Hedonic Seat Arrangement Problems*

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Abstract

In this paper, we study a variant of hedonic games, called SEAT ARRANGEMENT. The model is defined by a bijection from agents with preferences to vertices in a graph. The utility of an agent depends on the neighbors assigned in the graph. More precisely, it is the sum over all neighbors of the preferences that the agent has towards the agent assigned to the neighbor. We first consider the price of stability and fairness for different classes of preferences. In particular, we show that there is an instance such that the price of fairness (PoF) is unbounded in general. Moreover, we show an upper bound d(G) and an almost tight lower bound d(G) - 1/4of PoF, where d(G) is the average degree of an input graph. Then we investigate the computational complexity of problems to find certain "good" seat arrangements, say MAXIMUM WELFARE ARRANGEMENT, MAXIMIN UTILITY ARRANGEMENT, STABLE ARRANGEMENT, and ENVY-FREE ARRANGEMENT. We give dichotomies of computational complexity of four SEAT ARRANGEMENT problems from the perspective of the maximum order of connected components in an input graph. For the parameterized complexity, MAXIMUM WELFARE ARRANGEMENT can be solved in time $n^{O(\gamma)}$, while it cannot be solved in time $f(\gamma)^{o(\gamma)}$ under ETH, where γ is the vertex cover number of an input graph. Moreover, we show that MAXIMIN UTILITY AR-RANGEMENT and ENVY-FREE ARRANGEMENT are weakly NP-hard even on graphs of bounded vertex cover number. Finally, we prove that determining whether a stable arrangement can be obtained from a given arrangement by k swaps is W[1]-hard when parameterized by $k + \gamma$, whereas it can be solved in time $n^{O(k)}$.

1 Introduction

Given a set of n agents with preferences for each other and an n-vertex graph, called the *seat graph*, we consider to assign each agent to a vertex in the graph. Each agent has a utility that depends on the agents assigned to neighbors vertices in the graph. Intuitively, if a neighbor is preferable for the agent, his/her utility is high. This models several situations such as seat arrangements in classrooms, offices, restaurants, or vehicles. Here, a vertex corresponds to a seat and an assignment corresponds to a seat arrangement. If we arrange seats in a classroom, the seat graph is a grid. As

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another example, if we consider a round table in a restaurant, the seat graph is a cycle. We name the model SEAT ARRANGEMENT.

SEAT ARRANGEMENT is related to hedonic games [7]. If the seat graph in a SEAT ARRANGE-MENT instance is a disjoint union of cliques, then each clique may be viewed as a potential coalition. Hence an arrangement on that graph naturally corresponds to a coalition forming. In that sense, this model is considered a hedonic game of arrangement on topological structures.

In this paper, we consider the following problems to find four types of *desirable* seat arrangements: MAXIMUM WELFARE ARRANGEMENT (MWA), MAXIMIN UTILITY ARRANGEMENT (MUA), STABLE ARRANGEMENT (STA), and ENVY-FREE ARRANGEMENT (EFA). MWA is the problem to find a seat arrangement that maximizes the sum of utilities of agents, which is called the *social welfare*.

The concept of MWA is a macroscopic optimality, and hence it may ignore individual utilities. Complementarily, MUA is the problem to find a seat arrangement that maximizes the least utility of an agent. From the viewpoint of economics, the maximin utility of an arrangement can be interpreted as a measure of fairness [3, 10, 33].

Stability is one of the central topics in the field of hedonic games including STABLE MATCH-ING [24, 34, 2, 7]. Motivated by this, we define a *stable* arrangement as an arrangement with no pair of agents that has an incentive of swapping their seats (i.e., vertices), called a *blocking pair*. This corresponds to the definition of *exchange-stability* proposed by Alcalde in the context of stable matchings [1]. In SEAT ARRANGEMENT, STA is the problem of deciding whether there is a stable arrangement in a graph.

Finally, we consider the envy-freeness of SEAT ARRANGEMENT. The envy-freeness is also a natural and well-considered concept in hedonic games. In SEAT ARRANGEMENT, an agent p envies another agent q if p has an incentive of swapping its seat with q. Note that q may not have an incentive of swapping its seat with p. By definition, any envy-free arrangement is stable.

1.1 Our contribution

In this paper, we first investigate the price of stability (PoS) and the price of fairness (PoF) of SEAT ARRANGEMENT, which are defined as the ratio of the maximum social welfare over the social welfare of a maximum stable solution and a maximin solution, respectively. For the price of stability, we can say the PoS is 1 under symmetric preferences by a result in [32]. For the price of fairness, we show that there is a family of instances such that PoF is unbounded. For the binary case, we show an upper bound of $\tilde{d}(G)$ of PoF, where $\tilde{d}(G)$ is the average degree of the seat graph G. On the other hand, we present an almost tight lower bound $\tilde{d}(G) - 1/4$ of PoF. Furthermore, we give a lower bound $\tilde{d}(G)/2 + 1/12$ for the cases with symmetric preferences.

Next, we give dichotomies of computational complexity of four SEAT ARRANGEMENT problems from the perspective of the maximum order of connected components in the seat graph. For MWA, MUA, and symmetric EFA, we show that they are solvable in polynomial time if the order of each connected component in the seat graph is at most 2 whereas they are NP-hard even if the order of each connected component of the seat graph is 3. Since a maximum arrangement is always stable under symmetric preferences, symmetric STA can also be solved in polynomial time if the order of each connected component is at most 2. On the other hand, STA is NP-complete even if the order of each connected component in the seat graph is at most 2. Note that if each connected component in the seat graph is of order at most 1, it consists of only isolated vertices, and hence STA is trivially solvable.

For the parameterized complexity, we show that MWA can be solved in time $n^{O(\gamma)}$ whereas it is W[1]-hard with respect to vertex cover number γ of the seat graph and cannot be solved in time $n^{o(n)}$ and $f(\gamma)n^{o(\gamma)}$ under ETH. Moreover, we prove that MUA and symmetric EFA are weakly NP-hard even on seat graphs with $\gamma = 2$.

Finally, we study the parameterized complexity of local search of finding a stable arrangement. We show that determining whether a stable arrangement can be obtained from a given arrangement by k swaps is W[1]-hard when parameterized by $k + \gamma$, whereas it can be solved in time $n^{O(k)}$.

1.2 Related work

A *hedonic game* is a non-transferable utility game regarding coalition forming, where each agent's utility depends on the identity of the other agents in the same coalition [18, 6]. It includes the STABLE MATCHING problem [7]. SEAT ARRANGEMENT can be considered a hedonic game of arrangement on a graph.

Several graph-based variants of hedonic games have been proposed in the literature, see e.g. [8, 9, 22, 2, 27]. However, they typically utilize graphs to define the preferences of agents, and both the preferences and coalitions define the utilities of agents. On the other hand, in SEAT ARRANGEMENT, the preferences are defined independently of a graph and the utility of an agent is determined by an arrangement in a graph (more precisely, the preferences for the assigned neighbors in the graph).

A major direction of research about hedonic games is the computational complexity of the problems to find desirable solutions such as a solution with maximum social welfare and a stable solution [6, 2]. [35] and [26] investigate the parameterized complexity of hedonic games for several graph parameters (e.g., treewidth). For the local search complexity, Gairing and Savani study the PLS-completeness of finding a stable solution [22, 23]. In terms of mechanism design and algorithmic game theory, many researchers study the price of anarchy, the price of stability, and the price of fairness [34, 3, 7, 4].

Possibly the closest relative of SEAT ARRANGEMENT among hedonic games is STABLE MATCH-ING. It is the model where agents are partitioned into pairs under preferences [24, 34, 7]. STABLE MATCHING is a well studied problem, and polynomial-time solvability as well as several structural properties are known, for example, [24, 29, 28, 30, 7]. One might think that STABLE MATCHING could be modeled as SEAT ARRANGEMENT on a graph in which each of connected components is an edge. However, these two models are slightly different due to the difference of the definitions of blocking pairs. A blocking pair in STABLE MATCHING can deviate from their partners and then they match each other, whereas they can only swap their seats in SEAT ARRANGEMENT. Alcalde proposes the *exchange-stability* [1]. Under the exchange-stability, a blocking pair does not deviate from their partner, but they swap each other. Cechlárová and Manlove proved that STABLE MAR-RIAGE and STABLE ROOMMATES are NP-complete under exchange stability even if the preference list is complete and strict [11, 12].

Very recently, in the context of one-sided markets, Massand and Simon consider the problem of allocating indivisible objects to a set of rational agents where each agents final utility depends on the intrinsic valuation of the allocated item as well as the allocation within the agents local neighbourhood [32]. Although the problem is motivated from different contexts, it has a quite similar nature to SEAT ARRANGEMENT, and they also considered stable and envy-free allocation on the problem. In fact, the following results about SEAT ARRANGEMENT are immediately obtained from [32]: (1) The PoS of SEAT ARRANGEMENT is 1. (2) There is an instance of binary SEAT ARRANGEMENT with no stable arrangement. (3) The local search problem to find a stable solution under symmetric preferences by swapping two agents iteratively is PLS-complete. (4) EFA is NPcomplete. In this paper, we give further and deeper analyses of SEAT ARRANGEMENT.

2 The Model

We use standard terminologies on graph theory. Let G = (V, E) be a graph where n = |V| and m = |E|. For a directed graph G, we denote the set of in-neighbors (resp., out-neighbors) of v by $N_G^{\text{in}}(v)$ (resp., $N_G^{\text{out}}(v)$) and the *in-degree* (resp., *out-degree*) of v by $d_G^{\text{in}}(v) := |N^{\text{in}}(v)|$ (resp., $d_G^{\text{out}}(v) := |N^{\text{out}}(v)|$). For an undirected graph G, we denote the set of neighbors of v by $N_G(v)$ and the *degree* of v by $d_G(v) = |N_G(v)|$. We also define $\Delta(G) = \max_{v \in V} d_G(v)$ and $\tilde{d}(G) = 2m/n$

as the maximum degree and the average degree of G, respectively. A vertex cover X is the set of vertices such that for every edge, at least one endpoint is in X. The vertex cover number of G, denoted by $\gamma(G)$, is the size of a minimum vertex cover in G. For simplicity, we sometimes drop the subscript of G if it is clear. For the basic definition of parameterized complexity such as the classes FPT, XP and W[1], we refer the reader to the book [15].

We denote by \mathbf{P} the set of agents, and define an *arrangement* as follows.

Definition 1 (Arrangement). For a set of agents **P** and an undirected graph G, a bijection π : $\mathbf{P} \to V(G)$ is called an arrangement in G.

We denote by Π the set of all arrangements in G. Note that $|\Pi| = n!$. We call graph G the *seat graph*. The definition means that an arrangement assigns each agent to a vertex in G. When we specify that the seat graph G is in some graph class \mathcal{G} , we sometimes use term SEAT ARRANGEMENT on \mathcal{G} . Moreover, we define the (p, q)-swap arrangement for π .

Definition 2 ((p,q)-swap arrangement). Let **P** be a set of agents, G be a graph and π be an arrangement. For a pair of agents $p, q \in \mathbf{P}$, we say that π' is the (p,q)-swap arrangement if π' can obtained from π by swapping the arrangement of p and q, that is, π' satisfies that $\pi'(p) = \pi(q)$, $\pi'(q) = \pi(p)$, and $\pi'(r) = \pi(r)$ for every $r \in \mathbf{P} \setminus \{p,q\}$.

Next, we define the *preference* of an agent.

Definition 3 (Preference). The preference of $p \in \mathbf{P}$ is defined by $f_p : \mathbf{P} \setminus \{p\} \to \mathbb{R}$.

We denote by $\mathcal{F}_{\mathbf{P}}$ the set of preferences of all agents in \mathbf{P} . Here, we say the preferences are binary if $f_p : \mathbf{P} \setminus \{p\} \to \{0,1\}$ for every agent p, are nonnegative if $f_p : \mathbf{P} \setminus \{p\} \to \mathbb{R}_0^+$, and are positive if $f_p : \mathbf{P} \setminus \{p\} \to \mathbb{R}^+$. Furthermore, we say they are symmetric if $f_p(q) = f_q(p)$ holds for any pair of agents $p, q \in \mathbf{P}$ and strict if for any $p \in \mathbf{P}$ there is no pair of distinct $q, r \in \mathbf{P}$ such that $f_p(q) = f_p(r)$. The directed and weighted graph $G_{\mathcal{F}_{\mathbf{P}}} = (\mathbf{P}, E_{\mathcal{F}_{\mathbf{P}}})$ associated with the preferences $\mathcal{F}_{\mathbf{P}}$ is called the preference graph, where $E_{\mathcal{F}_{\mathbf{P}}} = \{(p,q) \mid f_p(q) \neq 0\}$ and the weight of (p,q) is $f_p(q)$. If the preferences are symmetric, we define the corresponding preference graph as an undirected graph.

Finally, we define the *utility* of an agent and the *social welfare* of an arrangement π .

Definition 4 (Utility and social welfare). Given an arrangement π and the preference of p, the utility of p is defined by $U_p(\pi) = \sum_{v \in N(\pi(p))} f_p(\pi^{-1}(v))$. Moreover, the social welfare of π for \mathbf{P} is defined by the sum of all utilities of agents and denoted by $\mathsf{sw}(\pi) = \sum_{p \in \mathbf{P}} U_p(\pi)$.

The function $U_p(\pi) = \sum_{v \in N(\pi(p))} f_p(\pi^{-1}(v))$ represents the sum over all neighbors of the preferences that the agent has towards the agent assigned to the neighbor. This function is often used in coalition formation games [8, 22, 2]. By the definition, if the seat graph is a complete graph, all the arrangements have the same social welfare.

In the following, we define four types of SEAT ARRANGEMENT problems. First, we define MAXIMUM WELFARE ARRANGEMENT. An arrangement π^* is maximum if it satisfies $sw(\pi^*) \ge sw(\pi)$ for any arrangement π . Then, MAXIMUM WELFARE ARRANGEMENT (MWA) is defined as follows.

Input: A graph G = (V, E), a set of agents **P**, and the preferences of agents $\mathcal{F}_{\mathbf{P}}$.

Task: Find a maximum arrangement in G.

An arrangement π^* is a maximin arrangement if π^* satisfies $\min_{p \in \mathbf{P}} U_p(\pi^*) \ge \min_{p \in \mathbf{P}} U_p(\pi)$ for any arrangement π . Then, MAXIMIN UTILITY ARRANGEMENT (MUA) is defined as follows.

Input: A graph G = (V, E), a set of agents **P**, and the preferences of agents $\mathcal{F}_{\mathbf{P}}$.

Task: Find a maximin arrangement in G.

Finally, we define the *stability* of SEAT ARRANGEMENT.

Definition 5 (STABLILITY). Given an arrangement π , a pair of agents p and q is called a blocking pair for π if it satisfies that $U_p(\pi') > U_p(\pi)$ and $U_q(\pi') > U_q(\pi)$ where π' is the (p,q)-swap arrangement for π . If there is no blocking pair in an arrangement, it is said to be stable.

Then, the STABLE ARRANGEMENT (STA) problem is as follows.

Input: A graph G = (V, E), a set of agents **P**, and the preferences of agents $\mathcal{F}_{\mathbf{P}}$.

Task: Decide whether there is a stable arrangement in G.

Finally, we define the *envy-freeness* of SEAT ARRANGEMENT.

Definition 6 (ENVY-FREE). An arrangement π is envy-free if there is no agent p such that there exists $q \in \mathbf{P} \setminus \{p\}$ that satisfies $U_p(\pi') > U_p(\pi)$ where π' is the (p,q)-swap arrangement for π .

By the definition of envy-freeness, we have:

Proposition 1. If an arrangement is envy-free, it is also stable.

Then, ENVY-FREE ARRANGEMENT (EFA) is defined as follows.

Input: A graph G = (V, E), a set of agents **P**, and the preferences of agents $\mathcal{F}_{\mathbf{P}}$.

Task: Decide whether there is an envy-free arrangement in G.

3 Stability and Fairness

In this section, we study the stability and the fairness of SEAT ARRANGEMENT. Let Π_s be the set of stable solutions and π^* be a maximum arrangement. Then, the price of stability (PoS) is defined as $\min_{\pi_s \in \Pi_s} sw(\pi^*)/sw(\pi_s)$ [34]. In other words, the price of stability is defined as the gap between the maximum social welfare and the social welfare of a maximum stable solution. From [32], we immediately obtain the following proposition.

Proposition 2 ([32]). In symmetric SEAT ARRANGEMENT, a maximum arrangement is stable, and thus, the PoS of symmetric SEAT ARRANGEMENT is 1. On the other hand, there is an instance of binary SEAT ARRANGEMENT with no stable arrangement.

Proposition 2 is easily shown by the potential function argument. On the other hand, there is an instance with no envy-free arrangement even if the preferences are symmetric. Consider three agents x, y, z such that $f_p(q) = f_q(p) = 1$ for $p, q \in \{x, y, z\}$. If we assign them to a path P_3 , two agents assigned to endpoints of P_3 envy the agent assigned to the center in P_3 .

Proposition 3. There is an instance of binary and symmetric SEAT ARRANGEMENT with no envy-free arrangement.

Next, we consider the price of fairness [3, 10, 7]. Let Π_f be the set of maximin solutions. Then, the price of fairness is defined as $\min_{\pi_f \in \Pi_f} \mathsf{sw}(\pi^*)/\mathsf{sw}(\pi_f)$, that is, the ratio between the maximum social welfare and the social welfare of a maximin arrangement.

Proposition 4. There is an instance such that the PoF of SEAT ARRANGEMENT is unbounded.

Proof. Let $x \ge y \ge 1$ be two integers and the seat graph G be a graph consisting of two edges. Finally, we set the preferences of four agents p_1, p_2, p_3, p_4 as follows: $f_{p_1}(p_3) = f_{p_2}(p_4) = f_{p_3}(p_2) = f_{p_4}(p_1) = x$, $f_{p_1}(p_2) = f_{p_2}(p_1) = f_{p_3}(p_4) = f_{p_4}(p_3) = y$, and $f_{p_1}(p_4) = f_{p_2}(p_3) = f_{p_3}(p_1) = f_{p_4}(p_2) = 0$.

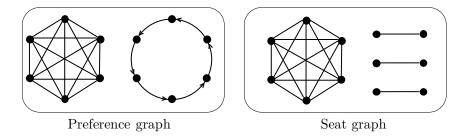


Figure 1: The preference graph (left) and the seat graph (right) in the proof of Proposition 6.

Since the seat graph consists of two edges, the instance has only three arrangements by symmetry. If p_1, p_2 (resp., p_3, p_4) are assigned to the same edge, the social welfare is 4y and each agent has the utility y. On the other hand, if p_1, p_3 (resp., p_2, p_4) or p_1, p_4 (resp., p_2, p_3) are assigned to the same edge, the social welfare is 2x and the least utility is 0. If x is arbitrary large integer and y = 1, the social welfare of a maximum arrangement is 2x. Then, we have PoF = 2x/4y = x/2y, and hence PoF is unbounded.

For the binary case, the PoF is bounded by the average degree $\tilde{d}(G)$ of the seat graph G. If the least utility is 0 for every arrangement, we can choose an arrangement with maximum social welfare. In this case, PoF is 1. Otherwise, the least utility is 1 and the social welfare of such an arrangement is at least n. Since the social welfare is at most 2m, PoF is bounded by $2m/n = \tilde{d}(G)$.

Proposition 5. For any instance, the PoF of binary SEAT ARRANGEMENT is at most d(G).

Finally, we give almost tight lower bounds of the price of fairness for binary SEAT ARRANGE-MENT.

Proposition 6. There is an instance such that the PoF of binary SEAT ARRANGEMENT is at least $\tilde{d}(G) - 1/4$. Furthermore, there is an instance such that the PoF of binary and symmetric SEAT ARRANGEMENT is at least $\tilde{d}(G)/2 + 1/12$.

Proof. We construct such an instance. Let \mathbf{P}_K and \mathbf{P}_C be sets of agents each having n members. The preference graph consists of an undirected clique and a directed cycle. The seat graph G consists of a clique of size n and n/2 disjoint edges. The number of edges in G is $n(n-1)/2+n/2 = n^2/2$ and the average degree of G is $\tilde{d}(G) = n/2$. See Figure 1.

In a maximum arrangement on G, every agent in \mathbf{P}_K is assigned to a clique and every agent in \mathbf{P}_C is assigned to disjoint edges. The social welfare is n(n-1) + n/2 = n(n-1/2) and at least one agent on an edge has the utility 0.

On the other hand, in a maximin arrangement, every agent in \mathbf{P}_C is assigned to a clique and every agent in \mathbf{P}_K is assigned to disjoint edges. Then the utility of any agent is 1 and the social welfare is n + n = 2n. Therefore, the price of fairness is $n(n - 1/2)/2n = n/2 - 1/4 \ge \tilde{d}(G) - 1/4$.

For the symmetric case, we modify the preference graph and the seat graph. For the preference graph, we change a directed cycle in the preference graph to an undirected cycle. For the seat graph, we replace n/2 disjoint edges by n/3 disjoint triangles K_3 . When every agent in \mathbf{P}_K is assigned to a clique, the social welfare is n(n-1) + 4n/3 = n(n+1/3) and it is maximum. The least utility of an agent is 1. On the other hand, when every agent in \mathbf{P}_C is assigned to a clique, the utility of any agent is 2. The social welfare is 2n + 6n/3 = 4n. Therefore, the price of fairness is $n(n+1/3)/4n = n/4 + 1/12 \ge \tilde{d}(G)/2 + 1/12$.

4 Computational Complexity

In this section, we give the dichotomy of computational complexity of three SEAT ARRANGEMENT problems in terms of the order of components in the seat graph.

4.1 Tractable case

In this subsection, we show that MWA, MUA, and symmetric EFA are solvable in polynomial time if each component of the seat graph is of order at most 2.

Theorem 1. MWA is solvable in polynomial time if each connected component of the seat graph has at most two vertices.

Proof. Let $K_n = (\mathbf{P}, E_{\mathbf{P}})$ be the weighted and undirected complete graph such that the weight of edge $(p, q) \in E_{\mathbf{P}}$ is defined by $f_p(q) + f_q(p)$. Also, let n' be the number of endpoints of edges in the seat graph. Notice that n' is always even. Then, we find a maximum weight matching $M_{n'/2}$ of size n'/2 in K_n . This can be done by using Edmonds's algorithm [19]. Next, we assign each pair of agents in $M_{n'/2}$ to an edge in G and the rest of agents in isolated vertices. Let π^* be such an arrangement.

In the following, we show that π^* is maximum. Suppose that there exists π' such that $\mathsf{sw}(\pi') > \mathsf{sw}(\pi^*)$. Let V_E be the set of endpoints of E and V_I be the set of isolated vertices. Since the size of each connected component in G is bounded by 2, for any π , we have $\mathsf{sw}(\pi) = \sum_{v \in V_E} U_{\pi^{-1}(v)}(\pi) + \sum_{v \in V_I} U_{\pi^{-1}(v)}(\pi) = \sum_{v \in V_E} U_{\pi^{-1}(v)}(\pi) = \sum_{(u,v) \in E} (f_{\pi^{-1}(u)}(\pi^{-1}(v)) + f_{\pi^{-1}(v)}(\pi^{-1}(u)))$. Now, the number of endpoints of edges is n', |E| = n'/2. Moreover, for any π , $M = \{(\pi^{-1}(u), \pi^{-1}(v)) \mid (u, v) \in E\}$ is a matching of size n'/2 with weight $\mathsf{sw}(\pi) = \sum_{(u,v) \in E} (f_{\pi^{-1}(u)}(\pi^{-1}(v)) + f_{\pi^{-1}(v)}(\pi^{-1}(u)) = \sum_{(p,q) \in M} (f_p(q) + f_q(p))$ in K_n . Thus, if there exists π' such that $\mathsf{sw}(\pi') > \mathsf{sw}(\pi^*)$, there exists a heavier matching of size n'/2 than $M_{n'/2}$. This contradicts that $M_{n'/2}$ is a maximum weight matching of size n'/2.

By using a maximin matching algorithm proposed in [21] instead of Edmonds's algorithm, we can solve MUA in polynomial time. We apply that to the weighted and undirected complete graph such that the weight of edge $\{p,q\} \in E_{\mathbf{P}}$ is defined by $\min\{f_p(q), f_q(p)\}$.

Theorem 2. MUA is solvable in polynomial time if each connected component of the seat graph has at most two vertices.

Proof. Let $K_n = (\mathbf{P}, E_{\mathbf{P}})$ be the weighted and undirected complete graph such that the weight of edge $(p,q) \in E_{\mathbf{P}}$ is defined by $\min\{f_p(q), f_q(p)\}$. Also, let n' be the number of endpoints of edges in the seat graph. Next, we find a maximum matching M of size n' in K_n , which is a matching of size n' such that the minimum weight of edges in M is maximum. It can be computed in time $O(m\sqrt{n\log n})$ [21]. Then we assign n'/2 pairs of agents in M to endpoints of an edge in G and the rest of agents to isolated vertices.

In the following, we confirm that such an arrangement, denoted by π , is a maximin arrangement. If the least utility on π is 0 and there is at least one isolate vertex, it is clearly a maximin arrangement because an agent with the least utility is on an isolate vertex. Otherwise, an agent with the least utility is on an edge. Here, we denote an agent with the least utility in π by p_l^{π} and its utility by $U_{p_l^{\pi}}(\pi)$.

Suppose that π is not a maximin arrangement. Then there exists an assignment π' and an agent $p_l^{\pi'}$ with the least utility in π' such that $U_{p_l^{\pi'}}(\pi') > U_{p_l^{\pi}}(\pi)$. We also denote an agent $p_{l_e}^{\pi'}$ with the least utility among agents on edges in π' . Note that $U_{p_{l_e}^{\pi'}}(\pi') \ge U_{p_l^{\pi'}}(\pi')$. Let q' be a partner of $p_{l_e}^{\pi'}$ on an edge. Since the seat graph consists of n'/2 edges, we have $U_{p_{l_e}^{\pi'}}(\pi') = f_{p_{l_e}^{\pi'}}(q')$. Here, we define $M' = \{(\pi'^{-1}(u), \pi'^{-1}(v)) \mid (u, v) \in E\}$. Then M' is a matching of size n'/2 in K_n . Because the weight of an edge in K_n is defined as $\min\{f_p(q), f_q(p)\}$, the least weight of an edge in M' is $U_{p_{l_e}^{\pi'}}(\pi') = f_{p_l^{\pi'}}(q')$. Since we have $U_{p_{l_e}^{\pi'}}(\pi) > U_{p_l^{\pi}}(\pi)$, this contradicts that M is a maximin matching of size n'/2.

For EFA, we show several cases that it can be solved in polynomial time. The following theorem can be shown by taking a perfect matching on the *best-preference graph* $G_{\mathcal{F}_{\mathbf{P}}}^{\text{best}} = (\mathbf{P}, E'_{\mathcal{F}_{\mathbf{P}}})$, where

 $E'_{\mathcal{F}_{\mathbf{P}}} = \{\{p,q\} \in E_{\mathcal{F}_{\mathbf{P}}} \mid f_p(q) \ge f_p(q') \text{ for all } q' \in P \setminus \{p\} \text{ and } f_q(p) \ge f_q(p') \text{ for all } p' \in P \setminus \{q\}\}.$ Note that $G^{\text{best}}_{\mathcal{F}_{\mathbf{P}}}$ is a bidirectional graph and hence it can be regarded as an undirected graph.

Theorem 3. EFA can be solved in polynomial time if each connected component of the seat graph is an edge.

Proof. We first observe that each agent must match to the most preferable agent on an edge. If not so, an agent that does not match to the most preferable agent envies the agent that matches to it. Thus, we consider the best preference graph $G_{\mathcal{F}_{\mathbf{P}}}^{\text{best}} = (\mathbf{P}, E'_{\mathcal{F}_{\mathbf{P}}})$ where $E'_{\mathcal{F}_{\mathbf{P}}} = \{(p,q) \in E_{\mathcal{F}_{\mathbf{P}}} \mid q =$ $\arg\max_{q' \in \mathbf{P} \setminus \{p\}} f_p(q)\}$. Note that there may exist a vertex (i.e., an agent) with $d_{G'_{\mathcal{F}_{\mathbf{P}}}}(p) \geq 2$. Here, we observe that p and q such that $f_p(q) \in E'_{\mathcal{F}_{\mathbf{P}}}$ but $f_q(p) \notin E'_{\mathcal{F}_{\mathbf{P}}}$ are matched, q envies the agent matched to the most preferable agent. Thus, any pair of agents p and q satisfies $f_p(q), f_q(p) \in E'_{\mathcal{F}_{\mathbf{P}}}$ in any envy-free arrangement. Therefore, we consider the undirected graph H'' such that each edge (p,q) corresponds to a bidirectional $edge(p,q) \in E'_{\mathcal{F}_{\mathbf{P}}}$. It is easily seen that there is a perfect matching in H'' if and only if there is an envy-free arrangement in G.

Theorem 4. Symmetric EFA can be solved in polynomial time if each connected component of the seat graph has at most two vertices.

Proof. Let $(G, \mathbf{P}, \mathcal{F}_{\mathbf{P}})$ be an instance of symmetric EFA. We denote by E and I the sets of edges and isolated vertices of G, respectively. By Theorem 3, we may assume that $I \neq \emptyset$. Now, if there is an agent p such that $f_p(q) < 0$ for all $q \in \mathbf{P} \setminus \{p\}$, it has to be assigned to an isolated vertex $v \in I$. If not so, p envies the agents assigned to isolated vertices. Thus, we can reduce the instance $(G, \mathbf{P}, \mathcal{F}_{\mathbf{P}})$ to $(G - v, \mathbf{P} \setminus \{p\}, \mathcal{F}_{\mathbf{P} \setminus \{p\}})$. In the following, we assume that there is no such agent.

Let $\mathbf{P}_0 = \{p \in \mathbf{P} \mid \max_{q \in \mathbf{P} \setminus \{p\}} f_p(q) = 0\}$ and $\mathbf{P}_+ = \{p \in \mathbf{P} \mid \max_{q \in \mathbf{P} \setminus \{p\}} f_p(q) > 0\}$. The assumption above implies that $\mathbf{P} = \mathbf{P}_0 \cup \mathbf{P}_+$. Let $H_0 = (\mathbf{P}_0, E_0)$ be the undirected graph with $E_0 = \{\{p,q\} \mid f_p(q) = f_q(p) = 0\}$. Similarly, let $H_+ = (\mathbf{P}_+, E_+)$ be the undirected graph with $E_+ = \{\{p,q\} \mid f_p(q) = f_q(p) > 0\}$.

Let C be a connected component of H_+ . Observe that if C contains an edge $\{u, v\}$ such that u is assigned to an endpoint of $e \in E$ and v is assigned to a vertex in I, then v envies the agent assigned to the other endpoint of e. Since C is connected, this implies that in every envy-free arrangement either all agents in C are assigned to E or all agents in C are assigned to I.

Observe also that if two agents with a negative mutual preference are assigned to an edge in E, then they envy the agents assigned to I. Therefore, for each $e \in E$, the agents assigned to e are adjacent in H_0 or H_+ . Furthermore, if an agent $p \in \mathbf{P}_+$ is assigned to an endpoint of $e \in E$, then all neighbors of p in H_+ are assigned to endpoints of some edges in E, and thus the agent assigned to the other endpoint of e has to be one of the most preferable ones to make p envy-free.

The discussion so far implies that there is an envy-free arrangement if and only E can be completely packed with some connected components of H_+ and some edges of H_0 so that the agents assigned to each $e \in E$ are best-preference pairs.

For each component C of H_+ , we check whether C has a perfect matching that uses only the best-preference edges. If C has no such matching, then it has to be packed into I. Let C_1, \ldots, C_h be the connected components of H_+ with such perfect matchings.

We now compute the maximum value k of the MAXIMUM 0-1 KNAPSACK instance with h items such that the weight and the value of the *i*th item are $|C_i|/2$ and the budget is m = |E|. This can be done in time polynomial in n [25]. We can see that there is an envy-free arrangement if and only if $k + k' \ge m$, where k' is the size of a maximum matching of H_0 . An envy-free arrangement can be constructed by first packing the best-preference perfect matchings in the components corresponding to the chosen items into E, and then packing a matching of size $m - k \le k'$ in H_0 into the unused part of E.

Theorem 5. EFA can be solved in polynomial time if each connected component has at most two vertices and the preferences are strict or positive.

Proof. If the preferences are strict, each agent must be matched with the agent that it prefers most. Thus, it is sufficient to check whether the most preferable agent of p likes p the most. Also, if the preferences are positive, whenever there is an isolated vertex in the seat graph, then an agent assigned to it envies other agents assigned to an edge. Thus, we can suppose that there is no isolated vertex and apply Theorem 2.

4.2 Intractable case

First, we show that STABLE ROOMMATES with the complete preference list under exchange stability can be transformed into STA. Let n be the number of agents in STABLE ROOMMATES. According to the complete preference order of agent p in EXCHANGE STABLE ROOMMATES, one can assign values from 1 to n - 1 to the preferences of p to other agents in STA. Moreover, let G be the seat graph consisting of n/2 disjoint edges. Then it is easily seen that there is a stable matching if and only if there is a stable arrangement in G. Since STABLE ROOMMATES with complete preference list under exchange stability is NP-complete [11, 12], STA is also NP-complete.

Theorem 6. STA is NP-complete even if the preferences are positive and each component of the seat graph is of order two.

Then we prove that symmetric EFA is NP-complete even if each connected component has at most three vertices.

Theorem 7. Symmetric EFA is NP-complete even if each connected component of the seat graph has at most three vertices.

Proof. We give a reduction from PARTITION INTO TRIANGLES: given a graph G = (V, E), determine whether V can be partitioned into 3-element sets $S_1, \ldots, S_{|V|/3}$ such that each S_i forms a triangle K_3 in G. The problem is NP-complete [25].

Given a graph G = (V, E), we construct the instance of EFA. First, we set $\mathbf{P} = V \cup \{x, y, z\}$. Three agents x, y, z are called *super agents*. Then we define the preferences as follows. For $p, q \in \{x, y, z\}$, we set $f_p(q) = f_q(p) = 2$. For $p \in \{x, y, z\}, q \in V$, we set $f_p(q) = f_q(p) = 1$. Finally, for $p, q \in V$, we set $f_p(q) = f_q(p) = 1$ if $(p, q) \in E$, and otherwise, $f_p(q) = f_q(p) = 0$. Clearly, the preferences are symmetric. The seat graph H consists of |V|/3 + 1 disjoint triangles.

Given a partition $S_1, \ldots, S_{|V|/3}$ of V, we assign them to triangles in the seat graph. Moreover, we assign $\{x, y, z\}$ to a triangle. Then the utilities of agents in V are 2 and the utilities of x, y, z are 4, respectively. Since these utilities are maximum for all agents, this arrangement is envy-free. Conversely, we are given an envy-free arrangement π .

Claim 7.1. In any envy-free arrangement π , $\{x, y, z\}$ is assigned to the same triangle in H.

Proof. Suppose that x is assigned to a triangle T_x and y is assigned to another triangle T_y . If z is assigned to T_x , y envies an the agent p in V assigned to T_x because the utility of y is increased from 2 to 4 by swapping y and p. Similarly, if z is assigned to another triangle T_z , y envies the agent in V assigned to T_x because the utility of y is increased from 2 to 3 by swapping y and p. Thus, $\{x, y, z\}$ must be assigned to the same triangle.

Then, if three agents $p, q, w \in V$ such that $(p, q) \notin E$ are assigned to the same triangle in H, p envies x because the utility of p is increased from 1 to 2 by swapping p and x. Since π is envy-free, for each triangle assigned to $p, q, r \in V$, they satisfy $(p, q), (q, r), (r, p) \in E$. This implies that there is a partition $S_1, \ldots, S_{|V|/3}$ of V in G.

By the definition, if the seat graph is a complete graph, the arrangement is all arrangements are equivalent. However, if the seat graph consists of a clique and an independent set, EFA is NP-complete. The reduction is from k-CLIQUE [25].

Theorem 8. EFA is NP-complete even if the seat graph consists of a clique and an independent set.

Proof. We give a reduction from k-clique. We are given a graph G = (V, E). For each $e \in E$, we use the corresponding agent p_e . Also, for each $v \in V$, we make K agents $p_v^{(1)}, \ldots, p_v^{(M)}$. The number of agents is |E| + M|V|. Then we define the preferences of agents. For p_e , we define $f_{p_e}(p_u^{(1)}) = f_{p_e}(p_v^{(1)}) = 1$ if e = (u, v), and otherwise $f_{p_e}(p_u^{(1)}) = f_{p_e}(p_v^{(1)}) = 0$. For each $v \in V$ and i, j, we set $f_{p_v^{(i)}}(p_v^{(j)}) = f_{p_v^{(j)}}(p_v^{(i)}) = 1$. Finally, we define the seat graph $G' = (I \cup C, E')$ as a graph consisting of an independent set I of size Mk + k(k-1)/2 and a clique C of size |E| + (|V| - k)M - k(k-1)/2.

In the following, we show that there is a clique of size k in G if and only if there is an envy-free arrangement in G'. Given a clique of size k, we assign all agents $p_v^{(i)}$ and p_e corresponding to a clique to vertices in I. Since the number of such agents is Mk + k(k-1)/2, the set of vertices not having a agent in G' is C. Thus, we assign other agents to vertices in C. Because the utility of an agent on I does not increase even if he is swapped for any agent on C, every agent on I is envy-free. Moreover, an agent on C is envy-free because the utility is at least 1 and C is a clique. Therefore, such an arrangement is envy-free.

Conversely, we are given an envy-free arrangement π . First, we observe the following fact.

Fact 1. If some $p_v^{(i)}$ is on *I*, all the $p_v^{(j)}$ for $j \neq i$ must be on *I*.

Otherwise, $p^{(i)}$ envy some agent on C because the utility increases by moving to C. Also, we have the following fact.

Fact 2. If p_e where e = (u, v) is on *I*, all the $p_u^{(i)}$ and $p_v^{(i)}$ must be on *I*.

If not so, $p_u^{(1)}$ is on C because every $p_u^{(i)}$ must be on C by Fact 1. However, this implies that p_e envies some agent on C. This is a contradiction.

Now, since |I| = Mk + k(k-1)/2, at most $Mk \ p_v^{(i)}$'s are on I from Fact 1. In other words, there are at most k vertices in V such that $p_v^{(i)}$ is on I for all i. The remaining vertices in I have p_e . From Facts 1 and 2, for every $e = (u, v) \in E$ such that $p_e \in I$, all $p_u^{(i)}$ and $p_v^{(i)}$ for all i must be on I. Because the number of p_e 's on I is at least k(k-1)/2, I must have exactly $k(k-1)/2 \ p_e$'s and $Mk \ p_v^{(i)}$'s such that v is an endpoint of e, so that π is envy-free. This implies that $\{v \mid p_v^{(1)} \in I\}$ is a clique of size k in G.

Finally, we show that EFA is NP-complete even if both the preference graph and the seat graph are restricted. The reduction is from 3-PARTITION [25].

Theorem 9. EFA is NP-complete even if the preference graph is a directed acyclic graph (DAG) and the seat graph is a tree.

Proof. We give a reduction from 3-PARTITION. The problem is strongly NP-complete [25] and defined as follows: Given a set of integers $A = \{a_1, \ldots, a_{3n}\}$, find a partition (A_1, \ldots, A_n) such that $|A_i| = 3$ and $\sum_{a \in A_i} a = B$ for each i where $B = \sum_{a \in A} a/n$. We call such a partition a 3-partition. First, we prepare n agents $P_T = \{p_{t_1}, \ldots, p_{t_n}\}$ corresponding to the resulting triples and 3n agents $P_A\{p_{a_1}, \ldots, p_{a_{3n}}\}$ corresponding to elements. Moreover, we use an agent p_r , called a root agent. Then we define the preferences and the seat graph as in Figure 2. For each p_{t_j} , we set $f_{p_{t_j}}(p_r) = B$ and $f_{p_t}(p_a) = a$. The remaining preference from p to q is 0. Note that the preference graph is a DAG. Then we define the seat graph G = (V, E), which is a tree with the root vertex v_r . The root vertex has n children and its children have exactly three children. The number of vertices is 4n + 1.

Given a 3-partition (A_1, \ldots, A_n) , we assign root agent p_r to v_r . Moreover, for three elements in A_i , we assign the three corresponding agents to leaves with the same parent in G. Finally, we assign p_t to an inner vertex in G arbitrarily.

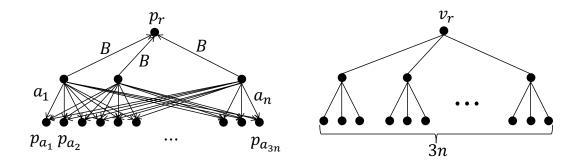


Figure 2: The preference graph (left) and the seat graph in the proof of Theorem 9

We show that this arrangement, denoted by π , is envy-free. By the definitions of preferences, the utilities of p_r and $p_a \in P_A$ are 0 for any arrangement. Thus, they are envy-free. Each $p_t \in P_T$ is also envy-free because every utility of p_t is $B^2 + B$ and the preferences of p_t to agents in $P_A \cup \{p_r\}$ are identical.

Conversely, we are given an envy-free arrangement π . Suppose that p_r is not assigned to the root vertex v_r . Then the degree of $\pi(r)$ is at most 4. Since there is $p_t \in P_T \setminus N(\pi(r))$ and $U_{p_t}(\pi) < B$, p_t envies a neighbor of p_r . Therefore, p_r must be assigned to v_r .

If an agent p_t in P_T is assigned to a leaf in G, p_t does not have p_r as a neighbor. Thus, the utility of p_t is less than B and p_t envies a neighbor of p_r . Since the number of inner vertices is n, every agent in P_A must be assigned to a leaf. If there is $p_t \in P_T$ with utility more than 2B, there is $p'_t \in P_T$ with utility less than 2B since $\sum_{a \in A} a = nB$ and $f_{p_{t_j}}(p_r) = B$ for every p_{t_j} . In this case, p'_t envies p_t . Therefore, the utility of each agent in P_T is exactly 2B. Because $f_{p_{t_j}}(p_r) = B$ for every p_{t_j} , if we partition A according to neighbors of p_t , the resulting partition is a 3-partition. \Box

Next, we show that MWA and MUA are NP-complete for several graph classes by reductions from SPANNING SUBGRAPH ISOMORPHISM. Here, we give the definition of SPANNING SUBGRAPH ISOMORPHISM as follows: given two graphs G = (V(G), E(G)) and H = (V(H), E(H)) where |V(G)| = |V(H)|, determine whether there is a bijection $g : V(G) \to V(H)$ such that $(g(u), g(v)) \in E(H)$ for any $(u, v) \in E(G)$.

SPANNING SUBGRAPH ISOMORPHISM is NP-complete even if G is a path and a cycle by a reduction from HAMILTONIAN PATH and HAMILTONIAN CYCLE [25]. Moreover, it is NP-complete if G is proper interval, trivially perfect, split, and bipartite permutation [31]. When G is disconnected, it is also NP-complete even if G is a forest and a cluster graph whose components are of size three [25, 5]. Here, if G is in some graph class \mathcal{G} in SPANNING SUBGRAPH ISOMORPHISM, we call the problem SPANNING SUBGRAPH ISOMORPHISM of \mathcal{G} . Then we give the following theorems.

Theorem 10. If SPANNING SUBGRAPH ISOMORPHISM of \mathcal{G} is NP-complete, then MWA on \mathcal{G} is NP-hard even if the preferences are binary and symmetric.

Proof. Given an instance of SPANNING SUBGRAPH ISOMORPHISM (G, H), we construct an instance of MWA as follows. Let $\mathbf{P} = V(H)$ be the set of agents and G be the seat graph. Then we set the preferences of agents as follows:

$$\begin{cases} f_p(q) = f_q(p) = 1 & \text{if } (p,q) \in E(H) \\ f_p(q) = f_q(p) = 0 & \text{otherwise.} \end{cases}$$

That is, the preference graph is H. By the definition, the preferences of agents are symmetric. We complete the proof by showing that there is a bijection g such that $(g(u), g(v)) \in E(H)$ for any $(u, v) \in E(G)$ if and only if there exists an arrangement π with social welfare 2|E(G)| in G.

Let $\pi = g^{-1}$. Since bijection g satisfies that $(g(u), g(v)) \in E(H)$ for any $(u, v) \in E(G)$ and $f_{g(u)}(g(v)) = f_{g(v)}(g(u)) = 1$ for $(g(u), g(v)) \in E(H)$, there exists (u, v) in E(G) for any $p, q \in \mathbf{P}(=V(H))$ such that $\pi(p) = u, \pi(q) = v$, and $f_p(q) = f_q(p) = 1$. Thus, $U_p(\pi) = \sum_{v \in N(\pi(p))} f_p(\pi^{-1}(v)) = d_G(\pi(p))$ for any $p \in \mathbf{P}(=V(H))$. Finally, we have the social welfare $\mathsf{sw}(\pi) = \sum_{p \in \mathbf{P}} U_p(\pi) = \sum_{p \in \mathbf{P}} d_G(\pi(p)) = \sum_{v \in V(G)} d_G(v) = 2|E(G)|$. Conversely, we are given π with social welfare 2|E(G)| in G. Let $g = \pi^{-1}$. Suppose that there is

Conversely, we are given π with social welfare 2|E(G)| in G. Let $g = \pi^{-1}$. Suppose that there is an edge $(u, v) \in E(G)$ such that $(\pi^{-1}(u), \pi^{-1}(v)) \notin E(H)$. Then, it holds that $f_{\pi^{-1}(u)}(\pi^{-1}(v)) =$ $f_{\pi^{-1}(v)}(\pi^{-1}(u)) = 0$ by the definition of the preferences. Thus, there exists an agent $p = \pi^{-1}(u) \in$ \mathbf{P} such that $U_p(\pi) < d_G(\pi(p))$ since it holds that $U_p(\pi) \leq d_G(\pi(p))$ for any $p \in \mathbf{P}$. This implies that $\mathsf{sw}(G) < \sum_{p \in \mathbf{P}} d_G(\pi(p)) = 2|E(G)|$. This is a contradiction. Thus, there exists a bijection $g = \pi^{-1}$ such that $(\pi^{-1}(u), \pi^{-1}(v)) \in E(H)$ for any $(u, v) \in E(G)$. This completes the proof. \Box

By assuming that the seat graph G is a regular graph, we obtain the following theorem.

Theorem 11. If SPANNING SUBGRAPH ISOMORPHISM of regular graphs is NP-complete, then MUA on regular graphs is NP-hard even if the preferences are binary and symmetric.

Proof. We give a reduction from SPANNING SUBGRAPH ISOMORPHISM to MUA. The setting is the same as MWA. Let G be an r-regular graph. Then we show that there is a bijection g such that $(g(u), g(v)) \in E(H)$ for any $(u, v) \in E(G)$ if and only if there exists an arrangement π such that the least utility of an agent is r in G. Given a bijection g such that $(g(u), g(v)) \in E(H)$ for any $(u, v) \in E(G)$, we set $\pi = g^{-1}$. Since bijection g satisfies that $(g(u), g(v)) \in E(H)$ for any $(u, v) \in E(G)$ and $f_{g(u)}(g(v)) = f_{g(v)}(g(u)) = 1$ for $(g(u), g(v)) \in E(H)$, there exists (u, v)in E(G) for any $p, q \in \mathbf{P}(=V(H))$ such that $\pi(p) = u, \pi(q) = v$, and $f_p(q) = f_q(p) = 1$. Thus, $U_p(\pi) = \sum_{v \in N(\pi(p))} f_p(\pi^{-1}(v)) = r$ for any $p \in \mathbf{P}(=V(H))$ since G is r-regular.

Conversely, we are given an arrangement π such that the least utility of an agent is r in G. Let $g = \pi^{-1}$. Suppose that there is an edge $(u, v) \in E(G)$ such that $(\pi^{-1}(u), \pi^{-1}(v)) \notin E(H)$. Then, it holds that $f_{\pi^{-1}(u)}(\pi^{-1}(v)) = f_{\pi^{-1}(v)}(\pi^{-1}(u)) = 0$ by the definition of the preferences. Thus, there exists an agent $p = \pi^{-1}(u) \in \mathbf{P}$ such that $U_p(\pi) < r$ since it holds that $U_p(\pi) \leq d_G(\pi(p))$ for any $p \in \mathbf{P}$. This is contradiction. Thus, there exists a bijection $g = \pi^{-1}$ such that $(\pi^{-1}(u), \pi^{-1}(v)) \in E(H)$ for any $(u, v) \in E(G)$.

Corollary 1. MWA and MUA are NP-hard on cycles and cluster graphs whose components are of order three. Furthermore, MWA is NP-hard on paths and linear forests whose components are paths of length three. These hold even if the preferences are binary and symmetric.

Moreover, SPANNING SUBGRAPH ISOMORPHISM cannot be solved in time $n^{o(n)}$ unless ETH is false [16]. Thus, we also have the following result.

Corollary 2. MWA cannot be solved in time $n^{o(n)}$ unless ETH is false.

5 Parameterized Complexity

MWA is NP-hard even on trees (i.e., treewidth 1), which implies that it admits no parameterized algorithm by treewidth if $P \neq NP$. Thus we consider to design an algorithm parameterized by a larger parameter: vertex cover number.

Theorem 12. MWA can be solved in time $O(n^{\gamma}\gamma!(n-\gamma)^3)$ where γ is the vertex cover number of the seat graph.

Proof. Given an instance $(G, \mathbf{P}, \mathcal{F}_{\mathbf{P}})$ of MWA, we first compute a minimum vertex cover S of size γ in time $O(1.2738^{\gamma} + \gamma n)$ [14]. Then we guess γ agents that are assigned to vertices in S. Let \mathbf{P}' be the set of agents assigned to S. Next, we guess all arrangements that assigns \mathbf{P}' to S. The number of candidates of arrangements is $O(n^{\gamma}\gamma!)$.

For each candidate, we consider how to assign the rest of the agents in $\mathbf{P} \setminus \mathbf{P'}$ to $V \setminus S$. Since $V \setminus S$ is an independent set, we can compute the utility of an agent $p \in \mathbf{P} \setminus \mathbf{P'}$ when p is assigned to $v \in V \setminus S$. Note that p does not affect the utility of other agents in $\mathbf{P} \setminus \mathbf{P'}$. Moreover, we can also compute the increase of the utilities of neighbors of p when p is assigned to $v \in V \setminus S$. Then we observe that the increase of the social welfare when p is assigned to $v \in V \setminus S$. Then we observe that p and the sum of the utilities of neighbors of p, that is, $\sum_{u \in N_G(v)} (f_p(\pi^{-1}(u)) + f_{\pi^{-1}(u)}(p))$. Thus, by computing a maximum weight perfect matching on a complete bipartite graph $(\mathbf{P} \setminus \mathbf{P'}, V \setminus S; E')$ with edge weight $w_{pv} = \sum_{u \in N_G(v)} (f_p(\pi^{-1}(u)) + f_{\pi^{-1}(u)}(p))$ for every candidate, we can obtain a maximum arrangement in G.

Since we can compute a maximum weight perfect matching in time $O((n-\gamma)^3)$ [20], the total running time is $O(n^{\gamma}\gamma!(n-\gamma)^3)$.

By Proposition 2, we obtain the following corollary.

Corollary 3. Symmetric STA can be solved in time $O(n^{\gamma}\gamma!(n-\gamma)^3)$ where γ is the vertex cover number of the seat graph.

Then we give the tight lower bound for MWA parameterized by the vertex cover number.

Theorem 13. MWA is W[1]-hard parameterized by the vertex cover number γ of the seat graph even if the preferences are binary. Furthermore, there is no $f(\gamma)n^{o(\gamma)}$ -time algorithm unless ETH fails where f is some computational function.

Proof. We give a parameterized reduction from k-CLIQUE: given a graph G = (V, E) and an integer k, determine whether there exists a clique of size k in G. The problem is W[1]-complete parameterized by k and admits no $f(k)n^{o(k)}$ -time algorithm unless ETH fails [17, 13].

Given an instance (G = (V, E), k) of k-CLIQUE, we construct the seat graph G' that consists of a clique $\{w_1, w_2, \ldots, w_k\}$ of size k and n - k isolated vertices w_{k+1}, \ldots, w_n . Clearly, the size of minimum vertex cover is k - 1. Let $\mathbf{P} = V$. Then we set the preferences of any pair of agents $u, v \in \mathbf{P}$ by $f_u(v) = f_v(u) = 1$ if $(u, v) \in E$, and otherwise $f_u(v) = f_v(u) = 0$. Note that the preferences are binary.

Finally, we show that k-CLIQUE is a yes-instance if and only if there exists an arrangement π with social welfare k(k-1) in G'. Given an instance (G, k) of k-CLIQUE, we give indices to each vertex v_1, v_2, \ldots, v_n arbitrarily. Given a k-clique, we denote it by $C = \{v_1, v_2, \ldots, v_k\}$ without loss of generality. Then we set $\pi(v_i) = w_i$ for any $i \in \{1, \ldots, n\}$. Since $(u, v) \in E$ for any pair of $u, v \in C$, we have $U_{v_i}(\pi) = k - 1$ for $i \in \{1, \ldots, k\}$. For each $i \in \{k + 1, \ldots, n\}$, w_i is an isolated vertex, and hence $U_{v_i}(\pi) = 0$. Therefore, $\mathsf{sw}(\pi) = k(k-1) + 0 = k(k-1)$.

For the reverse direction, we are given an arrangement π with social welfare k(k-1). Since w_{k+1}, \ldots, w_n are isolated vertices, $U_{\pi^{-1}(w_i)}(\pi) = 0$ for $i \in \{k+1, \ldots, n\}$. Moreover, because the preferences are binary, $U_{\pi^{-1}(w_i)}(\pi) \leq k-1$ for $i \in \{1, \ldots, k\}$. Thus, any agent $p = \pi^{-1}(w_i)$ for $i \in \{1, \ldots, k\}$ satisfies that $U_{\pi^{-1}(w_i)}(\pi) = k-1$ in order to achieve $\mathsf{sw}(\pi) = k(k-1)$. This implies that $(\pi^{-1}(w_i), \pi^{-1}(w_j)) \in E$ for any pair of w_i, w_j where $i, j \in \{1, \ldots, k\}$. Therefore, $\{\pi(w_1), \ldots, \pi(w_k)\}$ is a clique of size k.

For MUA, we show that it is weakly NP-hard even on a graph of vertex cover number 2, which again implies that it does not admit any parameterized algorithms by vertex cover number unless P=NP. We give a reduction from PARTITION: given a finite set of integers $A = \{a_1, a_2, \ldots, a_n\}$ and $W = \sum_{i=1}^{n} a_i$, determine whether there is partition (A_1, A_2) of A where $A_1 \cup A_2 = A$ and $\sum_{a \in A_1} a = \sum_{a \in A_2} a = W/2$. The problem is weakly NP-complete [25].

Theorem 14. MUA is weakly NP-hard even on a graph with $\gamma = 2$.

Proof. We are given a set of integers $A = \{a_1, a_2, \ldots, a_n\}$. We define two set of agents $\mathbf{A} = \{p_{a_1}, \ldots, p_{a_n}\}$ and $\mathbf{C} = \{c_1, c_2\}$. Each agent in \mathbf{A} corresponds to an element in A. For $p_{a_i} \in \mathbf{A}$, we define $f_{p_{a_i}}(q) = W/2$ if $q \in \mathbf{C}$, and otherwise $f_{p_{a_i}}(q) = 0$. Moreover, for $c \in \mathbf{C}$, we define

 $f_c(q) = a_i$ if $q \in \mathbf{A}$, and otherwise $f_c(q) = 0$. Finally, we define the seat graph G as a graph consisting of S_1 and S_2 , where S_i is a star of size n/2 + 1. Note that the vertex cover number of G is 2.

In the following, we show that there is a partition (A_1, A_2) where $\sum_{a \in A_1} a = \sum_{a \in A_2} a = W/2$ if and only if there is an arrangement such that the least utility is at least W/2 in G. Given a partition (A_1, A_2) , let \mathbf{A}_1 and \mathbf{A}_2 be the corresponding sets of agents in \mathbf{A} . We assign agents in \mathbf{A}_i to leaves of S_i and c_i to the center of S_i for $i \in \{1, 2\}$. In the arrangement, each utility is W/2.

Conversely, we are given an arrangement π such that the least utility is at least W/2. If $p \in \mathbf{A}$ is assigned to the center of a star, at least one agent in \mathbf{A} is adjacent to only p. Then its utility is 0. Thus, $c_i \in \mathbf{C}$ must be assigned to the center of S_i . By the definition of the preferences, the utilities of c_1 and c_2 are exactly W/2. Thus, two sets of agents in the leaves of S_1 and S_2 correspond to A_1 and A_2 .

Similarly, we show that (symmetric) EFA is weakly NP-hard even on a graph with vertex cover number 2. The reduction is from PARTITION and the reduced graph is the same as the one in the proof of Theorem 14. The preferences are defined as $f_p(q) = f_q(p) = a_i$ if $p = p_{a_i} \in \mathbf{A}$ and $q \in \mathbf{C}$, and otherwise $f_p(q) = f_q(p) = 0$.

Theorem 15. EFA is weakly NP-hard even if the preferences are symmetric and the vertex cover number of the seat graph is 2.

6 Parameterized Complexity of Local Search

As mentioned in Section 1.2, finding a stable solution under symmetric preferences by swapping two agents iteratively is PLS-complete. In this section, we investigate the parameterized complexity of local search of STABLE ARRANGEMENT by considering Local k-STA, which determines whether a stable arrangement can be obtained from any given arrangement by k swaps.

Given a set of agents \mathbf{P} with preferences $\mathcal{F}_{\mathbf{P}}$, a graph G with $|V(G)| = |\mathbf{P}|$, an arrangement $\pi : \mathbf{P} \to V(G)$, and an integer k, Local k-STA asks whether there is a stable arrangement π' that can be obtained from π in at most k swaps.

Theorem 16. Local k-STA is W[1]-hard parameterized by k even if the preferences are symmetric.

Proof. We give a reduction from the INDEPENDENT SET problem which is known to be W[1]complete [17]. Let (H, k) be an instance of INDEPENDENT SET where H is a graph on n vertices and k an integer, the parameter, and the question is whether H has an independent set of size k. Throughout the following, for convenience we will assume that n > k + 2.

We will construct an instance $\mathcal{I} = (\mathbf{P}, \mathcal{F}_{\mathbf{P}}, G, \pi, k)$ of Local k-STA such that \mathcal{I} is a YES-instance if and only of H has an independent set of size k.

We construct a set of n + 3k + 5 agents **P** which is partitioned into the following subsets: $\mathbf{P} = \mathbf{C}_1 \cup \mathbf{C}_2 \cup \mathbf{V} \cup \{\mathbf{x}_1\} \cup \mathbf{Y} \cup \{\mathbf{x}_2\}$. We have that $|\mathbf{C}_1| = k$, $|\mathbf{C}_2| = k + 2$, and $|\mathbf{Y}| = k + 1$. Finally, $\mathbf{V} = V(H)$ and we may refer to elements of the set $V(H) = \mathbf{V}$ both as vertices of H and of agents of **P**. The definition of the preferences $\mathcal{F}_{\mathbf{P}}$ is given in Table 1. Note that they are symmetric.

We construct a graph G as follows. G consists of one clique on 2k + 2 vertices whose vertices are $C_1 \cup C_2$ with $|C_1| = k$ and $|C_2| = k + 2$, one star on n + 1 vertices whose center is x_1 and whose leaves are called V_H , one star on k + 2 vertices whose center is x_2 and whose leaves are called Y.

We now let π be a map that maps bijectively \mathbf{C}_1 to C_1 , \mathbf{C}_2 to C_2 , \mathbf{V} to V_H , \mathbf{Y} to Y, \mathbf{x}_1 to x_1 and \mathbf{x}_2 to x_2 . This finishes the construction of our instance $\mathcal{I} = (\mathbf{P}, \mathcal{F}_{\mathbf{P}}, G, \pi, k)$. See the instance in Figure 3.

To prove the correctness of the reduction, we first show that given an independent set S in H of size k, we can swap the assignments (in π) of the agents corresponding to S with the agents in \mathbf{C}_1 to obtain a stable arrangement π' . Conversely, we will show that if there is a stable arrangement π' that is obtained from π in k swaps, then π' must have been obtained by swapping the assignments of the agents in \mathbf{C}_1 with some subset $S \subseteq \mathbf{V}$ which corresponds to an independent set in H.

				$q \in$			
		\mathbf{C}_1	\mathbf{C}_2	\mathbf{V}	Y	\mathbf{x}_1	\mathbf{x}_2
	\mathbf{C}_1	0	-n	-1	0	1	-1
	\mathbf{C}_2	-n	0	1	1	-n	-n
$p \in$	v	-1	1	$\begin{cases} -n, & \text{if } pq \in E(H) \\ 0, & \text{otherwise} \end{cases}$	0	-1	0
	Y	0	1	0	0	-n	1
	\mathbf{x}_1	1	-n	-1	-n		-n
	\mathbf{x}_2	-1	-n	0	1	-n	

Table 1: The preferences $\mathcal{F}_{\mathbf{P}}$ given in the proof of Theorem 16. For p, q from the corresponding sets, the entry shows $f_p(q)$.

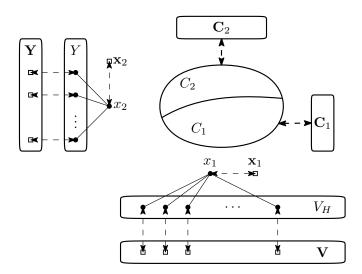


Figure 3: Illustration of the instance of Local k-STA constructed in the reduction of Theorem 16.

Claim 16.2. If H has an independent set S of size k, then $(\mathbf{P}, \mathcal{F}_{\mathbf{P}}, G, \pi, k)$ is a YES-instance.

Proof. Let π' be the arrangement obtained from π by swapping the assignments of the agents in S with the assignments of the agents in \mathbf{C}_1 . Since |S| = k, it is clear that π' can be reached from π by k swaps. We show that π' is a stable arrangement. Suppose not, then there is a blocking pair (p,q) for π' , i.e. if π'' is the (p,q)-swap arrangement of π' , then $U_p(\pi'') > U_q(\pi')$ and $U_q(\pi'') > U_q(\pi')$.

We conduct a case analysis on which sets of the above described partition of \mathbf{P} contain p and q. Throughout the following, we denote by π'' a (p,q)-swap arrangement of π' where p and q are defined depending on the below cases.

- **Case 1** $(p \in \mathbf{C}_1)$. Note that $\pi'(p) \in V_H$: in π , the agents in \mathbf{C}_1 were mapped to C_1 and π' is obtained from π by swapping the assignment of the agents in \mathbf{C}_1 with the agents in $S \subseteq \mathbf{V}$ which are mapped to V_H . This implies that $U_p(\pi') = 1$. As \mathbf{x}_1 is the only agent to which p has a positive preference, this is the maximum utility of p among all arrangements.
- **Case 2** $(p \in \mathbf{C}_2)$. Note that $\pi'(p) \in C_2$. Furthermore, $\pi'(p)$ has k neighbors to which an element of **V** is mapped, and k+1 neighbors to which an element of \mathbf{C}_2 is mapped. Hence, $U_p(\pi') = k$. We have that for any $q \in \mathbf{P} \setminus (\mathbf{V} \cup \mathbf{Y} \cup \{p\})$, $f_p(q) \leq 0$, and for $q \in \mathbf{V} \cup \mathbf{Y}$, $f_p(q) = 1$, so in order to increase the utility of p in any swap arrangement of π' , we would have to map p to a vertex that has more than k + 1 neighbors to which an element of $\mathbf{V} \cup \mathbf{Y}$ is mapped.

There are two such possibilities, namely either $q = \mathbf{x}_1$ or $q = \mathbf{x}_2$. Suppose $q = \mathbf{x}_1$. Note that $\pi'(\mathbf{x}_1) = x_1$, and x_1 has k neighbors to which an agent from \mathbf{C}_1 is mapped, and n-k neighbors to which an agent from \mathbf{V} is mapped. Hence, $U_q(\pi') = k - (n-k) = 2k - n$. As $\pi''(q) \in C_2$, $U_q(\pi'') = -(k+1)n - k = -(k+2)n$. As $U_q(\pi') = 2k - n > -(k+2)n = U_q(\pi'')$, (p,q) is not a blocking pair for π' . If $q = \mathbf{x}_2$, then we observe that $U_q(\pi') = k + 1 > -(k+1)n = U_q(\pi'')$, so (p,q) is not a blocking pair for π' either.

- **Case 3.1** $(p \in \mathbf{V}, p \in S)$. In this case, $\pi'(p) \in C_1$, and $U_p(\pi') = k + 2$: $\pi'(p)$ has k + 2 neighbors to which agents in \mathbf{C}_2 are mapped, and to the remaining neighbors of $\pi'(p)$, elements of Sare mapped. Since S is an independent set, the latter contributes with a total value of 0 to $U_p(\pi')$. We have that π' in fact achieves the maximum utility for p, among all arrangements: The only agents to which p has a positive preference are the ones in \mathbf{C}_2 , and in π' , all agents in \mathbf{C}_2 are mapped to neighbors of $\pi'(p)$.
- **Case 3.2** $(p \in \mathbf{V}, p \notin S)$. We observe that if $q \in \mathbf{C}_1 \cup (\mathbf{V} \setminus S)$, then the utility of p compared to the resulting swap arrangement does not change, since in this case, $\pi'(p) \in V_H \ni \pi'(q)$. The remaining ones are as follows:
 - If $q \in S$, then $U_q(\pi') = k + 2 > -1 = U_q(\pi'')$.
 - If $q \in \mathbf{C}_2$, then $U_q(\pi') = k > -n = U_q(\pi'')$.
 - If $q \in \mathbf{Y}$, then $U_q(\pi') = 1 > -n = U_q(\pi'')$.
 - If $q = \mathbf{x}_1$, then $U_p(\pi') = -1 > -k \ge U_p(\pi'')$.
 - If $q = \mathbf{x}_2$, then $U_q(\pi') = k + 1 > -n = U_q(\pi'')$.
- **Case 4** $(p \in \mathbf{Y})$. We have that $U_p(\pi') = 1$, as $x_2 = \pi'^{-1}(\mathbf{x}_2)$ is the only neighbor of $\pi'(p)$. There is only one way to increase the utility of p in one swap. It is to map p to a vertex to whose neighbors agents of \mathbf{C}_2 are mapped. This means that we would have to swap p with some $q \in \mathbf{C}_2 \cup S$. However, if $q \in \mathbf{C}_2$, we would have that $U_q(\pi') = k > -n = U_q(\pi'')$ in the (p,q)-swap arrangement π'' for π' . Moreover, if $q \in S$, then $\pi'(q) \in C_1$. Then we would have that $U_q(\pi') = k + 2 > 0 = U_q(\pi'')$ in the (p,q)-swap arrangement π'' .

Case 5 $(p \in \mathbf{x}_1)$. We observe that $U_{\mathbf{x}_1}(\pi') = 2k - n$. In the (p,q)-swap arrangement π'' , we have:

• If $q \in S$, $U_p(\pi') = 2k - n > -n(k+2) - (k-1) = U_p(\pi'')$.

- If $q \in \mathbf{V} \setminus S$, $U_q(\pi') = -1 > -k 1 \ge U_q(\pi'')$.
- If $q \in \mathbf{C}_1$, $U_q(\pi') = 1 > 1 (n-k) = U_q(\pi'')$.
- If $q \in \mathbf{C}_2$, $U_p(\pi') = 2k n > -n(k+1) k = U_p(\pi'')$.
- If $q \in \mathbf{Y}$, $U_p(\pi') = 2k n > -n = U_p(\pi'')$.
- If $q \in \mathbf{x}_2$, $U_p(\pi') = 2k n > -n(k+1) = U_p(\pi'')$.

Case 6 $(p \in \mathbf{x}_2)$. We observe that $U_{\mathbf{x}_2}(\pi') = k + 1$, and this is the maximum utility that \mathbf{x}_2 can achieve.

We have shown that there is no blocking pair in π' , so it is indeed a stable arrangement.

We now prove the converse direction of the correctness of this reduction. Suppose $(\mathbf{P}, \mathcal{F}_{\mathbf{P}}, G, \pi, k)$ is a YES-instance of Local k-STA, and let π' be an arrangement obtained from π in k swaps such that π' is stable. We will show that for π' to be stable, these k swaps need to swap the agents of \mathbf{C}_1 with k agents S in \mathbf{V} , and furthermore the agents in S need to correspond to an independent set in H.

First, we immediately observe that since $|\mathbf{C}_2| = k + 2$, and for each $p \in \mathbf{C}_2$, $\pi(p) \in C_2$, at least two agents of \mathbf{C}_2 remain assigned to a vertex in C_2 by π' . For a similar reason, namely $|\mathbf{Y}| = k + 1$, at least one agent in \mathbf{Y} is still mapped to a vertex in Y.

Observation 16.3. There are two distinct $p_1, p_2 \in \mathbf{C}_2$ such that $\pi'(p_1) \in C_2$ and $\pi'(p_2) \in C_2$. Furthermore, there is at least one agent $p \in \mathbf{Y}$ such that $\pi'(p) \in Y$.

Next, we show that in π' , neither \mathbf{x}_1 nor \mathbf{x}_2 can be mapped to a vertex in the clique $C_1 \cup C_2$.

Claim 16.4. $\pi'(\mathbf{x}_1) \notin C_1 \cup C_2$ and $\pi'(\mathbf{x}_2) \notin C_1 \cup C_2$.

Proof. Suppose $\pi'(\mathbf{x}_1) \in C_1 \cup C_2$. First, we have that $U_{\mathbf{x}_1}(\pi') \leq k - 2n$, since k is the maximum utility that \mathbf{x}_1 can have in any arrangement, and by Observation 16.3, there are at least two agents from \mathbf{C}_2 that are mapped to neighbors of $\pi'(\mathbf{x}_1)$. Again by Observation 16.3, there is one agent $y \in \mathbf{Y}$ that is still mapped to a vertex in Y. That vertex has only one neighbor (the center of the star to which the vertices in Y are leaves), so $U_y(\pi') \leq 1$. Now, if π'' is the (\mathbf{x}_1, y) -swap arrangement of π' , then $U_{\mathbf{x}_1}(\pi'') \geq -n$ ($\pi''(\mathbf{x}_1)$ having only one neighbor) and $U_y(\pi'') \geq 2$: \mathbf{x}_1 is not mapped to a neighbor of y in π'' , and y has non-negative preferences to the remaining agents; by Observation 16.3, $\pi''(y)$ has at least two neighbors to which an agent of \mathbf{C}_2 is mapped. So, $U_{\mathbf{x}_1}(\pi'') \geq -n > k - 2n \geq U_{\mathbf{x}_1}(\pi')$, as n > k, and $U_y(\pi'') \geq 2 > 1 \geq U_y(\pi')$, hence (\mathbf{x}_1, y) is a blocking pair in π' , a contradiction to π' being stable.

Similarly, if $\pi'(\mathbf{x}_2) \in C_1 \cup C_2$ and π'' is the (\mathbf{x}_2, y) -swap arrangement of π' , then we have that $U_{\mathbf{x}_2}(\pi'') \geq -n > k - 2n \geq U_{\mathbf{x}_2}(\pi')$ and $U_y(\pi'') \geq 2 > 1 \geq U_y(\pi')$, so (\mathbf{x}_2, y) is a blocking pair. \Box

Claim 16.5. There is no agent $p \in \mathbf{C}_1$ such that $\pi'(p) \in C_1 \cup C_2$.

Proof. Suppose there is. Then we have that $U_p(\pi') \leq -2 \cdot n$, since by Observation 16.3, p has at least two neighbors to which an agent of \mathbf{C}_2 is mapped, and \mathbf{x}_1 is the only agent that p has a positive preference to, and \mathbf{x}_1 is not assigned to a neighbor of $\pi'(p)$ by Claim 16.4.

By Observation 16.3, there is an agent $y \in \mathbf{Y}$ with $\pi'(y) \in Y$. We have that $U_y(\pi') \leq 1$. Now, (p, y) is a blocking pair of π' : let π'' be the (p, y)-swap arrangement of π' . Then, $U_p(\pi'') \geq -n$, since $\pi''(p) \in Y$ which only has one neighbor x_2 . Furthermore we can observe that $U_y(\pi'') \geq 2$: $\pi''(y)$ has at least two neighbors to which agents of \mathbf{C}_2 are mapped by Observation 16.3, \mathbf{x}_1 is the only agent to which y has a negative preference, and Claim 16.4 ensures that $\pi''(\mathbf{x}_1) = \pi'(\mathbf{x}_1) \notin C_1 \cup C_2$. Hence, $U_p(\pi'') \geq -n > -2 \cdot n \geq U_p(\pi')$, and $U_y(\pi'') \geq 2 > 1 \geq U_y(\pi')$.

Claim 16.5 yields the following information about the swaps that were executed to obtain π' from π .

Claim 16.6. The k swaps executed to obtain π' from π are $(p_1, \mathbf{c}_1), (p_2, \mathbf{c}_2), \ldots, (p_k, \mathbf{c}_k)$, where

- 1. $\mathbf{C}_1 = {\mathbf{c}_1, \dots, \mathbf{c}_k}$, and for all $i \in [k]$, $p_i \notin \mathbf{C}_1 \cup \mathbf{C}_2$.
- 2. for all $i, j \in [k]$ with $i \neq j, p_i \neq p_j$.

Proof. Part 1 is immediate from Claim 16.5 and the fact that $|\mathbf{C}_1| = k$: if there was one swap that did not remove an agent in \mathbf{C}_1 from $C_1 \cup C_2$, then for at least one agent $\mathbf{c} \in \mathbf{C}_1$, $\pi'(\mathbf{c}) \in C_1 \cup C_2$. For Part 2, suppose there are $i, j \in [k]$ with $i \neq j$ such that $p_i = p_j =: p$, and suppose wlog. that i < j. Then, after swapping \mathbf{c}_1 and p, we have that p is mapped to a vertex in C_1 . Then, when swapping p and \mathbf{c}_2 , \mathbf{c}_2 is still mapped to a vertex in C_1 . Since by Part 1, each agent is affected by at most one swap, we have that $\pi'(\mathbf{c}_2) \in C_1 \cup C_2$, a contradiction with Claim 16.5.

Now, combining Claim 16.6 with 16.4 tells us that \mathbf{x}_1 and \mathbf{x}_2 remain unaffected by the k swaps that yielded π' .

Observation 16.7. $\pi'(\mathbf{x}_1) = x_1 \text{ and } \pi'(\mathbf{x}_2) = x_2.$

Claim 16.6 ensures that each agent is affected by at most one swap and that each swap affects one unique agent in \mathbf{C}_1 . Furthermore, it rules out that the agents of \mathbf{C}_1 are swapped with agents from $\mathbf{C}_1 \cup \mathbf{C}_2$, and Observation 16.7 rules out that they are swapped with \mathbf{x}_1 or \mathbf{x}_2 . As our goal is to show that they are swapped with agents from \mathbf{V} , the only case that remains to be ruled out is when they are swapped with an agent from \mathbf{Y} .

Claim 16.8. There is no $i \in [k]$ such that $p_i \in \mathbf{Y}$.

Proof. Suppose there is, and let (p_i, \mathbf{c}_i) be the corresponding swap. Let π^* be the (p_i, \mathbf{c}_i) -swap arrangement of π and note that by Claim 16.6 and Observation 16.7, $U_{\mathbf{c}_i}(\pi^*) = U_{\mathbf{c}_i}(\pi') = -1$. As $|\mathbf{V}| > k$, there is at least one agent $q \in \mathbf{V}$ that π' assigns to V_H . Again by Observation 16.7, we have that $U_q(\pi') = -1$. Let π'' be the (\mathbf{c}_i, q) -swap arrangement of π' . Then, $U_{\mathbf{c}_i}(\pi'') = 1 > -1 = U_{\mathbf{c}_i}(\pi')$ and $U_q(\pi'') = 0 > -1 = U_q(\pi')$, so (\mathbf{c}_i, q) is a blocking pair for π' , a contradiction with π' being stable.

Claim 16.9. If π' is stable, then H contains an independent set of size k.

Proof. The above claims and observations lead us to the conclusion that the agents p_1, \ldots, p_k (in the notation of Claim 16.6) form a size-k subset, say S, of $\mathbf{V} = V(H)$. We argue that if π' is stable, then S is indeed an independent set in H. Suppose for the sake of a contradiction that there is an edge between p_i and p_j in H for some $i \neq j$. Then, $U_{p_i}(\pi') \leq (k+2) - n$. Consider again the agent $y \in \mathbf{Y}$ from Observation 16.3 which is such that $\pi'(y) \in Y$. We have that $U_y(\pi') = 1$ (together with Observation 16.7). Now let π'' be the (p_i, y) -swap arrangement of π' . Then, $U_{p_i}(\pi'') = 0 > (k+2) - n = U_{p_i}(\pi')$ (as by our initial assumption, n > k+2) and $U_y(\pi'') = k + 2 > 1 = U_y(\pi')$, so (p_i, y) is a blocking pair for π' .

This concludes the correctness proof of the reduction. We observe that $|\mathbf{P}| = n + 3k + 5 = \mathcal{O}(n)$, $|\mathcal{F}_{\mathbf{P}}| = \mathcal{O}(n^2)$, $|V(G)| = |\mathbf{P}| = \mathcal{O}(n)$, and $|E(G)| = \mathcal{O}(n + k^2)$ (G contains two stars with n and k+1 leaves, respectively and a clique on $\mathcal{O}(k)$ vertices). So, the size of the instance of Local k-STA is $\mathcal{O}(n^2)$ and the parameter k remained unchanged, which completes the proof.

We observe that the structure of the instance of Local k-STA is in fact quite restricted and yields the following stronger form of Theorem 16. First, in the preferences of the Local k-STA instance, we only have 4 different values of preferences. Second, the seat graph G obtained in the reduction above has a vertex cover of size 2k+3: take 2k+1 vertices from $C_1 \cup C_2$ and the vertices x_1 and x_2 . This means that even including the vertex cover number γ of the seat graph in the parameter, the problem remains W[1]-hard. **Corollary 4.** Local k-STA remains W[1]-hard parameterized by $k + \gamma$ where γ denotes the vertex cover number of the seat graph, even when the number of preference values is 4.

Moreover, it is not difficult to see that Local k-STA can be solved in time $n^{\mathcal{O}(k)}$ by brute force. We simply guess all sets of k pairs of agents, swap their assignments, and then verify whether or not the resulting assignment has a blocking pair. On the other hand, the value of the parameter $k + \gamma$ in the above reduction is *linear* in the value of the parameter of the INDEPENDENT SET instance. Since INDEPENDENT SET does not have an $n^{o(k)}$ time algorithm unless ETH fails [13], this implies that the runtime of this naive brute force algorithm is in some sense tight under ETH — even when the vertex cover number of the seat graph can be considered another component of the parameter, and even when we only have 4 different choices for values of preferences.

Corollary 5. Local k-STA can be solved in time $n^{\mathcal{O}(k)}$, and there is no $n^{o(k+\gamma)}$ time algorithm, where γ denotes the vertex cover number of the seat graph, even when the number of preference values is 4, unless ETH is false.

7 Conclusion

In this paper, we embark a new model of hedonic games, called SEAT ARRANGEMENT. The proposed model is powerful enough to treat real-world topological preferences. The results of the paper are summarized as follows: (1) We obtained basic results for the stability and fairness. In particular, we proved that the PoF is unbounded for the nonnegative case and we gave a upper bound $\tilde{d}(G)$ and an almost tight lower bound $\tilde{d}(G) - 1/4$ for the binary case. (2) We presented the dichotomies of computational complexity of four SEAT ARRANGEMENT problems in terms of the order of components. (3) We proved that MWA can be solved in time $n^{O(\gamma)}$ where γ is the vertex cover number whereas it is W[1]-hard for γ and cannot be solved in time $n^{o(n)}$ and $f(\gamma)n^{o(\gamma)}$, respectively, under ETH. Furthermore, MUA and symmetric EFA are weakly NP-hard even on graphs with $\gamma = 2$. (4) We proved that Local k-STA is W[1]-hard when parameterized by $k + \gamma$ and cannot be solved in time $n^{O(k)}$.

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