarrow{1 \times \tau \delta} M\lambda_f \xrightarrow{\rho \lambda_f} Mf$$

We have already seen that the two squares in this diagram commute. The composition $(\rho \lambda_f)(1 \times \tau \delta)$ is equal to the composition of the top and right sides of the diagram below.

The commutativity of the left-hand triangle above follows from the commutativity of the right-hand diagram in (3.36). The commutativity of the right-hand triangle above follows from the commutativity of the right-hand diagram in (3.37).

3.2.3 Symmetric relations

Definition 3.40. Say that a relation R on C is $\mathit{symmetric}$ if there exist morphisms

$$\nu_X: \Psi X_{\epsilon_0} \times_{\epsilon_0} \Psi X \to \Psi X$$

for every object X of C such that the following diagrams commute.

This might look very different from the strict symmetry defined previously. But notice that if one takes $\iota_X : \Psi X \to \Psi X$ to be the following composite,

$$\Psi X \xrightarrow{1 \times \eta \epsilon_0} \Psi X_{\epsilon_0} \times_{\epsilon_0} \Psi X \xrightarrow{\nu} \Psi X$$

then $\iota \eta = \eta$, $\epsilon_0 \iota = \epsilon_1$, and $\epsilon_1 \iota = \epsilon_0$ in the diagram below.

$$\begin{array}{c|cccc}
\Psi X & \xrightarrow{\iota} & \Psi X \\
\hline
\epsilon_0 & \uparrow & \downarrow \epsilon_1 & & \epsilon_1 & \uparrow & \downarrow \epsilon_0 \\
X & & & & X & & & X
\end{array}$$

Thus, ν begets a more familiar symmetry, ι . However, we need the full strength of the morphism ν to prove the following lemma.

Lemma 3.42. Consider a symmetric, transitive relation R on C. Then for every object X of C, the morphism

$$\Psi X \xrightarrow{\epsilon_0 \times \epsilon_1} X \times X$$

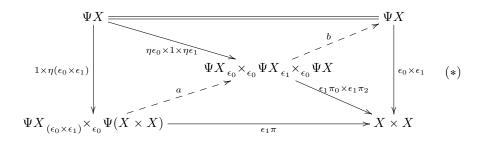
has a $\mathfrak{F}(R)$ -algebra structure.

Remark 3.43. Note that the following proof for this Lemma is identical to that for the strict version (Lemma 3.42) except that here we define b to be $\nu(1 \times \mu)$ instead of $\mu(\iota \times 1)(1 \times \mu)$.

Proof. We need to show that there is a solution to the following lifting problem.

$$\begin{array}{c|c} \Psi X = & \Psi X \\ \downarrow & \downarrow \\ \lambda(\epsilon_0 \times \epsilon_1) \downarrow & \downarrow \\ \Psi X_{(\epsilon_0 \times \epsilon_1)} \times_{\epsilon_0} \Psi (X \times X) \xrightarrow{\rho(\epsilon_0 \times \epsilon_1)} X \times X \end{array}$$

We will do this by finding two lifts a and b as illustrated below.



Let $u: \Psi(X \times X) \to \Psi X \times \Psi X$ denote the morphism induced by the universal property of $\Psi X \times \Psi X$. It makes the following diagram commute.

$$\begin{array}{c|c} \Psi X \xrightarrow{1 \times \eta \epsilon_0 \times \eta \epsilon_1} & \Psi X_{(\epsilon_0 \times \epsilon_1)} \times_{(\epsilon_0 \times \epsilon_0)} (\Psi X \times \Psi X) \\ 1 \times \eta (\epsilon_0 \times \epsilon_1) & \downarrow & \downarrow & \downarrow \\ \Psi X_{(\epsilon_0 \times \epsilon_1)} \times_{\epsilon_0} \Psi (X \times X) \xrightarrow{\epsilon_1 \pi_{\Psi(X \times X)}} & X \times X \end{array}$$

Note that the outside square of this diagram is isomorphic to the lower-left triangle of diagram (*). Therefore, $1 \times u$ is the lift a that we seek.

Now we let $b: \Psi X_{\epsilon_0} \times_{\epsilon_0} \Psi X_{\epsilon_1} \times_{\epsilon_0} \Psi X \to \Psi X$ be the following composite.

$$\Psi X _{\epsilon_0} \times_{\epsilon_0} \Psi X _{\epsilon_1} \times_{\epsilon_0} \Psi X \xrightarrow{1 \times \mu} \Psi X _{\epsilon_0} \times_{\epsilon_0} \Psi X \xrightarrow{\nu} \Psi X.$$

This b makes the upper right-hand portion of the above diagram commute.

Therefore, we have found a lift in the original diagram and shown that $\epsilon_0 \times \epsilon_1$ has a $\mathfrak{F}(R)$ -algebra structure.

Theorem 3.44. Consider a symmetric relation R on C such that $\mathfrak{F}(R)$ is a weak factorization structure and such that every morphism

$$\Psi X \xrightarrow{\epsilon_0 \times \epsilon_1} X \times X$$

is in $\mathfrak{F}(R)$ -alg. Then the class $\mathfrak{F}(R)$ -coalg is stable under pullback along $\mathfrak{F}(R)$ -alg.

Proof. The proof for this is identical to that for Theorem 3.24. \Box

Corollary 3.45. Consider a transitive, symmetric relation R on C such that the factorization $\mathfrak{F}(R)$ is a weak factorization structure. Then $\mathfrak{F}(R)$ is type-theoretic.

Proof. By the previous two results, we know that $\mathfrak{F}(R)$ -coalg is stable under pullback along $\mathfrak{F}(R)$ -alg. By Proposition 2.35, every object is fibrant. Thus, $\mathfrak{F}(R)$ is type-theoretic.

3.2.4 Summary

Now we have the following theorem.

Theorem 3.46. Consider a transitive and homotopical relation R on C. Then $\mathfrak{F}(R)$ is a weak factorization structure.

Proof. By Proposition 3.39, every morphism in the image of $\lambda_{\mathfrak{F}(R)}$ has a $\mathfrak{F}(R)$ -coalgebra structure, and by Proposition 3.33, every morphism in the image of $\rho_{\mathfrak{F}(R)}$ has a $\mathfrak{F}(R)$ -algebra structure.

Definition 3.47. A *Moore relation structure* on C is a relation R together with the structure given in the definitions of transitive, homotopical, and symmetric. A *Moore relation system* on C is a relation R together which is transitive, homotopical, and symmetric.

Now we have the following theorem.

Theorem 3.48. Consider a Moore relation system R on C. Then $\mathfrak{F}(R)$ is a type-theoretic weak factorization structure.

Proof. By the previous theorem, $\mathfrak{F}(R)$ is a weak factorization structure. Then by Corollary 3.45, $\mathfrak{F}(R)$ is type-theoretic.

4 Id-presentations from type-theoretic weak factorization systems

In this section, we consider a type-theoretic weak factorization structure W on a finitely complete category C . In the first section, 4.1, we show that the factorization $\mathfrak{FR}(W)$ is again a weak factorization structure equivalent to W. In the second section, 4.2, we show that the relation $\mathfrak{R}(W)$ is an Id-presentation of $[\mathfrak{FR}(W)] = [W]$. Combining these two results, we will have shown that any type-theoretic weak factorization system has an Id-presentation.

4.1 The main tool

Consider any type-theoretic weak factorization structure W on C. Our aim in this section is to show that $\mathfrak{FR}(W)$ is equivalent to W. However, we prove a slightly more general result which will become useful later (in Lemma 4.9, Proposition 5.6, and Proposition 6.2).

To that end, consider any relation R with the following components at each object X of C

$$X \stackrel{\epsilon_{0X}}{\underset{\epsilon_{1Y}}{\longleftarrow}} RX$$

such that each $\eta_X: X \to RX$ is in W-coalg and each $\epsilon_X = \epsilon_{0X} \times \epsilon_{1X}: RX \to X \times X$ is in W-alg. (We have in mind the relation $\mathfrak{R}(W)$ for our main result.)

Now we show that $\mathfrak{F}(R)$ is a weak factorization structure equivalent to W. For readability, we will let λ denote $\lambda_{\mathfrak{F}(R)}$ and ρ denote $\rho_{\mathfrak{F}(R)}$. We need to show that (1) $\mathfrak{F}(R)$ -coalg = W-coalg, (2) $\mathfrak{F}(R)$ -alg = W-alg, (3) $\lambda_f \in W$ -coalg, and (4) $\rho_f \in W$ -alg for every morphism f of C. These facts are all relatively straightforward to show except (3) which appears as Proposition 4.4.

The hypothesis that W is type-theoretic is integral to the proof below. In Lemma 4.1, where we show fact (4), we need every object in W to be fibrant. In Lemma 4.2, which is used to show fact (3) in Proposition 4.4, we need W-coalg to be stable under pullback along W-alg.

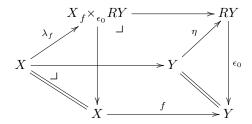
Lemma 4.1. For any morphism f of C, the morphism ρ_f is in W-alg.

Proof. Note first that $\pi_Y: X \times Y \to Y$ and $1_X \times \epsilon_1: X_f \times_{\epsilon_0} RY \to X \times Y$ are in W-alg because they are pullbacks of morphisms hypothesized to be in W-alg.

Since ρ_f is the composition of these two maps, it is also in W-alg.

Lemma 4.2. For any morphism f in W-alg, the morphism λ_f is in W-coalg,

Proof. The morphism λ_f is a pullback of $\eta \in W$ -coalg along $f \in W$ -alg,



and since W is type-theoretic, W-coalg is stable under pullback along W-alg. \square

Proposition 4.3. We have that W-coalg = $\mathfrak{F}(R)$ -coalg and W-alg = $\mathfrak{F}(R)$ -alg.

Proof. Consider a morphism f in $\mathfrak{F}(R)$ -alg. It is a retract of ρ_f . By Lemma 4.1, ρ_f is in W-alg. Since W-alg is closed under retracts [MP12, Prop. 14.1.8], f is in W-alg.

Now consider a morphism f in W-alg. Since λ_f is in W-coalg by Lemma 4.2, λ_f has the left lifting property against f. Therefore, f is in $\mathfrak{F}(R)$ -alg.

Thus, W-alg = $\mathfrak{F}(R)$ -alg.

Now consider $\ell \in W$ -coalg. Since ℓ has the left lifting property against W-alg, it has the left lifting property against ρ_{ℓ} in particular (Lemma 4.1). Thus it is in $\mathfrak{F}(R)$ -coalg.

Now suppose that $\ell \in \mathfrak{F}(R)$ -coalg. Then for any $r \in W$ -alg = $\mathfrak{F}(R)$ -alg, ℓ has the left-lifting property against r (Proposition 2.21). Thus, ℓ is in $\square(W$ -alg) = W-coalg.

Therefore,
$$W$$
-coalg = $\mathfrak{F}(R)$ -coalg.

Proposition 4.4. For any morphism f of C, the morphism $\lambda(f)$ is in W-coalg.

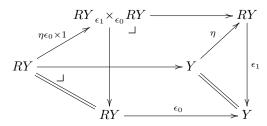
Proof. We need to show that λ_f has a λ -coalgebra structure, or that, equivalently, there is a solution to the following lifting problem.

$$X \xrightarrow{\lambda_{\lambda_f}} X_{\lambda_f} \times_{\epsilon_0} R(X_f \times_{\epsilon_0} RY)$$

$$\downarrow^{\lambda_f} \qquad \qquad \downarrow^{\rho_{\lambda_f}}$$

$$X_f \times_{\epsilon_0} RY = X_f \times_{\epsilon_0} RY$$

First we define a new morphism $\mu:RY_{\epsilon_1}\times_{\epsilon_0}RY\to RY$. Note that $\eta\epsilon_0\times 1:RY\to RY_{\epsilon_1}\times_{\epsilon_0}RY$ is in W-coalg since it is a pullback of a morphism in W-coalg along a morphism in W-alg, as shown below.



Then, we define μ to be a solution to the following lifting problem.

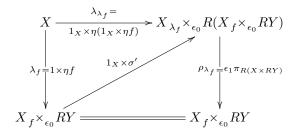
$$RY = RY$$

$$\downarrow_{\eta \epsilon_0 \times 1} \qquad \downarrow_{\epsilon_0 \times \epsilon_1} \qquad \downarrow_{\epsilon_0 \times \epsilon_1}$$

$$RY_{\epsilon_1} \times_{\epsilon_0} RY \xrightarrow{\epsilon_0 \pi_0 \times \epsilon_1 \pi_1} Y \times Y$$

Now we refer to figure Figure 1 on page 37. Since ρ_f is in W-alg, we know that λ_{ρ_f} is in W-coalg. Therefore, there is a lift σ as illustrated in the figure.

Let $\sigma': X_f \times_{\epsilon_0} RY \to R(X_f \times_{\epsilon_0} RY)$ be the composite $R(1_X \times \mu) \sigma(1_X \times \eta f \times 1_{RY})$ – that is, the composite from the bottom left to top right of the diagram in Figure 1. Then a rearrangement of Figure 1 produces the commutative diagram below, and $1_X \times \sigma'$ is our desired lift.



Therefore, λ_f is in W-coalg.

We put the preceding results together into the following theorems.

Theorem 4.5. Consider a type-theoretic weak factorization structure W on C. Consider a relation R on C which has components

$$X \stackrel{\epsilon_{0X}}{\underset{\epsilon_{1X}}{\longleftarrow}} RX,$$

such that η_X is W-coalg and $\epsilon_{0X} \times \epsilon_{1X} : R(X) \to X \times X$ is in W-alg at each object X of C. Then the factorization $\mathfrak{F}(R)$ is a weak factorization structure equivalent to W.

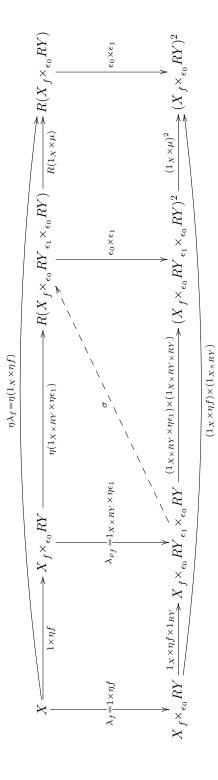


Figure 1: Lifting diagram

Proof. By Lemma 4.1 and Proposition 4.3, every morphism in the image of $\rho_{\mathfrak{F}(R)}$ is in $\mathfrak{F}(R)$ -alg. By Proposition 4.4 and Proposition 4.3, every morphism in the image of $\lambda_{\mathfrak{F}(R)}$ is in $\mathfrak{F}(R)$ -coalg. Thus, $\mathfrak{F}(R)$ is a weak factorization structure. By Propositions 2.24 and 4.3, it is equivalent to W.

The following corollary is the main result of this section.

Corollary 4.6. Consider a type-theoretic weak factorization structure W on C. The factorization $\mathfrak{FR}(W)$ is a weak factorization structure equivalent to W.

Proof. We need to show that the relation $\mathfrak{R}(W)$ can be substituted for R in the statement of the previous theorem, 4.5. In the notation of the previous theorem, 4.5, η_X is $\lambda_W(\Delta_X)$ and ϵ_X is $\rho_W(\Delta_X)$, and these are in the left and right class, respectively, as required.

The following corollary will become a useful technical device (in Proposition 5.6) and is the reason that we proved Theorem 4.5 in more generality than needed for Corollary 4.6.

Corollary 4.7. Consider a type-theoretic weak factorization structure W on C. Consider a relation on just one object Y of C with the following components

$$Y \overset{\epsilon_{0Y}}{\underset{\epsilon_{1Y}}{\longleftarrow}} RY,$$

such that η_Y is in the left class and $\epsilon_{0Y} \times \epsilon_{1Y} : RY \to Y \times Y$ is in the right class of W. Then for any morphism $f: X \to Y$ of C, in the following factorization

$$X \xrightarrow{1 \times \eta f} X \times_{\epsilon_0} RY \xrightarrow{\epsilon_1 \pi_{RY}} Y$$

the morphism $1 \times \eta f$ is in the left class, and $\epsilon_1 \pi_{RY}$ is in the right class of W.

Proof. Consider the relation $\mathfrak{R}(W)$. We construct a new relation S which coincides with $\mathfrak{R}(W)$ everywhere except at Y. So set $S(X) = \mathfrak{R}(W)(X)$ for every object $X \neq Y$ and set S(Y) = R. Then a lift of any morphism with domain or codomain Y can be extracted from the weak factorization structure W. That is, a lift of any morphism $f: X \to Y$ can be obtained as a solution to the following lifting problem.

$$X \xrightarrow{\eta f} \operatorname{Id}(Y)$$

$$\downarrow^{\eta} \qquad \downarrow^{\epsilon_0 \times \epsilon_1}$$

$$\operatorname{Id}(X) \xrightarrow{f \epsilon_0 \times f \epsilon_1} Y \times Y$$

A lift of any morphism $g: Y \to Z$ can be obtained analogously.

The relation S satisfies the hypotheses of Theorem 4.5 so $\mathfrak{F}(S)$ is a weak factorization structure equivalent to W. But $\mathfrak{F}(S)$ sends a morphism $f: X \to Y$ to the factorization in the statement. Thus $1 \times \eta f$ is in the left class and $\epsilon_1 \pi_{RY}$ is in the right class of W.

Remark 4.8. Some might be opposed to the reference of equality of objects. However, it is not strictly necessary. One can emulate the proof of Theorem 4.5, replacing λ_f with the $1 \times \eta f$ of the statement of Corollary 4.7 and replacing the other occurences of λ and ρ with $\lambda_{\mathfrak{FR}(W)}$ and $\rho_{\mathfrak{FR}(W)}$ (essentially, making the replacement of $\mathfrak{R}(W)(Y)$ by R(Y), as is done in the proof of Corollary 4.7, just where necessary in the proof of Theorem 4.5). Then we will obtain Corollary 4.7.

4.2 Id-presentations

Now we can show that every type-theoretic weak factorization system has an Id-presentation.

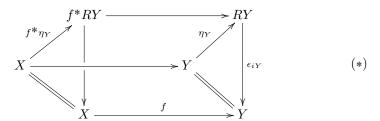
Lemma 4.9. Consider a relation R on C such that $\mathfrak{F}(R)$ is a type-theoretic weak factorization structure. Denote the components of R(X) for any object X of C by the following diagram.

$$X \xrightarrow[\epsilon_{1} x]{\epsilon_{0} x} RX$$

Then R is an Id-presentation of the weak factorization system $[\mathfrak{F}(R)]$ if and only if $\epsilon_{0X} \times \epsilon_{1X} : RX \to X \times X$ is in the right class for each object X.

Proof. Suppose that R is an Id-presentation. Then by definition, we must have that each $\epsilon_{0X} \times \epsilon_{1X} : RX \to X \times X$ is in $\mathfrak{F}(R)$ -alg.

Conversely, suppose that each $\epsilon_{0X} \times \epsilon_{1X} : RX \to X \times X$ is in $\mathfrak{F}(R)$ -alg. Then it remains to show that each $f^*\eta_Y$, as displayed in the diagram (*) below, is in $\mathfrak{F}(R)$ -coalg.



Note that when i=0 in the diagram (*) above, the morphism $f^*\eta_Y$ is isomorphic to $\lambda_{\mathfrak{F}(R)}(f)$ (i.e., it has the same universal property as $1_X \times \eta_Y f: X \to X \times_Y RY$). Thus, it must be in $\mathfrak{F}(R)$ -coalg.

There is an involution I on Rel(C) which sends $S(X)\epsilon_i$ to $S(X)\epsilon_{i+1}$ for any $S \in Rel(C)$, $X \in C$, and $i \in \mathbb{Z}/2$ and keeps all else constant. Then IR satisfies the hypotheses of Theorem 4.5, so $\mathfrak{F}(IR)$ is a weak factorization structure equivalent to $\mathfrak{F}(R)$. Now when i = 1, the morphism $f^*\eta_Y$ in the diagram (*) is isomorphic to $\lambda_{\mathfrak{F}(IR)}f$, so it is in $\mathfrak{F}(R)$ -coalg.

Therefore, every $f^*\eta_Y$ in the diagram (*) is in $\mathfrak{F}(R)$ -coalg, so R is an Idpresentation of this weak factorization system.

Now combining Corollary 4.6 with this lemma, 4.9, we see the following.

Theorem 4.10. Consider a type-theoretic weak factorization structure W on C. The relation $\Re(W)$ is an Id-presentation of the weak factorization system [W]. Thus, every type-theoretic weak factorization system has an Id-presentation.

Proof. By Corollary 4.6, $\Re(W)$ generates a type-theoretic weak factorization structure $\Re(W)$ equivalent to W. For any object X of C , the morphism

$$\Re(W)X\epsilon_0 \times \Re(W)X\epsilon_1 : \Re(W)X\Phi \to X \times X$$

is the right factor of the morphism $\Delta(X)$ in the factorization W. Thus, it is in the right class of the weak factorization system. Then this is an Id-presentation of [W] by Lemma 4.9.

Corollary 4.11. The functor $\mathfrak{R}: \mathsf{Fact}(\mathsf{C}) \to \mathsf{Rel}(\mathsf{C})$ restricts to a functor $\mathfrak{R}: \mathsf{ttWFS}(\mathsf{C}) \to \mathsf{IdPres}(\mathsf{C}),$ and the composition $|\mathfrak{FR}|: |\mathsf{ttWFS}(\mathsf{C})| \to |\mathsf{ttWFS}(\mathsf{C})|$ is isomorphic to the identity functor.

Proof. By the previous theorem, 4.10, all objects in the image of \mathfrak{R} : ttWFS(C) \rightarrow Rel(C) are Id-presentations. Thus, this functor restricts to \mathfrak{R} : ttWFS(C) \rightarrow IdPres(C).

By the previous theorem again, for any object $W \in \mathsf{ttWFS}(\mathsf{C})$, we have that W is equivalent to $\mathfrak{FR}(W)$. Thus, they are isomorphic as objects of $|\mathsf{ttWFS}(\mathsf{C})|$. Since $|\mathsf{ttWFS}(\mathsf{C})|$ is a proset, these isomorphisms assemble into a natural transformation $1 \cong |\mathfrak{FR}|$.

Example 4.12. Given any Cisinski model structure (C, W, F) on a topos M [Cis06], we claim that the weak factorization system $(C \cap W \cap M_F, F \cap M_F)$ restricted to the full subcategory M_F of fibrant objects is type-theoretic.

For this weak factorization system to be type-theoretic, all its objects must be fibrant, which we have satisfied by construction, and $C \cap W \cap M_F$ must be stable under pullback along $F \cap M_F$. In a Cisinski model structure, C is precisely the class of monomorphisms, so it is stable under pullback (along all morphisms) in M. Then, in particular, it is stable under pullback along $F \cap M_F$ in M_F . A standard result of model category theory says that $W \cap M_F$ is stable under pullback in M_F [Bro73, §1 Ex. 1,§4 Lem. 1]. Thus $C \cap W \cap M_F$ is stable under pullback along $F \cap M_F$, and the weak factorization system $(C \cap W \cap M_F, F \cap M_F)$ is type-theoretic.

Then we find many examples of type-theoretic weak factorization systems, including those in the categories of Kan complexes [Qui67], quasicategories [Joy08], and fibrant cubical sets [Cis06]. These all have Id-presentations.

5 Moore relations from type-theoretic weak factorization systems

In this section, we tie up the preceding sections by showing that a relation R on a finitely complete category ${\sf C}$ is a Moore relation system if and only if it is an Id-presentation.

We can immediately see from our previous results that any relation R which underlies a Moore relation system is an Id-presentation of the weak factorization system it generates.

Proposition 5.1. Consider a Moore relation system R on C. Then R is an Id-presentation.

Proof. By Theorem 3.48, R generates a type-theoretic weak factorization structure $\mathfrak{F}(R)$. By Lemma 3.42, every $RX\epsilon_0 \times RX\epsilon_1$ is in the right class. Then by Lemma 4.9, this is a Id-presentation of $[\mathfrak{F}(R)]$.

Now we prove the converse: that any Id-presentation is a Moore relation system. We consider a relation R on C which at an object X gives the following diagram.

$$X \stackrel{\epsilon_{0X}}{\underset{\epsilon_{1X}}{\longleftarrow}} \Psi X,$$

We let λ denote $\lambda_{\mathfrak{F}(R)}$ and ρ denote $\rho_{\mathfrak{F}(R)}$.

Proposition 5.2. Suppose that R is an Id-presentation of a weak factorization system. Then R is transitive.

Proof. For any object X of C, we let μ_X be a solution to the following lifting problem.

$$\begin{array}{c|c} \Psi X = & \Psi X \\ \downarrow^{\lambda_{\epsilon_1}} & \downarrow^{\mu_X} & \downarrow^{\epsilon_0 \times \epsilon_1} \\ \Psi X_{\epsilon_1} \times_{\epsilon_0} \Psi X = & \downarrow^{\epsilon_0 \pi_0 \times \epsilon_1 \pi_1} \\ & & \times X \end{array}$$

This makes R into a transitive relation.

Proposition 5.3. Suppose that R is an Id-presentation of a weak factorization system. Then R is symmetric.

Proof. For any object X of C , we let ν_X be a solution to the following lifting problem (where $\tau: \Psi X_{\epsilon_0} \times_{\epsilon_0} \Psi X \to \Psi X_{\epsilon_0} \times_{\epsilon_0} \Psi X$ is the standard twist involution).

$$\begin{array}{c|c} \Psi X = & \Psi X \\ \downarrow^{\tau \lambda_{\epsilon_0}} & \downarrow^{\nu_X} & \downarrow^{\epsilon_0 \times \epsilon_1} \\ \Psi X_{\epsilon_0} \times_{\epsilon_0} \Psi X \xrightarrow[\epsilon_1 \pi_0 \times \epsilon_1 \pi_1]{} & X \times X \end{array}$$

This makes R into a symmetric relation.

Theorem 5.4. Consider an Id-presentation R on C. The factorization $\mathfrak{F}(R)$ is a type-theoretic weak factorization structure.

Proof. By Proposition 5.3, R is symmetric. Then by Proposition 2.35 and Theorem 3.44, $\mathfrak{F}(R)$ is type-theoretic.

Corollary 5.5. The functor $\mathfrak{F}: Rel(C) \to Fact(C)$ restricts to a functor $\mathfrak{F}: IdPres(C) \to ttWFS(C)$.

Proof. The previous theorem, 5.4, tells us that every object in the image of $\mathfrak{F}: \mathsf{Rel}(\mathsf{C}) \to \mathsf{Fact}(\mathsf{C})$ is in the full subcategory $\mathsf{ttWFS}(\mathsf{C})$. Thus, this functor restricts to $\mathfrak{F}: \mathsf{IdPres}(\mathsf{C}) \to \mathsf{ttWFS}(\mathsf{C})$.

Proposition 5.6. Suppose that R is an Id-presentation of a weak factorization system. Then R is homotopical.

Proof. Let $\Psi^{\times 4}X$ denote the limit of the following diagram in C.

$$X \xleftarrow{\epsilon_0} \Psi X \xrightarrow{\epsilon_1} X$$

$$\uparrow_{\epsilon_0} \qquad \uparrow_{\epsilon_0}$$

$$\Psi X \qquad \qquad \Psi X$$

$$\downarrow_{\epsilon_1} \qquad \qquad \downarrow_{\epsilon_1}$$

$$X \xleftarrow{\epsilon_0} \Psi X \xrightarrow{\epsilon_1} X$$

There is a morphism $u: \Psi X \to \Psi^{\times 4} X$ which is induced by the following cone.

$$X \stackrel{\epsilon_0}{\longleftarrow} \Psi X \stackrel{\epsilon_1}{\longrightarrow} X$$

$$\uparrow^{\epsilon_0} \qquad \uparrow^{\eta\epsilon_0} \qquad \uparrow^{\epsilon_0}$$

$$\Psi X = = \Psi X = = \Psi X$$

$$\downarrow^{\epsilon_1} \qquad \downarrow^{\eta\epsilon_1} \qquad \downarrow^{\epsilon_1}$$

$$X \stackrel{\epsilon_0}{\longleftarrow} \Psi X \stackrel{\epsilon_1}{\longrightarrow} X$$

Now we factor $u: \Psi X \to \Psi^{\times 4} X$.

$$\Psi X \xrightarrow{\lambda_u} Mu \xrightarrow{\rho_u} \Psi^{\times 4} X$$

Let $\Psi^{\circ}X$ denote Mu, and let the following diagram denote the cone corresponding to ρ_u .

Note that the object $\Psi^{\square}X$ is defined to be the pullback $\Psi X_u \times_{\epsilon_0} \Psi(\Psi^{\times 4}X)$. Now we let $\delta_X : \Psi X \to \Psi^{\square}X$ be a solution to the following lifting problem.

For any $f:X\to Y$, we need to find a solution to the following lifting problem in order to define $\tau_f:X_{\eta f}\times_\zeta\Psi^\circ Y\to\Psi(X_f\times_{\epsilon_0}\Psi Y)$.

$$X \xrightarrow{\eta(1 \times \eta f)} \Psi(X_f \times_{\epsilon_0} \Psi Y)$$

$$\downarrow^{\tau_f} \qquad \qquad \downarrow^{\epsilon_0 \times \epsilon_1} \qquad \qquad \downarrow^{\epsilon_0 \times \epsilon_1} \qquad \qquad \downarrow^{\epsilon_0 \times \epsilon_1} \qquad \qquad \downarrow^{\tau_f} \qquad$$

Since R is an Id-presentation of $(\mathfrak{F}(R)\text{-coalg},\mathfrak{F}(R)\text{-alg})$, we know that the right hand map above, $\epsilon_0 \times \epsilon_1$, is in $\mathfrak{F}(R)\text{-alg}$. Thus, we need to show that $1 \times \lambda_u \eta f$ is in $\mathfrak{F}(R)\text{-coalg}$.

To see this, first observe that $\zeta_0 \times \zeta_1 : \Psi^{\times 4}Y \to \Psi Y \times \Psi Y$ is in $\mathfrak{F}(R)$ -alg, since it is given by the following pullback.

$$\Psi^{\times 4}Y \xrightarrow{\epsilon_0 \times \epsilon_1} \Psi Y \times \Psi Y$$

$$\downarrow^{\zeta_0 \times \zeta_1} \qquad \qquad \downarrow^{\epsilon_0 \times \epsilon_1 \times \epsilon_0 \times \epsilon_1}$$

$$\Psi Y \times \Psi Y \xrightarrow{\epsilon_0 \times \epsilon_0 \times \epsilon_1 \times \epsilon_1} Y \times Y \times Y \times Y$$

The right-hand map in the above diagram is in $\mathfrak{F}(R)$ -alg since it is the product of two maps in $\mathfrak{F}(R)$ -alg, and thus its pullback, $\zeta_0 \times \zeta_1$, is also in $\mathfrak{F}(R)$ -alg. Then the composition $(\zeta_0 \times \zeta_1)\rho_u : \Psi^{\circ}Y \to \Psi Y \times \Psi Y$, which we also denote by $\zeta_0 \times \zeta_1$, is in $\mathfrak{F}(R)$ -alg.

Thus, the following is a factorization of the diagonal $\Delta_{\Psi Y}$ into $\mathfrak{F}(R)$ -coalg and $\mathfrak{F}(R)$ -alg).

$$\Psi Y \xrightarrow{\lambda_u} \Psi^{\scriptscriptstyle \square} Y \xrightarrow{\zeta_0 \times \zeta_1} \Psi Y \times \Psi Y$$

By Corollary 4.7, in the following factorization of $\eta f: X \to \Psi Y$,

$$X \xrightarrow{1 \times \lambda_u \eta f} X_{\eta f} \times_{\zeta_0} \Psi^{\square} Y \xrightarrow{\zeta_1} \Psi Y$$

the morphism $1 \times \lambda_u \eta f$ is in $\mathfrak{F}(R)$ -coalg.

Thus, we obtain a lift τ_f as above.

Then τ and δ make R into a homotopical relation where the diagram

$$X \xrightarrow{\eta} \Psi^{\square} X \xrightarrow{\epsilon_0} \Psi X$$

of Definition 3.34 is given by the diagram

$$X \xrightarrow{\lambda_u \eta} \Psi^{\square} X \xrightarrow{\epsilon_0} \Psi X$$

that we have defined here.

Thus, we have the following theorem.

Theorem 5.7. Consider a relation R on C. It is an Id-presentation of a weak factorization system if and only if it is a Moore relation system.

Proof. By Proposition 5.1, a Moore relation system is an Id-presentation of the weak factorization system it generates.

By Propositions 5.2, 5.3, and 5.6, an Id-presentation of a weak factorization system is a Moore relation system. \Box

Now we can restate Theorem 4.10 in the following way.

Corollary 5.8. Consider a type-theoretic weak factorization structure W. Then the relation $\Re(W)$ is a Moore relation system which generates the weak factorization system represented by W. Thus, every type-theoretic weak factorization system can be generated by a Moore relation system.

Proof. This is Theorem 4.10 with 'Moore relation system' substituted for 'Idpresentation' as justified by Theorem 5.7. \Box

Example 5.9. Consider Example 4.12. Then given a Cisinski model structure (C,F,W) on a topos M, the weak factorization system $(C \cap W \cap M_F,F \cap M_F)$ is generated by a Moore relation system. In particular, the weak factorization systems in the category of Kan complexes, the category of quasicategories, and that of cubical sets are generated by Moore relation systems.

To conclude this section, we show that $|\mathfrak{RF}| : |IdPres(C)| \to |IdPres(C)|$ is isomorphic to the identity functor. We have shown that $|\mathfrak{FR}| : |ttWFS(C)| \to |ttWFS(C)|$ is also isomorphic to the identity functor (Corollary 4.11). Thus, this will show that $|\mathfrak{R}|$ and $|\mathfrak{F}|$ form an equivalence $|IdPres(C)| \simeq |ttWFS(C)|$.

Proposition 5.10. The functor $|\mathfrak{RF}| : |\mathsf{IdPres}(\mathsf{C})| \to |\mathsf{IdPres}(\mathsf{C})|$ is isomorphic to the identity functor.

Proof. We need to provide an equivalence between any R in $\mathsf{IdPres}(\mathsf{C})$ and $\mathfrak{RF}(R)$. Since $|\mathsf{IdPres}(\mathsf{C})|$ is a proset, this will automatically assemble into a natural isomorphism $1 \cong |\mathfrak{RF}|$.

Let RX be denoted by the following diagram for any X in C.

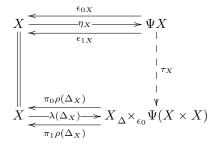
$$X \underbrace{\overset{\epsilon_{0X}}{-\eta_{X}}}_{\epsilon_{1X}} \Psi X$$

Then $\mathfrak{RF}(RX)$ gives the following diagram

$$X \underset{\pi_1 \rho(\Delta_X)}{\overset{\pi_0 \rho(\Delta_X)}{\longleftarrow}} X_{\Delta} \times_{\epsilon_0} \Psi(X \times X)$$

where λ denotes $\lambda_{\mathfrak{F}(R)}$ and ρ denotes $\rho_{\mathfrak{F}(R)}$.

Now a morphism $R \to \mathfrak{RF}(R)$ consists of a natural transformation $R(X) \to \mathfrak{RF}(R)(X)$ at each X, as displayed below, which, in turn, consists of the identity on X and a morphism $\tau_X : \Psi X \to X_{\Delta} \times_{\epsilon_0} \Psi(X \times X)$.



But we can obtain the morphism τ_X as a lift in the diagram below.

$$X \xrightarrow{\lambda(\Delta_X)} X_{\Delta} \times_{\epsilon_0} \Psi(X \times X)$$

$$\eta \downarrow \qquad \qquad \uparrow_{X} \qquad \qquad \downarrow \rho(\Delta_X)$$

$$\Psi X \xrightarrow{\epsilon_0 \times \epsilon_1} X \times X$$

since η is in $\mathfrak{F}(R)$ -coalg and $\rho(\Delta_X)$ is in $\mathfrak{F}(R)$ -alg.

Similarly, we can get a morphism $\mathfrak{RF}(R) \to R$ by solving the following lifting problem for each object X.

$$X \xrightarrow{\eta} \Psi X$$

$$\downarrow^{\lambda(\Delta_X)} \downarrow^{\epsilon_0 \times \epsilon_1}$$

$$X_{\Delta} \times_{\epsilon_0} \Psi(X \times X) \xrightarrow{\rho(\Delta_X)} X \times X$$

These lifts exist since $\lambda(\Delta_X)$ is in $\mathfrak{F}(R)$ -coalg and $\epsilon_0 \times \epsilon_1$ is in $\mathfrak{F}(R)$ -alg.

6 Models of Id-types

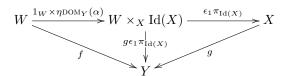
In this section, we show that a weak factorization system on a finitely complete category C models Id-types if and only if it models Id-types on objects. We use this result to also show that if a display map category (C, D) models Id-types,

then $(C, (^{\square}D)^{\square})$ also models Id-types regardless if they are functorial (this is not used in our main result, but was mentioned in Section 2.3 to motivate our work).

Consider Id-presentation R which at each object X of C has components denoted as follows.

$$X \stackrel{\stackrel{\epsilon_{0X}}{\longleftarrow}}{\stackrel{\eta_{X}}{\rightleftharpoons}} \Psi X,$$

For any object Y of C, let \mathcal{A}/Y denote the full subcategory of C/Y spanned by $\mathfrak{F}(R)$ -alg. Let $\mathrm{DOM}_Y: \mathcal{A}/Y \to C$ denote the domain functor. The weak factorization structure $\mathfrak{F}(R)$ induces a weak factorization structure $\mathfrak{F}(R)_Y$ [MP12, Thm. 15.3.6] which takes a morphism $\alpha: f \to g$ of \mathcal{A}/Y to the following.



The classes of coalgebras and algebras of this weak factorization structure in \mathcal{A}/Y are exactly the preimages of those in C: $\mathfrak{F}(R)_Y$ -coalg = $\mathrm{DOM}_Y^{-1}(\mathfrak{F}(R)$ -coalg) and $\mathfrak{F}(R)_Y$ -alg = $\mathrm{DOM}_Y^{-1}(\mathfrak{F}(R)$ -alg).

Lemma 6.1. The weak factorization structure $\mathfrak{F}(R)_Y$ on \mathcal{A}/Y is type-theoretic.

Proof. First, note that $1_Y \in \mathfrak{F}(R)$ -alg, so 1_Y is a terminal object of \mathcal{A}/Y . For any $f \in \mathcal{A}/Y$, the morphism to this terminal object is $f : f \to 1_Y$, whose image under DOM_Y is $f \in \mathfrak{F}(R)$ -alg. Thus, f is fibrant.

Now consider any $\ell \in \mathfrak{F}(R)_Y$ -coalg and $r \in \mathfrak{F}(R)_Y$ -alg who share a codomain. We have that $\mathrm{DOM}_Y(r^*\ell) = \mathrm{DOM}_Y(r)^*\mathrm{DOM}_Y(\ell) \in \mathfrak{F}(R)$ -coalg since $\mathfrak{F}(R)$ is typetheoretic (Theorem 5.4). Therefore, $r^*\ell \in \mathfrak{F}(R)_Y$ -coalg.

Proposition 6.2. Consider a weak factorization system with an Id-presentation R. Then $(C, \mathfrak{F}(R)\text{-}alg)$ models Id-types.

Proof. Since the weak factorization structure $\mathfrak{F}(R)_Y$ on C/Y is type-theoretic, it has an Id-presentation $\mathfrak{R}(\mathfrak{F}(R)_Y)$ which at each $f:X\to Y$ gives the following relation (depicted as a diagram in C).

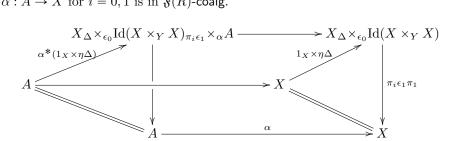
$$X \xrightarrow{\pi_0 \epsilon_1 \pi_1} X_{\Delta} \times_{\epsilon_0} \operatorname{Id}(X \times_Y X)$$

$$\xrightarrow{\pi_1 \epsilon_1 \pi_1} Y$$

$$(*)$$

Now we show that the collection of these Id-presentations is a model of Id-types in the display map category $(\mathsf{C},\mathfrak{F}(R)\text{-}\mathsf{alg})$. We just need to check that

any pullback $\alpha^*(1_X \times \eta \Delta)$ as shown below of $1_X \times \eta \Delta$ along any morphism $\alpha: A \to X$ for i = 0, 1 is in $\mathfrak{F}(R)$ -coalg.



Note that the image of diagram (*) above is a relation of X in C where $1_X \times \eta \Delta \in \mathfrak{F}(R)$ -coalg and $\epsilon_1 \pi_1 = \pi_0 \epsilon_1 \pi_1 \times \pi_1 \epsilon_1 \pi_1 \in \mathfrak{F}(R)$ -alg. Then by Corollary 4.7, the pullbacks shown above are in $\mathfrak{F}(R)$ -coalg when i=0. Similarly, by considering the involution I which swaps ϵ_0 and ϵ_1 (described in the proof of 4.9) applied to this relation, the pullbacks shown above are in $\mathfrak{F}(R)$ -coalg when i=1 by Corollary 4.7.

Corollary 6.3. Consider a display map category (C, D) which models Σ - and Id -types. Then $(C, (\square D)^{\square})$ also models Id -types.

Proof. Let $\{I_Y\}_{Y\in C}$ be the model of Id-types in (C,D). We obtain a model I_* of Id-types on objects in (C,D) (for a terminal object * in C). But I_* is trivially also a model of Id-types on objects in $(C,(^{\square}D)^{\square})$. By Proposition 6.2, $(C,(^{\square}D)^{\square})$ also models Id-types.

7 The main result

In Section 2, we defined the following diagram of categories.

$$\mathsf{fldPres}(\mathsf{C}) \overset{\mathfrak{F}}{\longleftarrow} \mathsf{fRel}(\mathsf{C}) \overset{\mathfrak{F}}{\overset{\mathfrak{F}}{\longleftarrow}} \mathsf{fFact}(\mathsf{C}) \overset{\mathsf{fttWFS}(\mathsf{C})}{\overset{\mathsf{F}}{\longleftarrow}} \mathsf{fttWFS}(\mathsf{C})$$

$$\mathsf{IdPres}(\mathsf{C}) \overset{\mathfrak{F}}{\longleftarrow} \mathsf{Rel}(\mathsf{C}) \overset{\mathfrak{F}}{\overset{\mathfrak{F}}{\longleftarrow}} \mathsf{Fact}(\mathsf{C}) \overset{\mathsf{fttWFS}(\mathsf{C})}{\overset{\mathsf{F}}{\longleftarrow}} \mathsf{fttWFS}(\mathsf{C})$$

In Section 3, we defined (strict) Moore relation systems. We showed that strict Moore relation systems generate type-theoretic, algebraic weak factorization systems, and that Moore relation systems generate type-theoretic weak factorization systems.

In Sections 4 and 5, we showed the following theorem.

Theorem 7.1. Consider a category C with finite limits. The functors \mathfrak{R} and \mathfrak{F} described above restrict to functors shown below.

Furthermore, after application of the proset truncation, these are equivalences.

Proof. The fact that \mathfrak{R} and \mathfrak{F} restrict to functors $\mathsf{ttWFS}(\mathsf{C}) \leftrightarrows \mathsf{IdPres}(\mathsf{C})$ is proven in Corollary 4.11 and Corollary 5.4. Then consider an object or morphism X of $\mathsf{fttWFS}(\mathsf{C})$. We know that $\mathfrak{R}(X) \in \mathsf{IdPres}(\mathsf{C}) \cap \mathsf{fRel}(\mathsf{C}) = \mathsf{fldPres}(\mathsf{C})$ so \mathfrak{R} restricts to a functor $\mathsf{fttWFS}(\mathsf{C}) \to \mathsf{fldPres}(\mathsf{C})$. Similarly, \mathfrak{F} restricts to a functor $\mathsf{fldPres}(\mathsf{C}) \to \mathsf{fttWFS}(\mathsf{C})$.

The fact that $|\mathfrak{R}|$ and $|\mathfrak{F}|$ constitute an equivalence $|\mathsf{IdPres}(\mathsf{C})| \simeq |\mathsf{ttWFS}(\mathsf{C})|$ is proven in Corollary 4.11 and Proposition 5.10. Since both squares in the diagram (*) above commute, this restricts to an equivalence $|\mathsf{fldPres}(\mathsf{C})| \simeq |\mathsf{fttWFS}(\mathsf{C})|$.

We then interpret this in the following theorem.

Theorem 7.2. Consider a category C with finite limits. The following properties of any weak factorization system (L, R) on C are equivalent:

- 1. it has an Id-presentation;
- 2. it is type-theoretic;
- 3. it is generated by a Moore relation system;
- 4. (C, R) is a display map category modeling Σ and Id-types.

Proof. The equivalence between (1) and (3) appears as Theorem 5.7.

That (2) implies (1) is Theorem 4.10.

That (3) implies (2) is Theorem 3.48.

Clearly, (4) implies (1).

By Theorem 2.35, (1) implies that all objects are fibrant in (L,R), and thus (C,R) is a display map category modeling Σ -types (Example 2.29). By Proposition 6.2, (1) implies that (C,R) models Id-types. Thus, (1) implies (4).

Theorem 7.3. Consider a category C with finite limits and a weak factorization system (L,R) satisfying the equivalent statements of the preceding theorem, 7.2. If (C,R) models pre- Π -types, then it models Π types. In particular, if C is locally cartesian closed, then (C,R) models Π -types.

Proof. By the previous theorem, 7.2, (L,R) is type-theoretic. Therefore, by Proposition 2.37, (C,R) models Π -types.

References

- [Bro73] K. S. Brown. Abstract homotopy theory and generalized sheaf cohomology. *Trans. Amer. Math. Soc.*, 186:419–458, 1973.
- [Cis06] D. Cisinski. Les préfaisceaux comme modèles des types d'homotopie. Astérisque, Paris, 2006.
- [Emm14] J. Emmenegger. A category-theoretic version of the identity type weak factorization system. ArXiv e-prints, 2014, 1412.0153.
- [GG08] N. Gambino and R. Garner. The identity type weak factorisation system. *Theoret. Comput. Sci.*, 409(1):94–109, 2008.
- [GT06] M. Grandis and W. Tholen. Natural weak factorization systems. Arch. Math. (Brno), 42(4):397–408, 2006.
- [Joy08] A. Joyal. Notes on quasi-categories. 2008. http://www.math.uchicago.edu/~may/IMA/Joyal.pdf.
- [Joy17] A. Joyal. Notes on Clans and Tribes. ArXiv e-prints, 2017, 1710.10238.
- [May75] J. P. May. Classifying spaces and fibrations. American Mathematical Society, Providence, RI, 1975.
- [MP12] J. P. May and K. Ponto. More concise algebraic topology. University of Chicago Press, Chicago, 2012.
- [Nor17] P. R. North. *Type theoretic weak factorization systems*. PhD thesis, University of Cambridge, 2017.
- [Nor19] P. R. North. Identity types and weak factorization systems in cauchy complete categories. *Math. Structures Comput. Sci.*, page 117, 2019.
- [Qui67] D. G. Quillen. Homotopical algebra. Springer-Verlag, Berlin, 1967.
- [RT02] J. Rosický and W. Tholen. Lax factorization algebras. J. Pure Appl. Algebra, 175(1-3):355–382, 2002.
- [Shu15] M. Shulman. Univalence for inverse diagrams and homotopy canonicity. *Math. Structures Comput. Sci.*, 25(5):1203–1277, 2015.

- [Str72] A. Strøm. The homotopy category is a homotopy category. Arch. Math.~(Basel),~23(1):435-441,~1972.
- $\begin{array}{ll} \hbox{[Tay99]} & \hbox{P. Taylor. } Practical \ foundations \ of \ mathematics. Cambridge \ University \ Press, \ Cambridge, \ UK, \ 1999. \end{array}$
- [vdBG12] B. van den Berg and R. Garner. Topological and simplicial models of identity types. $ACM\ Trans.\ Comput.\ Log.,\ 13(1):3:1–3:44,\ 2012.$