Complexity Framework for Forbidden Subgraphs IV: The Steiner Forest Problem

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──────── **Abstract** ────────

We study STEINER FOREST on H-subgraph-free graphs, that is, graphs that do not contain some fixed graph H as a (not necessarily induced) subgraph. We are motivated by a recent framework that completely characterizes the complexity of many problems on H-subgraph-free graphs. However, in contrast to e.g. the related STEINER TREE problem, STEINER FOREST falls outside this framework. Hence, the complexity of STEINER FOREST on H-subgraph-free graphs remained tantalizingly open.

In this paper, we make significant progress towards determining the complexity of STEINER FOREST on H-subgraph-free graphs. Our main results are four novel polynomial-time algorithms for different excluded graphs H that are central to further understand its complexity. Along the way, we study the complexity of STEINER FOREST for graphs with a small c-deletion set, that is, a small set S of vertices such that each component of $G - S$ has size at most c. Using this parameter, we give two noteworthy algorithms that we later employ as subroutines. First, we prove STEINER FOREST is FPT parameterized by $|S|$ when $c = 1$ (i.e. the vertex cover number). Second, we prove STEINER FOREST is polynomial-time solvable for graphs with a 2-deletion set of size at most 2. The latter result is tight, as the problem is NP-complete for graphs with a 3-deletion set of size 2.

1   Introduction

In this paper, we consider the complexity of a classical graph problem, STEINER FOREST, restricted to graphs that do not contain some fixed graph H as a subgraph. Such graphs are said to be *H-subgraph-free*, that is, they cannot be modified to H by a sequence of edge deletions and vertex deletions. A graph G is *H-free* if G cannot be modified into H by a sequence of vertex deletions only. Even though H-free graphs are more widely studied in the literature, H-subgraph-free graphs are also highly interesting, as was recently shown through the introduction of a large, general framework for the subgraph relation [19, 20, 21, 23].

Before giving our results on STEINER FOREST, we first explain the framework. For a set of graphs $\mathcal{H}$, a graph G is *$\mathcal{H}$-subgraph-free* if G is H-subgraph-free for every $H \in \mathcal{H}$. In order to unify and extend known classifications for INDEPENDENT SET [1], DOMINATING SET [1], LIST COLOURING [18], LONG PATH [1] and MAX-CUT [22] on $\mathcal{H}$-subgraph-free graphs (for finite $\mathcal{H}$), a systematic approach was followed in [19]. A class of graphs has bounded *treewidth* if there exists a constant c such that every graph in it has treewidth at most c. For an integer $k \geq 1$, the *k-subdivision* of an edge $e = uv$ of a graph replaces e by a path of length $k + 1$ with endpoints u and v (and k new vertices). The *k-subdivision* of a graph G is the graph obtained from G after k-subdividing each edge. For a graph class $\mathcal{G}$ and an integer k, let $\mathcal{G}^k$ consist of the k-subdivisions of the graphs in $\mathcal{G}$. A graph problem Π is NP-complete *under edge subdivision of subcubic graphs* if there is an integer $k \geq 1$ such that: if Π is NP-complete for the class $\mathcal{G}$ of subcubic graphs (graphs with maximum degree at most 3), then Π is NP-complete for $\mathcal{G}^{kp}$ for every $p \geq 1$. Now, Π is a *C123-problem* if:

C1. Π is polynomial-time solvable for every graph class of bounded treewidth,

C2. Π is NP-complete for the class of subcubic graphs, and

C3. Π is NP-complete under edge subdivision of subcubic graphs.

A *subdivided* claw is a graph obtained from a *claw* (4-vertex star) by subdividing each of its three edges zero or more times. The *disjoint union* of two vertex-disjoint graphs G_1 and G_2 is graph $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. The set $\mathcal{S}$ consists of all graphs that are disjoint unions of subdivided claws and paths. We can now state the complexity classification of [19].

▶ **Theorem 1** ([19]). *Let Π be a C123-problem. For a finite set $\mathcal{H}$, the problem Π on $\mathcal{H}$-subgraph-free graphs is polynomial-time solvable if $\mathcal{H}$ contains a graph from $\mathcal{S}$ (or equivalently, if the class of $\mathcal{H}$-subgraph-free graphs has bounded treewidth) and NP-complete otherwise.*

Examples of C123-problems include INDEPENDENT SET, DOMINATING SET, LONG PATH, MAX CUT, LIST COLOURING, DISJOINT PATHS, ODD CYCLE TRANSVERSAL, PERFECT MATCHING CUT, STEINER TREE, and many more; see a (long) table of problems in [19].

Nevertheless, there are many well-known graph problems that are not C123. For example, COLOURING [8], CONNECTED VERTEX COVER [24], FEEDBACK VERTEX SET [24], INDEPENDENT FEEDBACK VERTEX SET [21] and MATCHING CUT [12] do not satisfy C2, whereas HAMILTON PATH and k-INDUCED DISJOINT PATHS do not satisfy C3 [23].

There are also problems that only satisfy C2 and C3 but not C1. For example, SUBGRAPH ISOMORPHISM is NP-complete even for input pairs of path-width 1. A few years ago, Bodlaender et al. [6] settled the complexity of SUBGRAPH ISOMORPHISM for H-subgraph-free graphs, but only for connected graphs H and leaving open the case where $H = P_5$. This shows that the following question is challenging:

How do C23-*problems, i.e., that satisfy C2 and C3 but not C1, behave for H-subgraph-free graphs? Can we still classify their computational complexity?*

We consider this question for STEINER FOREST. A *Steiner forest* of a graph G, with a set $S = \{(s_1, t_1), \ldots, (s_p, t_p)\}$ of specified pairs of vertices called *terminals*, is a subgraph F of G, such that s_i and t_i, for every $i \in \{1, \ldots, p\}$, belong to the same connected component of F.

STEINER FOREST
  Instance: A graph G, a set S of terminal pairs and an integer k.
  Question: Does (G, S) have a Steiner forest F with $|E(F)| \leq k$?

In our problem definition, we consider unweighted graphs and the goal is to find Steiner forests with a small number of *edges*. Moreover, STEINER FOREST generalizes the C123-problem STEINER TREE, which is to decide whether for a given integer k, a graph G with some specified set S of vertices has a tree T with $|E(T)| \leq k$ containing every vertex of S: take all pairs of vertices of S as terminal pairs to obtain an equivalent instance of STEINER FOREST.

For a constant c, a *c-deletion set* of a graph $G = (V, E)$ is a set $T \subseteq V$ such that each connected component of $G - T$ has size at most c. The *c-deletion set number* of G is the size of a smallest c-deletion set; see e.g. [3, 9, 13, 15, 16] for results on this parameter and related ones, such as the vertex integrity, safe number and fracture number. Bateni, Hajiaghayi and Marx [4] proved the following (see also Section 2).[1]

▶ **Theorem 2** ([4]). STEINER FOREST *is polynomial-time solvable for graphs of treewidth at most* 2*, but* NP-*complete for graphs of treewidth* 3*, tree-depth* 4*, and* 3*-deletion set number* 2*.*

This shows in particular that STEINER FOREST does not satisfy C1, unlike STEINER TREE [2]. As STEINER TREE satisfies C2 and C3 [19], STEINER FOREST satisfies C2 and C3 and is thus a C23-problem, unlike STEINER TREE which is C123 [19]. This leaves the complexity of STEINER FOREST on H-subgraph-free graphs tantalizingly open.

Our Results

Recall that $\mathcal{S}$ is the class of disjoint unions of subdivided paths and claws. For two graphs H_1 and H_2, we write $H_1 \subseteq H_2$ if H_1 is a subgraph of H_2, i.e., $V(H_1) \subseteq V(H_2)$ and $E(H_1) \subseteq E(H_2)$. Our results on STEINER FOREST for H-subgraph-free graphs are:

▶ **Theorem 3.** *For a graph H,* STEINER FOREST *on H-subgraph-free graphs is*
- *polynomial-time solvable if $H \subseteq 2K_{1,3} + P_3 + sP_2, 2P_4 + P_3 + sP_2, P_9 + sP_2$ or $S_{1,1,4} + sP_2$ for each $s \geq 0$, and*
- NP-*complete if $H \supseteq 3K_{1,3}, 2K_{1,3} + P_4, K_{1,3} + 2P_4, 3P_4$ or if $H \notin \mathcal{S}$.*

As graphs of tree-depth 3 are P_8-subgraph-free, Theorems 2 and 3 yield a dichotomy:

▶ **Corollary 4.** *For a constant t,* STEINER FOREST *on graphs of tree-depth t is polynomial-time solvable if $t \leq 3$ and* NP-*compete if $t \geq 4$.*

The NP-hardness part of Theorem 3 follows from the gadget from Theorem 2 and NP-completeness of STEINER FOREST when $H \notin \mathcal{S}$ [5], as shown in Section 2. For the polynomial part, we first make some observations on 2-connectivity and high-degree vertices in Section 3. Then, in Section 4, we show that STEINER FOREST is FPT with respect to the *vertex cover number* (1-deletion set number) of a graph, and there we also prove the polynomial-part of the following dichotomy, in which the NP-completeness part is due to Theorem 2 [4].

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[1] Previously, NP-completeness for treewidth 3 was known for WEIGHTED STEINER FOREST [17].
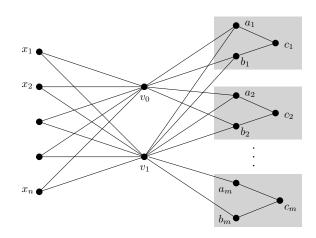
◼ **Figure 1** The graph G (gadget from [4]) used in the proof of Lemma 6.

▶ **Theorem 5.** *For a constant c,* STEINER FOREST *on graphs with a c-deletion set of size at most 2 is polynomial-time solvable if $c \leq 2$ and* NP*-complete if $c \geq 3$.*

Using the algorithms from Sections 3 and 4 as subroutines, we prove the polynomial part of Theorem 3 in Section 5.

Finally, a graph parameter p *dominates* a graph parameter q if there is a function f such that $p(G) \leq f(q(G))$ for every graph G. If p dominates q but q does not dominate p, then p is *more powerful* than q. From the definitions, it follows that treewidth is more powerful than treedepth, which is more powerful than c-deletion set number (fixed $c \geq 2$), which is more powerful than the vertex cover number (1-deletion set number). Given the hardness results in Theorems 2, Corollary 4 and Theorem 5, this also gives an indication to what extent our new FPT result for vertex cover number is best possible.

2 NP-Completeness Results

For a rooted forest F, the *closure* $C(F)$ is the graph with vertex set $V(F)$ with the property that two vertices u and v are adjacent in $C(F)$ if and only if u is an ancestor of v in F. The *depth* of F (number of "layers") is equal to the height of F plus 1. We say that F is a *tree-depth decomposition* of a graph G if G is a subgraph of $C(F)$. The *tree-depth* of G is the minimum depth over all tree-depth decompositions of G.

Bateni, Hajiaghayi and Marx [4] explicitly proved that STEINER FOREST is NP-complete for graphs of treewidth 3, see also Theorem 2. The additional properties in the lemma below can be easily verified from inspecting their gadget, which is displayed in Figure 1.

▶ **Lemma 6.** STEINER FOREST *is* NP*-complete for* $(3K_{1,3}, 2K_{1,3} + P_4, K_{1,3} + 2P_4, 3P_4)$*-subgraph-free graphs of tree-depth 4 with 3-deletion set number 2.*

Proof. For a given a so-called R-formula ϕ with n variables and m clauses, Bateni, Hajiaghayi and Marx [4] constructed an instance (G, S) of STEINER FOREST such that ϕ is satisfiable if and only if (G, S) has a solution of $n + 3m$ edges. The graph G (which has treewidth 3) is illustrated in Figure 1. It remains to make three observations. First, every P_4 or $K_{1,3}$ in G needs to use either v_0 or v_1. Second, the set $\{v_0, v_1\}$ is a 3-deletion set. Third, G has tree-depth 4; take $\{v_0\}$ as layer 0; $\{v_1\}$ as layer 1; the set of all c-type vertices and x-type vertices as layer 2; and the set of all a-type vertices and b-type vertices as layer 3. ◀

Proof of NP-*completeness part of Theorem 3.* It is known that STEINER TREE, and thus STEINER FOREST, is NP-complete for H-subgraph-free graphs if $H \notin \mathcal{S}$ [5]. The NP-completeness part of Theorem 3 now follows immediately from this observation and Lemma 6.

3   Basic Polynomial Results

A *minimum* Steiner forest for an instance (G, S) is one with the smallest number of edges. We denote the number of edges of such a forest by $\mathsf{sf}(G, S)$. We always assume that for any terminal pair $(s, t) \in S$, s and t are distinct (as any pair where $s = t$ can be removed without affecting the feasibility or the size of a minimum solution). A vertex v is a *terminal vertex* if there is a pair $(s, t) \in S$ with $v = s$ or $t = s$, we say that s and t are *(terminal) partners.*

We now describe several important subroutines for our later algorithms. A *cut vertex* of a connected graph is a vertex whose removal yields a graph with at least two connected components. A graph without a cut vertex is 2-*connected.* A 2-*connected component* or *block* of a graph is a maximal subgraph that is 2-connected.

▶ **Lemma 7.** *For every graph class $\mathcal{G}$, if* STEINER FOREST *is polynomial-time solvable for the subclass of* 2-*connected graphs of $\mathcal{G}$, then it is so for $\mathcal{G}$.*

Proof. Let G be any graph in $\mathcal{G}$ and $S \subseteq V(G) \times V(G)$ be any set of terminal pairs. If G is not connected, then we consider each connected component individually and for each individual instance, we restrict the set of terminal pairs to pairs of vertices of that component. If a terminal pair is split among two distinct components, the instance is obviously infeasible.

We may thus assume that the graph G is connected. We use the following recursive algorithm. Let A be the assumed algorithm for STEINER FOREST on 2-connected graphs of $\mathcal{G}$. If G is 2-connected, then we may simply call A on (G, S). Otherwise, G has a 2-connected component that contains only a single cut-vertex of G (i.e. any leaf of the block-cut tree of G). Let G_1 be any such component and let v be the cut vertex such that $G_1 - v$ is a connected component of $G - v$. Let $G_2 = G - (V(G_1) \setminus \{v\})$ be the remainder of G. We now construct two sets of terminal pairs. For $i = 1, 2$, let $S_i \subseteq V(G_i) \times V(G_i)$ initially be the set of all terminal pairs in G_i for which both terminals are in $V(G_i)$. Then add to S_i the following pairs: for each $(s_j, t_j) \in S$ such that $s_j \in V(G_i)$ and $t_j \notin V(G_i)$, add the pair (s_j, v) to S_i. By definition, G_1 is 2-connected. Apply A to (G_1, S_1). Recurse on (G_2, S_2).

It is immediate that this algorithm runs in polynomial time. Indeed, running A on G_1 takes polynomial time. As $|V(G_2)| < |V(G)|$, we apply A at most $|V(G)|$ times on graphs not larger than G and terminal sets not larger than S.

To see correctness, let F be a minimum Steiner forest for (G, S). We claim that for $i = 1, 2$, $F_i := (V(F) \cap V(G_i), E(F) \cap E(G_i))$ is a solution to (G_i, S_i). Indeed, for a pair $(s_j, t_j) \in S$ for which $s_j, t_j \in V(G_i)$, let P_j be an s_j-t_j path in F. Since v is a cut vertex, P_j cannot contain any edges of $E(G) - E(G_i)$. Hence, P_j exists in F_i. For a pair $(s_j, t_j) \in S$ for which $s_j \in V(G_i)$ and $t_j \notin V(G_i)$, let Q_j be an s_j-t_j path in F. As v is a cut vertex, Q_j must contain v and $E(Q_j) \cap E(G_i) \subseteq E(F_i)$ is a s_j-v path in G_i. Hence, F_i is a solution to (G_i, S_i) and thus $\mathsf{sf}(G, S) \geq \mathsf{sf}(G_1, S_1) + \mathsf{sf}(G_2, S_2)$.

Conversely, for $i = 1, 2$, let F_i be a minimum Steiner forest for (G_i, S_i). We claim that $F := (V(F_1) \cup V(F_2), E(F_1) \cup E(F_2))$ is a solution to (G, S). Consider any terminal pair $(s_j, t_j) \in S$. If $s_j, t_j \in V(G_i)$ for some $i \in \{1, 2\}$, then F_i contains an s_j-t_j path and thus so does F. Otherwise, without loss of generality, S_1 contains the pair (s_j, v) and S_2 contains the pair (v, t_j) by construction. Hence, F_1 contains an s_j-v path and F_2 contains a v-t_j path, and F contains the union of these two paths and thus a s_j-t_j path. Therefore, F is indeed a solution to (G, S) and $\mathsf{sf}(G, S) = \mathsf{sf}(G_1, S_1) + \mathsf{sf}(G_2, S_2)$. ◀

The *contraction* of an edge $e = (u, v)$ in a graph G replaces u and v by a new vertex w that is adjacent to all former neighbours of u and v in G. A 2-*path* in G is a path whose internal vertices have degree exactly 2 in G. A 2-path is *maximal* if both its ends have degree not equal to 2 in G (so they have degree 1 or at least 3). In particular, the ends may have degree 1 or at least 3 in G.

For a 2-path P in G, the following branching lemma considers the case that: 1) the whole path P is in the solution; and 2) parts of P are in the solution. In the latter case, we only distinguish between the parts of P in the solution that are incident on the ends of P and the remaining 'middle' part, and branch into relevant cases for each possible pair of edges splitting P in this manner. The lemma shows how to compute these branches, and that they are sufficient to solve the original problem.

▶ **Lemma 8.** *Let (G, S) be an instance of* STEINER FOREST. *Let P be a 2-path in G. Let G_1 be obtained from G by contracting all edges of P and let G_2 be obtained from G by removing all edges of P and any resulting isolated vertices. Then we can compute in polynomial time set $S_1 \subseteq V(G_1) \times V(G_1)$ and integer w_1, and sets $S_2^{e,f} \subseteq V(G_2) \times V(G_2)$ and integers $w_2^{e,f}$ for $e, f \in E(P)$ such that*

$$\mathsf{sf}(G, S) = \min\{\mathsf{sf}(G_1, S_1) + w_1, \min_{e, f \in E(P)}\{\mathsf{sf}(G_2, S_2^{e,f}) + w_2^{e,f}\}\}.$$

Proof. We first show how to compute S_1, S_2 and w_1, w_2. Let u, v be the ends of P and let x be the super-vertex in G_1 that is obtained after the contraction of P. To obtain S_1, start with $S_1 = S$ and remove any terminal pair $(s, t) \in S_1$ for which $s, t \in V(P)$. Note that $V(P)$ includes the ends of P. Then, for any terminal $(s, t) \in S_1$ for which $s \in V(P)$ and $t \notin V(P)$, replace s by x. Finally, let $w_1 = |E(P)|$.

Now let $e, f \in E(P)$. We explicitly allow $e = f$. Without loss of generality, e is closer (on P) to u than f or $e = f$. Let u' be the endpoint of e closer (on P) to u and let v' be the endpoint of f closer (on P) to v. Note that possibly $u = u'$ or $v = v'$. To compute $w_2^{e,f}$, we proceed as follows. We will compute a graph $G^{e,f}$ that will be a subpath of P. If $e = f$, let $G^{e,f}$ be the empty graph. Otherwise, let $G^{e,f}$ be the subpath of P starting at the endpoint of e that is not u' and ending at the endpoint of f that is not v'. Let $P^{e,f}$ and $Q^{e,f}$ be the two subpaths of $P - V(G^{e,f})$. Note that these subpaths both exclude e and f, as does $G^{e,f}$. If there exists a terminal pair $(s, t) \in S$ for which $s \in V(G^{e,f})$ and $t \notin V(G^{e,f})$, then set $S^{e,f} = \emptyset$ and $w_2^{e,f} = \infty$. Otherwise, let $S^{e,f}$ be the set of terminal pairs $(s, t) \in S$ for which $s, t \in V(G^{e,f})$ and let $w_2^{e,f} = \mathsf{sf}(G^{e,f}, S^{e,f}) + |E(P^{e,f})| + |E(Q^{e,f})|$. Since $G^{e,f}$ is a path, $\mathsf{sf}(G^{e,f}, S^{e,f})$ can be computed in polynomial time via Theorem 2. Finally, to obtain $S_2^{e,f}$, start with $S_2^{e,f} = S$ and remove any terminal pair $(s, t) \in S_2^{e,f}$ for which one of $s, t \in V(G^{e,f})$. Then, for any terminal $(s, t) \in S_2^{e,f}$ for which $s \in V(P^{e,f}) \cup V(Q^{e,f})$, replace s by u if $s \in V(P^{e,f})$ and by v otherwise. Then do the same with respect to t.

We now prove the lemma statement for the constructed S_1, S_2 and w_1, w_2. Let F be a minimum Steiner forest for (G, S). Suppose that $E(P) \subseteq E(F)$. Then let $F' = ((V(F) - V(P)) \cup \{x\}, E(F) \setminus E(P))$. Clearly, F' is a Steiner forest for (G_1, S_1): edge contractions in a forest cannot create a cycle and any two terminals connected via P are now connected via x. Hence, $\mathsf{sf}(G, S) \geq \mathsf{sf}(G_1, S_1) + w_1$. Suppose that $E(P) \nsubseteq E(F)$. Let e' be the edge of P that is not in F and is closest (on P) to u. Let f' be the edge of P that is not in F and is closest (on P) to v. Note that e', f' are properly defined and possibly $e' = f'$. By definition, $E(P^{e',f'}) \subseteq E(F)$ and $E(Q^{e',f'}) \subseteq E(F)$. Moreover, we can assume that $|E(F) - E(P^{e',f'}) - E(Q^{e',f'})| = \mathsf{sf}(G^{e,f}, S^{e,f})$, or we could replace $E(F) - E(P^{e',f'}) - E(Q^{e',f'})$ by any minimum Steiner forest for $(G^{e,f}, S^{e,f})$ to obtain a smaller Steiner forest for (G, S), a contradiction. It follows that $\mathsf{sf}(G, S) \geq \mathsf{sf}(G_2^{e,f}, S_2^{e,f}) + w_2^{e,f}$.

Conversely, suppose that the minimum is achieved by $\mathsf{sf}(G_1, S_1) + w_1$ and let F_1 be a minimum Steiner forest for (G_1, S_1). Let $F = (F_1 - x) \cup P = ((V(F_1) \setminus \{x\}) \cup V(P), E(F_1) \cup E(P))$; we only remove x if $x \in V(F_1)$, which is not necessarily the case. Observe that F is indeed a forest. Moreover, F is a Steiner forest for (G, S), because any terminal pair with both terminals in P is satisfied by P, any terminal pair with exactly one terminal in P is satisfied due to x being the replacement of the terminal by x in the construction of S_1, and for any other terminal pair, any path between them that uses x can be expanded using P. As F_1 and P are edge-disjoint, $\mathsf{sf}(G, S) \leq \mathsf{sf}(G_1, S_1) + w_1$.

Suppose that the minimum is achieved by $e', f' \in E(P)$ and $\mathsf{sf}(G_2^{e',f'}, S_2^{e',f'}) + w_2^{e',f'}$ and let F_2 be a minimum Steiner forest for $(G_2^{e',f'}, S_2^{e',f'})$. Let F' be a minimum Steiner forest for $G^{e',f'}$, which can be found in polynomial time via Theorem 2. Let $F = F_2 \cup F' \cup P^{e',f'} \cup Q^{e',f'} = (V(F_2) \cup V(F') \cup V(P^{e',f'}) \cup V(Q^{e',f'}), E(F_2) \cup E(F') \cup E(P^{e',f'}) \cup E(Q^{e',f'}))$. Observe that F is indeed a forest. Moreover, F is a Steiner forest for (G, S), because any terminal pair with both terminals in $G^{e',f'}$ is satisfied by F', any terminal pair with both terminals in $P^{e',f'}$ ($Q^{e',f'}$) is satisfied by $P^{e',f'}$ ($Q^{e',f'}$), any terminal pair with exactly one terminal in $P^{e',f'}$ ($Q^{e',f'}$) is satisfied by u (v) being the replacement terminal and $P^{e',f'} \cup F_2$ ($P^{e',f'} \cup F_2$), any terminal pair with exactly one terminal in $P^{e',f'}$ and exactly one in $Q^{e',f'}$ is satisfied by the combination of both, and any other terminal pair is satisfied by F_2. As F_2, F', $P^{e',f'}$, and $Q^{e',f'}$ are edge-disjoint, $\mathsf{sf}(G, S) \leq \mathsf{sf}(G_2, S_2^{e',f'}) + w_2^{e',f'}$.

From the above, $\mathsf{sf}(G, S) = \min\{\mathsf{sf}(G_1, S_1) + w_1, \min_{e,f \in E(P)}\{\mathsf{sf}(G_2, S_2^{e,f}) + w_2^{e,f}\}\}$. ◄

Using the above lemma, we can obtain the following result.

▶ **Lemma 9.** STEINER FOREST *is polynomial-time solvable for graphs with a bounded number of maximal 2-paths between vertices of degree more than* 2.

Proof. Let (G_0, S_0) be an instance of STEINER FOREST where G_0 has k_0 maximal 2-paths between vertices of degree more than 2. By Lemma 7, we may assume that G_0 has no vertices of degree 1. This reduction does not increase the number of vertices of degree more than 2 nor the number of 2-paths between them.

We apply the following branching algorithm on an instance (G, S) of STEINER FOREST. Let P be a maximal 2-path in G of which both ends have degree more than 2. Apply the construction of Lemma 8 to P. We compute $\mathsf{sf}(G_1, S_1)$ and w_1 and for all $e, f \in E(P)$, we compute $\mathsf{sf}(G_2^{e,f})$ and $w_2^{e,f}$. Then we return $\min\{\mathsf{sf}(G_1, S_1) + w_1, \min_{e,f \in E(P)}\{\mathsf{sf}(G_2, S_2^{e,f}) + w_2^{e,f}\}\}$. If G has no such path P, then G is a forest of subdivided stars and we can find a minimum Steiner forest using Theorem 2.

The correctness of the above algorithm is immediate from Lemma 8 and Theorem 2. It remains to analyze the running time of this algorithm when called on (G_0, S_0). In the branching algorithm, both in G_1 and in $G_2^{\cdot,\cdot}$, the number k of maximal 2-paths between vertices of degree more than 2 decreases by 1. Since the base of the branching, the time spent to determine the branches, and the number of branches in each step are all polynomial, the running time of the algorithm is $n^{O(k)}$. ◄

4 Vertex Covers and 2-Deletions Sets

In this section, we consider STEINER FOREST on graphs with a small deletion set. Recall that the 1-deletion number equals the vertex cover number. Here is our first result:

▶ **Theorem 10.** STEINER FOREST *is* FPT *by the vertex cover number of the input graph.*

Proof. Consider an instance of STEINER FOREST with a given vertex cover. That is, a graph $G = (V, E)$, a set $S \subseteq V \times V$ of terminal pairs, and a vertex cover C of G. We show that we can compute a Steiner forest for (G, S) of minimum size in time $2^{O(|C| \log |C|)} n^{O(1)}$. Here, size is measured in number of edges. Note that if the vertex cover is not given as part of the input, then we can compute one of size equal to the vertex cover number k in $2^{O(k)} n^{O(1)}$ time (see e.g. [11]). Let $R := V \setminus C$.

Throughout this description, we assume two things. First, no vertex of C is a terminal vertex. Indeed, if $v \in C$ is a terminal vertex, then we create a vertex v' only adjacent to v and move all terminals on v to v'. Note that C is still a vertex cover of the resulting graph, the resulting instance is feasible if and only if the original is, and the size of any solution is increased by exactly 1. Second, we assume that the set S is *transitive*, i.e., if $(u, v) \in S$ and $(v, w) \in S$, then $(u, w) \in S$ for any $u, v, w \in V$. This means that the terminals can be grouped, in what we call *schools*. Any school must be part of the same connected component of any Steiner forest for (G, S). Let $h - 1$ be the number of schools and let $\{L_1, \ldots, L_{h-1}\}$ be the set of schools. Note that $L_1, \ldots, L_{h-1}$ are disjoint. Let $L_h = R - \bigcup_{i=1}^{h} L_i$; by abuse of terminology, we also call L_h a school. Then $L_1, \ldots, L_h$ form a partition of R.

We use the following notation. Let F be any forest in G. Let $I = \{I_1, \ldots, I_\ell\}$ be the set of connected components of $F - R$; in other words, this is the set of connected components induced by the edges of F between vertices of C. Note that some of these connected components may be singletons, that not all vertices of C have to appear in a connected component, and that $\ell \leq |C|$. We call $I_1, \ldots, I_\ell$ the *islands* of F. Note that several islands of F may belong to the same connected component of F. For any connected component of F, we call the set of islands that are part of that component its *archipelago*. Clearly, F may have multiple archipelagos, denoted $A = \{A_1, \ldots, A_p\}$. We interpret A as a partition of $C - I_0$, where I_0 is the set of vertices of C not in any of $I_1, \ldots, I_\ell$.

In the remainder, let F^* be an arbitrary Steiner forest for (G, S) of minimum size. Let $I^* = \{I_1^*, \ldots, I_{\ell^*}^*\}$ be the islands of F^* and let I_0^* denote the set of vertices of C not in any of $I_1^*, \ldots, I_{\ell^*}^*$. Let $A^* = \{A_1^*, \ldots, A_{p^*}^*\}$ denote the set of archipelagos of F^*.

We first branch on all $2^{O(|C| \log |C|)}$ partitions of C. Then one of them has to be equal to $A^* \cup \{I_0^*\}$. Consider one such branch, letting I_0 be one part and $A = \{A_1, \ldots, A_p\}$ the set of remaining parts. That is, A is a partition of $C \setminus I_0$ (this effectively adds $p + 1$ further branches). We now aim to find a minimum-size Steiner forest F for (G, S) such that A are the archipelagos of F. For any such F, $|E(F)| \geq |E(F^*)|$. If the pair A, I_0 is equal to the pair A^*, I_0^*, then F^* is a possible solution and thus $|E(F)| = |E(F^*)|$.

The crux to computing F is to determine which schools connect to which archipelagos and in particular, whether they help in connecting some islands to form an archipelago. To this end, we perform a dynamic program. We first introduce some notation. For any two partitions $X = \{X_1, \ldots, X_p\}$ and $Y = \{Y_1, \ldots, Y_q\}$ of the same set, we use the notation $Y \sqsubseteq X$ to denote that for any $i \in \{1, \ldots, q\}$, $Y_i \subseteq X_j$ for some $j \in \{1, \ldots, p\}$. Note that this implies that $q \geq p$. We also say that $Y_{i_1}, \ldots, Y_{i_r}$ *build* X_j for some $i_1, \ldots, i_r \in \{1, \ldots, q\}$ and some some $j \in \{1, \ldots, p\}$ if $Y_{i_1} \cup \cdots \cup Y_{i_r} = X_j$.

Recall that I_0 and $A = \{A_1, \ldots, A_p\}$ are fixed by the branch we are in. For any partition $B = \{B_1, \ldots, B_q\}$ of $C \setminus I_0$ such that $B \sqsubseteq A$ and any integer $s \in \{0, 1, \ldots, h\}$, let $T[B, s]$ be the minimum size of any forest F' for G, such that the archipelagos of F' are exactly B and for any L_i with $i \leq s$, there exists an A_j for $j \in \{1, \ldots, p\}$ such that any terminal vertex in L_i has a neighbour in A_j in F'. We are looking to compute $T[A, h]$. After $T[B, h-1]$ has been computed, all schools have already been connected to parts of A. However, the vertices of L_h can still be used to connect archipelagos of B to form A. The school L_h is used for

this purpose. If the pair A, I_0 is equal to the pair A^*, I_0^*, then $T[A, h] = |E(F^*)|$.

We start with the base case: $T[B, 0]$ for some partition $B = \{B_1, \ldots, B_q\}$ of $C \setminus I_0$ such that $B \sqsubseteq A$. Let $T[B, 0]$ be equal to the sum over all $i \in \{1, \ldots, q\}$ of the size of a spanning tree of $G[B_i]$. If $G[B_i]$ has no spanning tree, we take the size to be ∞. This means that B is the set of islands of the (partial) solution. Note that $T[B, 0] = \infty$ if $G[B_i]$ is not connected for some i.

Let $s \in \{1, \ldots, h\}$. We describe the subproblem optimality property. If the pair A, I_0 is equal to the pair A^*, I_0^* and F'_* is an optimal solution to the subproblem indicated by $T[B, s]$, then let L' be the set of edges of F'_* incident on L_s. Then $F'_* - L'$ is a forest such that for any L_i with $i \leq s - 1$, there exists an A_j for $j \in \{1, \ldots, p\}$ such that any vertex in L_i has a neighbour in A_j in $F'_* - L'$. Let B' denote the archipelagos of $F'_* - L'$. Clearly, $B' \sqsubseteq B$ and by the preceding, $F'_* - L'$ is a possible solution to $T[B', s - 1]$. Moreover, the edges of L' connect the archipelagos described by B' to form B. Then any optimal solution to the subproblem indicated by $T[B', s - 1]$ union the edges of L' is again a solution for $T[B, s]$. The size of this union is no larger than F'_*, and thus optimal.

For the algorithm, we now compute $T[B, s]$ for some partition $B = \{B_1, \ldots, B_q\}$ of C such that $B \sqsubseteq A$. We consider each partition $B' = \{B'_1, \ldots, B'_{q'}\}$ of $C \setminus I_0$ such that: $B' \sqsubseteq B$; there is at most one part of A, say A_1, such that more parts of B' build A_1 than do parts of B (if this exists, called it a *distilled part*); and, for all other members A_j of A an equal number of parts of B' build A_j as do parts of B. We use $T[B', s - 1]$ to determine $T[B, s]$. Given B' and B, we compute how to connect parts of B' that build parts of B using vertices of the school L_s and to connect the remaining terminal vertices of L_s to vertices of a single part of A. We now formalize this idea.

Consider first the case that no part of A is distilled by B' and B. If $s = h$, then $T[B, s] = T[B', s - 1] = T[B, s - 1]$, because L_h has no terminal vertices by definition. If $s < h$, then all vertices of the school can connect to vertices of a single A_j for some $j \in \{1, \ldots, p\}$. We clearly pick one for which the added size is minimum. This can be found greedily. For every $j \in \{1, \ldots, p\}$, find a value g_j as follows. For every vertex $v \in L_s$, pick an edge to A_j and add its 1 to g_j; if no such edge exists, add ∞. Let $j' = \arg\min_{j \in \{1, \ldots, p\}} g_j$. If $g_{j'} = \infty$, then it is not possible to connect L_s via any single part of A, and we stop the dynamic program and proceed with the next branch for A and I_0. Otherwise, let $T[B, s] = g_{j'} + T[B', s - 1]$.

Consider the case that (without loss of generality) A_1 is the part of A that is distilled by B' and B. Suppose that $B_1, \ldots, B_z$ build A_1. For any $j \in \{1, \ldots, z\}$, suppose that $B'_{j_1}, \ldots, B'_{j_{z'}}$ build B_j (note that z' depends on j, but we drop this extra subscript for clarity when possible). If $z'_j > 1$, then the school L_s must connect $B'_{j_1}, \ldots, B'_{j_{z'}}$. We aim to compute a forest F'' such that: $E(F'') \subseteq E(G) \cap (L_s \times A_1)$; $L_s \subseteq V(F'')$; if $\hat{B}'$ are the archipelagos of a forest F''' of $G - L_s$, then B are the archipelagos of $F''' \cup F''$; there is no vertex $V(F'') \cap R$ incident in F'' on both a vertex of B_j and of $B_{j'}$ for any $j, j' \in \{1, \ldots, z\}$ with $j \neq j'$. Let F''_* be such an F'' of minimum size.

We now aim to 'guess' the pattern of how to connect parts of B' that build parts of B using vertices of the school L_s. The intuition of this pattern is as follows (see Figure 2). Suppose that an optimal solution F'_* to $T[B, s]$ is indeed formed by the union of F''_* and an optimal solution to $T[B', s - 1]$. Suppose we remove a vertex $v \in L_s$ from F'_* such that the number of archipelagos of the resulting forest is larger than of F'_*. Then some B_j that is an archipelago of F'_* is no longer an archipelago. Indeed, the sets $B'_{j_1}, \ldots, B'_{j_{z'}}$ that build B_j are now partitioned into a number of parts equal to the degree of v in F'_*; each of these parts forms an archipelago of the resulting forest. If we iteratively proceed, removing vertices of
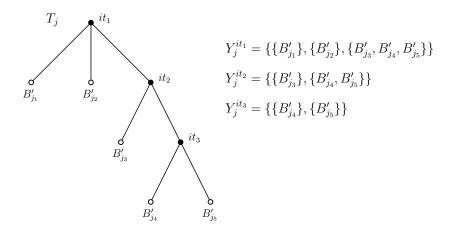
$$Y_j^{it_1} = \{\{B'_{j_1}\}, \{B'_{j_2}\}, \{B'_{j_3}, B'_{j_4}, B'_{j_5}\}\}$$

$$Y_j^{it_2} = \{\{B'_{j_3}\}, \{B'_{j_4}, B'_{j_5}\}\}$$

$$Y_j^{it_3} = \{\{B'_{j_4}\}, \{B'_{j_5}\}\}$$

■ **Figure 2** A possible pattern T_j where $z'_j = 5$. The tree has internal vertices it_1, it_2, it_3, and the associated B'_j are listed at the leafs. The sets Y_j^t are listed on the right, for all three of the internal tree vertices. Each internal vertex it corresponds to a vertex of L_s that connects the sets B'_j in Y_j^{it}. For example, it_2 corresponds to a vertex of L_s that has degree 2 in the solution: one edge to a vertex in B'_{j_3} and one edge to a vertex in B'_{j_4} or in B'_{j_5}. The perfect matching algorithm finds an optimal assignment of vertices in L_s to internal tree vertices of T_j, such that the pattern describes the connection structure between the sets B'_j.

L_s adjacent in F'_* to B_j such that the number of archipelagos increases, then we obtain a tree-like structure that is rooted. Each internal node of this tree (except possibly the root) has degree at least 3 and is associated with a partition of some subset of $\{B'_{j_1}, \ldots, B'_{j_{z'}}\}$; this subset is a part of the partition of its parent. The leafs of the tree are $B'_{j_1}, \ldots, B'_{j_{z'}}$. We seek this tree and its associated partitions.

We now turn the above intuition into a branching process (again, refer to Figure 2). For any $j \in \{1, \ldots, z\}$, let T_j be any rooted tree on z'_j labeled leafs $t_1, \ldots, t_{j_{z'}}$, where the leafs are identified with the sets $B'_{j_1}, \ldots, B'_{j_{z'}}$, such that all internal vertices (except possibly the root) have degree at least 3. For any internal vertex t of T_j, let Y_j^t be the set of, for each child t' of t, the subset of $\{B'_{j_1}, \ldots, B'_{j_{z'}}\}$ for which the corresponding leafs are descendants of t' in T_j (here, t' is a descendant of itself). Observe that this set is a partition of a subset of $\{B'_{j_1}, \ldots, B'_{j_{z'}}\}$. Let $Y_j = \bigcup_t \{Y_j^t\}$. We branch over all possible Y_j for all $j \in \{1, \ldots, z\}$. Note that an optimal such set Y_j^* and tree T_j^* may be obtained by the iterative process described above. Using Cayley's formula [7, 10], we note that there are $2^{O(|C| \log |C|)}$ branches.

We define some helpful variables. Let Γ be the set of indices $j \in \{1, \ldots, z\}$ for which $z'_j > 1$. Let $\beta = \sum_{j \in \Gamma} |Y_j|$; this is the number of distinct vertices of L_s that are needed to connect the parts of B' that build parts of B according to the pattern prescribed by the Y_j. If β is larger than $|L_s|$, then proceed to the next branch, as then there are not sufficient vertices in L_s to do this. Let σ be a bijection between $\{1, \ldots, \beta\}$ and the $\bigcup_{j \in \Gamma} Y_j$ so that we may speak of $Y^{\sigma(i)}$ to refer to the members of the Y_j. That is, each $Y^{\sigma(i)}$ corresponds to Y_j^t for some j and some internal vertex of T_j; in particular, it is a partition of a subset of $\{B'_{j_1}, \ldots, B'_{j_{z'}}\}$.

To create the assignment of vertices of L_s to the pattern, we construct an instance of minimum-weight perfect matching on an auxiliary bipartite graph H. Let $\alpha = |L_s|$. Create vertices $u_1, \ldots, u_\alpha$ (we identify these with the vertices in L_s); these form one side of the bipartition. Add vertices $v_1, \ldots, v_{\alpha-\beta}$ and vertices $x_1, \ldots, x_\beta$; these form the other side of the bipartition. Consider any $i \in \{1, \ldots, \beta\}$. For any u_k with $k \in \{1, \ldots, \alpha\}$ such that u_k is adjacent in G to a vertex in each of the sets $\{B'_{i_1}, \ldots, B'_{i_l}\}$ that build $Y^{\sigma(i)}$, add an edge

to H between u_k and x_i of weight equal to $|Y^{\sigma(i)}|$. This represents the cost of using u_k to connect the parts of the partition $Y^{\sigma(i)}$, thus helping to build the archipelagos. Finally, for each $k \in \{1, \dots, \alpha\}$ and each $i \in \{1, \dots, \alpha - \beta\}$, if u_k is adjacent in G to a vertex of A_1, then add an edge between u_k and v_i to H of weight equal to either 1 if $s < h$ and 0 otherwise. This represents the cost of directly connecting u_k to A_1.

Now find a minimum-weight perfect matching in H. This corresponds to a forest F''' that connects each of the parts of the Y_j for each $j \in \Gamma$ in a manner of minimum size. Hence, F''' is a candidate for a solution F''. We return the best solution we find over all branches. Since we encounter Y_j^* as a candidate branch, we find a solution of size at most the size of F_*''. Following all the previous arguments and the description of the algorithm, we return a minimum Steiner forest for (G, S).

To show that the running time of our algorithm is is $2^{O(|C| \log |C|)} n^{O(1)}$, note that there are $2^{O(|C| \log |C|)}$ candidates for I_0 and A and $2^{O(|C| \log |C|)}$ entries for the dynamic program. For each entry, we consider $2^{O(|C| \log |C|)}$ candidates for B' and branch on $2^{O(|C| \log |C|)}$ candidates for Y_j. Finding a minimum-weight perfect matching takes polynomial time in the size of the instance [14], which is $O(|L_s|^2)$. ◀

▶ Remark 11. The FPT algorithm extends straightforwardly to the setting with edge weights.

We also need the next result, which shows the polynomial part of Theorem 5.

▶ **Lemma 12.** STEINER FOREST *is polynomial-time solvable for graphs with a 2-deletion set of size at most 2.*

Proof. Let G be an n-vertex graph that, together with a set S of terminal pairs, is an instance of STEINER FOREST. If G has a 2-deletion set of size 1, then this vertex forms a cut vertex and we can apply Lemma 7 to reduce to case of graphs of size 2, which is trivial. So assume that G has a 2-deletion set C of size 2, say $C = \{u, v\}$. By Lemma 7 we may assume that G is 2-connected. If u and v are adjacent, then the problem is trivial. Indeed, either the edge uv is part of the solution and we may contract it to form a 2-deletion set of size 1, or the edge is not part of the solution and we may reduce to the case when u and v are not adjacent. Hence, we assume that u and v are not adjacent.

As G is 2-connected, there exists a path from u to v in G. As C is a 2-deletion set, every path from u to v has at most two inner vertices. We check every of the $O(n^2)$ paths from u to v (with at most two inner vertices). For each such a path P, it straightforward to find a minimum solution over all solutions that contain P (as we can contract the path P to a single vertex). We also compute a minimum solution over all solutions that contain u but not v, and a minimum solution over all solutions that contain v but not u. This takes polynomial time as well.

It remains to find a minimum solution over all solutions with two connected components, one containing u and the other one v, and to check this minimum solutions with the minimum solutions of the other types found above. As C is a 2-deletion set, any non-terminal vertex that is not adjacent to at least one of u, v will not be used in a minimum solution. Hence, we remove such non-terminal vertices from G. Moreover, we may restrict ourselves to solutions that contain at most three non-terminal vertices. Any solution with more than three non-terminal vertices has at least as many edges as a solution in which u and v are in the same connected component. Hence, we branch by considering all $O(n^3)$ options for the set of non-terminal vertices used in a minimum solution. We consider each of the $O(n^3)$ branches separately and as follows.

First we remove all non-terminal vertices that we did not guessed to be in the solution. For each of the at most three guessed non-terminal vertices we do as follows. By construction, such a guessed non-terminal vertex z is adjacent to u or v, and we contract the edge zu or zv, respectively. Afterwards, we apply Lemma 7 again such that we may assume that the resulting instance, which we denote by (G, S) again, is 2-connected. Now every vertex of $V(G) \setminus \{u, v\}$ is a terminal vertex. Hence, every vertex of $V(G) \setminus \{u, v\}$ will be in the solution we are trying to construct.

For every connected component D of $G - \{u, v\}$ we do as follows. As C is a 2-deletion set, D consists of exactly two vertices x and y. If x and y are both adjacent to both u and v, then we may remove the edge xy for the following reason. As x and y are terminal vertices, we need to connect x to either u or v, and we also need to connect y to either u or v. For doing this, we do not need to use the edge xy.

Now, suppose that one of x, y, say x, is adjacent to u and v, whereas the other one, y, is adjacent to only one of u and v, say to u. Again, as x and y are terminal vertices, we need to connect x to either u or v and we also need to connect y to either u or v. If we use xu, then we must also use yu, and in that case we can replace xu by xy. Hence, we may remove xu.

So, afterwards, we reduced the instance in polynomial time to a new instance, which we will also denote by (G, S), with the following properties. Every connected component of $G - \{u, v\}$ has at most two vertices. By 2-connectivity, for every connected component that contains exactly one vertex z we have the edges uz and vz. Moreover, for every connected component that contains exactly two vertices x and y, we have the edges ux and vy but not uy and not vx.

As every vertex in $V(G) \setminus \{u, v\}$ is a terminal vertex, we need one edge in the solution for each singleton connected component of $G - \{u, v\}$ and two edges for each two-vertex connected component. First suppose that one of u, v, say u, is not a terminal vertex. Then the solution that contains all vertices of $G - \{v\}$ is a minimum solution, so we can stop (note that we already found this solution before).

Now suppose that both u and v are terminal vertices. We discard the branch if u and v represent terminals of the same pair. Suppose this is not the case. If u represents s_i, then we connect the terminal vertex that represents t_i to u. This can only be done in one way: if the terminal vertex representing t_i is not adjacent to u, it contains a unique neighbour adjacent to u. Afterwards, we remove the terminal pairs that we connected in this way from G. If it was not possible to connect some terminal pair, then we discard the branch. Otherwise, we apply Lemma 7 again, such that we may assume that the resulting instance, which we denote by (G, S) again, is 2-connected. The other properties are maintained, and neither u nor v is a terminal vertex. Hence, we can take as solution the forest that has one connected component with vertex set $\{u\}$ and one connected component with vertex set $V(G) \setminus \{u\}$. ◀

For some of our results we need the following extension of Lemma 12.

▶ **Lemma 13.** STEINER FOREST *is polynomial-time solvable for graphs G with a set $X \subseteq E(G)$ of bounded size such that $G - X$ has a 2-deletion set C of size at most 2 and each end-point of every edge of X is either an isolated vertex of $G - X$ or a vertex of C.*

Proof. Let (G, S) be an instance of STEINER FOREST. Let X be a set that satisfies the conditions of the lemma. We apply the following branching algorithm recursively on the edges of X. Let $x \in X$. We use P to denote the 2-path consisting of x. Apply the construction of Lemma 8 to P. We compute $\mathsf{sf}(G_1, S_1)$ and w_1 and for all $e, f \in E(P)$, we compute $\mathsf{sf}(G_2^{e,f})$ and $w_2^{e,f}$. We return $\min\{\mathsf{sf}(G_1, S_1) + w_1, \min_{e,f \in E(P)}\{\mathsf{sf}(G_2, S_2^{e,f}) + w_2^{e,f}\}\}$. In each branch,

we have either contracted or deleted x. If every edge of X has been branched on, then C is still a 2-deletion set of size 2 in G. Hence, we can apply Lemma 12 to (G, S) respectively.

The correctness of the algorithm is immediate from Lemma 8 and Lemma 12. For the running time, note that we branch $|X|$ times in a constant number of options. Since Lemma 12 gives an algorithm with polynomial running time, the entire algorithm runs in polynomial time. ◀

5 Polynomial Cases

We start with a general lemma, using Theorem 10.

▶ **Lemma 14.** *For a graph H, if* STEINER FOREST *can be solved in polynomial time on the class of H-subgraph-free graphs, then* STEINER FOREST *can be solved in polynomial time on the class of $H + P_2$-subgraph-free graphs.*

Proof. Let (G, S) be an instance of STEINER FOREST such that G is $(H + P_2)$-subgraph-free. If G is H-subgraph-free, then STEINER FOREST can be solved in polynomial time by assumption. Hence, we may assume that G has a subgraph H' isomorphic to H. Then the connected components of $G - V(H')$ are P_2-subgraph-free and thus have size 1. Hence, H is a 1-deletion set and we apply Theorem 10. ◀

We now consider a number of graphs H. The first one is the case $H = 2K_{1,3}$.

▶ **Lemma 15.** STEINER FOREST *is polynomial-time solvable for $2K_{1,3}$-subgraph-free graphs.*

Proof. Let (G, S) be an instance of STEINER FOREST where G is $2K_{1,3}$-subgraph-free. First suppose that G is $K_{1,3}$-subgraph-free. Then, G has no vertices of degree 3 or more. Hence, G is a disjoint union of cycles and paths, and thus G has treewidth at most 2. We apply Theorem 2. Now suppose that G is not $K_{1,3}$-subgraph-free. We may assume that G is 2-connected by Lemma 7. Let v be a vertex of maximum degree. Since G is not $K_{1,3}$-subgraph-free, v has degree at least 3. Let v_1, v_2, v_3 be any three distinct neighbours of v. We call $A = \{v, v_1, v_2, v_3\}$ an *antares*.

We prove a sequence of claims about $2K_{1,3}$-subgraph-free graphs and antares.

▷ Claim 16. If v has degree more than 6, then every other vertex of G has degree at most 6.

Proof. For sake of contradiction, let $u, v \in V(G)$ both be vertices of degree more than 6. Pick three neighbours u_1, u_2, u_3 of u and three neighbours v_1, v_2, v_3 of v such that $u_1, u_2, u_3, v_1, v_2, v_3$ are distinct and not equal to u or v. Because $|(N(u)\backslash\{v\})\cup(N(v)\backslash\{u\})| \geq 6$ and $|N(u)|, |N(v)| \geq 7$, this is always possible. However, u, u_1, u_2, u_3 and v, v_1, v_2, v_3 form a subgraph isomorphic to $2K_{1,3}$ in G, a contradiction. ◁

▷ Claim 17. The graph $G - A$ has at most 15 connected components.

Proof. Since G is 2-connected, any component of $G - A$ has an edge to at least one of v_1, v_2, v_3. By Claim 16 and the fact that v has maximum degree in G, we find that v_1, v_2, v_3 each have degree at most 6. Recall that each of v_1, v_2, v_3 has an edge to v. Hence, $G - A$ has at most 15 connected components. ◁

▷ Claim 18. If v has degree more than 6, then $G - A$ is a union of paths. Moreover, only the ends of such a path can be adjacent to v_1, v_2, or v_3.

Proof. As $G - A$ is $K_{1,3}$-subgraph-free, $G - A$ is a disjoint union of cycles and paths. Suppose that $G - A$ has a connected component D that is a cycle. Since G is 2-connected, D is adjacent to at least one of v_1, v_2, v_3, say v_1. Let x be a vertex of D adjacent to v_1. By assumption and the definition of an antares, v has degree at least 7. Hence, it has three neighbours that are not v_1, x, or the two neighbours of x on D. Hence, G has a $2K_{1,3}$ as a subgraph, a contradiction. The same argument holds if D is a path and x is an internal vertex of this path adjacent to v_1, v_2, or v_3. ◁

We continue as follows. Firs suppose v (and thus any other vertex of G) has maximum degree at most 6. Then $G - A$ is $K_{1,3}$-subgraph-free, and all vertices of $G - A$ have degree at most 2. Hence, all vertices of degree more than 2 in G are in A or have an edge to a vertex of A. Hence, using the same argument as in Claim 17, there are at most $4 + (3 \cdot 5 + 3) = 22$ vertices of degree more than 2 in G. Each of them has degree at most 6, so there is a constant number of maximal 2-paths between them. We apply Lemma 9.

Now assume that v has degree more than 6. Let $A = \{v, v_1, v_2, v_3\}$ be an antares in G. We now apply a single but complex branching step. After this step, we will argue that all 2-connected components of the remaining graphs have treewidth at most 2 or have maximum degree at most 33. Note that in a 2-connected, $2K_{1,3}$-subgraph-free graph of maximum degree 33, it follows via an argument as in Claim 17 and a calculation as above that it has at most 49 vertices of degree more than 2. Each of them, except v, has degree at most 6 by Claim 16, so there is a constant number of maximal 2-paths between them. Then, by combining Theorem 2 and Lemma 9 with Lemma 7, we can solve each branch in polynomial time.

For the branching, we first observe that $G - A$ is a union of paths, $P_1, \ldots, P_a$, by Claim 18. By Claim 17, $a \le 15$. Moreover, by Claim 18, only the ends u_{P_i}, w_{P_i} of each such path P_i can be adjacent to v_1, v_2, or v_3. In any optimal solution F, the intersection of F and P_i consists of multiple subpaths $P_i^1, \ldots, P_i^{b_i}$ of P_i. We assume that these subpaths are numbered as they occur along P_i, where P_i^1 has its vertices closest to u_{P_i}. Note that possibly $b_i = 0$ and that possibly some subpaths consist of a single vertex. Observe that only P_i^1 and $P_i^{b_i}$ can be adjacent to v_1, v_2, v_3, if $u_{P_i} \in V(P_i^1)$ and $w_{P_i} \in V(P_i^{b_i})$ respectively. Moreover, there can be at most one edge between P_i^j and v in F. Denote these edges by $e_i^1, \ldots, e_i^{b_i}$, where e_i^j is the edge between P_i^j and v in F. Note that if some P_i^j is a connected component of the solution by itself, then e_i^j does not exist.

We now branch on what P_i^1, $P_i^{b_i}$, e_i^1, and $e_i^{b_i}$ are (if they exist). Note that P_i^1 and $P_i^{b_i}$ can be described by their ends; since $a \le 15$, the number of ends is at most 60 in total, which we choose among $O(n)$ vertices. Similarly, we need to choose at most 30 edges of the form e_i^1 and $e_i^{b_i}$ among $O(n)$ edges incident on v. Hence, the number of branches[2] is bounded by $O(n^{90})$. We explicitly allow the paths P_i^1 and $P_i^{b_i}$ to be empty (undefined) or to consist of a single vertex, and that the edges e_i^1 and $e_i^{b_i}$ are undefined.

Let Q_i^1, $Q_i^{b_i}$, f_i^1, and $f_i^{b_i}$ be the guessed paths and edges for all P_i in a branch B. Note that we do not guess the precise value of b_i, but use it as a placeholder here; we can have a branch in which $b_i = 0$ and Q_i^1, $Q_i^{b_i}$, f_i^1, and $f_i^{b_i}$ are undefined. We also discard a branch if f_i^1 is defined but not incident on a vertex of Q_i^1 and similarly if $f_i^{b_i}$ is defined but not incident on a vertex of $Q_i^{b_i}$. We apply substantial abuse of notation in the below, also if say $Q_i^1 = Q_i^{b_i}$ or $Q_i^1 = P_i$. However, this does not harm the overall argument.

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[2] This can be reduced to n^{60} by a more complex argument, but this is beyond the focus of our paper.

For each P_i, remove all edges incident both on v and on any vertex between u_{P_i} and the end of Q_i^1 that is furthest from u_{P_i}, except the edge f_i^1. Similarly, remove all edges incident both on v and on any vertex between w_{P_i} and the end of $Q_i^{b_i}$ that is furthest from w_{P_i}, except the edge $f_i^{b_i}$. Note that possibly $b_i = 0$, in which case all edges from v to P_i are removed. This is correct, because in any Steiner forest for (G, S) that contains Q_i^1 $(Q_i^{b_i})$ and f_i^1 $(f_i^{b_i})$ cannot also contain any of the edges that were removed, or either the forest would contain a cycle or Q_i^1 $(Q_i^{b_i})$ is not the first (last) path of solution restricted to P_i. If $b_i = 0$, then clearly any Steiner forest for (G, S) that does not contain any vertices of P_i cannot contain edges from v to P_i. In particular, this holds for the considered (but unknown) optimum solution F. Let G_B be the resulting graph for this branch. Note that G_B is a subgraph of G.

We claim that each 2-connected component of G_B has treewidth at most 2 or maximum degree 33. Let Q_i^m be the maximal subpath of P_i that is between and disjoint from Q_i^1 and $Q_i^{b_i}$. Note that Q_i^m does not exist if $b_i = 0$ or $V(Q_i^1) \cup V(Q_i^{b_i}) = V(P_i)$, but this does not harm the overall argument. Note that Q_i^m is only adjacent to v in G_B (and not to v_1, v_2, or v_3). Hence, they form a 2-connected component of G_B of treewidth at most 2. Any other 2-connected component of G_B contains a selection of v, v_1, v_2, v_3 and the vertices of the Q_i^1's and $Q_i^{b_i}$'s. However, each Q_i^1 and each $Q_i^{b_i}$ has at most one edge to v, and there are at most 30 of them. Hence, v has degree at most 33 in each such 2-connected component. The claim follows.

Note that if a 2-connected component has treewidth at most 2, the corresponding instance of STEINER FOREST can be solved in polynomial time by Theorem 2. If a 2-connected component has maximum degree 33, then using the same argument as in Claim 17, there are at most $4 + (3 \cdot 5 + 30) = 49$ vertices of degree more than 2 in G_B. Each of them, except v, has degree at most 6 by Claim 16, so there is a constant number of maximal 2-paths between them. Then the corresponding instance of STEINER FOREST can be solved in polynomial time by Lemma 9. Note that we can always reduce to 2-connected components by using Lemma 7.

Now it suffices to observe that there are $O(n^{90})$ branches and as argued above, in each branch B, the instance (G_B, S) can be solved in polynomial time. Hence, the entire algorithm runs in polynomial time. Since one of these branches corresponds to P_i^1, $P_i^{b_i}$, e_i^1, and $e_i^{b_i}$, the algorithm is correct. ◄

By using Lemma 13 (which generalized Lemma 12) we can extend Lemma 16 as follows.

▶ **Lemma 19.** STEINER FOREST *is polynomial-time solvable for* $(2K_{1,3} + P_3)$*-subgraph-free graphs.*

Proof. Let G be a $(2K_{1,3} + P_3)$-subgraph-free graph. If G is $2K_{1,3}$-subgraph-free, we use Lemma 15. Hence, we may assume G contains a $2K_{1,3}$. Let H be a subgraph of G isomorphic to $2K_{1,3}$.

Now every connected component of $G - H$ has size at most 2, as it is P_3-subgraph-free. If more than 12 such components are adjacent to the leafs of the $2K_{1,3}$'s, there must be a leaf of the $2K_{1,3}$ adjacent to at least 3 components. But then this leaf with those neighbours form a $K_{1,3}$, and the rest of its original $K_{1,3}$ in H is a P_3, and so the graph contains a $2K_{1,3} + P_3$. We conclude that at most 12 connected components of $G - H$ are adjacent to the leafs of the $2K_{1,3}$. As each of these components has size at most 2 and there are a constant number of them, the leafs of the $2K_{1,3}$ together with all components adjacent to them is in total a subgraph H' of constant size.

Observe that H' is only adjacent to the roots of the $2K_{1,3}$'s that make up H. Hence, a constant number of edges is incident on vertices of H'. The other components of $G - H$, each of size at most 2, are also only adjacent to these roots. Hence, the roots form a 2-deletion set of $G - E(H')$. Then, it follows that Steiner Forest can be solved in polynomial time by Lemma 13.   ◀

We now prove the case $H = S_{1,1,4}$.

▶ **Lemma 20.** Steiner Forest *is polynomial-time solvable for $S_{1,1,4}$-subgraph-free graphs.*

Proof. We prove that Steiner Forest is polynomial-time solvable for $S_{1,1,r}$-subgraph-free graphs for increasing r, $1 \leq r \leq 4$. In this proof, let (G, S) be an instance of Steiner Forest where G is a $S_{1,1,r}$-subgraph-free graph. By Lemma 7 we may assume that G is 2-connected.

Case 1. $r = 1$.
As $S_{1,1,1} = K_{1,3}$, we find that G is $K_{1,3}$-subgraph-free graph, so we can use Lemma 15.

Case 2. $r = 2$.
Assume G contains $S_{1,1,1}$ as a subgraph, else apply Case 1. Let s be the degree-3 vertex of the $S_{1,1,1}$ and let a, b, c be its degree-1 neighbours (these vertices may have higher degree in G). Any other vertex in G cannot be adjacent to any of a, b, c, as G is $S_{1,1,2}$-subgraph-free. If G has more than 4 vertices, then s is a cut-vertex, which contradicts the 2-connectivity of G. Hence, the instance is trivial.

Case 3. $r = 3$.
Assume G contains $S_{1,1,2}$ as a subgraph, else apply Case 2. Let s be the degree-3 vertex of the $S_{1,1,2}$ and let a_1, b, c be its neighbours and a_2 the neighbour of a_1 (these vertices may have higher degree in G). Consider the connected components of $G - \{s, a_1, b, c, a_2\}$. Let D be such a component containing an edge. Note that D cannot be adjacent to a_2 in G, as G is $S_{1,1,3}$-subgraph-free. Moreover, D cannot be adjacent to a_1, b, c in G, as G is $S_{1,1,3}$-subgraph-free and D contains an edge. But then s is a cut-vertex in G, which contradicts the 2-connectivity of G. Hence, no such component D with an edge exists, and all connected components of $G - \{s, a_1, b, c, a_2\}$ must be single vertices. But then G has a vertex cover of size 5, and we can solve Steiner Forest in polynomial time by Theorem 10.

Case 4. $r = 4$.
Assume G contains $S_{1,1,3}$ as a subgraph, else apply Case 3. Let s be the degree-3 vertex of the $S_{1,1,3}$, and let a_1, b, c be its neighbours and a_2 the neighbour of a_1, and a_3 the neighbour of a_2 (these vertices may have higher degree in G). Consider the connected components of $G' = G - \{s, a_1, b, c, a_2, a_3\}$.
   We prove the following claims.

▷ **Claim 21.**   Any connected component of G' of size at least 2 is not adjacent to a_2, a_3.

Proof. A connected component of size at least 2 contains an edge. Hence, if it is adjacent to either a_2 or a_3, there is a path of length 4 from s to the component, forming an $S_{1,1,4}$, a contradiction.   ◁

▷ **Claim 22.** b, c each are adjacent to at most one connected component of G'.

Proof. If say b is adjacent to more than one connected component of G', then $bsa_1a_2a_3$ is a length-4 path and forms a $S_{1,1,4}$ together with the adjacencies of the connected components, a contradiction.   ◁

▷ **Claim 23.** There are no two connected components D_1, D_2 of G' such that both are adjacent to a_1 and D_1 is adjacent to b or c.

Proof. Suppose there are two such components D_1, D_2. Let $d_1 \in V(D_1)$ and $d_2 \in V(D_2)$ be neighbors of a_1. If $d_1 b \in E(G)$, then $a_1 d_1 bsc$ is a length-4 path and a_1 has two other neighbours d_2, a_2, which together form a $S_{1,1,4}$, contradicting that G is $S_{1,1,4}$-subgraph-free. The argument straightforwardly extends to the case that b is only adjacent to the other vertex of D_1. ◁

▷ **Claim 24.** a_1 is not adjacent to at least two connected components D_1, D_2 of G', where $|V(D_1)| \geq 2$.

Proof. If there are two such components D_1, D_2, then by Claim 23, D_1 can only have edges to s other than having edges to a_1. By 2-connectivity of G, D_1 has an edge to s. If D_1 contains two vertices d_1^1, d_1^2 such that $(d_1^1, a_1), (d_1^2, s) \in E$, then G contains an $S_{1,1,4}$: $a_1 d_1^1 \ldots d_1^2 sb$ is at least a length-4 path, and a_1 has two other neighbours a_2, d_2. If D_1 contains no two such vertices, it contains a cut-vertex: there must be two adjacencies from D to a_1, s, but they cannot originate from different vertices of D_1. Hence, there cannot be two such connected components D_1, D_2 adjacent to a_1. ◁

▷ **Claim 25.** There are at most three connected components of G' of size at least 2: at most one adjacent to a_1, at most one adjacent to b, and at most one adjacent to c.

Proof. By Claim 24, a_1 is adjacent to at most one component of size at least 2. Hence, by Claim 21, all other components of size at least 2 are not adjacent to a_1, a_2, a_3. By 2-connectivity, any other component of G' is adjacent to at least two vertices of s, b, c. By the pigeonhole principle, if there are at least three other components of size at least 2, at least two are adjacent to either b or c, contradicting Claim 22. ◁

▷ **Claim 26.** Any connected component of G' has size at most 3.

Proof. Assume D is a component of G' of size at least 4. By Claim 21, D is not adjacent to either a_2 or a_3. By 2-connectivity and Claim 25, D is adjacent to exactly one of a_1, b, c. Now D must be a star with center d adjacent to that vertex among a_1, b, c, as otherwise D contains a P_3 attached with an endpoint to a_1, b, c which forms a length-4 path from s, and hence a $S_{1,1,4}$. If D is adjacent to b or c, say b, then $dbsa_1 a_2$ is a length-4 path and d has two other neighbours in D, forming a $S_{1,1,4}$. So $da_1 \in E(G)$. Note that D must have another neighbour among s, b, c by 2-connectivity and the origin f of this edge is not the center d of the star (otherwise, d is a cut vertex, contradicting 2-connectivity). But then $dfsa_1 a_2$, $dfbsa_1$, or $dfcsa_1$ is a length-4 path and d has two other neighbours in D as D is of size at least 4, which forms a $S_{1,1,4}$. Hence, in G', no connected component exists of size at least 4. ◁

By Claims 25 and 26 we know that in G' there are at most three components of size at least 2, and they have size at most 3. These components are adjacent only to s, a_1, b, c. All other components are adjacent only to s, a_1, b, c, a_2 and have size at most 1. It follows that s, a_1, b, c, a_2 together with the vertices of all components of size at least 2 form a vertex cover of constant size of the graph. We now apply Theorem 10. ◀

We use Lemma 20 to prove the case $H = P_9$. We also need Lemma 12 again.

▶ **Lemma 27.** STEINER FOREST *is polynomial-time solvable for P_9-subgraph-free graphs.*

Proof. Let $G = (V, E)$ be a P_9-subgraph-free graph that is part of an instance of STEINER FOREST. By Lemma 7 we may assume that G is 2-connected. Let $P = u_1 \cdots u_r$, for some $r \geq 2$, be a longest (not necessarily induced) path in G. As G is P_9-subgraph-free, we find that $r \leq 8$. If $r \leq 5$, then G is P_6-subgraph-free, and thus $S_{1,1,4}$-subgraph-free, and we can apply Lemma 20. Hence, $r \in \{6, 7, 8\}$.

Case 1. $r = 6$.
Then G is P_7-subgraph-free. Suppose $G - V(P)$ has a connected component D with more than one vertex. As G is P_7-subgraph-free and $|V(D)| \geq 2$, no vertex of D is adjacent to u_1, u_2, u_5 or u_6. As G is connected, at least one of u_3 or u_4, say u_3, has a neighbour v in D. Suppose $w \in V(D)$ is adjacent to u_4 (where $w = v$ is possible). Let vQw be a path from v to w in D; note that Q might be empty. Now the path $u_1u_2u_3vQwu_4u_5u_6$ has at least seven vertices, contradicting that G is P_7-subgraph-free. Hence, no vertex of D is adjacent to u_4. This means that u_3 is a cut-vertex of G, contradicting the 2-connectivity. We conclude that every connected component of $G - V(P)$ consists of one vertex. In other words, $\{u_1, \ldots, u_6\}$ is a vertex cover of G, and we can apply Theorem 10.

Case 2. $r = 7$.
Then G is P_8-subgraph-free. Suppose $G - V(P)$ has a connected component D with more than one vertex. As G is P_8-subgraph-free and $|V(D)| \geq 2$, no vertex of D is adjacent to u_1, u_2, u_6 or u_7. As G is connected, at least one of u_3, u_4 or u_5 has a neighbour v in D.

First assume that u_3 or u_5, say u_3, has a neighbour v in D. As $|V(D)| \geq 2$, we find that v has a neighbour w in D. If w has a neighbour x in D, then $xwvu_3u_4u_5u_6u_7$ is a path on eight vertices, contradicting that G is P_8-subgraph-free. Hence, v is the only neighbour of w in D. As G is 2-connected, this means that w has a neighbour on P. Recall that no vertex of D is not adjacent to u_1, u_2, u_6 or u_7. If w is adjacent to u_4, then $u_1u_2u_3vwu_4u_5u_6$ is a path on eight vertices. If w is adjacent to u_5, then $u_1u_2u_3vwu_5u_6u_7$ is a path on eight vertices. Hence, w must be adjacent to u_3 (and u_3 is the only neighbour of w on P). We now find that w is the only neighbour of v in D, as otherwise, if v has a neighbour $w' \neq w$ on D, then $w'vwu_3u_4u_5u_6u_7$ is a path on eight vertices. In other words, $V(D) = \{v, w\}$. By the same arguments, but now applied on v, we find that u_3 is the only neighbour of v on P. Hence, u_3 is a cut-vertex of G, contradicting the 2-connectivity of G.

From the above we conclude that no vertex of D is adjacent to u_3. By the same reason, no vertex of D is adjacent to u_5. We find that u_4 disconnects D from the rest of G, contradicting the 2-connectivity of G. We conclude that every connected component of $G - V(P)$ consists of one vertex. In other words, $\{u_1, \ldots, u_7\}$ is a vertex cover of G, and we can apply Theorem 10.

Case 3. $r = 8$.
Recall that G is P_9-subgraph-free, so this is the last case to consider. If every connected component of $G - V(P)$ consists of one vertex, then $\{u_1, \ldots, u_8\}$ is a vertex cover of G, and we can apply Theorem 10.

Now suppose that $G - V(P)$ has a connected component D with more than one vertex. As G is P_9-subgraph-free and $|V(D)| \geq 2$, no vertex of D is not adjacent to u_1, u_2, u_7 or u_8. As G is connected, at least one of u_3, u_4, u_5 or u_6 has a neighbour v in D.

First suppose that neither u_3 nor u_6 has a neighbour v in D. Then u_4 or u_5, say u_4, has a neighbour v in D. As G is 2-connected, there exists a path vQu_5 from v to u_5 that does not contain u_4. As no vertex from $\{u_1, u_2, u_3, u_6, u_7, u_8\}$ has a neighbour in D, the vertices of Q belong to D. Now, $u_1u_2u_3u_4vQu_5u_6u_7u_8$ is a path on nine vertices, contradicting that G is P_9-subgraph-free.

Hence, at least one of u_3 or u_6, say u_3, has a neighbour v in D. As $|V(D)| \geq 2$, we find

that v has a neighbour w in D. If w has a neighbour x in D, then $xwvu_3u_4u_5u_6u_7u_8$ is a path on nine vertices, contradicting that G is P_9-subgraph-free. Hence, v is the only neighbour of w in D. As G is 2-connected, this means that w has a neighbour on P. Recall that no vertex of D is not adjacent to u_1, u_2, u_7 or u_8. If w is adjacent to u_4, then $u_1u_2u_3vwu_4u_5u_6u_7$ is a path on nine vertices. If w is adjacent to u_5, then $u_1u_2u_3vwu_5u_6u_7u_8$ is a path on nine vertices. Hence, w must be adjacent to either or both u_3 and u_6 (and w has no other neighbours on P). We now find that w is the only neighbour of v in D for the following reason. Suppose v has a neighbour $w' \neq w$ on D. If w is adjacent to u_3, then $w'vwu_3u_4u_5u_6u_7u_8$ is a path on nine vertices. If w is adjacent to u_6, then $w'vwu_6u_5u_4u_3u_2u_1$ is a path on nine vertices. In other words, $V(D) = \{v, w\}$. By the same arguments, but now applied on v, we find that apart from u_3, it holds that v may have only u_6 as a neighbour of P.

Note that we have proven for any path $Q = q_1 \ldots q_8$ on eight vertices that every connected component of size at least 2 in $G - Q$ has size exactly 2, and moreover, q_3 and q_6 are the only vertices of Q with neighbours in such a connected component. By replacing P with $P' = u_1u_2u_3vwu_6u_7u_8$ we find that u_4 and u_5 have no neighbours outside P. We may assume without loss of generality that v is adjacent to u_3 and w is adjacent to u_6, as otherwise one of u_3, u_6, v, w is a cut-vertex of G, contradicting the 2-connectivity of G. By replacing P with $P' = u_5u_4u_3vwu_6u_7u_8$, we find that u_2 has no neighbours outside P (just like u_1). By symmetry, u_7 has no neighbours outside P (just like u_8). Hence, every connected component of $G - \{u_3, u_6\}$ has at most two vertices. In other words, $\{u_3, u_6\}$ is a 2-deletion set of size at most 2, and we can apply Lemma 12. ◀

We use Lemma 27 to show the case $H = 2P_4 + P_3$.

▶ **Lemma 28.** STEINER FOREST *is polynomial-time solvable for $(2P_4 + P_3)$-subgraph-free graphs.*

Proof. Let G be a $(2P_4 + P_3)$-subgraph-free graph. By Lemma 7 we may assume that G is 2-connected. If G is P_9-subgraph-free, we can solve STEINER FOREST in polynomial time by Lemma 27. Hence, we may assume G contains a P_9, say $P = (u_1, \ldots, u_9)$, as a subgraph. As a P_9 contains a $2P_4$, every connected component in $G - P$ is P_3-subgraph-free and hence of size at most 2.

Let $D = (d_1, d_2)$ be a component of size exactly 2 in $G - P$. Note that D cannot be adjacent to either u_1 or u_2, as then D and u_1, u_2 form a P_4, and $u_3u_4u_5u_6$ form a P_4, and $u_7u_8u_9$ form a P_3. Analogously, D cannot be adjacent to either u_8 or u_9. If D is adjacent to u_4 or u_6, D together with the adjacency and u_5 forms a P_4, and the rest of P contains a $P_4 + P_3$. Similarly, if D is adjacent to u_5, it forms a P_4 with u_5, u_6, and $u_7u_8u_9$ is a P_3 and $u_1u_2u_3u_4$ is a P_4. Hence, D can only be adjacent to u_3 or u_7.

If one of d_1, d_2 has u_3 as a neighbour, say d_1, and the other has u_7 as a neighbour, then $u_1u_2u_3d_1 + u_4u_5u_6 + d_2u_7u_8u_9$ form a $2P_4 + P_3$. So, either d_1 and d_2 are adjacent to only one of u_3 and u_7, or only one of d_1, d_2 has neighbours on P. But then there exists a cut vertex, which contradicts G being 2-connected. We conclude that such a component D of size 2 cannot exist in G.

It follows that every component in $G - P$ is of size 1, and P forms a vertex cover of size 9. By Theorem 10 we can solve STEINER FOREST in polynomial time. ◀

Proof of polynomial part of Theorem 3. Combine Lemma 14 with each of the others lemmas in this section.

6   Conclusions

The aim of this paper was to increase our understanding of the complexity of Steiner Forest and more generally, C23-problems, that is, graph problems that are not only NP-complete on subcubic graphs (C2) and under edge devision of subcubic graphs (C3), but also NP-complete for graphs of bounded treewidth (not C1). We therefore studied Steiner Forest for H-subgraph-free graphs. We significantly narrowed the number of open cases, thereby proving a number of of boundary cases (as can be seen in Theorem 3). However, we were not able to generalize Lemma 14 from P_2 to P_3:

▶ **Open Problem 1.** *Let H be a graph. Is* Steiner Forest *polynomial-time solvable on $(H + P_3)$-subgraph-free graphs if it is so for H-subgraph-free graphs?*

An affirmative answer to this question would reduce the number of open cases in Theorem 3 to a finite number. However, this requires a polynomial-time algorithm for Steiner Forest on graphs with a 2-deletion set of size d for any constant d. The question if such an algorithm exists turned out to be highly challenging. One important attempt that we made to solve this question is to reduce instances to highly structured instances. In particular, we are able to reduce it to the case where the vertices of the deletion set itself belong to different connected components of a minimum Steiner forest, and all vertices not in the deletion set are terminals. However, even solving such highly structured instances seems difficult. We managed to reduce it to a Constraint Satisfaction Problem (CSP). Interestingly, this CSP can be solved in polynomial time for 2-deletion sets of size 2 (cf. Lemma 12). The same CSP is NP-complete when we consider deletion sets of size 3. Unfortunately, our reduction is only one way, so this does not directly imply NP-completeness of Steiner Forest in this case. Still, this hints that the problem might actually be NP-complete for $H = sP_3$ for some $s \geq 4$.

It would also be interesting to know whether we can reduce the running time of our FPT algorithm by the vertex cover number vc of a graph to $2^{O(\text{vc})}n^{O(1)}$.

Finally, the C123 problem Steiner Tree is also classified with respect to the induced subgraph relation: it is polynomial-time solvable for H-free graphs if $H \subseteq_i sP_1 + P_4$ for some $s \geq 0$ and NP-complete otherwise [5]. The hardness part of this result immediately carries over to Steiner Forest. However, we do not know the complexity of Steiner Forest on $(sP_1 + P_4)$-free graphs.

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